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A New Approach to O&D Revenue Management Based on Scenario Trees*

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ABSTRACT. O&D revenue management (RM) – either leg-based or PNR-based – has become a standard in the airline industry. In this paper, we present a new approach to O&D RM which does not make any assumptions on demand distributions or on the correlations of the booking process. Protection levels are determined for all origin destination itineraries, fare classes, points of sale and data collection points (DCPs). This approach to the seat inventory problem is modelled as a multistage stochastic program, where its stages correspond to the DCPs of the booking horizon. The stochastic passenger demand process is approximated by a scenario tree generated from historical data by a recursive scenario reduction procedure. The stochastic program represents a specially structured large scale LP that may be solved by standard LP software (e.g. CPLEX). Preliminary numerical experience is reported.

KEYWORDS: O&D Revenue Management, Seat Inventory Control, Multistage Stochastic Programming, Scenario Tree Generation

INTRODUCTION

Revenue management (RM) refers to strategies to control the sale of (perishable) products or services in order to maximize revenue. It started in the early 70s with the work of Littlewood (1972) and was enforced after the deregulation of U.S. airline industry in 1979. For overviews we refer e.g. to (Weatherford, 1998; McGill and van Ryzin, 1999; Pak and Piersma, 2002; Klein and Petrick, 2003; Talluri

and van Ryzin, 2004).

The EMSRa and EMSRb methods (Belobaba, 1987; Belobaba, 1989) became most popular for single leg problems. They are commonly used under the assumption that demand for each fare class is independent and normally distributed. Extension for different types of distributions or dependencies may be found in (Curry, 1990; Wollmer, 1992; Brumelle and McGill, 1991; Brumelle *et al.*, 1990). In (Glover *et al.*, 1982) the first network formulation of the RM was given. Optimal booking limits were applied to the network problem in (Curry, 1990). Smith and Penn (1988) and Simpson (1989) proposed the bid price concept for network revenue management. An extensive study of bid prices in comparison with other methodologies was done in (Williamson, 1992). Theoretical properties of bid-price controls were provided by (Talluri and van Ryzin, 1999). In (van Ryzin and McGill, 2000) an adaptive scheme was used for updating protection levels based on frequencies of certain fill events and for solving some optimality conditions. The rate of occurrence of the fill events was determined directly from historical booking records. Neither assumptions about the distributions nor uncensoring was requested. General stochastic network models based on Markov decision processes and several types of approximations were developed and discussed in (van Ryzin and Talluri, 2003). Markov decision processes and mathematical programming approaches were combined in (Cooper and Homem-de-Mello, 2003).

In the present paper a scenario tree based stochastic programming approach to the O&D revenue management problem is developed. For this purpose the stochastic demand process is approximated by a scenario tree consisting of a finite number of scenarios. In the following section the stochastic programming model is established in scenario and node formulation. Furthermore, the generation of a booking and cancellation scenario tree from historical data is described. In the final sections, preliminary numerical experience is reported and concluding comments are given.

STOCHASTIC PROGRAMMING MODEL

Modelling

We consider an O&D network consisting of I origin-destination-itineraries, J fare classes, K points of sale, L legs with $M(l)$ compartments in each leg $l = 1, \dots, L$. We let the booking horizon be subdivided into T booking subintervals with data collection

points (dcps) $t = 0, \dots, T$. The booking process is controlled over time by decisions on protection levels $P_{i,j,k,t}$ for each fare class $j \in \{1, \dots, J\}$, itinerary $i \in \{1, \dots, I\}$, point of sale $k \in \{1, \dots, K\}$ and at each dcp $t = 0, \dots, T - 1$. The decisions at t are made for the next booking interval $(t, t + 1]$ based on the previous process of bookings and cancellations up to t and recursively over time. Protection levels are upper bounds for the inventory of booked, uncancelled seats.

We assume that the fares and the compartment capacities are given, i.e., they are deterministic input variables. The booking demand and the cancellation processes are regarded as a multivariate stochastic process $\{\xi_t\}_{t=0}^T$ over time, where ξ_0 represents a known deterministic starting value. The components of the random input vector ξ_t at t are the stochastic booking demands $d_{i,j,k,t}$ and stochastic cancellations $c_{i,j,k,t}$. Hence, ξ_t is a $2IJK$ -dimensional random vector whose components are statistically dependent and, furthermore, the random input vector ξ_t depends on its history $(\xi_0, \xi_1, \dots, \xi_{t-1})$.

To state the stochastic programming (SP) model we assume that S scenarios with probabilities $\pi^s > 0$, $s = 1, \dots, S$, of the booking demand and cancellations process are given. These scenarios may be obtained from stochastic demand models and by relying on expert knowledge, respectively.

Scenario-based SP Model

To set up the SP model we need some further notation. We denote the index set of itineraries containing leg l (i.e., the incidence set) by $\mathcal{I}_l \subset \{1, \dots, I\}$, the number of compartments on leg l by $M(l)$ and the index set of fare classes of compartment m on leg l by $\mathcal{J}_m(l) \subset \{1, \dots, J\}$. Further input data are the fares $f_{i,j,k,t}$ and the capacities $C_{l,m}$ of compartments $m \in \{1, \dots, M(l)\}$ and legs l .

The stochastic input variables are the booking demand $d_{i,j,k,t}^s$ and the cancellations $c_{i,j,k,t}^s$. The bookings $b_{i,j,k,t}^s$ and the inventories of booked seats $B_{i,j,k,t}^s$ represent the stochastic state variables of the model while the protection levels $P_{i,j,k,t}^s$ are the stochastic decisions. Here, the superscript s always refers to scenario s . As objective function we consider the expected total revenue, where total refers to the whole O&D network and booking horizon, all fare classes and points of sale.

Summarizing, our scenario-based stochastic programming model consists in maximizing

$$\sum_{s=1}^S \pi^s \sum_{t=1}^T \sum_{i=1}^I \sum_{j=1}^J \sum_{k=1}^K f_{i,j,k,t} (b_{i,j,k,t}^s - c_{i,j,k,t}^s) \quad (1)$$

subject to all protection levels $P_{i,j,k,t}^s$ satisfying

$$B_{i,j,k,t}^s \leq P_{i,j,k,t-1}^s, \quad (2)$$

where $B_{i,j,k,t}^s$ is the inventory of booked seats, i.e.,

$$\begin{aligned} B_{i,j,k,0}^s &:= \bar{B}_{i,j,k}^0 \\ B_{i,j,k,t}^s &:= B_{i,j,k,t-1}^s + b_{i,j,k,t}^s - c_{i,j,k,t}^s \end{aligned} \quad (3)$$

with $b_{i,j,k,t}^s$ satisfying the demand constraints

$$b_{i,j,k,t}^s \leq d_{i,j,k,t}^s, \quad (4)$$

and the leg capacity limits

$$\sum_{i \in \mathcal{I}_l} \sum_{j \in \mathcal{J}_m(l)} \sum_{k=1}^K P_{i,j,k,T-1}^s \leq C_{l,m}. \quad (5)$$

Finally, the integrality and nonnegativity constraints

$$b_{i,j,k,t}^s, P_{i,j,k,t}^s \in \mathbb{Z} \quad (6)$$

$$b_{i,j,k,t}^s, B_{i,j,k,t}^s, P_{i,j,k,t}^s \geq 0 \quad (7)$$

and the nonanticipativity constraints have to be satisfied, the latter meaning that

decisions at t only depend on the data until t . (8) Here, (1) corresponds to the total expected revenue, (3) describes the update of the inventories of booked seats and the constraint (5) expresses that, for each leg, the corresponding protection levels may not exceed the physical capacities of the compartments on the day of departure. While the protection levels, the number of bookings and the inventory of booked seats have to be nonnegative integers by nature, the constraint (8) expresses how the information flow evolves over time. (8) may be modelled by linear equations in various ways, see (Ruszczynski and Shapiro, 2003, Chapter 3.6) and (Römisch and Schultz, 2001). Altogether, the model (1) – (8) represents a large scale multistage stochastic integer program.

Input Scenario Trees

The nonanticipativity constraint (8) implies that the finitely many scenarios $\{\xi_t^s\}_{t=0}^T$, $s = 1, \dots, S$, can be represented in the form of a scenario tree. The scenario tree is based on a finite set $\mathcal{N} = \{0, 1, \dots, N\}$ of nodes that are arranged at the stages $t = 0, \dots, T$. The root node $n = 0$ is the only node at stage $t = 0$. The number of nodes at stage $t = 1$ corresponds to the number of different realizations of ξ_1 . Each of these nodes is connected with the root node by an arc. In general, each node $n \in \mathcal{N}$, $n \neq 0$, has a unique predecessor node denoted by n_- and a set $\mathcal{N}_+(n)$ of successor nodes. Each node in $\mathcal{N}_+(n)$ is

connected with n by an arc. The set $\{0, \dots, n_-, n\}$ of recursive predecessors of n is denoted by $\text{path}(n)$, which refers to the path from the root to n . $t(n)$ denotes the number of elements in $\text{path}(n)$ minus 1 and, thus, refers to the stage to which n is arranged, i.e., the nodes in $\mathcal{N}_t := \{n \in \mathcal{N} : t = t(n)\}$ correspond to the different realizations of ξ_t . Nodes n belonging to the set \mathcal{N}_T have the property $\mathcal{N}_+(n) \neq \emptyset$ and are called leaves. Hence, a scenario corresponds to a path from the root to some leaf, i.e., to $\text{path}(n)$ for some $n \in \mathcal{N}_T$, and its probability is renamed by π^n . We also say that π^n is the probability of the leaf n . Clearly, we have $\{\pi^n\}_{n \in \mathcal{N}_T} = \{\pi^s\}_{s=1}^S$. The probabilities of nodes $n \notin \mathcal{N}_T$ compute by the recursion $\pi^n := \sum_{n_+ \in \mathcal{N}_+(n)} \pi^{n_+}$. Clearly, we have that $\sum_{n \in \mathcal{N}_t} \pi^n = 1$ and $\xi_t(n) = \{\xi^n\}_{n \in \mathcal{N}_t}$ for each $t = 0, 1, \dots, T$.

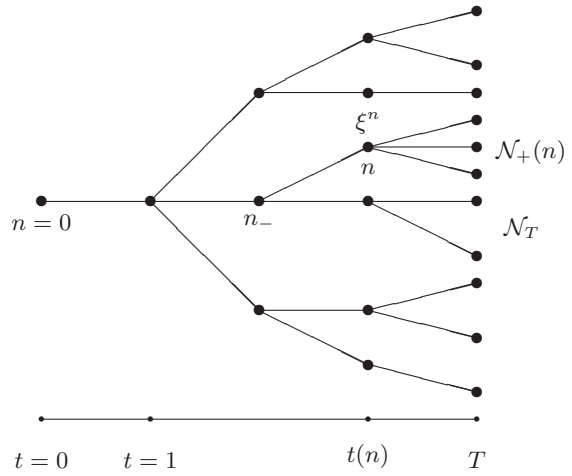


Figure 1: Scenario tree with $T = 4$, $N = 21$ and 11 leaves

The generation of scenario trees that approximate the stochastic input process $\{\xi_t\}_{t=0}^T$ is a challenging task when solving multistage stochastic programs. In (Dupačová *et al.*, 2000) an overview of scenario tree generation techniques is provided. More recent contributions are based on the moment-matching principle (Høyland and Wallace, 2001), the use of distances of probability distributions (Pflug, 2001) and (Gröwe-Kuska *et al.*, 2003), and Quasi-Monte Carlo methods (Pennanen, 2004), respectively.

Next, we briefly describe the scenario tree construction approach of (Gröwe-Kuska *et al.*, 2003). It starts with a finite number of individual scenarios $\{\xi_t^s\}_{t=0}^T$, $s = 1, \dots, S$, with probabilities and common root node, i.e., $\xi_0^1 = \dots = \xi_0^S$. This scenario fan is modified by recursive bundling and deletion of similar scenarios, respectively, leading to a tree structure.

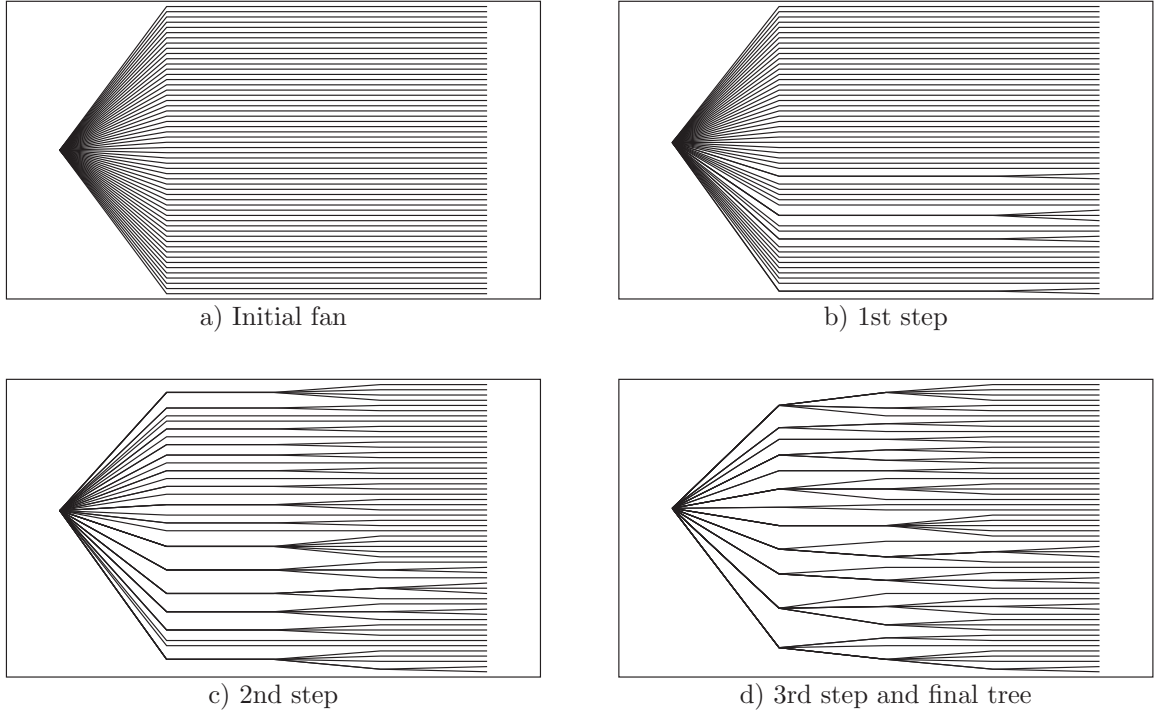


Figure 2: Construction of a scenario tree

Its methodology is based on the scenario reduction techniques developed in (Dupačová *et al.*, 2003) and (Heitsch and Römisch, 2003) and employs these techniques backward in time starting at $t = T$. The bundling and deletion process relies on computing and bounding the distance of the original distribution given by the individual scenarios with their weights and of the distributions of the approximate trees. Figure 2 illustrates the construction procedure starting from a fan of individual scenarios on a time horizon with $T = 4$. After three reduction steps at $t = 3, 2$ and 1 the final result is shown in d). The final scenario tree in d) exhibits possibly different branching structure at all stages, which is detected by the algorithm. The tree is constructed such that the set of its realizations $\{\xi^n\}_{n \in \mathcal{N}_t}$ at t is contained in the initial set $\{\xi_t^s\}_{s=1}^S$. The papers (Gröwe-Kuska *et al.*, 2003) and (Heitsch and Römisch, 2004) contain a detailed description of the algorithm and its theoretical justification. The important advantage of this tree generation method consists in its applicability to (highly) multivariate stochastic processes with finitely many scenarios without imposing further distribution assumptions. Hence, it also applies in the context of O&D revenue management and is used later to generate scenario trees for the passenger demand process (see Section Numerical Results).

SP Model in Node Form

Using the description of scenario trees, our SP model (1) – (8) may alternatively be represented in node formulation. To this end we introduce input, state and decision variables at all nodes using superscript $n = 0, \dots, N$. Making use of a mapping that assigns to each time-scenario pair (t, s) the corresponding node n with $t = t(n)$ and with $\text{path}(n)$ being a part of scenario s , we obtain the booking demands $d_{i,j,k}^n$, cancellations $c_{i,j,k}^n$, bookings $b_{i,j,k}^n$, booking inventories $B_{i,j,k}^n$ and protection levels $P_{i,j,k}^n$ at all nodes $n \in \mathcal{N}$ and all triples (i, j, k) . Then the node formulation of our SP model consists in maximizing

$$\sum_{n=1}^N \pi^n \sum_{i=1}^I \sum_{j=1}^J \sum_{k=1}^K f_{i,j,k,t(n)} (b_{i,j,k}^n - c_{i,j,k}^n) \quad (9)$$

subject to all protection levels $P_{i,j,k}^n$ satisfying

$$B_{i,j,k}^n \leq P_{i,j,k}^{n-}, \quad (10)$$

where $B_{i,j,k}^n$ is the inventory of booked seats, i.e.,

$$\begin{aligned} B_{i,j,k}^0 &:= \bar{B}_{i,j,k}^0 \\ B_{i,j,k}^n &:= B_{i,j,k}^{n-} + b_{i,j,k}^n - c_{i,j,k}^n \end{aligned} \quad (11)$$

with $b_{i,j,k}^n$ satisfying the demand constraints

$$b_{i,j,k}^n \leq d_{i,j,k}^n, \quad (12)$$

and for all $n \in \mathcal{N}_{T-1}$ the leg capacity limits

$$\sum_{i \in \mathcal{I}_l} \sum_{j \in \mathcal{J}_m(l)} \sum_{k=1}^K P_{i,j,k}^n \leq C_{l,m}. \quad (13)$$

Finally, we have the nonnegative integer constraints

$$b_{i,j,k}^n, P_{i,j,k}^n \in \mathbb{Z} \quad (14)$$

$$b_{i,j,k}^n, B_{i,j,k}^n, P_{i,j,k}^n \geq 0, \quad (15)$$

while the nonanticipativity constraints are satisfied by construction. Altogether, the model (9) – (15) represents a large scale structured integer program which is of smaller dimension compared to its scenario formulation. More precisely, it contains $IJK + 3IJK(N-1-S) + 2IJKS$ variables and $2IJK(N-1) + (\sum_{l=1}^L M(l))S$ constraints.

Since all variables are nonnegative and the bookings are bounded from above, the objective function is also bounded from above. Hence, the LP relaxation of the integer program, i.e., when the constraint (14) is ignored, is solvable. For its solution any standard LP solver may be used.

The protection levels $(P_{i,j,k}^n)_{n \in \mathcal{N}}$ as solutions of the multi-stage stochastic programs form a (multi-variate) stochastic process over time having the same tree structure as the input data.

- The (deterministic) protection levels of the first dcp may be taken directly for the reservation system CRS.
- At the next dcp's the information on the probability distribution (means, quantiles etc.) of the relevant stochastic protection levels may be used to determine an estimate of the protection levels.
- If the latter estimates do no longer reflect reality, say, at time $t_0 > 0$, the stochastic optimization model is restarted with a new input scenario tree having its root node at t_0 etc.

NUMERICAL RESULTS

In our preliminary numerical tests, the SP model was set up and solved for a single leg flight (namely, LH400, A340-300, Tue as day of departure). Table 1 shows the dimensions of the corresponding O&D RM problem. The passenger demand was modelled start-

L	I	J	K	T	$M(1)$
1	1	14	1	18	3

Table 1: Dimensions

ing from historical data of the corresponding flight

as follows. First the data was adjusted subject to a suitable demand model. Next a data and cluster analysis was processed, leading to four clusters {Mon, Tue, Wed, Thu}, {Fri}, {Sat}, {Sun} (Gröwe-Kuska and Römisch, 2001). Then 300 scenarios were drawn from the first cluster by resampling techniques. Starting from these scenarios a scenario tree consisting of 150 scenarios was generated by using the scenario reduction and tree construction algorithms described in (Gröwe-Kuska *et al.*, 2003; Heitsch and Römisch, 2004) (see Section Input Scenario Trees). The dimensions of the scenario tree and, thus, of the SP model (9) – (15) are shown in Table 2. The tree is shown in Figure 3. It contains branches at all dcps

S	N	#Variables	#Constraints
150	1190	47852	33742

Table 2: SP model dimensions

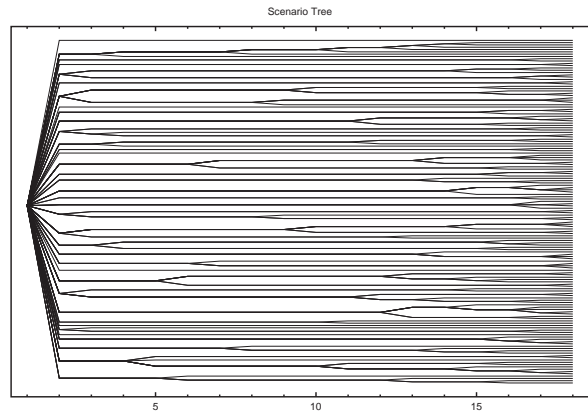


Figure 3: Scenario tree

and exhibits branches of varying degree, starting with many branches at the root node.

Ignoring the integrality constraints (14) the SP model was solved by CPLEX 8.1. An optimal solution was found by CPLEX 8.1 in less than 2 seconds on a Linux-PC equipped with a 2 GHz Intel Celeron processor. Figure 4 shows the optimal protection levels at the first stage, i.e., for the interval $[0, 1)$, and the corresponding fares. Figure 5 provides the trees of optimal protection levels over the whole booking horizon and the corresponding demand scenario trees for selected fare classes. Each picture also contains the mean value and the 5% and 95% quantile curves. The similar structure of both trees for each class indicates that the capacities of the compartments are not exploited by most of the scenarios. An interesting observation is that the optimal protection levels satisfy the integrality constraints although they were relaxed.

All in all, our results seem to be reasonable and raise the expectation that moderately sized O&D network problems may be solved in acceptable running times.

CONCLUSIONS

A stochastic programming approach to O&D revenue management is proposed. It is based on modelling scenario trees for the passenger demand and does not require any assumption on the underlying demand distributions or on the correlations of the booking process. The RM problem is modelled by a multistage stochastic program in node form and solved by standard LP software. Our numerical experience for a single leg model indicates that the approach bears potential for solving O&D network models in reasonable time. Our future work will be directed to the following issues:

- Analysis of O&D data, the generation of O&D demand scenarios and of demand scenario trees.
- Study of structural properties of the stochastic RM model and of the adaptability of decomposition approaches.
- Numerical tests on entire networks.
- Comparison with other approaches.
- Completion of the model (noshows, denied boarding cost).
- Study of modelling specific demand patterns (seasonal demand, special events).

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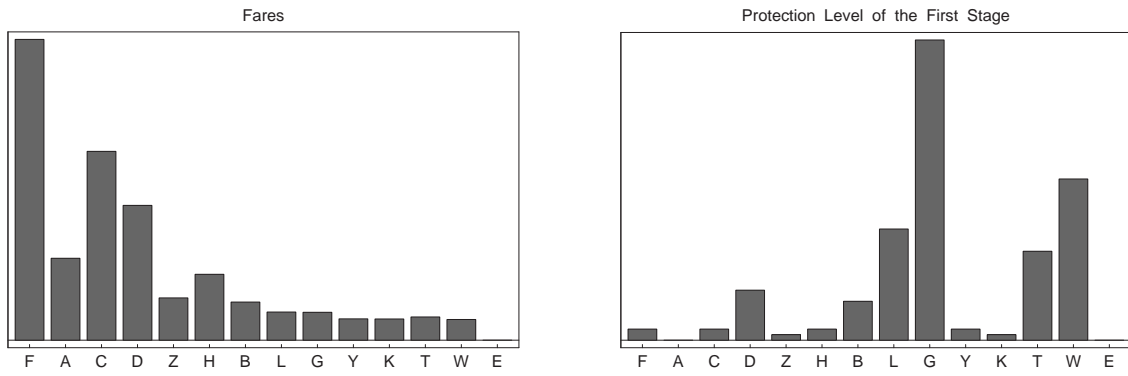


Figure 4: Fares and optimal first stage protection levels

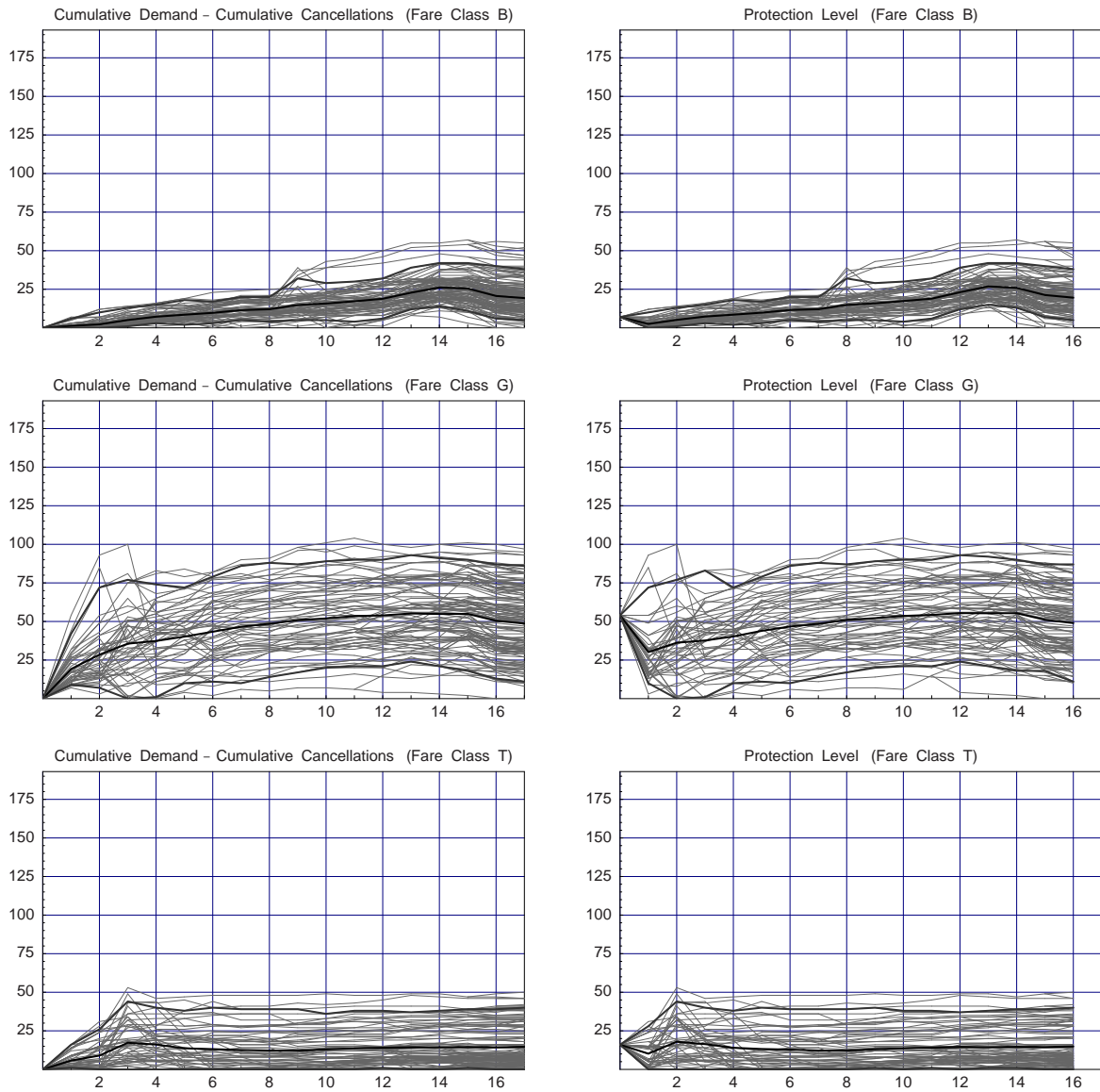


Figure 5: Cumulative passenger demand and protection level for selected fare classes.

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