WELL-POSEDNESS AND SMOOTH DEPENDENCE FOR A SEMILINEAR HYPERBOLIC SYSTEM WITH NONSMOOTH DATA [3]

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Abstract. We prove the existence, uniqueness, regularity and smooth dependence of the weak solution on the initial data for a certain class of semilinear first order dissipative hyperbolic systems with spacially discontinuous coefficients. Such kind of hyperbolic problems have succesfully been used to describe the dynamics of distributed feedback multisection semiconductor lasers in recent years. We show that in a suitable function space of continuous functions the weak solutions generate a smooth semiflow.

1. Introduction

We investigate a model for the dynamics of distributed feedback multisection semiconductor lasers based on the traveling wave equations. This is a semilinear hyperbolic system of partial differential equations. It consists of two coupled traveling wave equations describing a forward and backward propagating complex light wave which are nonlinearly coupled to a spacially parametrized ODE corresponding to carrier rate equations. Generally speaking this is a semilinear hyperbolic (degenerated) system with spacially discontinuous coefficients of the structure (2.1)-(2.3). The spacial discontinuities are natural and appear in the equations due to the significantly different electrical and optical properties of each section in a multisection semiconductor laser. Discontinuities with respect to time occur when the laser is subject to step-like forcings through optical injection. This kind of hyperbolic problem with discontinuous data has received remarkable attention in physics literature, see e.g. [3, 16, 20].

There are many examples of semilinear parabolic or hyperbolic equations *with smooth coefficients* where it is possible to reduce the dynamics onto a lower dimensional exponentially attractive smooth invariant manifold. In order to obtain a smooth system of reduced equations it is of basic importance to find an appropriate function space setting such that the solutions depend smoothly on the initial data and form a smooth semiflow. In contrast to the well understood parabolic problems no general method is known to construct a function space such that the solutions form a smooth semiflow. In our hyperbolic case, due to the lack of a smoothing property of the semigroup compensating possible discontinuities in the coefficients appearing in the nonlinear Nemytskij or superposition operators, we

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can not expect the smooth dependence of the weak or mild solution on the initial data in a Sobolev type function space. Here it is necessary to work in larger function spaces imposing less regularity. Now the problem arises that in large function spaces, for example the L^p scale for $1 \leq p < \infty$, nonlinear Nemytskij operators are not smooth [10], and in smaller function spaces, for example the classical Sobolev classes $W^{1,p}$, the operators will not even be defined due to the discontinuous coefficients. Therefore it is natural to study the weak solutions in the space of bounded measurable functions L^∞, where nonlinear superposition operators are both well defined and smooth, see [6].

In this work we will study the existence, uniqueness, regularity and smooth dependence of the weak solution in the space L^∞. After establishing the variation of constants formula we prove by means of the implicit function theorem that the L^∞ solutions depend smoothly on the initial data in Theorem 2.2. However, on L^∞ the class of C_0 semigroups is very restricted. It can be shown [12] that any C_0 semigroup on L^∞ has a bounded operator. The reason for this strong restriction on the generator is the continuity in 0. In particular any semigroup on L^∞ generated by an unbounded operator is discontinuous, i.e it is not C_0. As a consequence it is necessary to impose more regular initial data in order to obtain a smooth semiflow. In Theorem 2.6 we remark that in a function space of continuous functions the weak solutions form a smooth semiflow and process in the autonomous and nonautonomous case with dynamic boundary data, respectively. Theorem 2.3 and Theorem 2.5 state basic regularity results for the weak solutions and Theorem 2.4 contains a priori estimates. We note that the proofs of Theorems 2.2, 2.3, 2.6 also work for more general hyperbolic systems of arbitary size as long as a priori estimates are known which guarantee the boundedness of the solutions.
As a special case our work include the results of [8] even though our proofs are different since we avoid weak boundary homogenization and establish the variation of constants formula also in the case of inhomogeneous boundary data. We remark that the assumption that the boundary data is of bounded variation required in [8] is not necessary using our proofs.

Followig the ideas of [14, 17, 19] our results can be used to prove the existence of smooth normally hyperbolic exponentially attracting invariant manifolds of the model (5.1) using persistence theorems for *locally noncompact* invariant manifolds for smooth semiflows in Banach spaces obtained by Bates, Lu and Zeng in [2]. However, the normal hyperbolicity of the unperturbed manifold in the non-Hilbert space C is not obvious. The reason is that it is not known if the spectral mapping property is valid for hyperbolic PDEs where the semigroup does not possess a certain kind of regularity. The only result regarding the spectrum of hyperbolic systems strongly related to our type of equation seems to be the work [13] where L^p spaces are used for $1 \leq p < \infty$. In an upcoming article we will present resolvent estimates for such kind of hyperbolic systems. This will enable us to prove Theorem B in [13] also for the smaller admissible space C equipped with the L^∞ norm using the characterization of hyperbolicity for semigroups in Banach spaces in terms of the resolvent first obtained by Kaashoek and Lunel [9] and recently improved by Latushkin [11]. On these center manifolds the solutions become infinitely differentiable with respect to time t. All the interesting dynamics

and bifurcations which are observable in numerical and real world experiments with semiconductor lasers occur there [3, 15, 17, 18, 20].
In [14, 17, 18] the dynamics of a linearized and Galerkin projected version of our more general nonlinear model has been studied. In this simplified model the carrier rate equations have been transformed into an ODE after a very low dimensional Galerkin projection using Steklov average step functions. This linear averaged model neglects important physical effects such as longitudinal spacial hole burning, nonlinear gain compression or current redistribution. For applications in order to have not only a qualtitative, but also quantitative good agreement with real world measurements performed by engineers, it is important to work with the more advanced nonlinear model (5.1) which does not neglect these physical effects. In the linear model used in [14, 17, 18] the problem can be formulated in the usual manner as an abstract evolution equation of the form $\frac{d}{dt}u = Au + f(u(t))$ in the Hilbert space $V := L^2(]0, L[, \mathbb{C}^2) \times \mathbb{R}^m$ with $f \in C^\infty(V, V)$, where A is the generator of a C^0 semigroup $T(t)$, $t \in \mathbb{R}$, $t \geq 0$, on V. There the assumption that f is linear with respect to the optical variable, that is f operates linearly on L^2, is a crucial assumption guaranteeing the smoothness of the Nemytskij operator f. As a consequence the mild solutions, that is solutions to the integral equation $u(t) = T(t)u_0 + \int_0^t T(t-s)f(u(s))\,ds$, generate a smooth semiflow in the Hilbert space V. Also, since the resolvent of the linear hyperbolic operator A is bounded on vertical stripes, it follows by the Gearhart [5] and Herbst [7] theorem that the semigroup generated by A has the spectral mapping property in the Hilbert space L^2, this in particular implies the normal hyperbolicity of certain center manifolds corresponding to mode reduced equations. Hence, the previously assumed linearity was essential for the center manifold reduction performed in [14, 17, 18]. We note that in the general nonlinear setting considered here the flow will not be smooth in the Hilbert space V because any *nonlinear* superposition operator f is not smooth as a map from V into itself [10]. Our results here show that by chosing a different non Hilbert space we will obtain a smooth semiflow not only in the linear case. Thus the previously obtained results can be generalized to remain valid for the nonlinear and nonprojected model (5.1).

We have divided this work into four sections. First we state our general assumptions and results in Section 2. Then in Sec. 3 we establish the variation of constants formula for the weak solutions defined in Sec. 2 and prove our results for the problem with truncated nonlinearities. In Sec. 4 we show a priori estimates which are independend of the truncation parameter. This shows that all results hold for the original nontruncated problem. In the final Section 5 we introduce the model equations from the physics literature for which our theory applies.

The results presented in sections 1-4 have been obtained by the first author Mark Lichtner as part of his phd work.

2. Assumptions and Results

The system we consider is of the following form:

$$\left\{\begin{array}{lcl} \partial_t\psi(t,x) &=& (-\partial_x\psi_1(t,x),\partial_x\psi_2(t,x)) + G\left(x,\psi(t,x),n(t,x)\right) \\ \partial_t n(t,x) &=& I(t,x) + H\left(x,\psi(t,x),n(t,x)\right) \\ & & + \sum_{k=1}^m b_k\chi_{S_k}(x)\left(⨍_{S_k} n(t,y)\,dy - n(t,x)\right) \end{array}\right. \tag{2.1}$$

with the inhomogeneous, dynamic boundary conditions

$$\left\{\begin{array}{lcl} \psi_1(t,0) &=& r_0\psi_2(t,0) + \alpha(t) \\ \psi_2(t,L) &=& r_L\psi_1(t,L) \end{array}\right. \tag{2.2}$$

and the initial values

$$\psi(0,x) = \psi^0(x), \quad n(0,x) = n^0(x). \tag{2.3}$$

The function n is real valued, $\psi = (\psi_1,\psi_2)$ is $\mathbb{C}^2$ valued. They depend on the time $t \in \mathbb{R}$ and space variable $x \in [0,L]$. The interval $[0,L] = \cup_{k=1}^m \overline{S_k}$ is divided into m subsectional intervals $S_k :=]x_{k-1},x_k[$, $x_{k-1} < x_k$, $k = 1,...,m$. By χ_{S_k} we denote the characteristic function of S_k, that is $\chi_{S_k}(x) := 1$ for $x \in S_k$, $\chi_{S_k}(x) := 0$ if $x \notin S_k$. The symbol $⨍_{S_k} := \frac{1}{x_k - x_{k-1}}\int_{S_k}$ denotes the integral average on the subinterval S_k. The nonlinearities $G :]0,L[\times\mathbb{C}^2\times\mathbb{R} \to \mathbb{C}^2$ and $H :]0,L[\times\mathbb{C}^2\times\mathbb{R} \to \mathbb{R}$ are differentiable with respect to the phase variables (ψ,n), but only measurable and bounded with respect to the spacial variable $x \in [0,L]$. We now list the general assumptions required precisely and refer to Section 5 for an example from semiconductor laser dynamics fulfilling all our assumptions:

(I) The functions G and H are C^k-Carathéodory functions (see Definition 2.7) on $]0,L[$ from $\mathbb{C}^2\times\mathbb{R}$ into $\mathbb{C}^2$ and $\mathbb{R}$, respectively.

(II) There exist constants $0 < \nu_1 < \nu_2$ and $c_1,c_2,d_1,d_2 > 0$ such that for all $\psi \in \mathbb{C}^2$ and *a.a.* $x \in]0,L[$ the relations
$H(x,\psi,n) \geq -c_1 n$, if $n \leq \nu_1$,
$H(x,\psi,n) \leq -c_2 n$, if $n \geq \nu_2$,
$H(x,\psi,n) + d_1\mathfrak{Re}\,\langle G(x,\psi,n),\psi\rangle \leq -d_2\left(n + |\psi|^2\right)$ for all $n \in \mathbb{R}$
hold.

(III) For every compact $K \subset \mathbb{R}$ there exists $M > 0$ such that for all $n \in K$, $\psi \in \mathbb{C}^2$ and *a.a* $x \in]0,L[$ we have
$\left\|G(x,\psi,n)\right\| \leq M\left(\left\|\psi\right\| + 1\right)$.

(IV) $I \in L^\infty\left(]0,T[\times]0,L[,\mathbb{R}\right)$, $I(t,x) \geq 0$ for *a.a.* $(t,x) \in]0,T[\times]0,L[$.

(V) $\alpha \in L^\infty\left(]0,T[;\mathbb{C}\right)$.

(VI) $r_0, r_L \in \mathbb{C}$, $|r_0| < 1$, $|r_L| \leq 1$.

(VII) $n^0 \in L^\infty(]0,L[;\mathbb{R})$, $n^0(x) \geq 0$ for *a.a.* $x \in]0,L[$, $\psi^0 \in L^\infty(]0,L[;\mathbb{C}^2)$.

(VIII) $b_k \in \mathbb{R}, b_k \geq 0$ for $1 \leq k \leq m$.

We assume that $T > 0$ is arbitrarily chosen but fixed. The abbreviation "*a.a.*" stands for "almost all" in the sense of Lebesgue's measure, $\mathfrak{Re}$ denotes the real part of a complex number, $\langle\cdot,\cdot\rangle$ the canonical scalar product in $\mathbb{C}^2$ and $\|\cdot\|$ its corresponding norm.

Definition 2.1. A pair $(\psi, n) \in L^\infty\left(]0,T[\times]0,L[\,;\mathbb{C}^2\times\mathbb{R}\right)$ is a weak solution to (2.1), (2.2), (2.3) if

$$\begin{aligned}&\int_0^L \Big\langle \psi(t,x)-\psi^0(x),\varphi(x)\Big\rangle\,dx\\ &=\int_0^t\Bigg\{\int_0^L\Big[\psi_1(s,x)\overline{(\partial_x\varphi_1)(x)}-\psi_2(s,x)\overline{(\partial_x\varphi_2)(x)}\\ &\quad+\langle G(x,\psi(s,x),n(s,x)),\varphi(x)\rangle\Big]dx+\alpha(s)\overline{\varphi_1(0)}\Bigg\}ds\end{aligned}\tag{2.4}$$

for all $t\in[0,T]$ and all $\varphi\in W^{1,2}\left(]0,L[,\mathbb{C}^2\right)$ with $\varphi_2(0)=\overline{r_0}\varphi_1(0)$ and $\varphi_1(L)=\overline{r_L}\varphi_2(L)$ and if

$$\begin{aligned}n(t,x) \;=\; & n^0(x)+\int_0^t\Bigg\{I(s,x)+H(x,\psi(s,x),n(s,x))\\ &+\sum_{k=1}^m b_k\chi_{S_k}(x)\left[\fint_{S_k} n(s,y)\,dy-n(s,x)\right]\Bigg\}ds\end{aligned}\tag{2.5}$$

for all $t\in[0,T]$ and a.a. $x\in]0,L[$.

Theorem 2.2 (Existence, Uniqueness and smooth Dependence)**.** *Assume* (I) – (VIII)*. There exists a unique weak solution* (ψ,n) *to* (2.1), (2.2), (2.3)*. Moreover, the map*

$$\begin{aligned}&(\psi_0,n_0,I,\alpha)\in L^\infty\left(]0,L[\,;\mathbb{C}^2\times\mathbb{R}\right)\times L^\infty\left(]0,T[\times]0,L[\,,\mathbb{R}\right)\times L^\infty\left(]0,T[\,;\mathbb{C}\right)\\ &\mapsto(\psi,n)\in L^\infty\left(]0,T[\times]0,L[\,;\mathbb{C}^2\times\mathbb{R}\right)\end{aligned}$$

is C^k*-smooth.*

We denote the closed subspace in $L^\infty(]0,L[\,,\mathbb{R})$ of section-wise uniformly continuous functions

$$C_P:=\{n\in L^\infty(]0,L[;\mathbb{R})\mid n_{|S_k}\ \text{uniformly continuous for } k=1,2,\dots,m\}.$$

Theorem 2.3 (Solution Regularity I)**.** *Assume* (I) – (VIII)*. Let* (ψ,n) *be the weak solution. Then the following holds:*

ı) $\psi\in C\left([0,T]\,;L^2\left(]0,L[\,;\mathbb{C}^2\right)\right)$, $n\in W^{1,\infty}\left(]0,T[\,;L^\infty\left(]0,L[\,;\mathbb{R}\right)\right)$.

ıı) For $t\in[0,T]$ *denote* $\tilde\psi(t):=\int_0^t\psi(s)\,ds$*.*
Then for all $t\in[0,T]$ *we have* $\tilde\psi(t)\in W^{1,2}\left(]0,L[\,;\mathbb{C}^2\right)$ *and*

$$\tilde\psi_1(t)(0)=r_0\tilde\psi_2(t)(0)+\int_0^t\alpha(s)ds,\ \ \tilde\psi_2(t)(L)=r_L\tilde\psi_1(t)(L).$$

ıı) Let $\alpha\in W^{1,2}\left(]0,T[\,;\mathbb{C}\right)$, $\psi^0\in W^{1,2}(]0,L[;\mathbb{C}^2)$ *and suppose*

$$\psi_1^0(0)=r_0\psi_2^0(0)+\alpha(0),\ \ \psi_2^0(L)=r_L\psi_1^0(L).\tag{2.6}$$

Then

$$\psi\in C\left([0,T];W^{1,2}\left(]0,L[\,;\mathbb{C}^2\right)\right)\cap C^1\left([0,T]\,;L^2\left(]0,L[\,;\mathbb{C}^2\right)\right)$$

and (2.1), (2.2) *hold for* $t\in[0,T]$ *in the classical sense.*
If $I\in C\left([0,T];L^\infty(]0,T[\,;\mathbb{R})\right)$ *then* $n\in C^1([0,T];L^\infty(]0,L[;\mathbb{R}))$*.*

iv) Suppose $\psi^0 \in C\left([0,L];\mathbb{C}^2\right)$, $\alpha \in C([0,T];\mathbb{C})$ *and* (2.6). *Then*

$$\psi \in C\left([0,T]\times[0,L];\mathbb{C}^2\right) \text{ and (2.2) is satisfied pointwise.}$$

Further assume $n^0 \in C_P$, $I(t) \in C_P$ *for a.a.* $t \in [0,T]$ *and*

(IX) $$H(\cdot,\psi,n) \in C_P \text{ for all } \psi \in \mathbb{C}^2 \text{ and } n \in \mathbb{R}.$$

Then $n \in C([0,T];C_P)$. *If* $I \in C([0,T];C_P)$, *then* $n \in C^1([0,T];C_P)$.

Theorem 2.4 (A priori estimates)**.** *Suppose* (I) – (VIII). *Let* (ψ,n) *denote the weak solution.*

For all $t \in [0,T]$

$$\int_0^L n(t,x)dx + \frac{d_1}{2}\left\|\psi(t)\right\|_{L^2}^2 \le \mu + \max\left\{\int_0^L n^0(x)dx + \frac{d_1}{2}\left\|\psi^0\right\|_{L^2}^2 - \mu, 0\right\}e^{-ct}, \quad (2.7)$$

where

$$c := \min\left\{d_2, \frac{2d_2}{d_1}\right\}, \quad \mu := c^{-1}\left(\frac{d_1}{2(1-|r_0|^2)}\|\alpha\|_{L^\infty}^2 + L\|I\|_{L^\infty}\right).$$

Moreover, for all $t \in [0,T]$ *and a.a.* $x \in]0,L[$

$$\min\left\{n^0(x),\nu_1\right\}e^{-(c_1+b)t} \le n(t,x) \le N + \max\left\{n^0(x)-N,0\right\}e^{-c_2 t}, \quad (2.8)$$

where

$$N := \max\left\{\nu_2, c_2^{-1}\left(\|I\|_{L^\infty} + \max_{1\le k\le m}\left(\frac{b_k}{|S_k|}\right)\cdot\max\left\{\mu, \int_0^L n^0(x)dx + \frac{d_1}{2}\left\|\psi^0\right\|_{L^2}^2\right\}\right)\right\}$$

and

$$b := \max_{1\le k\le m}(b_k).$$

If the data ψ^0 and α are $W^{1,2}$-smooth, then Theorem 2.3, *iii*), states that the weak solution ψ will be $W^{1,2}$-smooth with respect to the spacial variable x. Of course, under natural assumptions of piecewise smoothness for the data entering the equation for n, this smoothness of ψ carries over to n via the coupling of ψ and n in (2.1). Theorem 2.5 states this precisely. Let

$$W_P^{1,2} := \left\{n \in L^\infty(]0,L[;\mathbb{R}) \mid n_{|S_k} \in W^{1,2}(S_k;\mathbb{R})\ k = 1,2,\dots,m\right\}$$

denote the Hilbert space of piecewise $W^{1,2}$ functions.

Theorem 2.5 (Solution Regularity II, piecewise smoothness of n)**.** *Suppose* (I)–(VIII) *and*

(X) $H_{|S_k\times\mathbb{C}^2\times\mathbb{R}} \in C^1\left(\overline{S_k}\times\mathbb{C}^2\times\mathbb{R};\mathbb{R}\right)$ *for* $1 \le k \le m$.

(XI) *For all compact* $K \subset \mathbb{R}$ *there exists* $\Lambda > 0$ *such that* $\left\|DH(x,\psi,n_1) - DH(x,\psi,n_2)\right\| \le \Lambda|n_1 - n_2|$ *for* $x \in S_k$, $\psi \in \mathbb{C}$ *and* $n_1, n_2 \in K$.

(XII) *There exists a constant* $\tau > 0$ *such that for all compact* $K \subset \mathbb{R}$ *there exists* $R > 0$ *with*

$$\partial_x H(x,\psi,n)\tilde{n} + \partial_n H(x,\psi,n)\tilde{n}^2 + \partial_\psi H(x,\psi,n)\tilde{\psi}\tilde{n} \le R\left(1 + |\tilde{n}| + \left\|\tilde{\psi}\right\| + \left\|\tilde{\psi}\right\||\tilde{n}| + \left\|\tilde{\psi}\right\|^2\right) - \tau\tilde{n}^2$$

for all $x \in S_k$, $1 \le k \le m$, $\psi \in \mathbb{C}^2$, $\tilde{\psi} \in \mathbb{C}^2$, $n \in K$ *and* $\tilde{n} \in \mathbb{R}$.

If $\alpha \in W^{1,2}(]0,T[;\mathbb{C})$, $\psi^0 \in W^{1,2}(]0,L[;\mathbb{C}^2)$, (2.6) *is satisfied,* $n^0 \in W_P^{1,2}$ *and* $I \in C\left([0,T];W_P^{1,2}\right)$, *then*

$$n \in C^1([0,T];W_P^{1,2}).$$

In (XI) the symbol DH denotes the total derivative of H with respect to all variables (x,ψ,n). We note that all assumptions (I) – (XII) are fulfilled in applications, see Section 5.

Define the phase space

$$\mathfrak{P} := \{\psi \in C([0,L],\mathbb{C}^2) \mid \psi_1(0) = r_0\psi_2(0), \psi_2(L) = r_L\psi_1(L)\} \times C_P.$$

Then we have the following simple consequence of Theorems 2.2-2.4:

Theorem 2.6 (C^k-Semiflow property)**.** *Suppose* (I) – (IX).
(i) *In the autonomous case, that is* $\alpha = 0$ *and* $I = 0$, *the weak solutions generate a smooth semiflow in the function space* $\mathfrak{P}$. *The operator* $S^t : \mathfrak{P} \to \mathfrak{P}$, *defined through*

$$S^t\left(\psi^0,n^0\right) := (\psi(t),n(t))$$

for $t \ge 0$ *and* $\left(\psi^0,n^0\right) \in \mathfrak{P}$, *where* $(\psi(t),n(t))$ *denotes the weak solution corresponding to the initial values* $\left(\psi^0,n^0\right)$, *has the following properties*

- *i)* $(t,\psi,n) \mapsto S^t(\psi,n)$ *is continuous from* $[0,\infty[\times\mathfrak{P}$ *into* $\mathfrak{P}$,
- *ii)* $S^t : \mathfrak{P} \to \mathfrak{P}$ *is* C^k *smooth,*
- *iii)* $S^{t+s} = S^t \circ S^s$, $t,s \in \mathbb{R}$, $t,s \ge 0$,
- *iv)* S^0 *is the identity operator on* $\mathfrak{P}$.

(ii) *Assume* $\alpha \in C(\mathbb{R};\mathbb{C})$ *and* $I \in L^\infty(\mathbb{R};C_P)$. *Let* $\mathcal{A} \in C(\mathbb{R};C([0,L];\mathbb{C}^2) \times C_P)$ *be such that* $\mathcal{A}$ *satisfies the inhomogeneous boundary condition* $\mathcal{A}(t)_1(0) = r_0\mathcal{A}(t)_2(0) + \alpha(t)$ *and* $\mathcal{A}(t)_2(L) = r_L\mathcal{A}(t)_1(L)$ *for* $t \in \mathbb{R}$. *Let* $X\left(t,s,\left(\psi^0,n^0\right)\right)$, $s \le t$, *denote the weak solution corresponding to the initial condition* $\psi(s,x) = \psi^0(x)$, $n(s,x) = n^0(x)$, *for a.a.* $x \in]0,L[$. *Then the operator* $Z(t,s) : \mathfrak{P} \to \mathfrak{P}$, *defined through*

$$Z(t,s)\left(\psi^0,n^0\right) := X\left(t,s,\left(\psi^0,n^0\right) + \mathcal{A}(s)\right) - \mathcal{A}(t)$$

for $t \ge s$ *and* $\left(\psi^0,n^0\right) \in \mathfrak{P}$, *is a* C^k *smooth two parameter nonautonomous process, that is*

- *i)* *for* $t \ge s$ *the map* $p \in \mathfrak{P} \mapsto Z(t,s,p) \in \mathfrak{P}$ *is* C^k *smooth,*
- *ii)* *the map* $(t,s,p) \mapsto Z(t,s,p)$ *is continuous from* $\{(t,s) \in \mathbb{R}^2 \mid s \le t\} \times \mathfrak{P}$ *into* $\mathfrak{P}$,
- *iii)* $Z(s,s,\cdot)$ *is the identity operator on* $\mathfrak{P}$,
- *iv)* *for* $t \ge s \ge r$ *the process property* $Z(t,s,Z(s,r,p)) = Z(t,r,p)$ *holds.*

The function $\mathcal{A}$ homogenizes the boundary condition (2.2). In applications one will have to choose it in an appropriate manner. In an upcoming paper we will give an example of a homogenization $\mathcal{A}$ suitable for a smooth center manifold reduction of (5.1) in the case of dynamic boundary conditions $\alpha \neq 0$ using Theorem 2.6.

In assumption (I) we require that both G and H are C^k-Carathéodory functions, which we define next.

Definition 2.7 (C^k-Carathéodory functions)**.** Let V, W be finite dimensional vector spaces and $k \in \mathbb{N}$. A function $S :]0, L[\times V \to W$, $S = S(x, v)$, $x \in]0, L[$, $v \in V$, is called a C^k Carathéodory function iff S satisfies the following three conditions:

i) For *a.a.* $x \in]0, L[$ $S(x, \cdot) \in C^k(V; W)$ and $S(\cdot, v)$ is measurable for all $v \in V$.
ii) For all compact $K \subset V$ there exists a constant $M > 0$ such that $\left\| \frac{\partial^i S(x,v)}{\partial v^i} \right\| \leq M$ for $0 \leq i \leq k$, all $v \in K$ and *a.a.* $x \in]0, L[$.
iii) For all compact $K \subset V$ and $\epsilon > 0$ there exists a $\delta > 0$ such that for all $v_1 \in K$, $v_2 \in V$ with $\|v_1 - v_2\| < \delta$ and *a.a.* $x \in]0, L[$ we have $\left\| \frac{\partial^k S(x,v_1)}{\partial v^k} - \frac{\partial^k S(x,v_2)}{\partial v^k} \right\| < \epsilon$.

3. Variation of constants formula and proofs for the truncated problem

Let $S :]0, L[\times V \to W$ be a C^k Carathéodory function. Denote the corresponding superposition operator

$$\mathfrak{S} : \mathcal{M}(]0, L[; V) \to \mathcal{M}(]0, L[; W), \ \mathfrak{S}(v)(x) := S(x, v(x)), \ a.a.\, x \in]0, L[, \tag{3.1}$$

where $\mathcal{M}(]0, L[; V)$ denotes the linear space of measurable functions defined almost everywhere on $]0, L[$ with values in V. We need the following easy to prove differentiability property of $\mathfrak{S}$.

Proposition 3.1. *(see* [6]*) The suporposition operator* $\mathfrak{S}$ *maps* $L^\infty(]0, L[; V)$ C^k*-smoothly into* $L^\infty(]0, L[; W)$*.*

In the following we will frequently make use of the superposition operators

$$\mathfrak{G} \in C^k(L^\infty(]0, L[; \mathbb{C}^2 \times \mathbb{R}), L^\infty(]0, L[; \mathbb{C}^2))$$
$$\mathfrak{H} \in C^k(L^\infty(]0, L[; \mathbb{C}^2 \times \mathbb{R}), L^\infty(]0, L[; \mathbb{R}))$$

generated by G and H through (3.1). Also the following operators

$$\mathfrak{B} \in \mathcal{L}(L^\infty(]0, L[; \mathbb{R})) , \ \mathfrak{I} \in L^\infty(]0, T[, L^\infty(]0, L[; \mathbb{R}))$$

will appear which are defined through

$$\begin{aligned} \mathfrak{B}(n)(x) &:= \sum_{k=1}^{m} b_k \chi_{S_k}(x) \left(\fint_{S_k} n(t, y)\, dy - n(t, x) \right) \text{ for a.a. } x \in]0, L[\\ \mathfrak{I}(t)(x) &:= I(t, x) \text{ for } a.a.\ x \in]0, L[. \end{aligned}$$

Here $\mathcal{L}(L^\infty(]0, L[; \mathbb{R}))$ denotes the space of bounded linear mappings of $L^\infty(]0, L[; \mathbb{R})$ into itself.

For establishing the variation of constants formula for our notion of weak solution we first need some definitions:

For $\eta \in \mathbb{R}$ let

$$L^2_\eta(]0, \infty[, \mathbb{C}) := \left\{ f :]0, \infty[\to \mathbb{C} \mid f \text{ measurable } \int_0^\infty |f(x)|^2 (1 + x^2)^\eta dx < \infty \right\}$$

denote the Hilbert space of complex valued weighted square integrable functions on $]0, \infty[$ with weight $(1 + x^2)^\eta$ with respect to the Lebesque measure on $]0, \infty[$. We denote its scalar product by $\langle f, g \rangle_{L^2_\eta} := \int_0^\infty f(x) \overline{g(x)} (1 + x^2)^\eta dx$. Let $W^{1,2}_\eta$ denote

the corresponding Sobolev space of functions $f \in L^2_\eta(]0,\infty[,\mathbb{C})$ with distributional derivative in $L^2_\eta(]0,\infty[,\mathbb{C})$. Define the extended space

$$X_e := L^2(]0,L[;\mathbb{C}^2) \times L^2(]0,L[;\mathbb{R}) \times L^2_\eta(]0,\infty[;\mathbb{C}) \tag{3.2}$$

with some fixed $\eta < -0.5$. This choice of η guarantees that $L^\infty(]0,\infty[;\mathbb{C})$ is continuously embedded in $L^2_\eta(]0,\infty[;\mathbb{C})$. Put

$$T_e(t)\left(\psi_1^0,\psi_2^0,n^0,a\right) := \left(\psi_1(t),\psi_2(t),n^0,\tau_t a\right),$$

where $\tau_t a(x) := a(t+x)$ denotes the left translation of a by t and ψ_1,ψ_2 are given by

$$\begin{aligned}
\psi_1(t,x) &:= \begin{cases} \psi_1^0(x-t) & , \text{ for } a.a.\ x \in]t,L[\\ r_0\psi_2^0(t-x)+a(t-x) & , \text{ for } a.a.\ x \in]0,t[\end{cases} \\
\psi_2(t,x) &:= \begin{cases} \psi_2^0(x+t) & , \text{ for } a.a.\ x \in]0,L-t[\\ r_L\psi_1^0(2L-x-t) & , \text{ for } a.a.\ x \in]L-t,L[\,. \end{cases}
\end{aligned} \tag{3.3}$$

Extend $T_e(t), t \in [0,L]$ to the whole positive axis $[0,\infty[$ by defining for $t > L$ inductively $T_e(t) := T_e(t-L)T_e(L)$. Then it is easy to verify that $T_e(\cdot)$ is a C_0 semigroup of bounded operators in X_e with infinitesimal generator

$$A_e := \operatorname{diag}(-\partial_x, \partial_x, 0, \partial_x)$$

having the domain

$$\begin{aligned}
\mathfrak{D}(A_e) := \{(\psi,n,a) \in W^{1,2}(]0,L[;\mathbb{C}^2) \times L^2(]0,L[;\mathbb{R}) \times W^{1,2}_\eta(]0,\infty[;\mathbb{C}) \mid \\
\psi_1(0) = r_0\psi_2(0) + a(0), \psi_2(L) = r_L\psi_1(L)\}.
\end{aligned}$$

Set

$$T(t)(\psi^0) := \Pi_\psi T_e(t)(\psi^0,0,0)$$

for $t \geq 0$ and $\psi \in L^2(]0,L[\,,\mathbb{C}^2)$, where Π_ψ is the projection onto the first variable ψ. Then $T(t)$ is a C_0 semigroup of contractions in $L^2(]0,L[\,,\mathbb{C}^2)$ with infinitesimal generator

$$A := \operatorname{diag}(-\partial_x, \partial_x)$$

and domain

$$\mathfrak{D}(A) := \left\{\psi \in W^{1,2}(]0,L[\,;\mathbb{C}^2) \mid \psi_1(0) = r_0\psi_2(0),\ \psi_2(L) = r_L\psi_1(L)\right\}.$$

Let $\prod_{(\psi,n)}$ denote the projection of X_e onto $L^2(]0,L[;\mathbb{C}^2 \times \mathbb{R})$ by dropping the trivial last component. Then the following Lemma holds

Lemma 3.2. *The pair (ψ,n) is a weak solution to* (2.1), (2.2), (2.3) *iff (ψ,n) satisfies the variation of constants formula*

$$\begin{pmatrix}\psi(t)\\ n(t)\end{pmatrix} = \prod_{(\psi,n)} T_e(t)\begin{pmatrix}\psi^0\\ n^0\\ \alpha\end{pmatrix} + \int_0^t \begin{pmatrix} T(t-s)\mathfrak{G}(\psi(s),n(s)) \\ \mathfrak{I}(s) + \mathfrak{B}n(s) + \mathfrak{H}(\psi(s),n(s))\end{pmatrix} ds \tag{3.4}$$

for all $t \in [0,T]$.

Proof. Straightforward calculations yield that the adjoint A_e^* of A_e is the closed densely defined operator

$$A_e^*(\psi,n,a) = (\partial_x\psi_1, -\partial_x\psi_2, 0, -(1+x^2)^{-\eta}\partial_x(a(x)\cdot(1+x^2)^\eta)) =: (A_\psi^*, 0, A_a^*)$$

with the domain

$$\begin{aligned}\mathfrak{D}(A_e^*) &= \Big\{(\psi,n,a)\in W^{1,2}(]0,L[;\mathbb{C}^2)\times L^2(]0,L[;\mathbb{R})\\ &\quad \times W_\eta^{1,2}(]0,\infty[;\mathbb{C})\mid \psi_2(0)=\overline{r_0}\psi_1(0),\psi_1(L)=\overline{r_L}\psi_2(L),a(0)=\psi_1(0)\Big\}.\end{aligned}$$

We trivially extend α on the whole axis $[0,\infty[$ by setting α to zero on $[T,\infty[$. Then define $a\in C([0,\infty[;L^2_\eta([0,\infty[;\mathbb{C}))$, $a(t):=\tau_t\alpha, t\in[0,\infty[$. By definition (ψ,n) is a weak solution iff $(\psi,n)\in L^\infty\left(]0,T[\times]0,L[\,;\mathbb{C}^2\times\mathbb{R}\right)$ and for all $(\varphi,0,\varphi_a)\in D(A_e^*)$ the equation

$$\begin{aligned}&\langle\psi(t)-\psi_0,\varphi\rangle_{L^2}+\langle a(t)-a(0),\varphi_a\rangle_{L^2_\eta}\\ =\;&\lim_{\rho\to0}\Big\{\int_0^t\Big(\langle\psi(s),A_\psi^*\varphi\rangle_{L^2}+\langle\mathfrak{G}(\psi(s),n(s)),\varphi\rangle_{L^2}+\alpha_\rho(s)\overline{\varphi_1(0)}\Big)ds\\ &+\int_0^t\langle(\partial_x\alpha_\rho)(s+\cdot),\varphi_a\rangle_{L^2_\eta}ds\Big\}\\ =\;&\lim_{\rho\to0}\Big\{\int_0^t\Big(\langle\psi(s),A_\psi^*\varphi\rangle_{L^2}+\langle\mathfrak{G}(\psi(s),n(s)),\varphi\rangle_{L^2}+\langle\alpha_\rho(s+\cdot),A_a^*\varphi_a\rangle_{L^2_\eta}\Big)ds\Big\}\\ =\;&\int_0^t\Big(\langle\psi(s),A_\psi^*\varphi\rangle_{L^2}+\langle\mathfrak{G}(\psi(s),n(s)),\varphi\rangle_{L^2}+\langle a(s),A_a^*\varphi_a\rangle_{L^2_\eta}\Big)ds\end{aligned}$$

holds and (2.5) is satisfied for n. Here

$$\alpha_\rho(x):=\int_0^T m_\rho(x-y)\alpha(y)dy,\quad m_\rho(y):=\frac{m_0(\rho y)}{\rho}\quad(x,y\in\mathbb{R})$$

denotes the mollification of α with parameter $\rho>0$ with respect to some mollifier $m_0\in C^\infty(\mathbb{R})$, $m_0\geq0$, supp $m_0\subset B_1$, $\int_{-\infty}^{\infty}m_0(y)dy=1$. It was used above in order to perform partial integraton. For the first equality one should note that for $\alpha\in L^2_\eta,\alpha_\rho\in W^{1,2}_\eta$ and $\lim_{x\to\infty}\alpha_\rho(x)\left(1+x^2\right)^\eta=0$. The above calculations together with [1] proves: (ψ,n) is a weak solution iff (3.4) holds for $t\in[0,T]$. □

We now define the truncated problem to (2.1)-(2.3):

Definition 3.3. Let $\delta\in]0,\infty[$ be arbitrary. Let $T_1^\delta:\mathbb{R}\to\mathbb{R}$ be a C^∞ function with $T_1^\delta(n)=n$ for $|n|\leq\delta^{-1}$ and $T_1^\delta(n)=2\delta^{-1}|n|^{-1}n$ for $|n|\geq2\delta^{-1}$. Similarly let $T_2^\delta:\mathbb{C}^2\to\mathbb{C}^2$ be C^∞ with $T_2^\delta(v)=v$ for $\|v\|\leq\delta^{-1}$ and $T_2^\delta(v)=2\delta^{-1}\|v\|^{-1}v$ for $\|v\|\geq2\delta^{-1}$. Define the truncated nonlinearities

$$\begin{aligned}G^\delta:]0,L[\times\mathbb{C}^2\times\mathbb{R}\to\mathbb{C}^2,\quad G^\delta(x,\psi,n):=G(x,T_2^\delta(\psi),T_1^\delta(n)),\\ H^\delta:]0,L[\times\mathbb{C}^2\times\mathbb{R}\to\mathbb{R},\quad H^\delta(x,\psi,n):=H(x,T_2^\delta(\psi),T_1^\delta(n)).\end{aligned}$$

Then G^δ,H^δ are C^k-smooth Carathéodory functions generating the smooth superposition operators $\mathfrak{G}^\delta,\mathfrak{H}^\delta$. The truncated problem reads:

$$\left\{\begin{array}{rcl}\partial_t\psi^\delta(t,x)&=&\left(-\partial_x\psi_1^\delta(t,x),\partial_x\psi_2^\delta(t,x)\right)+G^\delta(x,\psi^\delta(t,x),n^\delta(t,x))\\ \partial_t n^\delta(t,x)&=&I(t,x)+H^\delta(x,\psi^\delta(t,x),n^\delta(t,x))\\ &&+\sum_{k=1}^m b_k\chi_{S_k}(x)\left(\fint_{S_k}n^\delta(t,y)\,dy-n^\delta(t,x)\right)\end{array}\right.\tag{3.5}$$

with the same boundary conditions and initial values:

$$\psi_1^\delta(t,0)=r_0\psi_2^\delta(t,0)+\alpha(t),\ \psi_2^\delta(t,L)=r_L\psi_1^\delta(t,L)\tag{3.6}$$

$$\psi^\delta(0,x)=\psi^0(x),\ n^\delta(0,x)=n^0(x).\tag{3.7}$$

Weak solutions to (3.5)-(3.7) are defined analogously to Def. 2.1.

Remark 3.4. *After truncation G^δ and H^δ satisfy condition ıı) of Definition 2.7 globally. In particular G^δ and H^δ become globally Lipschitz uniformly with respect to $x \in]0, L[$, that is for each $\delta > 0$ there exists a constant Λ such that for all $\psi_1, \psi_2 \in \mathbb{C}^2$, $n_1, n_2 \in \mathbb{R}$ and a.a. $x \in]0, L[$*

$$\begin{aligned}\left\|G^\delta(x, \psi_1, n_1) - G^\delta(x, \psi_2, n_2)\right\| + |H^\delta(x, \psi_1, n_1) - H^\delta(x, \psi_2, n_2)| \\ \leq \Lambda\left(\left\|\psi_1 - \psi_2\right\| + |n_1 - n_2|\right).\end{aligned}$$

The superposition operators $\mathfrak{G}^\delta$ and $\mathfrak{H}^\delta$ become globally Lipschitz from $L^p(]0, L[\,; \mathbb{C}^2 \times \mathbb{R})$ into $L^p(]0, L[\,; \mathbb{C}^2)$ and $L^p(]0, L[\,; \mathbb{R})$, respectively, for any $p \in [1, \infty]$.

Lemma 3.5. *For each $\delta > 0$ the Theorems 2.2 and 2.3 hold for the weak solution $\left(\psi^\delta, n^\delta\right)$ to the truncated problem* (3.5)-(3.7)

Proof. Denote the weak solution space

$$\mathfrak{X} := L^\infty\left(]0, T[\times]0, L[\,; \mathbb{C}^2 \times \mathbb{R}\right).$$

Extend it to

$$\mathfrak{X}_e := \mathfrak{X} \times L^\infty(]0, L[; \mathbb{C}^2 \times \mathbb{R}) \times L^\infty(]0, T[; \mathbb{C}) \times L^\infty(]0, T[\times]0, L[; \mathbb{R})$$

by attaching the corresponding spaces of the initial data ψ^0, n^0 and the dynamic data α, I. Both $\mathfrak{X}$ and $\mathfrak{X}_e$ are equipped with the corresponding L^∞ norms. Define the operator $\mathfrak{F} : \mathfrak{X}_e \to \mathfrak{X}$,

$$\mathfrak{F}\begin{pmatrix}\psi \\ n \\ \psi^0 \\ n^0 \\ \alpha \\ I\end{pmatrix}(t) := \begin{pmatrix}\psi(t) \\ n(t)\end{pmatrix} - \prod_{(\psi,n)}\left\{T_e(t)\begin{pmatrix}\psi^0 \\ n^0 \\ \alpha\end{pmatrix} + \int_0^t T_e(t-s)\begin{pmatrix}\mathfrak{G}^\delta(\psi(s), n(s)) \\ \mathfrak{I}(s) + \mathfrak{B}n(s) + \mathfrak{H}^\delta(\psi(s), n(s)) \\ 0\end{pmatrix} ds\right\}.$$

For fixed ψ^0, n^0, α, I denote $\mathfrak{F}_0 : \mathfrak{X} \to \mathfrak{X}$,

$$\mathfrak{F}_0(\psi, n)(t) := (\psi(t), n(t)) - (\mathfrak{F}(\psi, n, \psi^0, n^0, \alpha, I))(t).$$

By Lemma 3.2 the truncated problem (3.5)-(3.7) has a unique weak solution (ψ^δ, n^δ) corresponding to the data ψ^0, n^0, α, I iff $\mathfrak{F}_0$ has a unique fixed point in $\mathfrak{X}$. By Remark 3.4 $\mathfrak{G}^\delta$ and $\mathfrak{H}^\delta$ are globally Lipschitz from $L^\infty\left(]0, L[\,; \mathbb{C}^2 \times \mathbb{R}\right)$ into $L^\infty\left(]0, L[\,, \mathbb{C}^2\right)$ and $L^\infty(]0, L[\,, \mathbb{R})$, respectively, with some Lipschitz constant Λ depending on the truncation parameter δ. Thus from the explicit formula (3.3) for the semigroup $T_e(t)$ it follows by induction that for $l \in \mathbb{N}$, $(\psi_a, n_a), (\psi_b, n_b) \in \mathfrak{X}$

$$\left\|\mathfrak{F}_0^l(\psi_a, n_a) - \mathfrak{F}_0^l(\psi_b, n_b)\right\|_{\mathfrak{X}} \leq \frac{(\Lambda T)^l}{l!}\left\|(\psi_a, n_a) - (\psi_b, n_b)\right\|_{\mathfrak{X}}.$$

Hence, for l sufficiently large $\mathfrak{F}_0^l$ is a contraction in the Banach space $\mathfrak{X}$. By a generalization of Banachs fixed point theorem $\mathfrak{F}_0$ has a unique fixed point (ψ^δ, n^δ) in $\mathfrak{X}$. This proves the existence and uniqueness part of Theorem 2.2.

From the assumptions that G, H are C^k Caratheódory functions (Definition 2.7) and Proposition 3.1 we get that $\mathfrak{F}$ maps $\mathfrak{X}_e$ C^k-smoothly into $\mathfrak{X}$. The existence and uniqueness of the weak solutions just proved is equivalent to saying that for any

ψ^0, n^0, α, I there exists a unique $(\psi, n) \in \mathfrak{X}$ such that $\mathfrak{F}(\psi, n, \psi^0, n^0, \alpha, I) = 0$. The partial derivative of $\mathfrak{F}$ with respect to (ψ, n) operating on $v = (v_\psi, v_n) \in \mathfrak{X}$ satisfies the formula

$$\left(\frac{\partial \mathfrak{F}}{\partial(\psi, n)}\begin{pmatrix}\psi\\ n\\ \psi^0\\ n^0\\ \alpha\\ I\end{pmatrix}\begin{pmatrix}v_\psi\\ v_n\end{pmatrix}\right)(t) = \begin{pmatrix}v_\psi(t)\\ v_n(t)\end{pmatrix} - \prod_{(\psi,n)}\int_0^t T_e(t-s)\begin{pmatrix}\left(\partial\mathfrak{G}^\delta(\psi(s), n(s))\right)v(s)\\ \mathfrak{B}v_n(s) + \partial\mathfrak{H}^\delta(\psi(s), n(s))v(s)\\ 0\end{pmatrix}ds.$$

Again it follows by Banachs fixed point theorem that for any $w \in \mathfrak{X}$ there exists a unique $v \in \mathfrak{X}$ such that

$$v(t) = \prod_{(\psi,n)}\int_0^t T_e(t-s)\begin{pmatrix}\left(\partial\mathfrak{G}^\delta(\psi(s), n(s))\right)v(s)\\ \mathfrak{B}v_n(s) + \partial\mathfrak{H}^\delta(\psi(s), n(s))v(s)\\ 0\end{pmatrix}ds + w(t) \quad (t \in [0, T]).$$

Banachs open mapping theorem implies that $\partial_{(\psi,n)}\mathfrak{F}$ is an isomorphism from $\mathfrak{X}$ onto $\mathfrak{X}$. Hence Theorem 2.2 is a consequence of the implicit function theorem.

Statement *ı*) of Theorem 2.3 follows directly from Definition 2.1 and the variation of constants formula.

We now prove *ıı*): As in the proof of Lemma 3.2 trivially extend α to the whole $[0, \infty[$ by setting α almost everywhere to zero on $[T, \infty[$ and define

$$a \in C\left(]0, \infty[\,; L^2_\eta(]0, \infty[\,; \mathbb{C})\right),\ a(s)(x) := \tau_s\alpha(x),$$

for $s \geq 0$ and *a.a.* $x \in]0, \infty[$, where τ_s denotes the left translation of α again. Integrating the variation of constants formula (3.4) with respect to time yields

$$\int_0^t \begin{pmatrix}\psi(s)\\ n(s)\\ a(s)\end{pmatrix}ds = \int_0^t T_e(s)\begin{pmatrix}\psi^0\\ n^0\\ \alpha\end{pmatrix}ds$$
$$+ \int_0^t\int_0^s T_e(s-r)\begin{pmatrix}\mathfrak{G}^\delta(\psi(r), n(r))\\ \mathfrak{I}(r) + \mathfrak{B}n(r) + \mathfrak{H}^\delta(\psi(r), n(r))\\ 0\end{pmatrix}drds \ \ (t \in [0, T]).$$

From this formula and the uniform continuity $(t, p) \mapsto T_e(t)p$ of the C_0 semigroup T_e one easily proves that the limit

$$\lim_{h\downarrow 0}\frac{T_e(h) - I}{h}\int_0^t (\psi(s), n(s), a(s))ds$$

exists in X_e (see (3.2)) for each $t \in [0, T]$. This is equivalent to $\int_0^t (\psi(s), n(s), a(s))ds \in \mathfrak{D}(A_e)$ or statement *ıı*).

Now assume $\alpha \in W^{1,2}(]0, T[\,; \mathbb{C})$, $\psi^0 \in W^{1,2}(]0, L[; \mathbb{C}^2)$ and (2.6). Extend α to the whole $]0, \infty[$ such that the extension lies in $W^{1,2}_\eta(]0, \infty[\,; \mathbb{C})$. Then (ψ^0, n^0, α) belongs to $\mathfrak{D}(A_e)$. Since X_e is reflexive it follows from Proposition 4.3.9 in [4] that

$$(\psi, n, \tau_t\,\alpha) \in C([0, T]; \mathfrak{D}(A_e)) \cap C^1([0, T]; X_e),$$

which proves *iii*).

We prove Theorem 2.3, *iv*). Choose sequences $\psi_i^0 \in W^{1,2}(]0,L[;\mathbb{C}^2)$, $\alpha_i \in W^{1,2}(]0,T[;\mathbb{C})$, $i \in \mathbb{N}$, which satisfy the boundary condition $\psi_{i\,1}^0(0) = r_0\psi_{i\,2}^0(0) + \alpha_i(0)$ and $\psi_{i\,2}^0(L) = r_L\psi_{i\,1}^0(L)$, and have the property that $\psi_i^0 \to \psi^0$ in $L^\infty(]0,L[;\mathbb{C}^2)$ and $\alpha_i \to \alpha$ in $L^\infty(]0,T[;\mathbb{C})$. By Theorem 2.3 *iii*) $\psi_i \in C([0,T]\times[0,L];\mathbb{C}^2)$, and by Theorem 2.2 the solution sequences (ψ_i, n) converge to (ψ, n) in $\mathfrak{X}$. Thus $\psi \in C([0,T]\times[0,L];\mathbb{C}^2))$ and ψ satisfies (2.2) pointwise in $[0,T]$. By assumption (IX) on H the superposition operator $\mathfrak{H}^\delta$ keep the space C_P invariant. The ψ-part of the fixed point (ψ, n) of the operator $\mathfrak{F}_0$ is uniformly continuous on $[0,T]\times[0,L]$. Since $n^0 \in C_P$ and the part n can be obtained by a fixed point iteration in the space $C([0,T];C_P)$ alone, keeping ψ unchanged, we obtain that $n \in C([0,T];C_P)$. The relation $n \in C^1([0,T],C_P)$ follows directly from (2.5) if $I \in C([0,T];C_P)$. □

Remark 3.6. *(Lipschitz dependence of solutions with respect to L^2) Because of Remark 3.4 Gronwall's Lemma applied to* (3.4) *easily shows that there exists a constant $C = C(\delta, T)$ such that*

$$\left\|(\psi,n)-(\tilde\psi,\tilde n)\right\|_{C([0,T];L^2(]0,L[;\mathbb{C}^2\times\mathbb{R})} \le$$

$$C\left(\left\|(\psi^0,n^0)-(\tilde\psi^0,\tilde n^0)\right\|_{L^2(]0,L[;\mathbb{C}^2\times\mathbb{R})} + \|\alpha-\tilde\alpha\|_{L^2(]0,T[;\mathbb{C})}\right)$$

where (ψ, n) and $(\tilde\psi, \tilde n)$ denote the weak solution with initial data $\left(\psi^0, n^0, \alpha\right)$ and $\left(\tilde\psi^0, \tilde n^0, \tilde\alpha\right)$, respectively.

4. apriori estimates

We will use the following elementary inequality:

Proposition 4.1. *Let $u : [0,b] \to \mathbb{R}$ be absolutely continuous and $u^* \in \mathbb{R}$. Suppose there are constants $r_1, r_2 > 0$ such that $u'(t) \le -r_1u(t) + r_2$ for a.a. $t \in [0,b]$ with $u(t) \ge u^*$. Then $u(t) \le \bar u + \max\{u(0) - \bar u, 0\}\, e^{-r_1t}$ for $t \in [0,b]$ with $\bar u := \max\left\{\frac{r_2}{r_1}, u^*\right\}$.*

Proof. Define $h : \mathbb{R} \to \mathbb{R}, h(x) := (\max\{x - \bar u, 0\})^2$. Set $f(t) := h(u(t))$. Then f is absolutely continuous and

$$f'(t) = h'(u(t))u'(t) \le -h'(u(t))r_1\left(u(t) - \frac{r_2}{r_1}\right) \le -2r_1f(t)$$

for *a.a.* $t \in [0,b]$. Therefore $f(t) \le e^{-2r_1t}f(0)$ for $t \in [0,b]$ and taking the square root yields the inequality. □

Lemma 4.2. *Let (ψ^δ, n^δ) be the weak solution to the truncated problem (3.5), (3.6), (3.7). There exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ estimate* (2.7) *holds for $t \in [0,T]$ and the bounds* (2.8) *are satisfied for $t \in [0,T]$ and a.a. $x \in]0,L[$. Moreover, there exists a constant B not dependning on $\delta > 0$ such that*

$$\left\|\psi^\delta(t)\right\|_{L^\infty} \le B \quad \text{for all } t \in [0,T].$$

Proof. Let $t_0 \in [0,T]$ be arbitrary and assume first that $\fint_{S_k} n^\delta(t,y)\,dy \ge 0$ for all $t \in [0,t_0]$ and all $1 \le k \le m$. Let $k \in \mathbb{N}$, $1 \le k \le m$. Suppose $0 < \delta \le \nu_1^{-1}$. Then

for *a.a.* $x \in S_k$ assumptions (II), (IV), (VIII) imply that for *a.a.* $t \in [0, t_0]$ which satisfy $n^\delta(t,x) \le \nu_1$ the inequality

$$\frac{d}{dt} n^\delta(t,x) \ge (-c_1 - b_k)\, n^\delta(t,x)$$

holds. Put

$$h(t,x) := \min\left\{n^\delta(t,x), \nu_1\right\} \quad \text{and} \quad \tau_k(n) := \begin{cases} 1 & , n \le \nu_1 \\ 0 & , n > \nu_1 \end{cases}.$$

Then for *a.a.* $x \in S_k$ and *a.a.* $t \in [0, t_0]$

$$\begin{aligned} \frac{d}{dt} h(t,x) &= \tau_k\left(n^\delta(t,x)\right) \frac{d}{dt} n^\delta(t,x) \\ &\ge (-c_1 - b_k)\, \tau_k\left(n^\delta(t,x)\right) n^\delta(t,x) \\ &\ge (-c_1 - b_k)\, h(t,x). \end{aligned}$$

Therefore for *a.a.* $x \in S_k$ and all $t \in [0, t_0]$

$$n^\delta(t,x) \ge h(t,x) \ge h(0,x) e^{-(c_1+b_k)t} = \min\left\{n^0(x), \nu_1\right\} e^{-(c_1+b_k)t} \quad (\ge 0). \tag{4.1}$$

Now we show that $⨍_{S_k} n^\delta(t,y)\, dy \ge 0$ for all $t \in [0,T]$ and all $1 \le k \le m$. Assume the contrary. Then there exists a $k \in \mathbb{N}$, $1 \le k \le m$, such that

$$t_0 := \sup\left\{t \in [0,T] \mid ⨍_{S_k} n^\delta(s,y) dy \ge 0 \text{ for } s \in [0,t]\right\} < T. \tag{4.2}$$

By (4.1) we have $n^\delta(t_0, x) \ge 0$ for *a.a.* $x \in \,]0, L[$ and by (4.2) $\int_{S_k} n^\delta(t_0, y)\, dy = 0$. Therefore $n^\delta(t_0, x) = 0$ for *a.a* $x \in S_k$. Hence, by continuity, there exists $0 < \epsilon < T - t_0$ such that for all $t \in [t_0, t_0 + \epsilon[$ and *a.a.* $x \in S_k$ we have $n^\delta(t,x) \le \nu_1$. Thus from the assumptions (II) and (IV), definition of H^δ and due to the choice $\delta \le \nu_1^{-1}$ we have for *a.a* $t \in [t_0, t_0 + \epsilon[$

$$\frac{d}{dt} ⨍_{S_k} n^\delta(t,y) dy = ⨍_{S_k} \left(I(t,y) + H(y, \psi^\delta(t,y), n^\delta(t,y))\right) dy \ge -c_1 ⨍_{S_k} n^\delta(t,y) dy.$$

This yields $⨍_{S_k} n^\delta(t,y) dy \ge ⨍_{S_k} n^\delta(t_0,y)\, dy \cdot e^{-c_1(t-t_0)} = 0$ for $t \in [t_0, t_0 + \epsilon[$ which contradicts the choice of t_0 from which there exist infinitely many points $s \in \,]t_0, t_0 + \epsilon[$ with $⨍_{S_k} n^\delta(s,y) dy < 0$ accumulating in t_0. This proves (4.1) for all $t \in [0,T]$ and the lower bound for n^δ in (2.8).

Now define

$$T_\delta := \sup\left\{t \in [0,T] \mid \left\|\psi^\delta(s)\right\|_{L^\infty} \le \delta^{-1} \text{ and } \left\|n^\delta(s)\right\|_{L^\infty} \le \delta^{-1} \text{ for } s \in [0,t]\right\}.$$

Suppose $\delta > 0$ is sufficiently small such that $T_\delta > 0$. Assume $\alpha \in W^{1,2}(]0,T[; \mathbb{C})$ and $\psi_0 \in W^{1,2}(]0,L[; \mathbb{C}^2)$ together with (2.6). Denote

$$h(t) := \int_0^L n^\delta(t,x)\, dx + \frac{d_1}{2} \int_0^L \left\|\psi^\delta(t,x)\right\|^2 dx.$$

From (I), (II), (VI) and Theorem 2.3 *iii*), proved for the truncated problem in Lemma (3.5), it follows by partial integration that for *a.a* $t \in [0, T_\delta]$

$$\begin{aligned}
\frac{d}{dt}h(t) &= d_1 \,\mathfrak{Re} \int_0^L \left[-\partial_x \psi_1^\delta(t,x)\overline{\psi_1^\delta(t,x)} + \partial_x \psi_2^\delta(t,x)\overline{\psi_2^\delta(t,x)}\right] dx \\
&\quad + \int_0^L \left[I(t,x) + H(x,\psi^\delta(t,x), n^\delta(t,x)) + d_1 \mathfrak{Re}\left\langle G(x,\psi^\delta(t,x), n^\delta(t,x)), \psi^\delta(t,x)\right\rangle\right] dx \\
&\leq \frac{d_1}{2}\left(-\left|\psi_1^\delta(t,L)\right|^2 + \left|\psi_1^\delta(t,0)\right|^2 + \left|\psi_2^\delta(t,L)\right|^2 - \left|\psi_2^\delta(t,0)\right|^2\right) \\
&\quad + \int_0^L I(t,x)\,dx - d_2\left(\int_0^L n^\delta(t,x)\,dx + \int_0^L \left\|\psi^\delta(t,x)\right\|^2 dx\right) \\
&\leq \frac{d_1}{2}\left(\left(|r_0|^2-1\right)|\psi_2^\delta(t,0)|^2 + |\alpha(t)|^2 + 2|r_0||\psi_2^\delta(t,0)||\alpha(t)| + \left(|r_L|^2-1\right)|\psi_1^\delta(t,L)|^2\right) \\
&\quad + L\,\|I\|_{L^\infty} - c\cdot h(t) \\
&\leq L\,\|I\|_{L^\infty} + \frac{d_1}{2}\,\|\alpha\|_{L^\infty}^2 + d_1 \max_{\rho\in\mathbb{R}}\left(\frac{|r_0|^2-1}{2}\rho^2 + |r_0|\,\|\alpha\|_{L^\infty}\rho\right) - c\cdot h(t) \\
&= \frac{d_1}{2(1-|r_0|^2)}\,\|\alpha\|_{L^\infty}^2 + L\,\|I\|_{L^\infty} - c\cdot h(t).
\end{aligned}$$

Therefore the δ-independend estimate (2.7) for $\left(\psi^\delta, n^\delta\right)$ and $t \in [0, T_\delta]$ follows from Proposition 4.1. Because of Remark 3.6 this remains valid by density if $\alpha \in L^\infty(]0,T[\,;\mathbb{C})\backslash W^{1,2}(]0,T[\,;\mathbb{C})$ or $\psi_0 \in L^\infty(]0,L[\,;\mathbb{C}^2)\backslash W^{1,2}(]0,L[\,;\mathbb{C}^2)$. By Definition 2.1 $n^\delta(\cdot,x)$ is absolutely continuous on $[0,T]$ for *a.a* $x \in]0,L[$. From assumption (II) it follows that for *a.a* $t \in [0,T_\delta]$ with $n^\delta(t,x) \geq \nu_2$ the inequality

$$\frac{d}{dt}n^\delta(t,x) \leq \|I\|_{L^\infty} + \max_{1\leq k\leq m}\left(\frac{b_k}{|S_k|}\right)\cdot \max\left\{\mu, \int_0^L n^0(x)dx + \frac{d_1}{2}\left\|\psi^0\right\|_{L^2}^2\right\} - c_2 n^\delta(t,x)$$

holds. Proposition 4.1 yields the δ-independend upper bound for n^δ and $t \in [0,T_\delta]$ in (2.8).

From the explicit formula (3.3) we have the following decay rates for the semigroups T and T_e: For $t \geq 0$

$$\left\|\Pi_\psi T_e(t)\begin{pmatrix}\psi^0\\ n^0\\ \alpha\end{pmatrix}\right\|_{L^\infty} \leq D_0 e^{-\gamma t}\left\|\psi^0\right\|_{L^\infty} + 2\left(1-|r_0 r_l|\right)^{-1}\|\alpha\|_{L^\infty}, \tag{4.3}$$

where $D_0 := \begin{cases} |r_0 r_L|^{-1} & , r_0 r_L \neq 0 \\ e & , r_0 r_L = 0 \end{cases}$ and $\gamma := \begin{cases} -(2L)^{-1}\log|r_0 r_L| & , r_0 r_L \neq 0 \\ (2L)^{-1} & , r_0 r_L = 0 \end{cases}$.

Let M_0 be a constant in assumption (III) for $K = \left[0, N + \left\|n^0\right\|_{L^\infty}\right]$. From (4.3), (3.4), (2.8) and (III) we get for $t \in [0,T_\delta]$

$$\begin{aligned}
\left\|\psi^\delta(t)\right\|_{L^\infty} &\leq \left\|\Pi_\psi T_e(t)\begin{pmatrix}\psi^0\\ n^0\\ \alpha\end{pmatrix}\right\|_{L^\infty} + \int_0^t \left\|T(t-s)\mathfrak{G}^\delta(\psi^\delta(s), n^\delta(s))\right\|_{L^\infty} ds \\
&\leq D_0 e^{-\gamma t}\left\|\psi^0\right\|_{L^\infty} + 2\left(1-|r_0 r_l|\right)^{-1}\|\alpha\|_{L^\infty} + M_0 T + \int_0^t M_0\left\|\psi^\delta(s)\right\|_{L^\infty} ds.
\end{aligned}$$

Gronwall's Lemma yields the existence of a constant B independent on $\delta > 0$ such that $\left\|\psi^\delta(t)\right\|_{L^\infty} \leq B$ for $t \in [0, T_\delta]$.
Moreover, since assumption (III) is valid also for the truncated nonlinearity G^δ and n^δ is continuous from $[0, T]$ to L^∞ by chosing a possibly larger M_0 corresponding to a larger set K than above we can find a constant B independend of $\delta > 0$ such that for each $\delta > 0$ there exists a neighbourhood U_δ of T_δ so that $\left\|\psi^\delta(t)\right\|_{L^\infty} \leq B$ for $t \in [0, T_\delta] \cup U_\delta$. This proves that $T_\delta = T$ if δ is chosen sufficiently small. □

We have shown that for sufficiently small $\delta > 0$ the weak solutions of the truncated problem coincide with the original weak solutions of the nontruncated problem. Hence the proof of Theorems 2.2-2.4 is complete. We are only left with the

Proof of Theorem 2.5. Let (ψ, n) be the weak solution. From the differentiability assumption (X) on H the map $w \mapsto \mathfrak{H}(\psi(s), w)$ is well defined from $W_P^{1,2}$ into itself for $s \in [0, T]$ since $\psi \in C([0, T], W^{1,2})$. Furthermore condition (XI) implies that this map is Lipschitz on bounded subsets of $W_P^{1,2}$ uniformly in $s \in [0, T]$. By truncation we can make it globally Lipschitz: for $\eta > 0$ let $T_\eta : W_P^{1,2} \to W_P^{1,2}$ be globally Lipschitz with $T_\eta(w) = w$, if $\|w\|_{W_P^{1,2}} \leq \eta^{-1}$, $T_\eta(w) = 2\eta^{-1} w \|w\|_{W_P^{1,2}}^{-1}$, if $\|w\|_{W_P^{1,2}} \geq 2\eta^{-1}$. Define the following truncated operators

$$\mathfrak{H}_\eta(p, w) := \mathfrak{H}(p, T_\eta(w)) \text{ for } p \in W^{1,2} \text{ and } w \in W_P^{1,2}.$$

Then for all $p \in W^{1,2}$ the map $w \mapsto \mathfrak{H}_\eta(p, w)$ is globally Lipschitz in $W_P^{1,2}$ where the Lipschitz constant depends only on η and $\left\|p\right\|_{W^{1,2}}$.

Define $\mathfrak{F} : C([0, T], W_P^{1,2}) \to C([0, T], W_P^{1,2})$,

$$(\mathfrak{F}m)(t) := n^0 + \int_0^t \left(\mathfrak{I}(s) + \mathfrak{B}m(s) + \mathfrak{H}_\eta(\psi(s), m(s))\right) ds \quad (t \in [0, T]).$$

Then $\mathfrak{F}$ has a unique fixed point n_η in $C([0, T], W_P^{1,2})$ by a generalization of Banachs fixed point theorem since sufficient high iterates of $\mathfrak{F}$ become contractive. In particular $n_\eta \in C^1([0, T], W_P^{1,2})$.

Set $T_\eta := \sup\left\{t \in [0,T] \mid \left\|n_\eta(s)\right\|_{W_p^{1,2}} \le \eta^{-1} \text{ for } 0 \le s \le t\right\}$. By (XII) and the Hölder-Young inequalities we have for all $t \in [0, T_\eta]$

$$
\begin{aligned}
& \partial_t \frac{1}{2} \left\|\partial_x n_\eta(t)\right\|^2_{L^2(S_k)} \\
= & \int_{S_k} \partial_x \Big(I(t,x) - b_k n_\eta(t,x) + H(x,\psi(t,x),n_\eta(t,x))\Big) \partial_x n_\eta(t,x)\, dx \\
\le & \int_{S_k} \left|\partial_x I(t,x) \partial_x n_\eta(t,x)\right| dx + \int_{S_k} \Big(\partial_x H(x,\psi(t,x),n_\eta(t,x)) \partial_x n_\eta(t,x) \\
& + \partial_\psi H(x,\psi(t,x),n_\eta(t,x)) \partial_x \psi(t,x) \partial_x n_\eta(t,x) \\
& + \partial_n H(x,\psi(t,x),n_\eta(t,x)) \left(\partial_x n_\eta(t,x)\right)^2\Big) dx \\
\le & \frac{3}{2\tau} \left\|\partial_x I(t)\right\|^2_{L^2(S_k)} - \tau \frac{5}{6} \left\|\partial_x n_\eta(t)\right\|^2_{L^2(S_k)} \\
& + R_0 \left(\|1\|_{L^1(S_k)} + \left\|\partial_x n_\eta(t)\right\|_{L^1(S_k)} + \left\|\partial_x n_\eta(t)\right\|_{L^2(S_k)} \left\|\partial_x \psi(t)\right\|_{L^2(S_k)} + \left\|\partial_x \psi(t)\right\|^2_{L^2(S_k)}\right) \\
\le & \frac{3}{2\tau} \sup_{t\in[0,T]} \left\|\partial_x I(t)\right\|^2_{L^2} + R_0 L + \frac{3}{2\tau} R_0^2 L + \left(\frac{3R_0^2}{2\tau} + 1\right) \left\|\partial_x \psi(t)\right\|^2_{L^2(S_k)} - \tau \frac{1}{2} \left\|\partial_x n_\eta(t)\right\|^2_{L^2}.
\end{aligned}
$$

Hence (see Prop. 4.1) we get the following η independent bound

$$
\left\|\partial_x n_\eta(t)\right\|^2_{L^2(S_k)} \le \frac{3}{2\tau^2} \sup_{t\in[0,T]} \left\|\partial_x I(t)\right\|^2_{L^2(S_k)} + \frac{R_0 L}{\tau} + \frac{3R_0^2 L}{2\tau^2} + \left(\frac{3R_0^2}{2\tau^2} + \frac{1}{\tau}\right) \sup_{s\in[0,T_\eta]} \left\|\partial_x \psi(s)\right\|^2_{L^2}
$$

which is valid for $t \in [0, T_\eta]$.

Since the a priori estimates of Theorem 2.4 must hold for n_η as long as $t \in [0, T_\eta]$ we see that $T_\eta = T$ and $n_\eta = n$ if η is chosen sufficiently small. □

5. Example

The system of equations (2.1)-(2.3) is a general form of the Traveling-Wave model used to simulate temporal-longitudinal behaviour of slowly varying complex amplitudes of counterpropagating optical fields and carriers in multisection semiconductor lasers [3, 16, 18, 20]. Different dynamical behaviour of properly designed lasers can be effectively used in different technological applications. Examples are wavelength tuning, chirp reduction, enhanced modulation bandwidths, mode-locking of short pulses, and frequency-tunable self-pulsations for high-speed data transmission in optical communication systems (see, e.g., technology references in [3, 16]).

In the nonnormalized form the Traveling Wave model equations can be written as

follows:

$$\left\{\begin{array}{lcl} \partial_t\psi(t,x) & = & v_{gr}\left[(-\partial_x\psi_1,\partial_x\psi_2)+(\beta\psi_1+i\kappa(x)\psi_2, i\kappa(x)\psi_1+\beta\psi_2)\right] \\ \beta(x,\psi,n) & := & -i\left(\delta(x)+\beta_{th}(x)I(x)\right)-\frac{\alpha_0(x)}{2}+\frac{(1-i\alpha_H)\tilde{g}(x,n)}{2(1+\epsilon_G(x)\|\psi\|^2)} \\ \partial_t n(t,x) & = & \frac{I(x)+I_M(t,x)}{e\sum_{k=1}^m \chi_{S_k}(x)|S_k|}+H(x,\psi,n) \\ & & +\sum_{k=1}^m \frac{\chi_{S_k}(x)}{e|S_k|R_k}\left(\fint_{S_k} n(t,y)\,dy - n(t,x)\right) \\ H(x,\psi,n) & := & -\left[A(x)n+B(x)n^2+C(x)n^3\right]-\frac{v_{gr}\tilde{g}(x,n)\|\psi\|^2}{1+\epsilon_G(x)\|\psi\|^2}. \end{array}\right. \tag{5.1}$$

Moreover, field function ψ satisfies boundary conditions (2.2) and (ψ, n) satisfy the initial value condition (2.3).

The group velocity v_{gr} is assumed to be positive and constant within all laser. It can be easily eliminated from the above equations by simple scaling of time or space. Due to this elimination the equations (5.1) are a particular case of the eqs. (2.1).

Functions $\|\psi(t,x)\|^2 = \langle\psi,\psi\rangle$ and $n(t,x)$ are local photon and carrier densities. When multiplying $|\psi_j(t,x)|^2$ by factor $v_{gr}\frac{\hbar c_0}{\lambda_0}$ and by local crossection area of active zone one gets the local power of the forward ($j=1$) or backward ($j=2$) propagating field. The function $\tilde{g}(x,n)$ denotes the gain function. It is increasing in n, that is

$$\text{for } a.a.\ x\in\,]0,L[\quad\text{and all}\quad n\in\mathbb{R}\quad \partial_n\tilde{g}(x,n)\geq 0.$$

In the following simulations we assume a frequently used linear in n approximation of gain function: $\tilde{g}(x,n)\simeq g^d(x)\,(n-n_{tr})$.

In the equations above the used physical constants e, c_0 and $\hbar$ denote electron charge, speed of light in vacuum and Planck's constant, respectively. The remaining parameters specifying the considered laser are described below in Table 1.

After elimination of v_{gr} and taking into account the dependence of the used functions and parameters to the functional spaces indicated in Table 1 one can easily check the validity of all assumptions (I)-(XII) taken in Section 2. The operators β and H are physically meaningless for $n<0$. In particular assumption (II) may not be satisfied for $n<0$. However, since our apriori estimates guarantee that n will always stay positive, we are free to extend the definitions of the operators G (or β) and H so that for all $n\in\mathbb{R}$ our required assumptions are satisfied. In our example condition (II) holds with $d_1=2$ and $d_2=\operatorname{ess\,inf}_{x\in]0,L[}\min\{\alpha_0(x),A(x)\}>0$, since the internal absorption and inverse linear carrier life time are both >0. Assumptions (XI) and (XII) are satisfied due to nonlinear gain compression $\operatorname{essinf}_{x\in]0,L[}\epsilon_G(x)>0$. Hence, the results described in previous sections fits our system (5.1, 2.2, 2.3) originated from the real world applications.

In the four right columns of Table 1 we specify typical parameters of a 3 section (i.e. $m=3$) distributed feedback (DFB) laser schematically depicted in Fig. 1 and considered in more details in, e.g., [16]. The symbol

$$C_P^1 \quad := \quad \left\{n\in L^\infty(]0,L[\,;\mathbb{R})\mid \forall k\ n_{|S_k}\in C^1\left(\overline{S_k},\mathbb{R}\right)\right\}$$

TABLE 1. Parameters used in simulations.

symbol	allowed spaces for (I)-(XIV)	explanation	values S_1	values S_2	values S_3	units
$\kappa(x)$	$L^\infty(]0,L[,\mathbb{R})$	field coupling coefficients	15	0	5	10^3/m
$\delta(x)$	$L^\infty(]0,L[,\mathbb{R})$	static detuning	35	0	0	10^3/m
$\beta_{th}(x)$	$L^\infty(]0,L[,\mathbb{R})$	thermal detuning	0	1	0	10^{-7}m/A
$\alpha_0(x)$	$L^\infty(]0,L[,\mathbb{R})$ $\operatorname{essinf}_{x\in]0,L[}\alpha_0>0$	internal absorption	3	1	2	10^3/m
$\alpha_H(x)$	$L^\infty(]0,L[,\mathbb{R})$	Henry factor	-4	0	-4	
$\epsilon_G(x)$	$C^1_P, >0$	nonlinear gain saturation	3	3	3	10^{-24}m^3
$A(x)$	$C^1_P, >0$	inverse carrier life time	3	5	3	10^8/s
$B(x)$	$C^1_P, \geq 0$	bimolecular recombination	1	0	1	10^{-16}m^3/s
$C(x)$	$C^1_P, \geq 0$	auger recombination	1	0	1	10^{-40}m^6/s
$g^d(x)$	$C^1_P, \geq 0$	differential gain	1	0	1	10^{-20}m^2
$n_{tr}(x)$	$C^1_P, \geq 0$	transparency density	1	1	1	10^{24}/m^3
$I(x)$	$W^{1,2}_P$	current injection density	12	4(2)	3	10^{10}A/m^2
$I_M(t,x)$	$C([0,T],W^{1,2}_P)$	modulated current density	0	0	0	A/m^2
R_k	$\mathbb{R}, >0$	series resistance factor in section S_k	5	5	5	10^{-39}/Am
$\lvert S_k\rvert$	$\mathbb{R}, >0$	length of the section S_k	3	2	1	10^{-4}m
λ_0	$\mathbb{R}, >0$	central wavelength		1.54		10^{-6}m
c_0/v_{gr}	$\mathbb{R}, >0$	group velocity factor		3.6		
(r_0, r_L)	$\mathbb{C}^2$	facet reflectivities		0,0		

appearing in the first column denotes the space of on each laser section S_k, $1 \leq k \leq m$, C^1 functions.

The first section (DFB_1) of this laser contains Bragg grating which couples coun-

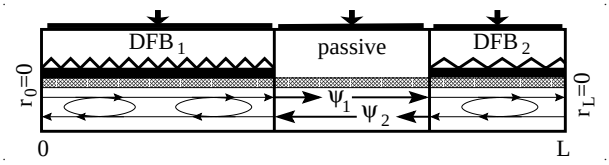

FIGURE 1. Scheme of 3-section DFB laser

terpropagating fields (nozero $\kappa(x)_{|S_1}$), is active (sufficiently large positive $I(x)_{|S_1}$ and strictly positive $g^d(x)_{|S_1}$), generating optical field output from this section and from full laser. The third section (DFB_2) is similar, but here an applied current is low. This section operates mainly as a wavelength dependent reflector. The middle section is passive ($g^d(x)_{|S_2} = 0$), has no coupling ($\kappa(x)_{|S_2} = 0$) and provides an additional possibility of control in experiments via the thermal detuning term $\beta_{th}(x)I(x)_{|S_2}$. We have performed a test simulation of the laser described above using the software LDSL-tool (abbreviation for "longitudinal dynamics in semiconductor lasers"). Fig. 2 shows a induced transition of the simulated solution from the quasi-stationary state to quasi-periodic state after a kick of the current injection in the middle section. We write "quasi", since only amplitudes $|\psi_j(t,x)|$, $j = 1,2$, of the complex optical fields are independent on time, or periodic, respectively. In Fig. 3 we draw spatial-temporal distributions of some functions in already established quasi-periodic state. These figures indicate continuity of optical fields

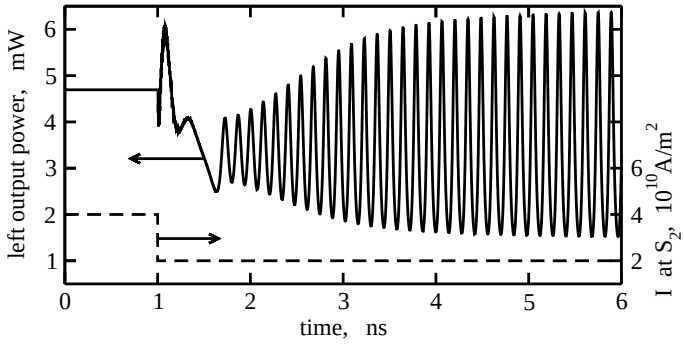

FIGURE 2. Simulated response of the laser (function $|\psi_2(t,0)|^2$) to the change of current.

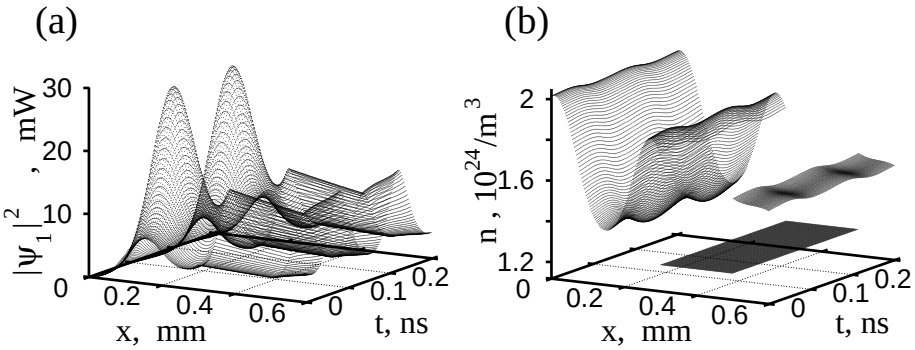

FIGURE 3. Spatial-temporal distributions of functions $|\psi_2(t,x)|^2$ (a) and $n(t,x)$ (b).

and discontinuity of the carrier densities at the interfaces between the sections. We also point out a strong nonuniformity (spatial hole burning) of the carrier densities within the first section, while in other sections its variation is less pronounced.

These observed quasi-periodic, or "pulsating" states with $\sim$ 7 GHz repetition frequency are of particular interest for high speed data transmission in optical communication systems (see [16] and references therein).

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