Risk-Averse PDE-Constrained Optimization Using the Conditional Value-At-Risk

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D. P. KOURI* AND T. M. SUROWIEC†

Abstract. Uncertainty is inevitable when solving science and engineering application problems. In the face of uncertainty, it is essential to determine robust and risk-averse solutions. In this work, we consider a class of PDE-constrained optimization problems in which the PDE coefficients and inputs may be uncertain. We introduce two approximations for minimizing the conditional value-at-risk for such PDE-constrained optimization problems. These approximations are based on the primal and dual formulations of the conditional value-at-risk. For the primal problem, we introduce a smooth approximation of the conditional value-at-risk in order to utilize derivative-based optimization algorithms and to take advantage of the convergence properties of quadrature-based discretizations. For this smoothed conditional value-at-risk, we prove differentiability as well as consistency of our approximation. For the dual problem, we regularize the inner maximization problem, rigorously derive optimality conditions, and demonstrate the consistency of our approximation. Furthermore, we propose a fixed-point iteration that takes advantage of the structure of the regularized optimality conditions and provides a means of calculating worst-case probability distributions based on the given probability level. We conclude with numerical results.

Key words. PDE Optimization, Conditional Value-At-Risk, Uncertainty Quantification

AMS subject classifications. 49M15, 65K05, 65N35, 90C15

1. Introduction.

In this work, we develop approximations and theory for the conditional value-at-risk (CVaR) applied to optimization problems constrained by partial differential equations (PDEs). PDE-constrained optimization problems arise in numerous engineering applications. Often in these applications, PDE coefficients and inputs are unknown and estimated from empirical data, adding uncertainty to the optimization problem. In the face of uncertainty, it is essential to manage the risk associated with an optimal design or control. There are a multitude of constructs to measure risk: for example, one can control large deviations or rare-events in the objective function to be minimized. Such optimization problems are considered in [7, 8, 23, 22, 39].

In this work, we focus on sample-based discretizations of the PDE with uncertain coefficients. Sample-based discretizations allow us to more easily reuse existing deterministic PDE solvers. Discretizations for PDEs with uncertain inputs are typically classified as either projection-based or sample-based. Projection-based methods include polynomial chaos and stochastic Galerkin methods [4, 5, 19, 45], as well as the more recent best N-term approximation and sparse-tensor discretization [20, 35]; sample-based methods include (quasi-)Monte Carlo and stochastic collocation or deterministic quadrature methods [3, 26, 25, 44].

Optimizing the conditional value-at-risk is common practice in financial mathematics to determine risk-averse investment strategies, see e.g. [13, 32, 31, 37]. At

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a specified probability level, $0 < \beta < 1$, the value-at-risk (VaR) is defined as the β-quantile of a random variable. Building on the concept of VaR, the conditional value-at-risk is the expected value of the β-tail distribution, see, e.g. [32, Def. 3]. Thus, CVaR emphasizes rare and low probability events when $\beta \gg 0$. In engineering terms, tail-probability events often correspond to failure of the engineered system. As such, it is important to conservatively manage these risks, e.g., by minimizing CVaR.

CVaR is a proper and coherent risk measure in the sense of [1]. Namely, CVaR is convex, monotonic, translation equivariant, and positive homogeneous. Coherent risk measures preserve desirable properties of the original objective function such as convexity. Moreover, proper, coherent risk-averse optimization problems have equivalent dual, minimax formulations [33, Th. 2]. In this paper, we investigate both the primal and dual formulations of the CVaR problem for PDE-constrained optimization.

For the primal formulation, we approximate the expected value in the CVaR objective function using a sample-based discretization [21]. Such sample-based discretizations include (quasi-)Monte Carlo and deterministic quadrature. Whereas Monte Carlo methods exhibit a constant convergence rate of $Q^{-1/2}$ where Q denotes the number of samples, deterministic quadrature exploits any regularity of the integrand to accelerate convergence. Once discretized, our goal is to employ standard gradient-based optimization algorithms to solve the CVaR problem. To ensure the existence of derivatives and the regularity of the integrand, we smooth the conditional value-at-risk. We analyze a general smoothing technique for CVaR based on the previous work in [9, 29] and prove that the resulting smoothed CVaR is monotonic, convex, and translation equivariant. In addition, we prove that the smoothed CVaR function is differentiable and determine rigorous upper bounds on the error in the optimal control variables associated with smoothing. In [21], the first author proves explicit quadrature error bounds for certain risk-averse optimal controls. Combining the smoothing error bound with similar quadrature error bounds completely characterizes the errors in the optimal controls.

For the dual approach, we treat the minimax reformulation as a nonsmooth optimization problem. We derive differential sensitivity results of the nonsmooth objective functional and, using these results, we prove rigorous optimality conditions for the minimax problem. After suitable regularization, the specific structure of the problem allows the solution of the inner maximization to be reduced to a single nonsmooth equation in one-dimension. Finally, we derive a primal-dual optimality system. This system has two purposes. First, it can be used to define an alternating fixed-point scheme to solve the optimal control problem. In the event that the fixed-point mapping is not contractive, we can still use the additional information to easily calculate a worst-case probability measure associated with a given control value.

The remainder of the paper is outlined as follows. First, we present the problem formulation for risk-averse PDE-constrained optimization. The problem we focus on is the optimal control of a nonlinear PDE with a tracking-type objective function. Next, we introduce the conditional value-at-risk and review some basic properties of CVaR. Following this review, we present and analyze the primal approach to solve the CVaR optimization problem in which we discuss the well-posedness and consistency of our approximations. Subsequently, we present the dual problem formulation and discuss theoretical and numerical aspects associated with this approach. Finally, we present numerics confirming our theoretical results and provide some concluding remarks.

2. Problem Formulation. Let $\Omega \subset \mathbb{R}^d$ ($d \in \{1, 2, 3\}$), an open and bounded subset with Lipschitz boundary $\partial\Omega$, denote the physical domain and let $(\Pi, \mathcal{F}, P)$

be a complete probability space. Here, Π is the set of outcomes, $\mathcal{F} \subseteq 2^{\Pi}$ is a σ-algebra of events, and $P : \mathcal{F} \to [0,1]$ is a probability measure. Although many of the results in this paper apply to more general PDEs, we motivate this work by considering optimization problems governed by the following nonlinear elliptic PDE with uncertain coefficients,

$$-\nabla_x \cdot (\widehat{\epsilon}(\omega,x)\nabla_x u(\omega,x)) + \widehat{N}(u(\omega,x),\omega) = \widehat{f}(\omega,x), \qquad x \in \Omega \tag{2.1a}$$

$$u(\omega,x) = 0, \qquad x \in \partial\Omega. \tag{2.1b}$$

Here, we assume $\widehat{\epsilon}$, $\widehat{N}$, and $\widehat{f}$ are sufficiently regular. The state variables are denoted by u and are random fields. The control variables are denoted by z and are deterministic. The controls will enter (2.1) as a combination of distributed controls (appearing on the right-hand side of (2.1a)) and Neumann controls (appearing as a boundary flux, augmenting (2.1b)). Equation 2.1 is required to hold P-almost everywhere in Π. For the optimization formulation, we consider (2.1) in the weak form. To formulate the weak form of (2.1) we require $\widehat{\epsilon} \in L_P^\infty(\Pi; L^\infty(\Omega))$, $\widehat{f} \in L_P^q(\Pi; H^{-1}(\Omega))$ for some $q \in (1,2]$, and the nonlinear term satisfies $\widehat{N} : H_0^1(\Omega) \times \Pi \to H^{-1}(\Omega)$ such that

$$\widehat{N}(u(\cdot,x),\cdot) \in L_P^q(\Pi; H^{-1}(\Omega)), \quad \forall\, u \in L_P^p(\Pi; H_0^1(\Omega)),$$

where $\frac{1}{p} + \frac{1}{q} = 1$, i.e., $p \in [2, +\infty)$. The weak form is: Find $u \in \widehat{\mathcal{U}} = L_P^p(\Pi; \mathcal{V})$ with $\mathcal{V} = H_0^1(\Omega)$ such that

$$\begin{aligned}\int_\Pi \int_\Omega \widehat{\epsilon}(\omega,x)\nabla_x u(\omega,x) \cdot \nabla_x v(\omega,x)\, \mathrm{d}x\mathrm{d}P(\omega) + \int_\Pi \left\langle \widehat{N}(u(\omega),\omega), v(\omega)\right\rangle_{\mathcal{V}^*,\mathcal{V}} \mathrm{d}P(\omega) \\ = \int_\Pi \left\langle \widehat{f}(\omega), v(\omega)\right\rangle_{\mathcal{V}^*,\mathcal{V}} \mathrm{d}P(\omega), \quad \forall\, v \in \widehat{\mathcal{U}}.\end{aligned} \tag{2.2}$$

To simplify notation, we define the stochastic weak-form PDE operator as $\widehat{a} : \mathcal{V} \times \mathcal{V} \times \Pi \to \mathbb{R}$ such that

$$\widehat{a}(u,v,\omega) = \int_\Omega \widehat{\epsilon}(\omega,x)\nabla_x u(x) \cdot \nabla_x v(x)\, \mathrm{d}x + \left\langle \widehat{N}(u,\omega), v\right\rangle_{\mathcal{V}^*,\mathcal{V}} \tag{2.3}$$

and we define the stochastic forcing term on the right-hand side of (2.2) as $\widehat{b} : \mathcal{V} \times \Pi \to \mathbb{R}$ such that

$$\widehat{b}(v,\omega) = \left\langle \widehat{f}(\omega), v\right\rangle_{\mathcal{V}^*,\mathcal{V}}. \tag{2.4}$$

The weak-form (2.2) is compactly written as: Find $u \in \widehat{\mathcal{U}}$ such that

$$\int_\Pi \left\{ \widehat{a}(u(\omega),v(\omega),\omega) - \widehat{b}(v(\omega),\omega) \right\} \mathrm{d}P(\omega) = 0 \quad \forall\, v \in \widehat{\mathcal{U}}.$$

As is common in the PDEs with uncertain coefficients literature [4, 3], we employ the finite-dimensional noise assumption to facilitate the numerical solution of (2.2).

ASSUMPTION 2.1. *There exists an M-dimensional random vector $\xi : \Pi \to \Xi = \prod_{k=1}^M \Xi_k$ with $\Xi_k \subseteq \mathbb{R}$ and joint density $\rho : \Xi \to [0,+\infty) \cup \{+\infty\}$ such that*

$$\widehat{\epsilon}(\omega,\cdot) \equiv \epsilon(\xi(\omega),\cdot), \quad \widehat{N}(\cdot,\omega) \equiv N(\cdot,\xi(\omega)), \quad \textit{and} \quad \widehat{f}(\omega,\cdot) \equiv f(\xi(\omega),\cdot)$$

for some $\epsilon \in L^\infty_\rho(\Xi; L^\infty(\Omega))$, $f \in L^q_\rho(\Xi; \mathcal{V}^*)$, *and* $N : \mathcal{V} \times \Xi \to \mathcal{V}^*$ *satisfying*

$$N(u(\cdot), \cdot) \in L^q_\rho(\Xi; \mathcal{V}^*), \quad \forall\, u \in \mathcal{U} = L^p_\rho(\Xi; \mathcal{V}).$$

One can achieve Assumption 2.1, for example, using a Karhunen-Loève expansion [18, 24, 3]. Furthermore, Assumption 2.1 permits the following change of variables for (2.2)

$$\begin{aligned}\int_\Xi \rho(\xi) \int_\Omega \epsilon(\xi, x) \nabla_x u(\xi, x) \cdot \nabla_x v(\xi, x) \,\mathrm{d}x\mathrm{d}\xi + \int_\Xi \rho(\xi) \left\langle N(u(\xi), \xi), v(\xi) \right\rangle_{\mathcal{V}^*, \mathcal{V}} \mathrm{d}\xi \\ = \int_\Xi \rho(\xi) \left\langle f(\xi), v(\xi) \right\rangle_{\mathcal{V}^*, \mathcal{V}} \mathrm{d}\xi \quad \forall\, v \in \mathcal{U}. \end{aligned} \tag{2.5}$$

As in (2.3) and (2.4), we have the parametrized state and forcing operators $\widehat{a}(\cdot, \cdot, \omega) \equiv a(\cdot, \cdot, \xi(\omega))$ and $\widehat{b}(\cdot, \omega) \equiv b(\cdot, \xi(\omega))$ where $a : \mathcal{V} \times \mathcal{V} \times \Xi \to \mathbb{R}$ and $b : \mathcal{V} \times \Xi \to \mathbb{R}$ are defined analogously to (2.3) and (2.4), respectively.

We consider the optimal control problem

$$\underset{z \in L^2(\Omega_c)}{\text{minimize}} \quad \widehat{J}(z) = \frac{1}{2} \sigma \left[\int_{\Omega_o} (u(\cdot, x; z) - \bar{u}(x))^2 \,\mathrm{d}x \right] + \frac{\alpha}{2} \int_{\Omega_c} z(x)^2 \,\mathrm{d}x \tag{2.6}$$

where $\Omega_o \subseteq \Omega$, $\Omega_c \subseteq \Omega \cup \partial\Omega$, $\bar{u} \in L^2(\Omega_o)$, $\alpha > 0$, and $u(\cdot, \cdot; z) = u \in \mathcal{U}$ solves

$$\int_\Xi \rho(\xi) \left\{ a(u(\xi), v(\xi), \xi) - b(v(\xi), \xi) - \int_{\Omega_c} z(x) v(\xi, x) \,\mathrm{d}x \right\} \mathrm{d}\xi = 0, \quad \forall\, v \in \mathcal{U}. \tag{2.7}$$

We consider controls, z, that are possibly a combination of distributed and Neumann controls. To simplify notation, we denote the control space as $\mathcal{Z} = L^2(\Omega_c)$. The operator $\sigma : L^{p/2}_\rho(\Xi) \to \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ is a risk measure, see e.g., [33, 37]. Risk measures are motivated by the need in science and industry to control large deviations, tail probability, quantiles, and rare events. In this paper, we focus our attention on the conditional value-at-risk (CVaR) [31, 42].

Throughout, we use the following notation to represent the weak form (2.7)

$$e(u, z, \xi) = 0 \in \mathcal{V}^* \quad \text{a.e. in } \Xi,$$

for $u \in \mathcal{V}$ and $z \in \mathcal{Z}$ where $e : \mathcal{V} \times \mathcal{Z} \times \Xi \to \mathcal{V}^*$ is defined as

$$\left\langle e(u, z, \xi), v \right\rangle_{\mathcal{V}^*, \mathcal{V}} = a(u, v, \xi) - b(v, \xi) - \int_{\Omega_c} z(x) v(x) \,\mathrm{d}x.$$

Note, the assumptions that follow are independent of (2.7) and only refer to the abstract nonlinear equation $e(u, z, \xi) = 0$. Thus, the resulting theory applies to more general PDEs than (2.7). The following assumption ensures the existence of solutions to (2.7) and that (2.6) is well-defined, cf. [23].

Assumption 2.2. *For each* $z \in \mathcal{Z}$, *there exists a unique* $u = u(z) \in \mathcal{U} = L^p_\rho(\Xi; \mathcal{V})$ *which solves* (2.7) *and satisfies*

$$\|u(\xi; z)\|_{\mathcal{V}} \leq \kappa(1 + \|z\|_{\mathcal{Z}}) \quad \textit{a.e. in } \Xi$$

for some $\kappa > 0$ *independent of* $z \in \mathcal{Z}$ *and* $\xi \in \Xi$. *Moreover, if* $\{z_n\} \subset \mathcal{Z}$ *such that* $z_n \rightharpoonup z \in \mathcal{Z}$, *then* $u(\xi; z_n) \rightharpoonup u(\xi; z)$ *in* $\mathcal{V}$ *for almost every* $\xi \in \Xi$.

One class of solution techniques discussed in this paper are gradient-based optimization algorithms. In order to ensure the gradient of $\widehat{J}(z)$ exists, we require the following assumption.

ASSUMPTION 2.3. *The nonlinear PDE residual operator is continuously Fréchet differentiable with respect to $u \in \mathcal{V}$ and $z \in \mathcal{Z}$ with derivatives*

$$e_u(u,z,\xi) \in \mathcal{L}(\mathcal{V},\mathcal{V}^*) \quad \text{and} \quad e_z(u,z,\xi) \in \mathcal{L}(\mathcal{Z},\mathcal{V}^*) \quad \text{a.e. in } \Xi,$$

respectively. Moreover, the PDE solution is continuously Fréchet differentiable as a function from $\mathcal{Z}$ into $\mathcal{U}$ with derivative

$$u'(z) \in \mathcal{L}(\mathcal{Z},\mathcal{U}),$$

which uniquely solves the sensitivity equation $s = u'(z)h \in \mathcal{U}$ for $h \in \mathcal{Z}$ and

$$\int_\Xi \rho(\xi)\, \langle (e_u(u(\xi),z,\xi)s(\xi) + e_z(u(\xi),z,\xi)h), v(\xi)\rangle_{\mathcal{V}^*,\mathcal{V}}\ \mathrm{d}\xi = 0,$$

for all $v \in \mathcal{U}$. Finally, there exists a unique solution $\lambda \in \mathcal{U}$ of the adjoint equation

$$\int_\Xi \rho(\xi) \left(\langle e_u(u(\xi),z,\xi)^*\lambda(\xi), v\rangle_{\mathcal{V}^*,\mathcal{V}} + \int_{\Omega_o} (u(\xi,x) - \bar{u}(x))v(\xi,x)\ \mathrm{d}x \right) \mathrm{d}\xi = 0,$$

for all $v \in \mathcal{U}$.

If the risk measure $\sigma : L_\rho^{p/2}(\Xi) \to \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ is Hadamard differentiable, then $\widehat{J}(z)$ is also Hadamard differentiable and Fubini's theorem [12, Th. 2.39] implies the following explicit gradient,

$$\nabla \widehat{J}(z) = \alpha z + \int_\Xi \rho(\xi) \nabla\sigma \left[\int_{\Omega_o} (u(\cdot,x;z) - \bar{u}(x))^2\ \mathrm{d}x \right](\xi)\ e_z(u(\xi),z,\xi)^*\lambda(\xi)\ \mathrm{d}\xi,$$

where $\lambda \in \mathcal{U}$ solves the adjoint equation and $\nabla\sigma[X] \in L_\rho^{p/(p-2)}(\Xi)$ for $X \in L_\rho^{p/2}(\Xi)$ denotes the Hadamard derivative of the risk measure.

3. The Conditional Value-At-Risk. The conditional value-at-risk controls uncertainty by minimizing the expected value of the random variable tracking term over its quantiles. We denote the tracking term as

$$\xi \mapsto T(\xi;z) = \int_{\Omega_o} (u(\xi,x;z) - \bar{u}(x))^2\ \mathrm{d}x \in L_\rho^{p/2}(\Xi) \subseteq L_\rho^1(\Xi).$$

To define the conditional value-at-risk, we first define the value-at-risk (VaR). For some $0 < \beta < 1$, the β-VaR corresponds to the β-quantile of a random variable X,

$$\mathrm{VaR}_\beta[X] = \inf\ \{\ t \in \mathbb{R}\ :\ \Pr[X \le t] \ge \beta\ \}.$$

Here, $\Pr[X \le t]$ denotes the probability that the random variable X is less than or equal to t. That is,

$$\Pr[X \le t] = \int_{\{\ \xi \in \Xi\ :\ X(\xi) \le t\ \}} \rho(\xi)\ \mathrm{d}\xi.$$

In [31, 32], Rockafellar and Uryasev show that the conditional value-at-risk is

$$\mathrm{CVaR}_\beta[X] = \inf_{t \in \mathbb{R}}\ F_\beta(t,X),$$

where $F_\beta : \mathbb{R} \times L^1_\rho(\Xi) \to \mathbb{R}$ is given by

$$F_\beta(t, X) = t + \frac{1}{1-\beta}\mathbb{E}\left[(X-t)^+\right] \quad \text{and} \quad (x)^+ = \max\{0, x\}.$$

Here, we denote the expected value of $X \in L^1_\rho(\Xi)$ as $\mathbb{E}[X] = \int_\Xi \rho(\xi)X(\xi)\,\mathrm{d}\xi$. Using these definitions, our risk-averse optimal control problem (2.6) is

$$\underset{z\in\mathcal{Z}}{\text{minimize}} \quad \widehat{J}(z) = \frac{1}{2}\text{CVaR}_\beta\left[\int_{\Omega_o}(u(\cdot,x;z)-\bar{u}(x))^2\,\mathrm{d}x\right] + \frac{\alpha}{2}\int_{\Omega_c} z(x)^2\,\mathrm{d}x. \tag{3.1}$$

Therefore, Theorem 14 in [32] ensures that minimizing the β-CVaR in (3.1) is equivalent to solving the augmented optimization problem

$$\begin{aligned}\underset{t\in\mathbb{R},\, z\in\mathcal{Z}}{\text{minimize}} \quad J(t,z) = \frac{1}{2}\left(t + \frac{1}{1-\beta}\int_\Xi \rho(\xi)\left(\int_{\Omega_o}(u(\cdot,x;z)-\bar{u}(x))^2\,\mathrm{d}x - t\right)^+\mathrm{d}\xi\right)\\ + \frac{\alpha}{2}\int_{\Omega_c} z(x)^2\,\mathrm{d}x.\end{aligned} \tag{3.2}$$

The conditional value-at-risk is a *coherent* risk measure in the sense of [1] and has domain $L^1_\rho(\Xi)$. That is, $\sigma = \text{CVaR}_\beta$ satisfies the following four axioms: For Y, $Z \in L^1_\rho(\Xi)$ and $t \in \mathbb{R}$

1. *Monotonicity:* If $Y \le Z$ a.e., then $\sigma[Y] \le \sigma[Z]$
2. *Translation Equivariant:* $\sigma[Y + t] = \sigma[Y] + t$
3. *Convexity:* $\sigma[tY + (1-t)Z] \le t\sigma[Y] + (1-t)\sigma[Z]$ for $t \in (0,1)$
4. *Positive Homogeneity:* $\sigma[tY] = t\sigma[Y]$ for $t \ge 0$.

A risk measure satisfying Axioms 1 and 2 is called a *monetary* risk measure while a monetary risk measure also satisfying Axiom 3 is called a *convex* risk measure [13]. Note that if the random variable tracking term is convex with respect to z, then $\widehat{J}(z)$ is also convex with respect to z.

In this work, we develop two approaches for solving the CVaR optimal control problem (3.2). The first approach is to approximate the expected value in (3.2) using a sample-based discretization and solve the resulting PDE-constrained optimization problem [21, 23, 22]. The second approach uses the Fenchel-Moreau duality theory for convex risk functions [33, 37] to transform (3.2) into the minimax problem

$$\min_{z\in\mathcal{Z}} \widehat{J}(z) = \sup_{\vartheta\in\mathcal{A}} \quad \frac{1}{2}\int_\Xi \vartheta(\xi)\rho(\xi)\int_{\Omega_o}(u(\xi,x;z)-\bar{u}(x))^2\,\mathrm{d}x\mathrm{d}\xi + \frac{\alpha}{2}\int_{\Omega_c} z(x)^2\,\mathrm{d}x, \tag{3.3}$$

where the feasible set of density functions $\mathcal{A}$ is defined as

$$\mathcal{A} = \left\{\, \vartheta \in (L^1_\rho(\Xi))^* \;:\; 0 \le \vartheta(\xi) \le \frac{1}{1-\beta} \text{ a.e. in } \Xi,\ \int_\Xi \vartheta(\xi)\rho(\xi)\,\mathrm{d}\xi = 1 \right\}. \tag{3.4}$$

4. Primal Approach. The primal approach to minimizing the conditional value-at-risk is to solve (3.2) directly. In this section, we first prove existence of solutions to (3.2) and present the standard smooth, constrained reformulation of (3.2). We then discuss a sample-based approach to discretizing (3.2) in sample space.

THEOREM 4.1. *Let Assumption 2.2 hold. Then there exists a solution to* (3.2).

Proof. We first prove coercivity of $J(t,z)$. Let $X \in L^1_\rho(\Xi)$ be such that $X \ge 0$ a.e. in Ξ and notice that the monotonicity of $(\cdot)^+$ implies

$$F^\beta(t,X) \ge F^\beta(t,0) = t + (1-\beta)^{-1}\max\{0,-t\} = \begin{cases} \left(\frac{\beta}{1-\beta}\right)|t| & \text{for } t \le 0 \\ t & \text{for } t > 0. \end{cases}$$

Hence, $F^\beta(t,0)$ is convex, continuous, and nonnegative. Since $T(\xi;z) = \int_{\Omega_o}(u(\xi,x;z)-\bar{u}(x))^2\,\mathrm{d}x \geq 0$ a.e. in Ξ, we have

$$F^\beta\left(t, \int_{\Omega_o}(u(\xi,x;z)-\bar{u}(x))^2\,\mathrm{d}x\right) \geq F^\beta(t,0).$$

Thus, $J(t,z) \geq F^\beta(t,0) + \frac{\alpha}{2}\int_{\Omega_c} z(x)^2\,\mathrm{d}x$. Therefore, $J(t,z)$ is coercive, i.e.,

$$\left(|t_n|^2 + \|z_n\|_{\mathcal{Z}}^2\right)^{\frac{1}{2}} \to +\infty \quad \Longrightarrow \quad J(t_n,z_n) \to +\infty.$$

We now prove weak lower semicontinuity of $J(t,z)$. Note that since $\partial\Omega$ is Lipschitz and by Assumption 2.2, $\mathcal{V} \hookrightarrow L^2(\Omega)$ is compact and the tracking term, $T(\xi,z) = \int_{\Omega_o}(u(\xi,x;z)-\bar{u}(x))^2\,\mathrm{d}x$, is weakly continuous with respect to $z \in \mathcal{Z}$ a.e. in Ξ. Moreover, the penalty term, $R(z) = \int_{\Omega_c} z(x)^2\,\mathrm{d}x$ is also weakly lower semicontinuous. Now, let $\{(t_n,z_n)\} \subset \mathbb{R}\times\mathcal{Z}$ be such that $(t_n,z_n) \rightharpoonup (t,z) \in \mathbb{R}\times\mathcal{Z}$. Thus, the subadditivity of $(\,\cdot\,)^+$ implies

$$(T(\xi,z)-t)^+ \leq (T(\xi,z)-T(\xi,z_n))^+ + (T(\xi,z_n)-t_n)^+ + (t_n-t)^+ \quad \text{a.e. in } \Xi$$

and the a.e. weak continuity of $T(\xi,z)$ implies

$$(T(\xi,z)-t)^+ \leq \liminf_{n\to\infty}\ (T(\xi,z_n)-t_n)^+ \quad \text{a.e. in } \Xi.$$

Taking expected values and applying Fatou's Lemma [12, L. 2.18] yields

$$\mathbb{E}\left[(T(\cdot,z)-t)^+\right] \leq \mathbb{E}\left[\liminf_{n\to\infty}\ (T(\cdot,z_n)-t_n)^+\right] \leq \liminf_{n\to\infty}\ \mathbb{E}\left[(T(\cdot,z_n)-t_n)^+\right].$$

Therefore, combining this result with the weak lower semicontinuity of the penalty term, $R(z)$ proves that $J(t,z)$ is weakly lower semicontinuous.

Since $\mathcal{Z} = L^2(\Omega_c)$ is a Hilbert space (namely, $\mathcal{Z}$ is reflexive) and $J(t,z)$ is a coercive, weakly lower semicontinuous function we can apply standard existence theorems such as [2, Th. 3.2.5]. □

Before delving into solution techniques, we point out that (3.2) can be reformulated into a smooth, constrained optimization problem by introducing the auxiliary variables $\zeta \in L^1_\rho(\Xi)$:

$$\underset{t\in\mathbb{R},\ z\in\mathcal{Z},\zeta\in L^1_\rho(\Xi)}{\text{minimize}} \quad \frac{1}{2}\left(t + \frac{1}{1-\beta}\int_\Xi \rho(\xi)\zeta(\xi)\,\mathrm{d}\xi\right) + \frac{\alpha}{2}\int_{\Omega_c} z(x)^2\,\mathrm{d}x \tag{4.1a}$$

$$\text{subject to} \quad \zeta(\xi) \geq \int_{\Omega_o}(u(\xi,x;z)-\bar{u}(x))^2\,\mathrm{d}x - t \quad \text{a.e. in } \Xi \tag{4.1b}$$

$$\zeta(\xi) \geq 0 \quad \text{a.e. in } \Xi. \tag{4.1c}$$

Though the objective function is smooth, (4.1) has a nonlinear, nonconvex system of constraints with the form: $G(z,t,\zeta,\xi) \geq 0$, a.e. in Ξ. For first-order optimality conditions, one must show that the linearization of G at a solution is surjective onto $L^{p/2}_\rho(\Xi)$. For complex solution operators $u(z)$, this can be quite challenging. In addition, the resulting system contains multiple integral operators, thereby adding extra challenge to the numerics.

In finite dimensional stochastic programming, this reformulation is typically solved using sample average approximation (SAA). The SAA approach to minimizing CVaR

has been extensively studied in [36, 37] in the context of, e.g., portfolio optimization, but this approach has not been studied in the context of function-space or PDE-constrained optimization. Although the discretized problem is a (large-scale) nonlinear program, the associated KKT system may be inconsistent with the true first-order optimality system (assuming it even exists). In deterministic PDE-constrained optimization problems with state constraints, numerical schemes developed for the solution of the discrete problems typically exhibit mesh-dependent behavior as the discretization is refined if the true Lagrange multipliers are not sufficiently regular. Finally, we also note that, for many high-dimensional quadrature rules, the smooth reformulation (4.1) may become ill-posed due to the presence of negative quadrature weights; see the discussion below.

4.1. Sample-Based Approximation. As mentioned, our primal approach employs sample-based approximation to discretize the expected value in CVaR. Given an independent and identically distributed set of samples with size Q, a typical Monte Carlo approach achieves a convergence rate of $Q^{-1/2}$. Thus, a potentially large number of samples are required to get significant reduction in the discretization error. On the other hand, deterministic quadrature discretizations exploit any regularity of the tracking term

$$\xi \mapsto T(\xi; z) = \int_{\Omega_o} (u(\xi, x; z) - \bar{u}(x))^2 \, \mathrm{d}x \in L^1_\rho(\Xi).$$

with respect to $\xi \in \Xi$ to achieve fast convergence. One downside to deterministic quadrature is that one often has to restrict Ξ and ρ to tensor product probability spaces, i.e., Ξ is a tensor product of intervals and ρ is a tensor product of 1D probability densities. Once discretized, our goal is to use standard gradient-based optimization algorithms to solve (3.2). In addition, the convergence analysis for deterministic quadrature approximation is strongly dependent on regularity. To obtain this regularity, we can either reformulate the sample-discretized problem as in (4.1) or we can smooth $(\cdot)^+$.

Let $\{(\xi_k, \omega_k)\}_{k=1}^Q \subset \Xi \times \mathbb{R}$ denote a system of Q samples and their associated weights. The discretized optimal control problem is

$$\begin{aligned} \underset{t\in\mathbb{R},\, z\in\mathcal{Z}}{\text{minimize}} \quad & \frac{1}{2}\left(t + \frac{1}{(1-\beta)} \sum_{k=1}^Q \omega_k \left(\int_{\Omega_o} (u(\xi_k, x; z) - \bar{u}(x))^2 \, \mathrm{d}x - t \right)^+ \right) \\ & + \frac{\alpha}{2} \int_{\Omega_c} z(x)^2 \, \mathrm{d}x. \end{aligned} \tag{4.2}$$

If the sample weights are all positive, i.e. $\omega_k > 0$ for $k = 1, \ldots, Q$, then we have the equivalent reformulation

$$\underset{t\in\mathbb{R},\, z\in\mathcal{Z},\, \zeta_k\in\mathbb{R}}{\text{minimize}} \quad \frac{1}{2}\left(t + \frac{1}{(1-\beta)} \sum_{k=1}^Q \omega_k \zeta_k \right) + \frac{\alpha}{2} \int_{\Omega_c} z(x)^2 \, \mathrm{d}x \tag{4.3a}$$

$$\text{subject to} \quad \zeta_k \geq \int_{\Omega_o} (u(\xi_k, x; z) - \bar{u}(x))^2 \, \mathrm{d}x - t \quad k = 1, \ldots, Q \tag{4.3b}$$

$$\zeta_k \geq 0 \quad k = 1, \ldots, Q. \tag{4.3c}$$

Although the reformulation (4.3) is smooth, for deterministic quadrature and high dimensional Ξ we often cannot satisfy the positive weight assumption. One common

class of quadrature rules with non-positive weights is sparse grids [14, 15, 27]. To circumvent this issue, we avoid the smooth reformulation (4.3) and smooth $(\cdot)^+$.

4.1.1. Smoothing the CVaR Risk Measure. In this subsection, we discuss a general approach for smoothing $(\,\cdot\,)^+$ motivated by the work in [9]. Let $\delta : \mathbb{R} \to \mathbb{R}$ satisfy the following assumptions:

ASSUMPTION 4.2.

1. $\delta \in C(\mathbb{R})$ *and there exists* $0 < K < \infty$ *such that* $|\delta(x)| \leq K$ *for all* $x \in \mathbb{R}$;
2. $\delta(x) \geq 0$ *for all* $x \in \mathbb{R}$ *and* $\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1$;
3. $\int_{-\infty}^{\infty} \delta(x)|x|\,\mathrm{d}x < \infty$;
4. *Either* $\int_{-\infty}^{\infty} \delta(x)x\,\mathrm{d}x \leq 0$ *or* $\int_{-\infty}^{0} \delta(x)|x|\,\mathrm{d}x = 0$;
5. *The support of* δ, $\mathrm{supp}(\delta) = \{\, x \in \mathbb{R} : \delta(x) > 0 \,\}$, *is connected.*

We define the parametrized family of smoothed plus functions depending on $\varepsilon > 0$ as

$$(x)_\varepsilon^+ = \int_{-\infty}^{x} \mathbf{G}_\varepsilon(\tau)\,\mathrm{d}\tau \quad \text{where} \quad \mathbf{G}_\varepsilon(x) = \int_{-\infty}^{x} \frac{1}{\varepsilon}\delta\left(\frac{\tau}{\varepsilon}\right)\,\mathrm{d}\tau.$$

By Proposition 2.2 in [9], $(x)_\varepsilon^+$ is at least twice continuously differentiable and satisfies

$$-\varepsilon\Delta_2 \leq (x)_\varepsilon^+ - (x)^+ \leq \varepsilon\Delta_1 \tag{4.4}$$

where

$$\Delta_1 = \int_{-\infty}^{0} \delta(x)|x|\,\mathrm{d}x \quad \text{and} \quad \Delta_2 = \max\left\{ \int_{-\infty}^{\infty} \delta(x)x\,\mathrm{d}x,\, 0 \right\}.$$

Moreover, equation (4.4) implies the error bound

$$|(x)_\varepsilon^+ - (x)^+| \leq c\varepsilon \tag{4.5}$$

for some $c \leq \max\{\Delta_1, \Delta_2\}$. In addition, $(x)_\varepsilon^+$ is nondecreasing and convex with $0 \leq \partial_x(x)_\varepsilon^+ = \mathbf{G}_\varepsilon(x) \leq 1$ and $0 \leq \partial_{xx}(x)_\varepsilon^+ = \frac{1}{\varepsilon}\delta\left(\frac{x}{\varepsilon}\right) \leq \frac{K}{\varepsilon}$.

Some examples of such smoothed plus functions are

$$(x)_{\varepsilon,1}^+ = x + \varepsilon \log\left(1 + \exp\left(\frac{-x}{\varepsilon}\right)\right) \tag{4.6}$$

$$(x)_{\varepsilon,2}^+ = \begin{cases} 0 & \text{if } x \leq 0 \\ \left(\frac{x^3}{\varepsilon^2} - \frac{x^4}{2\varepsilon^3}\right) & \text{if } x \in (0, \varepsilon) \\ x - \frac{\varepsilon}{2} & \text{if } x \geq \varepsilon \end{cases} \tag{4.7}$$

$$(x)_{\varepsilon,3}^+ = \left(x + \frac{\varepsilon}{2}\right)_{\varepsilon,2}^+. \tag{4.8}$$

The first smoothed plus function $(\cdot)_{\varepsilon,1}^+$ is infinitely differentiable while the second and third are only twice continuously differentiable. For fixed $\varepsilon > 0$, we have:

$$(x)_{\varepsilon,1}^+ > (x)_{\varepsilon,3}^+ \geq (x)^+ \geq (x)_{\varepsilon,2}^+ \quad \forall\, x \in \mathbb{R}. \tag{4.9}$$

Moreover, the constant in (4.5) for each $\ell = 1, 2, 3$ is $c_1 = \log(2)$, $c_2 = \frac{1}{2}$, and $c_3 = \frac{3}{32}$.

For a given $\delta \in C(\mathbb{R})$ satisfying the above assumptions, we define the smoothed auxiliary function $F_\varepsilon^\beta : \mathbb{R} \times L_\rho^1(\Xi) \to \mathbb{R}$ as

$$F_\varepsilon^\beta(t, X) = t + \frac{1}{1-\beta}\mathbb{E}\left[(X - t)_\varepsilon^+\right],$$

which permits the following definition of smoothed β-CVaR:

$$\sigma_\varepsilon^\beta[X] = \inf \left\{ F_\varepsilon^\beta(t, X) \ : \ t \in \mathbb{R} \right\}.$$

LEMMA 4.3. *Let $X \in L^1_\rho(\Xi)$ and suppose $\delta : \mathbb{R} \to \mathbb{R}$ satisfies Assumptions 4.2. Then the smoothed β-CVaR satisfies*

$$\left|\sigma_\varepsilon^\beta[X] - \mathrm{CVaR}_\beta[X]\right| \leq \frac{c}{1-\beta}\varepsilon.$$

Proof. Fix $X \in L^1_\rho(\Xi)$. By Theorem 10 in [32], there exists $t^* = \mathrm{VaR}_\beta[X] \in \mathbb{R}$ which is finite and minimizes $F^\beta(t, X)$. By (4.5) and the proof of Theorem 10 in [32], we can show that the level sets of $t \mapsto F_\varepsilon^\beta(t, X)$ are bounded and thus the conclusion of the proof of Theorem 10 in [32] applies. Hence, there exists a finite minimizer $t_\varepsilon^* \in \mathbb{R}$ of $F_\varepsilon^\beta(t, X)$.

Now, suppose $\int_{-\infty}^{\infty} \delta(x)x \,\mathrm{d}x \leq 0$. By (4.4), $(x)_\varepsilon^+ \geq (x)^+$. Therefore, the optimality of t^* and t_ε^* implies

$$\begin{aligned} t^* + \frac{1}{1-\beta}\mathbb{E}\left[(X - t^*)_\varepsilon^+\right] &\geq t_\varepsilon^* + \frac{1}{1-\beta}\mathbb{E}\left[(X - t_\varepsilon^*)_\varepsilon^+\right] = \sigma_\varepsilon^\beta[X] \\ &\geq t_\varepsilon^* + \frac{1}{1-\beta}\mathbb{E}\left[(X - t_\varepsilon^*)^+\right] \\ &\geq t^* + \frac{1}{1-\beta}\mathbb{E}\left[(X - t^*)^+\right] = \mathrm{CVaR}_\beta[X]. \end{aligned}$$

On the other hand, suppose $\int_{-\infty}^{0} \delta(x)|x| \,\mathrm{d}x = 0$. By (4.4), $(x)_\varepsilon^+ \leq (x)^+$. Again, the optimality of t^* and t_ε^* implies

$$\begin{aligned} t_\varepsilon^* + \frac{1}{1-\beta}\mathbb{E}\left[(X - t_\varepsilon^*)^+\right] &\geq t^* + \frac{1}{1-\beta}\mathbb{E}\left[(X - t^*)^+\right] = \mathrm{CVaR}_\beta[X] \\ &\geq t^* + \frac{1}{1-\beta}\mathbb{E}\left[(X - t^*)_\varepsilon^+\right] \\ &\geq t_\varepsilon^* + \frac{1}{1-\beta}\mathbb{E}\left[(X - t_\varepsilon^*)_\varepsilon^+\right] = \sigma_\varepsilon^\beta[X]. \end{aligned}$$

Additionally, the smoothed plus function error bound (4.5) gives

$$\left|\left(t + \frac{1}{1-\beta}\mathbb{E}\left[(X-t)_\varepsilon^+\right]\right) - \left(t + \frac{1}{1-\beta}\mathbb{E}\left[(X-t)^+\right]\right)\right| \leq \frac{c}{1-\beta}\varepsilon, \quad \forall\, t \in \mathbb{R}.$$

Combining this error bound with the above inequalities yields the desired result. ∎

Smoothed CVaR is a Convex Risk Measure. The smoothed β-CVaR risk measure satisfies the three axioms for a convex risk measure: Monotonicity, translation equivariance, and convexity. In what follows, we prove these three properties. Throughout, we assume the smoothed plus function is built on a function $\delta : \mathbb{R} \to \mathbb{R}$ satisfying Assumptions 4.2.

We first show that the smoothed β-CVaR satisfies the monotonicity property.

PROPOSITION 4.4. *For any $\varepsilon > 0$, the smoothed β-CVaR, σ_ε^β, satisfies*

$$\sigma_\varepsilon^\beta[X] \geq \sigma_\varepsilon^\beta[Y] \quad \textit{whenever} \quad X \geq Y \quad \textit{a.e. in } \Xi, \quad X, Y \in L^1_\rho(\Xi).$$

Proof. Fix arbitrary $\varepsilon > 0$. For arbitrary X, $Y \in L^1_\rho(\Xi)$ satisfying $X \geq Y$ ρ-a.e. in Ξ and arbitrary $t \in \mathbb{R}$, we have

$$\sigma^\beta_\varepsilon[Y] \leq F^\beta_\varepsilon(t, Y).$$

Since $(\cdot)^+_\varepsilon : \mathbb{R} \to \mathbb{R}$ is nondecreasing, we have that

$$F^\beta_\varepsilon(t, Y) \leq F^\beta_\varepsilon(t, X).$$

Since t is arbitrary, we pass to the infimum to obtain the desired result. ∎

In the subsequent proposition, we show that the smoothed β-CVaR is convex.

PROPOSITION 4.5. *For any $\varepsilon > 0$, the smoothed β-CVaR, σ^β_ε, satisfies*

$$\sigma^\beta_\varepsilon[\tau X + (1-\tau)Y] \leq \tau\sigma^\beta_\varepsilon[X] + (1-\tau)\sigma^\beta_\varepsilon[Y] \quad \forall\, X, Y \in L^1_\rho(\Xi) \quad \textit{and} \quad \tau \in (0,1).$$

Proof. Fix arbitrary $\varepsilon > 0$. For arbitrary X, $Y \in L^1_\rho(\Xi)$, $\tau \in (0,1)$, and $t \in \mathbb{R}$, we have

$$\sigma^\beta_\varepsilon[\tau X + (1-\tau)Y] \leq F^\beta_\varepsilon(t, \tau X + (1-\tau)Y).$$

Since $(\cdot)^+_\varepsilon : \mathbb{R} \to \mathbb{R}$ is convex, we have that

$$F^\beta_\varepsilon(t, \tau X + (1-\tau)Y) \leq \tau F^\beta_\varepsilon(t, X) + (1-\tau)F^\beta_\varepsilon(t, Y).$$

Since t is arbitrary, we pass to the infimum to obtain the desired result. ∎

Finally, we prove that the smoothed β-CVaR risk measure is translation equivariant.

PROPOSITION 4.6. *For any $\varepsilon > 0$, the smoothed β-CVaR, σ^β_ε, satisfies*

$$\sigma^\beta_\varepsilon[X + \tau] = \sigma^\beta_\varepsilon[X] + \tau \quad \forall\, X \in L^1_\rho(\Xi) \quad \textit{and} \quad \tau \in \mathbb{R}.$$

Proof. Fix arbitrary $\varepsilon > 0$. For arbitrary $X \in L^1_\rho(\Xi)$ and $\tau \in \mathbb{R}$, we have

$$\begin{aligned}
\sigma^\beta_\varepsilon[X+\tau] &= \inf\left\{ t + \frac{1}{1-\beta}\mathbb{E}\left[(X+\tau-t)^+_\varepsilon\right] \;:\; t \in \mathbb{R} \right\} \\
&= \inf\left\{ \tau + (t-\tau) + \frac{1}{1-\beta}\mathbb{E}\left[(X-(t-\tau))^+_\varepsilon\right] \;:\; t \in \mathbb{R} \right\} \\
&= \tau + \inf\left\{ t + \frac{1}{1-\beta}\mathbb{E}\left[(X-t)^+_\varepsilon\right] \;:\; t \in \mathbb{R} \right\} \\
&= \sigma^\beta_\varepsilon[X] + \tau.
\end{aligned}$$

This proves the desired result. ∎

Due to the nature of the smoothed plus-functions $(\cdot)^+_\varepsilon$, the smoothed β-CVaR functions are not positive homogeneous and, therefore, are convex but not coherent risk measures.

Smoothed CVaR is Differentiable. For the primal sample-based approach, objective function evaluations require a PDE solve at each sample. Thus, it is ideal to use rapidly converging derivative-based optimization algorithms if possible. Using several classical results on the differentiability of convex integral functionals, we

immediately obtain the following result. Note that a direct proof using the data assumptions in conjunction with the Lebesgue Dominated Convergence Theorem [12] is also possible.

THEOREM 4.7. *For fixed $t \in \mathbb{R}$, $X \mapsto F_\varepsilon^\beta(t, X)$ is Hadamard differentiable as a function from $L_\rho^1(\Xi)$ to $\mathbb{R}$, where*

$$\partial_X F_\varepsilon^\beta(t, X, h) = (1-\beta)^{-1}\mathbb{E}\left[\mathbf{G}_\varepsilon(X-t)h\right].$$

Moreover, for fixed $X \in L_\rho^1(\Xi)$, $t \mapsto F_\varepsilon^\beta(t, X)$ is continuously differentiable as a function from $\mathbb{R}$ to $\mathbb{R}$, where

$$\partial_t F_\varepsilon^\beta(t, X, h) = \left(1 - (1-\beta)^{-1}\mathbb{E}\left[\mathbf{G}_\varepsilon(X-t)\right]\right)h.$$

Proof. Given the data assumptions, it follows from [30, 3E], see also the discussion in [10, Sect. 2.7], that the associated subdifferentials are singletons. Therefore, the mappings in the statement of the theorem are Gâteaux differentiable with the asserted formulae. In light of the local Lipschitz continuity, we appeal to [6, Prop. 2.49] in order to prove that both mappings are in fact Hadamard differentiable. □

Theorem 4.7 ensures the smoothed CVaR objective function is Hadamard differentiable. We can prove additional differentiability properties in the case that $X \in L_\rho^2(\Xi)$.

COROLLARY 4.8. *For fixed $t \in \mathbb{R}$, $X \mapsto F_\varepsilon^\beta(t, X)$ is continuously Fréchet differentiable as a function from $L_\rho^2(\Xi)$ to $\mathbb{R}$.*

Proof. Since $L_\rho^2(\Xi) \subset L_\rho^1(\Xi)$, $F_\varepsilon^\beta(t, \cdot)$ is well-defined as a function from $L_\rho^2(\Xi)$ to $\mathbb{R}$. Now, for any X, $h \in L_\rho^2(\Xi)$ and any $t \in \mathbb{R}$ we have

$$F_\varepsilon^\beta(t, X+h) - F_\varepsilon^\beta(t, X) = (1-\beta)^{-1}\mathbb{E}[(X+h-t)_\varepsilon^+ - (X-t)_\varepsilon^+].$$

Since $(\,\cdot\,)_\varepsilon^+$ is continuously differentiable, the Integral Mean Value Theorem ensures

$$F_\varepsilon^\beta(t, X+h) - F_\varepsilon^\beta(t, X) = (1-\beta)^{-1}\mathbb{E}\left[\int_0^1 \mathbf{G}_\varepsilon(X+\tau h - t)h \,\mathrm{d}\tau\right].$$

Since $(\,\cdot\,)_\varepsilon^+$ has bounded second derivatives, we have

$$\begin{aligned}
&|F_\varepsilon^\beta(t, X+h) - F_\varepsilon^\beta(t, X) - (1-\beta)^{-1}\mathbb{E}[\mathbf{G}_\varepsilon(X-t)h]| \\
&\qquad \leq (1-\beta)^{-1}\mathbb{E}\left[\int_0^1 |(\mathbf{G}_\varepsilon(X+\tau h - t) - \mathbf{G}_\varepsilon(X-t))h| \,\mathrm{d}\tau\right] \\
&\qquad \leq (1-\beta)^{-1}\frac{K}{4\varepsilon}\mathbb{E}[h^2] \qquad (4.10)
\end{aligned}$$

where $K > 0$ is the uniform bound on $\delta(x)$. Equation (4.10) is sufficient to prove Fréchet differentiability and the Fréchet derivative is

$$\partial_X F_\varepsilon^\beta(t, X) = (1-\beta)^{-1}\mathbf{G}_\varepsilon(X-t).$$

Moreover, given any $\eta > 0$, we have that if $\|h\|_{L_\rho^2(\Xi)} < \gamma = 2(1-\beta)\varepsilon K^{-1}\eta$, then the boundedness of the second derivative of $(\,\cdot\,)_\varepsilon^+$ combined with the Integral Mean Value Theorem ensures that

$$\begin{aligned}
\|(1-\beta)^{-1}\mathbf{G}_\varepsilon(X+h-t) - (1-\beta)^{-1}\mathbf{G}_\varepsilon(X-t)\|_{L_\rho^2(\Xi)} &\leq \frac{K}{2\varepsilon}(1-\beta)^{-1}\|h\|_{L_\rho^2(\Xi)} \\
&< \frac{K}{2\varepsilon}(1-\beta)^{-1}\gamma \leq \eta.
\end{aligned}$$

Therefore, $X \mapsto F_\varepsilon^\beta(t,\cdot) : L^2_\rho(\Xi) \to \mathbb{R}$ is continuously Fréchet differentiable. ☐

Under additional assumptions on δ, we can prove that F_ε^β is twice continuously differentiable in the case that $X \in L^2_\rho(\Xi)$.

COROLLARY 4.9. *Suppose $\delta : \mathbb{R} \to \mathbb{R}$ satisfies Assumptions 4.2 and, additionally, δ is Lipschitz continuous with Lipschitz constant $\Delta > 0$. Then, $F_\varepsilon^\beta(t,X)$ is twice continuously Fréchet differentiable.*

Proof. By Corollary 4.8, $X \mapsto F_\varepsilon^\beta(t,X)$ is continuously Fréchet differentiable as a function from $L^2_\rho(\Xi)$ to $\mathbb{R}$ with derivative

$$\partial_X F_\varepsilon^\beta(t,X) = (1-\beta)^{-1}\mathbf{G}_\varepsilon(X-t).$$

Now, similar to the proof of Corollary 4.8, for any X, $h \in L^2_\rho(\Xi)$ and any $t \in \mathbb{R}$ the Integral Mean Value Theorem implies

$$\mathbf{G}_\varepsilon(X+h-t) - \mathbf{G}_\varepsilon(X-t) = \frac{1}{\varepsilon}\int_0^1 \delta\left(\frac{X+\tau h - t}{\varepsilon}\right) h \,\mathrm{d}\tau.$$

Since δ is Lipschitz continuous, we have

$$\begin{aligned}
&(1-\beta)^{-1}\|\mathbf{G}_\varepsilon(X+h-t) - \mathbf{G}_\varepsilon(X-t) - \varepsilon^{-1}\delta(\varepsilon^{-1}(X-t))h\|_{L^2_\rho(\Xi)} \\
&\quad = (1-\beta)^{-1}\varepsilon^{-1}\left\|\int_0^1 \delta\left(\frac{X+\tau h-t}{\varepsilon}\right) - \delta\left(\frac{X-t}{\varepsilon}\right)\,\mathrm{d}\tau\, h\right\|_{L^2_\rho(\Xi)} \\
&\quad \le (1-\beta)^{-1}\varepsilon^{-2}\frac{\Delta}{2}\|h\|^2_{L^2_\rho(\Xi)}. \qquad (4.11)
\end{aligned}$$

Equation (4.11) is sufficient to prove the Fréchet differentiability and the second Fréchet derivative is

$$\partial_{XX} F_\varepsilon^\beta(t,X) = (1-\beta)^{-1}\varepsilon^{-1}\delta(\varepsilon^{-1}(X-t)).$$

Moreover, given any $\eta > 0$, we have that if $\|h\|_{L^2_\rho(\Xi)} < \gamma = 2(1-\beta)\varepsilon^2\Delta^{-1}\eta$, then the Lipschitz continuity of δ implies

$$\begin{aligned}
(1-\beta)^{-1}\varepsilon^{-1}\|\delta(\varepsilon^{-1}(X+h-t)) - \delta(\varepsilon^{-1}(X-t))\|_{L^2_\rho(\Xi)} &\le \frac{\Delta}{2}(1-\beta)^{-1}\varepsilon^{-2}\|h\|_{L^2_\rho(\Xi)} \\
&< \frac{\Delta}{2}(1-\beta)^{-1}\varepsilon^{-2}\gamma \le \eta.
\end{aligned}$$

Therefore, $X \mapsto F_\varepsilon^\beta(t,X) : L^2_\rho(\Xi) \to \mathbb{R}$ is twice continuously Fréchet differentiable. Similar arguments for $\partial_{tX}F_\varepsilon^\beta(t,X)$, $\partial_{Xt}F_\varepsilon^\beta(t,X)$, and $\partial_{tt}F_\varepsilon^\beta(t,X)$ give

$$\begin{aligned}
\partial_{tX}F_\varepsilon^\beta(t,X) = \partial_{Xt}F_\varepsilon^\beta(t,X) &= -(1-\beta)^{-1}\varepsilon^{-1}\delta(\varepsilon^{-1}(X-t)), \\
\partial_{tt}F_\varepsilon^\beta(t,X) &= (1-\beta)^{-1}\varepsilon^{-1}\mathbb{E}[\delta(\varepsilon^{-1}(X-t))].
\end{aligned}$$

This completes the proof. ☐

4.1.2. The Smoothed Optimal Control Problem. Since F_ε^β, is Hadamard differentiable, the smoothed conditional value-at-risk objective function

$$J_\varepsilon(t,z) = \frac{1}{2}F_\varepsilon^\beta\left(t, \int_{\Omega_o}(u(\cdot,x;z) - \bar{u}(x))^2\,\mathrm{d}x\right) + \frac{\alpha}{2}\int_{\Omega_c} z(x)^2\,\mathrm{d}x$$

is continuously differentiable with respect to $t \in \mathbb{R}$ and Hadamard differentiable with respect to $z \in \mathcal{Z}$. This differentiability follows because

$$z \mapsto \int_{\Omega_o} (u(\cdot, x; z) - \bar{u}(x))^2 \, \mathrm{d}x$$

is Fréchet differentiable by Assumption 2.3. Moreover, the smoothed objective functions are convex with respect to the t variable. In addition, the inequalities (4.5) imply that each smoothed objective function converges to $J(t, z)$ linearly with respect to ε. That is,

$$|J_\varepsilon(t, z) - J(t, z)| \leq \frac{c}{2(1-\beta)}\varepsilon \tag{4.12}$$

for all $t \in \mathbb{R}$ and $z \in \mathcal{Z}$.

We now prove existence of solutions to the optimization problem

$$\underset{t\in\mathbb{R},\, z\in\mathcal{Z}}{\text{minimize}} \quad J_\varepsilon(t, z). \tag{4.13}$$

THEOREM 4.10. *Let Assumption 2.2 hold. Then there exists a solution to* (4.13).

Proof. Since $J(t, z)$ is coercive and $J_\varepsilon(t, z) \geq J(t, z) - \frac{c}{2(1-\beta)}\varepsilon$ for all $(t, z) \in \mathbb{R}\times\mathcal{Z}$, $J_\varepsilon(t, z)$ is also coercive.

We now prove weak lower semicontinuity of $J_\varepsilon(t, z)$. Again, since $\partial\Omega$ is Lipschitz and by Assumption 2.2, the tracking term, $T(\xi, z) = \int_{\Omega_o}(u(\xi, x; z) - \bar{u}(x))^2 \, \mathrm{d}x$, is weakly continuous with respect to $z \in \mathcal{Z}$ a.e. in Ξ and the penalty term, $R(z) = \int_{\Omega_c} z(x)^2 \, \mathrm{d}x$ is also weakly lower semicontinuous. Now, let $\{(t_n, z_n)\} \subset \mathbb{R} \times \mathcal{Z}$ be such that $(t_n, z_n) \rightharpoonup (t, z) \in \mathbb{R} \times \mathcal{Z}$. Define the sequence $T_n(\xi) = \inf_{k\geq n}(T(\xi, z_k) - t_k)$ a.e. in Ξ. Then, $T_n(\xi) \leq (T(\xi, z_k) - t_k)$ a.e. in Ξ for $k \geq n$ and since $(\,\cdot\,)_\varepsilon^+$ is monotonically increasing,

$$(T_n(\xi))_\varepsilon^+ \leq (T(\xi, z_k) - t_k)_\varepsilon^+ \quad \text{a.e. in } \Xi,\ k \geq n.$$

This implies that

$$(T_n(\xi))_\varepsilon^+ \leq \inf_{k\geq n} (T(\xi, z_k) - t_k)_\varepsilon^+ \quad \text{a.e. in } \Xi.$$

Now, since $(\,\cdot\,)_\varepsilon^+$ is continuous from $\mathbb{R}$ to $\mathbb{R}$,

$$(\liminf_{n\to\infty} (T(\xi, z_n) - t_n))_\varepsilon^+ = \lim_{n\to\infty} (T_n(\xi))_\varepsilon^+ \leq \liminf_{n\to\infty} (T(\xi, z_n) - t_n)_\varepsilon^+ \quad \text{a.e. in } \Xi.$$

The a.e. weak continuity of $T(\xi, z)$ and continuity of $(\,\cdot\,)_\varepsilon^+$ imply

$$(T(\xi, z) - t)_\varepsilon^+ \leq \liminf_{n\to\infty} (T(\xi, z_n) - t_n)_\varepsilon^+ \quad \text{a.e. in } \Xi.$$

Therefore, Fatou's Lemma [12, L. 2.18] implies

$$\mathbb{E}\left[(T(\xi, z) - t)_\varepsilon^+\right] \leq \mathbb{E}\left[\liminf_{n\to\infty} (T(\xi, z_n) - t_n)_\varepsilon^+\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[(T(\xi, z_n) - t_n)_\varepsilon^+\right].$$

Since $R(z)$ is weakly lower semicontinuous, $J_\varepsilon(t, z)$ is also weakly lower semicontinuous.

Since $\mathcal{Z} = L^2(\Omega_c)$ is a Hilbert space (namely, $\mathcal{Z}$ is reflexive) and $J_\varepsilon(t,z)$ is a coercive, weakly lower semicontinuous function, standard existence theorems such as [2, Th. 3.2.5] apply. □

For smoothing to be a feasible solution approach, we must be able to show that the minimizers of $J_\varepsilon(t,z)$ converge to a minimizer of $J(t,z)$. We should also be able to quantify the error committed by approximating $(\,\cdot\,)^+$ with $(\,\cdot\,)^+_\varepsilon$. Throughout this section, we employ the following inner-product notation: for $x = (t_1, z_1)$, $y = (t_2, z_2) \in \mathbb{R} \times \mathcal{Z}$,

$$(x, y) = t_1 t_2 + \langle z_1, z_2 \rangle_{\mathcal{Z}}.$$

Moreover, the associated inner-product induced norm is denoted $\|\cdot\|$. We begin by proving a technical lemma which gives an explicit lower bound on the smoothed value-at-risk approximation.

LEMMA 4.11. *Let $x^*_\varepsilon = (t^*_\varepsilon, z^*_\varepsilon) \in \mathbb{R} \times \mathcal{Z}$ be a minimizer of $J_\varepsilon(t,z)$. Then, $t^*_\varepsilon \geq -\ell(\varepsilon,\beta)$, where*

$$\ell(\varepsilon,\beta) = \max\{\, \ell^*(\varepsilon,\beta),\, 0 \,\},$$

and $\ell^*(\varepsilon,\beta) = \inf\{\, \tau \,:\, \mathbf{G}_\varepsilon(\tau) \geq (1-\beta) \,\}$ is finite.

Proof. First, since $\mathbf{G}_\varepsilon(x)$ is continuous and nondecreasing, and since $\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1$, we have that $\ell(\varepsilon,\beta)$ is well-defined and finite.

Now by definition, $T(\xi,z) = \int_{\Omega_o} (u(\xi,x;z) - \bar{u}(x))^2\,\mathrm{d}x \geq 0$ ρ-a.e. in Ξ for all $z \in \mathcal{Z}$. Suppose $t^*_\varepsilon < -\ell(\varepsilon,\beta)$. Then, $(T(\xi,z^*_\varepsilon) - t^*_\varepsilon) > \ell(\varepsilon,\beta)$ ρ-a.e. in Ξ. Note that since $\mathrm{supp}(\delta)$ is connected, $\mathbf{G}_\varepsilon(x)$ is strictly increasing for all $x \in \mathbf{G}_\varepsilon^{-1}((0,1))$. Moreover, by continuity of $\mathbf{G}_\varepsilon$, $\mathbf{G}_\varepsilon^{-1}((0,1))$ is open and, in fact, $\mathbf{G}_\varepsilon^{-1}((0,1)) = \varepsilon\,\mathrm{supp}(\delta)$. If $\ell^*(\varepsilon,\beta) \geq 0$ or if $\ell^*(\varepsilon,\beta) < 0$ and $0 \in \mathbf{G}_\varepsilon^{-1}((0,1))$, then differentiating $(\,\cdot\,)^+_\varepsilon$ and applying the previous results yields

$$\mathbf{G}_\varepsilon(T(\xi,z^*_\varepsilon) - t^*_\varepsilon) > \mathbf{G}_\varepsilon(\ell(\varepsilon,\beta)) \geq \mathbf{G}_\varepsilon(\ell^*(\varepsilon,\beta)) \geq 1-\beta \quad \rho\text{-a.e. in } \Xi,$$

but optimality of x^*_ε, implies

$$1-\beta = \mathbb{E}\left[\mathbf{G}_\varepsilon(T(\xi,z^*_\varepsilon) - t^*_\varepsilon)\right] > (1-\beta).$$

This is a contradiction. Similarly, if $\ell^*(\varepsilon,\beta) < 0$ and $0 \notin \mathbf{G}_\varepsilon^{-1}((0,1))$, then

$$\mathbf{G}_\varepsilon(T(\xi,z^*_\varepsilon) - t^*_\varepsilon) \geq \mathbf{G}_\varepsilon(\ell(\varepsilon,\beta)) = 1 \quad \rho\text{-a.e. in } \Xi,$$

but, again, optimality of x^*_ε, implies

$$1-\beta = \mathbb{E}\left[\mathbf{G}_\varepsilon(T(\xi,z^*_\varepsilon) - t^*_\varepsilon)\right] \geq 1.$$

This, again, is a contradiction since $\beta > 0$. Therefore, $t^*_\varepsilon \geq -\ell(\varepsilon,\beta)$. □

Using Lemma 4.11, we prove consistency for the smoothed approximations.

THEOREM 4.12. *Let $\{\varepsilon_k\}_{k=1}^{\infty} \subset (0,\infty)$ be a decreasing sequence of smoothing parameters satisfying $\varepsilon_k \to 0$ as $k \to \infty$ and let $x^*_k = (t^*_{\varepsilon_k}, z^*_{\varepsilon_k}) \in \mathbb{R} \times \mathcal{Z}$ be a minimizer of $J_{\varepsilon_k}(t,z)$. Then there exists subsequence of $\{x^*_k\}$ that converges weakly to some $x^* = (t^*, z^*) \in \mathbb{R} \times \mathcal{Z}$ where x^* is a minimizer of $J(t,z)$.*

Proof. The optimality of $x^*_{\varepsilon_k}$ implies

$$J_{\varepsilon_k}(0,0) \geq J_{\varepsilon_k}(t^*_{\varepsilon_k}, z^*_{\varepsilon_k}) \geq \frac{1}{2} t^*_{\varepsilon_k} + \frac{\alpha}{2} \|z^*_{\varepsilon_k}\|^2_{\mathcal{Z}}.$$

Therefore, Lemma 4.11 gives the bounds

$$|t^*_{\varepsilon_k}| \leq \max\{\ell(\varepsilon_k,\beta), 2|J_{\varepsilon_k}(0,0)|\}$$

and

$$\|z^*_{\varepsilon_k}\|_{\mathcal{Z}} \leq \sqrt{2\alpha^{-1}\left(\ell(\varepsilon_k,\beta) + |J_{\varepsilon_k}(0,0)|\right)}.$$

Note that $\varepsilon \mapsto \mathbf{G}_\varepsilon(\tau)$ for fixed $\tau \geq 0$ is a nonincreasing function and, therefore, $\varepsilon \mapsto \ell(\varepsilon,\beta)$ is nondecreasing. Thus, since $\{\varepsilon_k\}_{k=1}^\infty \subset (0,\infty)$ is decreasing, $\varepsilon_1 \geq \varepsilon_k$ and $\ell(\varepsilon_1,\beta) \geq \ell(\varepsilon_k,\beta)$ for all $k \geq 1$. Moreover, by (4.12), we have that

$$|J_{\varepsilon_k}(0,0)| \leq \frac{c}{2(1-\beta)}\varepsilon_1 + |J(0,0)|.$$

Thus, the sequence x^*_k is bounded in the $\mathbb{R}\times\mathcal{Z}$ norm. Since $\mathbb{R}\times\mathcal{Z}$ is a Hilbert space, there exists a subsequence that converges weakly to some limit point $x^* = (t^*, z^*) \in \mathbb{R}\times\mathcal{Z}$. We denote the subsequence as $\{x^*_k\}$.

Now, by (4.12), we have that for all $(t,z) \in \mathbb{R}\times\mathcal{Z}$

$$\frac{c}{2(1-\beta)}\varepsilon_k + J(t,z) \geq J_{\varepsilon_k}(t,z) \geq J_{\varepsilon_k}(t^*_{\varepsilon_k}, z^*_{\varepsilon_k}) \geq J(t^*_{\varepsilon_k}, z^*_{\varepsilon_k}) - \frac{c}{2(1-\beta)}\varepsilon_k.$$

As shown in the proof of Theorem 4.1, $J(t,z)$ is weakly lower semicontinuous. Hence, passing to the limit-inferior yields

$$J(t,z) \geq \liminf_{k\to\infty}\ J_{\varepsilon_k}(t^*_{\varepsilon_k}, z^*_{\varepsilon_k}) \geq \liminf_{k\to\infty}\ J(t^*_{\varepsilon_k}, z^*_{\varepsilon_k}) \geq J(t^*,z^*) \quad \forall\ (t,z) \in \mathbb{R}\times\mathcal{Z}.$$

Therefore $x^* = (t^*, z^*)$ is a minimizer of $J(t,z)$. □

Theorem 4.12 demonstrates that as the smoothing parameter ε decreases, the minimizers of the smoothed objective function become more accurate. We now show that, under certain assumptions, we can quantify the rate of convergence with respect to the smoothing parameter.

THEOREM 4.13. *Let $x^* = (t^*, z^*) \in \mathbb{R}\times\mathcal{Z}$ be a minimizer of $J(x) = J(t,z)$ and $x^*_\varepsilon = (t^*_\varepsilon, z^*_\varepsilon) \in \mathbb{R}\times\mathcal{Z}$ be a minimizer of $J_\varepsilon(x) = J_\varepsilon(t,z)$ for fixed $\varepsilon > 0$. Suppose*

$$\langle \nabla J_\varepsilon(x) - \nabla J_\varepsilon(x^*_\varepsilon), x - x^*_\varepsilon\rangle \geq \alpha\|x - x^*_\varepsilon\|^2 \tag{4.14}$$

for all $x = x^*_\varepsilon + \tau(x^* - x^*_\varepsilon)$ with $0 < \tau < 1$. Then, there exists $C > 0$ such that

$$\|x^* - x^*_\varepsilon\| = \left(|t^* - t^*_\varepsilon|^2 + \|z^* - z^*_\varepsilon\|^2_{\mathcal{Z}}\right)^{\frac{1}{2}} \leq C\varepsilon^{\frac{1}{2}}.$$

Proof. Note that (4.12) implies

$$|J(x^*_\varepsilon) - J_\varepsilon(x^*_\varepsilon)| \leq \widehat{C}\varepsilon$$

where $\widehat{C} = \frac{c}{2(1-\beta)}$. Using this inequality, we have

$$\begin{aligned} J(x^*) - J_\varepsilon(x^*_\varepsilon) &= J(x^*) - J_\varepsilon(x^*) + J_\varepsilon(x^*) - J_\varepsilon(x^*_\varepsilon) \\ &\geq -\widehat{C}\varepsilon + \int_0^1 \langle \nabla J_\varepsilon(x^*_\varepsilon + \tau(x^* - x^*_\varepsilon)), (x^* - x^*_\varepsilon)\rangle\ \mathrm{d}\tau. \end{aligned}$$

By the optimality of x_ε^* and our assumptions on x^* and J_ε, we have

$$\begin{aligned}\langle \nabla J_\varepsilon(x_\varepsilon^* + \tau(x^* - x_\varepsilon^*)), (x^* - x_\varepsilon^*)\rangle \\ &= \tau^{-1}\langle \nabla J_\varepsilon(x_\varepsilon^* + \tau(x^* - x_\varepsilon^*)) - \nabla J_\varepsilon(x_\varepsilon^*), \tau(x^* - x_\varepsilon^*)\rangle \\ &\geq \tau\alpha\|x^* - x_\varepsilon^*\|^2.\end{aligned}$$

Therefore, integrating this quantity gives us

$$\frac{\alpha}{2}\|x^* - x_\varepsilon^*\|^2 \leq \widehat{C}\varepsilon + J(x^*) - J_\varepsilon(x_\varepsilon^*).$$

Now, since x^* is optimal, $J(x^*) - J(x_\varepsilon^*) \leq 0$ and

$$\begin{aligned}J(x^*) - J_\varepsilon(x_\varepsilon^*) &= J(x^*) - J(x_\varepsilon^*) + J(x_\varepsilon^*) - J_\varepsilon(x_\varepsilon^*) \\ &\leq J(x_\varepsilon^*) - J_\varepsilon(x_\varepsilon^*) \leq \widehat{C}\varepsilon.\end{aligned}$$

Hence, combining these results gives

$$\|x^* - x_\varepsilon^*\|^2 \leq \frac{4\widehat{C}}{\alpha}\varepsilon.$$

Taking the square root of both sides and defining $C = \left(\frac{4\widehat{C}}{\alpha}\right)^{\frac{1}{2}} = \left(\frac{2c}{\alpha(1-\beta)}\right)^{\frac{1}{2}}$ gives

$$\|x^* - x_\varepsilon^*\| \leq C\varepsilon^{\frac{1}{2}}.$$

□

Theorem 4.13 demonstrates that, under the directional convexity assumption (4.14), the smoothing error decays at a rate of $\varepsilon^{\frac{1}{2}}$. For example, if the state equation (2.1) is a linear PDE (i.e. $u \mapsto N(u, \cdot)$ is linear), then the penalty term in $J_\varepsilon(t, z)$ ensures that $J_\varepsilon(t, z)$ is uniformly convex with respect to $z \in \mathcal{Z}$. Moreover, the proof of Theorem 4.12 ensures that the set of minimizers of $J_\varepsilon(t, z)$ and $J(t, z)$ is bounded. Therefore, $J_\varepsilon(t, z)$ is uniformly convex with respect to $(t, z) \in \mathbb{R} \times \mathcal{Z}$ on this bounded set. In this case, the assumptions of Theorem 4.13 are satisfied.

5. Dual Approach. In order to develop a primal-dual approach to solving the minimax problem, we first consider a more general setting.

5.1. Formulation, Setting and Regularization. As mentioned earlier, the CVaR problem enjoys an equivalent reformulation as a minimax problem. In order to develop tractable numerical methods, we add a strongly concave regularization term to the objective functional of the maximum problem:

$$\underset{\vartheta \in L_\rho^\infty(\Xi)}{\text{maximize}} \quad \int_\Xi \vartheta(\xi)\rho(\xi)c(z, \xi)\, \mathrm{d}\xi + d(z) - \frac{\varepsilon}{2}\int_\Xi \vartheta(\xi)^2\rho(\xi)\, \mathrm{d}\xi \tag{5.1a}$$

$$\text{s.t.} \qquad 0 \leq \vartheta(\xi) \leq \frac{1}{1-\beta},\ a.e.\,\Xi, \quad \int_\Xi \vartheta(\xi)\rho(\xi)\, \mathrm{d}\xi = 1 \tag{5.1b}$$

Here, $\varepsilon \geq 0$, $c(z, \cdot) \in L_\rho^1(\Xi)$, $d : Y \to \mathbb{R}$, where Y is a reflexive Banach space and $\beta \in \mathbb{R}$ with $0 < \beta < 1$. We note that the vast majority of the following results do not require $\varepsilon > 0$ to be valid. As before, we let $\mathcal{A}$ be the feasible set in (5.1). This leads us to consider the class of problems:

$$\underset{z \in Y}{\text{minimize}} \quad D_\varepsilon(z), \tag{5.2}$$

where D_ε, for $\varepsilon \geq 0$, is the optimal value function associated with (5.1).

5.2. Sensitivity of the Optimal Value Function. Our goal is to derive first-order optimality conditions for (5.2). This requires knowledge of the differentiability of D_ε. We define the possibly set-valued mapping $S_\varepsilon : Y \to 2^{\mathcal{A}}$ by

$$S_\varepsilon(z) := \arg\max_{\vartheta} \left\{ \int_\Xi \vartheta(\xi)\rho(\xi)c(z,\xi)\,\mathrm{d}\xi + d(z) - \frac{\varepsilon}{2}\int_\Xi \vartheta(\xi)^2\rho(\xi)\,\mathrm{d}\xi \;:\; \vartheta \in \mathcal{A} \right\}. \tag{5.3}$$

The following result is inspired by a well-known theorem by Danskin [11, Ch. 3], see also [6, Sec. 4.3.1, Pr. 4.12, Th. 4.13]. Unfortunately, these results require certain compactness or smoothness assumptions that are not amenable to our setting in which the weak* sequential topology on $L^\infty_\rho(\Xi)$ plays a role. For this reason, we provide our own full proof.

THEOREM 5.1. *In addition to the standing assumptions, suppose that*

1. *$d : Y \to \mathbb{R}$ is directionally differentiable in the sense of Gâteaux.*
2. *For all z, $\eta \in Y$ and ρ-almost every $\xi \in \Xi$ there exists a $r(\cdot,\xi)$ and a positively homogeneous mapping $c'(z,\xi;\cdot) : Y \to L^1_\rho(\Xi)$ such that*

$$c(z+t\eta,\xi) = c(z,\xi) + tc'(z,\xi;\eta) + r(t,\xi)$$

   for which it holds that $r(t,\cdot) \in L^1_\rho(\Xi)$ and $r(t,\cdot)/t \to 0$ in $L^1_\rho(\Xi)$ as $t \downarrow 0$.
3. *$c(z+\eta,\cdot) \to c(z,\cdot)$ strongly in $L^1_\rho(\Xi)$ for any $\eta \in Y$ with $\eta \to 0$.*
4. *d is continuous on Y.*

Then D_ε is directionally differentiable and the directional derivative at z in the direction η is

$$D'_\varepsilon(z;\eta) = \sup_{\vartheta^* \in S_\varepsilon(z)} \int_\Xi \vartheta^*(\xi)\rho(\xi)c'(z,\xi;\eta)\,\mathrm{d}\xi + d'(z;\eta). \tag{5.4}$$

Moreover, if in addition

5. *$\varepsilon > 0$,*
6. *$c'(z,\xi;\eta)$, is linear in η, ρ-a.e.,*
7. *$d'(z;\eta) = d'(z)\eta$,*

then $S_\varepsilon(z)$ is a singleton and D_ε is Gâteaux differentiable with

$$D'_\varepsilon(z)\eta = \int_\Xi \vartheta^*(\xi)\rho(\xi)c'(z,\xi)\eta\,\mathrm{d}\xi + d'(z)\eta,\ \vartheta^* = S_\varepsilon(z). \tag{5.5}$$

Proof. Begin by fixing an arbitrary $t > 0, \varepsilon \geq 0$, $z, \eta \in Y$. Since $\vartheta^* \in S_\varepsilon(z) \subset \mathcal{A}$, it is feasible for the optimization problem associated with $D_\varepsilon(z+t\eta)$. It follows that

$$\begin{aligned}\frac{D_\varepsilon(z+t\eta) - D_\varepsilon(z)}{t} &\geq \frac{1}{t}\left(\int_\Xi \vartheta^*(\xi)\rho(\xi)c(z+t\eta,\xi)\,\mathrm{d}\xi + d(z+t\eta) - \right.\\ &\left.\frac{\varepsilon}{2}\int_\Xi (\vartheta^*(\xi))^2\rho(\xi)\,\mathrm{d}\xi - \int_\Xi \vartheta^*(\xi)\rho(\xi)c(z,\xi)\,\mathrm{d}\xi - d(z) - \frac{\varepsilon}{2}\int_\Xi (\vartheta^*(\xi))^2\rho(\xi)\,\mathrm{d}\xi\right)\\ &= \int_\Xi \vartheta^*(\xi)\rho(\xi)c'(z,\xi;\eta)\,\mathrm{d}\xi + d'(z;\eta) + \frac{o(t)}{t} + \int_\Xi \frac{r(t,\xi)}{t}\vartheta^*(\xi)\rho(\xi)\,\mathrm{d}\xi\end{aligned}$$

Hence,

$$\liminf_{t\downarrow 0} \frac{D_\varepsilon(z+t\eta) - D_\varepsilon(z)}{t} \geq \int_\Xi \vartheta^*(\xi)\rho(\xi)c'(z,\xi;\eta)\,\mathrm{d}\xi + d'(z;\eta),\ \forall \vartheta^* \in S_\varepsilon(z),$$

from which we deduce

$$\liminf_{t\downarrow 0} \frac{D_\varepsilon(z+t\eta) - D_\varepsilon(z)}{t} \geq \sup_{\vartheta^* \in S_\varepsilon(z)} \int_\Xi \vartheta^*(\xi)\rho(\xi)c'(z,\xi;\eta)\,\mathrm{d}\xi + d'(z;\eta). \tag{5.6}$$

Now fix an arbitrary $\vartheta_t^* \in S_\varepsilon(z+t\eta)$. By definition, $\vartheta_t^* \in \mathcal{A}$. Therefore, $0 \leq \vartheta_t^*(\xi) \leq \frac{1}{1-\beta}$ for ρ-almost every ξ in Ξ and every $t > 0$, i.e., the set $\{\vartheta_t^*\}_{t>0}$ is uniformly bounded in $L_\rho^q(\Xi)$ with $1 \leq q \leq \infty$. Moreover, we have the following relation

$$\begin{aligned}
&\frac{D_\varepsilon(z+t\eta) - D_\varepsilon(z)}{t} \leq \\
&\quad \frac{1}{t}\left(\int_\Xi \vartheta_t^*(\xi)\rho(\xi)c(z+t\eta,\xi)\,\mathrm{d}\xi + d(z+t\eta) - \int_\Xi \vartheta_t^*(\xi)\rho(\xi)c(z,\xi)\,\mathrm{d}\xi - d(z)\right) \\
&\quad = \int_\Xi \vartheta_t^*(\xi)\rho(\xi)c'(z,\xi;\eta)\,\mathrm{d}\xi + d'(z;\eta) + t^{-1}o(t) + \int_\Xi t^{-1}r(t,\xi)\vartheta_t^*(\xi)\rho(\xi)\,\mathrm{d}\xi \\
&\quad \leq \int_\Xi \vartheta_t^*(\xi)\rho(\xi)c'(z,\xi;\eta)\,\mathrm{d}\xi + d'(z;\eta) + t^{-1}o(t) + \int_\Xi |t^{-1}r(t,\xi)||\vartheta_t^*(\xi)||\rho(\xi)|\,\mathrm{d}\xi \\
&\quad \leq \int_\Xi \vartheta_t^*(\xi)\rho(\xi)c'(z,\xi;\eta)\,\mathrm{d}\xi + d'(z;\eta) + t^{-1}o(t) + \frac{1}{1-\beta}\int_\Xi |t^{-1}r(t,\xi)|\rho(\xi)\,\mathrm{d}\xi
\end{aligned}$$

Hence, the limit superior of the difference quotient exists. Let $t_n \downarrow 0$ be some sequence such that

$$\begin{aligned}
&\limsup_{t\downarrow 0} \frac{1}{t}\left(\int_\Xi \vartheta_t^*(\xi)\rho(\xi)c(z+t\eta,\xi)\,\mathrm{d}\xi + d(z+t\eta) - \int_\Xi \vartheta_t^*(\xi)\rho(\xi)c(z,\xi)\,\mathrm{d}\xi - d(z)\right) = \\
&\lim_{n\to\infty} \frac{1}{t_n}\left(\int_\Xi \vartheta_{t_n}^*(\xi)\rho(\xi)c(z+t_n\eta,\xi)\,\mathrm{d}\xi + d(z+t_n\eta) - \int_\Xi \vartheta_{t_n}^*(\xi)\rho(\xi)c(z,\xi)\,\mathrm{d}\xi - d(z)\right)
\end{aligned}$$

According to the analysis above, the sequence $\{\vartheta_{t_n}^*\}_n$ contains a weak* convergent subsequence $\{\vartheta_{t_{n_l}}^*\}_l$ in $L_\rho^\infty(\Xi)$ such that $\vartheta_{t_{n_l}}^* \overset{*}{\rightharpoonup} \vartheta^* \in \mathcal{A}$. The feasibility of ϑ^* follows from the weak* compactness of $\mathcal{A}$ in $L_\rho^\infty(\Xi)$, see e.g. [38, L. 4.2]. Moreover, one can easily demonstrate that $\vartheta^* \in S_\varepsilon(z)$. Therefore,

$$\begin{aligned}
&\limsup_{t\downarrow 0} \frac{1}{t}\left(\int_\Xi \vartheta_t^*(\xi)\rho(\xi)c(z+t\eta,\xi)\,\mathrm{d}\xi + d(z+t\eta) - \int_\Xi \vartheta_t^*(\xi)\rho(\xi)c(z,\xi)\,\mathrm{d}\xi - d(z)\right) = \\
&\lim_{n\to\infty} \frac{1}{t_n}\left(\int_\Xi \vartheta_{t_n}^*(\xi)\rho(\xi)c(z+t_n\eta,\xi)\,\mathrm{d}\xi + d(z+t_n\eta) - \int_\Xi \vartheta_{t_n}^*(\xi)\rho(\xi)c(z,\xi)\,\mathrm{d}\xi - d(z)\right) = \\
&\lim_{l\to\infty} \frac{1}{t_{n_l}}\left(\int_\Xi \vartheta_{t_{n_l}}^*(\xi)\rho(\xi)c(z+t_{n_l}\eta,\xi)\,\mathrm{d}\xi + d(z+t_{n_l}\eta) - \int_\Xi \vartheta_{t_{n_l}}^*(\xi)\rho(\xi)c(z,\xi)\,\mathrm{d}\xi - d(z)\right).
\end{aligned}$$

Expanding and passing to the limit in the latter term yields:

$$\begin{aligned}
\limsup_{t\downarrow 0} \frac{D_\varepsilon(z+t\eta) - D_\varepsilon(z)}{t} &\leq \int_\Xi \vartheta^*(\xi)\rho(\xi)c'(z,\xi;\eta)\,\mathrm{d}\xi + d'(z;\eta) \\
&\leq \sup_{\vartheta^* \in S_\varepsilon(z)} \int_\Xi \vartheta^*(\xi)\rho(\xi)c'(z,\xi;\eta)\,\mathrm{d}\xi + d'(z;\eta).
\end{aligned} \tag{5.7}$$

Combining (5.6) and (5.7) yields (5.4). The final assertion follows from (5.4) under the assumptions. □

COROLLARY 5.2. *Let $\varepsilon \geq 0$ and z^* an optimal solution of* (5.2). *If assumptions 1-4 of Theorem 5.1 hold, then*

$$\sup_{\vartheta^* \in S_\varepsilon(z^*)} \int_\Xi \rho(\xi)\vartheta^*(\xi)c'(z^*,\xi;\eta)\,\mathrm{d}\xi + d'(z^*;\eta) \geq 0,\ \forall \eta \in Y. \tag{5.8}$$

If, in addition, 5-7 hold, then

$$\int_\Xi \vartheta^*(\xi)\rho(\xi)c'(z^*,\xi)\eta\,\mathrm{d}\xi + d'(z^*)\eta = 0,\ \forall \eta \in Y,\ \vartheta^* = S_\varepsilon(z). \tag{5.9}$$

REMARK 5.3. (5.9) *implies that $d'(z^*)$ is equal to the expected valued of $c'(z^*,\cdot)$ with respect to the new "risk-adjusted" measure $\vartheta^*\rho$.*

Conditions (5.8) and (5.9) are the first step towards workable optimality conditions. In order to make full use of them, we will need to better characterize solutions of the inner problem.

5.3. Optimality Conditions for the Max Problem. In this section, we use a standard technique based on constraint qualifications, cf. [6, Ch. 3], to derive multiplier-based (necessary and sufficient) first-order optimality conditions for (5.1)

PROPOSITION 5.4. *Let $\varepsilon \geq 0$, $z \in Y$, and $\vartheta^* \in L^\infty_\rho(\Xi)$. Moreover, define $Q \subset L^\infty_\rho(\Xi)$ by*

$$Q := \left\{ q \in L^\infty_\rho(\Xi) \ :\ 0 \leq q(\xi) \leq \frac{1}{1-\beta} \ \text{a.e. in } \Xi \right\}. \tag{5.10}$$

If ϑ^* is an optimal solution of (5.1), *then there exists a multiplier $\mu \in \mathbb{R}$ such that the following relations hold:*

$$\int_\Xi \rho(\xi)(\mu + \varepsilon\vartheta^*(\xi) - c(z,\xi))(\vartheta'(\xi) - \vartheta^*(\xi))\,\mathrm{d}\xi \geq 0,\ \forall \vartheta' \in Q, \tag{5.11}$$

$$\vartheta^* \in Q :\ \mathbb{E}[\vartheta^*] = 1. \tag{5.12}$$

Conversely, if there exists a $\mu \in \mathbb{R}$ and $\vartheta^* \in \mathcal{A}$ such that (5.11)-(5.12) *hold, then ϑ^* is an optimal solution of* (5.1).

Proof. We start by noting that the structure of the pointwise constraints in Q forces any ρ-measurable function that satisfies them to be in $L^\infty_\rho(\Xi)$. Therefore, we can consider (5.1) in $L^q_\rho(\Xi)$ with $1 \leq q \leq \infty$ and still obtain the same solution. The choice of $q < \infty$ effects the definition of the tangent and normal cones to Q. In particular, we may use the standard duality pairing between L^s_ρ-spaces with $1 \leq s < \infty$. Therefore, we assume that (5.1) is defined in $L^s_\rho(\Xi)$ with $1 \leq s < \infty$.

Let $G : L^s_\rho(\Xi) \to \mathbb{R}$ be defined simply by $G(q) := \mathbb{E}[q] - 1$ and $K := 0 \in \mathbb{R}$. Then Robinson's constraint qualification is said to hold at a feasible point $\vartheta_0 \in \mathcal{A}$ provided there exists some $\delta > 0$ such that

$$\mathbb{B}_\delta(0) \subset G(\vartheta_0) + DG(\vartheta_0)(Q - \vartheta_0) - K. \tag{RCQ}$$

In our case, (RCQ) reduces to the requirement: There exists some $\tau > 0$ such that for any $\tau' \in (-\tau, \tau)$ there exists $q' \in Q$ for which it holds that $\tau' = \mathbb{E}[q'] - 1$. This can be easily verified. Indeed, if $\tau' \in [0, \frac{1}{1-\beta} - 1]$, then $q' = 1 + \tau'$ is an element of Q.

Hence, $\mathbb{E}[q'] - 1 = \tau'$. Taking $\tau' \in [-1, 0)$, we again set $q' = 1 + \tau'$, for which it holds that $q' \in Q$ and $\mathbb{E}[q'] - 1 = \tau'$. Taking $\tau = \frac{1}{1-\beta} - 1$ completes the proof.

Robinson's constraint qualification guarantees the existence of a Lagrange multiplier. In our setting, this means: ϑ^* is an optimal solution of (5.1) if and only if there exists $\mu \in \mathbb{R}$ such that:

$$0 \in c(z, \cdot) - \varepsilon\vartheta^* - \mu\nabla G(\vartheta^*) - N_Q(\vartheta^*), \quad \mathbb{E}[\vartheta^*] = 1. \tag{5.13}$$

Note that $N_Q(\vartheta^*)$ is the standard normal cone from convex analysis, in the current setting it is defined as follows: If $\vartheta^* \in Q$, then

$$N_Q(\vartheta^*) := \{\lambda \in X^* \mid \langle \lambda, \vartheta' - \vartheta^* \rangle \leq 0,\ \forall \vartheta' \in Q\}, \tag{5.14}$$

otherwise $N_Q(\vartheta^*) = \emptyset$. Here, $\langle \cdot, \cdot \rangle$ is understood as the duality pairing for $X = L^s_\rho(\Xi)$ with its (strong) topological dual $X^* = L^t_\rho(\Xi)$ with $1/t + 1/s = 1$. It follows that the optimality condition (5.13) can be written:

$$\int_\Xi \rho(\xi)(\mu + \varepsilon\vartheta^*(\xi) - c(z, \xi))(\vartheta'(\xi) - \vartheta^*(\xi))\, \mathrm{d}\xi \geq 0,\ \forall \vartheta' \in Q, \text{ and } \vartheta^* \in Q :\ \mathbb{E}[\vartheta^*] = 1.$$

□

The next step is to rewrite (5.11) and (5.12) in a more convenient form. However, this may not always be possible. In the case when $\rho > 0$ almost everywhere with respect to the Lebesgue measure on Ξ and $\varepsilon > 0$, then the following result holds. Such an assumption is reasonable, indeed if a subset $\mathcal{B} \subset \Xi$ exists such that $\rho(\mathcal{B}) = 0$, but the Lebesgue measure is positive, then we have chosen the "wrong" sample space Ξ. One could then replace Ξ with $\widetilde{\Xi} := \Xi \setminus \mathcal{B}$ and ρ with $\tilde{\rho} = \rho|_{\widetilde{\Xi}}$.

PROPOSITION 5.5. *Let $\varepsilon > 0$ and $z \in Y$. If $\rho > 0$ almost everywhere on Ξ with respect to the Lebesgue measure, then the first-order necessary and sufficient optimality conditions* (5.11)-(5.12) *reduce to the following: If ϑ^* is the optimal solution of* (5.1)*, then there exists a multiplier $\mu \in \mathbb{R}$ such that*

$$\vartheta^*(\cdot) = \varepsilon^{-1}(c(z, \cdot) - \mu) + (\varepsilon^{-1}(\mu - c(z, \cdot)))_+ - (\varepsilon^{-1}(c(z, \cdot) - \mu) - 1/(1 - \beta))_+. \tag{5.15}$$

and $\mathbb{E}[\vartheta^*] = 1$. Conversely, if there exists $\mu \in \mathbb{R}$ such that

$$1 = \mathbb{E}\left[\varepsilon^{-1}(c(z) - \mu) + (\varepsilon^{-1}(\mu - c(z)))_+ - (\varepsilon^{-1}(c(z) - \mu) - 1/(1 - \beta))_+\right]. \tag{5.16}$$

then ϑ^* defined by (5.15) *is the optimal solution of* (5.1)*.*

Proof. This result follows from standard results found in [40, L. 2.26, Th. 2.27, Th. 2.28]. □

Using the results derived above, we return to our initial setting and derive the following corollary.

COROLLARY 5.6. *Let $\varepsilon > 0$, $Y = \mathcal{Z}$, $c(\cdot, z) := \int_{\Omega_o} (u(\cdot, x; z) - \bar{u}(x))^2\, \mathrm{d}x \in L^1_\rho(\Xi)$, $d(z) = \frac{\alpha}{2} \int_{\Omega_c} z(x)^2\, \mathrm{d}x$, and u be the solution operator of the PDE* (2.1) *under the standing assumptions. If z^* is an optimal solution, then*

$$z^* = \frac{1}{\alpha} \int_\Xi \rho(\xi)\, \vartheta^*(\xi)\, e_z(u^*(\xi), z^*, \xi)^* \lambda^*(\xi)\, \mathrm{d}\xi, \tag{5.17}$$

where ϑ^* (along with some $\mu \in \mathbb{R}$) satisfies (5.15) *and λ^* solves the adjoint equation associated with $u^* = u(z^*)$ in Assumption* (2.3)*.*

Note that (5.16) can be conveniently rewritten as a fixed point problem in one dimension: Find $\mu \in \mathbb{R}$ such that

$$F(\mu) - \mu = 0, \tag{5.18}$$

where $F(\mu) := \mathbb{E}[c(z)] + \mathbb{E}[(\mu - c(z))_+] - \mathbb{E}[(c(z) - \mu - \varepsilon(1/(1-\beta)))_+] - \varepsilon$. In general, F is Lipschitz continuous with Lipschitz constant 1. In the following result, we provide a formula for the directional derivative.

PROPOSITION 5.7. *The functional F is directionally differentiable with*

$$F'(\mu;\eta) = \eta \Pr[\mathcal{B}_1(\mu)] + (\eta)_+ \Pr[\mathcal{B}_2(\mu)] - \eta \Pr[\mathcal{B}_3(\mu)] - (\eta)_+ \Pr[\mathcal{B}_4(\mu)],$$

where

$$\mathcal{B}_1(\mu) = \{\, \xi \in \Xi \,:\, \mu - c(z) > 0 \,\}, \tag{5.19}$$
$$\mathcal{B}_2(\mu) = \{\, \xi \in \Xi \,:\, \mu - c(z) = 0 \,\}, \tag{5.20}$$
$$\mathcal{B}_3(\mu) = \{\, \xi \in \Xi \,:\, c(z) - \mu - \varepsilon(1/(1-\beta)) > 0 \,\}, \tag{5.21}$$
$$\mathcal{B}_4(\mu) = \{\, \xi \in \Xi \,:\, c(z) - \mu - \varepsilon(1/(1-\beta)) = 0 \,\}, \tag{5.22}$$

and, as before, $\Pr[B] = \int_B \rho(\xi)\mathrm{d}\xi$ *denotes the probability of the event* $B \subseteq \Xi$.

REMARK 5.8. *It follows that F is Gâteaux differentiable whenever* $\Pr[\mathcal{B}_2(\mu)] = \Pr[\mathcal{B}_4(\mu)] = 0$. *In particular, a result by Uryasev [41, 32] shows that if the cumulative distribution,* $\Psi_{c(z)}(t) = \Pr[c(z) \le t]$ *is continuous with respect to t, then* $\Pr[\mathcal{B}_2(\mu)] = \Pr[\mathcal{B}_4(\mu)] = 0$ *and F is continuously differentiable.*

Proof. The proof is standard: one formulates the difference quotients and applies the Lebesgue Dominated Convergence Theorem after exploiting the Lipschitz continuity of the integrand. □

Proposition 5.7, in light of Remark 5.8, shows that the one-dimensional fixed point equation (5.18) is solvable via a standard Newton step (barring certain pathological cases). Moreover, since $F'(\mu;\eta) \in (0,1)$, reaching 1 only in pathological settings, one could also solve (5.18) using a standard fixed point iteration. In future work, we will consider an algorithm based on the simultaneous solution of the optimality conditions. However, this goes beyond the scope of the current paper.

5.4. Consistency. It is important that we investigate the behavior of solutions and stationary points associated with the regularization as $\varepsilon \to 0$. We begin with the following lemma.

LEMMA 5.9. *Let $z \in Y$ and $\varepsilon > 0$. Let $\vartheta_\varepsilon \in \mathcal{A}$ be the (unique) solution of the problem:*

$$\underset{\vartheta \in \mathcal{A}}{\text{maximize}} \quad \int_\Xi \vartheta(\xi)\rho(\xi)c(z,\xi)\,\mathrm{d}\xi + d(z) - \frac{\varepsilon}{2}\int_\Xi \vartheta(\xi)^2\rho(\xi)\,\mathrm{d}\xi.$$

For any sequence $\varepsilon_k \to 0$, there exists a subsequence of solutions $\left\{\vartheta_{\varepsilon_{k_l}}\right\}$ with $\vartheta_{\varepsilon_{k_l}} \overset{*}{\rightharpoonup} \vartheta^*$ in $L^\infty_\rho(\Xi)$, where $\vartheta^* \in \arg\max\left\{ \int_\Xi \vartheta(\xi)\rho(\xi)c(z,\xi)\mathrm{d}\xi + d(z) \,:\, \vartheta \in \mathcal{A} \right\}$.

Proof. Let ε_k and $\vartheta_k := \vartheta_{\varepsilon_k}$ be defined as above. Then $\vartheta_k \in \mathcal{A}$ implies $\{\vartheta_k\}$ is uniformly bounded in $L^\infty_\rho(\Xi)$. It follows that there exists a $\vartheta^* \in L^\infty_\rho(\Xi)$ along with a subsequence $\{\vartheta_{k_l}\}$ such that $\vartheta_{k_l} \to \vartheta^*$ weak* in $L^\infty_\rho(\Xi)$. Since $\mathcal{A}$ is convex and closed in $L^\infty_\rho(\Xi)$, it is sequentially weak* closed. In which case $\vartheta^* \in \mathcal{A}$ [38, L. 4.2].

Turning now to the definition of ϑ_k, we have

$$\int_\Xi \vartheta_k(\xi)\rho(\xi)c(z,\xi)\,\mathrm{d}\xi + d(z) - \frac{\varepsilon_k}{2}\int_\Xi \vartheta_k(\xi)^2\rho(\xi)\,\mathrm{d}\xi \\ \geq \int_\Xi \vartheta(\xi)\rho(\xi)c(z,\xi)\,\mathrm{d}\xi + d(z) - \frac{\varepsilon_k}{2}\int_\Xi \vartheta(\xi)^2\rho(\xi)\,\mathrm{d}\xi,\ \forall \vartheta \in \mathcal{A}$$

Since $\rho \geq 0$ almost everywhere, we have for every $\vartheta \in \mathcal{A}$:

$$\int_\Xi \vartheta_k(\xi)\rho(\xi)c(z,\xi)\,\mathrm{d}\xi + d(z) \geq \int_\Xi \vartheta(\xi)\rho(\xi)c(z,\xi)\,\mathrm{d}\xi + d(z) - \frac{\varepsilon_k}{2}\int_\Xi \vartheta(\xi)^2\rho(\xi)\,\mathrm{d}\xi.$$

Replacing k with k_l and passing to the limit proves the assertion. □

Continuing, we prove the following technical lemma.

LEMMA 5.10. *Let $\varepsilon > 0$, D_ε be defined as in (5.2), and z_ε a minimizer of D_ε. If $c(\cdot,\xi) \geq 0$ a.e. in Ξ for all $z \in Y$ and d satisfies there exists a constant $\tau > 0$ such that $d(z) \geq \tau||z||^2$ for all $z \in Y$. Then the set $\{z_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in Y.*

REMARK 5.11. *The assumptions on c and d are consistent with the motivating tracking-type objective function.*

Proof. By definition,

$$d(z_\varepsilon) + \sup\left\{ \int_\Xi \vartheta(\xi)\rho(\xi)c(z_\varepsilon,\xi)\,\mathrm{d}\xi - \frac{\varepsilon}{2}\int_\Xi \vartheta(\xi)^2\rho(\xi)\,\mathrm{d}\xi \ :\ \vartheta \in \mathcal{A} \right\} \\ \leq d(z) + \sup\left\{ \int_\Xi \vartheta(\xi)\rho(\xi)c(z,\xi)\,\mathrm{d}\xi - \frac{\varepsilon}{2}\int_\Xi \vartheta(\xi)^2\rho(\xi)\,\mathrm{d}\xi \ :\ \vartheta \in \mathcal{A} \right\},\ \forall z \in Y.$$

Then by setting $z = 0$ on the right hand side of the inequality and using the fact that $\mathcal{A}$ is closed and bounded, we deduce the existence of a constant $K \in \mathbb{R}$ such that

$$d(z_\varepsilon) + \sup\left\{ \int_\Xi \vartheta(\xi)\rho(\xi)c(z_\varepsilon,\xi)\,\mathrm{d}\xi - \frac{\varepsilon}{2}\int_\Xi \vartheta(\xi)^2\rho(\xi)\,\mathrm{d}\xi \ :\ \vartheta \in \mathcal{A} \right\} \leq K.$$

Thus, the coercivity of d, the almost everywhere nonnegativity of c, and the bounds on $\vartheta \in \mathcal{A}$ imply

$$\kappa||z_\varepsilon||_Y^2 \leq K + \varepsilon K',$$

where $K' = \frac{(1-\beta)^2}{2}$, from which the assertion follows. □

In order to prove the convergence of (weak) accumulation points to minimizers of the minimax problem, we employ a well-known result from the theory of variational convergence.

THEOREM 5.12. *Define the family of functions $\{D_\varepsilon\}_{\varepsilon>0}$ as in (5.2). If $z \mapsto c(z,\cdot) : Y \to L^1_\rho(\Xi)$ is weakly continuous and d is sequentially weakly lower semicontinuous on Y. Then D_ε epiconverges in the sense of Mosco to the function D defined by:*

$$D(z) := \sup\left\{ \int_\Xi \vartheta(\xi)\rho(\xi)c(z,\xi)\,\mathrm{d}\xi + d(z) \ :\ \vartheta \in \mathcal{A} \right\}. \tag{5.23}$$

REMARK 5.13. *One could perhaps weaken the assumptions, however, for our application they are sufficient.*

Proof. Fix an arbitrary $z \in Y$ and let $z_k \rightharpoonup z$ and $\varepsilon_k \downarrow 0$. Then

$$D_{\varepsilon_k}(z_k) \geq \int_\Xi \vartheta(\xi)\rho(\xi)c(z_k,\xi)\,\mathrm{d}\xi - \frac{\varepsilon_k}{2}\int_\Xi \vartheta(\xi)^2\rho(\xi)\,\mathrm{d}\xi + d(z_k), \forall \vartheta \in \mathcal{A}$$

In particular, we may select any $\vartheta^* \in \mathcal{A}$ such that $D(z) = \int_\Xi \vartheta^*(\xi)\rho(\xi)c(z,\xi)\,\mathrm{d}\xi + d(z)$. Therefore, $\liminf_{k\to\infty} D_{\varepsilon_k}(z_k) \geq D(z)$. Finally, let $z \in Y$ be arbitrary, $\varepsilon_k \downarrow 0$, and $z_k \equiv z$. Then $D_{\varepsilon_k}(z_k) = D_{\varepsilon_k}(z) \leq \int_\Xi \vartheta_k(\xi)\rho(\xi)c(z,\xi)\,\mathrm{d}\xi + d(z)$ along with the closed and boundedness of $\mathcal{A}$ in $L^\infty_\rho(\Xi)$, implies there exists some $D^* \in \mathbb{R}$ such that

$$\limsup_{k\to\infty} D_{\varepsilon_k}(z_k) = D^*.$$

Let $\{k_m\} \subset \mathbb{N}$ be a subsequence such that

$$\limsup_{k\to\infty} D_{\varepsilon_k}(z_k) = \lim_{m\to\infty} D_{\varepsilon_{k_m}}(z_{k_m}) = D^*$$

According to Lemma 5.9, there exists a subsequence $\{k_{m_l}\} \subset \mathbb{N}$ such that

$$\begin{aligned} D_{\varepsilon_{k_{m_l}}}(z_{k_{m_l}}) &\to \int_\Xi \vartheta^*(\xi)\rho(\xi)c(z,\xi)\,\mathrm{d}\xi + d(z) \\ &= \max\left\{ \int_\Xi \vartheta(\xi)\rho(\xi)c(z,\xi)\,\mathrm{d}\xi + d(z) \;:\; \vartheta \in \mathcal{A} \right\} = D(z). \end{aligned}$$

But then $\limsup_{k\to\infty} D_{\varepsilon_k}(z_k) = \lim_{m\to\infty} D_{\varepsilon_{k_m}}(z_{k_m}) = D^* = D(z)$. ∎

COROLLARY 5.14. *Under the standing data assumptions, let $\varepsilon_k \downarrow 0$ and let $z_{\varepsilon_k} \in Y$ be a minimizer of D_{ε_k}. Then there exists a subsequence $\{k_l\}$ such that $z_{k_l} \overset{Y}{\rightharpoonup} z^*$ where z^* is a minimizer of D.*

Proof. According to Lemma 5.10, the sequence $\{z_k\}$ is uniformly bounded in Y. Since Y is assumed to be a reflexive Banach space, there exists a subsequence $\{z_{k_l}\}$ and $z^* \in Y$ with $z_{k_l} \overset{Y}{\rightharpoonup} z^*$. The assertion follows immediately in light of Theorem 5.12 and the definition of Mosco epiconvergence. ∎

This concludes our study of the consistency of the regularized minimax problem. In the future, we plan to investigate the convergence analysis of the optimality conditions and path-following as in [17, 16] which may be required for the development of efficient numerical methods.

6. Numerical Results.

6.1. Optimal Control of an Elliptic Equation with Discontinuous Coefficients. We consider the optimal control of a linear elliptic PDE with discontinuous conductivity. For this problem, the location of the discontinuity is uncertain. This problem was studied in [23]. Let $\alpha = 10$, $\Omega_o = \Omega_c = \Omega = (-1,1)$, and $\bar{u} \equiv 1$. Consider the optimal control problem

$$\underset{z\in L^2(-1,1)}{\text{minimize}} \quad J(z) = \frac{1}{2}\mathrm{CVaR}_\beta\left[\int_{-1}^1 (u(\cdot,x;z)-1)^2\,\mathrm{d}x\right] + \frac{\alpha}{2}\int_{-1}^1 z(x)^2\,\mathrm{d}x$$

where $u = u(z) \in L^2_\rho(\Xi; H^1_0(-1,1))$ solves the weak form of

$$-\partial_x\left(\epsilon(\xi,x)\partial_x u(\xi,x)\right) = f(\xi,x) + z(x) \qquad (\xi,x) \in \Xi\times\Omega, \tag{6.1a}$$

$$u(\xi,-1) = 0, \quad u(\xi,1) = 0 \qquad \xi \in \Xi. \tag{6.1b}$$

$\Xi = [-0.1, 0.1] \times [-0.5, 0.5]$ is endowed with the uniform density $\rho(\xi) \equiv 5$, and the random field coefficients are

$$\epsilon(\xi, x) = \epsilon_l \chi_{(-1,\xi_1)} + \epsilon_r \chi_{(\xi_1,1)}, \quad \text{with} \quad \epsilon_l = 0.1,\ \epsilon_r = 10,$$

and $f(\xi, x) = \exp(-(x-\xi_2)^2)$. One can show that for any $z \in L^2(-1,1)$, the solution to (6.1) has a unique solution $u = u(z) \in L^2_\rho(\Xi; H^1_0(\Xi))$ and $\xi \mapsto u(\xi)$ is continuous (see [23]). Hence, $u(z) \in L^\infty_\rho(\Xi; H^1_0(\Xi))$ and the tracking term $\int_{-1}^1 (u(\cdot, x; z) - 1)^2 \, \mathrm{d}x \in L^\infty_\rho(\Xi) \subset L^2_\rho(\Xi)$. Therefore, Corollaries 4.8 and 4.9 imply the smoothed CVaR objective function is twice continuously Fréchet differentiable for certain smoothing distributions, δ.

To discretize the PDE in space, we use continuous piecewise linear finite elements built on a mesh of 256 intervals. The mesh for the state variable changes for each random sample. The state mesh is subdivided into to two uniform meshes of 128 intervals, one for $(-1, \xi_1)$ and one for $(\xi_1, 1)$. The mesh for the control variable is uniform on the subdomains $(-1, -0.75)$, $(-0.75, -0.25)$, $(-0.25, 0.25)$, and $(0.25, 1)$. The first subdomain contains $\frac{1}{4}$ of the mesh elements, the second contains $\frac{1}{16}$ of the mesh elements, the third contains $\frac{9}{16}$ of the mesh elements, and the final contains the remaining $\frac{1}{8}$ of the mesh elements. We chose this subdivision because of the nature of the optimal solution.

For the primal approach, we consider two discretizations of the Ξ variables. First, we discretize Ξ using $Q = 10{,}000$ Monte Carlo samples. For our second discretization, we use level 8 isotropic sparse grids built on one-dimensional Gauss-Patterson quadrature points and weights. This sparse grid is exact for polynomials of total degree 35 (see [28, Th. 1] and [14, Sect. 3.3] for more details). Moreover, note that small smoothing parameters ε may have an adverse effect on the quadrature error. Nevertheless, sparse grids lead to a considerable reduction in computational effort for fixed ε. For example, solving the sparse-grid problem with $\beta = 0.95$ and $\varepsilon = 10^{-2}$ requires 147,026 PDE solves as opposed to 790,000 PDE solves for the Monte Carlo problem.

We solve the resulting discretized, smoothed CVaR optimization problem for $\beta \in \{0.05, 0.5, 0.95\}$ using a classic trust-region algorithm with truncated conjugate gradients (CG). For the trust-region iteration, we terminated when the norm of the gradient was less than 10^{-8}. For the truncated CG iteration, we terminated when one of the following occurred: the maximum number of iterations (100) was exceeded, the step was larger than the trust-region radius, negative curvature was detected or the relative residual was less than 10^{-4}. The iteration counts for the trust-region procedure applied to the sparse-grid discretized problem with probability levels $\beta \in \{0.05, 0.5, 0.95\}$ are listed in Table 6.1. The average number of CG iterations per trust-region iteration rounded to the nearest integer is included in parentheses. Note that when the number of trust-region iterations is large, the number of CG iterations is typically small. This is due to the algorithm detecting unsatisfactory steps and shrinking the trust-region radius.

Figure 6.1 displays the computed error in the value-at-risk, the error in the optimal control, and the total error. The slope of the error in the value-at-risk is $v = -1.685$, the slope of the error in the control is $v = -1.203$, and the slope of the total error is $v = -1.198$. To compute these errors, we employ the aforementioned Monte Carlo discretization and then solve the nonsmooth discretized CVaR problem using a bundle-type method for nonsmooth optimization [34]. We terminate the bundle method when the aggregate subgradient and aggregate linearization error are below 10^{-8}, resulting

		log(ε)						
		0	-1	-2	-3	-4	-5	-6
	0.05	6(4)	9(3)	10(3)	13(3)	20(2)	30(1)	33(1)
β	0.5	7(3)	11(3)	12(3)	15(3)	21(2)	35(2)	54(2)
	0.95	7(3)	13(2)	15(2)	22(1)	24(1)	39(1)	46(1)

Table 6.1: The iteration counts for the primal approach. The numbers in parentheses are the average number of truncated CG iterations per trust-region iteration rounded to the nearest integer.

in an approximate solution that satisfies

$$J(x_k) \leq J(x) + 10^{-8}\|x - x_k\| + 10^{-8} \quad \forall\, x \in \mathbb{R} \times \mathcal{Z}_h$$

where $\mathcal{Z}_h$ denotes the finite-dimensional control approximation space [34, L. 2.2]. We then solve the discretized smooth CVaR problems using the previously mentioned trust-region algorithm. We terminate the trust-region iteration when the norm of the gradient is below 10^{-8}. Since we are using Monte Carlo, we can consider the discretization of Ξ as replacing the continuous probability measure with a sum of point masses centered at the sample points. Thus, the results of Theorem 4.13 directly apply. As can be seen in Figure 6.1, our numerical results confirm the theoretical results proved in Theorem 4.13.

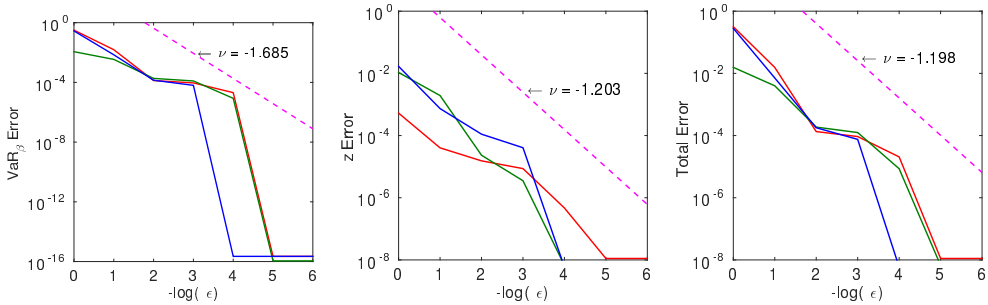

Fig. 6.1: Left to right: a) Error in the value-at-risk for $\beta = 0.05$ (blue), $\beta = 0.5$ (green), and $\beta = 0.95$ (red). b) Error in the optimal control for $\beta = 0.05$ (blue), $\beta = 0.5$ (green), and $\beta = 0.95$ (red). c) Total error for $\beta = 0.05$ (blue), $\beta = 0.5$ (green), and $\beta = 0.95$ (red).

For the dual approach, we solve the minimax problem using an alternating fixed-point iteration. Our fixed-point algorithm consists of the following steps:

1. Given the current control, solve the state equation;
2. Solve (5.16) for μ using bisection;
3. Solve the adjoint equation and update the control using equation (5.17).

Since this is a fixed-point iteration, α must be sufficiently large to ensure the optimality system is contractive. Moreover, since the dual approach requires a high-dimensional integral of a nonsmooth function, deterministic quadrature may not be a suitable discretization for the Ξ variables. Instead we use Monte Carlo with $Q = 10{,}000$ samples. We terminated bisection when the residual given by (5.16) was

less than 10^{-10}. We terminated the fixed point iteration when $\|\alpha(z_{k-1} - z_k)\|_{\mathcal{Z}}$ was less than 10^{-8}. Note that the difference between consecutive steps scaled by α is

$$\alpha(z_{k-1} - z_k) = \alpha z_{k-1} - \int_\Xi \rho(\xi)\vartheta_{k-1}(\xi)\lambda_{k-1}(\xi)\,\mathrm{d}\xi$$

which is the Gâteaux derivative of D_ε (recall Theorem 5.1). Thus, the alternating fixed-point iteration is exactly steepest descent with step scaling α^{-1}. Here, λ_{k-1} and ϑ_{k-1} denote the adjoint variable and dual distribution corresponding to z_{k-1}, respectively. Table 6.2 contains the iteration counts for our dual approach. The numbers in parentheses are the average number of bisection iterations per dual iteration rounded to the nearest integer. Notice that as the regularization parameter ε decreases, the number of bisection iterations also decreases. This is because the initial lower bound used in the bisection iteration increases as ε decreases, while the initial upper bound remains fixed. That is, the effective search area decreases with ε. Figure 6.2 depicts

		$\log(\varepsilon)$								
		0	-1	-2	-3	-4	-5	-6	-7	-8
β	0.05	9(30)	9(30)	9(27)	9(27)	9(30)	9(28)	9(9)	9(7)	9(7)
	0.5	9(32)	10(33)	9(32)	9(32)	9(32)	9(29)	9(26)	9(9)	9(9)
	0.95	9(34)	11(34)	9(34)	9(32)	9(32)	9(7)	9(5)	9(5)	9(5)

Table 6.2: The iteration counts for the dual approach. The numbers in parentheses are the average number of bisection iterations rounded to the nearest integer.

the dual distributions ϑ^* sampled at the $Q = 10{,}000$ Monte Carlo samples for varying probability levels $\beta \in \{0.05, 0.5, 0.95\}$. The distributions only take values at the lower and upper bounds, 0 and $(1-\beta)^{-1}$, respectively. This observation points to one possible advantage of an algorithm based on the dual approach. The advantage is that the computation of the dual distribution only depends on the state variable, thus to update the controls one only needs to solve the adjoint equation for those values of ϑ which are nonzero. This could drastically reduce the number of PDE solves required to solve the optimization problem. This could further lead to significant computational savings in the case of optimization governed by large-scale time-dependent PDEs in which checkpointing is required. The red dots correspond to ϑ^* values of $(1-\beta)^{-1}$ while the blue dots correspond to ϑ^* values of 0. For each β, the number of red dots is $(1-\beta)Q$.

6.2. Optimal Control of Burger's Equation. In this section, we consider the optimal control of the steady Burger's equation with uncertain coefficients. This problem was studied in [23]. Let $\alpha = 10^{-3}$, $\Omega_o = \Omega_c = \Omega = (0,1)$, and $\bar{u} \equiv 1$. Consider the optimal control problem

$$\underset{z \in L^2(0,1)}{\text{minimize}} \quad J(z) = \frac{1}{2}\mathrm{CVaR}_\beta\left[\int_0^1 (u(\cdot,x;z) - 1)^2\,\mathrm{d}x\right] + \frac{\alpha}{2}\int_0^1 z(x)^2\,\mathrm{d}x$$

where $u = u(z) \in L^3_\rho(\Xi; H^1(0,1))$ solves the weak form of

$$-\nu(\xi)\partial_{xx}u(\xi,x) + u(\xi,x)\partial_x u(\xi,x) = f(\xi,x) + z(x) \qquad (\xi,x) \in \Xi \times \Omega, \tag{6.2a}$$

$$u(\xi,0) = d_0(\xi), \quad u(\xi,1) = d_1(\xi) \qquad \xi \in \Xi. \tag{6.2b}$$

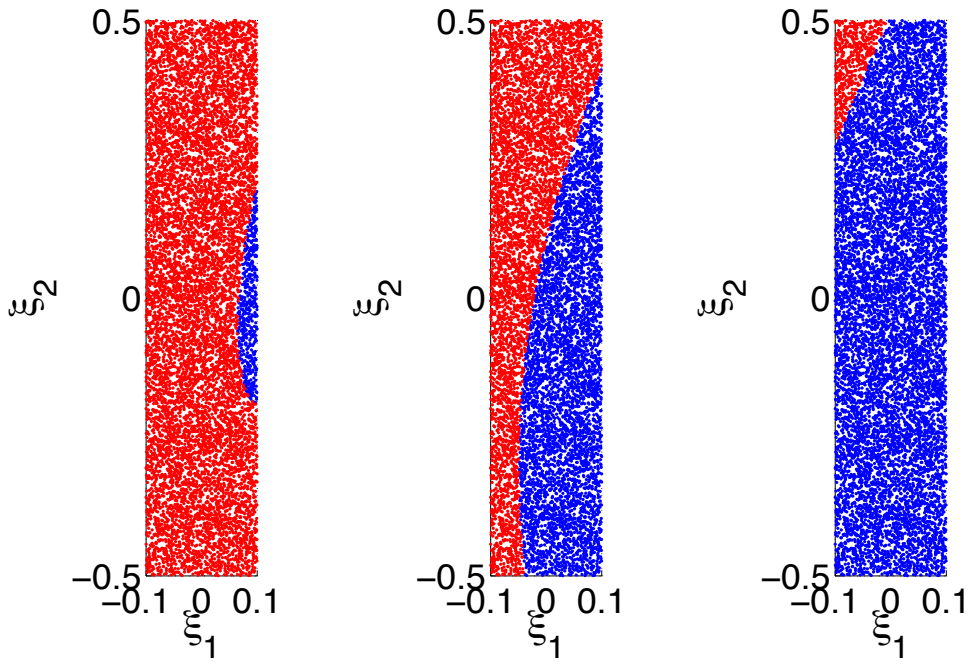

Fig. 6.2: The resulting optimal probability density from the dual approach, ϑ^* for $\beta = 0.05$ (left), $\beta = 0.5$ (center), and $\beta = 0.95$ (right). The red dots correspond to ϑ^* values of $(1-\beta)^{-1}$ and the blue dots correspond to ϑ^* values of 0.

$\Xi = [-1,1]^4$ is endowed with the uniform density $\rho(\xi) \equiv 2^{-4}$, and the random field coefficients are

$$\nu(\xi) = 10^{\xi_1 - 2}, \quad f(\xi, x) = \frac{\xi_2}{100}, \quad d_0(\xi) = 1 + \frac{\xi_3}{1000}, \quad \text{and} \quad d_1(\xi) = \frac{\xi_4}{1000}.$$

Fix $\xi \in \Xi$, then [43, Th. 2.13] ensures that equation (6.2) has a solution for any $z \in L^2(0,1)$ and this solution is unique if $\nu(\xi)$ is sufficiently large. Moreover, if $z \in L^2(0,1) \cap C[0,1]$, then [43, Th. 2.17] ensures that $u(\xi) \in C^2(0,1) \cap C[0,1]$ for all $\xi \in \Xi$ and $u_0(x) \leq u(\xi, x) \leq u_1(x)$ where

$$\begin{cases} u_0(x) = -0.001 - \|z\|_{C[0,1]} + x \\ u_1(x) = \max\left\{0.001 + \|z\|_{C[0,1]}, 1.001\right\} + x. \end{cases}$$

Therefore, if $z \in L^2(0,1) \cap C[0,1]$, then $u \in L^\infty_\rho(\Xi; H^1(0,1)) \subset L^3_\rho(\Xi; H^1(0,1))$. Moreover, this implies that the tracking term $\int_0^1 (u(\cdot, x; z) - 1)^2 \, \mathrm{d}x \in L^2_\rho(\Xi)$ and Corollaries 4.8 and 4.9 ensure the smoothed CVaR objective function is twice continuously Fréchet differentiable for certain smoothing distributions, δ.

The optimal control z^* exhibits boundary layers near $x = 0$ and $x = 1$ while the optimal state $u^* = u(z^*)$ exhibits a boundary layer near $x = 1$. To reduce the effects of these layers, we discretize Burger's equation using continuous piecewise linear finite elements built on a piecewise uniform mesh of 256 intervals. To build the mesh, we partition the domain into three subdomains $\Omega \cup \partial\Omega = [0, 0.2] \cup (0.2, 0.8) \cup$

$[0.8, 1]$. We mesh the first subdomain, $[0, 0.2]$, with 80 uniform intervals; we mesh the second subdomain, $(0.2, 0.8)$, 16 uniform intervals; and we mesh the third subdomain, $[0.8, 1]$, with 160 uniform intervals. We adopt the same discretization for the control. As with the previous example, we investigate two different discretizations for the Ξ variables: Monte Carlo with $Q = 10{,}000$ samples and a level 8 isotropic Smolyak sparse grid built on one-dimensional Clenshaw-Curtis quadrature. This sparse grid is exact for polynomials of total degree 9 (see [28, Th. 1] and [14, Sect. 3.2] for more information). Again note that small smoothing parameters ε may have adverse effects on the quadrature error. We then solve the discretized nonlinear PDE at each quadrature point using Newton's method globalized with a backtracking line search.

For the primal approach, we solve the smoothed optimization problem using a classic trust-region algorithm with truncated conjugate gradients. We vary β between 0.1 and 0.9 in increments of 0.1. Moreover, we vary $\log_{10}(\varepsilon)$ between -5 and 0 in increments of 1. The optimal controls, value-at-risks, and the cumulative distribution function of the random variable tracking term

$$\xi \mapsto T(\xi; z) = \int_0^1 (u(\xi, x; z_\beta^*) - 1)^2 \, \mathrm{d}x$$

are plotted in Figure 6.3. We generated these results using the aforementioned sparse-grid discretization. We plot the optimal controls for $\beta \in \{0.1, 0.5, 0.9\}$ and $\varepsilon = 10^{-5}$ in the left image and the value-at-risk for $\varepsilon = 10^{-5}$ in the center. The CDFs are computed using sample average approximation. We increase the number of Monte Carlo samples by 10^3 until the ℓ^∞-norm error between subsequent approximations of the CDFs is less than 10^{-4}. For $\beta = 0.1$ we required 19,000 samples; for $\beta = 0.5$ we required 17,000 samples; and for $\beta = 0.9$ we required 23,000 samples. Notice that the optimal controls corresponding to increasing β reduce the variability in the tracking term.

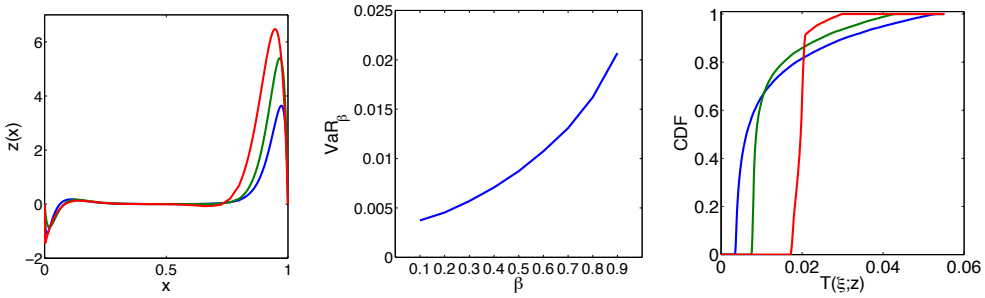

Fig. 6.3: Left: Optimal controls for $\beta = 0.1$ (blue), $\beta = 0.5$ (green), and $\beta = 0.9$ (red). Center: Value-at-risk for varying β. Right: Cumulative distribution function of tracking term for $\beta = 0.1$ (blue), $\beta = 0.5$ (green), and $\beta = 0.9$ (red).

In Figure 6.4, we plot the error for decreasing ε. Figure 6.4-a depicts the value-at-risk error which is decreasing at a rate of $v = -1.116$ with respect to ε. Figure 6.4-b depicts the control error which is decreasing at a rate of $v = -0.840$. Figure 6.4-c depicts the total error which is decreasing at a rate of $v = -0.871$. To compute the error, we use the previously mentioned Monte Carlo discretization and then solve the nonsmooth discretized CVaR problem using a bundle-type method for nonsmooth

optimization [34]. As in the prior example, we terminate the bundle method when the aggregate subgradient and aggregate linearization error are below 10^{-8}. We then solve the discretized smooth CVaR problems using the trust-region algorithm (we terminate when the norm of the gradient is below 10^{-8}). As can by seen in Figure 6.4, our numerical results confirm the theoretical results proved in Theorem 4.13.

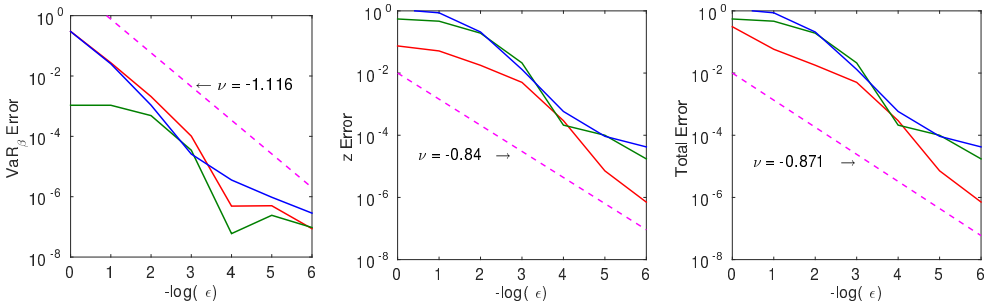

Fig. 6.4: Left to right: a) Error in the value-at-risk for $\beta = 0.1$ (blue), $\beta = 0.5$ (green), and $\beta = 0.9$ (red). b) Error in the optimal control for $\beta = 0.1$ (blue), $\beta = 0.5$ (green), and $\beta = 0.9$ (red). c) Total error for $\beta = 0.1$ (blue), $\beta = 0.5$ (green), and $\beta = 0.9$ (red).

The parameter configuration for this problem is not amenable to the fixed-point iteration proposed in Section 6.1. We can instead use the control computed via the primal approach along with the dual analysis to generate the dual distribution ϑ^*. To generate ϑ^*, we solved the state equation at 100,000 Monte Carlo samples and then solved (5.16) with dual regularization parameter $\varepsilon = 10^{-8}$ to determine the distribution. As in Section 6.1, the dual distribution takes either the value $\vartheta^*(\xi) = 0$ or the value $\vartheta^*(\xi) = (1-\beta)^{-1}$. Figure 6.5 plots the histograms of the samples corresponding to $\vartheta^*(\xi) = (1-\beta)^{-1}$. The columns correspond to $log_{10}(\nu(\xi))$ (first), $f(\xi)$ (second), $d_0(\xi)$ (third), and $d_1(\xi)$ (fourth). The rows correspond to $\beta = 0.1$ (top), $\beta = 0.5$ (middle), and $\beta = 0.9$ (bottom). The histogram for $log_{10}(\nu(\xi))$ clearly demonstrates that the most challenging scenarios to control correspond to $\nu(\xi) \approx 10^{-3}$ and $\nu(\xi) \approx 10^{-1}$. Similarly the histogram for $f(\xi)$ shows more emphasis on larger values of $f(\xi)$. Finally, the histograms for $d_0(\xi)$ and $d_1(\xi)$ appear to be uniformly distributed. This suggests that the optimal control problems is less sensitive to the uncertainty in the boundary conditions.

7. Conclusions. In engineering optimization, it is essential to determine controls and designs which are, in some sense, robust to uncertainty. The conditional value-at-risk is a natural vehicle for producing such controls. In this paper, we have developed and analyzed two approaches for minimizing the conditional value-at-risk for PDE-constrained optimization problems. The primal approach is motivated by the desire to use gradient-based optimization and involves smoothing the plus function $(\cdot)^+$. We have demonstrated that this results in a consistent approximation, proving that smoothed CVaR is a convex risk measure and is at least Hadamard differentiable. We have also demonstrated that the minimizers of our smoothed approximations converge to minimizers of the true CVaR optimization problem, giving an explicit error bound under a convexity assumption. We do not discuss efficient numerical optimization methods for solving this problem although some methods have been developed in

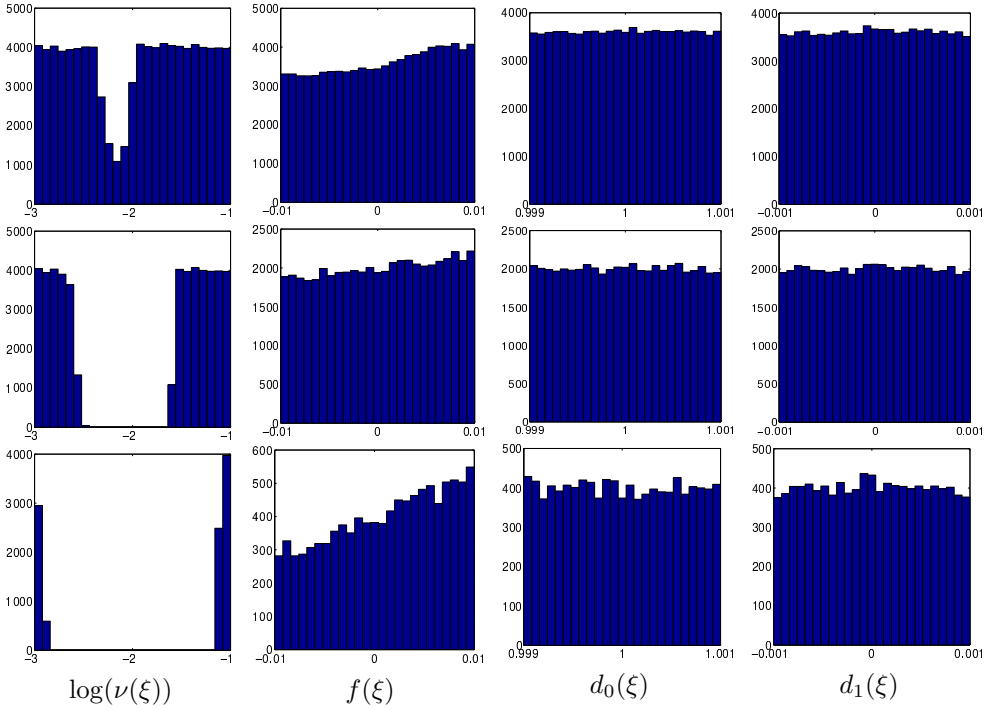

Fig. 6.5: The rows correspond to $\beta = 0.1$ (top), $\beta = 0.5$ (middle), $\beta = 0.9$ (bottom). The columns contain the histograms of the samples with $\vartheta^* = (1-\beta)^{-1}$. From left to right: a) $\log_{10}(\nu(\xi))$; b) $f(\xi)$; c) $d_0(\xi)$; d) $d_1(\xi)$.

[23, 22]. On the other hand, the dual approach is based on Fenchel-Moreau duality. We derive rigorous optimality conditions for the resulting minimax problem and show that to solve the inner maximization problem we only need to solve a one-dimensional nonlinear equation. This result is a possible backbone for numerical methods for solving the minimax problem. We have provided a simple numerical scheme using these conditions. Further development of numerical methods for the dual approach will be investigated in future work.

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