Topological analysis of qualitative features in electrical circuit theory

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Abstract

Several qualitative properties of equilibria in electrical circuits are analyzed in this paper. Specifically, non-singularity, hyperbolicity, and asymptotic stability are addressed in terms of the circuit topology, which is captured through the use of Modified Nodal Analysis (MNA) models. The differential-algebraic or semistate nature of these models drives the analysis of the spectrum to a matrix pencil setting, and puts the results beyond the ones already known for state-space models, unfeasible in many actual problems. The topological conditions arising in this qualitative study are proved independent of those supporting the index, and therefore they apply to both index-1 and index-2 configurations. The analysis combines results coming from graph theory, matrix analysis, matrix pencil theory, and Lyapunov theory for DAEs. The study is restricted to problems with independent sources; qualitative properties of circuits including controlled sources are the focus of future research.

Keywords: asymptotic stability, hyperbolicity, differential-algebraic equation, circuit model, modified nodal analysis, graph topology, matrix pencil.

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1 Introduction

The time-domain analysis of different features of (linear or non-linear) electrical circuits often requires a strong interaction between differential equation theory, graph theory, and matrix analysis. The time-domain model of a lumped (non-distributed) circuit with reactive devices will take the form of a differential equation, either an explicit ODE in so-called *state-space* models (see e.g. [5, 8, 13, 36] and references therein), or a differential-algebraic equation (DAE) in *semistate* models [6, 7, 16, 17, 25, 27, 29, 33]. Graphs [2, 3] naturally describe the topology of lumped circuits [8]. Matrices arise not only in the analytical description of graphs [3], but also in the linearization of the characteristics of (coupled) electrical devices. Additionally, matrix pencils [4, 12] are a key tool in the analysis of semistate models.

Qualitative properties of equilibria and operating points have been mainly addressed within the framework of state (ODE) models [5, 13, 14, 15, 35, 36]. In particular, Chua's paper [5] provides a nice compilation of the main results in this direction up to 1980. The use of state models requires a careful distinction between operating points and equilibria, as well as *local solvability* hypotheses (see Theorem 12 in [5]). The local stability analysis in [5] is supported on linearization; different results based upon Lyapunov function methods can be found in [10, 36] and references therein.

The operating point/equilibrium distinction was pursued further in [13, 14, 15], where operating point *potential stability* and *instability* are defined and analyzed in terms of the DC equations only. Broadly speaking, an operating point of a given DC circuit is unstable if no insertion of shunt capacitors and series inductors results in a structurally stable equilibrium of the corresponding dynamic circuit. Related results can be found in [35].

Besides other limitations of state models [16], the main drawback of state-space-based qualitative analyses is the need for assumptions allowing for the derivation of an explicit ODE model. Qualitative properties should ideally reflect features of the circuit itself, regardless of the model. Therefore, assumptions needed for the formulation of a state model are in many cases unnecessarily restrictive from the qualitative point of view. For instance, local solvability hypotheses in [5, Theorem 4] can be seen as local index-1 conditions on a certain semistate model [31], and therefore the stability results in [5] are restricted to certain (index-1, in DAE terms) configurations. As a simple example, a C-loop (displayed for instance in many MOS transistor models [31, 36, 38]) would put the circuit out of the scope of these results. No state-space-based general approach to the study of hyperbolicity or stability problems for these or other degenerate configurations [18, 28, 34] is known to the authors.

A semistate approach seems therefore to be of interest in this regard, since the formulation of differential-algebraic models such as Tableau Analysis or Modified Nodal Analysis ones [9, 16, 17, 38] (the latter being used in SPICE or TITAN) do actually need not much more than the lumped hypothesis. A framework based on matrix pencil theory has been recently proposed for the local stability analysis of semistate-modeled nonlinear circuits [31], supported on previous results concerning stability of DAEs [20, 21, 26, 30, 37]. The results in [31], however, are not stated in terms of the topology of the circuit or the specific features of the devices; we extend in the present paper those results by addressing qualitative properties in these circuit-theoretic terms, with particular attention to topological aspects. These features will be captured here through the use of MNA models. Background in this direction is given in Section 2.

Specifically, under certain local passivity assumptions on resistive devices, we will state topological conditions guaranteeing non-singularity as well as hyperbolicity of equilibria. This is carried out in Section 3 (Theorems 1 and 3), and only requires reciprocity in reactive devices. These topological conditions, milder than those of [5], will be proved independent of those supporting the index, showing that local solvability hypotheses can be actually removed. V-C loops and I-L cutsets can be accommodated without difficulty in this context, so that these results hold for both index-1 and index-2 configurations.

Requiring additionally local passivity in the reactances, these topological conditions will be shown in Theorem 4 of Section 4 to guarantee *linearized* asymptotic stability of equilibria. In addition, Theorem 5 shows that the property Re $\lambda < 0$ on the matrix pencil eigenvalues derived in the previous result indeed guarantees asymptotic stability of the equilibrium; in other words, we show that Lyapunov asymptotic stability holds via linearization in index-1 and index-2 circuit DAEs. The focus will be placed on *equilibrium* stability, since the location of reactive elements will be assumed to be fixed and known. Rather natural consequences of this analysis follow for periodically-forced circuits, and the approach might open another way for the study of additional qualitative aspects involving bifurcations, quasiperiodic excitations, etc., which are beyond the scope of the present work.

Finally, this framework is applied in Section 5 to a specific nonlinear Josephson circuit which displays several different configurations depending on certain circuit parameters.

2 Background

2.1 MNA models

The conventional MNA equations for an RLC circuit without controlled sources can be written as follows (see [38] and the bibliography therein):

$$A_C(\psi(A_C^{\mathrm{T}}e))' + A_R\gamma(A_R^{\mathrm{T}}e) + A_Li_l + A_Vi_v = -A_Ii_s(t) \tag{1a}$$

$$(\varphi(i_l))' - A_L^{\mathrm{T}}e = 0 \tag{1b}$$

$$-A_V^{\mathrm{T}}e = -v_s(t), \tag{1c}$$

the "$-$" sign in the last equation owing to later convenience. Here, A_R (resp. A_L, A_C, A_V, A_I) describes the *incidence* between resistive (resp. inductive, capacitive, voltage source, current source) branches and nodes in the circuit, once a reference node has been chosen. Specifically, the incidence matrix $A = (a_{ij}) \in \mathbb{R}^{(n-1)\times b}$ (n and b being the number of nodes and branches in the circuit, respectively) is given by

$$a_{ij} = \begin{cases} 1 & \text{if branch } j \text{ leaves node } i \\ -1 & \text{if branch } j \text{ enters node } i \\ 0 & \text{if branch } j \text{ is not incident with node } i. \end{cases}$$

The vector e stands for node voltages; i_l, i_v represent currents in inductors and voltage sources, respectively, and $i_s(t)$, $v_s(t)$ denote currents and voltages in the (independent) sources. Capacitors and resistors are assumed to be voltage-controlled through the relations $q = \psi(A_C^{\mathrm{T}} e)$, $i_r = \gamma(A_R^{\mathrm{T}} e)$, whereas inductors are supposed to be current-controlled by $\phi = \varphi(i_l)$. Note that these assumptions allow for coupling effects within each one of these three sets of devices.

System (1) may be rewritten as a quasilinear standard-form DAE under C^1 assumptions on ψ and φ. Assuming also that γ is C^1, let us denote the capacitance, inductance, and conductance matrices as

$$\begin{aligned} C(u) &= \psi'(u) && \text{(2a)} \\ L(i) &= \varphi'(i) && \text{(2b)} \\ G(u) &= \gamma'(u), && \text{(2c)} \end{aligned}$$

the last one being aimed at later use. In circuit-theoretic terms, symmetric capacitance or inductance matrices will be said to describe *reciprocal* devices, whereas positive definite capacitance, inductance or conductance matrices will be said to yield *strictly locally passive* elements [5]; positive definiteness of an $n \times n$ matrix B means in this paper that $x^{\mathrm{T}} B x > 0$ for any $x \in \mathbb{R}^n - \{0\}$, not implying that B is symmetric.

In the light of (2), system (1) can be rewritten as

$$\begin{aligned} A_C C(A_C^{\mathrm{T}} e) A_C^{\mathrm{T}} e' + A_R \gamma(A_R^{\mathrm{T}} e) + A_L i_l + A_V i_v &= -A_I i_s(t) && \text{(3a)} \\ L(i_l) i_l' - A_L^{\mathrm{T}} e &= 0 && \text{(3b)} \\ -A_V^{\mathrm{T}} e &= -v_s(t), && \text{(3c)} \end{aligned}$$

which is a quasilinear DAE

$$A(x)x' + f(x) = s(t), \tag{4}$$

where x is the semistate vector of conventional MNA,

$$x = \begin{pmatrix} e \\ i_l \\ i_v \end{pmatrix},$$

and

$$A = \begin{pmatrix} A_C C(A_C^{\mathrm{T}} e) A_C^{\mathrm{T}} & 0 & 0 \\ 0 & L(i_l) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} A_R \gamma(A_R^{\mathrm{T}} e) + A_L i_l + A_V i_v \\ -A_L^{\mathrm{T}} e \\ -A_V^{\mathrm{T}} e \end{pmatrix}, \quad s = \begin{pmatrix} -A_I i_s \\ 0 \\ -v_s \end{pmatrix}.$$

Many analytical and numerical features of a DAE rely upon its *index*: see [9, 38] for a discussion of this notion in the context of MNA models. In [38], topological criteria for the index of MNA-modeled circuits without controlled sources are discussed; a central role is played by certain configurations such as V-C loops (loops formed exclusively by capacitors and/or voltage sources) and I-L cutsets (cutsets formed exclusively by inductors and/or current sources). Remark that V-loops and I-cutsets are excluded if the circuit is well-posed.

We compile in Proposition 1 below Theorems 4 and 5 of the above-mentioned paper. It is worth emphasizing that, in the first assertion, positive definiteness of L in [38] is replaced by the milder assumption that L is non-singular: this owes to the fact that in the proof of [38, Theorem 4] nothing more than non-singularity is actually needed on L.

Proposition 1. *Assume that the capacitance and conductance matrices are positive definite, and that the inductance matrix is non-singular.*

1. *If the network contains neither I-L cutsets nor V-C loops (except for C-loops), then the MNA system (3) has index ≤ 1.*
2. *Assume additionally that the inductance matrix is positive definite. If the network contains an I-L cutset or a V-C loop (with at least one voltage source), then the MNA system (3) has index 2.*

Note that the "index $\leq$ 1" condition in the first item can be rewritten as an "index 1" condition except in cases in which there are no voltage sources and every node has a capacitive path to a reference node, what characterizes index-0 problems.

2.2 Linearization and matrix pencils

Assume that a given circuit has only independent DC sources, so that s in (4) is a constant vector. We may hence rewrite this equations as the quasilinear autonomous DAE

$$A(x)x' + g(x) = 0, \tag{5}$$

with $g(x) = f(x) - s$.

Equilibrium points of (5) are defined by the condition $g(x^*) = 0$, and the linearization of the DAE at equilibrium leads to the *matrix pencil* [4, 12]

$$\lambda A(x^*) + g'(x^*) = \lambda \begin{pmatrix} A_C C(A_C^{\mathrm{T}} e^*) A_C^{\mathrm{T}} & 0 & 0 \\ 0 & L(i_l^*) & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} A_R G(A_R^{\mathrm{T}} e^*) A_R^{\mathrm{T}} & A_L & A_V \\ -A_L^{\mathrm{T}} & 0 & 0 \\ -A_V^{\mathrm{T}} & 0 & 0 \end{pmatrix}. \tag{6}$$

Several qualitative properties of equilibria can be characterized in terms of the spectrum $\sigma\{A(x^*), g'(x^*)\} = \{\lambda \in \mathbb{C} \ / \ \det(\lambda A(x^*) + g'(x^*)) = 0\}$ of the matrix pencil depicted in (6). These results often depend on the structure and/or the index of the equation; the reader is referred to [20, 21, 22, 26, 30, 31, 37] for specific results concerning the relation between the pencil spectrum and the qualitative properties of equilibria. Specifically, we prove in Section 4 that certain results in [20, 21, 22], guaranteeing that the property Re $\lambda < 0$ on the matrix pencil eigenvalues yields asymptotic stability of the equilibrium, hold for both index-1 and index-2 MNA equations.

Based upon this background, the purpose of the present work is to characterize the spectrum of (6) and several related properties of equilibria in terms of the topology of the circuit. In this regard, several results from graph theory will be needed; such results are compiled below.

2.3 Some graph-theoretic properties

Lemma 1. *If $\mathcal{G}$ has n nodes and k connected components, then* $\mathrm{rk}A = n - k$ [3, 3.16].

Corollary 1. *$\mathcal{G}$ is connected if and only if A has full row rank $(n-1)$.*

Let $\mathcal{K}$ be a subset of the set of branches of a connected graph $\mathcal{G}$. Denote as $A_{\mathcal{K}}$ (resp. $A_{\mathcal{G}-\mathcal{K}}$) the submatrix of A formed by the columns corresponding to the branches in $\mathcal{K}$ (resp. not in $\mathcal{K}$). The following is a particular case of [3, 3.17]:

Lemma 2. *A subset $\mathcal{K}$ of the set of branches of a connected graph $\mathcal{G}$ are the branches of a tree if and only if $A_{\mathcal{K}}$ is a non-singular $((n-1) \times (n-1))$ matrix.*

Cutsets. A subset $\mathcal{K}$ of the set of branches of a connected graph is a *cutset* if the deletion of $\mathcal{K}$ results in a disconnected graph, and it is minimal with respect to this property (*i.e.* the deletion of any proper subset of $\mathcal{K}$ does not disconnect the graph).

Lemma 3. *A subset $\mathcal{K}$ of the set of branches of a connected graph $\mathcal{G}$ includes at least one cutset if and only if $A_{\mathcal{G}-\mathcal{K}}$ does not have full row rank, namely, the rows of $A_{\mathcal{G}-\mathcal{K}}$ are linearly dependent.*

This follows from the fact that, according to Corollary 1, $A_{\mathcal{G}-\mathcal{K}}$ does not have full row rank if and only if it is disconnected, *i.e.*, if $\mathcal{K}$ disconnects the graph, which in turn is equivalent to the fact that $\mathcal{K}$ includes at least one cutset.

Corollary 2. *Let $\mathcal{K}$ be a subset of the set of branches of a connected graph $\mathcal{G}$. $\mathcal{K}$ does not contain cutsets if and only if $x^{\mathrm{T}} A_{\mathcal{G}-\mathcal{K}} = 0 \Rightarrow x = 0$ or, equivalently, $A^{\mathrm{T}}_{\mathcal{G}-\mathcal{K}} x = 0 \Rightarrow x = 0$, that is,* $\mathrm{Ker}\ A^{\mathrm{T}}_{\mathcal{G}-\mathcal{K}} = \{0\}$.

Loops. A set of branches forming a loop yields linearly dependent columns in the incidence matrix [3, 3.13 and p. 145]. Conversely, if a set $\mathcal{K}$ does not contain a loop, then there exists a tree $\mathcal{T}$ containing all the branches in $\mathcal{K}$ [2, 2.29 and pp. 219-220]. Owing to Lemma 2, the columns of $A_{\mathcal{T}}$ are linearly independent, and since they include the columns of $A_{\mathcal{K}}$, these are also linearly independent. Hence:

Lemma 4. *A subset $\mathcal{K}$ of the set of branches of a connected graph $\mathcal{G}$ includes at least one loop if and only if $A_{\mathcal{K}}$ does not have full column rank, namely, the columns of $A_{\mathcal{K}}$ are linearly dependent.*

Corollary 3. *Let $\mathcal{K}$ be a subset of the set of branches of a connected graph $\mathcal{G}$. $\mathcal{K}$ does not contain loops if and only if $A_{\mathcal{K}} y = 0 \Rightarrow y = 0$, that is,* $\mathrm{Ker}\ A_{\mathcal{K}} = \{0\}$.

3 Hyperbolicity

3.1 Non-singularity

As indicated in 2.2, equilibrium points of (5) are defined by the vanishing of $g(x)$. For several reasons, it is of interest to assure that a given equilibrium x^* is non-singular in the sense that the Jacobian $g'(x^*)$ is invertible. For instance, in linear problems (defined by $A(x) = A$, $g(x) = Bx - s$, for certain constant matrices A, B) there exists a unique equilibrium point if and only if $g' = B$ is non-singular (invertible), regardless of the vector s. In non-linear cases, assumed an equilibrium point x^* is given, the non-singularity of $g'(x^*)$ guarantees the isolation of this equilibrium. Furthermore, these non-singular equilibria are well-conditioned for Newton-based computations [32].

Non-singularity of equilibria in circuits with definite conductance can be guaranteed if the topological conditions of Theorem 1 below are satisfied; these topological conditions are entirely independent of those characterizing the index in Proposition 1. Note that this result is consistent with the fact that a V-L loop or a I-C cutset would yield a bad-posed resistive DC circuit after short-circuiting inductors and open-circuiting capacitors. It is also coherent with the results in [23], but in its present form the result is stated without the need to refer to operating points of this resistive DC circuit.

Theorem 1. *Let $x^* = (e^*, i_l^*, i_v^*)$ be an equilibrium point of (5). Denote $G = G(A_R^{\mathrm{T}}e^*)$, and assume that G is (positive or negative) definite. Then x^* is non-singular (equivalently, $0 \notin \sigma\{A(x^*), g'(x^*)\}$) if and only if there are neither V-L loops nor I-C cutsets in the circuit.*

Proof: note that

$$g'(x^*) = \begin{pmatrix} A_R G A_R^{\mathrm{T}} & A_L & A_V \\ -A_L^{\mathrm{T}} & 0 & 0 \\ -A_V^{\mathrm{T}} & 0 & 0 \end{pmatrix}.$$

Write $M = A_R G A_R^{\mathrm{T}}$, $N = (A_L \ A_V)$, so that

$$g'(x^*) = \begin{pmatrix} M & N \\ -N^{\mathrm{T}} & 0 \end{pmatrix}.$$

By construction, $y^{\mathrm{T}}My = 0 \Rightarrow A_R^{\mathrm{T}}y = 0$, since $y^{\mathrm{T}}My = y^{\mathrm{T}}A_R G A_R^{\mathrm{T}}y = (A_R^{\mathrm{T}}y)^{\mathrm{T}}G(A_R^{\mathrm{T}}y) = 0 \Rightarrow A_R^{\mathrm{T}}y = 0$ in virtue of the definiteness of G. This results in the property

$$\begin{pmatrix} A_R G A_R^{\mathrm{T}} & N \\ -N^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = 0 \Leftrightarrow \left\{ \begin{array}{c} A_R^{\mathrm{T}}y = 0 \\ N^{\mathrm{T}}y = 0 \\ Nz = 0 \end{array} \right\}, \tag{7}$$

which is non-trivial only in the "$\Rightarrow$" sense: $My + Nz = 0$ yields $y^{\mathrm{T}}My + y^{\mathrm{T}}Nz = 0$, but $y^{\mathrm{T}}N = (N^{\mathrm{T}}y)^{\mathrm{T}} = 0$ implies that $y^{\mathrm{T}}My = 0$ and, as indicated above, $A_R^{\mathrm{T}}y = 0$. $Nz = 0$ follows.

The equivalence depicted in (7) can be also written as

$$\text{Ker } g'(x^*) = \text{Ker } (A_R \;\; N)^{\mathrm{T}} \times \text{Ker } N, \tag{8}$$

and, therefore, Ker $g'(x^*) = \{0\}$ if and only if Ker $(A_R \;\; N)^{\mathrm{T}} =$ Ker $(A_R \;\; A_L \;\; A_V)^{\mathrm{T}} = \{0\}$ and Ker $N =$ Ker $(A_L \;\; A_V) = \{0\}$. But, owing to Corollaries 2 and 3, these two conditions are equivalent to the absence of I-C cutsets and V-L loops.

□

3.2 Non-vanishing, purely imaginary eigenvalues

Non-trivial, purely imaginary eigenvalues are important in linear circuit applications since they characterize the existence of periodic solutions describing oscillations. In non-linear circuits, a pair of purely imaginary eigenvalues may be responsible for a Hopf bifurcation phenomenon also yielding oscillations [24].

Lemma 5. *Let K be positive (resp. negative) definite. Denoting as $z^\star$ the conjugate transpose of z, then $z^\star \frac{K+K^{\mathrm{T}}}{2} z$ is real and positive (resp. negative) for any non-vanishing complex vector z.*

This well-known property can be easily checked by writing $z = x + y\sqrt{-1}$, for real vectors x, y. Then,

$$z^\star \frac{K+K^{\mathrm{T}}}{2} z = x^{\mathrm{T}} \frac{K+K^{\mathrm{T}}}{2} x + y^{\mathrm{T}} \frac{K+K^{\mathrm{T}}}{2} y > 0 \;\; (\text{resp.} < 0),$$

since the purely imaginary terms cancel due to the symmetry of $\frac{K+K^{\mathrm{T}}}{2}$, and x and y do not vanish simultaneously.

Theorem 2. *If G is (positive or negative) definite, both $C = C(A_C^{\mathrm{T}} e^*)$ and $L = L(i_l^*)$ are symmetric and non-singular, and any one of the conditions*

a) there are no I-C-L cutsets; or

b) there are no V-C-L loops;

is satisfied, then there are no purely imaginary eigenvalues $\lambda = \alpha j$ with $\alpha \in \mathbb{R} - \{0\}$.

Proof: Note that $\lambda \in \mathbb{C}$ is an eigenvalue if and only if there exists a nonvanishing vector $w = (w_e, w_l, w_v)$ such that $(\lambda A(x^*) + g'(x^*))w = 0$, that is,

$$\begin{pmatrix} \lambda A_C C A_C^{\mathrm{T}} + A_R G A_R^{\mathrm{T}} & A_L & A_V \\ -A_L^{\mathrm{T}} & \lambda L & 0 \\ -A_V^{\mathrm{T}} & 0 & 0 \end{pmatrix} \begin{pmatrix} w_e \\ w_l \\ w_v \end{pmatrix} = 0,$$

or

$$\lambda A_C C A_C^{\mathrm{T}} w_e + A_R G A_R^{\mathrm{T}} w_e + A_L w_l + A_V w_v = 0 \tag{9a}$$

$$-A_L^{\mathrm{T}} w_e + \lambda L w_l = 0 \tag{9b}$$

$$-A_V^{\mathrm{T}} w_e = 0. \tag{9c}$$

Multiplying (9a) by the conjugate transpose $w_e^\star$, we get

$$\lambda w_e^\star A_C C A_C^{\mathrm{T}} w_e + w_e^\star A_R G A_R^{\mathrm{T}} w_e + w_e^\star A_L w_l + w_e^\star A_V w_v \quad = \quad 0. \tag{10}$$

Note that (9b) yields $w_e^\star A_L = \overline{\lambda} w_l^\star L$, where we have made use of the symmetry of L. On the other hand, from (9c), it follows that $w_e^\star A_V = 0$. Therefore, (10) reads

$$\lambda w_e^\star A_C C A_C^{\mathrm{T}} w_e + w_e^\star A_R G A_R^{\mathrm{T}} w_e + \overline{\lambda} w_l^\star L w_l \quad = \quad 0. \tag{11}$$

Applying the * operator to (11), we are led to

$$\overline{\lambda} w_e^\star A_C C A_C^{\mathrm{T}} w_e + w_e^\star A_R G^{\mathrm{T}} A_R^{\mathrm{T}} w_e + \lambda w_l^\star L w_l \quad = \quad 0, \tag{12}$$

since C is also symmetric. The semisum of (11) and (12) reads

$$(\mathrm{Re}\lambda) w_e^\star A_C C A_C^{\mathrm{T}} w_e + w_e^\star A_R \frac{G + G^{\mathrm{T}}}{2} A_R^{\mathrm{T}} w_e + (\mathrm{Re}\lambda) w_l^\star L w_l \quad = \quad 0. \tag{13}$$

Let λ be a non-vanishing eigenvalue with $\mathrm{Re}\lambda = 0$. Equation (13) then leads to $A_R^{\mathrm{T}} w_e = 0$, owing to Lemma 5. Note also that $A_V^{\mathrm{T}} w_e = 0$, as displayed in (9c).

Assume that condition a) is satisfied. The exclusion of I-C-L cutsets, together with $A_R^{\mathrm{T}} w_e = 0$ and $A_V^{\mathrm{T}} w_e = 0$, implies that $w_e = 0$. From (9b), the assumption $\lambda \neq 0$, and the non-singularity of L, we get $w_l = 0$. Then, from (9a), we get $A_V w_v = 0$, and the exclusion of V-loops in well-posed circuits would yield $w_v = 0$.

Assume now that condition b) is satisfied, and write (9a) as

$$A_C(\lambda C A_C^{\mathrm{T}} w_e) + A_L w_l + A_V w_v \quad = \quad 0,$$

since $A_R^{\mathrm{T}} w_e = 0$. From the V-C-L loop exclusion property, it follows that $\lambda C A_C^{\mathrm{T}} w_e = 0$, $w_l = 0$, $w_v = 0$. From the first identity, the non-vanishing of λ, and the non-singularity of C, we get $A_C^{\mathrm{T}} w_e = 0$. On the other hand, $w_l = 0$ yields, in the light of (9b), $A_L^{\mathrm{T}} w_e = 0$. Together with the conditions $A_C^{\mathrm{T}} w_e = 0$, $A_R^{\mathrm{T}} w_e = 0$, $A_V^{\mathrm{T}} w_e = 0$, and the exclusion of I cutsets in well-posed circuits, we would get $w_e = 0$.

□

3.3 Hyperbolicity

An equilibrium point x^* is said to be *hyperbolic* if the spectrum of the linearization has no purely imaginary eigenvalues. Null eigenvalues have been considered in terms of the non-singularity of $g'(x^*)$ in 3.1, whereas non-trivial, purely imaginary eigenvalues are ruled out by the discussion performed in 3.2. Hence, Theorems 1 and 2 together provide a sufficient condition for the hyperbolicity of the matrix pencil. Merging the topological conditions, and with some additional effort (Propositions 2 and 3), we may improve the result in order to allow for the existence of I-L cutsets and V-C loops, so that the resulting topological conditions be entirely independent of the index conditions appearing in Proposition 1. Therefore, Theorem 3 will naturally apply to both index-1 and index-2 problems.

Proposition 2. *Let $\mathcal{K}_1,\ \mathcal{K}_2$ be two sets of branches of a connected graph $\mathcal{G}$, $\mathcal{K}_1 \subseteq \mathcal{K}_2$. If all cutsets within $\mathcal{K}_2$ are actually contained in $\mathcal{K}_1$, then*

$$w^{\mathrm{T}} A_{\mathcal{G}-\mathcal{K}_2} = 0 \Rightarrow w^{\mathrm{T}} A_{\mathcal{K}_2-\mathcal{K}_1} = 0.$$

Equivalently, Ker $A^{\mathrm{T}}_{\mathcal{G}-\mathcal{K}_2} =$ Ker $A^{\mathrm{T}}_{\mathcal{G}-\mathcal{K}_1}$.

Proof: From the working assumption we may derive that $\mathrm{rk}A_{\mathcal{G}-\mathcal{K}_2} = \mathrm{rk}A_{\mathcal{G}-\mathcal{K}_1}$. This can be seen as follows: assume $r_2 = \mathrm{rk}A_{\mathcal{G}-\mathcal{K}_2} < \mathrm{rk}A_{\mathcal{G}-\mathcal{K}_1} = r_1$. If $r_1 < n-1$, that is, if $\mathcal{G} - \mathcal{K}_1$ is not connected, add to $\mathcal{G} - \mathcal{K}_1$ a set $\mathcal{M}$ of $(n-1) - r_1$ branches from within $\mathcal{K}_1$ to get a connected graph, that is, complete minimally the columns of $A_{\mathcal{G}-\mathcal{K}_1}$ so that full row rank is achieved in $(A_{\mathcal{G}-\mathcal{K}_1}\ A_{\mathcal{M}})$.

From the assumption $r_2 < r_1$, $(A_{\mathcal{G}-\mathcal{K}_2}\ A_{\mathcal{M}})$ cannot meet the maximal rank $n-1$, so that $(\mathcal{G}-\mathcal{K}_2)\cup\mathcal{M} = \mathcal{G}-(\mathcal{K}_2-\mathcal{M})$ is not connected. This means that there is a "disconnecting set" of branches (and therefore a cutset) within the set $\mathcal{K}_2 - \mathcal{M}$. This cutset cannot be formed exclusively by elements of $\mathcal{K}_1 - \mathcal{M}$ since their removal from $\mathcal{G}$ would result in a connected set (because it would comprise $\mathcal{G}-(\mathcal{K}_1-\mathcal{M})$, which satisfies $\mathrm{rk}(A_{\mathcal{G}-\mathcal{K}_1}\ A_{\mathcal{M}}) = n-1$). This means that there is at least one branch of this cutset belonging to $(\mathcal{K}_2-\mathcal{M}) - (\mathcal{K}_1-\mathcal{M}) = \mathcal{K}_2-\mathcal{K}_1$, against the main assumption in the statement.

Hence, from the already proved identity $\mathrm{rk}A_{\mathcal{G}-\mathcal{K}_2} = \mathrm{rk}A_{\mathcal{G}-\mathcal{K}_1}$, it follows that all columns in $\mathcal{K}_2 - \mathcal{K}_1 = (\mathcal{G}-\mathcal{K}_1) - (\mathcal{G}-\mathcal{K}_2)$ can be written as a linear combination of those in $\mathcal{G}-\mathcal{K}_2$, that is, there exists a matrix F such that

$$A_{\mathcal{K}_2-\mathcal{K}_1} = A_{\mathcal{G}-\mathcal{K}_2} F.$$

Therefore, from the assumption $w^{\mathrm{T}} A_{\mathcal{G}-\mathcal{K}_2} = 0$ we get $w^{\mathrm{T}} A_{\mathcal{K}_2-\mathcal{K}_1} = w^{\mathrm{T}} A_{\mathcal{G}-\mathcal{K}_2} F = 0$.

□

Proposition 3. *Let $\mathcal{J}_1,\ \mathcal{J}_2$ be two sets of branches of a connected graph $\mathcal{G}$, $\mathcal{J}_1 \subseteq \mathcal{J}_2$. If all loops within $\mathcal{J}_2$ are actually contained in $\mathcal{J}_1$, then*

$$A_{\mathcal{J}_1} w_1 + A_{\mathcal{J}_2-\mathcal{J}_1} w_2 = 0 \Rightarrow w_2 = 0.$$

Equivalently, letting the first columns of $A_{\mathcal{J}_2}$ be those of $A_{\mathcal{J}_1}$, Ker $A_{\mathcal{J}_2} =$ Ker $A_{\mathcal{J}_1} \times \{0\}$.

Proof: If $w_2 \neq 0$, then it is possible to construct a vanishing, non-trivial linear combination of columns of $A_{\mathcal{J}_2}$, corresponding to two sets of branches $\mathcal{R} \subseteq \mathcal{J}_1$ and $\mathcal{L} \subseteq \mathcal{J}_2 - \mathcal{J}_1$, in a way such that the columns in $A_{\mathcal{R}}$ are linearly independent and $\mathcal{L}$ is non-empty. The existence of such a linear combination implies that $(A_{\mathcal{R}}\ A_{\mathcal{L}})$ does not have full column rank, and then Lemma 4 leads to the existence of a loop formed by branches within the set $\mathcal{R}\cup\mathcal{L} \subseteq \mathcal{J}_2$. On the other hand, loops exclusively formed by branches in $\mathcal{R}$ are precluded from the linear independence of columns in $A_{\mathcal{R}}$, so that there must actually exist one loop containing some branch $l \in \mathcal{J}_2 - \mathcal{J}_1$.

□

Theorem 3. *If G is (positive or negative) definite, both C and L are symmetric and non-singular, and any one of the two pairs of conditions*

a) there are neither V-L loops nor I-C-L cutsets (except maybe I-L cutsets); or

b) there are neither I-C cutsets nor V-C-L loops (except maybe V-C loops);

is satisfied, then $\text{Re}\lambda \neq 0, \ \forall \lambda \in \sigma\{A(x^*), g'(x^*)\}$.

Proof: Since I-C-L cutsets include in particular I-C cutsets, and so do V-C-L loops with regard to V-L loops, the only cases which do not follow automatically from Theorem 1 and Theorem 2 are those in which either I-L cutsets or V-C loops are present. We have to show that purely imaginary non-vanishing eigenvalues may not exist in this situation.

Let us first consider case a). Proceeding as in the proof of Theorem 2, we get $A_R^{\mathrm{T}} w_e = 0$ and $A_V^{\mathrm{T}} w_e = 0$. But now the condition $w_e = 0$ derived there does not follow from the cutset exclusion property, since I-L cutsets may be present. In contrast, we will make use of the exclusion of V-L loops, which did not hold there.

Denote as $\mathcal{K}_1$ the set of branches corresponding to inductors and current sources, and as $\mathcal{K}_2$ the ones corresponding to capacitors, inductors and current sources. If $\mathcal{G}$ stands for the graph of the circuit, the branches in $\mathcal{G} - \mathcal{K}_2$ correspond to resistors and voltage sources, whereas those in $\mathcal{K}_2 - \mathcal{K}_1$ are the capacitive ones. With this notation, and in the light of Proposition 2, we get that $w_e^{\mathrm{T}}(A_R \ A_V) = 0 \Rightarrow w_e^{\mathrm{T}} A_C = 0$, that is, $A_C^{\mathrm{T}} w_e = 0$. From this property, (9a) reads $A_L w_l + A_V w_v = 0$, and the exclusion of V-L loops in a) yields $w_l = 0$, $w_v = 0$. Additionally, (9b) implies $A_L^{\mathrm{T}} w_e = 0$, and the absence of I cutsets in well-posed circuits implies $w_e = 0$.

Now consider case b). Again, $A_R^{\mathrm{T}} w_e = 0$ and $A_V^{\mathrm{T}} w_e = 0$ hold, but the reasoning in Theorem 2 cannot be applied to derive $A_C^{\mathrm{T}} w_e = 0$ since now V-C loops may be present. The proof in this case will now rely upon the non-existence of I-C cutsets.

Using $A_R^{\mathrm{T}} w_e = 0$, equation (9a) reads

$$\lambda A_C C A_C^{\mathrm{T}} w_e + A_L w_l + A_V w_v \quad = \quad 0. \tag{14}$$

Let $\mathcal{J}_1$ stand for the capacitor and voltage source branches, and assume that $\mathcal{J}_2$ includes these and, additionally, the inductive branches. Based upon the absence of V-C-L loops except for V-C loops, a straightforward application of Proposition 3 shows that (14) yields $w_l = 0$. In virtue of (9b), it is $A_L^{\mathrm{T}} w_e = 0$, and the properties $A_R^{\mathrm{T}} w_e = 0$, $A_V^{\mathrm{T}} w_e = 0$, together with the exclusion of I-C cutsets, lead to $w_e = 0$. Finally, $w_v = 0$ from (9a) and the absence of V-loops in well-posed circuits.

□

Note that both a) and b) in Theorem 3 are (in a certain sense) minimal topological extensions of the conditions in Theorem 1 excluding null eigenvalues.

4 Asymptotic stability

Proposition 4. *If G is positive definite, and both C and L are symmetric positive definite, then* $\text{Re}\lambda \leq 0,\ \forall \lambda \in \sigma\{A(x^*), g'(x^*)\}$.

Proof: The derivation of (13) in Theorem 2 is still valid under the current working assumptions. Let λ be an eigenvalue with $\text{Re}\lambda > 0$. From Lemma 5 and the assumption of symmetry and positive definiteness of both C and L, it follows that

$$w_e^\star A_C C A_C^{\mathrm{T}} w_e = w_e^\star A_R \frac{G+G^{\mathrm{T}}}{2} A_R^{\mathrm{T}} w_e = w_l^\star L w_l \quad = \quad 0, \tag{15}$$

so that $A_C^{\mathrm{T}} w_e = 0$, $A_R^{\mathrm{T}} w_e = 0$, $w_l = 0$ and (using (9b)) $A_L^{\mathrm{T}} w_e = 0$. Additionally, $A_V^{\mathrm{T}} w_e = 0$ as displayed in (9c). Since current source cutsets are forbidden in well-posed circuits, it follows that $w_e = 0$. From (9a), we get $A_V w_v = 0$ and, since voltage source loops are also excluded in well-posed circuits, it follows that $w_v = 0$. This would yield the contradiction $w = 0$, meaning that it must be $\text{Re}\lambda \leq 0$.

□

It is worth emphasizing that Proposition 4 does not require topological conditions at all, apart from the exclusion of V-loops and I-cutsets guaranteeing that the circuit is well-posed. In particular, it holds independently of the topological index conditions. Note also that the assumptions on the devices imply in particular those used in the hyperbolicity criterion of Theorem 3. Therefore, adding the topological conditions assumed there, we get the following sufficient condition to guarantee that $\text{Re}\lambda < 0$ for $\lambda \in \sigma\{A(x^*), g'(x^*)\}$.

Theorem 4. *Assume that:*

1) G is positive definite, and both C and L are symmetric positive definite.

2) At least one of the two pairs of topological conditions holds:

2a) There are neither V-L loops nor I-C-L cutsets (except maybe I-L cutsets); or

2b) There are neither I-C cutsets nor V-C-L loops (except maybe V-C loops).

Then, all eigenvalues in the spectrum $\sigma\{A(x^*), g'(x^*)\}$ verify $\text{Re}\lambda < 0$. □

Again, the topological conditions here are independent of those characterizing the index in Proposition 1. This provides a significant improvement with respect to [5, Theorem 12], supported on a state-space approach, since the local solvability and topological requirements there restrict the scope of Chua's result to certain index-1 configurations.

It remains to show that the condition $\text{Re}\lambda < 0$ for $\lambda \in \sigma\{A(x^*), g'(x^*)\}$ guarantees indeed asymptotic stability of the equilibrium. Loosely speaking, the difficulty in the linearization does not rely on the "dynamics" (since this is essentially given by Lyapunov's theorem [1]) but on the DAE structure. See [20, 21, 22, 26, 30, 31, 37] for details. In our case, we need to show that Lyapunov's theorem can be applied to index-1 and index-2 MNA equations.

We first remark in this direction that we are allowed to work with the charge-oriented model

$$\begin{aligned} A_C q' + A_R \gamma(A_R^{\mathrm{T}} e) + A_L i_l + A_V i_v + A_I i_s &= 0 && \text{(16a)} \\ \phi' - A_L^{\mathrm{T}} e &= 0 && \text{(16b)} \\ -A_V^{\mathrm{T}} e + v_s &= 0 && \text{(16c)} \\ q - \psi(A_C^{\mathrm{T}} e) &= 0 && \text{(16d)} \\ \phi - \varphi(i_l) &= 0 && \text{(16e)} \end{aligned}$$

instead of the conventional one, by virtue of Proposition 5 below. We have written this system focusing on the autonomous case. Note that the index of this model is proved in [38, Th. 7] to coincide with that of the conventional system as far as both are not higher than 2. System (16) has a semilinear structure

$$\hat{A} z' + \hat{g}(z) = 0, \tag{17}$$

in which the coefficient matrix $\hat{A}$ is constant. This property will make the analysis easier.

Proposition 5. *If $x^* = (e^*, i_l^*, i_v^*)$ is an equilibrium of the conventional MNA model (5) and $z^* = (x^*, q^*, \phi^*)$ with $q^* = \psi(A_C^{\mathrm{T}} e^*)$, $\phi^* = \varphi(i_l^*)$ is the corresponding equilibrium of the charge-oriented system (17), then the spectrum of the matrix pencil $\lambda A(x^*) + g'(x^*)$ (6) coincides with that of the charge-oriented pencil*

$$\lambda \hat{A} + \hat{g}'(z^*) = \lambda \begin{pmatrix} 0 & 0 & 0 & A_C & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} A_R G A_R^{\mathrm{T}} & A_L & A_V & 0 & 0 \\ -A_L^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ -A_V^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ -C A_C^{\mathrm{T}} & 0 & 0 & I & 0 \\ 0 & -L & 0 & 0 & I \end{pmatrix}, \tag{18}$$

regardless of the index.

Proof: Note that in (18) we denote $G = G(A_R^{\mathrm{T}} e^*) = \gamma'(A_R^{\mathrm{T}} e^*)$, $C = C(A_C^{\mathrm{T}} e^*) = \psi'(A_C^{\mathrm{T}} e^*)$, $L = L(i_l^*) = \varphi'(i_l^*)$. The spectrum of the matrix pencil (18) is the set of values of λ for which the matrix

$$M = \left(\begin{array}{ccc:cc} A_R G A_R^{\mathrm{T}} & A_L & A_V & \lambda A_C & 0 \\ -A_L^{\mathrm{T}} & 0 & 0 & 0 & \lambda I \\ -A_V^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ \hdashline -C A_C^{\mathrm{T}} & 0 & 0 & I & 0 \\ 0 & -L & 0 & 0 & I \end{array}\right) \tag{19}$$

is singular. We have partitioned M in (19) as

$$M = \left(\begin{array}{c:c} D & E \\ \hdashline J & K \end{array} \right) \tag{20}$$

to make apparent that $D - EK^{-1}J$ is the Schur reduction [19] of the identity K in M. Therefore, M is singular if and only if so it is $D - EK^{-1}J$, which reads

$$\begin{pmatrix} A_R G A_R^{\mathrm{T}} & A_L & A_V \\ -A_L^{\mathrm{T}} & 0 & 0 \\ -A_V^{\mathrm{T}} & 0 & 0 \end{pmatrix} - \begin{pmatrix} \lambda A_C & 0 \\ 0 & \lambda I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -CA_C^{\mathrm{T}} & 0 & 0 \\ 0 & -L & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} A_R G A_R^{\mathrm{T}} & A_L & A_V \\ -A_L^{\mathrm{T}} & 0 & 0 \\ -A_V^{\mathrm{T}} & 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} A_C C A_C^{\mathrm{T}} & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is exactly the matrix whose singularities define the eigenvalues of the matrix pencil (6), what proves the fact that both spectra are identical.

□

Theorem 5. *Assume that the conventional system (5) has index ≤ 2. Let x^* be an equilibrium point of (5), i.e. $g(x^*) = 0$, and assume that the spectrum $\sigma\{A(x^*), g'(x^*)\}$ of the matrix pencil (6) has only eigenvalues with negative real part. Let ψ, φ and γ be C^1 and, in the index-2 case, assume additionally that ψ', φ' and γ' are positive definite and that ψ'', φ'' and γ'' exist and are uniformly bounded. Then x^* is asymptotically stable in the sense of Lyapunov.*

Proof: Note that for index-0 cases it suffices to apply Lyapunov's theorem for ODEs [1]. Let again (x^*, q^*, ϕ^*), with $q^* = \psi(A_C^{\mathrm{T}} e^*)$, $\phi^* = \varphi(i_l^*)$, denote the corresponding equilibrium of the charge-oriented system (17). In the index-1 case, its semilinear structure allows for the direct application of [21, Th. 2.1] to conclude that (x^*, q^*, ϕ^*) is asymptotically stable for the dynamics of the charge-oriented MNA model. Alternatively, we may rewrite the semilinear charge-oriented equation as a semiexplicit one: if $\mathrm{rk}\hat{A} = r$, there exist non-singular matrices E, F such that [11]

$$E\hat{A}F = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Now, if we premultiply (17) by E and perform the coordinate change $z = Fu$, the semilinear system in z adopts a semiexplicit form in u, and the index-1 condition is preserved. We may then directly apply [31, Th. 1].

For the index-2 case, we want to apply Theorem 3.3 in [22]. Introducing

$$y = \begin{pmatrix} \begin{pmatrix} y_e \\ y_L \\ y_V \end{pmatrix} \\ y_q \\ y_\phi \end{pmatrix} = \begin{pmatrix} x - x^* \\ q - q^* \\ \phi - \phi^* \end{pmatrix},$$

the equation system (16) can be rewritten as

$$\hat{A}y' + \hat{B}y + \hat{h}(y) = 0 \tag{21}$$

with

$$\hat{A} = \begin{pmatrix} 0 & 0 & 0 & A_C & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} A_R\gamma'(A_R^{\mathrm{T}}e^*)A_R^{\mathrm{T}} & A_L & A_V & 0 & 0 \\ -A_L^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ -A_V^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ -\psi'(A_C^{\mathrm{T}}e^*)A_C^{\mathrm{T}} & 0 & 0 & I & 0 \\ 0 & -\varphi'(i_l^*) & 0 & 0 & I \end{pmatrix}$$

and

$$\hat{h}(y) = \begin{pmatrix} A_R\gamma(A_R^{\mathrm{T}}(y_e + e^*)) - A_R\gamma(A_R^{\mathrm{T}}e^*) - A_R\gamma'(A_R^{\mathrm{T}}e^*)A_R^{\mathrm{T}}y_e \\ 0 \\ 0 \\ -\psi(A_C^{\mathrm{T}}(y_e + e^*)) + \psi(A_C^{\mathrm{T}}e^*) + \psi'(A_C^{\mathrm{T}}e^*)A_C^{\mathrm{T}}y_e \\ -\phi(y_L + i_l^*) + \phi(i_l^*) + \varphi'(i_l^*)y_L. \end{pmatrix}.$$

Then, $\hat{h}(y)$ is twice continuously differentiable with $\hat{h}(0) = 0$ and $\hat{h}'(0) = 0$. Additionally, $\hat{h}''(y)$ is uniformly bounded. Consequently, the assumptions (A) and (B) of Theorem 3.3 in [22] are satisfied. Next, we will see that also assumption (C) of Theorem 3.3 in [22] is true. Using Lemma 3.4 in [22], it is sufficient to show

$$\hat{h}(y) = \hat{h}(Py + UQy)$$

for projectors P, Q and U satisfying

$$\operatorname{im} Q = \ker \hat{A}, \quad P = I - Q, \quad \ker U = \ker \hat{A} \cap \{z : \ \hat{B}z \in \operatorname{im} \hat{A}\}.$$

Regarding the special structure of $\hat{A}$, we may choose

$$Q := \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_C & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with any projector $\bar{Q}_C$ onto $\ker A_C$. Taking additionally any projector Q_C onto $\ker A_C^{\mathrm{T}}$, the space $\ker \hat{A} \cap \{z : \ \hat{B}z \in \operatorname{im} \hat{A}\}$ equals

$$\{z : \ A_C^{\mathrm{T}}z_e = 0, \ A_R^{\mathrm{T}}z_e = 0, \ A_V^{\mathrm{T}}z_e = 0, \ z_L = 0, \ Q_C^{\mathrm{T}}A_Vz_V = 0, \ z_q = 0, \ z_\phi = 0\}.$$

Consequently,

$$U := \begin{pmatrix} I - Q_{CRV} & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I - \bar{Q}_{C-V} & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}$$

projects onto $\ker \hat{A} \cap \{z : \ \hat{B}z \in \operatorname{im} \hat{A}\}$ for any projector Q_{CRV} onto $\ker(A_C, A_R, A_V)^{\mathrm{T}}$ and any projector $\bar{Q}_{C-V}$ onto $\ker Q_C^{\mathrm{T}} A_V$. This leads to

$$Py + UQy = \begin{pmatrix} (I - Q_{CRV})y_e \\ y_L \\ (I - \bar{Q}_{C-V})y_V \\ y_q \\ y_\phi \end{pmatrix},$$

which implies $\hat{h}(y) = \hat{h}(Py + UQy)$.

Therefore, we may apply Theorem 3.3 in [22], showing that the the equilibrium (x^*, q^*, ϕ^*) of (17) is asymptotically stable in the sense of Lyapunov for the change-oriented index-2 case.

Finally, since the dynamics of the charge-oriented and the conventional system are linked by the smooth functions $q = \psi(A_C^{\mathrm{T}} e)$, $\phi = \varphi(i_l)$, it follows that in both index-1 and index-2 cases the equilibrium x^* is also asymptotically stable for the conventional MNA system (5).

□

5 A Josephson junction circuit

Consider the nonlinear circuit depicted in Figure 1. The device labeled as L_2 is a *Josephson junction*, consisting of two superconductors separated by an insulating layer [8]. This junction can be treated as a nonlinear inductor with a current-flux characteristic $i_2 = I_0 \sin k\phi_2$, where $I_0 > 0$ is a device parameter, and $k = 4\pi e/h$ (e and h denoting electron charge and Planck's constant, respectively). The incremental inductance of this device is $L_2 = (I_0 k \cos k\phi_2)^{-1}$.

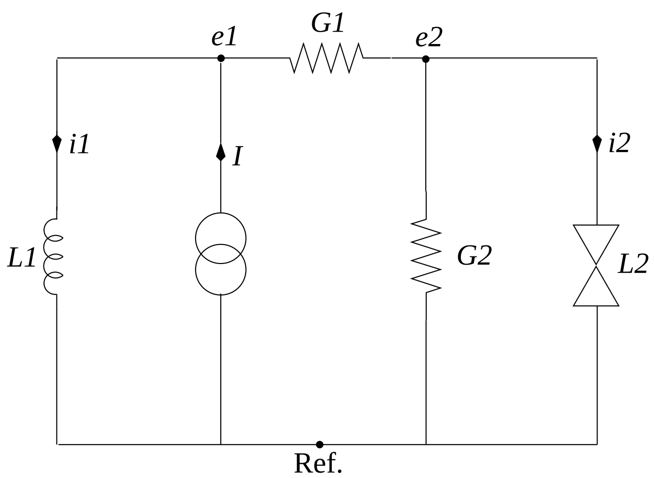

Figure 1: Nonlinear circuit.

The two resistors are linear with conductances G_1, G_2, and the inductor L_1 is a linear one.

MNA equations for this circuit are easily shown to read

$$
\begin{aligned}
L_1 i_1' &= e_1 && \text{(22a)} \\
L_2 i_2' &= e_2 && \text{(22b)} \\
0 &= i_1 + G_1(e_1 - e_2) - I && \text{(22c)} \\
0 &= i_2 - G_1(e_1 - e_2) + G_2 e_2. && \text{(22d)}
\end{aligned}
$$

Equilibrium points are given by $e_1 = e_2 = 0,\ i_1 = I,\ i_2 = 0$. The last condition yields $\sin k\phi_2 = 0$, that is, $\phi_2 = n\pi/k,\ n \in \mathbb{Z}$, so that the incremental inductance L_2 at equilibrium is $(I_0 k)^{-1}$ for $\phi_2 = 2m\pi/k,\ m \in \mathbb{Z}$, and $-(I_0 k)^{-1}$ for $\phi_2 = (2m+1)\pi/k,\ m \in \mathbb{Z}$. In both cases, the Josephson junction is (locally, around equilibria) current-controlled.

Stability properties of these equilibria have been analyzed in [31] through the explicit computation of the spectrum arising in a certain differential-algebraic model of the circuit. Our present purpose is to illustrate that such a local qualitative analysis can be performed checking only device characteristics and circuit topology, without making explicit use of (22) or any other semistate model.

Following [31], we will assume that $G_1 > 0$, $L_1 > 0$, and will consider two different cases: $G_2 > 0$, and $G_2 = 0$. The latter describes a situation in which the resistor defined by G_2 is open-circuited, yielding a change in the circuit configuration. The conductance matrix G reads $\mathrm{diag}(G_1, G_2)$ if $G_2 > 0$, and amounts to (G_1) if $G_2 = 0$, that is, if the branch associated with G_2 is removed. In both cases, the conductance matrix is (symmetric) positive definite. In contrast, the (symmetric) inductance matrix $L = \mathrm{diag}(L_1, L_2)$ is positive definite at equilibria yielding $L_2 = (I_0 k)^{-1}$ (that is, for $\phi_2 = 2m\pi/k,\ m \in \mathbb{Z}$), whereas the case $L_2 = -(I_0 k)^{-1}$ (displayed for $\phi_2 = (2m+1)\pi/k,\ m \in \mathbb{Z}$) yields a non-definite matrix.

Index. Since neither capacitors nor voltage sources appear in the circuit, according to Proposition 1 it suffices to check for I-L cutsets in order to compute the index of the MNA model (22). It is easy to see that no I-L cutset is displayed when $G_2 > 0$; since the index-1 conditions in Proposition 1 only require a non-singularity assumption in the inductance matrix L, we may conclude that $G_2 > 0$ yields an index-1 configuration around any equilibrium, regardless of the sign of the Josephson junction inductance L_2.

In contrast, the case $G_2 = 0$ yields an I-L cutset defined by the linear inductor, the current source and the Josephson junction. In this situation, Proposition 1 only allows one to conclude that the index is 2 if L is positive definite, that is, around equilibria in which $L_2 = (I_0 k)^{-1} > 0$. Using (22), it is not difficult to check that, at the remaining equilibria (for which $L_2 = -(I_0 k)^{-1} < 0$), the index is 2 if and only if the additional condition $L_1 \neq -L_2$ is satisfied.

Non-singularity and hyperbolicity. The absence of capacitors and voltage sources make the topological conditions in Theorem 3 amount to the absence of L-loops, which is verified for all equilibria independently of the value of G_2. Note that this topological property is independent of the topological conditions characterizing the index. Since this theorem only requires L to be symmetric and non-singular, this means that all equilibria are hyperbolic, regardless of the sign of L_2. In more detail, zero eigenvalues are ruled out by Theorem 1, showing that equilibria are non-singular, whereas non-trivial, purely imaginary eigenvalues are precluded by case b) of Theorem 2.

Asymptotic stability. Finally, in the light of the absence of L-loops, Theorems 4 and 5 make it possible to guarantee that equilibria with $L_2 = (I_0 k)^{-1} > 0$ are asymptotically stable, since for them the inductance matrix is (symmetric) positive definite. This property is again independent of the topological index conditions.

Asymptotic stability of equilibria with $L_2 = -(I_0 k)^{-1} < 0$ cannot be assessed in terms of Theorems 4 and 5. An explicit spectrum computation shows that, when $G_2 > 0$, these equilibria are unstable; in contrast, if $G_2 = 0$, these equilibria are asymptotically stable if $-L_2 = (I_0 k)^{-1} < L_1$, and unstable if $-L_2 = (I_0 k)^{-1} > L_1$. Again, a more detailed study is necessary for non-passive circuits; note, in particular, that the two different situations depicted by the case $G_2 = 0$ show that the inertia of the conductance, capacitance and inductance matrices is not always sufficient to characterize the stability properties of equilibria in non-passive problems, even in hyperbolic cases.

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