

# From structural assumptions to a link between assets and interest rates

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**Abstract:** We derive a link between the short rate and a new index constructed in a multi-asset economy. This uses two structural assumptions: The volatility structure of the assets is *rigidly spherical*, and the short rate function is *homogeneous* of degree 0. We give clear motivations for the assumptions, and our main result is economically intuitive and testable from observed data. A preliminary empirical study illustrates how one can test such results.

**Key words:** relations between assets and interest rates, asset indices, homogeneity, multi-asset economy

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# 0 Introduction

This paper presents an approach to a derivation of links between assets and interest rates. The basic idea is to build a framework where clearly motivated structural hypotheses are exploited to yield natural economic results. This gives a transparent view how economic phenomena can be explained as consequences from simple modelling assumptions.

Our starting point is a standard framework for a multi-asset economy. We then impose two structural assumptions, one on the asset volatility matrix and one on the short rate (function), and we derive from these model properties a close link between assets and interest rates. The two assumptions are motivated in detail, and the derived result is both economically intuitive and testable from observed data. Our main focus is on the theoretical analysis of our model, but we also provide a short empirical part to illustrate how one could start to test our results.

In more detail, the paper is structured as follows. We present in Section 1 the basic setup where the assets  $X$  and the pricing kernel are described by general Itô processes. Section 2 defines indices (= numeraire) as the value processes of self-financing portfolios with unit initial capital and uses them to introduce the concepts of a spherical and a rigid volatility structure. The volatility matrix  $\sigma$  of  $X$  is called *spherical* if one can find an index  $I$  such that all asset prices expressed in the numeraire  $I$  have volatilities of equal magnitude. If these intrinsic volatility vectors are even constant,  $\sigma$  is called *rigid*. This corresponds to a market which fluctuates in a Black-Scholes type fashion around a suitable (spherical) index  $I$ .

Section 3 is mainly computational. We specialize the general setup to an autonomous Markovian situation and compute the dynamics of the short rate  $r(X)$  under the assumption that the function  $r$  is homogeneous of degree 0. This is motivated from a postulate of scale-invariance for asset prices; see Section 3 for a more detailed discussion. Combining this with a rigidly spherical volatility structure  $\sigma$ , we derive in Section 4 a very intuitive link between the short rate  $r(X)$  and the spherical index  $I$ . In the special case where the short rate volatility is constant, we obtain a distinction into two basic regimes: The instantaneous correlation between index  $I$  and short rate  $r(X)$  is positive or negative, depending on whether the forward rate curve at the short end is downward or upward sloping. Finally, we provide in Section 5 a preliminary study to illustrate how one could test such results on the basis of empirical data.

Of course, many questions remain. Our model is a stylized economy whose basic assumptions may be criticized. Even if one accepts them, a thorough empirical study still needs to be done. We have used in our statistical analysis very crude methods; dealing with nonstationary situations will require more sophisticated tools. However, our main message is that fairly general *structural* assumptions in a standard model can be enough to explain natural and observed economic phenomena.

## 1 The basic setup

Our ultimate goal is to derive links between the dynamics of assets and interest rates from simple structural assumptions in a general model. This section provides the basic framework.

We start with a probability space  $(\Omega, \mathcal{F}, P)$ , a time horizon  $T_\infty \in (0, \infty)$  and a filtration  $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T_\infty}$  satisfying the usual conditions of right-continuity and completeness. As in [HK], a *unit* is any strictly positive  $\mathbb{F}$ -adapted RCLL process  $N = (N_t)_{0 \leq t \leq T_\infty}$  with  $N_0 = 1$ . For  $T \in (0, T_\infty]$ , a  $T$ -claim is an  $\mathcal{F}_T$ -measurable random variable  $C_T \geq 0$ ; it describes the payoff at time  $T$  of some financial instrument. We denote its price at time  $t \leq T$  by  $C_t$  and assume that there exists a unit  $N$  such that

$$C_t = N_t E \left[ \frac{C_T}{N_T} \middle| \mathcal{F}_t \right] \quad (1.1)$$

for a sufficiently large class of  $T$ -claims  $C_T$ ; so  $\frac{1}{N}$  is a pricing kernel. More precisely, (1.1) should hold for at least  $n$  processes  $X^1, \dots, X^n$  on  $[0, T_\infty]$ . We think of  $X$  as a vector of basic asset prices and take this as a given fundamental ingredient. Note that if we postulate (1.1) also for any  $T$  and  $C_T = 1$ , we get from  $N$  the prices of all zero coupon bonds and thus an entire term structure model within the same framework. But for the moment  $X$  is all we need.

For  $N$ , we assume that its dynamics are given by

$$\frac{dN_t}{N_t} = (r_t + |\lambda_t|^2) dt + \lambda_t^\top dW_t \quad (1.2)$$

with an  $\mathbb{R}^m$ -valued Brownian motion  $W$  and predictable processes  $r$  (real-valued) and  $\lambda$  ( $\mathbb{R}^m$ -valued) such that  $\int_0^{T_\infty} (|r_u| + |\lambda_u|^2) du < \infty$   $P$ -a.s. This weak assumption is for instance satisfied if  $\mathbb{F}$  is the  $P$ -augmented filtration generated by  $W$  and  $N$  is a positive semimartingale with an absolutely continuous finite variation part.

**Remarks.** 1) The processes  $X^1, \dots, X^n$  describe the primary tradable assets in our market. They are observable and their evolution will eventually determine everything else; we formalize this later by imposing a Markovian structure.

2) One could view our setup as having  $n$  primary assets and  $m$  Brownian motions. If  $\mathbb{F}$  is generated by  $W$ , one might then distinguish between a complete ( $m < n$ ) and an incomplete ( $m \geq n$ ) market. But our focus is on links between assets and interest rates, and so issues of completeness versus incompleteness do not matter for us.  $\diamond$

To obtain more structure, we assume that the dynamics of our basic assets  $X$  on  $[0, T_\infty]$  are given by the SDEs

$$\frac{dX_t^i}{X_t^i} = dR_t^i = \mu_t^i dt + \sum_{j=1}^m \sigma_t^{ij} dW_t^j, \quad X_0^i = x_0^i > 0, \quad i = 1, \dots, n. \quad (1.3)$$

The drift vector  $\mu$  and volatility matrix  $\sigma$  are  $\mathbb{F}$ -predictable processes such that  $\int_0^{T_\infty} |\mu_u^i| du < \infty$   $P$ -a.s. and the  $\mathbb{R}^m$ -valued process  $\sigma^i = (\sigma^{ij})_{j=1, \dots, m}$  is  $W$ -integrable for each  $i$ . To exclude local redundancies between assets, we suppose that

**(FR)**  $\sigma_t$  has full rank  $P$ -a.s. for every  $t \in [0, T_\infty]$ .

If  $n \geq m$ , this implies that  $\sigma_t^\top \sigma_t$  is invertible  $P$ -a.s. for every  $t$ . Note that we do not assume the existence of a locally riskless asset; hence all the processes  $\sigma^i$  can be nonzero.

Trading in  $X$  by self-financing strategies is here modelled by pairs  $(v_0, a)$ , where  $v_0 \in (0, \infty)$  is the initial capital one starts with and the  $\mathbb{R}^n$ -valued predictable  $R$ -integrable process  $a = (a_t)_{0 \leq t \leq T_\infty}$  describes the fractions of total wealth held over time in the available assets. More precisely, the current wealth at time  $t$  must be positive and is

$$V_t(v_0, a) = v_0 \mathcal{E} \left( \int a^\top dR \right)_t = v_0 \exp \left( \int_0^t a_s^\top \sigma_s dW_s + \int_0^t \left( a_s^\top \mu_s - \frac{1}{2} |\sigma_s^\top a_s|^2 \right) ds \right), \quad (1.4)$$

and the fraction  $a_t^i$  of  $V_t(v_0, a)$  is currently invested in asset  $i$ . Fractions can be negative (we do not exclude short sales), but must of course sum to 1 so that we have in addition the restriction that  $a_t^\top \mathbf{1} \equiv 1$ , where  $\mathbf{1} = (1 \dots 1)^\top \in \mathbb{R}^n$ . Again as in [HK], the value process  $V(a)$  of such a strategy  $(1, a)$  with  $v_0 = 1$  is called a *numeraire* (because it is a unit obtained by self-financing trading) or a *generalized index*; see Section 2 for more details.

In combination with the pricing mechanism (1.1), our assumptions (1.2) and (1.3) about the dynamics of  $N$  and  $X$  automatically enforce some relations between the parameters  $r, \lambda$  and  $\mu, \sigma$ . This is the well-known no-arbitrage restriction that the drift  $\mu$  must be in the span of the vector  $\mathbf{1}$  and the range of the volatility matrix  $\sigma$ .

**Lemma 1** *Assume (1.2), (1.3) and the pricing relation (1.1) for  $C = X^i$ ,  $i = 1, \dots, n$ . Then*

**(DC)** *there exist  $\mathbb{F}$ -predictable processes  $\alpha$  ( $\mathbb{R}$ -valued) and  $\beta$  ( $\mathbb{R}^m$ -valued) with  $\beta$   $W$ -integrable,  $\int_0^{T_\infty} |\alpha_u| du < \infty$   $P$ -a.s. and such that*

$$\mu_t = \alpha_t \mathbf{1} + \sigma_t \beta_t \quad P\text{-a.s. for every } t \in [0, T_\infty]. \quad (1.5)$$

*In fact, we can take  $\alpha = r$  and  $\beta = \lambda$ .*

**Proof.** This is a standard argument. By (1.1) the product  $XN$  is a  $P$ -martingale. Combining (1.2) and (1.3) gives its dynamics via the product rule, and the martingale property implies that the drift part is zero. Written out, this gives (DC).  $\blacksquare$

Conversely, (DC) plus the dynamics (1.3) of  $X$  essentially give (1.2) and (1.1). More precisely:

**Lemma 2** *Suppose that  $X$  is given by (1.3). If (DC) holds, then*

**(AA)** *there exist a numeraire  $V(a)$  and a local  $P$ -martingale  $Z > 0$  with  $Z_0 = 1$  such that  $Z \frac{X}{V(a)}$  is a local  $P$ -martingale.*

**Proof.** The self-financing strategy  $(1, a)$  with  $a := (1 \ 0 \ \dots \ 0)^\top \in \mathbb{R}^n$  yields the numeraire  $V(a) = \frac{X}{X_0^1}$ . If we set  $U := - \int \alpha_u du - \int \beta_u^\top dW_u$  and  $Z := V(a) \mathcal{E}(U)$ , then  $Z > 0$  with  $Z_0 = 1$ , and (DC) readily implies that  $Z$  and  $XZ/V(a)$  are local  $P$ -martingales. Hence we get (AA).  $\blacksquare$

**Remarks.** 1) The *absence-of-arbitrage* type condition (AA) is up to some integrability issues almost as good as (1.1) plus (1.2). To see this, suppose that  $Z$  and  $V(a)$  have been constructed from (DC) as in the proof of Lemma 2. Then it is straightforward to check that  $N := \frac{V(a)}{Z} = 1/\mathcal{E}(-\int \alpha_u du - \int \beta^\top dW)$  satisfies (1.2) with  $r = \alpha$  and  $\lambda = \beta$ . Moreover, we also get (1.1) if  $\frac{ZX}{V(a)}$  is not only a local, but a true  $P$ -martingale. The reason for mentioning (AA) here is that it is equivalent to (DC) if  $\mathbb{F}$  is generated by  $W$ ; see [RSS] for a proof of this result.

2) Our setup and the above results are very similar to Jamshidian [Ja]; see [RSS] for a detailed discussion. But in the sequel, our objectives are different. Jamshidian [Ja] is mainly interested in a particular class of derivatives; he proves that homogeneous payoffs (e.g., LIBOR derivatives) can always be hedged under fairly general conditions. In contrast, our goal is to study dynamic links between assets and interest rates under extra structural assumptions.  $\diamond$

Under one additional assumption that will come up again later,  $\alpha$  and  $\beta$  in (DC) are unique and have a nice economic interpretation.

**Lemma 3** *Suppose that  $X$  is given by (1.3) and that (1.1) and (1.2) hold. (As a matter of fact, (1.3) plus (DC) is also enough.) If we have  $m < n$ , (FR) and*

$$(R1) \quad \mathbf{1} \notin \text{range}(\sigma_t) \quad P\text{-a.s. for every } t \in [0, T_\infty],$$

*then there is at most one pair  $(\alpha, \beta)$  satisfying (DC). Moreover,  $\bar{N} := \exp(\int r_u du)$  is the unique numeraire of finite variation.*

**Proof.** We omit this argument; see Theorem 1 of [RSS].  $\blacksquare$

Because  $\bar{N}$  is tradable, Lemma 3 implies that (under the stated assumptions)  $r$  can be interpreted as the instantaneous *short rate* for the classical savings account  $\bar{N}$ , and  $\lambda$  is the *market price of risk*. As shown in Corollary 3 of [RSS],  $\lambda$  can then be explicitly written as

$$\lambda = (\sigma^\top \sigma)^{-1} \sigma^\top (\mu - r\mathbf{1}), \quad (1.6)$$

and there is also an explicit expression for the short rate  $r$  in terms of  $\mu$  and  $\sigma$  alone. Intuitively, the setting of Lemma 3 corresponds to a complete market situation where the basic assets  $X$  determine the prices of all other assets, including the entire term structure of interest rates.

In general (i.e., without the additional assumptions in Lemma 3), we can still interpret  $r$  as a *virtual short rate* and  $\lambda$  as a *virtual market price of risk* in the following sense. If we define the (possibly not tradable) unit  $\hat{N} := \mathcal{E}(\int r_u du)$  and assume that the local  $P$ -martingale  $\hat{Z} := \frac{\hat{N}}{N} = \mathcal{E}(-\int \lambda^\top dW)$  is a true  $P$ -martingale, we can rewrite (1.1) as

$$C_t = \hat{N}_t E_{\hat{P}} \left[ \frac{C_T}{\hat{N}_T} \middle| \mathcal{F}_t \right]$$

with the probability measure  $\hat{P}$  defined by  $d\hat{P} = \hat{Z}_{T_\infty} dP$ . Hence prices are obtained by discounting with  $\hat{N}$  and then taking conditional expectations under  $\hat{P}$ , and so  $\hat{P}$  is an equivalent martingale measure for  $X$  with respect to the unit  $\hat{N}$ . Moreover,  $\hat{W} := W + \int \lambda_u du$  is then a Brownian motion under  $\hat{P}$ .

## 2 Asset indices and volatility structures

In this section, we look at general asset indices (= numeraires) and construct a particular one under a structural condition on the volatility matrix  $\sigma$  of  $X$ . We start with any strategy  $(1, a)$  and consider the process  $I^a = V(a)$ . Explicitly,  $a = (a_t)_{0 \leq t \leq T_\infty}$  is thus an  $\mathbb{R}^n$ -valued  $\mathbb{F}$ -predictable  $R$ -integrable process with  $a_t^\top \mathbf{1} \equiv 1$ , and the numeraire  $I^a = \mathcal{E} \left( \int a^\top dR \right)$  is obtained by starting with one unit of initial capital and holding at each date  $t$  the fraction  $a_t^i$  of total wealth in asset  $i$  for  $i = 1, \dots, n$ . The dynamics of  $I^a$  are very simple:

$$\frac{dI_t^a}{I_t^a} = a_t^\top dR_t = \bar{\mu}_t(a) dt + \bar{\sigma}_t(a)^\top dW_t \quad (2.1)$$

with  $\bar{\mu}(a) = a^\top \mu$  and the  $\mathbb{R}^m$ -valued process  $\bar{\sigma}(a) = \sigma^\top a$ . If we rewrite (2.1) as

$$\frac{dI_t^a}{I_t^a} = \sum_{i=1}^n a_t^i \frac{dX_t^i}{X_t^i}, \quad (2.2)$$

we see that  $I^a$  is observable from  $X$  if the strategy  $a$  is, and that at least at the instantaneous level, the return of  $I^a$  is a generalized convex combination of the returns of the  $X^i$ . We call  $I^a$  the (*generalized*) *index* associated to  $a$ . Because  $X$  and  $I^a$  are both stochastic exponentials, the  $I^a$ -discounted assets  $\tilde{X}(a) := X/I^a$  are readily seen to follow the SDEs

$$\frac{d\tilde{X}_t^i(a)}{\tilde{X}_t^i(a)} = (\mu_t^i - \bar{\mu}_t(a) - \bar{\sigma}_t(a)^\top \tilde{\sigma}_t^i(a)) dt + (\tilde{\sigma}_t^i(a))^\top dW_t$$

with  $\tilde{\sigma}^{ij}(a) := \sigma^{ij} - \bar{\sigma}^j(a)$  for  $i = 1, \dots, n$  and  $j = 1, \dots, m$ . Intuitively,  $\tilde{X}^i(a)$  describes the multiplicative fluctuations of asset  $X^i$  around the index  $I^a$ , and  $\tilde{\sigma}(a)$  is the matrix of *intrinsic volatilities* (with respect to  $I^a$ , to be accurate). Like the exchange prices in Platen [P00], the  $\tilde{X}^i(a)$  are ratios of two Itô processes and thus have a specific volatility structure.

To construct a particular index with good properties, we now introduce some terminology.

**Definition.** The volatility structure  $\sigma$  of  $X$  is called *spherical* if there exists an index  $I^a$  such that all corresponding intrinsic volatilities  $\tilde{\sigma}^i(a)$  are of equal magnitude, i.e.,  $|\sigma_t^i - \bar{\sigma}_t(a)| = \text{const}(t, \omega)$ ,  $P$ -a.s. for all  $t \in [0, T_\infty]$  and  $i = 1, \dots, n$ . Any such index  $I^a$  is called a *spherical index* (for the volatility structure  $\sigma$ ).

Put differently,  $\sigma$  is spherical if all volatility vectors  $\sigma^i$  lie on a sphere with center of the form  $\bar{\sigma}(a) = \sigma^\top a$  for some  $a$  with  $a^\top \mathbf{1} \equiv 1$ . This formulation is more convenient to work with and is equivalent to the above definition. In fact, due to (1.1) – (1.3), the drift  $\bar{\mu}(a)$  of any spherical index  $I^a$  is uniquely determined from  $\bar{\sigma}(a)$ , because (DC) gives  $\mu = r\mathbf{1} + \sigma\lambda$  and therefore

$$\bar{\mu}(a) = \mu^\top a = ra^\top \mathbf{1} + \lambda^\top \sigma^\top a = r + \lambda^\top \bar{\sigma}(a). \quad (2.3)$$

Hence a spherical index  $I^a$  is unique as soon as its volatility vector  $\bar{\sigma}(a)$  is unique.

**Proposition 4** Assume (1.1) – (1.3) and (FR), and recall that  $m = \dim W$ ,  $n = \dim X$ . In addition, assume

- 1) if  $m > n - 1$ : nothing extra.
- 2) if  $m = n - 1$ : that (R1) holds.
- 3) if  $m < n - 1$ : that  $\sigma$  is spherical.

Then there exists a unique spherical index.

**Proof.** a) Existence follows if  $\sigma$  is spherical. This is true in case 3) by assumption and clear in case 1) because due to (FR), the  $n$  points  $\sigma^1, \dots, \sigma^n$  in  $\mathbb{R}^m$  always lie on a sphere in  $\mathbb{R}^m$  if  $m \geq n$ . For case 2), we have to use (R1). Generally,  $n = m + 1$  vectors in  $\mathbb{R}^m$  lie on a sphere if and only if they are not in a hyperplane of dimension  $\leq m - 1$ . However, the latter cannot happen for  $\sigma^1, \dots, \sigma^n$ , because (R1) excludes the case where the vectors lie in a hyperplane not containing the origin and (FR) the case of a hyperplane through the origin.

b) If we have two spherical indices  $I^{a_1}, I^{a_2}$ , then  $|\sigma^i - \bar{\sigma}(a_k)| = |\sigma^1 - \bar{\sigma}(a_k)|$  for all  $i$  and thus

$$2(\sigma^i - \sigma^1)^\top \bar{\sigma}(a_k) = |\sigma^i|^2 - |\sigma^1|^2 \quad \text{for } i = 2, \dots, n \text{ and } k = 1, 2. \quad (2.4)$$

Hence  $\bar{\sigma}(a_1) - \bar{\sigma}(a_2)$  is orthogonal to  $\sigma^i - \sigma^1$  for  $i = 2, \dots, n$ . But we also know that  $\bar{\sigma}(a_k) = \sigma^\top a_k$  and  $a_k^\top \mathbf{1} \equiv 1$  and therefore

$$\bar{\sigma}(a_k) = (\sigma^\top - \sigma^1 \mathbf{1}^\top) a_k + \sigma^1 \mathbf{1}^\top a_k = \sigma^1 + (\sigma - \sigma^1 \mathbf{1})^\top a_k.$$

Hence  $\bar{\sigma}(a_1) - \bar{\sigma}(a_2) = (\sigma - \sigma^1 \mathbf{1})^\top (a_1 - a_2)$  is also in the span of the vectors  $\sigma^i - \sigma^1$ ,  $i = 2, \dots, n$ , and so we must have  $\bar{\sigma}(a_1) - \bar{\sigma}(a_2) = 0$ . Uniqueness follows because  $\bar{\sigma}(a)$  determines  $\bar{\mu}(a)$ . ■

The importance of a spherical index  $I^a$  is that it gives a numeraire in which relative asset prices  $\tilde{X}(a) = X/I^a$  have a simple volatility structure: the intrinsic volatilities  $\tilde{\sigma}^i(a) = \sigma^i - \bar{\sigma}(a)$  always lie on a (random and time-dependent) sphere. This will become even simpler when we assume that  $\tilde{\sigma}^i(a)$  does not depend on  $\omega$  and  $t$ , because we then have a multidimensional Black-Scholes type fluctuation around the reference level  $I^a$ . Note that if the number  $n$  of assets is fixed, we can always ensure existence of a spherical index by increasing the number  $m$  of driving factors. This is especially useful if  $n$  is small, e.g., if we think of a situation with 3 or 4 representative assets that each summarize one market segment.

To construct a spherical index  $I^a$ , we need a generating strategy  $a$ . Although  $a$  need not be unique,  $I^a$  always is (under the assumptions of Proposition 4) so that the choice of  $a$  does not matter. We need  $a$  itself later to generate  $I^a$  from  $X$  via (2.2). Due to (2.4),  $a$  can be found as a solution of the equations

$$|\sigma^i|^2 - |\sigma^1|^2 = 2a^\top \sigma(\sigma^i - \sigma^1) = 2 \sum_{\ell=1}^n a^\ell ((\sigma^\ell)^\top \sigma^i - (\sigma^\ell)^\top \sigma^1) \quad \text{for } i = 2, \dots, n, \quad (2.5)$$

with the constraint that  $a^\top \mathbf{1} \equiv 1$ . Note that (2.5) only involves quantities that are  $\omega$ -wise computable from the asset price data  $X$  since we need the volatilities  $|\sigma_t^i|^2$  (only the lengths, not the entire vectors) and the instantaneous return covariances

$$(\sigma_t^\ell)^\top \sigma_t^i = \frac{d}{dt} \left\langle \int \frac{dX^\ell}{X^\ell}, \int \frac{dX^i}{X^i} \right\rangle_t. \quad (2.6)$$

As these can all be estimated from asset price data,  $I^a$  is observable from  $X$ .

**Definition.** A volatility structure  $\sigma$  is called *rigid* if there exists an  $\mathbb{R}^m$ -valued process  $\sigma_0$  such that  $\sigma^i - \sigma_0$  is constant (in  $\omega, t$ ) for  $i = 1, \dots, n$ .

**Lemma 5** *A volatility structure  $\sigma$  is rigid if and only if for all constant vectors  $b \in \mathbb{R}^n$  with  $b^\top \mathbf{1} = 1$ , the difference  $\sigma^i - \sigma_{(b)}$  is constant for  $i = 1, \dots, n$ , where  $\sigma_{(b)} := \sum_{i=1}^n b^i \sigma^i$ .*

**Proof.** For any  $b \in \mathbb{R}^n$  with  $b^\top \mathbf{1} = 1$  and any process  $\sigma_0$ , we have

$$\sigma^i - \sigma_{(b)} = \sigma^i - \sigma_0 + \sigma_0 - \sigma_{(b)} = \sigma^i - \sigma_0 - \sum_{j=1}^n b^j (\sigma^j - \sigma_0).$$

This shows the “only if” part, and the “if” part is obvious if we take as  $\sigma_0$  any  $\sigma_{(b)}$ .  $\blacksquare$

**Corollary 6** *If the relative prices  $X^i/X^k$  have constant volatility vectors for  $i = 1, \dots, n$  and for at least one asset  $X^k$ , then all relative price processes  $X^i/X^j$  have constant volatility vectors for  $i, j = 1, \dots, n$ , and then  $\sigma$  is rigid. Hence*

- 1) *a rigid volatility structure may be seen as a Black-Scholes volatility structure for relative prices.*
- 2) *the structural property of being rigid does not depend on the choice of discounting numeraire  $I^b$  as long as  $b$  is constant.*

**Definition.** A volatility structure  $\sigma$  is called *rigidly spherical* if it is spherical and rigid.

**Theorem 7** *If  $\sigma$  is rigidly spherical, the unique spherical index can be generated by a constant strategy  $a \in \mathbb{R}^n$  with  $a^\top \mathbf{1} = 1$ . The corresponding relative prices  $\tilde{X}^i(a) = X^i/I^a$  then have constant volatility vectors  $\tilde{\sigma}^i(a) = \sigma^i - \bar{\sigma}(a)$  whose length is the same for all  $i$ .*

**Proof.** Because  $\sigma$  is spherical, the sphere center  $\bar{\sigma}$  exists, satisfies  $|\sigma^i - \bar{\sigma}| = |\sigma^1 - \bar{\sigma}|$  for  $i = 2, \dots, n$  and may be written as  $\bar{\sigma} = \sum_{i=1}^n a^i \sigma^i$  with a process  $a$  satisfying  $a^\top \mathbf{1} \equiv 1$ . Hence

$$|\sigma^i - \bar{\sigma}|^2 - |\sigma^1 - \bar{\sigma}|^2 = (\sigma^i + \sigma^1 - 2\bar{\sigma})^\top (\sigma^i - \sigma^1) = 0, \quad i = 2, \dots, n$$

and so

$$\sum_{j=1}^n a^j (\sigma^i - \sigma^j + \sigma^1 - \sigma^j)^\top (\sigma^i - \sigma^1) = 0, \quad i = 2, \dots, n.$$



This is a system of linear equations for  $a$  whose coefficients are constant because of rigidity and Lemma 5. Hence the solution  $a$  (which exists since  $\sigma$  is spherical) may also be taken to be constant. The rest follows from Corollary 6.  $\blacksquare$

The spherical index obtained from Theorem 7 will be linked to interest rates under a second structural assumption on  $X$ .

### 3 Dynamics under homogeneity

From this section on, we specialize our general model to an autonomous Markovian diffusion for  $X$ . This formalizes the idea that everything is determined by the basic assets  $X$ . We then explore the consequences of imposing two structural assumptions: the volatility structure is rigidly spherical, and the virtual short rate function is homogeneous of degree 0.

Instead of processes,  $\mu : \mathbb{R}_{++}^n \rightarrow \mathbb{R}^n$  and  $\sigma : \mathbb{R}_{++}^n \rightarrow \mathbb{R}^{n \times m}$  are now functions assumed sufficiently smooth and such that there exists a continuous  $\mathbb{R}_{++}^n$ -valued process  $X$  on  $[0, T_\infty]$  satisfying the SDEs

$$\frac{dX_t^i}{X_t^i} = \mu^i(X_t) dt + \sum_{j=1}^m \sigma^{ij}(X_t) dW_t^j, \quad i = 1, \dots, n. \quad (3.1)$$

(DC) and (FR) translate into the conditions that for every  $x \in \mathbb{R}_{++}^n$ ,

$$\mu(x) \in \text{span}\left(\mathbf{1}, \text{range}(\sigma(x))\right), \quad (3.2)$$

$$\text{rank}(\sigma(x)) = \min(m, n). \quad (3.3)$$

We keep the standing assumption (1.1) and make (1.2) more precise by assuming that

$$\frac{dN_t}{N_t} = (r(X_t) + |\lambda(X_t)|^2) dt + \lambda(X_t)^\top dW_t, \quad (3.4)$$

where  $r : \mathbb{R}_{++}^n \rightarrow \mathbb{R}$  and  $\lambda : \mathbb{R}_{++}^n \rightarrow \mathbb{R}^m$  are also not processes, but sufficiently smooth functions. As explained in Section 1, we think of  $r(X_t)$  as the (virtual) instantaneous short rate at time  $t$  on a savings account.

Our first main assumption is now that

$$r \text{ is homogeneous of degree } 0; \quad (3.5)$$

so  $r(\gamma x) = r(x)$  for all  $\gamma > 0$  and  $x \in \mathbb{R}_{++}^n$ . How do we motivate this condition?

The basic idea behind (3.5) is a postulate of *scale-invariance*. A first line of argument uses this in the situation of Lemma 3, a complete market where all prices and in particular the term structure of interest rates are determined by  $X$ . The resulting short rate function then coincides with the abstract function  $r$  from the dynamics (3.4) of  $N$  and can be explicitly expressed in terms of  $\mu$  and  $\sigma$ ; see Corollary 4 of [RSS]. Now we exploit the idea of scale-invariance: Because

all prices are relative to some unit, a simultaneous scaling of all asset prices should not affect the return dynamics. Hence  $\mu$  and  $\sigma$  should be homogeneous of degree 0, and this entails the same for  $r$  which is directly given from  $\mu$  and  $\sigma$ . By (1.6), the same is true for  $\lambda$ , although we do not use this here.

Alternatively, start with the pricing mechanism (1.1) and the dynamics (3.4) for  $N$ . Conceptually, the coefficients  $r$  and  $\lambda$  of  $N$  should only depend on the relevant asset prices, i.e., on  $X$  to start with. If we now scale all prices by a factor  $\gamma > 0$ ,  $C$  becomes  $\gamma C$  in (1.1), the new relevant asset prices are  $\gamma X$  instead of  $X$ , and these (instead of the obsolete values  $X$ ) should be plugged into  $r$  and  $\lambda$ . But economically, scaling prices does not change anything so that the pricing rule (1.1) should still hold. Hence the dynamics of  $N$  should remain unchanged, and thus the coefficients  $r$  (and  $\lambda$ ) should be homogeneous of degree 0.

**Remark.** The idea of scale-invariance also appears in [HN1, HN2], and even the condition that  $\mu$  and  $\sigma$  should be homogeneous of degree 0 can be found there. The main thrust of [HN1, HN2] is that “any payoff function should be representable by a homogeneous function of degree one in tradables”, and this is then exploited to give alternative derivations for a number of well-known option pricing results. The same idea of homogeneity earlier already appears in [Ja] who even proves that homogeneous payoffs can always be hedged under more general conditions. Our thrust here is in a different direction, and in contrast to [HN1, HN2], we clearly distinguish between economic intuition and mathematical derivation.  $\diamond$

For our subsequent analysis, it does not matter how we justify the assumed homogeneity of  $r$ . We now proceed to explore its consequences, starting with a simple analytic lemma.

**Lemma 8** *If  $h : \mathbb{R}_{++}^n \rightarrow \mathbb{R}$  is  $C^2$  and homogeneous of degree 0, then*

$$\sum_{i=1}^n x^i \frac{\partial h}{\partial x^i} \equiv 0, \quad \frac{\partial h}{\partial x^k} + \sum_{i=1}^n x^i \frac{\partial^2 h}{\partial x^i \partial x^k} \equiv 0, \quad \sum_{i,k=1}^n x^i x^k \frac{\partial^2 h}{\partial x^i \partial x^k} \equiv 0.$$

**Proof.** By homogeneity,  $\gamma \mapsto h(\gamma x)$  is constant on  $(0, \infty)$  for each  $x \in \mathbb{R}_{++}^n$ . Differentiate to get the first result, differentiate that with respect to  $x^k$  to get the second one, and multiply by  $x^k$  and sum over  $k$  to get the third result by using the first one.  $\blacksquare$

Now recall from Section 1 and Lemma 1 that  $\hat{W} = W + \int \lambda_u du$  and

$$\mu = r\mathbf{1} + \sigma\lambda. \tag{3.6}$$

Itô's formula, (3.6), the first property in Lemma 8 and the definition of  $\hat{W}$  yield

$$\begin{aligned}
dr(X_t) &= \sum_{i=1}^n \frac{\partial r}{\partial x^i} \mu^i X_t^i dt + \frac{1}{2} \sum_{i,k=1}^n \frac{\partial^2 r}{\partial x^i \partial x^k} X_t^i X_t^k (\sigma \sigma^\top)^{ik} dt + \sum_{i=1}^n \frac{\partial r}{\partial x^i} X_t^i (\sigma dW_t)^i \\
&= \frac{1}{2} \sum_{i,k=1}^n \frac{\partial^2 r}{\partial x^i \partial x^k} X_t^i X_t^k (\sigma \sigma^\top)^{ik} dt + \sum_{i=1}^n \frac{\partial r}{\partial x^i} X_t^i (\sigma d\hat{W}_t)^i \\
&= \hat{c}(X_t) dt + b(X_t)^\top d\hat{W}_t \\
&= c(X_t) dt + b(X_t)^\top dW_t
\end{aligned}$$

for the dynamics of  $r(X)$ , where we drop the argument  $X_t$  in most functions and set

$$b(x) := \sum_{i=1}^n x^i \frac{\partial r}{\partial x^i}(x) \sigma^i(x), \quad (3.7)$$

$$\hat{c}(x) := c(x) - (b^\top \lambda)(x) := \frac{1}{2} \sum_{i,k=1}^n x^i x^k \frac{\partial^2 r}{\partial x^i \partial x^k}(x) (\sigma \sigma^\top)^{ik}(x). \quad (3.8)$$

Now fix a function  $\sigma_{\text{ref}} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , set  $\tilde{\sigma}^{ij} := \sigma^{ij} - \sigma_{\text{ref}}^j$  for  $i = 1, \dots, n$  and  $j = 1, \dots, m$ , plug  $\sigma^i = \tilde{\sigma}^i + \sigma_{\text{ref}}$  into (3.7) and use Lemma 8 and the  $\mathbb{R}^m$ -valued functions  $\tilde{\sigma}^i := (\tilde{\sigma}^{ij})_{j=1, \dots, m}$  to get

$$b(x) = \sum_{i=1}^n x^i \frac{\partial r}{\partial x^i}(x) \tilde{\sigma}^i(x) \quad (3.9)$$

Differentiating with respect to  $x^k$ , multiplying by  $x^k (\tilde{\sigma}^k)^\top$  and summing over  $k$  gives

$$\sum_{k=1}^n x^k (\tilde{\sigma}^k)^\top \frac{\partial b}{\partial x^k} = \sum_{i,k=1}^n x^i x^k \frac{\partial^2 r}{\partial x^i \partial x^k} (\tilde{\sigma} \tilde{\sigma}^\top)^{ik} + \sum_{i=1}^n x^i \frac{\partial r}{\partial x^i} |\tilde{\sigma}^i|^2 + \sum_{i,k=1}^n x^i x^k \frac{\partial r}{\partial x^i} (\tilde{\sigma}^k)^\top \frac{\partial \tilde{\sigma}^i}{\partial x^k}. \quad (3.10)$$

On the other hand, we can also plug  $\sigma^i = \tilde{\sigma}^i + \sigma_{\text{ref}}$  into (3.8) and use the first property in Lemma 8, then the second one and then (3.9) to get

$$\hat{c} = c - b^\top \lambda = -b^\top \sigma_{\text{ref}} + \frac{1}{2} \sum_{i,k=1}^n x^i x^k \frac{\partial^2 r}{\partial x^i \partial x^k} (\tilde{\sigma} \tilde{\sigma}^\top)^{ik}. \quad (3.11)$$

Finally, replace the second derivatives in (3.11) via (3.10) and use the first property in Lemma 8 to get

$$\begin{aligned}
\hat{c} &= c - b^\top \lambda \\
&= -b^\top \sigma_{\text{ref}} + \frac{1}{2} \sum_{i=1}^n x^i (\tilde{\sigma}^i)^\top \frac{\partial b}{\partial x^i} - \frac{1}{2} \sum_{i,k=1}^n x^i x^k \frac{\partial r}{\partial x^i} (\tilde{\sigma}^k)^\top \frac{\partial \tilde{\sigma}^i}{\partial x^k} - \frac{1}{2} \sum_{i=1}^n x^i \frac{\partial r}{\partial x^i} (|\tilde{\sigma}^i|^2 - \tilde{\sigma}_{\text{av}}^2)
\end{aligned} \quad (3.12)$$

with  $\tilde{\sigma}_{\text{av}}^2 := \frac{1}{n} \sum_{i=1}^n |\tilde{\sigma}^i|^2$ . This is a *general result* on the structure of the virtual short rate dynamics in homogeneous Markovian models.

**Definition.** A Markovian volatility structure function  $\sigma(x)$  is called (*rigidly*) *spherical* if the process  $\sigma(X)$  is (rigidly) spherical.

If  $\sigma$  is spherical with spherical index  $I^a$ , (2.5) makes it clear that  $a_t(\omega)$  is like  $\sigma(X_t(\omega))$  a function of  $X_t(\omega)$ , and so the same is true for  $\bar{\sigma}_a = \sigma^\top a$ . If we then choose for  $\sigma_{\text{ref}}$  the volatility function  $\bar{\sigma}_a$  of  $I^a$ , we see that the vectors  $\tilde{\sigma}_a^i = \sigma^i - \bar{\sigma}_a$  all have the same length (because  $\sigma$  is spherical) so that the last term in (3.12) vanishes. If  $\sigma$  is also rigid,  $a$  can by Theorem 7 be chosen constant in  $x$  and so  $\tilde{\sigma}_a^i$  then also becomes constant in  $x$  due to Lemma 5. Hence the double sum in (3.12) vanishes as well and we get (dropping all arguments  $X_t$ )

**Proposition 9** *Assume (1.1) and (3.1) – (3.5). If  $\sigma$  is rigidly spherical, we have*

$$\hat{c} = -b^\top \bar{\sigma}_a + \frac{1}{2} \sum_{i=1}^n X_t^i (\tilde{\sigma}_a^i)^\top \frac{\partial b}{\partial x^i} = -\frac{d}{dt} \left\langle \int \frac{dI^a}{I^a}, r \right\rangle + \frac{1}{2} \sum_{i=1}^n X_t^i (\tilde{\sigma}_a^i)^\top \frac{\partial b}{\partial x^i}. \quad (3.13)$$

**Remark.** Models of rigidly spherical Markovian markets do exist, even if we add the condition that the function  $b$  (the “short rate volatility” from (3.7)) should have the form  $b(x) = b_0(r(x))$  for a function  $b_0 : \mathbb{R} \rightarrow \mathbb{R}^m$ . See Appendix A.  $\diamond$

## 4 The main result

We continue under the same assumptions as in Section 3. Proposition 9 then tells us how the  $\hat{P}$ -drift  $\hat{c}(X)$  of the virtual short rate  $r(X)$  is related to the spherical index  $I^a$ , the constant intrinsic volatilities  $\tilde{\sigma}_a^i$ , the assets  $X$  and the instantaneous volatility vector  $b(X)$  of  $r(X)$ . Let us now show how to transform this into a result with a clear economic interpretation.

First observe a well-known fact: In any term structure model with nice coefficients, the risk-neutral short rate drift is equal to the slope of the forward rate curve at the short end, i.e.,

$$\hat{c}_{t_0} = \frac{\partial}{\partial T} f_{t_0, T} \Big|_{T=t_0}. \quad (4.1)$$

A proof of this result is given in Appendix B.

The second step is a minor additional assumption. Because the vector  $b(X)$  of instantaneous volatilities is (as a vector) not observable, we assume that

$$b(x) = b_0(r(x)) \quad \text{for some smooth function } b_0 : \mathbb{R} \rightarrow \mathbb{R}^n. \quad (4.2)$$

One possible motivation is that this goes towards a Markovian short rate model which is a popular assumption in the literature. We provide below an alternative characterization of (4.2) as a condition on the structure of the homogeneous function  $r$ .

**Theorem 10** Assume (1.1), (3.1) – (3.5) and that  $\sigma$  is rigidly spherical. If (4.2) holds, then

$$\hat{c}(X_{t_0}) = -b_0(r(X_{t_0}))^\top \bar{\sigma}_a(X_{t_0}) + \frac{1}{2}(|b_0||b_0'|)(r(X_{t_0})) \quad (4.3)$$

$$= -\frac{d}{dt} \left\langle \int \frac{dI^a}{I^a}, r(X) \right\rangle_t \Big|_{t=t_0} + \frac{1}{2}(|b_0||b_0'|)(r(X_{t_0})) \quad \text{for each } t_0. \quad (4.4)$$

If  $r(X)$  is the short rate on a savings account and the term structure has nice coefficients,

$$\begin{aligned} \frac{\partial}{\partial T} f_{t_0, T} \Big|_{T=t_0} &= -b_0(r(X_{t_0}))^\top \bar{\sigma}_a(X_{t_0}) + \frac{1}{2}(|b_0||b_0'|)(r(X_{t_0})) \\ &= -\frac{d}{dt} \left\langle \int \frac{dI^a}{I^a}, r(X) \right\rangle_t \Big|_{t=t_0} + \frac{1}{2}(|b_0||b_0'|)(r(X_{t_0})). \end{aligned} \quad (4.5)$$

**Proof.** By (4.2) and (3.9),

$$\sum_{i=1}^n x^i \tilde{\sigma}_a^i \frac{\partial b}{\partial x^i}(x) = \left( b_0'(r(x)) \right)^\top b_0(r(x)).$$

Since  $(b_0')^\top b_0 = \frac{1}{2} \frac{d}{dr} b_0^\top b_0 = \frac{1}{2} \frac{d}{dr} |b_0|^2 = (|b_0||b_0'|)$ , (4.3) and (4.4) follow from (3.13). Using (4.1) then gives (4.5).  $\blacksquare$

(4.5) is our main result on the link between interest rates and the spherical index  $I^a$ . The economic interpretation is as follows. Think of a fixed short rate and consider the effect of a change in expected interest rates in the near future. Because the last term in (4.5) is then constant, we see that *higher expectations about future interest rates*, in the form of an increased slope of the initial forward rate curve, *go with a decrease of correlation between the short rate and the spherical index*, and vice versa.

The central relation (4.5) is (almost) *testable* in the sense that (almost) all its ingredients can be computed  $\omega$ -wise from observable data. For the left-hand side, we only need the initial forward rate curve  $T \mapsto f_{t_0, T}$  near  $t_0$ . The first term on the right-hand side can be written as (dropping the argument  $X_{t_0}$ )

$$-(b_0 \circ r)^\top \bar{\sigma}_a = -|b_0 \circ r| |\bar{\sigma}_a| \rho_{I^a, r},$$

where

$$\rho_{I^a, r} := \frac{(b_0 \circ r)^\top \bar{\sigma}_a}{|b_0 \circ r| |\bar{\sigma}_a|} = \frac{\frac{d}{dt} \left\langle \int \frac{dI^a}{I^a}, r(X) \right\rangle_t}{\sqrt{\left( \frac{d}{dt} \left\langle \int \frac{dI^a}{I^a} \right\rangle_t \right) \left( \frac{d}{dt} \langle r(X) \rangle_t \right)}} \Big|_{t=t_0} \quad (4.6)$$

is the instantaneous correlation at time  $t_0$  between the return of the index  $I^a$  and the short rate  $(r(X))$ . Since both  $I^a$  (see the discussion in Section 2) and  $r(X)$  are observable, so are  $\rho_{I^a, r}$  and the volatilities  $|b_0 \circ r|$  of  $r$  and  $|\bar{\sigma}_a|$  of  $I^a$ . The final term in (4.5) becomes observable under an auxiliary parametric assumption on  $b_0$ ; for instance, we could try  $|b_0(r)| = \beta r^{1/2}$  if we believe in a CIR-like model. Alternatively, we could use (4.5) to estimate the parameters in a model for the function  $b_0$ .

The simplest case of (4.2) occurs if  $b(x)$  is a constant vector  $b_*$ ; this corresponds to a “semi-Vasiček” type model for the short rate with constant volatility  $|b_*|$ . The last term in (4.5) then vanishes and we are left with the simplified relation

$$\frac{\partial}{\partial T} f_{t_0, T} \Big|_{T=t_0} = - \frac{d}{dt} \left\langle \int \frac{dI^a}{I^a}, r(X) \right\rangle_t \Big|_{t=t_0}. \quad (4.7)$$

This also has a very appealing and plausible economic interpretation: *If the forward rate curve is upward (downward) sloping, the short rate is negatively (positively) correlated with the spherical index  $I^a$ .* In Section 5, we show some results from a very simple empirical study of (4.7).

To conclude this section, we provide the promised characterization of condition (4.2).

**Proposition 11** *If  $\sigma$  is rigidly spherical, a sufficient condition for*

$$b(x) = b_0(r(x)) \neq 0 \quad (4.8)$$

is that

$$r(x) = \varphi(J_\pi(x)) \quad (4.9)$$

for some  $\pi$  and some strictly monotone  $C^1$  function  $\varphi : [0, \infty) \rightarrow \mathbb{R}$ , where  $J_\pi : \mathbb{R}^n \rightarrow \mathbb{R}$  is the homogeneous function

$$J_\pi(x) := \prod_{i=1}^n (x^i)^{\pi^i} \quad \text{with } \pi \in \mathbb{R}^n \setminus \{0\} \text{ satisfying } \sum_{i=1}^n \pi^i = 0.$$

Conversely, (4.9) is also necessary for (4.8) if we have (3.3) (the analogue of (FR)) and either  $m \geq n$  or the combination of  $m = n - 1$  plus (the analogue of (R1))

$$\mathbf{1} \notin \text{range}(\sigma(x)) \quad \text{for every } x \in \mathbb{R}_{++}^n. \quad (4.10)$$

**Proof.** The sufficiency part is easy. Denote by  $\varphi^{\text{inv}}$  the inverse function of  $\varphi$ , differentiate (4.9) and use (3.9) to get

$$b(x) = J_\pi(x) \varphi'(J_\pi(x)) \sum_{i=1}^n \pi^i \tilde{\sigma}_a^i = \varphi^{\text{inv}}(r(x)) \varphi'(\varphi^{\text{inv}}(r(x))) \sum_{i=1}^n \pi^i \tilde{\sigma}_a^i = b_0(r(x)). \quad (4.11)$$

Note that  $\bar{v}_\pi := \sum_{i=1}^n \pi^i \tilde{\sigma}_a^i$  is a constant vector since  $\sigma$  is rigidly spherical. The necessity part is more involved; its proof can be found in Appendix C. ■

The proof of Proposition 11 shows in particular that  $b(x)$  is a scalar  $r(x)$ -dependent multiple of a constant vector  $\bar{v}_\pi$  and gives with (4.11) an expression for the function  $b_0$  in terms of  $\varphi$ . Simple examples are  $\varphi(z) = \log z$  which leads to  $b_0(r) = b^*$  for a constant vector  $b^*$ , or  $\varphi(z) = (\log z)^{\frac{1}{1-\beta}}$  which gives  $b_0(r) = Cr^\beta$  with another constant vector  $C$ . Hence our setup contains a rich class of term structure models, and these have a Markovian short rate as soon as (4.8) holds and the projection of  $\bar{\sigma}_a(x)$  on  $\bar{v}_\pi$  is a function of  $r(x)$  only; see (4.3).

## 5 A first empirical study

This section presents a first empirical study for the simplified relation (4.7) deduced from Theorem 10. We indicate how one could test (4.7) on the basis of market data. But this is only very preliminary work, and a detailed statistical analysis still remains to be done.

### 5.1 Estimation of the required quantities

Our data come from the Euro asset market, but we do not consider all stocks. We take the German index DAX, the French index CAC and the Dutch index AEX as representative price processes and regard these three stock indices as our basic assets. We have daily close data for them, and we also have data of riskless Euro yields for 1, 2 and 3 months time to maturity.

In order to test the relation (4.7), we construct for the above data the spherical index, obtain the correlation between this index and the short rate, and determine the initial slope of the forward yield curve. Since Theorem 10 is a local result, we carry out our estimations over rather short time periods of four months. We outline the procedure in three steps below.

#### Step 1. Estimation of the spherical index

We first estimate the covariance matrix of the assets. We observe the assets  $X_k^j$ ,  $j = 1, \dots, n$  (with  $n = 3$  in our example) at  $N + 1$  (daily) dates  $T_k$ ,  $k = 1, \dots, N + 1$ . In view of (2.6), we estimate the local return covariance rates by

$$(\widehat{\sigma^\ell})^\top \sigma^i = \frac{1}{N} \sum_{k=1}^N \frac{1}{T_{k+1} - T_k} \frac{X_{k+1}^\ell - X_k^\ell}{X_k^\ell} \frac{X_{k+1}^i - X_k^i}{X_k^i}. \quad (5.1)$$

We solve the system of linear equations (2.5) using the estimates from (5.1) for  $(\sigma^\ell)^\top \sigma^i$  and obtain the estimated weights  $\hat{a}^\ell$  of the spherical index. The estimate for the squared volatility norm of the spherical index is then computed via

$$|\widehat{\sigma_a}|^2 = \sum_{i,\ell=1}^n \hat{a}^i \hat{a}^\ell (\widehat{\sigma^\ell})^\top \sigma^i. \quad (5.2)$$

After computing the index weight estimates, we construct from the asset returns via (2.2) a time series of estimates  $\hat{I}_k^a$  of the spherical index. The result is shown in Figure 3.

#### Step 2. Covariance and correlation between short rate and spherical index

In view of the available data, we use the 1 month spot yield as approximation for the short rate. Denote by  $r_k$  a time series of this process observed at the dates  $T_k$  and let  $b_*$  be the (constant) volatility vector of the short rate. The local covariance between the short rate and the spherical index  $I^a$  is next estimated by

$$\widehat{b_*^\top \sigma_a} = \frac{1}{N} \sum_{k=1}^N \frac{1}{T_{k+1} - T_k} (r_{k+1} - r_k) \frac{\hat{I}_{k+1}^a - \hat{I}_k^a}{\hat{I}_k^a}. \quad (5.3)$$

After computing an estimate for the squared short rate volatility norm by

$$|\widehat{b}_*|^2 = \frac{1}{N} \sum_{k=1}^N \frac{(r_{k+1} - r_k)^2}{T_{k+1} - T_k}, \quad (5.4)$$

we may estimate the instantaneous correlation between  $I^a$  and  $r$  by

$$\widehat{\rho}_{I^a, r} = \frac{\widehat{b}_*^\top \widehat{\sigma}_a}{|\widehat{b}_*| |\widehat{\sigma}_a|}. \quad (5.5)$$

### Step 3. Estimation of the initial forward yield slope

In order to estimate at date  $t_0$  the slope of the instantaneous forward yield curve at the short end, we use riskless continuously compounded bond yields  $R_{t_0, T}$ . These are connected with the zero coupon bond prices  $B_{t_0, T}$  via

$$R_{t_0, T} = -\frac{\log B_{t_0, T}}{T - t_0}. \quad (5.6)$$

We estimate the value of  $\lim_{T \searrow t_0} \frac{\partial R_{t_0, T}}{\partial T}$  by the slope of the regression line through the three points given by the 1, 2 and 3 month yields and then use the relation

$$\left. \frac{\partial f_{t_0, T}}{\partial T} \right|_{T=t_0} = -\lim_{T \searrow t_0} \frac{\partial^2 \log B_{t_0, T}}{\partial T^2} = 2 \lim_{T \searrow t_0} \frac{\partial R_{t_0, T}}{\partial T}, \quad (5.7)$$

where the second equality follows from (5.6) by standard calculus.

**Remark.** As already mentioned, (4.7) is a local result that involves instantaneous volatilities and correlations. Steps 1 and 2 use time averages from time series to generate estimates for these local quantities, and this usually works well only if the time series under consideration come from stationary processes. The same comment applies to Step 3 where we have estimated the desired slope by averaging over all observation dates the slopes obtained at each date. Once the stationarity assumption is not met, the reliability of such crude estimates is drastically reduced and more sophisticated methods are called for.  $\diamond$

## 5.2 Empirical results

On the basis of the previous description, we performed an analysis of the data in the periods Jan – Apr 2001 and Jan – Apr 2002. During the period in 2001, the yield curve was downward sloping; Figures 1 and 2 show the yield curves of the first and the last day of this period.



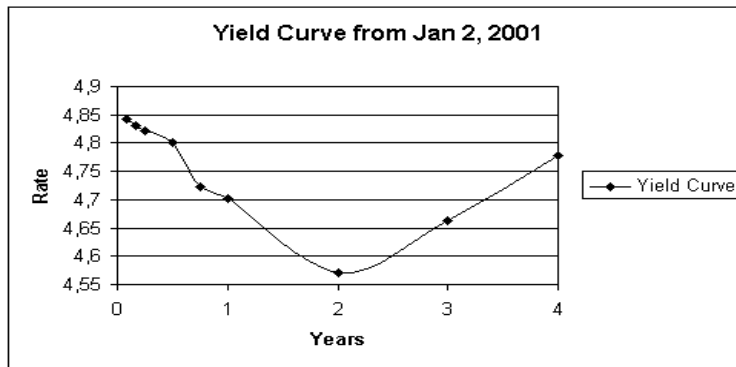


Figure 1: Euro yield curve on Jan 2, 2001. Estimated initial slope is  $\approx -0.0012$

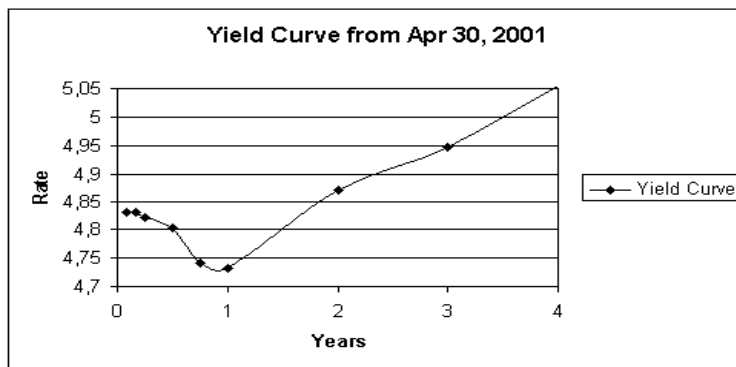


Figure 2: Euro yield curve on Apr 30, 2001. Estimated initial slope is  $\approx -0.0006$

From Step 1, we obtain estimates for the covariances of the three assets, the spherical index weights and the spherical index volatility. The assets and the corresponding spherical index estimates, both scaled by their initial values, are shown in Figure 3.

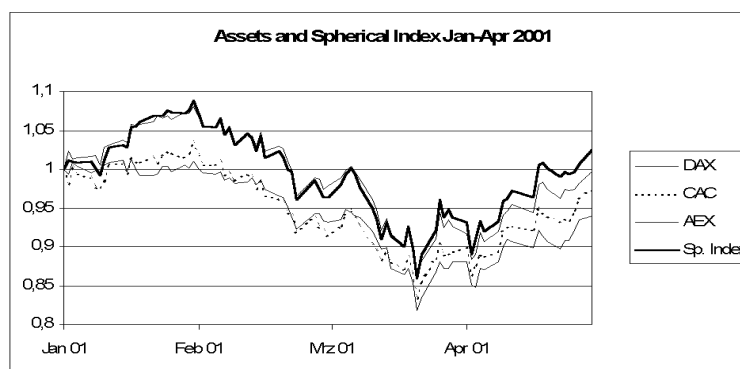


Figure 3: Estimated weights of spherical index are  $\hat{a}_{DAX} = 0.8908$ ,  $\hat{a}_{CAC} = 0.6554$  and  $\hat{a}_{AEX} = -0.5462$ .

For the period Jan – Apr 2001 we then obtain the following estimates:

Short rate volatility $ \widehat{b}_* $ :	0.0061
Spherical index volatility $ \widehat{\sigma}_a $ :	0.3097
Correlation $\widehat{\rho}_{I^a,r}$ between index and short rate:	0.2169
Average initial slope of yield curve $T \mapsto R_{t_0,T}$ :	-0.0040
Average initial slope of forward rate curve $T \mapsto f_{t_0,T}$ : $2 \times -0.0040 =$	-0.0080

According to (4.7), the product of the first three numbers,  $0.0061 \times 0.3097 \times 0.2169 \approx 0.0004$ , should be equal to minus the last one, 0.0080. So we might conclude that the empirical results for this example are roughly consistent with (4.7) with respect to sign and order of magnitude.

During the period in 2002, the situation on the interest rate market was quite different from 2001. As we see in Figures 4 and 5, the sign of the initial slope of the yield curve was changing from negative to positive. This already shows that an assumption of stationarity is here probably violated and casts doubt on the reliability of the subsequent estimates.

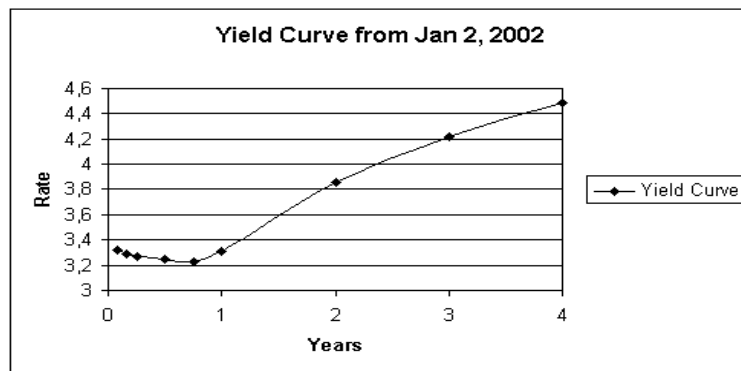


Figure 4: Euro yield curve on Jan 2, 2002. Estimated initial slope is  $\approx -0.0025$

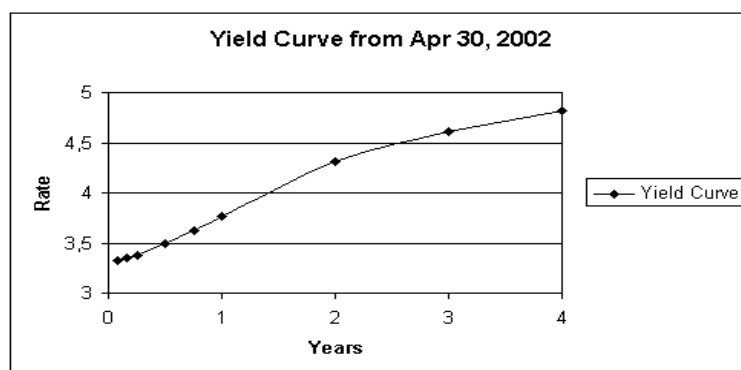


Figure 5: Euro yield curve on Apr 30, 2002. Estimated initial slope is  $\approx +0.0029$

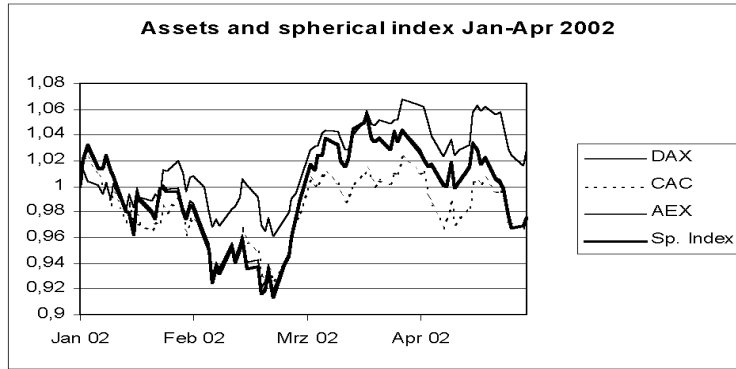


Figure 6: Estimated weights of spherical index are  $\hat{a}_{\text{DAX}} = 1.0251$ ,  $\hat{a}_{\text{CAC}} = 0.0778$  and  $\hat{a}_{\text{AEX}} = -0.1029$ .

The resulting estimates for the period Jan – Apr 2002 are:

Short rate volatility $ \widehat{b}_* $ :	0.0021
Spherical index volatility $ \widehat{\sigma}_a $ :	0.2306
Correlation $\widehat{\rho}_{I^a, r}$ between index and short rate:	0.0489
Average initial slope of yield curve $T \mapsto R_{t_0, T}$ :	0.0018
Average initial slope of forward rate curve $T \mapsto f_{t_0, T}$ : $2 \times 0.0018 =$	0.0035

The product of the first three numbers is  $0.0021 \times 0.2306 \times 0.0489 \approx 0.0000$  and thus has no unambiguous sign. Moreover, the forward rate curve is changing so much during that period that we clearly cannot assume stationarity. Hence the estimate for the average slope is also not reliable and we cannot say much about (4.7) on the basis of this second data set.

## Appendix A: Existence of models

We show here that many rigidly spherical markets with a short rate volatility of the form  $b(x) = b_0(r(x))$  exist. First choose a homogeneous rigidly spherical asset volatility structure with (3.3) and set  $r(x) := \varphi(J_\pi(x))$  as in (4.9). Then  $r(x)$  is also homogeneous, and Proposition 11 shows that we have  $b(x) = b_0(r(x))$ . If we then choose some homogeneous function  $\lambda(x)$ , we obtain a homogeneous  $\mu(x)$  from the drift condition (DC). This construction also illustrates that we usually have enough freedom in the choice of our parameters to produce a model with (for instance) a desired short rate process as output.

## Appendix B: Proof of (4.1)

In this appendix, we prove the representation

$$\hat{c}_{t_0} = \frac{\partial}{\partial T} f_{t_0, T} \Big|_{T=t_0} \quad (4.1)$$

for the risk-neutral short rate drift  $\hat{c}$ . We start with the risk-neutral forward rate dynamics

$$df_{t,T} = \gamma_{t,T} dt + \delta_{t,T}^\top d\hat{W}_t$$

to obtain for  $T = t > t_0$

$$r_t = f_{t,t} = f_{t_0,t} + \int_{t_0}^t \gamma_{u,t} du + \int_{t_0}^t \delta_{u,t}^\top d\hat{W}_u.$$

If we write a dot  $\dot{\cdot}$  for partial derivatives with respect to the second argument, we get

$$dr_t = \dot{f}_{t_0,t} dt + \gamma_{t,t} dt + \int_{t_0}^t \dot{\gamma}_{u,t} du dt + \delta_{t,t}^\top d\hat{W}_t + \int_{t_0}^t \dot{\delta}_{u,t}^\top d\hat{W}_u dt \quad (\text{B.1})$$

because all quantities are sufficiently smooth. Now the HJM drift condition says that

$$\gamma_{u,t} = \delta_{u,t}^\top \int_u^t \delta_{u,s} ds$$

since  $\gamma$  is the risk-neutral forward rate drift. Plugging this into (B.1) yields

$$dr_t = \dot{f}_{t_0,t} dt + \left( \int_{t_0}^t \dot{\delta}_{u,t}^\top \int_u^t \delta_{u,s} ds du + \int_{t_0}^t |\delta_{u,t}|^2 du + \int_{t_0}^t \dot{\delta}_{u,t}^\top d\hat{W}_u \right) dt + \delta_{t,t}^\top d\hat{W}_t.$$

Letting  $t \searrow t_0$ , we obtain (4.1).

## Appendix C: Proof of Proposition 11

In this appendix, we prove that (4.9) is necessary for (4.8).

**Step 1:** For each  $x \in \mathbb{R}_{++}^n$ ,  $x^j \frac{\partial r}{\partial x^j}(x)$  is uniquely determined by  $b(x) = b_0(r(x))$  so that

$$x^j \frac{\partial r}{\partial x^j}(x) = f_j(r(x)) \quad (\text{C.1})$$

for functions  $f_1, \dots, f_n : \mathbb{R} \rightarrow \mathbb{R}$ . To see this, suppose that we have the representation

$$b(x) = \sum_{i=1}^n x^i \frac{\partial r}{\partial x^i}(x) \sigma^i(x) = \sum_{i=1}^n x^i \frac{\partial \bar{r}}{\partial x^i}(x) \sigma^i(x)$$

from (3.7) for two homogeneous functions  $r$  and  $\bar{r}$ . Then we get for  $h := r - \bar{r}$  that

$$\sum_{i=1}^n x^i \frac{\partial h}{\partial x^i}(x) \sigma^i(x) = 0 \quad \text{and} \quad \sum_{i=1}^n x^i \frac{\partial h}{\partial x^i}(x) = 0 \quad (\text{C.2})$$

by Lemma 8. Now if  $m \geq n$ , the full rank condition (3.3) yields  $\text{range}(\sigma(x)) = \mathbb{R}^n$  and therefore  $x^i \frac{\partial h}{\partial x^i}(x) = 0$  for  $i = 1, \dots, n$  due to (C.2). If  $m = n - 1$ , combining (3.3) with (4.10) implies that  $\text{span}(\mathbf{1}, \text{range}(\sigma(x))) = \mathbb{R}^n$  and we see again from (C.2) that  $x^i \frac{\partial h}{\partial x^i}(x) = 0$ . This proves the assertion.

**Step 2:** Because of (C.1) and

$$\sum_{i=1}^n x^i \frac{\partial r}{\partial x^i}(x) \tilde{\sigma}_a^i = b(x) = b_0(r(x)) \neq 0,$$

there exists for any  $x_0 \in \mathbb{R}_{++}^n$  an open ball  $U_0$  around  $x_0$  and an index  $j_0$  such that

$$x^{j_0} \frac{\partial r}{\partial x^{j_0}}(x) = f_{j_0}(r(x)) \neq 0 \quad \text{for all } x \in U_0.$$

This implies the existence of constants  $\pi^i$  with  $\pi^{j_0} = 1$  and  $\sum_{i=1}^n \pi^i = 0$  such that

$$f_i \circ r = \pi^i f_{j_0} \circ r \quad \text{on } U_0 \text{ for all } i. \quad (\text{C.3})$$

To prove this, we may assume that  $j_0 = 1$  so that  $f_1 \circ r \neq 0$  on  $U_0$ . For  $i \neq j$ , (C.1) implies

$$\frac{\partial^2 r}{\partial x^i \partial x^j} = \frac{\partial}{\partial x^i} \frac{f_j}{x^j} = \frac{1}{x^j} f_j' \frac{\partial r}{\partial x^i} = \frac{f_j' f_i}{x^i x^j} = \frac{f_i' f_j}{x^i x^j},$$

hence

$$f_i'(r(x)) = \frac{f_1'(r(x))}{f_1(r(x))} f_i(r(x)) \quad \text{on } U_0.$$

This ODE has for each  $i$  the unique solution

$$f_i(r(x)) = \pi^i f_1(r(x))$$

for a constant  $\pi^i$ , and homogeneity of  $r$  enforces by Lemma 8 that  $\sum_{i=1}^n \pi^i = 0$ . This proves (C.3).

**Step 3:** The representation (C.3) holds a priori only on  $U_0$ , and the constants  $\pi^i$  could depend on  $x_0$ . We claim now that the  $\pi^i$  are global constants and that (C.3) holds on all of  $\mathbb{R}_{++}^n$ . Since  $\mathbb{R}_{++}^n$  is  $\sigma$ -compact, both assertions follow once we prove the following result: If we have two representations

$$f_i \circ r = \pi_\ell^i f_{j_\ell} \circ r \quad \text{with} \quad f_{j_\ell} \circ r \neq 0 \quad \text{on } U_\ell, \text{ for } \ell = 1, 2$$

with open balls  $U_1, U_2$  such that  $U_1 \cap U_2 \neq \emptyset$ , then

$$f_i \circ r = \pi_1^i f_{j_1} \circ r = \pi_2^i f_{j_2} \circ r \quad \text{holds on } U_1 \cup U_2. \quad (\text{C.4})$$

To see this, use the shorthand  $g := f \circ r$  and note that we have  $g_i = \pi_1^i g_{j_1} = \pi_2^i g_{j_2}$  on  $U_1 \cap U_2$ ; hence  $\pi_1^{j_2} g_{j_1} = g_{j_2} \neq 0$  on  $U_1 \cap U_2$ , so  $\pi_1^{j_2} \neq 0$ , and then it follows that  $\pi_1^i g_{j_1} = \pi_2^i \pi_1^{j_2} g_{j_1}$  on  $U_1 \cap U_2$  so that  $\pi_1^i = \pi_2^i \pi_1^{j_2}$ . Using this, we have on  $U_1$  that

$$g_i = \pi_1^i g_{j_1} = \pi_2^i \pi_1^{j_2} g_{j_1} = \pi_2^i g_{j_2},$$

and so (C.4) follows.

**Step 4:** Now define the function  $\tilde{r} : \mathbb{R}_{++}^n \rightarrow \mathbb{R}$  by

$$\tilde{r}(y) := r(\exp[y]) = r(x)$$

with  $x = \exp[y] := (\exp(y_1), \dots, \exp(y_n))^\top$ . Then we have from (C.3)

$$\frac{\partial \tilde{r}}{\partial y^j}(y) = x^j \frac{\partial r}{\partial x^j}(x) = \pi^j f_{j_0}(r(x)) = \pi^j f_{j_0}(\tilde{r}(y)). \quad (\text{C.5})$$

Choose  $H : \mathbb{R} \rightarrow \mathbb{R}$  with  $H' = 1/f_{j_0}$  and integrate (C.5) with respect to  $y^j$  for a fixed  $j$  to get

$$H(\tilde{r}(y)) = C_j(y) + \pi^j y^j, \quad (\text{C.6})$$

where  $C_j(y)$  does not depend on  $y^j$ . Now differentiate (C.6) with respect to  $y^i$  for  $i \neq j$  and use (C.5) to get  $\frac{\partial C_j(y)}{\partial y^i} = \pi^i$ . This yields  $C_j(y) = \pi^i y^i + C_{i,j}(y)$ , where  $C_{i,j}(y)$  now depends neither on  $y^i$  nor on  $y^j$ , and

$$H(\tilde{r}(y)) = C_{i,j}(y) + \pi^i y^i + \pi^j y^j.$$

Iterating this argument finally gives

$$H(\tilde{r}(y)) = C + \sum_{i=1}^n \pi^i y^i = C + \sum_{i=1}^n \pi^i \log x^i = C + \log J_\pi(x)$$

with a constant  $C$ , and inverting  $H$  yields

$$r(x) = \tilde{r}(y) = H^{\text{inv}}(C + \log J_\pi(x))$$

which is of the form (4.9).

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