



DFG-Research Center MATHEON
Mathematics for Key Technologies

Parameter-dependent rank-one perturbations of singular Hermitian or symmetric pencils

Christian Mehl

Volker Mehrmann

Michal Wojtylak

Preprint 02/05/2016

MATHEON preprint

<http://opus4.kobv.de/opus4-matheon>

February 05, 2016

Parameter-dependent rank-one perturbations of singular Hermitian or symmetric pencils

C. Mehl ^{‡*} V. Mehrmann ^{‡*} M. Wojtylak ^{†§}

February 5, 2016

Abstract

Structure-preserving generic low-rank perturbations are studied for classes of structured matrix pencils, including real symmetric, complex symmetric, and complex Hermitian pencils. For singular pencils it is analyzed which characteristic quantities stay invariant in the perturbed canonical form, and it is shown that the regular part of a structured matrix pencil is not affected by generic perturbations of rank one. When the rank one perturbations involve a scaling parameter, the behavior of the canonical forms in dependence of this parameter is analyzed as well.

Keywords. Hermitian pencils, symmetric pencils, singular pencils, structured Kronecker canonical form, generic low-rank perturbations

AMS subject classification. 15A18, 15A21, 15A22, 65F15, 65F22

1 Introduction

In this paper we study low-rank perturbations in the coefficients of linear differential-algebraic equations (DAEs) of the form

$$E\dot{x} + Ax = f, \tag{1}$$

[‡]Institut für Mathematik, MA 4-5, TU Berlin, Str. des 17. Juni 136, D-10623 Berlin, FRG. {mehl,mehrmann}@math.tu-berlin.de.

[†]Institut Matematyki, Wydział Matematyki i Informatyki, Uniwersytet Jagielloński, Kraków, ul. Łojasiewicza 6, 30-348 Kraków, Poland michal.wojtylak@uj.edu.pl.

[§]Supported by the Alexander von Humboldt Foundation with a Scholarship for Experienced Scientists (carried out at TU Berlin) and by a Return Home Scholarship.

*Partially supported by the *Einstein Stiftung Berlin* through the Research Center MATHEON *Mathematics for key technologies* in Berlin.

which arise as linearizations around stationary solutions of general nonlinear DAEs of the form $F(t, x, \dot{x}) = 0$, see, e.g., [3]. The analysis of the solution behavior of (1) can be characterized via the *Kronecker canonical form* of the matrix pencil $\lambda E + A$, see [2, 7, 11]. It is well known that small perturbations can drastically change the canonical form and hence also the solution behavior of (1). This is particularly unfortunate if perturbations make the pencil $\lambda E + A$ singular, because then the perturbed system may not be (uniquely) solvable any more. In view of these observations, a major motivation for our work comes from structured pencils arising in stability analysis.

Example 1 In the finite element analysis of disc brake squeal [8, 17], large scale second order differential equations arise that have the form

$$M\ddot{q} + (D + G)\dot{q} + (K + N)q = f,$$

where $M = M^T > 0$ is the mass matrix, $D = D^T \geq 0$ models material and friction induced damping, $G = -G^T$ models gyroscopic effects, $K = K^T > 0$ models the stiffness and $N = -N^T$, is a nonsymmetric matrix modeling circulatory effects. (Here $>$ (\geq) denotes positive (semi)-definiteness of a matrix). An appropriate first order formulation is associated with the linear pencil

$$\lambda E + A + L := \lambda \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix} + \begin{bmatrix} D & \frac{1}{2}N \\ \frac{1}{2}N & 0 \end{bmatrix} + \begin{bmatrix} G & K + \frac{1}{2}N \\ -(K + \frac{1}{2}N) & 0 \end{bmatrix},$$

where E is real symmetric, L is skew-symmetric, and A is symmetric.

The classical modal truncation approach [4] used in commercial finite element packages computes the eigenvalues closest to 0 and the associated eigenvectors of the symmetric eigenvalue problem $\lambda E + A$, and projects the full problem into the subspace spanned by these eigenvectors. In the analysis presented in [8], it was noticed that the matrix pencil $\lambda E + A$ was close to a singular pencil, and this effect was traced back to the introduction of a small number of stiff springs instead of rigid connections. A similar behavior was observed in [10]. These low-rank perturbations in the modeling process lead to pencils that are close to being singular. This creates large difficulties in the numerical methods, because these pencils numerically behave as if they were singular pencils. Our analysis is motivated by the desire to understand the effect of low rank perturbations in these situations.

The smallest perturbation (in some norm) that makes a pencil singular is called the *distance to singularity* and it is a long time open problem [1] to

determine this distance. Some progress in the solution of this problem has been made recently in [15] for the structured pencils and the case that the perturbations are restricted to be of rank-one. In this paper, we consider a related, but slightly different point of view, by starting from a singular pencil $\lambda E + A$ and investigating perturbations of the form

$$\lambda E + A + \tau(\lambda e + a)uv^\top, \quad (2)$$

In this paper we consider a related but slightly different point of view, by starting from a singular pencil $\lambda E + A$. For the case of pencils without additional symmetry structure, the effect of low-rank perturbations for small values of τ has been studied in great detail in [5, 6]. In the present work we consider structured pencils and rank-one perturbations of the form

$$\lambda E + A + \tau(\lambda e + a)uv^\top,$$

where u, v are vectors and a, e are scalars and τ goes through the whole real line. We focus on the following structures: In this paper we consider structured pencils and we focus on the following structures:

- Hermitian pencils, i.e., $A, E \in \mathbb{C}^{n,n}$ and $A^* = A, E^* = E$,
- real symmetric pencils, i.e., $A, E \in \mathbb{R}^{n,n}$ and $A^\top = E^\top$,
- complex symmetric pencils, i.e., $A, E \in \mathbb{C}^{n,n}$ and $A^\top = E^\top$,

where $^\top$ denotes the transpose and $*$ the conjugate transpose. Other important structures include real or complex T -alternating pencils $\lambda E + A$, where $A = A^\top$ and $E = -E^\top$, or where $A = -A^\top$ and $E = E^\top$. We will not consider T -alternating pencils in the main part of this paper, but for the sake of future reference, some preliminary results are formulated in a very general fashion so that they also cover T -alternating pencils. We will also mainly consider perturbations of the form (2) with $a = 1$ and $e = 0$; the general case can, however, always be reduced to this case by applying an adequate Möbius transformation [13].

As particular rank-one perturbations may have very specialized effects, we will mainly consider *generic* rank-one perturbations, where genericity is understood in the following sense. For $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$, a set $\mathcal{A} \subseteq \mathbb{F}^n$ is called *algebraic*, if it is a set of common zeros of finitely many polynomials p_1, \dots, p_k in n variables. An algebraic set \mathcal{A} is called *proper* if $\mathcal{A} \neq \mathbb{F}^n$. A set $\Omega \subseteq \mathbb{F}^n$ is called *generic* if its complement $\mathbb{F}^n \setminus \Omega$ is contained in a proper algebraic set. Throughout the paper, we use the following convention. The statement

“for generic $u \in \mathbb{F}^n$ property X is satisfied” precisely means “there exists a generic set $\Omega \subseteq \mathbb{F}^n$ such that for all $u \in \Omega$ property X is satisfied”.

It is known that for a perturbed singular pencil as in (2), the eigenvalues and their algebraic multiplicities are generically constant in the parameter $\tau \neq 0$, see [15]. Surprisingly, this need not be the case for the corresponding partial multiplicities which may depend on τ , as the following example shows.

Example 2 Consider the real symmetric singular pencil

$$P(\lambda) = \lambda \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} \right] + \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array} \right].$$

Letting $u = e_1 + e_4 = [1 \ 0 \ 0 \ 1 \ 0]^\top$ we obtain that

$$P_\tau(\lambda) := P(\lambda) + \tau uu^* = \left[\begin{array}{ccccc} \tau & 0 & 0 & 1+\tau & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \\ 1+\tau & \lambda & 0 & \tau & 0 \\ 0 & 1 & \lambda & 0 & 0 \end{array} \right].$$

For $\tau \neq 0$ and

$$S_\tau^T = \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\frac{1+\tau}{\tau} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right],$$

we then obtain

$$S_\tau^T P_\tau(\lambda) S_\tau = \left[\begin{array}{cc|cc} \tau & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & \lambda & -\frac{1}{\tau}(1+2\tau) & 0 \\ 0 & \lambda & 1 & 0 & 0 \end{array} \right].$$

Thus, for all $\tau \neq 0$, the pencil $P_\tau(\lambda)$ is regular and has the eigenvalue infinity with algebraic multiplicity one and the eigenvalue zero with algebraic multiplicity four. Note that for $\tau \neq -1/2$ the pencil has a block of size four corresponding to the eigenvalue zero in the structured Kronecker canonical form, see (3) below, while for $\tau = -1/2$ there are two blocks of size two.

Although having an eigenvalue of multiplicity four is a non-generic property, we will show that such change in the canonical form for a specific value of the parameter τ is generic in the classes of real or complex symmetric pencils. More precisely, we will show that if a singular pencil becomes regular after a rank-one perturbation, then its regular part generically is not affected by the perturbation. Furthermore, there appear new eigenvalues whose location is independent of the norm of the perturbation as their algebraic multiplicities are constant in τ , but their partial multiplicities will only be constant in τ except for a specific value τ_0 . In the case of real symmetric pencils, this discontinuity of the canonical form is accompanied by a switch of one sign from the so-called *sign characteristic* of the pencil.

The paper is organized as follows. In Section 2 we present structured canonical forms for Hermitian, real symmetric, and complex symmetric pencils. Section 3 contains preparatory results that are applicable to several kinds of structured matrix pencils and contain as an important result that the regular part of a structured matrix pencil is not affected by generic perturbations of rank-one. In Section 4, we investigate Hermitian and real or complex symmetric singular pencils that become regular after a structure-preserving rank-one perturbation. In particular, the behavior of their canonical forms in dependence of a scaling parameter in the rank-one perturbation is analyzed.

2 Canonical forms of pencils

In this section, we recall basic decompositions for all three classes of structured pencils considered in this paper. We start with the canonical form for Hermitian pencils, see e.g. [12, 18].

Theorem 3 (Hermitian canonical form) *Let $A, E \in \mathbb{C}^{n \times n}$ be Hermitian. Then there exists an invertible matrix $S \in \mathbb{C}^{n \times n}$ such that the pencil $S^*(\lambda E + A)S$ is block-diagonal with diagonal blocks of one of the following forms:*

i) *blocks corresponding to a real eigenvalue $\gamma \in \mathbb{R}$:*

$$\mathcal{J}_{k,\gamma}^s(\lambda) := s \begin{bmatrix} & & & \lambda - \gamma \\ & & \ddots & 1 \\ & & \ddots & \\ \lambda - \gamma & 1 & & \end{bmatrix} \in \mathbb{R}^{k \times k}, \quad s \in \{-1, 1\}; \quad (3)$$

ii) blocks corresponding to a pair of conjugate complex eigenvalues $\gamma, \bar{\gamma}$, where $\gamma \in \mathbb{C}^+ := \{z \in \mathbb{C} \mid \text{Im } z > 0\}$:

$$\mathcal{J}_{k,k,\gamma}(\lambda) := \begin{bmatrix} 0 & \mathcal{J}_{k,\gamma}^1(\lambda) \\ \mathcal{J}_{k,\bar{\gamma}}^1(\lambda) & 0 \end{bmatrix} \in \mathbb{C}^{2k \times 2k}, \quad (4)$$

where $\mathcal{J}_{k,\gamma}^1(\lambda)$ and $\mathcal{J}_{k,\bar{\gamma}}^1(\lambda)$ are defined as in (3);

iii) blocks corresponding to the eigenvalue infinity:

$$\mathcal{N}_k^s(\lambda) := s \begin{bmatrix} & & & 1 \\ & & \ddots & \lambda \\ & \ddots & \ddots & \\ 1 & \lambda & & \end{bmatrix} \in \mathbb{R}^{k \times k}, \quad s \in \{-1, 1\}; \quad (5)$$

iv) singular blocks:

$$\mathcal{L}_{2k+1}(\lambda) := \begin{bmatrix} 0 & \mathcal{G}_k(\lambda) \\ \mathcal{G}_k^\top(\lambda) & 0 \end{bmatrix} \in \mathbb{R}^{(2k+1) \times (2k+1)}, \quad (6)$$

where

$$\mathcal{G}_k(\lambda) = \begin{bmatrix} 1 & & & \\ \lambda & \ddots & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda \end{bmatrix} \in \mathbb{R}^{(k+1) \times k}. \quad (7)$$

The parameters $\gamma \in \mathbb{C}$, $s \in \{-1, 1\}$, and $k \in \mathbb{N}$ (for blocks of type i)–iii) we have $k \geq 1$) depend on the particular block and hence may be different in different blocks. Moreover, the canonical form is unique up to permutation of diagonal blocks.

A Hermitian pencil is *singular* if and only if it contains blocks of the form (6) and *infinity* is an eigenvalue if and only if it contains a block of the form (5). The collection of the signs s appearing in the blocks associated with a fixed real eigenvalue or the eigenvalue infinity, respectively, is called the *sign characteristic* of the corresponding eigenvalue.

Note that the canonical form as we presented it here is consistent with interpreting a Hermitian pencil as a degree one matrix polynomial $\lambda E + A$.

In the literature, Hermitian pencils are also written in the form $\lambda E - A$ and then instead of $\mathcal{N}_k^s(\lambda)$ a block of the form

$$\tilde{\mathcal{N}}_k^{\tilde{s}}(\lambda) := \tilde{s} \begin{bmatrix} & & & -1 \\ & & \ddots & \lambda \\ & \ddots & \ddots & \\ -1 & \lambda & & \end{bmatrix} \in \mathbb{R}^{k \times k}, \tilde{s} \in \{-1, 1\}$$

is occurring in the canonical form. This has an effect on the definition of the sign characteristic at infinity via the canonical form, because $\mathcal{N}_k^s(\lambda)$ and $\tilde{\mathcal{N}}_k^{\tilde{s}}(\lambda)$ are congruent, if k is odd, but if k is even, then $\mathcal{N}_k^s(\lambda)$ and $\tilde{\mathcal{N}}_k^{-\tilde{s}}(\lambda)$ are congruent, see also [16] for a detailed discussion of this issue.

Next, we recall a corresponding theorem for the case of real symmetric pencils, see [19]. Note that most of the blocks in the canonical form in Theorem 3 are already real. Only for blocks of the form (4) a different representation is needed.

Theorem 4 (Real symmetric canonical form) *Let $A, E \in \mathbb{R}^{n \times n}$ be symmetric. Then there exists an invertible matrix $S \in \mathbb{R}^{n \times n}$ such that the pencil $S^\top(\lambda E + A)S$ is block-diagonal with diagonal blocks of one of the following forms:*

- i) *blocks corresponding to a real eigenvalue $\gamma \in \mathbb{R}$: $\mathcal{J}_{k,\gamma}^s(\lambda)$, where $\mathcal{J}_{k,\gamma}^s(\lambda)$ is as in (3) and $s \in \{-1, 1\}$;*
- ii) *blocks corresponding to a pair of conjugate complex eigenvalues $\alpha \pm i\beta$, where $\alpha, \beta \in \mathbb{R}$ and $\beta > 0$:*

$$\mathcal{J}_{k,k,\alpha,\beta}(\lambda) := \begin{bmatrix} & & & \lambda R_2 - M_{\alpha,\beta} \\ & & \ddots & R_2 \\ & \ddots & \ddots & \\ \lambda R_2 - M_{\alpha,\beta} & R_2 & & \end{bmatrix} \in \mathbb{R}^{2k \times 2k}, \quad (8)$$

where

$$M_{\alpha,\beta} := \begin{bmatrix} -\beta & \alpha \\ \alpha & \beta \end{bmatrix} \in \mathbb{R}^{2,2} \quad \text{and} \quad R_2 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad (9)$$

- iii) *blocks corresponding to the eigenvalue infinity: $\mathcal{N}_k^s(\lambda)$, where $\mathcal{N}_k^s(\lambda)$ is as in (5) and $s \in \{-1, 1\}$;*

iv) *singular blocks*: $\mathcal{L}_{2k+1}(\lambda)$, where $\mathcal{L}_{2k+1}(\lambda)$ is as in (6).

The parameters $\alpha, \beta, \gamma \in \mathbb{R}$, $s \in \{-1, 1\}$, and $k \geq 0$ depend on the particular block and hence may be different in different blocks. Moreover, the canonical form is unique up to permutation of diagonal blocks.

The third class considered in the paper are complex symmetric matrices. Here, we have the following canonical form, see [19].

Theorem 5 (Complex symmetric canonical form) *Let $A, E \in \mathbb{C}^{n \times n}$ be symmetric. Then there exists an invertible matrix $S \in \mathbb{C}^{n \times n}$ such that the pencil $S^\top(\lambda E + A)S$ is block-diagonal with diagonal blocks of one of the following forms:*

i) *blocks corresponding to a complex or real eigenvalue $\gamma \in \mathbb{C}$:*

$$\mathcal{J}_{k,\gamma}(\lambda) := \begin{bmatrix} & & & \lambda - \gamma \\ & & \ddots & 1 \\ & & \ddots & \\ \lambda - \gamma & 1 & & \end{bmatrix} \in \mathbb{C}^{k \times k}, \quad (10)$$

iii) *blocks corresponding to the eigenvalue infinity: $\mathcal{N}_k(\lambda) := \mathcal{N}_k^1(\lambda)$, where $\mathcal{N}_k^1(\lambda)$ is as in (5);*

iv) *singular blocks*: $\mathcal{L}_{2k+1}(\lambda)$, where $\mathcal{L}_{2k+1}(\lambda)$ is as in (6).

The parameters $\gamma \in \mathbb{C}$ and $k \geq 0$ depend on the particular block and hence may be different in different blocks. Moreover, the canonical form is unique up to permutation of diagonal blocks.

With the discussed structured canonical forms under congruence available, we will now discuss how these structures change under generic low-rank perturbations.

3 Preparatory results

In this section we show that the regular part of a pencil of one of the structures mentioned in the introduction stays intact under generic rank-one perturbations. We present Theorems 6 and 7 below in a very general setting. Note that they both cover not only the three main classes considered in the paper, i.e., Hermitian pencils, real symmetric pencils, and complex symmetric pencils, but also many other structures as real or complex T -alternating pencils.

Theorem 6 Let $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$, let $\star \in \{\ast, \top\}$, and let $A, E \in \mathbb{F}^{n,n}$ be such that $A^\star = \delta_A A$ and $E^\star = \delta_E E$ with $\delta_A, \delta_E \in \{+1, -1\}$. Then there exists a nonsingular matrix $U \in \mathbb{F}^{n,n}$ such that

$$\lambda U^\star E U + U^\star A U = \begin{bmatrix} S(\lambda) & 0 & 0 \\ 0 & R_f(\lambda) & 0 \\ 0 & 0 & R_i(\lambda) \end{bmatrix},$$

where $R_f(\lambda) = \lambda E_f + A_f$ with E_f nonsingular (this part contains the finite eigenvalues of the pencil $\lambda E + A$), $R_i(\lambda) = \lambda E_i + A_i$ with A_i nonsingular and E_i being nilpotent (this part contains the infinite eigenvalues of $\lambda E + A$), and

$$S(\lambda) = \text{diag}(\mathcal{L}_{2k_1+1}(\lambda), \dots, \mathcal{L}_{2k_\ell}(\lambda))$$

with $k_1, \dots, k_\ell \in \mathbb{N}$ and

$$\mathcal{L}_{2k_j+1}(\lambda) = \begin{bmatrix} 0 & \mathcal{G}_{k_j}(\lambda) \\ \delta_A \mathcal{G}_{k_j}^\top(\delta_A \delta_E \lambda) & 0 \end{bmatrix} = \begin{bmatrix} 0 & G_u + \lambda G_\ell \\ \delta_A G_u^\top + \lambda \delta_E G_\ell^\top & 0 \end{bmatrix}, \quad (11)$$

where $\mathcal{G}_{k_j}(\lambda)$ is as in (7) and

$$G_u =: \begin{bmatrix} I_{k_j} \\ 0 \end{bmatrix}, \quad G_\ell := \begin{bmatrix} 0 \\ I_{k_j} \end{bmatrix}. \quad (12)$$

for $j = 1, \dots, \ell$. In particular, $S(\lambda)$ is uniquely determined up to a permutation of the ℓ singular blocks on its block diagonal.

Proof. The proof follows immediately by inspection from the canonical form of pairs of Hermitian matrices in [18] or of real or complex pairs of matrices that are either symmetric or skew-symmetric given in [19]. \square

We call $R(\lambda) = \text{diag}((R_f(\lambda), R_i(\lambda)))$ the regular part and $S(\lambda)$ the singular part of the pencil $\lambda E + A$. Note that we have omitted the dependence on δ_A and δ_E in the notation $\mathcal{L}_{2k+1}(\lambda)$. It will become clear from the context which of the possible structures is meant.

In the following, we will present a general result about the effect of generic structure-preserving rank-one perturbations on the regular part of a structured pencil $\lambda E + A$. As before, we will present this theorem in a very general setting by simultaneously considering real and complex cases and in the complex case symmetry structures with respect to both the transpose and the conjugate transpose. Instead of expressing the next theorem in terms of rank-one perturbations in the form

$$\lambda(E + e u u^\star) + (A + a u u^\star), \quad (13)$$

we rather consider the unperturbed pencil and the vector u separately, thus interpreting (13) as a pair $(P(\lambda), u)$, consisting of a structured pencil $P(\lambda) = \lambda E + A$ with $A, E \in \mathbb{F}^{n,n}$ and a “perturbation vector” $u \in \mathbb{F}^n$. A congruence transformation on the pencil (13) will then take the form $(P(\lambda), u) \mapsto (U^*P(\lambda)U, U^*u)$, where $U \in \mathbb{F}^{n,n}$ is nonsingular.

Theorem 7 *Let $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$, let $\star \in \{*, \top\}$, and let $P(\lambda) = \lambda E + A$ with $A, E \in \mathbb{F}^{n,n}$ be such that $A^\star = \delta_A A$ and $E^\star = \delta_E E$ with $\delta_A, \delta_E \in \{+1, -1\}$. Furthermore, let $P(\lambda)$ have at least one singular block $\mathcal{L}_{2m+1}(\lambda)$, $m \geq 0$ in its canonical form. Then for a generic $u \in \mathbb{F}^n$, there exists a nonsingular matrix $U \in \mathbb{F}^{n \times n}$ (depending on u) such that*

$$U^*P(\lambda)U = \begin{matrix} & n_s & n_r \\ \begin{matrix} n_s \\ n_r \end{matrix} & \begin{bmatrix} S(\lambda) & 0 \\ 0 & R(\lambda) \end{bmatrix} \end{matrix}, \quad U^*u = \begin{matrix} n_s \\ n_r \end{matrix} \begin{bmatrix} u_s \\ 0 \end{bmatrix}, \quad (14)$$

where $R(\lambda)$ and $S(\lambda)$ are the regular and singular parts of the pencil, respectively.

Proof. Without loss of generality we may assume that $P(\lambda)$ has the form discussed in Theorem 6. Furthermore, we may assume that it has one singular block $\mathcal{L}_{2m+1}(\lambda)$ only. Similarly, we can consider $R_f(\lambda)$ and $R_i(\lambda)$ from Theorem 6 separately, so we may assume without loss of generality that $R(\lambda) = \lambda R_E + R_A$, where $R_A, R_E \in \mathbb{F}^{k \times k}$ and where either R_A or R_E is invertible. By interchanging the roles of A and E if necessary, we may further assume that it is R_E which is invertible. Note that interchanging A and E may change the actual structure of the pencil, e.g., in the case of an alternating pencil from even to odd, but this is not of importance. Indeed, if (14) is shown to hold for the pencil $\lambda A + E$, then it obviously also holds for the pencil $\lambda E + A$.

Let $u = [u_1^\top, u_2^\top, u_3^\top]^\top \in \mathbb{F}^n$, where $u_1 \in \mathbb{F}^{m+1}$, $u_2 \in \mathbb{F}^m$, $u_3 \in \mathbb{F}^k$, and let $D \in \mathbb{F}^{k \times (m+1)}$ be a matrix satisfying

$$u_3 = -Du_1, \quad (15)$$

with entries still to be determined. Then with

$$\mathcal{D} := \begin{bmatrix} I_{m+1} & 0 & 0 \\ 0 & I_m & 0 \\ D & 0 & I_k \end{bmatrix}^\star,$$

we obtain that $\mathcal{D}^*u = [u_1^\top, u_2^\top, 0]^\top$, and we have that $\tilde{A} := \mathcal{D}^*AD$ and $\tilde{E} := \mathcal{D}^*ED$ take the forms

$$\tilde{A} = \begin{bmatrix} 0 & G_u & 0 \\ \delta_A G_u^\top & 0 & \delta_A G_u^\top D^* \\ 0 & DG_u & R_A \end{bmatrix}, \quad \tilde{E} = \begin{bmatrix} 0 & G_\ell & 0 \\ \delta_E G_\ell^\top & 0 & \delta_E G_\ell^\top D^* \\ 0 & DG_\ell & R_E \end{bmatrix},$$

respectively. Thus, the transformed vector \mathcal{D}^*u has the desired form, but we have introduced unwanted nonzero entries in the (2, 3)- and (3, 2)-blocks of \tilde{A} and \tilde{E} . We eliminate these blocks with the help of a transformation of the form

$$\mathcal{X} := \begin{bmatrix} I_m & 0 & 0 \\ 0 & I_{m-1} & X \\ 0 & 0 & I_k \end{bmatrix}^*,$$

where $X \in \mathbb{F}^{(m-1) \times k}$ still has to be determined. Note that $\mathcal{X}^* \mathcal{D}^*u = \mathcal{D}^*u$ and

$$\begin{aligned} \hat{A} &:= \mathcal{X}^* \tilde{A} \mathcal{X} \\ &= \begin{bmatrix} 0 & G_u & 0 \\ \delta_A G_u^\top & XDG_u + \delta_A G_u^\top D^* X^* + XR_A X^* & \delta_A G_u^\top D^* + XR_A \\ 0 & DG_u + R_A X^* & R_A \end{bmatrix} \end{aligned}$$

as well as

$$\begin{aligned} \hat{E} &:= \mathcal{X}^* \tilde{E} \mathcal{X} \\ &= \begin{bmatrix} 0 & G_\ell & 0 \\ \delta_E G_\ell^\top & XDG_\ell + \delta_E G_\ell^\top D^* X^* + XR_E X^* & \delta_E G_\ell^\top D^* + XR_E \\ 0 & DG_\ell + R_E X^* & R_E \end{bmatrix}. \end{aligned}$$

Suppose that X and D can be chosen such that

$$DG_u + R_A X^* = 0 \quad \text{and} \quad DG_\ell + R_E X^* = 0, \quad (16)$$

(we will show below that this is indeed possible). Then the (3, 2)-block entries of \hat{A} and \hat{E} are zero and by (skew-)symmetry this also holds for the (2, 3)-block-entry. In that case, we can define the transformation

$$\mathcal{Y} = Y \oplus I_k,$$

where Y is chosen in such a way that the block upper 2×2 block of $\mathcal{Y}^*(\lambda \hat{E} + \hat{A})\mathcal{Y}$ is in the reduced form of Theorem 6. Then, due to (16),

the pencil $\mathcal{Y}^* \mathcal{X}^* \mathcal{D}^* P(\lambda) \mathcal{D} \mathcal{X} \mathcal{Y}$ is in the corresponding form of Theorem 6 as well. Hence,

$$Y^* \left(\begin{bmatrix} 0 & G_u \\ G_u^\top & * \end{bmatrix} + \lambda \begin{bmatrix} 0 & G_l \\ G_l^\top & * \end{bmatrix} \right) Y = \begin{bmatrix} 0 & G_u \\ G_u^\top & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & G_l \\ G_l^\top & 0 \end{bmatrix},$$

because $\mathcal{Y}^*(\lambda \widehat{E} + \widehat{A}) \mathcal{Y}$ and $P(\lambda)$ are congruent and thus $\mathcal{Y}^*(\lambda \widehat{E} + \widehat{A}) \mathcal{Y}$ must contain exactly one singular block \mathcal{L}_{2m+1} . Thus, choosing $U = \mathcal{D} \mathcal{X} \mathcal{Y}$, finishes the proof under the assumptions (15) and (16).

It remains to show that X and D can be chosen such that both (15) and (16) are satisfied. Since we have assumed that R_E is invertible, the second equation in (16) can be solved for X^* which gives $X^* = -R_E^{-1} D G_l$. Inserting this into the first equation in (16), we obtain

$$D G_u = -R_A X^* = R_A R_E^{-1} D G_l,$$

or, equivalently,

$$R_E^{-1} D G_u = R_E^{-1} R_A R_E^{-1} D G_l. \quad (17)$$

Note that the matrix on the left hand side of (17) consists of the first $m-1$ columns of $R_E^{-1} D$ while the matrix on the right hand side of (17) consists of the last $m-1$ columns of $R_E^{-1} R_A R_E^{-1} D$. Denoting $R_E^{-1} D = [d_1, \dots, d_m]$, we thus obtain

$$R_E^{-1} D = \left[(R_E^{-1} R_A)^{m-1} d_m \quad \dots \quad R_E^{-1} R_A d_m \quad d_m \right], \quad (18)$$

i.e., the matrix is uniquely determined by its last column d_m . Conversely, for every choice of d_m the matrix D defined by (18) satisfies (16). Thus, we aim to choose d_m in such a way that (15) is satisfied which translates into the condition

$$R_E^{-1} u_3 = -R_E^{-1} D u_1.$$

Note that

$$-R_E^{-1} D u_1 = -M d_m,$$

where

$$M = \left(\sum_{j=0}^{m-1} u_{1j} (R_E^{-1} R_A)^j \right)$$

and the u_{1j} , $j = 1, \dots, m-1$, denote the components of u_1 . Thus, if M is invertible then the choices $d_m = -M^{-1} R_E^{-1} u_3$,

$$D = R_E \left[(R_E^{-1} R_A)^{m-1} d_m \quad \dots \quad R_E^{-1} R_A d_m \quad d_m \right],$$

and

$$X^\star = -R_E^{-1}DG_\ell$$

satisfy both equations (15) and (16). The assertion follows by noting that $\det M$ is a nonzero polynomial in the entries of u_1 which shows that for a generic vector u the matrix M is invertible. \square

As a direct corollary we obtain an invariance result for the regular part under generic rank-one perturbations.

Corollary 8 *Let $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$, let $\star \in \{*, \top\}$, and let $P(\lambda) = \lambda E + A$ with $A, E \in \mathbb{F}^{n,n}$ be such that $A^\star = \delta_A A$ and $E^\star = \delta_E E$ with $\delta_A, \delta_E \in \{+1, -1\}$. Furthermore, let $P(\lambda)$ have at least one singular block $\mathcal{L}_{2m+1}(\lambda)$, $m \geq 0$ in its canonical form. Then for generic $u \in \mathbb{F}^n$ and for all $(e, a) \in \mathbb{F}^2$, the regular part of $\lambda E + A$ is contained in the regular part of the pencil $\lambda(E + e u u^\star) + (A + a u u^\star)$ in the following sense: There exists nonsingular matrices $U_1, U_2 \in \mathbb{F}^{n,n}$ such that*

$$U_1^\star(\lambda E + A)U_1 = \begin{bmatrix} R_1(\lambda) & 0 \\ 0 & S_1(\lambda) \end{bmatrix},$$

$$U_2^\star(\lambda E + A + (\lambda e + a)u u^\star)U_2 = \begin{bmatrix} R_2(\lambda) & 0 \\ 0 & S_2(\lambda) \end{bmatrix},$$

and

$$R_2(\lambda) = \begin{bmatrix} R_1(\lambda) & 0 \\ 0 & \tilde{R}(\lambda) \end{bmatrix}$$

for some $\tilde{R}(\lambda)$, where $R_1(\lambda)$ and $R_2(\lambda)$ are the regular parts of $\lambda E + A$ and $\lambda(E + e u u^\star) + (A + a u u^\star)$, respectively.

Note that for the sake of structure-preservation, we may have to restrict the sets of possible values for a and e in Corollary 8 depending on the considered field and the corresponding involution \star in order to guarantee that the perturbed matrix pencil is from the same set of structured pencils as the unperturbed one. For example, if $\mathbb{F} = \mathbb{C}$, $A^\star = A$ and $E^\star = E$, i.e., the pencil $\lambda E + A$ is Hermitian, then we must have $a, e \in \mathbb{R}$ to ensure that also $\lambda(E + e u u^\star) + (A + a u u^\star)$ is Hermitian. If, on the other hand, we have, e.g., $\mathbb{F} = \mathbb{R}$, $A^\top = -A$ and $E^\top = E$, i.e., the pencil $\lambda E + A$ is real T -odd, then we must require that $a = 0$ and $e \in \mathbb{R}$. However, note that the statement of Corollary 8 remains true even in the case that a and e have been chosen such that the perturbed pencil does no longer have the same structure as the unperturbed pencil.

4 Perturbation of a single singular block

For the remainder of the paper, we will focus on Hermitian, complex symmetric, and real symmetric pencils. We will investigate what happens to a matrix pencil under a generic structure-preserving rank-one perturbation when the given pencil has exactly one singular block in the structured canonical form. We also focus on perturbations that perturb the constant coefficient matrix of the pencil only, which corresponds to setting $e = 0$ in (13). As we have seen in the previous section that generically the regular part of the given singular pencil will not be affected by a structure-preserving rank-one perturbation, we can focus our attention on the singular part of the pencil only.

We start the analysis with the following lemma on the roots of a certain polynomial. Here and in the following, we use the notation \bar{q} for the polynomial whose coefficients are the complex conjugates of the coefficients of the polynomial q , i.e., if $q(\lambda) = a_k\lambda^k + \dots + a_1\lambda + a_0$, then

$$\bar{q}(\lambda) := \bar{a}_k\lambda^k + \dots + \bar{a}_1\lambda + \bar{a}_0.$$

Lemma 9 *Let $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$. Then for a generic vector $u \in \mathbb{F}^k$ the polynomial*

$$q(\lambda) = \lambda^k - \frac{u_2}{u_1}\lambda^{k-1} + \dots + (-1)^k \frac{u_{k+1}}{u_1}\lambda^0 \quad (19)$$

has simple complex roots only. Furthermore, if $\mathbb{F} = \mathbb{C}$ is interpreted as \mathbb{R}^2 then for generic $(\operatorname{Re} u, \operatorname{Im} u) \in \mathbb{R}^{2k}$ and for each nonreal root λ_0 , the conjugate $\bar{\lambda}_0$ is not a root of $q(\lambda)$. If $\mathbb{F} = \mathbb{R}$, then for all $u \in \mathbb{R}^k$ and for each nonreal root λ_0 , the conjugate $\bar{\lambda}_0$ is a root of $q(\lambda)$.

Proof. For the proof let us recall that two polynomials p_1, p_2 have a common root if and only if their Sylvester resultant matrix $S(p_1, p_2)$ is singular, see, e.g., [9, Theorem 5.7].

Under the generic assumption that $u_1 \neq 0$, the zeros of $q(\lambda)$ coincide with the zeros of $u_1q_1(\lambda) = u_1\lambda^k - u_2\lambda^{k-1} + \dots + (-1)^k u_{k+1}\lambda^0$. To prove the first assertion, note that $\det S(u_1q_1(\lambda), u_1q_1'(\lambda))$ is a nonzero polynomial in u_1, \dots, u_k .

To see the “furthermore” part, note that $|\det S(u_1q_1(\lambda), \bar{u}_1\bar{q}_1(\lambda))|^2$ is a real nonzero polynomial in the entries of $(\operatorname{Re} u, \operatorname{Im} u)$ in the case $\mathbb{F} = \mathbb{C}$. The statement for the case $\mathbb{F} = \mathbb{R}$ is elementary. \square

Remark 10 Observe that the set \mathbb{R}^k can be divided into two disjoint subsets. The first set contains all vectors $u \in \mathbb{R}^k$ defined by the property that

the polynomial q made of the components of u as in (19) has a real root, while the second subset consists of all $u \in \mathbb{R}^k$ such that q has only complex roots. Note that if k is even, then both sets have a nonempty interior while if k is odd, then the latter set is empty.

4.1 Hermitian case

In this section, we will assume that the given singular pencil is Hermitian.

Proposition 11 *Let $u \in \mathbb{C}^{2k+1}$, with $u_1 \neq 0$. If $k \geq 1$, then for $\tau \in \mathbb{R} \setminus \{0\}$ we have*

$$\det(\mathcal{L}_{2k+1}(\lambda) + \tau uu^*) = \tau \cdot (q\bar{q})(\lambda),$$

where

$$q(\lambda) = \lambda^k - \frac{u_2}{u_1} \lambda^{k-1} + \dots + (-1)^k \frac{u_{k+1}}{u_1} \lambda^0.$$

Furthermore, infinity is an eigenvalue of $\mathcal{L}_{2k+1}(\lambda) + \tau uu^*$ with the corresponding single block $\mathcal{N}_1^s(\lambda)$, where $s = \text{sgn}\tau$. Also, if $k = 0$ then

$$\det(\mathcal{L}_{2k+1}(\lambda) + \tau uu^*) = \tau |u|^2.$$

Proof. Consider

$$S := \begin{bmatrix} \frac{1}{u_1} & & & & \\ -\frac{u_2}{u_1} & 1 & & & \\ \vdots & & \ddots & & \\ -\frac{u_{2k+1}}{u_1} & & & & 1 \end{bmatrix}^*$$

such that $S^*u = e_1$. Then

$$S^*(\mathcal{L}_n(\lambda) + \tau uu^*)S = \left[\begin{array}{c|cccc} \tau & & & & \\ & \frac{1}{u_1} & & & \\ & \lambda - \frac{u_2}{u_1} & 1 & & \\ & -\frac{u_3}{u_1} & \lambda & \ddots & \\ & \vdots & & \ddots & 1 \\ & -\frac{u_{k+1}}{u_1} & & & \lambda \end{array} \right],$$

$$\left[\begin{array}{cccc|cccc} \frac{1}{\bar{u}_1} & \lambda - \frac{\bar{u}_2}{\bar{u}_1} & -\frac{\bar{u}_3}{\bar{u}_1} & \dots & -\frac{\bar{u}_{k+1}}{\bar{u}_1} & -2\text{Re} \frac{u_{k+2}}{u_1} & -\frac{\bar{u}_{k+3}}{\bar{u}_1} & \dots & -\frac{\bar{u}_{2k+1}}{\bar{u}_1} \\ & 1 & \lambda & & & -\frac{u_{k+3}}{u_1} & & & \\ & & \ddots & \ddots & & \vdots & & & \\ & & & 1 & \lambda & -\frac{u_{2k+1}}{u_1} & & & \end{array} \right],$$

with the non-indicated entries being zeros. Eliminating the entries in the positions $(1, k+2)$ and $(k+2, 1)$, by adding a multiple of the first row and column, respectively, we obtain a congruent pencil

$$[\tau] \oplus \left[\begin{array}{cccc|cccc} & & & & \lambda - \frac{u_2}{u_1} & 1 & & & \\ & & & & -\frac{u_3}{u_1} & \lambda & \ddots & & \\ & & & & \vdots & & \ddots & & 1 \\ & & & & -\frac{u_{k+1}}{u_1} & & & & \lambda \\ \hline \lambda - \frac{\bar{u}_2}{u_1} & -\frac{\bar{u}_3}{u_1} & \dots & -\frac{\bar{u}_k}{u_1} & f(u) & -\frac{u_{k+3}}{\bar{u}_1} & \dots & -\frac{u_{2k+1}}{\bar{u}_1} \\ & 1 & \lambda & & -\frac{u_{k+3}}{u_1} & & & \\ & & \ddots & \ddots & \vdots & & & \\ & & & 1 & \lambda & -\frac{u_{2k+1}}{u_1} & & \end{array} \right], \quad (20)$$

where

$$f(u) = -2\operatorname{Re} \frac{u_{k+2}}{u_1} - \frac{1}{\tau|u_1|^2}.$$

As the determinants of the upper-right and lower left-block are equal to $q(\lambda)$ and $\bar{q}(\lambda)$, respectively, the proof is finished. \square

Corollary 12 *For every monic complex polynomial q of degree $k \geq 1$ there exists a vector $u \in \mathbb{C}^{2k+1}$ with $u_1 \neq 0$ such that the characteristic polynomial of the pencil $\mathcal{L}_{2k+1}(\lambda) + uu^*$ equals $(q\bar{q})(\lambda)$.*

We now present the main theorem for Hermitian rank-one perturbations of Hermitian pencils having exactly one singular block.

Theorem 13 *Let $A, E \in \mathbb{C}^n$ be Hermitian and such that the pencil $\lambda E + A$ has precisely one singular block in its canonical form, say $\mathcal{L}_{2k+1}(\lambda)$, $k \geq 0$. Then for $u \in \mathbb{C}^n$ with $(\operatorname{Re} u, \operatorname{Im} u) \in \mathbb{R}^{2n}$ being generic, the canonical form of $A + \tau uu^* + \lambda E$ is constant for $\tau \in \mathbb{R} \setminus \{0\}$ and equals*

$$\mathcal{N}_1^s(\lambda) \oplus \bigoplus_{j=1}^k \mathcal{J}_{1,1,\lambda_j}(\lambda) \oplus R_f(\lambda) \oplus R_i(\lambda), \quad (21)$$

where $s = \operatorname{sgn} \tau$, where $\lambda_1, \dots, \lambda_k \in \mathbb{C}^+$ are mutually distinct eigenvalues, and where $R_f(\lambda) \oplus R_i(\lambda)$ is the (canonical form of) the regular part of $\lambda E + A$. In the case $k = 0$, the form (21) reduces to $\mathcal{N}_1^s(\lambda) \oplus R_f(\lambda) \oplus R_i(\lambda)$.

Proof. By Theorem 7 we may assume without loss of generality that we have $\lambda E + A = \mathcal{L}_{2k+1}(\lambda)$. By Proposition 11, the statement is trivial for the case $k = 0$ and for $k > 0$ the eigenvalues of $\lambda E + A + \tau uu^*$ are the roots of $q(\lambda)$ and $\bar{q}(\lambda)$. By Lemma 9 all roots of the polynomial q are generically simple and different from their conjugates, which determines the canonical form to be as in (21). \square

4.2 Real symmetric case

In this section, we assume that the given pencil $\lambda E + A$ is real symmetric. As a direct consequence of Proposition 11 we get the following result.

Proposition 14 *Let $u \in \mathbb{R}^{2k+1}$, with $u_1 \neq 0$. If $k \geq 1$, then for $\tau \in \mathbb{R} \setminus \{0\}$ we have*

$$\det(\mathcal{L}_{2k+1}(\lambda) + \tau uu^\top) = \tau q^2(\lambda),$$

where

$$q(\lambda) = \lambda^k - \frac{u_2}{u_1} \lambda^{k-1} + \dots + (-1)^k \frac{u_{k+1}}{u_1} \lambda^0.$$

Furthermore, infinity is an eigenvalue of $\mathcal{L}_{2k+1}(\lambda) + \tau uu^\top$ with the corresponding single block $\mathcal{N}_1^s(\lambda)$, where $s = \text{sgn}\tau$.

Also, if $k = 0$ then

$$\det(\mathcal{L}_{2k+1}(\lambda) + \tau uu^\top) = \tau u^2.$$

Note that by Proposition 14 every root of q is precisely a double eigenvalue of $\mathcal{L}_{2k+1} + uu^\top$, and in addition for each nonreal root of q there is a corresponding complex conjugate root, see Lemma 9 and Remark 10.

We will study now the generic canonical structure corresponding to these real and nonreal double eigenvalues. Recalling that the *inertia index* $\text{ind}(H) = (n_+, n_-, n_0)$ of a Hermitian or real symmetric matrix H consists of the numbers n_+ of positive eigenvalues, n_- of negative eigenvalues, and n_0 of zero eigenvalues, we introduce the following lemma.

Lemma 15 *Let $\lambda_0 \in \mathbb{R}$ be an eigenvalue of algebraic multiplicity two of the real regular symmetric matrix pencil*

$$P(\lambda) := \begin{bmatrix} 0 & \lambda C + B \\ \lambda C^\top + B^\top & \lambda F + D \end{bmatrix},$$

where $B, C, D, F \in \mathbb{R}^{n,n}$ and $D = D^\top$, $F = F^\top$. Then exactly one of the following statements hold:

(i) The geometric multiplicity of λ_0 is equal to one and the sign s of λ_0 in the sign characteristic of $P(\lambda)$ is given by

$$s = \begin{cases} +1 & \text{if } \text{ind}(P(\lambda_0)) = (n, n-1, 1), \\ -1 & \text{if } \text{ind}(P(\lambda_0)) = (n-1, n, 1). \end{cases}$$

(ii) The geometric multiplicity of λ_0 is equal to two and the signs in the sign characteristic of λ_0 are $+1$ and -1 .

Proof. For the first part of the proof, we will ignore that the pencil $P(\lambda)$ is real and consider it as particular complex Hermitian pencil. Clearly, there exists unitary matrices $U, V \in C^{n,n}$ such that $U^*(\lambda C + B)V$ is in anti-triangular form. (This follows by computing the generalized Schur decomposition of $\lambda C + B$ and then applying a row permutation.) Thus, $\text{diag}(U^*, V^*)P(\lambda)\text{diag}(U, V)$ is in anti-triangular form as well. Then it follows from [14, Remark 16 and Theorem 17] that for each real eigenvalue γ of $P(\lambda)$ in the (complex) canonical form, the number of blocks associated with γ and having odd size is even, say $2m$, and exactly m of them have sign $+1$ in the sign characteristic (and the other m have the sign -1 in the sign characteristic). To be more precise, if

$$\mathcal{J}_{k_1, \gamma}^{s_1}(\lambda), \dots, \mathcal{J}_{k_{2m}, \gamma}^{s_{2m}}(\lambda), \mathcal{J}_{k_{2m+1}, \gamma}^{s_{2m+1}}(\lambda), \dots, \mathcal{J}_{k_{2m+\ell}, \gamma}^{s_{2m+\ell}}(\lambda)$$

are exactly the blocks associated with γ in the canonical form in Theorem 3, where k_1, \dots, k_{2m} are odd and $k_{2m+1}, \dots, k_{2m+\ell}$ are even, then exactly m of the signs s_1, \dots, s_{2m} are equal to $+1$ and exactly m of them are equal to -1 . In the following we will call this condition the “*sign restriction for γ* ”.

If we insert a value $\mu \in \mathbb{R} \setminus \{\gamma\}$ in the pencil $\mathcal{J}_{k_j, \gamma}^{s_j}(\lambda)$, then the inertia index of the matrix $\mathcal{J}_{k_j, \gamma}^{s_j}(\mu)$ is given by

$$\begin{cases} (\frac{k_j}{2}, \frac{k_j}{2}, 0) & \text{if } k_j \text{ is even,} \\ (\frac{k_j+1}{2}, \frac{k_j-1}{2}, 0) & \text{if } k_j \text{ is odd and } s_j(\gamma - \mu) > 0, \\ (\frac{k_j-1}{2}, \frac{k_j+1}{2}, 0) & \text{if } k_j \text{ is odd and } s_j(\gamma - \mu) < 0. \end{cases}$$

Thus, it follows that the sign restriction for γ has the effect that the inertia index of the matrix

$$\mathcal{J}_\gamma(\mu) := \bigoplus_{j=1}^{2m+\ell} \mathcal{J}_{k_j, \gamma}^{s_j}(\mu) \tag{22}$$

is equal to $(\frac{a_\gamma}{2}, \frac{a_\gamma}{2}, 0)$ for all $\mu \in \mathbb{C} \setminus \{\gamma\}$, where a_γ denotes the algebraic multiplicity of the eigenvalue γ . A similar observation shows that we also have

$$\text{ind}(\mathcal{J}_\infty(\mu)) = (\frac{a_\infty}{2}, \frac{a_\infty}{2}, 0),$$

where a_∞ is the algebraic multiplicity of infinity as the eigenvalue of $P(\lambda)$, and where $\mathcal{J}_\infty(\mu)$ is defined analogously to (22). Thus, if $\mathcal{J}_\infty(\lambda)$ is the matrix pencil consisting of all blocks associated with the eigenvalue infinity of the pencil $P(\lambda)$, then $\mathcal{J}_\infty(\mu)$ is the matrix obtained by evaluating $\mathcal{J}_\infty(\lambda)$ at $\lambda = \mu$.

For the remainder of the proof, let us consider the real canonical form $S^\top P(\lambda)S$ of $P(\lambda)$ as in Theorem 4. Although we have been arguing via the complex canonical form so far, note that the blocks associated with real eigenvalues or with the eigenvalue infinity coincide in the real and complex canonical form. Thus, by our considerations above and observing that each block of the form $\mathcal{J}_{k,k,\alpha,\beta}(\lambda)$ as in (8) has inertia index $(k, k, 0)$ for all $\lambda \in \mathbb{R}$, we see that the matrix $P(\lambda_0)$ obtained by evaluating the pencil $P(\lambda)$ at $\lambda = \lambda_0$ has the inertia index

$$\text{ind}(P(\lambda_0)) = (n - 1, n - 1, 0) + \text{ind}(\mathcal{J}_{\lambda_0}(\lambda_0)).$$

Here the sum of inertia index is taken componentwise. By assumption, the algebraic multiplicity of λ_0 as an eigenvalue of $P(\lambda)$ is equal to two and thus its geometric multiplicity is either one or two. If it is one, then

$$J_{\lambda_0}(\lambda) = J_{2,\lambda_0}^s(\lambda) = s \begin{bmatrix} 0 & \lambda_0 - \lambda \\ \lambda_0 - \lambda & 1 \end{bmatrix}$$

for some sign $s \in \{+1, -1\}$ and evaluating this at $\lambda = \lambda_0$ it immediately follows that

$$\text{ind}(\mathcal{J}_{\lambda_0}(\lambda_0)) = (\frac{1+s}{2}, \frac{1-s}{2}, 1) = \begin{cases} (1, 0, 1) & \text{if } s = 1, \\ (0, 1, 1) & \text{if } s = -1 \end{cases}$$

which proves (i).

If on the other hand the geometric multiplicity of λ_0 is two, then we have $J_{\lambda_0}(\lambda) = J_{1,\lambda_0}^{s_1}(\lambda) \oplus J_{1,\lambda_0}^{s_2}(\lambda)$ for some signs $s_1, s_2 \in \{+1, -1\}$. Since we have two blocks of odd size 1 in this case, the sign restriction for λ_0 requires that the signs s_1, s_2 are opposite which proves (ii). \square

Using these lemmas we can prove the following theorem.

Theorem 16 *Let $A, E \in \mathbb{R}^n$ be symmetric and such that the pencil $\lambda E + A$ has precisely one singular block in its canonical form, say $\mathcal{L}_{2k+1}(\lambda)$, $k \geq 0$. Then for generic $u \in \mathbb{R}^n$ the canonical form of $\lambda E + A + \tau uu^\top$ is as follows:*

$$\mathcal{N}_1^s(\lambda) \oplus \bigoplus_{j=1}^{k'} \mathcal{J}_{2,2,\alpha_j,\beta_j}(\lambda) \oplus \bigoplus_{j=1}^{k''} \mathcal{S}_{2,\lambda_j}(\tau, \lambda) \oplus R_f(\lambda) \oplus R_i(\lambda),$$

where

- 1) $s = \text{sgn}\tau$,
- 2) $\alpha_j + i\beta_j$, $j = 1, \dots, k'$ are mutually distinct nonreal eigenvalues with $\alpha_j, \beta_j \in \mathbb{R}$ and $\beta_j > 0$,
- 3) λ_j , $j = 1, \dots, k''$ are mutually distinct real eigenvalues of algebraic multiplicity two and the corresponding blocks have the form

$$\mathcal{S}_{2,\lambda_j}(\tau, \lambda) = \begin{cases} \mathcal{J}_{2,\lambda_j}^1(\lambda) & \text{if } \tau^{-1} < \tau_j, \\ \mathcal{J}_{1,\lambda_j}^1(\lambda) \oplus \mathcal{J}_{1,\lambda_j}^{-1}(\lambda) & \text{if } \tau^{-1} = \tau_j, \\ \mathcal{J}_{2,\lambda_j}^{-1}(\lambda) & \text{if } \tau^{-1} > \tau_j, \end{cases}$$

where $\tau_1, \dots, \tau_{k''} \in \mathbb{R}$ are some real numbers depending on u ,

$$4) \quad 2k = 4k' + 2k'',$$

- 5) $R_f(\lambda) \oplus R_i(\lambda)$ is (the canonical form of) the regular part of $\lambda E + A$.

In the case $k = 0$, the canonical form reduces to $\mathcal{N}_1^s(\lambda) \oplus R_f(\lambda) \oplus R_i(\lambda)$.

Proof. By Theorem 7 we may assume without loss of generality that

$$\lambda E + A = \mathcal{L}_{2k+1}(\lambda).$$

The case $k = 0$ is trivial, so we may assume $k > 0$ and we will continue the congruence transformations from the proof of Proposition 11 starting from (20). Note that all transformations appearing therein were real for $u \in \mathbb{R}^{2k+1}$. We may simply assume first that $u_1 = 1$, which gives that the second diagonal block of the pencil in (20) has the form

$$R(\lambda, \tau) := \begin{bmatrix} 0 & \lambda I - A_1 \\ \lambda I - A_1^\top & B(\tau) \end{bmatrix}$$

$$:= \left[\begin{array}{cccc|cccc} & & & & \lambda - u_2 & 1 & & & \\ & & & & -u_3 & \lambda & \ddots & & \\ & & & & \vdots & & \ddots & & 1 \\ & & & & -u_{k+1} & & & & \lambda \\ \hline \lambda - u_2 & -u_3 & \cdots & -u_{k+1} & f(\tau, u) & -u_{k+3} & \cdots & -u_{2k+1} \\ 1 & \lambda & & & -u_{k+3} & & & \\ & & \ddots & \ddots & \vdots & & & \\ & & & 1 & \lambda & & & \\ & & & & & -u_{2k+1} & & \end{array} \right],$$

where

$$f(\tau, u) = -2u_{k+2} - \frac{1}{\tau}$$

is the only entry depending on τ .

First let us consider the case $k = 1$. We make a generic assumption that $u_3 \neq 0$. In this case

$$\begin{bmatrix} 0 & \lambda I - A_1 \\ \lambda I - A_1^\top & B(\tau) \end{bmatrix} = \begin{bmatrix} 0 & \lambda - u_2 \\ \lambda - u_2 & -2u_3 - \tau^{-1} \end{bmatrix},$$

which, since inserting u_2 in $R(\lambda, \tau)$ gives $R(u_2, \tau) = \text{diag}(0, -2u_3 - \tau^{-1})$, has the canonical form

$$\begin{cases} J_{2, u_2}^1(\lambda) & \text{for } \tau^{-1} < -2u_3, \\ J_{1, u_2}^1(\lambda) \oplus J_{1, u_2}^{-1}(\lambda) & \text{for } \tau^{-1} = -2u_3, \\ J_{2, u_2}^{-1}(\lambda) & \text{for } \tau^{-1} > -2u_3 \end{cases}$$

by Lemma 15. Now let us deal with the case $k \geq 2$. Let λ_0 be an eigenvalue of A_1 . (For the moment, we do not fix this eigenvalue to be real or nonreal, but if it is nonreal, we may assume $\text{Im } \lambda_0 > 0$ by otherwise switching to its conjugate which must also be an eigenvalue of $P(\lambda)$.) Clearly, λ_0 is then also an eigenvalue of A_1^\top . Furthermore, by our initial assumption that q has simple roots only, λ_0 is a simple, nonzero eigenvalue of both A_1 and A_1^\top . Let us further transform the matrix

$$R(\lambda_0, \tau) = \begin{bmatrix} 0 & \lambda_0 I - A_1 \\ \lambda_0 I - A_1^\top & B(\tau) \end{bmatrix}. \quad (23)$$

As λ_0 is an eigenvalue of A_1 , there exists a nonsingular matrix $T_1 \in \mathbb{C}^{k \times k}$ such that

$$(\lambda_0 - A_1)T_1 = \begin{bmatrix} 0 & 1 & & \\ 0 & \lambda_0 & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & & & \lambda_0 \end{bmatrix} =: A_2.$$

Indeed, let

$$T_1 = \begin{bmatrix} 1 & & & \\ x_3 & 1 & & \\ \vdots & & \ddots & \\ x_{k+1} & & & 1 \end{bmatrix},$$

with

$$\begin{aligned} x_{k+1} &= \frac{u_{k+1}}{\lambda_0} \\ x_{j-1} &= \frac{u_{j-1}}{\lambda_0} - \frac{x_j}{\lambda_0}, \quad j = k+1, \dots, 4, \end{aligned}$$

so that the entries $2, \dots, k$ of the first column of $(\lambda_0 - A_1)T_1$ are eliminated. Furthermore,

$$x_3 = \frac{u_3}{\lambda_0} - \frac{u_4}{\lambda_0^2} + \dots + (-1)^k \frac{u_{k+1}}{\lambda_0^{k-1}},$$

and hence, with q being the characteristic polynomial of the matrix A_1 , we have

$$\lambda_0 - u_2 + x_3 = \lambda_0^{-k+1} q(\lambda_0) = 0,$$

which gives a zero in the $(1, 1)$ -entry of $(\lambda_0 I - A_1)T$.

Obviously, there exists a matrix $T_2 \in \mathbb{C}^{k \times k}$ such that

$$T_2 A_2 = \begin{bmatrix} 0 & 1 & & \\ 0 & & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} =: A_3.$$

Note that we can take

$$T_2 = \begin{bmatrix} 1 & & & \\ -\lambda_0 & & \ddots & \\ \vdots & & \ddots & \ddots \\ (-1)^k \lambda_0^k & \dots & -\lambda_0 & 1 \end{bmatrix}.$$

Setting $\mathcal{T} := \text{diag}(T_2^\top, T_1)$, we obtain

$$\begin{aligned} \mathcal{T}^\top \begin{bmatrix} 0 & \lambda_0 I - A_1 \\ \lambda_0 I - A_1^\top & B(\tau) \end{bmatrix} \mathcal{T} &= \begin{bmatrix} 0 & T_2(\lambda_0 I - A_1)T_1 \\ T_1^\top(\lambda_0 I - A_1^\top)T_2^\top & T_1^\top B(\tau)T_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & A_3 \\ A_3^\top & T_1^\top B(\tau)T_1 \end{bmatrix}. \end{aligned}$$

Observe that

$$T_1^\top B(\tau)T_1 = \begin{bmatrix} g(\tau, u) & -u_{k+3} & \cdots & -u_{2k+1} \\ -u_{k+3} & & & \\ \vdots & & & \\ -u_{2k+1} & & & \end{bmatrix},$$

with

$$g(\tau, u) = -2u_{k+2} - \frac{1}{\tau} - 2u_{k+3}x_3 - \cdots - 2u_{2k+1}x_{k+1}. \quad (24)$$

Let

$$\mathcal{S}^\top := \left[\begin{array}{cccc|cccc} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & \ddots & & & & & \\ & & & \ddots & & & & \\ & & & & 1 & & & \\ \hline 0 & \cdots & \cdots & 0 & 1 & & & \\ & u_3 & & \vdots & & \ddots & & \\ & & \ddots & \vdots & & & \ddots & \\ & & & u_{2k+1} & 0 & & & 1 \end{array} \right],$$

then

$$\mathcal{S}^\top \mathcal{T}^\top \begin{bmatrix} 0 & \lambda_0 I - A_1 \\ \lambda_0 I - A_1^\top & B(\tau) \end{bmatrix} \mathcal{T} \mathcal{S} = \begin{bmatrix} & & I_{k-1} \\ & 0 & 0 \\ & 0 & g(\tau, u) \\ I_{k-1} & & \end{bmatrix}, \quad (25)$$

where the indicated zeros in the matrix on the right hand side are scalar entries.

First, consider the case that λ_0 is nonreal. Recall that we work under the generic assumption $u_1 \neq 0$. We add another generic assumption that $u_{k+1} \neq 0$. Observe that x_3, \dots, x_{k+1} depend polynomially on u_1, \dots, u_{n+1} and that x_{k+1} is nonreal. Hence, for generic u the expression $g(1, u)$ is

nonreal which implies that the expression $g(\tau, u)$ is nonzero for all $\tau \in \mathbb{R} \setminus \{0\}$. Thus, for generic u we have

$$\dim \ker R(\lambda_0, \tau) = 1, \quad \tau \in \mathbb{R} \setminus \{0\},$$

i.e., λ_0 is an eigenvalue of $\mathcal{L}(\lambda_0) + \tau uu^*$ of geometric multiplicity one for all $\tau \in \mathbb{R} \setminus \{0\}$. Hence, if we set $\lambda_0 = \alpha + i\beta$, then there is a block $J_{2,2,\alpha,\beta}(\lambda)$ in the real canonical form of the pencil $\mathcal{L}(\lambda_0) + \tau uu^*$ for all $\tau \in \mathbb{R} \setminus \{0\}$.

Next, consider the case that λ_0 is real. Then for generic u we have that

$$\tau_0 := -2u_{k+2} - 2u_{k+3}x_3 - \cdots - 2u_{2k+1}x_{k+1} \neq 0. \quad (26)$$

Indeed, τ_0 depends polynomially on the entries of u and is nonzero as a polynomial, because for $u_{k+2} \neq 0$ and $u_j = 0$, $j = k+3, \dots, 2k+1$, we have $\tau_0 \neq 0$. It is now clear that for $\tau^{-1} \neq \tau_0$ there exists a Jordan chain of length two corresponding to the zero eigenvalue of the matrix $R(\lambda_0, \tau)$ and thus one block of size two $J_{2,\lambda_0}^{s(\tau)}(\lambda)$ corresponding to λ_0 occurs in the canonical form of $R(\lambda, \tau)$. By Lemma 15 applied to the pencil $R(\lambda, \tau)$, in view of (25), we have that the sign $s(\tau)$ is equal to the sign of $g(\tau, u)$ which clearly is

$$s(\tau) = \begin{cases} +1 & \text{if } g(\tau, u) > 0, \quad \text{i.e. } \tau^{-1} < \tau_0 \\ -1 & \text{if } g(\tau, u) < 0, \quad \text{i.e. } \tau^{-1} > \tau_0. \end{cases}$$

Finally, $\dim \ker R(\lambda_0, \tau_0^{-1}) = 2$ and therefore, taking into account Lemma 15 again, there are two blocks $J_{1,\lambda_0}^1(\lambda)$ and $J_{1,\lambda_0}^{-1}(\lambda)$ in the canonical form of $P(\lambda)$. \square

4.3 Complex symmetric case

Finally, we consider a complex symmetric rank-one perturbation of a complex symmetric pencil: $\lambda E + A + \tau uu^\top$, where $u \in \mathbb{C}^n$ is a generic vector and the parameter τ is complex. Similarly to Proposition 11 we obtain the following result.

Proposition 17 *Let $u \in \mathbb{C}^{2k+1}$, with $u_1 \neq 0$. If $k \geq 1$, then for $\tau \in \mathbb{C} \setminus \{0\}$ we have*

$$\det(\mathcal{L}_{2k+1}(\lambda) + \tau uu^\top) = \tau q^2(\lambda),$$

where

$$q(\lambda) = \lambda^k - \frac{u_2}{u_1} \lambda^{k-1} + \cdots + (-1)^k \frac{u_{k+1}}{u_1} \lambda^0.$$

Furthermore, infinity is a simple eigenvalue of $\mathcal{L}_{2k+1}(\lambda) + \tau uu^\top$. Also, if $k = 0$ then

$$\det(\mathcal{L}_{2k+1}(\lambda) + \tau uu^\top) = \tau u^2.$$

Proposition 17 states that each finite eigenvalue of $\mathcal{L}_{2k+1}(\lambda) + \tau uu^\top$ is a double eigenvalue. Now let us investigate the block structure for these double eigenvalues.

Theorem 18 *Let $A, E \in \mathbb{C}^n$ be symmetric and such that the pencil $\lambda E + A$ has precisely one singular block in its canonical form, say $\mathcal{L}_{2k+1}(\lambda)$, $k \geq 0$. Then for generic $u \in \mathbb{R}^n$ the canonical form of $\lambda E + A + \tau uu^\top$ ($\tau \in \mathbb{C}$) is as follows:*

$$\mathcal{N}_1(\lambda) \oplus \bigoplus_{j=1}^k \mathcal{S}_{2,\lambda_j}(\tau, \lambda) \oplus R_f(\lambda) \oplus R_i(\lambda),$$

where

- 1) $\lambda_j, j = 1, \dots, k'$ are mutually distinct complex eigenvalues of algebraic multiplicity two and the corresponding blocks have the form

$$\mathcal{S}_{2,\lambda_j}(\tau, \lambda) = \begin{cases} \mathcal{J}_{2,\lambda_j}(\lambda) & \text{if } \tau^{-1} \neq \tau_j, \\ \mathcal{J}_{1,\lambda_j}(\lambda) \oplus \mathcal{J}_{1,\lambda_j}(\lambda) & \text{if } \tau^{-1} = \tau_j, \end{cases}$$

where τ_1, \dots, τ_k are some complex numbers depending on u ,

- 2) $R_f(\lambda) \oplus R_i(\lambda)$ is (the canonical form of) the regular part of $\lambda E + A$.

In the case $k = 0$, the canonical form reduces to $\mathcal{N}_1(\lambda) \oplus R_f(\lambda) \oplus R_i(\lambda)$.

Proof. We repeat the congruence transformations from the proof of Theorem 16 to get (25) and (24), having in mind that now they are complex congruences. Note that the same argument as in the real case applies to prove that $\tau_0 \neq 0$, where τ_0 is defined as in (26). The remainder of the proof follows the same lines as the proof of Theorem 16 with the simplification that now no sign characteristic is involved. \square

Conclusions

We have analyzed the behavior of the structured Kronecker canonical form of singular structured matrix pencils under generic structured rank-one perturbations. In this case the regular Kronecker structure of the pencil is not affected by the perturbation and the behavior of the canonical form of the newly generated regular blocks can be characterized when the perturbations involve a scalar parameter.

References

- [1] R. Byers, C. He, and V. Mehrmann. Where is the nearest non-regular pencil. *Linear Algebra Appl.*, 285:81–105, 1998.
- [2] S. L. Campbell. *Singular Systems of Differential Equations I*. Pitman, San Francisco, CA, 1980.
- [3] S. L. Campbell. Linearization of DAE’s along trajectories. *Z. Angew. Math. Phys.*, 46:70–84, 1995.
- [4] R. Craig and M. Bampton. Coupling of substructures for dynamic analyses. *AIAA Journal*, 6:1313–1319, 1968.
- [5] F. De Terán and F. Dopico. First order spectral perturbation theory of square singular matrix polynomials. *Linear Algebra Appl.*, 432:892–910, 2010.
- [6] F. De Terán, F. Dopico, and J. Moro. First order spectral perturbation theory of square singular matrix pencils. *Linear Algebra Appl.*, 429:548–576, 2008.
- [7] F. R. Gantmacher. *Theory of Matrices*. Chelsea, New York, 1959.
- [8] N. Gräßner, V. Mehrmann, S. Quraishi, C. Schröder, and U. von Wagner. Numerical methods for parametric model reduction in the simulation of disc brake squeal. Preprint 16/2015, Institut für Mathematik, TU Berlin, 2015.
- [9] N. Jacobson. *Basic Algebra I*. W.H. Freeman and Co., San Francisco, 1974.
- [10] R. Kannan, S. Hendry, N.J. Higham, and F. Tisseur. Detecting the causes of ill-conditioning in structural finite element models. *Comput. Struct.*, 133:79–89, 2014.
- [11] P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations. Analysis and Numerical Solution*. EMS Publishing House, Zürich, Switzerland, 2006.
- [12] P. Lancaster and L. Rodman. Canonical forms for Hermitian matrix pairs under strict equivalence and congruence. *SIAM Review*, 47:407–443, 2005.

- [13] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Möbius transformations of matrix polynomials. *Linear Algebra Appl.*, 470:120–184, 2015.
- [14] C. Mehl. Anti-triangular and anti-m-Hessenberg forms for Hermitian matrices and pencils. *Linear Algebra Appl.*, 317:143–176, 2000.
- [15] C. Mehl, V. Mehrmann, and M. Wojtylak. On the distance to singularity via low rank perturbations. *Operators and Matrices*, 9:733–772, 2015.
- [16] V. Mehrmann, V. Noferini, F. Tisseur, and H. Xu. On the sign characteristics of Hermitian matrix polynomials. Preprint 32/2015, Institut für Mathematik, TU Berlin, 2015.
- [17] V. Mehrmann and C. Schröder. Eigenvalue analysis and model reduction in the treatment of disk brake squeal. *SIAM News*, 49:1, 2016.
- [18] R.C. Thompson. The characteristic polynomial of a principal subpencil of a Hermitian matrix pencil. *Linear Algebra Appl.*, 14:135–177, 1976.
- [19] R.C. Thompson. Pencils of complex and real symmetric and skew matrices. *Linear Algebra Appl.*, 147:323–371, 1991.