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Spectra of Laplacian matrices of weighted graphs: structural genericity properties

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Abstract

This article deals with the spectra of Laplacians of weighted graphs. In this context, two objects are of fundamental importance for the dynamics of complex networks: the second eigenvalue of such a spectrum (called algebraic connectivity) and its associated eigenvector, the so-called Fiedler vector. Here we prove that, given a Laplacian matrix, it is possible to perturb the weights of the existing edges in the underlying graph in order to obtain simple eigenvalues and a Fiedler vector composed of only non-zero entries. These structural genericity properties with the constraint of not adding edges in the underlying graph are stronger than the classical ones, for which arbitrary structural perturbations are allowed. These results open the opportunity to understand the impact of structural changes on the dynamics of complex systems.

Keywords: Spectra of Graphs, Laplacians, Perturbations of Eigenvalues

AMS 2010 classification: 35PXX, 05C50, 47AXX

1 Introduction

Many dynamical phenomena observed in real world networks take place on weighted graphs [3, 2]. Synchronization in networks of lasers is an important example as it increases laser stability [18]. Another example is given by stability in power grids, where certain coupling configurations can lead to the desynchronization of generators and thereby to major blackouts [25]. Recently, random walks on graphs have attracted much attention [22, 9, 13] which has been of interest for graph community detection (see [11] and [12] as well). Further important examples can be found in neuroscience [31, 7] and in epidemiology [21]. These dynamical processes are intimately related to the spectrum of the Laplacian matrix associated with the graph.

Simplicity of eigenvalues plays a major role as it guarantees good properties of the underlying dynamics such as exponentially and uniformly fast convergence towards synchronization in diffusively coupled networks [29] and convergence to the stationary measure in random walks [1]. In fact, when the eigenvalues fail to be simple the underlying dynamics can be pathological [26]. The eigenvectors of the Laplacian matrix also have a fundamental impact on the system. In particular, the so-called Fiedler vector associated with the algebraic connectivity (i.e the second eigenvalue) of this matrix is of importance for spectral graph partitioning [23, 33] and for synchronization [28, 29]. Indeed, when the Fiedler vector has non-zero entries one can design structural changes that make synchronization unstable [27, 19].

The purpose of this article is to deal with generic properties of graph Laplacians spectra. Since the 70's and the seminal work of Fiedler, graph spectra have attracted a great deal of attention [16, 10, 24, 30, 5, 14, 6].

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Here we focus on generic properties under the constraint of keeping the graph structure unchanged and only slightly modifying the positive weights.

In general, given any Laplacian matrix (in fact any matrix) it is always possible to perturb its entries to obtain simple eigenvalues. Typically, such perturbations will change the underlying graph drastically. In fact, these perturbations usually cascade to the whole graph resulting in a massively connected graph or even a fully connected one. However, in the context of many dynamical networks it is important to keep the initial graph topology. For instance, adding new connections to power grids is very costly and can lead to instabilities [35]. In general, the links have assigned physical meaning and while we can slightly change the weights, we cannot introduce new links. In other words, we look for generic properties when we don't add new edges to the initial graph and only slightly change the weights on already existing edges.

Hence, given a connected weighted and undirected graph our main goal is to address the two questions:

(Q1) *Can we perturb the weights of the existing edges to obtain a Laplacian matrix with simple eigenvalues?*

(Q2) *Can we perturb the weights of the existing edges to obtain a Laplacian matrix with Fiedler vector having non-zero entries?*

The manuscript gives positive answer to these two questions. In fact, we prove stronger results. The set of graph Laplacians such that these two questions have positive answer has full Lebesgue measure. We will show that for directed graphs the answer to **(Q1)** is also positive.

The manuscript is organized as follows. In Section 2, we introduce our notations and the notion of structural genericity for sets of symmetric zero-row sum real matrices. This notion permits us to formulate a density result with constraint for undirected graphs. In Section 3 we state our main results (Theorem 3.1 and Theorem 3.2) providing positive answer to **(Q1)** and **(Q2)** by showing that having a simple spectrum and eigenvectors with non-zero entries are structurally generic properties. Then, in Section 4 we prove Theorem 3.1 and in Section 5 we prove Theorem 3.2. Finally, in Section 6 we generalize our first result on simplicity of eigenvalues to directed graphs (Theorem 6.6).

2 Notations and Definitions

2.1 Weighted and Undirected Graphs

Let us recall some basic notions of algebraic graph theory (we refer the reader to [4, 10] and the references therein for a detailed introduction on the subject).

A *simple graph* with n vertices or nodes is a pair (\mathcal{V}, E) formed by a vertex set $\mathcal{V} = (v_1, \dots, v_n)$ and an edge set $E \subset \{1, \dots, n\} \times \{1, \dots, n\}$, where a pair (i, j) belonging to E is called an *edge* or *link* between vertex v_i and vertex v_j (or in a short way, (i, j) is an edge between node i and node j). In particular, there is no *loop* linking a node to itself. A graph is *weighted* if any of its edges (i, j) is associated to a number $w_{i,j} > 0$ (called *weight*). When (i, j) is not in E , we set $w_{i,j} = 0$. A graph is said to be *undirected* when for any pair (i, j) belonging to its edge set E we have that (j, i) belongs to E as well and $w_{i,j} = w_{j,i}$.

Connected Graphs. An undirected graph $\mathcal{G} = (\mathcal{V}, E)$ is connected, if for any two nodes i and j of \mathcal{G} , there exists a *path* $\{i = i_1, \dots, i_p = j\}$ of connected nodes (i.e nodes successively connected by some edges of E) between node i and node j .

A *tree* is an undirected graph for which any two nodes i and j are connected by exactly one path. Geometrically, it means there is no cycle in such a graph: for this reason trees are also referred to as acyclic graphs. A *spanning tree* is a tree of which set of nodes equals \mathcal{V} , and for which the set of edges is included in E . A *rooted spanning tree* is a spanning tree for which one node has been designated as a root.

2.2 Notations on matrices

We will denote by $\mathcal{M}_n(\mathbb{R})$ the set of square real matrices of size n , by $\mathcal{Z}_n(\mathbb{R})$ the vector subspace of $\mathcal{M}_n(\mathbb{R})$ formed by zero-row sum real matrices of size n , and by $\mathcal{S}_n(\mathbb{R})$ the vector subspace of $\mathcal{M}_n(\mathbb{R})$ formed by symmetric real matrices of size n . Given a subset \mathcal{M} of $\mathcal{M}_n(\mathbb{R})$, \mathcal{M}^c will be its complement set. The notation $\|\cdot\|$ will stand for a norm over the set of matrices of $\mathcal{M}_n(\mathbb{R})$. Any norm can be used in our study.

Given a matrix M in $\mathcal{M}_n(\mathbb{R})$, the notation $\det(M)$ will stand for the determinant of M , χ_M will stand for the characteristic polynomial of the matrix M , tM will stand for the transpose matrix of M , and $\mathbf{Com}(M)$ for the comatrix of M , formed by the cofactors of these matrices. In other words the element $c_{i,j}$ of $\mathbf{Com}(M)$ is given by:

$$c_{i,j} = (-1)^{i+j} \det(M(i|j)),$$

where $M(i|j)$ is the submatrix of M obtained by deleting the i -th row and the j -th column.

\mathbb{R}_+^n will denote the subset of vectors for which the coordinates are positive or null, and \mathbb{R}_+^{*n} the one for which the coordinates are all strictly positive. Given a vector X in \mathbb{R}^n , its i -th entry will be denoted by $X_{(i)}$. Given a subspace A of \mathbb{R}^n we will denote by $\mathcal{L}|_A$ the Lebesgue measure restricted to A .

2.3 The set of Laplacian matrices of weighted graphs

Given an undirected weighted graph \mathcal{G} its Laplacian matrix L is the square matrix of size n defined by

$$L = D - W,$$

where W is the weighted *adjacency matrix* representing the weights of the edge set E i.e $W = (w_{i,j})_{i,j=1}^n$, and where $D = \text{diag}(D_1, \dots, D_n)$ with $D_i = \sum_{j=1}^n w_{i,j}$, is the matrix of degrees of \mathcal{G} . The Laplacian matrix L is a positive semi-definite operator:

Proposition 2.1. *Let L a Laplacian matrix of an undirected graph with n nodes. Then, the spectrum $\mathfrak{S}(L)$ has the form:*

$$\mathfrak{S}(L) = \{0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n\}.$$

Moreover, the multiplicity of the eigenvalue 0 equals the number of connected components of the graph.

The first part of this result follows from the Gerschgorin's disk theorem. The multiplicity of the zero eigenvalue is readily obtained by noticing that constant vectors are eigenvectors associated with this eigenvalue. The eigenvalue λ_2 (possibly equal to zero) is called *algebraic connectivity* of the graph. It is closely related to constants which are important for characterizing the topology of a graph, such as the diameter or the isoperimetric number.

Notation 2.2. *We will denote by \mathcal{W} the set of Laplacian matrices associated to connected undirected weighted graphs with n nodes. We will denote by:*

\mathcal{W}_s *the subset of \mathcal{W} formed by Laplacian matrices having only simple eigenvalues,*

\mathcal{W}_0 *the subset of \mathcal{W} formed by Laplacian matrices whose Fiedler vector has at least one zero entry.*

In Theorem 3.1 we prove a *structural density* property. Before stating this result, we need to introduce the following notion of perturbations of symmetric zero-row sum matrices:

Definition 2.3 (Structural Perturbations). For any matrix M in $\mathcal{S}_n(\mathbb{R}) \cap \mathcal{Z}_n(\mathbb{R})$ and any tuple $\mathcal{E} = (\epsilon_{i,j})_{1 \leq i < j \leq n}$ in $\mathbb{R}^{\frac{n(n-1)}{2}}$ we define the matrix $\underline{M}(\mathcal{E})$ in $\mathcal{S}_n(\mathbb{R}) \cap \mathcal{Z}_n(\mathbb{R})$ by:

$$\forall 1 \leq i < j \leq n, \underline{M}(\mathcal{E})_{i,j} = \begin{cases} -\epsilon_{i,j} & \text{if } M_{i,j} \neq 0 \\ 0 & \text{else} \end{cases}$$

$$\underline{M}(\mathcal{E})_{j,i} = \underline{M}(\mathcal{E})_{i,j}$$

$$\underline{M}(\mathcal{E})_{i,i} = - \sum_{\substack{j=1 \\ j \neq i}}^n \underline{M}(\mathcal{E})_{i,j}.$$

For example, consider the following Laplacian matrix in $\mathcal{S}_4(\mathbb{R})$:

$$L = \begin{bmatrix} 3/2 & -1 & 0 & -1/2 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 4/3 & -1/3 \\ -1/2 & 0 & -1/3 & 5/6 \end{bmatrix}.$$

Then given any tuple $\mathcal{E} = (\epsilon_{i,j})_{1 \leq i < j \leq 4}$ in \mathbb{R}^6 , the matrix $\underline{L}(\mathcal{E})$ is equal to:

$$\underline{L}(\mathcal{E}) = \begin{bmatrix} \epsilon_{1,2} + \epsilon_{1,4} & -\epsilon_{1,2} & 0 & -\epsilon_{1,4} \\ -\epsilon_{1,2} & \epsilon_{1,2} + \epsilon_{2,3} & -\epsilon_{2,3} & 0 \\ 0 & -\epsilon_{2,3} & \epsilon_{2,3} + \epsilon_{3,4} & -\epsilon_{3,4} \\ -\epsilon_{1,4} & 0 & -\epsilon_{3,4} & \epsilon_{1,4} + \epsilon_{3,4} \end{bmatrix}.$$

Remark 2.4. Given a Laplacian matrix L , the matrix $\underline{L}(\mathcal{E})$ may not be Laplacian, for its off-diagonal entries may not be negative. However, $\underline{L}(\mathcal{E})$ is a Laplacian matrix for \mathcal{E} in $\mathbb{R}_+^{\frac{n(n-1)}{2}}$. We emphasize the fact that, even for a tuple \mathcal{E} in $\mathbb{R}_+^{\frac{n(n-1)}{2}}$, the graph associated to the Laplacian matrix $\underline{L}(\mathcal{E})$ is a subgraph of \mathcal{G} (or \mathcal{G} itself). It does not have more edges than the graph \mathcal{G} .

To formulate Questions (Q1) and (Q2) we need a new notion of density with constraint for sets of real matrices, namely the constraint of keeping the graph structure of a given matrix.

Definition 2.5 (Structural density). Let $\mathcal{M}_0 \subset \mathcal{M}$ be two subsets of $\mathcal{S}_n(\mathbb{R}) \cap \mathcal{Z}_n(\mathbb{R})$. We say that \mathcal{M}_0 is structurally dense in \mathcal{M} if the following holds:

$$\forall M \in \mathcal{M}, \forall \epsilon_0 > 0, \exists \mathcal{E} \in \mathbb{R}^{\frac{n(n-1)}{2}}, \text{ such that } M + \underline{M}(\mathcal{E}) \in \mathcal{M}_0 \text{ and } \|\underline{M}(\mathcal{E})\| < \epsilon_0.$$

Obviously, one could extend the definition of structural density to the entire set $\mathcal{S}_n(\mathbb{R})$: to do this, it suffices to delete the condition

$$\underline{M}(\mathcal{E})_{i,i} = - \sum_{\substack{j=1 \\ j \neq i}}^n \underline{M}(\mathcal{E})_{i,j}$$

in Definition 2.3 above. We don't consider this extended definition here as our article focuses exclusively on Laplacian matrices, for which the structural perturbations correspond precisely to Definition 2.3. Concerning the non symmetric case, the notion of structural density for matrices in the complement set $\mathcal{S}_n(\mathbb{R})^c$ of real non symmetric matrices is dealt with in Section 6, where we consider non symmetric graph Laplacians. To this *structural density* notion, corresponds a new notion of *structural genericity* property:

Definition 2.6 (Structural genericity). A property \mathcal{R}_0 for matrices belonging to a subset \mathcal{M} of $\mathcal{S}_n(\mathbb{R}) \cap \mathcal{Z}_n(\mathbb{R})$ is said to be structurally generic in \mathcal{M} if:

- The set of matrices in \mathcal{M} satisfying \mathcal{R}_0 is structurally dense in \mathcal{M} .
- Given a matrix $M \in \mathcal{M}$, the set of tuples \mathcal{E} in $\mathbb{R}^{\frac{n(n-1)}{2}}$ such that $\underline{M}(\mathcal{E})$ belongs to \mathcal{M} and does not satisfy \mathcal{R}_0 is of Lebesgue measure zero in $\mathbb{R}^{\frac{n(n-1)}{2}}$.

3 Main Results

We can now enunciate the first result of this paper, which gives a positive answer to **(Q1)**:

Theorem 3.1. *The property of having only simple eigenvalues is structurally generic in the set \mathcal{W} , that is:*

- *The set \mathcal{W}_s is structurally dense in \mathcal{W} .*
- *For any Laplacian matrix L in \mathcal{W} , the set of tuples \mathcal{E} in $\mathbb{R}^{\frac{n(n-1)}{2}}$ such that $\underline{L}(\mathcal{E})$ belongs to $\mathcal{W}_s^c \cap \mathcal{W}$ is of Lebesgue measure zero in $\mathbb{R}^{\frac{n(n-1)}{2}}$.*

Hence, changing only the existing weights of a graph Laplacian in \mathcal{W} leads to simple spectrum. Actually, in Section 6 we will prove a similar result in the case of directed graphs (see Theorem 6.6). Our next results provides an affirmative answer to **(Q2)**, which asks how big the complement set $\mathcal{W}_0^c \cap \mathcal{W}$ is in \mathcal{W} :

Theorem 3.2. *The property of having a Fiedler vector with only non-zero components is structurally generic in \mathcal{W} :*

- *The set $\mathcal{W}_0^c \cap \mathcal{W}$ is structurally dense in the set \mathcal{W} .*
- *For any Laplacian matrix L in \mathcal{W} , the set of tuples $\mathcal{E} \in \mathbb{R}^{\frac{n(n-1)}{2}}$ such that $\underline{L}(\mathcal{E})$ belongs to \mathcal{W}_0 is of Lebesgue measure zero.*

Actually, we prove a more general extension of Theorem 3.2: the property of having a basis of eigenvectors having only non-zero components is structurally generic in \mathcal{W} . This is stated in Corollary 5.1.

3.1 Sketches of the proofs

To prove Theorem 3.1 we follow two main steps. First, we construct an iterative process to obtain a Laplacian matrix with simple eigenvalues $\underline{L}(\mathcal{E})$ over a graph with the same structure as the original graph. We start from the longest path inside one of its spanning trees, and iteratively include new branches while controlling of the spectrum. Second, we consider the map $D_L : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $(\alpha, \beta) \mapsto \text{Discr}(\chi_{\alpha L + \beta \underline{L}(\mathcal{E})})$, and we apply a classical algebraic variety result on non identically null polynomial maps in several variables: the sets of zeros of such maps are of Lebesgue measure zero, and their complement is dense (see [15]). This permits us to conclude.

The proof of Theorem 3.2 is slightly more involved. Starting from a Laplacian L over a graph with n nodes and the corresponding eigenvector equation $Lv = \lambda v$, we decompose $v = (X, z)$ and L in blocks matrices, one of these blocks being given by the Laplacian over the first $n - 1$ nodes of the initial graph. Perturbing L we can make the real number z distinct from zero. Then we obtain an equation of the form $UX = b$, where U is an operator depending on the perturbation and on the graph. We invert this operator and show that all entries of X are polynomials in the eigenvalue. Finally, using this observation the conclusion follows from the regularity of algebraic varieties.

4 Structural genericity of graph Laplacians with simple spectrum: Proof of Theorem 3.1

Here we deal with Question **(Q1)** for undirected graphs. In this case the Laplacian matrices are symmetric and their eigenvalues are real. We aim at proving that having simple eigenvalues is structurally generic in \mathcal{W} .

Proof of Theorem 3.1. (a) First we prove that \mathcal{W}_s is structurally dense in the set \mathcal{W} . Let us fix an element L of \mathcal{W} , and \mathcal{G} its associated weighted, undirected, connected graph. The idea of the proof is to construct a graph with the same structure as \mathcal{G} and simple spectrum. We will do this iteratively: we start from a path, then we will include step-by-step nodes and edges to recover the topology of \mathcal{G} . At each step of the process, we will impose smaller and smaller weights in the edges added, so as to keep a simple spectrum.

As \mathcal{G} is connected, it admits a spanning tree \mathcal{T} . Then let's consider the longest path \mathcal{P} in \mathcal{T} (see Figure 1). Without loss of generality we can reorder if necessary the nodes of \mathcal{G} so that \mathcal{T} is rooted at the node 1 and $\mathcal{P} = \{1, 2, \dots, p-1, p\}$ where $p \leq n$: indeed, reordering the nodes of \mathcal{G} thanks to a permutation σ comes to considering the matrix $P_\sigma^{-1}LP_\sigma$ instead of L (where P_σ stands for the permutation matrix defined by σ), which does not restrict the generality since two similar matrices have the same spectra.

For any tuple $\mathcal{A} = (a_{1,2}, \dots, a_{p-1,p})$ in \mathbb{R}_+^{p-1} , the Laplacian matrix $L_{\mathcal{P}}(\mathcal{A})$ (of the path \mathcal{P}) weighted by \mathcal{A} has the following form:

$$L_{\mathcal{P}}(\mathcal{A}) = \begin{bmatrix} a_{1,2} & -a_{1,2} & 0 & \cdots & \cdots & \cdots & 0 \\ -a_{1,2} & a_{1,2} + a_{2,3} & -a_{2,3} & 0 & \cdots & \cdots & 0 \\ 0 & -a_{2,3} & a_{2,3} + a_{3,4} & -a_{3,4} & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \ddots & \\ 0 & \cdots & \cdots & 0 & -a_{p-2,p-1} & a_{p-2,p-1} + a_{p-1,p} & -a_{p-1,p} \\ 0 & \cdots & \cdots & 0 & 0 & -a_{p-1,p} & a_{p-1,p} \end{bmatrix}.$$

The spectrum of such tridiagonal matrices of size p has been intensively studied. In particular, it is well known that if \mathcal{A} is only composed of two-by-two distinct reals, $L_{\mathcal{P}}(\mathcal{A})$ has only simple eigenvalues. Let us take one such tuple \mathcal{A} with positive distinct two-by-two elements, we can then write:

$$\mathfrak{S}(L_{\mathcal{P}}(\mathcal{A})) = \{0 < \alpha_2 < \cdots < \alpha_p\},$$

where the positive real numbers α_i depend on the a_i .

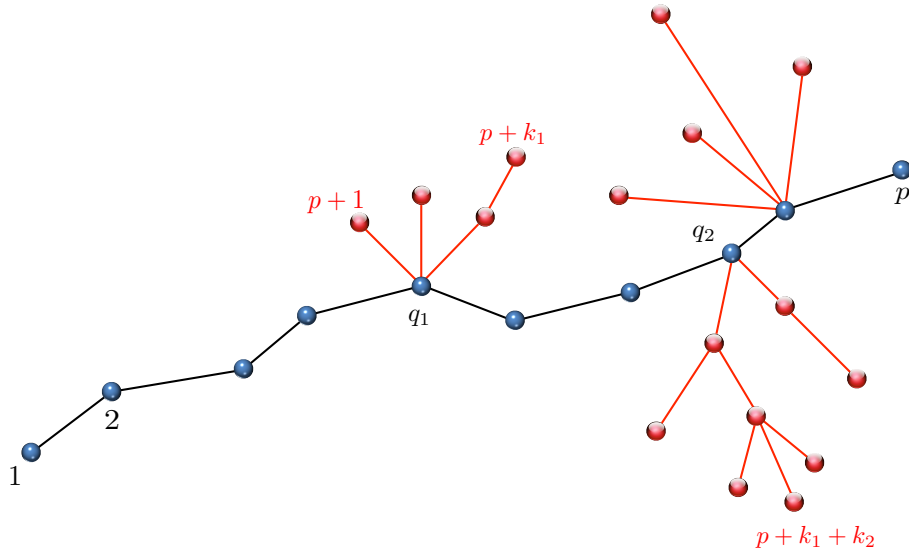


Figure 1: The path \mathcal{P} (in black) inside the spanning tree \mathcal{T} , both of them subgraphs of \mathcal{G} . We have represented in red the out-branches of \mathcal{T} , notably \mathcal{B}_1 and \mathcal{B}_2 rooted at nodes q_1 and q_2 .

Now let us consider the subgraph $\mathcal{P} \cup \{p+1, \dots, n\}$, i.e the disconnected graph composed of the path \mathcal{P} and of the other nodes of \mathcal{G} (Figure 1). Then the Laplacian matrix of this subgraph is the following square

matrix of size n :

$$L_{\mathcal{P} \cup \{p+1, \dots, n\}}(\mathcal{A}) = \left(\begin{array}{c|c} L_{\mathcal{P}}(\mathcal{A}) & (0) \\ \hline (0) & (0) \end{array} \right),$$

for which the spectrum is:

$$\mathfrak{S}(L_{\mathcal{P} \cup \{p+1, \dots, n\}}(\mathcal{A})) = \{0^{\otimes(n-p+1)} < \alpha_2 < \dots < \alpha_p\}.$$

Now, we look at the edges of the tree \mathcal{T} that do not belong to the path \mathcal{P} : let us consider the first of these edges, starting from the root 1 of this tree. This first edge is adjacent to a node say q_1 in \mathcal{P} , and without loss of generality we can assume it is the edge $\mathcal{E}_1 = (q_1, p+1)$. Re-ordering the nodes if necessary we can assume the first out-branch \mathcal{B}_1 starting at this edge \mathcal{E}_1 is composed of the nodes $q_1, p+1, \dots, p+k_1$, where k_1+1 is the number of nodes of \mathcal{B}_1 (Figure 1). Then, let's take $\epsilon_1 > 0$, small enough compared to the a_i : the subgraph $\mathcal{P} \cup \mathcal{E}_1 \cup \{p+2, \dots, n\}$ with weights given by the tuple $(\mathcal{A}, \epsilon_1)$ has one component less than the graph $\mathcal{P} \cup \{p+1, \dots, n\}$, and satisfies the relation:

$$L_{\mathcal{P} \cup \mathcal{E}_1 \cup \{p+2, \dots, n\}}(\mathcal{A}, \epsilon_1) = L_{\mathcal{P} \cup \{p+1, \dots, n\}}(\mathcal{A}) + L_{\{1, \dots, p\} \cup \mathcal{E}_1 \cup \{p+2, \dots, n\}}(\epsilon_1).$$

Therefore the spectrum of $\mathcal{P} \cup \mathcal{E}_1 \cup \{p+2, \dots, n\}$ has the following form:

$$\mathfrak{S}(L_{\mathcal{P} \cup \mathcal{E}_1 \cup \{p+2, \dots, n\}}(\mathcal{A}, \epsilon_1)) = \{0^{\otimes(n-p)} < \beta_1(\epsilon_1) < \alpha_{2,1} < \dots < \alpha_{p,1}\},$$

where $\beta_1(\epsilon_1)$ is a small perturbation of 0, and the $\alpha_{i,1}$ are small perturbations of the α_i .

Now we repeat this process with the second edge \mathcal{E}_2 of the branch \mathcal{B}_1 . Let us consider the subgraph $\mathcal{P} \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \{p+3, \dots, n\}$ weighted by the tuple $(\kappa(\epsilon_1)[\mathcal{A}, \epsilon_1], \epsilon_2)$ where $\kappa(\epsilon_1) > 0$ is chosen large enough so that the eigenvalues $\kappa(\epsilon_1)\beta_1(\epsilon_1), \kappa(\epsilon_1)\alpha_{2,1}, \dots, \kappa(\epsilon_1)\alpha_{p,1}$ are very large numbers compared to 0, and where $\epsilon_2 > 0$ is small enough compared to the entries of $\kappa(\epsilon_1)[\mathcal{A}, \epsilon_1]$.

Then this subgraph $\mathcal{P} \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \{p+3, \dots, n\}$ has one connected component less than the subgraph $\mathcal{P} \cup \mathcal{E}_1 \cup \{p+2, \dots, n\}$.

Therefore the spectrum $\mathfrak{S}(L_{\mathcal{P} \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \{p+3, \dots, n\}}(\kappa(\epsilon_1)[\mathcal{A}, \epsilon_1], \epsilon_2))$ is of the form:

$$\{0^{\otimes(n-p-1)} < \beta_2(\epsilon_2) < \beta_{1,2} < \alpha_{2,2} < \dots < \alpha_{p,2}\},$$

where $\beta_2(\epsilon_2)$ is a small perturbation of 0, and the terms $\beta_{1,2}, \alpha_{i,2}$ are small perturbations of the real numbers $\beta_1(\epsilon_1), \alpha_{i,1}$.

Repeating this process for the rest of the nodes $p+3, \dots, p+k_1$ of \mathcal{B}_1 and for the other out-branches \mathcal{B}_2 (rooted at a node say q_2), \mathcal{B}_3, \dots of the tree \mathcal{T} , we get the existence of a tuple \mathcal{E} in \mathbb{R}_+^{*n-1} for which the spectrum of the Laplacian matrix $L_{\mathcal{T}}(\mathcal{E})$ with weights given by \mathcal{E} has only simple eigenvalues:

$$\mathfrak{S}(L_{\mathcal{T}}(\mathcal{E})) = \{0 < \lambda_2(\mathcal{E}) < \dots < \lambda_n(\mathcal{E})\}.$$

We can now complete the tree \mathcal{T} with edges weighted by a tuple \mathcal{E}' in $\mathbb{R}_+^{\frac{n(n-3)}{2}}$, so as to recover the complete graph \mathcal{G} weighted by a tuple $\mathcal{E}_0 = (\mathcal{E}, \mathcal{E}') \in \mathbb{R}_+^{\frac{n(n-1)}{2}}$: choosing the real numbers of \mathcal{E}' very small compared to \mathcal{E} , we get that the Laplacian matrix $\underline{L}(\mathcal{E}_0)$ weighted by \mathcal{E}_0 belongs to \mathcal{W}_s .

Finally, to get our structural density result, it just remains to apply a classical argument on algebraic varieties. Indeed, consider the map D_L defined by:

$$D_L : \mathbb{R}^2 \rightarrow \mathbb{R} \\ (\alpha, \beta) \mapsto \text{Discr}(\chi_{\alpha L + \beta \underline{L}(\mathcal{E}_0)})'$$

where $Discr$ stands for the Discriminant map over the field $\mathbb{R}_n[X]$ of real polynomials of degree n . Recall the map $Discr$ is itself a polynomial map (in the coefficients of the element of $\mathbb{R}_n[X]$ considered) and that it satisfies the following relation:

$$\forall P \in \mathbb{R}_n[X] : Discr(P) = c(P) \prod_{i < j} (\alpha_i - \alpha_j)^2,$$

where $c(P)$ is a constant and the α_i are the roots of the polynomial P . (Notice this formula gives us that $Discr(P) = 0$ if and only if P admits at least one multiple eigenvalue).

Then D_L is a polynomial map in the entries of the matrix $\alpha L + \beta \underline{L}(\mathcal{E}_0)$. Moreover, this map D_L is not identically null over \mathbb{R}^2 since we have proved above that $D_L(0, 1) \neq 0$.

Therefore the complement of the algebraic variety $D_L^{-1}(\{0\})$ is dense in \mathbb{R}^2 . Thus we have:

$$\forall a > 0, \exists 0 < \beta < \alpha \text{ such that } \frac{\beta}{\alpha} < a \text{ and } L + \frac{\beta}{\alpha} \underline{L}(\mathcal{E}_0) \in \mathcal{W}_s.$$

We have proved that \mathcal{W}_s is structurally dense in \mathcal{W} .

(b) Now let's fix L in \mathcal{W} and prove the set of \mathcal{E} in $\mathbb{R}^{\frac{n(n-1)}{2}}$ such that $\underline{L}(\mathcal{E})$ belongs to $\mathcal{W}_s^c \cap \mathcal{W}$ is of Lebesgue measure zero in $\mathbb{R}^{\frac{n(n-1)}{2}}$. Actually the proof of this fact is already given by the first point we established above.

Indeed, it suffices to consider the map:

$$E_L : \begin{array}{ccc} \mathbb{R}^{\frac{n(n-1)}{2}} & \rightarrow & \mathbb{R} \\ \mathcal{E} & \mapsto & Discr(\chi_{\underline{L}(\mathcal{E})}) \end{array},$$

that we proved to be a non-null polynomial map. As before, we conclude the algebraic variety $E_L^{-1}(0)$ is of Lebesgue measure zero in $\mathbb{R}^{\frac{n(n-1)}{2}}$. QED. \square

5 Structural genericity of graph Laplacians having a Fiedler vector with non-zero entries: Proof of Theorem 3.2

In this section we tackle Question (Q2), which asks how big the complement set $\mathcal{W}_0^c \cap \mathcal{W}$ is in the set \mathcal{W} .

Proof of Theorem 3.2. (a) First let us prove that $\mathcal{W}_0^c \cap \mathcal{W}$ is structurally dense in \mathcal{W} .

Let us fix L in \mathcal{W} a Laplacian matrix of a graph \mathcal{G} , and L_{n-1} the Laplacian matrix of the subgraph of \mathcal{G} induced by the first $n-1$ nodes.

As \mathcal{W}_s is structurally dense in \mathcal{W} , we can assume that L is in \mathcal{W}_s . Denote by (α, a) the non identically null tuple in $\mathbb{R}_+^{\frac{(n-1)(n-2)}{2}} \times \mathbb{R}_+^{n-1}$ defining the weights of L , where α is the tuple in $\mathbb{R}_+^{\frac{(n-1)(n-2)}{2}}$ defining the weights of L_{n-1} . The matrix L has the following form:

$$L = \left(\begin{array}{cccc|ccc} & & & & & -a_1 & \\ & & & & & \vdots & \\ & & & & & -a_i & \\ & & & & & \vdots & \\ & & & & & -a_{n-1} & \\ \hline -a_1 & \cdots & -a_i & \cdots & -a_{n-1} & \sum_{k=1}^{n-1} a_k & \end{array} \right), \quad (1)$$

where the matrix $D(a)$ is the diagonal matrix of size $n-1$ containing the elements of the non identically null tuple $a = (a_1, \dots, a_{n-1})$. In accordance with our definition of structural density, we are going to prove that, up to a perturbation of the weights of L , i.e of the non-zero parameters in (α, a) , the matrix L belongs to \mathcal{W}_0 .

Consider the algebraic connectivity $\lambda_2 = \lambda_2(\alpha, a)$ of L : for a generic choice of the non-zero parameters in the tuple (α, a) , the real number λ_2 does not belong to the spectrum of L_{n-1} (this fact is proved in Lemma 7.1 in the Appendix). Therefore perturbing our non-null parameters if necessary, we can assume λ_2 is not in this spectrum.

For an eigenvector $(X, z) = (X(\alpha, a), z(\alpha, a))$ in \mathbb{R}^n associated to λ_2 we have:

$$\begin{bmatrix} L_{n-1}(\alpha) X \\ 0 \end{bmatrix} + \begin{bmatrix} a_1 (X_{(1)} - z) \\ \vdots \\ a_{n-1} (X_{(n-1)} - z) \\ -\sum_{k=1}^{n-1} a_k (X_{(k)} - z) \end{bmatrix} = \lambda_2 \begin{bmatrix} X \\ z \end{bmatrix}.$$

(i) Now, assume that in the tuple a , only the real numbers a_1 and a_i are non zero (this case does not restrict the generality as is shown later in the proof).

In this case the eigenvector equation above can be written as:

$$\begin{bmatrix} L_{n-1}(\alpha) X \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} a_1 (X_{(1)} - z) \\ \mathbf{0} \\ a_i (X_{(i)} - z) \\ \mathbf{0} \end{bmatrix} = \lambda_2 \begin{bmatrix} X \\ z \end{bmatrix} \quad (2)$$

and:

$$a_i (X_{(i)} - z) = -a_1 (X_{(1)} - z) - \lambda_2 z. \quad (3)$$

1st case: Assume we have $X_{(i)} \neq z$.

In this case, the theory of perturbations of eigenvalues (see [34, 32]) tells us that the algebraic connectivity map $x \mapsto \lambda_2(\alpha, a_1, x)$ is differentiable in $x = a_i$ (since $\lambda_2(\alpha, a_1, a_i)$ is simple) and its derivative is given by:

$$\frac{\partial}{\partial x}|_{x=a_i} \lambda_2(\alpha, a_1, x) = \frac{\mathbf{t} \begin{bmatrix} X \\ z \end{bmatrix} \cdot L(0, \dots, 1, 0, \dots, 0) \cdot \begin{bmatrix} X \\ z \end{bmatrix}}{\| \begin{bmatrix} X \\ z \end{bmatrix} \|^2} = (X_{(i)} - z)^2 > 0.$$

Thus, perturbing the parameter a_i if necessary, we have:

$$\lambda_2 = \lambda_2(\alpha, a_1, a_i) \notin \mathfrak{S}(L_{n-1}(\alpha) + M(a_1)),$$

where:

$$M(a_1) = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ -a_1 & 0 & \dots & 0 \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix}.$$

Under this possible small perturbation of a_i , the new algebraic connectivity $\lambda_2(\alpha, a_1, a_i)$ and the new Fiedler vector (X, z) still satisfies $X_{(i)} \neq z$. From Equation (2) we then obtain the following relation:

$$(L_{n-1}(\alpha) + M(a_1) - \lambda_2 I_{n-1}) X = \begin{bmatrix} a_1 z \\ \mathbf{0} \\ \lambda_2 z - a_1 z \\ \mathbf{0} \end{bmatrix},$$

in which only λ_2 depends on the parameter a_i . From this we get $z \neq 0$ and:

$$\begin{aligned} X &= \frac{1}{\mu(\alpha, a_1, a_i)} \mathbf{tCom} [L_{n-1}(\alpha) + M(a_1) - \lambda_2 I_{n-1}] \begin{bmatrix} a_1 z \\ \mathbf{0} \\ \lambda_2 z - a_1 z \\ \mathbf{0} \end{bmatrix} \\ &= \frac{1}{\mu(\alpha, a_1, a_i)} \mathbf{Com} [L_{n-1}(\alpha) + \mathbf{t}M(a_1) - \lambda_2 I_{n-1}] \begin{bmatrix} a_1 z \\ \mathbf{0} \\ \lambda_2 z - a_1 z \\ \mathbf{0} \end{bmatrix} \end{aligned}$$

where $\mu(\alpha, a_1, a_i)$ denotes the determinant of the matrix $L_{n-1}(\alpha) + M(a_1) - \lambda_2 I_{n-1}$. We are thus led to look at the comatrix of $L_{n-1}(\alpha) + \mathbf{t}M(a_1) - \lambda_2 I_{n-1}$. Let us denote by $(c_{i,j})_{1 \leq i, j \leq n-1}$ the entries of this comatrix and let us first compute all the coefficients $c_{k,1}$: to do this it suffices to develop with respect to the first column in the matrix $L_{n-1}(\alpha) + \mathbf{t}M(a_1) - \lambda_2 I_{n-1}$. We get:

$$\begin{aligned} c_{11} &= \prod_{j=1}^{n-2} (\lambda'_j(1,1) - \lambda_2), \\ \forall k > 1, c_{k1} &= (-1)^{k+1} \left(\prod_{j=1}^{n-2} (\lambda'_j(k,1) - \lambda_2) - a_1 (-1)^{i+1} \prod_{j=1}^{n-3} (\lambda''_j(k,1,1,i) - \lambda_2) \right), \end{aligned}$$

where $\lambda'_j(1,1)$ denote the eigenvalues of the submatrix $L_{n-1}(1|1)$ of size $n-2$, obtained by deleting the 1th row and the 1th column in $L_{n-1}(\alpha)$, $\lambda'_j(k,1)$ denote the eigenvalues of the submatrix $L_{n-1}(k|1)$ of size $n-2$, obtained by deleting the k th row and the 1th column, and where the terms $\lambda''_j(k,1,1,i)$ denote the eigenvalues of the submatrix (of size $n-3$) of $L_{n-1}(k|1)$, obtained by deleting the 1th row and the i th column in $L_{n-1}(k|1)$. With the same notations, we have:

$$\begin{aligned} c_{1i} &= (-1)^{1+i} \prod_{j=1}^{n-2} (\lambda'_j(1,i) - \lambda_2), \\ \forall k > 1, c_{ki} &= (-1)^{k+i} \left(\prod_{j=1}^{n-2} (\lambda'_j(k,i) - \lambda_2) + a_1 \prod_{j=1}^{n-3} (\lambda''_j(k,i,1,1) - \lambda_2) \right). \end{aligned}$$

We notice that all the eigenvalues involved above $\lambda'_j(1,1)$, $\lambda'_j(k,1)$, $\lambda'_j(1,i)$, $\lambda'_j(k,i)$ and $\lambda''_j(k,1,1,i)$, $\lambda''_j(k,i,1,1)$ are eigenvalues of submatrices of $L_{n-1}(\alpha)$: they do not depend on the coefficient a_i , and thus changing slightly a_i if necessary, we have that (for a generic choice of the parameter a_i) all those eigenvalues are distinct from the algebraic connectivity $\lambda_2 = \lambda_2(\alpha, a_1, a_i)$.

Therefore, for every integer $1 \leq k \leq n$ the coefficients c_{k1} and c_{ki} are (non identically null) polynomials of degree $n-2$ in the eigenvalue $\lambda_2 = \lambda_2(\alpha, a_1, a_i)$, each of the coefficients of these polynomials expressions being independent on the coefficient a_i . Thus we conclude that the Fiedler vector X has the form:

$$X = \frac{z}{\mu(\alpha, a_1, a_i)} \begin{bmatrix} P_1(\lambda_2) \\ \vdots \\ P_{n-1}(\lambda_2) \end{bmatrix},$$

Where each P_i are non identically null polynomials of degree $n-1$ in the eigenvalue λ_2 . Therefore, we get that, up to a perturbation of the (non-null) parameter a_i the eigenvector (X, z) has only non-zero components.

2nd case: Assume we have $X_{(i)} = z$.

Then we have $a_1 (X_{(1)} - z) = -\lambda_2 z$, with $X_{(1)} \neq z$, otherwise : $\lambda_2 \in \mathfrak{S}(L_{n-1}(\alpha))$. Thus $z \neq 0$ and:

$$\begin{aligned} X &= \frac{1}{\mu(\alpha, a_1, a_i)} \mathbf{Com}[L_{n-1}(\alpha) - \lambda_2 I_{n-1}] \begin{bmatrix} \lambda_2 z \\ \mathbf{0} \end{bmatrix} \\ &= \frac{1}{\mu(\alpha, a_1, a_i)} \mathbf{Com}[L_{n-1}(\alpha) - \lambda_2 I_{n-1}] \begin{bmatrix} \lambda_2 z \\ \mathbf{0} \end{bmatrix}. \end{aligned}$$

Now the derivative of the map $x \mapsto \lambda_2(\alpha, x, a_i)$ satisfies:

$$\frac{\partial}{\partial x|_{x=a_1}} \lambda_2(\alpha, x, a_i) = (X_{(1)} - z)^2 > 0,$$

so, perturbing a_1 if necessary, we have this time the relation

$$c_{k1} = (-1)^{k+1} \prod_{j=1}^{n-2} (\lambda'_j(k, 1) - \lambda_2) \neq 0.$$

Under this small perturbation of a_1 , either the new eigenvalue λ_2 and the new eigenvector (X, z) satisfy $X_{(i)} \neq z$ in which case we are led to the 1st case, or they satisfy $X_{(i)} = z$ in which case we get again $z \neq 0$ and the relation:

$$X = \frac{z}{\mu(\alpha, a_1, a_i)} \begin{bmatrix} c_{1,1} \lambda_2 \\ \vdots \\ c_{n-1,1} \lambda_2 \end{bmatrix},$$

which implies the eigenvector (X, z) has only non-zero coordinates.

So we proved that, up to a perturbation of the non-null parameters in the tuple (α, a_1, a_i) , the Fiedler vector $(X(\alpha, a_1, a_i), z(\alpha, a_1, a_i))$ has only non-zero coordinates.

(ii) The general situation where a is not identically null is completely similar. Indeed, either there is only one non-zero coefficient say a_i (in which case the situation is easier than in (i), since all the entries $c_{i,j}$ of the comatrix appeared above are equal in absolute value to $|\prod_{j=1}^{n-2} (\lambda'_j(k, l) - \lambda_2)|$), or more than two of them are distinct from zero: in this former case the exactly same reasoning as in (i) applies, from which we get again that the coordinates of the vector X can be expressed as non-null polynomials in the eigenvalue λ_2 .

Thus in any case, for a generic choice of the non-null parameters in (α, a) , i.e for a generic choice of the non-null weights of our initial Laplacian matrix L , the Fiedler vector $(X(\alpha, a), z(\alpha, a))$ associated to the algebraic connectivity $\lambda_2(\alpha, a)$ has only non-zero components.

In other words, for any Laplacian matrix L in \mathcal{W} , there exists a tuple \mathcal{E} in $\mathbb{R}_+^{\frac{n(n-1)}{2}}$ of norm as small as we want, such that the matrix $L + \underline{L}(\mathcal{E})$ belongs to \mathcal{W}_0 . QED.

(b) Secondly, let us fix L in \mathcal{W} and prove the set of tuples \mathcal{E} in $\mathbb{R}_+^{\frac{n(n-1)}{2}}$ for which $\underline{L}(\mathcal{E})$ belongs to \mathcal{W}_0 is of Lebesgue measure zero.

Let L a Laplacian matrix in \mathcal{W}_0 with weights given by the tuple (α, a) , and denote by $\lambda_2 = \lambda_2(\alpha, a)$ its second eigenvalue and by $X = X(\alpha, a)$ its Fiedler vector. Then in the same notations as in the proof of structural density we have just done above, either we have:

$$\lambda_2 \in \mathfrak{S}(L_{n-1}) \cap \mathfrak{S}(L)$$

which implies:

$$\mathfrak{S}(L_{n-1}) \cap \mathfrak{S}(L) \neq \{0\},$$

or λ_2 does not belong to both of these spectra: in this last case the algebraic connectivity $\lambda_2(\alpha, a)$ is a non constant map in one of the parameters of the tuple (α, a) , say a_i , and we have:

$$X = \begin{bmatrix} Q_1(\lambda_2) \\ \vdots \\ Q_{n-1}(\lambda_2) \end{bmatrix},$$

with $Q_i(\lambda_2) = 0$ for at least one index i , where the Q_i are one variable polynomials of degree $n - 1$, of which coefficients depend on all the parameters of the tuple (α, a) except a_i . Thus we have:

$$\begin{aligned} \mathcal{L}_{|\mathbb{R}^{\frac{n(n-1)}{2}}}(\{\underline{L}(\mathcal{E}) \in \mathcal{W}_0\}) &\leq \mathcal{L}_{|\mathbb{R}^{\frac{n(n-1)}{2}}}(\{(\mathcal{E}_1, \mathcal{E}_2) \in \mathbb{R}^{\frac{n(n-1)}{2}} : \mathfrak{S}(\underline{L}_{n-1}(\mathcal{E}_1)) \cap \mathfrak{S}(\underline{L}(\mathcal{E}_1, \mathcal{E}_2)) \neq \{0\}\}) \\ &+ \mathcal{L}_{|\mathbb{R}}(\cup_{i=1}^n \text{Zeros}(Q_i)). \end{aligned}$$

It is actually proven in Lemma 7.1 of the Appendix that the first term of this sum is 0. And as a set of zeros of any non null polynomial in one variable is a finite set in \mathbb{R} , therefore:

$$\mathcal{L}_{|\mathbb{R}^{\frac{n(n-1)}{2}}}(\{\underline{L}(\mathcal{E}) \in \mathcal{W}_0\}) = 0,$$

as desired. □

The reader might have observed that the proof of Theorem 3.2 can be applied to all non-zero simple eigenvalues of a Laplacian matrix in \mathcal{W} , and not only to the algebraic connectivity. And as the eigenvector of the eigenvalue 0 is $(1, \dots, 1)$, the following corollary holds:

Corollary 5.1. *The property of having a basis of eigenvectors with only non-zero entries is structurally generic in \mathcal{W} . In other words, denoting by \mathcal{W}_a the subset of \mathcal{W} formed by Laplacian matrices for which there exists a basis of eigenvectors, one of them admitting at least one zero component, we have:*

- The set $\mathcal{W}_a^c \cap \mathcal{W}$ is structurally dense in the set \mathcal{W} .
- For any L in \mathcal{W} , the set of tuples \mathcal{E} in $\mathbb{R}^{\frac{n(n-1)}{2}}$ such that $\underline{L}(\mathcal{E})$ belongs to \mathcal{W}_a is of Lebesgue measure zero in $\mathbb{R}^{\frac{n(n-1)}{2}}$.

6 Structural genericity of graph Laplacians with simple spectrum: the directed case

Section 4 was devoted to prove our first structural genericity result in the undirected case: actually Theorem 3.1 extends to the case of directed graphs (i.e simple graphs where edges have been assigned an orientation), provided we consider directed graphs presenting a particular kind of ‘‘connectedness’’. Let us first recall the notion of connectedness in the directed case.

Definition 6.1. *A directed graph (or in a short way a digraph) $\mathcal{G} = (\mathcal{V}, E)$ is strongly connected, if for any of its nodes i and j , there exists a directed path between i and j . Moreover, A digraph is said to be weakly connected if its underlying undirected graph, obtained by ignoring the orientations of the edges, is connected.*

Given a digraph, a *directed rooted spanning tree* is a rooted spanning tree for which the edges are directed towards the root. A *diverging rooted spanning tree* is a rooted spanning tree for which the edges are directed away from the root.

As a consequence, there does not always exist a directed rooted spanning tree for a weakly connected digraph; nor does there always exist a diverging rooted spanning tree (see Figure 2). Notice that other names are given in the literature for characterizing these objects: for instance diverging rooted spanning trees are sometimes called *arborescence* or *out-trees*, while directed rooted spanning trees are sometimes called *anti-arborescence* or *in-trees*.

Notation 6.2. We will denote by \mathcal{W}' the set of Laplacian matrices over weakly connected weighted digraphs, and by \mathcal{W}'_s the subset of \mathcal{W}' of Laplacian matrices for which the eigenvalues are simple.

As in the symmetric case, 0 is always an eigenvalue for Laplacian matrices over directed graphs: moreover, 0 is simple if and only if the underlying digraph admits a diverging spanning rooted tree. Other results can be found in the literature on the multiplicity of 0 in terms of number of diverging spanning trees contained in diverging spanning forests of digraphs (see [20, 8]).

Now, we aim at dealing with Question **(Q2)** for digraphs: actually Theorem 3.1 cannot be extended to the whole set \mathcal{W}' : indeed if we take the directed graph \mathcal{G}_1 (with four vertices) given in Figure 2, then \mathcal{G}_1 does not have a directed rooted spanning tree nor a diverging rooted spanning tree. Its corresponding Laplacian matrix is:

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -a_{2,1} & a_{2,1} + a_{2,4} & 0 & -a_{2,4} \\ -a_{3,1} & 0 & a_{3,1} + a_{3,4} & -a_{3,4} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

for which the eigenvalue 0 is double, and this for any values of the non-null parameters $a_{i,j}$. Similarly for the graph \mathcal{G}_2 with five vertices (see Figure 2) for which the eigenvalue 0 of the corresponding Laplacian matrix L_2 is double as well.

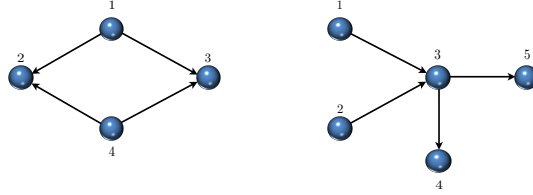


Figure 2: Two weakly connected digraphs \mathcal{G}_1 (on the left), \mathcal{G}_2 (on the right): both do not have a directed spanning tree nor a diverging spanning tree.

In fact our method used in the proof of Theorem 3.1 works for digraphs which have a diverging rooted spanning tree.

Notation 6.3. We will denote by \mathcal{U}' the subset of \mathcal{W}' of Laplacian matrices over weighted digraphs having a diverging rooted spanning tree. We will denote by \mathcal{U}'_s the subset of \mathcal{U}' formed by Laplacian matrices with only simple eigenvalues: $\mathcal{U}'_s = \mathcal{U}' \cap \mathcal{W}'_s$

Definition 6.4. For any matrix M in $\mathcal{S}_n(\mathbb{R})^c \cap \mathcal{Z}_n(\mathbb{R})$ and any tuple $\mathcal{E} = (\epsilon_{i,j})_{1 \leq i \neq j \leq n}$ in $\mathbb{R}^{n(n-1)}$, we define the matrix $\underline{M}(\mathcal{E})$ in $\mathcal{Z}_n(\mathbb{R})$ by:

$$\forall i \neq j, \underline{M}(\mathcal{E})_{i,j} = \begin{cases} -\epsilon_{i,j} & \text{if } M_{i,j} \neq 0 \\ 0 & \text{else} \end{cases}$$

$$\underline{M}(\mathcal{E})_{i,i} = - \sum_{\substack{j=1 \\ j \neq i}}^n \underline{M}(\mathcal{E})_{i,j}.$$

This definition holds only for non symmetric matrices M in $\mathcal{S}_n(\mathbb{R})^c \cap \mathcal{Z}_n(\mathbb{R})$: indeed, in the case where M is symmetric, it would not be interesting to consider a weighted matrix $\underline{M}(\mathcal{E})$ that would not be symmetric, for this would come to lose a structural information on M . Actually, for symmetric matrices M , the definition that we must consider is $\underline{M}(\mathcal{E})$ (see definition 2.3), which is the one fitting well with the undirected setting.

Definition 6.5. Let $\mathcal{M}_0 \subset \mathcal{M}$ two subsets of $\mathcal{S}_n(\mathbb{R})^c \cap \mathcal{Z}_n(\mathbb{R})$. We say that \mathcal{M}_0 is structurally dense in \mathcal{M} if the following holds:

$$\forall M \in \mathcal{M}, \forall \epsilon_0 > 0, \exists \mathcal{E} \in \mathbb{R}^{n(n-1)}, \text{ such that } M + \underline{M}(\mathcal{E}) \in \mathcal{M}_0 \text{ and } \|\underline{M}(\mathcal{E})\| < \epsilon_0.$$

From this definition of structural density, we deduce the corresponding notion of structural genericity for matrices in $\mathcal{S}_n(\mathbb{R})^c \cap \mathcal{Z}_n(\mathbb{R})$. The result now enunciates as:

Theorem 6.6. *The property of having only simple eigenvalues is structurally generic in the set of Laplacian matrices \mathcal{U}' over digraphs for which exists a diverging rooted spanning tree:*

- *The set \mathcal{U}'_s is structurally dense in the set \mathcal{U}' .*
- *For any L in \mathcal{U}' , the set of tuples \mathcal{E} in $\mathbb{R}^{n(n-1)}$ such that $\underline{L}(\mathcal{E})$ belongs to $\mathcal{U}'_s \cap \mathcal{U}'$ is of Lebesgue measure zero in $\mathbb{R}^{n(n-1)}$.*

Proof. The proof works in a completely similar as in the undirected case.

(a) First we prove that \mathcal{U}'_s is structurally dense in the set \mathcal{U}' . Let L be an element of \mathcal{U}' , and \mathcal{G} its associated weighted, weakly connected digraph.

By definition of \mathcal{U}' , \mathcal{G} admits a diverging spanning tree \mathcal{T} , rooted at a node, say node 1. Then let's consider the longest directed path \mathcal{P} in \mathcal{T} , say $\mathcal{P} = \{1, 2, \dots, p-1, p\}$ where $p \leq n$. For any tuple $\mathcal{A} = (a_{2,1}, \dots, a_{p,p-1})$ of distinct elements in \mathbb{R}_+^{p-1} , the Laplacian matrix $L_{\mathcal{P}}(\mathcal{A})$ (of the path \mathcal{P}) weighted by \mathcal{A} is now lower triangular:

$$L_{\mathcal{P}}(\mathcal{A}) = \begin{bmatrix} 0 & 0 & 0 & \dots & \dots & \dots & 0 \\ -a_{2,1} & a_{2,1} & 0 & 0 & \dots & \dots & 0 \\ 0 & -a_{3,2} & a_{3,2} & 0 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & & \ddots & \\ & & & & & & 0 \\ 0 & \dots & \dots & 0 & -a_{p-1,p-2} & a_{p-1,p-2} & 0 \\ 0 & \dots & \dots & 0 & 0 & -a_{p,p-1} & a_{p,p-1} \end{bmatrix},$$

and therefore it has a simple spectrum precisely equal to: $\mathfrak{S}(L_{\mathcal{P}}(\mathcal{A})) = \{0, a_{2,1}, \dots, a_{p,p-1}\}$. Now, as the tree \mathcal{T} is diverging, there exists an out-branch \mathcal{B}_1 starting at a node say $1 \leq q_1 \leq p$, composed of the nodes $q_1, p+1, \dots, p+k_1$. The edges of \mathcal{B}_1 being directed away from q_1 , the Laplacian matrix $L_{\mathcal{P} \cup \mathcal{B}_1 \cup \{p+k_1+1, \dots, n\}}(\mathcal{A}, \mathcal{E}_1)$ is still lower triangular (and this for any tuple \mathcal{E}_1 in $\mathbb{R}_+^{k_1}$). Therefore, its spectrum is directly given by the tuple $(\mathcal{A}, \mathcal{E}_1)$, from which we deduce again that the non-zero eigenvalues of $L_{\mathcal{P} \cup \mathcal{B}_1 \cup \{p+k_1+1, \dots, n\}}(\mathcal{A}, \mathcal{E}_1)$ are all distinct for a generic choice of \mathcal{E}_1 in $\mathbb{R}_+^{k_1}$.

Repeating the process, we get at the end the existence of a tuple $\mathcal{E}_0 \in \mathbb{R}_+^{n(n-1)}$, for which the Laplacian non symmetric matrix $\underline{L}(\mathcal{E}_0)$ weighted by \mathcal{E}_0 belongs to \mathcal{U}'_s . The rest of the proof is similar to Item (a) in the proof of Theorem 3.1.

(b) The exactly same reasoning applies, using the Discriminant map. □

7 Conclusion and discussions

We have defined and studied structural genericity properties for Laplacian matrices over weighted graphs, where only perturbations of non-zero entries were allowed, i.e, where the constraint of preserving the initial graph topology is imposed. Thanks to constructive methods, we proved that having simple spectrum and having a Fiedler vector with only non-zero components are both structurally generic for Laplacians of connected graphs.

According to a theorem by Fiedler on graph partitioning (see [17]), given a connected undirected weighted graph \mathcal{G} , if the coordinates v_i of the Fiedler vector v are all distinct from zero, then the set \mathcal{C} of edges (i, j) for which we have $v_i v_j < 0$ forms a cut of \mathcal{G} into two connected components, namely the set \mathcal{G}_1 of nodes i for which $v(i) > 0$ and the set \mathcal{G}_2 of nodes i for which $v(i) < 0$. In this case the partitioning of \mathcal{G} corresponding to the cut \mathcal{C} is uniquely determined (which is not always true) by $\mathcal{G}_1, \mathcal{G}_2$.

Therefore, our result given by Theorem 3.2 adds a more precise information to this graph partitioning theorem: namely the fact that generically (in the structural meaning we have defined in this paper), the coordinates of the Fiedler vector always provide such a partition into two connected components. This refinement of the Fiedler's partitioning result has applications to the theory of synchronization of diffusively coupled dynamical systems. We will show in a future paper how we can use these genericity results to understand the effects of structural changes on the synchronizability of complex systems.

Lastly, we conjecture that for a generic choice of the non-null weights of a given graph, the Fiedler vector has only distinct coordinates. This would be another extension of Theorem 3.2, useful as well for studying synchronization loss or synchronization enhancement in coupled networks. Such a direction will be followed in some other future works.

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Appendix

Here, we prove the following result used in the proof of Theorem 3.2 above, which asserts that given a Laplacian matrix L over a connected undirected weighted graph \mathcal{G} , then for a generic choice of the weights of L , the non-null eigenvalues of this matrix are not eigenvalues of the Laplacian matrix of the subgraph of \mathcal{G} formed by the nodes $1, \dots, n-1$:

Lemma 7.1. *Let L a Laplacian matrix in \mathcal{W} over a graph \mathcal{G} with n nodes. Let L_{n-1} the Laplacian matrix of the subgraph of \mathcal{G} formed by the first $n-1$ nodes. Then, for a generic choice of the weights of L , the following holds:*

$$\mathfrak{S}(L_{n-1}) \cap \mathfrak{S}(L) = \{0\}.$$

More precisely, the set of parameters $(\mathcal{E}_1, \mathcal{E}_2)$ in $\mathbb{R}^{\frac{(n-1)(n-2)}{2}} \times \mathbb{R}^{n-1}$ for which we have

$$\mathfrak{S}(\underline{L}_{n-1}(\mathcal{E}_1)) \cap \mathfrak{S}(\underline{L}(\mathcal{E}_1, \mathcal{E}_2)) = \{0\}$$

is dense and its complement is of Lebesgue measure zero in $\mathbb{R}^{\frac{n(n-1)}{2}}$.

Before we prove this lemma, we need the following auxiliary result:

Lemma 7.2. *Let \mathcal{P} be a path over p nodes. Then, there exists a tuple $b = (b_1, \dots, b_{p-1})$ in \mathbb{R}_+^{*p-1} such that for the Laplacian matrix $L_{\mathcal{P}}(b)$ weighted by b , the eigenvalues are simple:*

$$\mathfrak{S}(L_{\mathcal{P}}(b)) = \{0 < \lambda_2(b) < \dots < \lambda_p(b)\}$$

and the eigenvectors:

$$(\mathbf{1}, X_2(b), \dots, X_p(b))$$

satisfy:

$$\forall k \in \{2, p\} : \prod_{i=1}^p X_k(b)_{(i)} \neq 0.$$

Proof. Let \mathcal{P} be a path over p nodes, with all weights equal to one and $L_{\mathcal{P}}$ be its associated Laplacian matrix. Then denoting by L_{p-1} the Laplacian matrix of the subpath over the nodes $1, \dots, p-1$, with weights equal to one, we can write:

$$L_{\mathcal{P}} = \left(\begin{array}{cccc|c} & & & & 0 \\ & & & & \vdots \\ & & L_{p-1} + D(0, \dots, 0, 1) & & 0 \\ & & & & \vdots \\ & & & & -1 \\ \hline 0 & \dots & 0 & \dots & -1 & 1 \end{array} \right),$$

where $D(0, \dots, 0, 1)$ stands for the diagonal matrix of size $p-1$ of which entries are given by the tuple $(0, \dots, 0, 1)$. Some of the components of the eigenvectors of $L_{\mathcal{P}}$ can be zero (see Remark 7.3): the idea is to obtain the desired tuple b from a perturbation of the tuple $(1, \dots, 1)$.

It is well known that:

$$\begin{aligned} \mathfrak{S}(L_{\mathcal{P}}) &= \left\{ 0 < 2 - 2 \cos\left(\frac{\pi}{p}\right) < 2 - 2 \cos\left(\frac{2\pi}{p}\right) < \dots < 2 - 2 \cos\left(\frac{\pi(p-1)}{p}\right) \right\} \\ \mathfrak{S}(L_{p-1}) &= \left\{ 0 < 2 - 2 \cos\left(\frac{\pi}{p-1}\right) < 2 - 2 \cos\left(\frac{2\pi}{p-1}\right) < \dots < 2 - 2 \cos\left(\frac{\pi(p-2)}{p-1}\right) \right\} \end{aligned}$$

and consequently:

$$\mathfrak{S}(L_{\mathcal{P}}) \cap \mathfrak{S}(L_{p-1}) = \{0\}.$$

Now, if λ is a strictly positive eigenvalue of $L_{\mathcal{P}}$, there exists a non-null vector $\begin{bmatrix} X \\ z \end{bmatrix}$ such that:

$$\begin{cases} L_{p-1}X + \begin{bmatrix} (0) \\ -\lambda z \end{bmatrix} = \lambda X \\ -(X_{(p-1)} - z) = \lambda z \end{cases}$$

and as $\lambda \notin \mathfrak{S}(L_{p-1})$, thus $z \neq 0$ and

$$X = \frac{1}{\det[L_{p-1} - \lambda I_{p-1}]} \mathbf{tCom}[L_{p-1} - \lambda I_{p-1}] \begin{bmatrix} (0) \\ \lambda z \end{bmatrix}.$$

We now use the same argument as in the proof of Theorem 3.2: as $z \neq 0$, we have $X_{(p-1)} \neq z$ and so

$\frac{\partial}{\partial x_{|x_p=1}} \lambda(1, \dots, 1, x_p) = \frac{(X_{(p-1)} - z)^2}{\|X\|^2 + z^2} > 0$. Thus perturbing the value $x_p = 1$ in $1 + \epsilon_p$ we have that the

new eigenvalue $\lambda(1, \dots, 1, 1 + \epsilon_p)$ (corresponding to a new eigenvector $\begin{bmatrix} X(1, \dots, 1, 1 + \epsilon_p) \\ z(1, \dots, 1, 1 + \epsilon_p) \end{bmatrix}$) is not in the

spectra of the submatrices of L_{p-1} , and still does not belong to the spectrum of $\mathfrak{S}(L_{p-1})$. Therefore, as in the proof of Theorem 3.2, the coordinates of this new eigenvector (associated to the eigenvalue $\lambda(1, \dots, 1, 1 + \epsilon_p)$) are all non-zero polynomial expressions in $\lambda(1, \dots, 1, 1 + \epsilon_p)$. Perturbing again the value $1 + \epsilon_p$ if necessary, we get that all these polynomial expressions are distinct from zero:

$$\prod_{i=1}^{p-1} X(1, \dots, 1, 1 + \epsilon_p)_{(i)} z(1, \dots, 1, 1 + \epsilon_p) \neq 0.$$

And this holds for any eigenvector of any strictly positive eigenvalue of $L_{\mathcal{P}}(1, \dots, 1, 1 + \epsilon_p)$. QED. \square

Remark 7.3. Notice that the eigenvectors of the Laplacian matrix of the undirected unweighted path with p nodes are $(\mathbf{1}, X_2, \dots, X_p)$, with:

$$X_{k(i)} = \cos\left(\frac{\pi(k-1)i}{p} - \frac{\pi(k-1)}{2p}\right).$$

It can happen that an entry of X_k is zero: for instance if $p = 10$, $k = 3$ and $i = 3$. Therefore Lemma 7.2 is not true for the tuple $(1, \dots, 1)$: we needed to perturb this tuple to prove Lemma 7.2.

With Lemma 7.2 we can now prove Lemma 7.1:

Proof of Lemma 7.1. Let \mathcal{T}_n a spanning tree of \mathcal{G} and \mathcal{P} the longest path inside \mathcal{T}_n : reordering the nodes if necessary we can assume $\mathcal{P} = \{1, 2, \dots, p\}$. Applying Lemma 7.2, let us take a tuple $b = (b_1, \dots, b_{p-1})$ of strictly positive numbers such that:

$$\mathfrak{S}(L_{\mathcal{P}}(b)) = \{0 < \lambda_2(b) < \dots < \lambda_p(b)\}$$

with eigenvectors:

$$(\mathbf{1}, X_2(b), \dots, X_p(b))$$

satisfying:

$$\forall k \in \{2, p\} : \prod_{i=1}^p X_k(b)_{(i)} \neq 0.$$

As in the proof of Theorem 3.1 we look at the edges of the tree \mathcal{T}_n that do not belong to the path \mathcal{P} : starting from the root 1 of the tree, let us consider the first out-branch \mathcal{B}_1 of \mathcal{T}_n , and inside \mathcal{B}_1 , the first of these edges, say $\mathcal{E}_1 = (q_1, p+1)$, weighted by a real number α_p .

We have:

$$\mathfrak{S}(L_{\mathcal{P} \cup \mathcal{E}_1}(b, \alpha_p)) = \{0, \lambda_2(b, \alpha_p), \dots, \lambda_{p+1}(b, \alpha_p)\},$$

with:

$$\{\lambda_2(b, 0), \lambda_3(b, 0) \dots, \lambda_{p+1}(b, 0)\} = \{0, \lambda_2(b), \dots, \lambda_p(b)\}.$$

For $\alpha_p > 0$ small enough the second eigenvalue $\lambda_2(b, \alpha_p) > 0$ is a simple one, since it is a very small perturbation of the double eigenvalue 0 of $L_{\mathcal{P} \cup \{p+1\}}$. Moreover, since $\lambda_k(b, 0)$ is simple we can apply again the perturbation formula to get:

$$\forall k \in \{3, \dots, p+1\} : \frac{\partial}{\partial x_p} \lambda_k(b, x_p)|_{x_p=0} = \frac{X_{k-1}(b)_{(q_1)}^2}{\|X_{k-1}(b)\|^2} > 0.$$

Therefore the spectrum of $L_{\mathcal{P} \cup \mathcal{E}_1}(b, \alpha_p)$ is entirely perturbed, i.e it does not intersect the spectrum of $L_{\mathcal{P}}(b)$ (except in the zero eigenvalue):

$$\mathfrak{S}(L_{\mathcal{P} \cup \mathcal{E}_1}(b, \alpha_p)) \cap \mathfrak{S}(L_{\mathcal{P}}(b)) = \{0\}.$$

Besides, for $3 \leq k \leq p+1$, let us write the eigenvectors of $L_{\mathcal{P} \cup \mathcal{E}_1}(b, \alpha_p)$ as:

$$\{\mathbf{1}, X_2(b, \alpha_p), \dots, X_{p+1}(b, \alpha_p)\} = \left\{ \begin{bmatrix} \mathbf{1} \\ 1 \end{bmatrix}, \begin{bmatrix} U_2(b, \alpha_p) \\ z_2(b, \alpha_p) \end{bmatrix}, \dots, \begin{bmatrix} U_{p+1}(b, \alpha_p) \\ z_{p+1}(b, \alpha_p) \end{bmatrix} \right\}.$$

It now suffices to use the same argument as in the proof of Theorem 3.2 and of Lemma 7.2. Writing the eigenvalue equation $L_{\mathcal{P} \cup \mathcal{E}_1}(b, \alpha_p) X_k(b, \alpha_p) = \lambda_k(b, \alpha_p) X_k(b, \alpha_p)$ we obtain the relation $X_k(b, \alpha_p)_{(q_1)} \neq z_k(b, \alpha_p)$ otherwise $\lambda_k(b, \alpha_p)$ would belong to the spectrum $\mathfrak{S}(L_{\mathcal{P}}(b))$. Therefore $z_k(b, \alpha_p) \neq 0$ and we have:

$$U_k(b, \alpha_p) = \frac{1}{\det[L_{\mathcal{P}}(b) - \lambda_k(b, \alpha_p) I_p]} \mathbf{tCom}[L_{\mathcal{P}}(b) - \lambda_k(b, \alpha_p) I_p] \begin{bmatrix} (0) \\ \lambda_k(b, \alpha_p) z_k(b, \alpha_p) \\ (0) \end{bmatrix},$$

from which we conclude again that perturbing the value α_p if necessary, $U_k(b, \alpha_p)$ has only non-zero coordinates, and the same holds for all the eigenvectors $X_2(b, \alpha_p), \dots, X_{p+1}(b, \alpha_p)$.

Repeating the process for the other edges of the first out-branch \mathcal{B}_1 and for the other out-branches of the tree \mathcal{T}_n , we get the existence of a tuple \mathcal{D} in \mathbb{R}_+^{n-2} for the subtree \mathcal{T}_{n-1} (formed by the nodes $(1, \dots, n-1)$) of \mathcal{T}_n and of a real number $\alpha_{k_n} > 0$ such that for the tuple $\mathcal{E} = (\mathcal{D}, \alpha_{k_n})$ we have:

$$\begin{aligned} \mathfrak{S}(L_{\mathcal{T}_n}(\mathcal{E})) &= \{0 < \lambda_2(\mathcal{E}) < \dots < \lambda_n(\mathcal{E})\} \\ \forall k \in \{2, n\} : \prod_{i=1}^n X_k(\mathcal{E})_{(i)} &\neq 0, \end{aligned}$$

and:

$$\mathfrak{S}(L_{\mathcal{T}_{n-1}}(\mathcal{D})) \cap \mathfrak{S}(L_{\mathcal{T}_n}(\mathcal{E})) = \{0\}$$

where $\{\mathbf{1}, X_2(\mathcal{E}), \dots, X_n(\mathcal{E})\}$ are the eigenvectors associated to the eigenvalues $\lambda_k(\mathcal{E})$. Consequently, denoting by Res the resultant map (recall that for two polynomials P, Q we have $Res(P, Q) = 0$ if and only if P and Q have a common root), we conclude the map R_L defined by:

$$\begin{aligned} R_L : \mathbb{R}^{\frac{(n-1)(n-2)}{2}} \times \mathbb{R}^{n-1} &\rightarrow \mathbb{R} \\ (\mathcal{E}_1, \mathcal{E}_2) &\mapsto Res\left(\frac{\chi_{L_{n-1}(\mathcal{E}_1)}}{X}, \frac{\chi_{L(\mathcal{E}_1, \mathcal{E}_2)}}{X}\right), \end{aligned}$$

is a non-null polynomial map. Indeed let us complete the tuple \mathcal{E} found above by adding some 0 so as to obtain a tuple $(\mathcal{E}_0, \mathcal{E}'_0)$ in $\mathbb{R}_+^{\frac{n(n-1)}{2}}$. We have proved above that:

$$R_L(\mathcal{E}_0, \mathcal{E}'_0) = Res\left(\frac{\chi_{L_{\mathcal{T}_{n-1}}(\mathcal{D})}}{X}, \frac{\chi_{L_{\mathcal{T}_n}(\mathcal{E})}}{X}\right) \neq 0.$$

Finally, as in the proof of Theorem 3.1, we thus conclude that the set of tuples $(\mathcal{E}_1, \mathcal{E}_2)$ for which we have $R_L(\mathcal{E}_1, \mathcal{E}_2) = 0$, is of Lebesgue measure 0 in $\mathbb{R}^{\frac{n(n-1)}{2}}$ and is of complement dense in $\mathbb{R}^{\frac{n(n-1)}{2}}$. QED. \square

References

- [1] D. Aldous and J. Fill. Reversible markov chains and random walks on graphs, 2002.
- [2] A. Arenas, A. Daz-Guilera, J. Kurths, Y. Moreno, and C. Zhou. Synchronization in complex networks. *Phys. Rep.*, 469:93–153, 2008.
- [3] A. Barrat, M. Barthelemy, and A. Vespignani. *Dynamical processes on complex networks*. Cambridge university press, 2008.

- [4] N. Biggs. *Algebraic Graph Theory*. Cambridge University Press, 2nd edition, 1993.
- [5] T. Biyikoğlu, P. F. Stadler, and J. Leydold. *Laplacian eigenvectors of graphs*. Springer, 2007.
- [6] J. C. Bronski and L. DeVille. Spectral theory for dynamics on graphs containing attractive and repulsive interactions. *SIAM Journal of Applied Mathematics*, 74:83–105, 2014.
- [7] E. Bullmore and O. Sporns. Complex brain networks: graph theoretical analysis of structural and functional systems. *Nature Reviews Neuroscience*, 10:186–198, 2009.
- [8] P. Y. Chebotarev and R. P. Agaev. Coordination in multiagent systems and laplacian spectra of digraphs. *Autom. Remote Control*, 70:469–483, 2009.
- [9] F. R. K. Chung. *Spectral Graph Theory*. American Mathematical Society, 1997.
- [10] D. M. Cvetković, M. Doob, and H. Sachs. *Spectra of graphs: theory and application*, volume 87. Academic Press, New York, 1980.
- [11] J.-C. Delvenne, S. N. Yaliraki, and M. Barahona. Stability of graph communities across time scales. *Proceedings of the National Academy of Sciences*, 107:12755–12760, 2010.
- [12] E. Estrada. Community detection based on network communicability. *Chaos*, 21, 2011.
- [13] E. Estrada and F. Arrigo. Predicting triadic closure in networks using communicability distance functions. *SIAM Journal on Applied Mathematics*, 75:1725–1744, 2015.
- [14] E. Estrada and J. A. Rodríguez-Velázquez. Spectral measures of bipartivity in complex networks. *Phys. Rev. E*, 72:046105, 2005.
- [15] H. Federer. *Geometric measure theory*. Springer-Verlag Berlin Heidelberg, 1996.
- [16] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal*, 23:298–305, 1973.
- [17] M. Fiedler. A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory. *Czechoslovak Mathematical Journal*, 25:619–633, 1975.
- [18] V. Flunkert, S. Yanchuk, T. Dahms, and E. Schöll. Synchronizing distant nodes: a universal classification of networks. *Physical review letters*, 105:254101, 2010.
- [19] J. D. Hart, J. P. Pade, T. Pereira, T. E. Murphy, and R. Roy. Adding connections can hinder network synchronization of time-delayed oscillators. *Phys. Rev. E*, 92:022804, 2015.
- [20] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *Automatic Control, IEEE Transactions on*, 48:988–1001, 2003.
- [21] W. Lin and D. Guan-Zhong. Global stability of virus spreading in complex heterogeneous networks. *SIAM Journal on Applied Mathematics*, 68(5):1495–1502, 2008.
- [22] L. Lovász. Random walks on graphs. *Combinatorics, Paul erdos is eighty*, 2:1–46, 1993.
- [23] R. Merris. Laplacian graph eigenvectors. *Linear algebra and its applications*, 278:221–236, 1998.
- [24] B. Mohar. Some applications of laplace eigenvalues of graphs. In *Graph symmetry*, pages 225–275. Springer, 1997.
- [25] A. E. Motter, S. A. Myers, M. Anghel, and T. Nishikawa. Spontaneous synchrony in power-grid networks. *Nature Physics*, 9:191–197, 2013.

- [26] T. Nishikawa and A. E. Motter. Network synchronization landscape reveals compensatory structures, quantization, and the positive effect of negative interactions. *Proceedings of the National Academy of Sciences*, 107:10342–10347, 2010.
- [27] J. P. Pade and T. Pereira. Improving network structure can lead to functional failure. *Nature Scientific Reports*, 5:09968, 2015.
- [28] L. M. Pecora and T. L. Carroll. Master stability functions for synchronized coupled systems. *Physical Review Letters*, 80:2109, 1998.
- [29] T. Pereira, J. Eldering, M. Rasmussen, and A. Veneziani. Towards a theory for diffusive coupling functions allowing persistent synchronization. *Nonlinearity*, 27:501, 2014.
- [30] D. L. Powers. Graph partitioning by eigenvectors. *Linear Algebra and its Applications*, 101:121–133, 1988.
- [31] O. Sporns and G. Tononi. Classes of network connectivity and dynamics. *Complexity*, 7:28–38, 2001.
- [32] J. G. Sun. Multiple eigenvalue sensitivity analysis. *Linear Algebra and its Applications*, 137:183 – 211, 1990.
- [33] U. Von Luxburg. A tutorial on spectral clustering. *Statistics and computing*, 17:395–416, 2007.
- [34] D. S. Watkins. *Fundamentals of Matrix Computations*. John Wiley & Sons, Inc., New York, NY, USA, 1991.
- [35] D. Witthaut and M. Timme. Braess’s paradox in oscillator networks, desynchronization and power outage. *New journal of physics*, 14:083036, 2012.