Evaluation of Sylvester type determinants using block-triangularization

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Abstract

It is shown that the values of Sylvester type determinants considered by Askey in [1] can be ascertained inductively using simple block-triangularization schemes.

1 Introduction

Richard Askey in [1] shows two ways, one matrix-theoretic and another based on orthogonal polynomials, to evaluate determinants

$$D_{N+1}(x) := \begin{vmatrix} x & 1 & & & & & \\ N & x & 2 & & & 0 & \\ & N-1 & x & 3 & & & \\ & & \dots & & & & \\ & 0 & & & 2 & x & N \\ & & & & 0 & 1 & x \end{vmatrix},$$

which were first considered by Sylvester [2]. In addition, he obtains several generalizations of Sylvester's determinants and explores their connection to orthogonal polynomials.

The purpose of this note is to show how the determinants from [1] can be evaluated in yet another way, based on partial information about left or right eigenvectors of the corresponding matrices coupled with a simple similarity trick. In all cases except one, only one (the most obvious) eigenvector is used. The exceptional case is the Sylvester determinant itself, where two eigenvectors are readily available and hence used to derive the result.

2 Sylvester's determinant and two close variants

Let us start with the Sylvester determinant. We want to prove that

$$D_{N+1}(x) = \prod_{j=0}^{N} (x + N - 2j) \tag{1}$$

(which is formulas (2.3), (2.4) from [1]). Since the values of D_1 and D_2 agree with (1), it is enough to show that

$$D_{N+1}(x) = (x - N)(x + N) D_{N-1}(x) \tag{2}$$

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for $N \geq 1$. The Sylvester determinant D_{N+1} is the characteristic polynomial of the matrix $-\mathcal{D}_{N+1}$, i.e., $D_{N+1}(x) = \prod_{\lambda \in \sigma(D_{N+1})}(x+\lambda)$, where

$$\mathcal{D}_{N+1} := \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ N & 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & N-1 & 0 & 3 & \cdots & 0 & 0 \\ 0 & 0 & N-2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & N \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Note that $(1,1,1,\ldots,1)$ is a left eigenvector of $\mathcal{D}_{N+1}$ corresponding to eigenvalue N and $(1,-1,1,\ldots,(-1)^N)$ is its left eigenvector corresponding to eigenvalue $-N$. The similarity transformation $\mathcal{D}_{N+1} \mapsto \mathcal{T}_{N+1}\mathcal{D}_{N+1}\mathcal{T}_{N+1}^{-1}$ where

$$\mathcal{T}_{N+1} := \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 1 & -1 & \cdots & (-1)^N \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

therefore reduces $\mathcal{D}_{N+1}$ to a block lower-triangular form

$$\begin{bmatrix} \operatorname{diag}(N,-N) & 0 \\ * & \mathcal{M}_{N-1} \end{bmatrix},$$

where

$$\mathcal{M}_{N-1} := \begin{bmatrix} 0 & 3+(-N+1) & 0 & -N+1 & 0 & -N+1 & \cdots \\ N-2 & 0 & 4 & 0 & 0 & 0 & \cdots \\ 0 & N-3 & 0 & 5 & 0 & 0 & \cdots \\ 0 & 0 & N-4 & 0 & 6 & 0 & \cdots \\ 0 & 0 & 0 & N-5 & 0 & 7 & \cdots \\ 0 & 0 & 0 & 0 & N-6 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Now it remains to show that $\mathcal{M}_{N-1}$ is similar to $\mathcal{D}_{N-1}$. But this is indeed so, since $\mathcal{S}_{N-1}^{-1}\mathcal{M}_{N-1}\mathcal{S}_{N-1} = \mathcal{D}_{N-1}$ where

$$\mathcal{S}_{N-1} := \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & -1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & -1 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

This proves (2) and therefore (1). ∎

A related determinant

$$B_{N+1}(x) := \begin{vmatrix} x & a & 0 & 0 & \cdots & 0 & 0 & 0 \\ N(a-1) & x-1 & 2a & 0 & \cdots & 0 & 0 & 0 \\ 0 & (N-1)(a-1) & x-2 & 3a & \cdots & 0 & 0 & 0 \\ & & \cdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 2(a-1) & x-(N-1) & Na \\ 0 & 0 & 0 & 0 & \cdots & 0 & a-1 & x-N \end{vmatrix}$$

can be evaluated analogously and even more simply. We use the notation

$$\mathcal{B}_{N+1} := \begin{bmatrix} 0 & a & 0 & 0 & \cdots & 0 & 0 & 0 \\ N(a-1) & -1 & 2a & 0 & \cdots & 0 & 0 & 0 \\ 0 & (N-1)(a-1) & -2 & 3a & \cdots & 0 & 0 & 0 \\ & & \cdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 2(a-1) & -(N-1) & Na \\ 0 & 0 & 0 & 0 & \cdots & 0 & a-1 & -N \end{bmatrix}$$

for the matrix that satisfies $B_{N+1}(x) = \prod_{\lambda \in \sigma(\mathcal{B}_{N+1})}(x+\lambda)$.

To prove that

$$B_{N+1}(x) = \prod_{j=0}^{N} [x + (N-2j)a - N + j] \tag{3}$$

(formula (2.8) from [1]), we will show that

$$B_{N+1}(x) = (x + Na - N) B_N(x-a). \tag{4}$$

The vector $(1, 1, 1, \ldots, 1)$ is a left eigenvector of $\mathcal{B}_{N+1}$ corresponding to eigenvalue $Na - N$. The similarity transformation $\mathcal{B}_{N+1} \mapsto \mathcal{T}_{N+1}\mathcal{B}_{N+1}\mathcal{T}_{N+1}^{-1}$ where

$$\mathcal{T}_{N+1} := \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

reduces $\mathcal{B}_{N+1}$ to a block lower triangular form

$$\begin{bmatrix} Na - N & 0 \\ * & \mathcal{M}_N \end{bmatrix},$$

where

$$\mathcal{M}_N := \begin{bmatrix} -1 - N(a-1) & 2a - N(a-1) & -N(a-1) & -N(a-1) & -N(a-1) & \cdots \\ (N-1)(a-1) & -2 & 3a & 0 & 0 & \cdots \\ 0 & (N-2)(a-1) & -3 & 4a & 0 & \cdots \\ 0 & 0 & (N-3)(a-1) & -4 & 5a & \cdots \\ 0 & 0 & 0 & (N-4)(a-1) & -5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Finally, $\mathcal{M}_N$ is similar to $\mathcal{B}_N - aI$:

$$\mathcal{M}_N \mathcal{S}_N = \mathcal{S}_N(\mathcal{B}_N - aI),$$

where

$$\mathcal{S}_N := \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

This proves (3). □

As is shown in [1], the formula (3) for B_{N+1} implies the formula (1) for the Sylvester determinant D_{N+1}, since

$$D_{N+1}(x) = \lim_{a\to\infty} \frac{B_{N+1}(ax)}{a^{N+1}}.$$

Also, (3) gives a formula for another related determinant,

$$A_{N+1}(x) := \begin{vmatrix} x & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -N & x-2 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -(N-1) & x-4 & 3 & & & & \\ & & \cdots & & & & & \\ 0 & 0 & 0 & 0 & & -2 & x-2(N-1) & N \\ 0 & 0 & 0 & 0 & \cdots & & -1 & x-2N \end{vmatrix}$$

via the relation

$$A_{N+1}(x) = 2^{N+1} B_{N+1}\left(\frac{x}{2}\right)\Big|_{a=\frac{1}{2}}.$$

3 Determinants for Krawtchouk and dual Hahn polynomials

The determinant for the Krawtchouk polynomial $K_{N+1}(x;p,N)$ is given by

$$K_{N+1}(x;p,N) := \begin{vmatrix} -x+pN & pN & 0 & 0 & \cdots & 0 & 0 \\ (1-p) & -x+pN+(1-2p) & p(N-1) & 0 & \cdots & 0 & 0 \\ 0 & 2(1-p) & -x+pN+2(1-2p) & p(N-2) & \cdots & 0 & 0 \\ & & \cdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & N(1-p) & -x+pN+N(1-2p) \end{vmatrix}.$$

With the usual notation $(a)_k$ for the shifted factorial (see [1, formula (3.10)]), we need to prove [1, (3.25)]:

$$K_{N+1}(x;p,N) = (-x)_{N+1}. \tag{5}$$

For that, it is enough to show that

$$K_{N+1}(x;p,N) = (-x)K_N(x;p,N-1). \tag{6}$$

As usual, we need to find an 'obvious' eigenvector of the matrix

$$\mathcal{K}_{N+1} := \begin{bmatrix} pN & pN & 0 & 0 & \cdots & 0 & 0 \\ (1-p) & pN+(1-2p) & p(N-1) & 0 & \cdots & 0 & 0 \\ 0 & 2(1-p) & pN+2(1-2p) & p(N-2) & \cdots & 0 & 0 \\ & & \cdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & N(1-p) & pN+N(1-2p) \end{bmatrix}$$

corresponding to eigenvalue 0. In this case, it happens to be the right eigenvector $(1,-1,1,-1,\ldots,(-1)^N)^T$. We then use the transformation

$$\mathcal{T}_{N+1} := \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^N & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{7}$$

to transform $\mathcal{K}_{N+1}$ into a block upper-triangular form

$$\mathcal{T}_{N+1}^{-1}\mathcal{K}_{N+1}\mathcal{T}_{N+1} = \begin{bmatrix} 0 & * \\ 0 & \mathcal{M}_N \end{bmatrix},$$

where

$$\mathcal{M}_N := \begin{bmatrix} pN+PN+(1-2p) & p(N-1) & 0 & \cdots & 0 & 0 \\ -pN+2(1-p) & pN+2(1-2p) & p(N-2) & \cdots & 0 & 0 \\ pN & 0 & 3(1-p) & \cdots & 0 & 0 \\ -pN & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^N pN & 0 & 0 & \cdots & N(1-p) & pN+N(1-2p) \end{bmatrix}.$$

Now, $\mathcal{M}_N$ is similar to the matrix $\mathcal{K}_N + I$, viz.

$$\mathcal{S}_N\mathcal{M}_N = (\mathcal{K}_N + I)\mathcal{S}_N$$

via the transformation

$$\mathcal{S}_N := \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{8}$$

This proves (6) and hence (5). □

The determinant for dual Hahn polynomials,

$$R_{N+1}(\lambda(x);\gamma,\delta,N) := \begin{vmatrix} \lambda(x)+N(\gamma+1) & N(\gamma+1) & 0 & 0 & \\ (N+\delta) & \lambda(x)+N(\gamma+3) \\ -(\gamma-\delta+2) & (N-1)(\gamma+2) & 0 & \cdots & \\ 0 & 2(N+\delta-1) & \lambda(x)+N(\gamma+5) \\ -2(\gamma-\delta+4) & (N-2)(\gamma+3) & & \\ & & \cdots & & \\ 0 & 0 & \cdots & N(\delta+1) & \left\{ \begin{array}{r} \lambda(x)+N(2N+\gamma+1) \\ -N(2N+\gamma-\delta) \end{array} \right\} \end{vmatrix},$$

where

$$\lambda(x) := -x(x+\gamma+\delta+1),$$

appears in [1] as the first example beyond Krawtchouk polynomials in terms of simplicity of recurrence coefficients. This determinant seems much harder to evaluate, but this in fact requires just an additional shift of parameters. The formula for R_{N+1} is [1, (4.5)]:

$$R_{N+1}(\lambda(x);\gamma,\delta,N) = (-x)_N (x+\gamma+\delta+1)_{N+1}. \tag{9}$$

It can be proved from the relation

$$R_{N+1}(\lambda(x);\gamma,\delta,N) = \lambda(x) R_N(\lambda(x)+(\gamma+\delta+2);\gamma+1,\delta+1,N-1). \tag{10}$$

The latter can be checked exactly as above, i.e., by applying the transformation $\mathcal{T}_{N+1}$ of the form (7) to the matrix

$$\mathcal{R}_{N+1}(\gamma,\delta) := \begin{bmatrix} N(\gamma+1) & N(\gamma+1) & 0 & 0 & \\ (N+\delta) & \begin{matrix} N(\gamma+3) \\ -(\gamma-\delta+2) \end{matrix} & (N-1)(\gamma+2) & 0 & \cdots \\ 0 & 2(N+\delta-1) & \begin{matrix} N(\gamma+5) \\ -2(\gamma-\delta+4) \end{matrix} & (N-2)(\gamma+3) & \\ & & \cdots & & \\ 0 & 0 & \cdots & N(\delta+1) & \left\{ \begin{matrix} N(2N+\gamma+1) \\ -N(2N+\gamma-\delta) \end{matrix} \right\} \end{bmatrix},$$

obtaining a block upper-triangular matrix

$$\mathcal{T}_{N+1}^{-1} \mathcal{R}_{N+1}(\gamma,\delta) \mathcal{T}_{N+1} = \begin{bmatrix} 0 & * \\ 0 & \mathcal{M}_N \end{bmatrix},$$

where

$$\mathcal{M}_N(\gamma,\delta) := \begin{bmatrix} \begin{matrix} N(\gamma+1)+N(\gamma+3) \\ -(\gamma-\delta-2) \end{matrix} & (N-1)(\gamma+2) & 0 & 0 & \cdots \\ \begin{matrix} -N(\gamma+1) \\ +2(N+\delta-1) \end{matrix} & \begin{matrix} N(\gamma+5) \\ -2(\gamma-\delta+4) \end{matrix} & (N-2)(\gamma+3) & 0 & \cdots \\ N(\gamma+1) & 3(N+\delta-2) & \begin{matrix} N(\gamma+7) \\ -3(\gamma-\delta+6) \end{matrix} & (N-3)(\gamma+4) & \cdots \\ -N(\gamma+1) & 0 & 4(N+\delta+3) & \begin{matrix} N(\gamma+9) \\ -4(\gamma-\delta+8) \end{matrix} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and then transforming $\mathcal{M}_N(\gamma,\delta)$ into $\mathcal{R}_N(\gamma+1,\delta+1)+(\gamma+\delta+2)I$ by

$$\mathcal{M}_N(\gamma,\delta) \mapsto \mathcal{S}_N \mathcal{M}_N(\gamma,\delta) \mathcal{S}_N^{-1},$$

where $\mathcal{S}_N$ has the form (8). This proves (10) and (9). □

4 Determinants for Hahn, Racah, and q-Racah polynomials

The last examples appearing in [1] are three types of orthogonal polynomials, whose recurrence coefficients are fractional rather than polynomial. Recall that the recurrence coefficients appear as entries on the main diagonal and the first sub- and super- diagonals of the tridiagonal determinant that gives the value of the corresponding polynomial. The alternating sums of these coefficients are still zero, just like in the two examples we just considered in section 3. In other words, the determinants we now consider all have the form

$$\begin{vmatrix} \lambda(x)-b_0 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ c_1 & \lambda(x)-b_1 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & c_2 & \lambda(x)-b_2 & \cdots & 0 & 0 & 0 \\ & \cdots & & & & & \\ 0 & 0 & 0 & \cdots & c_{N-1} & \lambda(x)-b_{N-1} & a_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & c_N & \lambda(x)-b_N \end{vmatrix} =: \det(\lambda(x)+\mathcal{G}_{N+1}), \qquad (11)$$

where

$$b_n = -a_n - c_n \qquad \text{for all} \quad n. \qquad (12)$$

The coefficients a_n and c_n for Hahn polynomials are

$$\begin{aligned} a_n &= \frac{(n+\alpha+\beta+1)(n+\alpha+1)(N-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} \\ c_n &= \frac{n(n+\alpha+\beta+N+1)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}, \end{aligned}$$

with $\lambda(x) = -x$, while the coefficients for Racah polynomials are

$$\begin{aligned} a_n &= \frac{(n+\alpha+1)(n+\alpha+\beta+1)(n+\gamma+1)(N-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} \\ c_n &= \frac{-n(n+\alpha+\beta+N+1)(n+\alpha+\beta-\gamma)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}, \end{aligned}$$

with $\beta+\gamma+1 = -N$ and $\lambda(x) = -x(x+\gamma+\delta)$, and the coefficients for q-Racah polynomials are

$$a_n = \frac{(1-abq^{n+1})(1-q^{n+1})(1-q^{n-N})(1-cq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})} \qquad (13)$$

$$c_n = \frac{cq^{-N}}{b}\frac{(1-q^n)(1-bq^n)(1-abc^{-1}q^n)(1-abq^{n+N+1})}{(1-abq^{2n})(1-abq^{2n+1})}, \qquad (14)$$

with $bdq = q^{-N}$ and $\lambda(x) = -(-q^{-x})(1-q^{x+1}cd)$ (see [1, Sec. 4]).

Recall that Hahn and Racah polynomials are just limiting cases of q-Racah polynomials [1, Sec. 4]. Precisely, if we denote Hahn polynomials by $H_{N+1}(\lambda(x);\alpha,\beta,N)$, Racah polynomials by $RA_{N+1}(\lambda(x);\alpha,\beta,\gamma,N)$, and q-Racah polynomials by $QRA_{N+1}(\lambda(x);q,a,b,c,N)$, then we see that

$$\begin{aligned} RA_{N+1}(\lambda(x);\alpha,\beta,\gamma,N) &= \lim_{q\to 1}(-1)^{N+1}\frac{QRA_{N+1}(\lambda(x);q,q^\alpha,q^\beta,q^\gamma,N)}{(1-q)^{2N+2}}, \\ H_{N+1}(\lambda(x);\alpha,\beta,N) &= \lim_{\gamma\to\infty}\frac{RA_{N+1}(\lambda(x);\alpha,\beta,\gamma,N)}{\gamma^{N+1}}. \end{aligned}$$

So, it is enough to evaluate q-Racah polynomials. The formula is given in [1] in the form

$$QRA_{N+1}(\lambda(x);q,a,b,c,N) = (-1)^{N+1}(q^{-x};q)_{N+1}(q^{x+1}cd;q)_{N+1},$$

where $(\ ;\ )_k$ is the q-analogue of the shifted factorial [1, formula (4.13)]:

$$(a;q)_k := \prod_{j=0}^{k-1}(1-aq^j).$$

Its equivalent form that is more suitable for an inductive proof is

$$QRA_{N+1}(\lambda(x);q,a,b,c,N) = \prod_{n=0}^{N}(\lambda(x)-\lambda(n)). \tag{15}$$

This formula will be proved once we show that

$$QRA_{N+1}(\lambda(x);q,a,b,c,N) = \lambda(x)q^{-N}QRA_N(\widetilde{\lambda}(x-1);q,aq,b,cq,N-1), \tag{16}$$

where $\widetilde{\lambda}(x)$ corresponds to the parameters $(q,aq,b,cq,N-1)$ so that

$$\widetilde{\lambda}(x-1) = -(1-q^{-x+1})(1-\frac{cq}{bq^N}q^x) = q\left(\lambda(x)+1+\frac{c}{bq^N}-\frac{1}{q}-\frac{cq}{bq^N}\right). \tag{17}$$

To prove (16), let us start with an observation about our ansatz matrices $\mathcal{G}_{N+1}$ satisfying (11) and (12). Suppose that such a $\mathcal{G}_{N+1}$ is transformed using the matrix $\mathcal{T}_{N+1}$ given in (7). Then, as we already saw, $\mathcal{T}_{N+1}^{-1}\mathcal{G}_{N+1}\mathcal{T}_{N+1}$ is block upper triangular:

$$\begin{bmatrix} 0 & * \\ 0 & \mathcal{M}_N \end{bmatrix}.$$

Next, the transformation $\mathcal{S}_N$ given by (8) reduces $\mathcal{M}_N$ to the tridiagonal form

$$\mathcal{S}_N\mathcal{M}_N\mathcal{S}_N^{-1} = \begin{bmatrix} a_0+c_1 & a_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ c_1 & a_1+c_2 & a_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & c_2 & a_2+c_3 & a_3 & \cdots & 0 & 0 & 0 \\ & & \cdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & c_{N-2} & a_{N-2}+c_{N-1} & a_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & c_{N-1} & a_{N-1}+c_N \end{bmatrix}. \tag{18}$$

Proving (16) therefore amounts to showing that the matrix (18), with the entries a_n and c_n coming from the determinant $QRA_{N+1}(\lambda(x);q,a,b,c,N)$, is similar to the matrix $\frac{1}{q}(\mathcal{G}_N+\tilde{\lambda}(x-1)I_N)-\lambda(x)I_N$, where $\mathcal{G}_N$ is an $N\times N$-matrix of the form (11)–(12), with the entries a_n and c_n coming from the determinant $QRA_N(\lambda(x-1);q,aq,b,cq,N-1)$. (Note that, due to (17), the difference $\lambda(x)-\frac{1}{q}\tilde{\lambda}(x-1)$ does not depend on x.) The needed similarity is realized by the diagonal matrix

$$\Lambda_N := \operatorname{diag}\left((1-abq^2),\frac{1}{q}(1-abq^4),\frac{1}{q^2}(1-abq^6),\ldots,\frac{1}{q^{N-1}}(1-abq^{2N})\right).$$

so that

$$\Lambda_N^{-1}\mathcal{S}_N\mathcal{M}_N\mathcal{S}_N^{-1}\Lambda_N = \frac{1}{q}(\mathcal{G}_N+\tilde{\lambda}(x-1)I_N)-\lambda(x)I_N. \tag{19}$$

Verification of (19) is straightforward for off-diagonal entries. For diagonal entries, it reduces to verification of the identity

$$\left(a_n+c_{N+1}-1-\frac{c}{bq^N}+\frac{1}{q}+\frac{cq}{bq^N}\right)q = a_{n+1}\frac{1-abq^{2n+4}}{1-abq^{2n+2}}+q^2c_n\frac{1-abq^{2n}}{1-abq^{2n+2}},$$

where a_n are given by (13) and c_n by (14). This last identity can be checked using MATLAB Symbolic Math Toolbox. This finishes the proof of (16) and (15). □

The determinant for Racah polynomials is therefore

$$RA_{N+1}(\lambda(x);\alpha,\beta,\gamma,N) = (-x)_{N+1}(x+\gamma+\delta+1)_{N+1}$$

and the determinant for Hahn polynomials is

$$H_{N+1}(\lambda(x);\alpha,\beta,N) = (-x)_{N+1}.$$

References

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[2] J. J. Sylvester, Nouvelles Annales de Mathématiques, XIII (1854), 305, Reprinted in Collected Mathematical Papers, vol. II, 28.