

OPTIMAL CONTROL OF LOW-FREQUENCY ELECTROMAGNETIC FIELDS IN MULTIPLY CONNECTED CONDUCTORS *

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Abstract. Several classes of optimal control of electromagnetic fields are considered. Special emphasis is laid on a non-standard H -based formulation of the equations of electromagnetism in multiply connected conductors. By this technique, the Maxwell equations can be solved with reduced computational complexity. While the magnetic field H in the conductor is obtained from an elliptic equation with the curl σ^{-1} curl operator, an elliptic equation with the $\operatorname{div} \mu \nabla$ operator is set up for a potential ψ in the isolator. Both equations are coupled by appropriate interface conditions. In all problems, the electrical current is controlled in the conducting domain. Several types of control functions are discussed. In particular, the problem of sparse optimal control is investigated in a package of electrical wires. For all problems, the associated sensitivity analysis is performed.

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1. Introduction. Our paper is a contribution to the fast developing numerical analysis of optimal control of electromagnetic fields. In associated mathematical models, often a vector potential ansatz is used for the magnetic induction B , namely $B = \operatorname{curl} A$. In this case, the associated Maxwell equations have to be solved for a 3D vector formulation in the whole computational domain. This domain should be taken sufficiently large, so that the choice of standard boundary conditions will guarantee a sufficiently precise solution.

The potential ansatz has the advantage of a certain simplicity but suffers in some sense from a quite large computational complexity. Instead, a complicated geometry of the conducting domain does not lead to essential difficulties.

Another way of modeling is the H -based eddy current formulation, where a standard scalar elliptic equation is given in the insulator and a vector formulation is only needed in the conductor. This approach is theoretically slightly more complicated, since several additional conditions must be required, if the conductor is not simply connected. On the other hand, the computational savings can be considerable, if the domain Ω that contains the whole setting has to be chosen large. Think of a torus of moderate thickness with very big radius. The numerical analysis of underlying models in E - or H -formulation is discussed extensively in [2].

The main aim of our paper is to study such an H -based formulation in the control of electric and magnetic fields. In this way, our problems will be close to the setting in [24], [25], but the associated mathematical analysis is essentially different.

Optimal control of electromagnetic fields is a quite active subject, important for

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various applications. We mention the control of induction heating as in [16], [17], [30], heat sources such as in [29], the optimal control of MHD processes as in [4], [10], [12], [13], [15], [18], [19], [26], optimal control problems for time-harmonic eddy current problems as in [21], [22], inverse problems for electromagnetic fields as in [3], or the control of magnetic fields in flow measurement as in [24], [25].

2. Models of electromagnetism.

2.1. Time-harmonic Maxwell and eddy current equations. For establishing our eddy current formulation, we follow [2] and begin with the standard Maxwell system

$$\frac{\partial D}{\partial t} + J = \operatorname{curl} H \quad (2.1)$$

$$\frac{\partial B}{\partial t} + \operatorname{curl} E = 0 \quad (2.2)$$

$$\operatorname{div} D = \rho \quad (2.3)$$

$$\operatorname{div} B = 0, \quad (2.4)$$

where B , H , D , and E denote the magnetic induction, the magnetic field, the electric induction, and the electric field, respectively.

These fields are related through some constitutive equations. A linear dependence of the form $D = \varepsilon E$, $B = \mu H$ is usually assumed, where the symmetric and (uniformly) positive definite matrices ε and μ are called electric permittivity and magnetic permeability, respectively. We assume that the entries of ε and μ are bounded and measurable real functions on Ω .

The (total) current J is the sum of the generated current and an impressed current J_e . By the generalized Ohm's law, we have

$$J = \sigma E + J_e, \quad (2.5)$$

where σ is the electrical conductivity, that is assumed to be a symmetric and (uniformly) positive definite matrix in the conducting region and to vanish in the insulating region. Again, we assume that the entries of σ are bounded and measurable real functions on Ω_C .

In time-harmonic models, it is assumed that J_e is an alternating current of the form

$$J_e(\mathbf{x}, t) = J(\mathbf{x}) \cos(\omega t + \phi),$$

where J is a real vector function that accounts for direction and strength of the current, ω is the angular frequency and ϕ is the phase angle. Expressing these quantities in a complex setting, we have

$$J_e(\mathbf{x}, t) = \operatorname{Re} [J(\mathbf{x}) e^{i\omega t + i\phi}] = \operatorname{Re} [\mathbf{J}_e(\mathbf{x}) e^{i\omega t}].$$

The complex vector function \mathbf{J}_e will be our control; we assume that it is supported in the conducting region, namely, it is vanishing inside the non-conducting region.

This time-periodic impressed current J_e generates associated time-periodic solutions in the form

$$E(\mathbf{x}, t) = \operatorname{Re} [\mathbf{E}(\mathbf{x}) e^{i\omega t}], \quad H(\mathbf{x}, t) = \operatorname{Re} [\mathbf{H}(\mathbf{x}) e^{i\omega t}].$$

Inserting these quantities in the Maxwell equations, and using $D = \varepsilon E$ and $B = \mu H$, we finally arrive in a standard way at the equations of the *time-harmonic Maxwell system*

$$\begin{aligned} \operatorname{curl} \mathbf{H} - (i\omega\varepsilon + \sigma)\mathbf{E} &= \mathbf{J}_e \\ \operatorname{curl} \mathbf{E} + i\omega\mu\mathbf{H} &= \mathbf{0}. \end{aligned}$$

We shall assume that the term $i\omega\varepsilon\mathbf{E}$ can be neglected (this is often the case for low-frequency problems). Thus we end up with the *time-harmonic eddy current system*

$$\begin{aligned} \operatorname{curl} \mathbf{H} - \sigma\mathbf{E} &= \mathbf{J}_e \\ \operatorname{curl} \mathbf{E} + i\omega\mu\mathbf{H} &= \mathbf{0} \end{aligned} \tag{2.6}$$

that holds in the whole space \mathbb{R}^3 .

2.2. Eddy current formulation in weak and strong form.

Assumption 2.1 (Geometry). In the paper, $\Omega \subset \mathbb{R}^3$ is a bounded and simply connected Lipschitz domain with connected boundary Γ ; Ω is the “holdall” computational domain containing all conductors. The subdomain $\Omega_C \subset \Omega$ that denotes the conductor is a bounded Lipschitz set. We require that Ω_C is the union of finitely many disjoint open and connected sets $(\Omega_C)_l$, $l \in \{1, \dots, k\}$, the so-called (connected) components of Ω_C . Assume further that $\operatorname{cl} \Omega_C \cap \partial\Omega = \emptyset$. The set $\Omega_I := \Omega \setminus \operatorname{cl} \Omega_C$ stands for the non-conducting domain. For simplicity, it is assumed to be connected.

DEFINITION 2.2. Let $g \in \mathbb{N} \cup \{0\}$ be the number of all “handles” of Ω_I (precisely, the rank of the first homology group of $\operatorname{cl} \Omega_I$, or, equivalently, the first Betti number of Ω_I). Due to our assumption on Ω , it is also the number of “handles” of Ω_C . If all the components $(\Omega_C)_l$ are simply connected, we have $g = 0$.

This assumption allows fairly general forms of conductors (see Figure 2.1). For instance, the conducting domain can include finitely many tori which might form together more complicated geometrical figures like the Borromean rings. Also any knot (for example, a trefoil knot) is allowed as a conducting domain. We mention these geometrical examples, since their shape requires additional mathematical conditions for the well-posedness of our equations.

The function spaces used in our paper are defined over the field of complex numbers. For instance, $L^p(D)$, $1 \leq p < \infty$, is defined as the space of all complex valued functions $v : D \rightarrow \mathbb{C}$ such that $|v|^p$ is integrable on $D \subset \mathbb{R}^3$. To distinguish this space from the one with real-valued functions, we introduce

$$L_{\mathbb{R}}^p(D) = \{v : D \rightarrow \mathbb{R}, |v|^p \text{ is integrable}\}.$$

The spaces $L^\infty(D)$ (complex) and $L_{\mathbb{R}}^\infty(D)$ (real) are defined accordingly.

DEFINITION 2.3. We denote by ρ_j , $j \in \{1, \dots, g\}$, a basis of the space of μ -harmonic fields

$$\mathcal{H}_I^\mu = \{\mathbf{v} : \Omega_I \rightarrow \mathbb{R}^3 : \operatorname{curl} \mathbf{v} = \mathbf{0} \text{ in } \Omega_I, \operatorname{div}(\mu\mathbf{v}) = 0 \text{ in } \Omega_I, \mu\mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_I\}, \tag{2.7}$$

where \mathbf{n} is the unit outward normal vector on $\partial\Omega_I$. Classical result of algebraic topology assure that the dimension of \mathcal{H}_I^μ is indeed equal to the first Betti number of Ω_I (for a more detailed presentation of this aspect, see [2]).

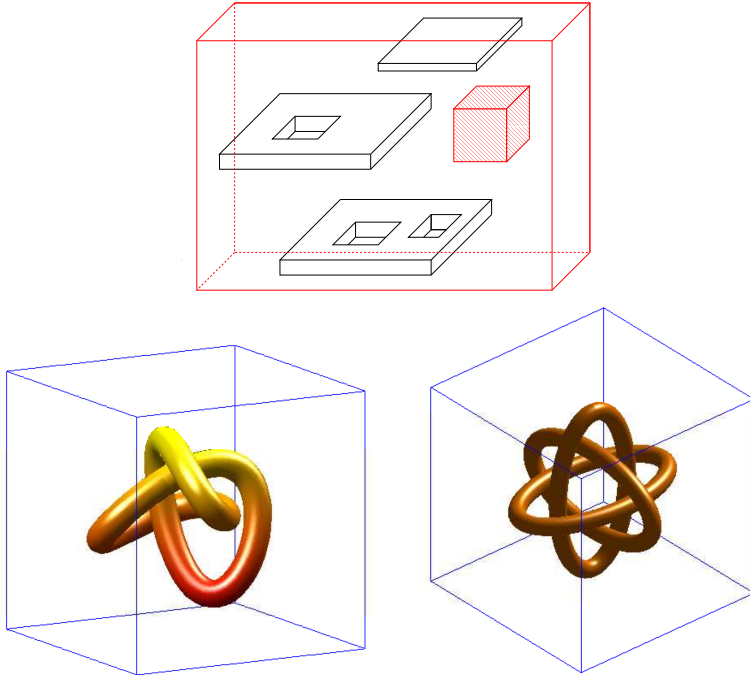


FIG. 2.1. Geometrical configurations (courtesy of Ana Alonso Rodríguez). Top: three internal conductors of different topological shape are drawn, while the red domain is not a part of Ω (the first Betti number of Ω_I is $g = 3$). Bottom: the conductor Ω_C is a trefoil knot (left; the first Betti number of Ω_I is $g = 1$) or the union of the three Borromean rings (right; the first Betti number of Ω_I is $g = 3$).

The functions $\boldsymbol{\rho}_j$ are determined as follows. First, one constructs a basis \mathbf{T}_j^0 , $j \in \{1, \dots, g\}$, of the first de Rham cohomology group as done in [1]. Then one projects them on the subspace μ -orthogonal to the gradients, namely, one considers $\boldsymbol{\rho}_j = \mathbf{T}_j^0 - \text{grad } \eta_j$, where η_j is the solution to

$$\eta_j \in H^1(\Omega_I)/\mathbb{C} : \int_{\Omega_I} \mu \text{grad } \eta_j \cdot \text{grad } \xi = \int_{\Omega_I} \mu \mathbf{T}_j^0 \cdot \text{grad } \xi \quad \forall \xi \in H^1(\Omega_I)/\mathbb{C}.$$

It is easily checked that $\boldsymbol{\rho}_j \in \mathcal{H}_I^\mu$; moreover, recalling that the loop fields \mathbf{T}_j^0 satisfy

$$\oint_{\sigma_n} \mathbf{T}_j^0 \cdot d\mathbf{s} = k_{nj},$$

where the cycles $\{\sigma_n\}$, $n \in \{1, \dots, g\}$, are a basis of the first homology group of $\text{cl } \Omega_I$ and $K = (k_{nj})$ is a non-singular matrix, it is easy to see that the fields $\boldsymbol{\rho}_j$ thus defined are linearly independent (just compute the line integral of a linear combination of them on each cycle σ_n).

Remark 2.4. The functions $\boldsymbol{\rho}_j$ can be computed once “offline” before the numerical solution of the optimal control problem is started. They are only needed when at least one of the conducting subdomains $(\Omega_C)_i$ is not simply connected (such as a torus). Instead, when all the components of Ω_C are simply connected (e.g., balls, cubes, balls with holes) these functions $\boldsymbol{\rho}_j$ are not necessary (in fact, they are vanish-

ing). However, we recall that Ω_I is assumed to be connected and this excludes that the components of Ω_C are tori with interior holes or balls with interior holes.

From the Ampère equation (2.6)₁ we see that the magnetic field satisfies $\text{curl } \mathbf{H} = \mathbf{0}$ in Ω_I (remember that σ and \mathbf{J}_e are vanishing in Ω_I). Therefore, $\mathbf{H}|_{\Omega_I}$ can be written as $\nabla\psi + \sum_{j=1}^g \alpha_j \boldsymbol{\rho}_j$ (see, e.g., [2, Appen. A.3]). We are thus led to write the weak formulation of our eddy current system in the state space

$$\mathbf{V}_0 = \{(\mathbf{H}, \psi, \boldsymbol{\alpha}) \in \mathbf{V} \text{ that satisfy the interface conditions (2.8) below}\},$$

where

$$\mathbf{V} = H(\text{curl}; \Omega_C) \times H^1(\Omega_I)/\mathbb{C} \times \mathbb{C}^g$$

and

$$\mathbf{H} \times \mathbf{n} - \nabla\psi \times \mathbf{n} - \sum_{j=1}^g \alpha_j \boldsymbol{\rho}_j \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma. \quad (2.8)$$

Both spaces \mathbf{V} and \mathbf{V}_0 are equipped with the norm

$$\|(\mathbf{H}, \Psi, \boldsymbol{\alpha})\|_{\mathbf{V}} = \left(\|\mathbf{H}\|_{H(\text{curl}; \Omega_C)}^2 + \|\psi\|_{H^1(\Omega_I)/\mathbb{C}}^2 + |\boldsymbol{\alpha}|^2 \right)^{1/2},$$

where

$$\|\mathbf{H}\|_{H(\text{curl}; \Omega_C)} = \left(\int_{\Omega_C} (\text{curl } \mathbf{H} \cdot \text{curl } \overline{\mathbf{H}} + \mathbf{H} \cdot \overline{\mathbf{H}}) \right)^{1/2}$$

and

$$\|\psi\|_{H^1(\Omega_I)/\mathbb{C}} = \left(\int_{\Omega_I} \nabla\psi \cdot \nabla\overline{\psi} \right)^{1/2}.$$

In $H^1(\Omega_I)/\mathbb{C}$, this H^1 -seminorm is equivalent to the standard norm of $H^1(\Omega_I)$ (see, e.g., [9, Chap. IV, Sect. 7.2]). The space \mathbf{V}_0 defined above is a (complex) Hilbert space, because it is closed in \mathbf{V} . Notice that the trace mappings $\mathbf{H} \mapsto \mathbf{H} \times \mathbf{n}$ and $\psi \mapsto \nabla\psi \times \mathbf{n}$ are continuous from $H(\text{curl}; \Omega_C)$ to $H^{-1/2}(\text{div}_\tau; \Gamma)$ and from $H^1(\Omega_I)$ to $H^{-1/2}(\text{div}_\tau; \Gamma)$, respectively, where, for a smooth surface Γ , the trace space $H^{-1/2}(\text{div}_\tau; \Gamma)$ is defined as

$$H^{-1/2}(\text{div}_\tau; \Gamma) := \{\boldsymbol{\lambda} \in H^{-1/2}(\Gamma)^3 : \boldsymbol{\lambda} \cdot \mathbf{n} = 0 \text{ on } \Gamma, \text{div}_\tau \boldsymbol{\lambda} \in H^{-1/2}(\Gamma)\}$$

(see, e.g., [2, Appen. A.1], where also a more general characterization is discussed, when Γ is a Lipschitz closed surface).

We also define the norms

$$\|\mathbf{Q}\|_{\Omega_C} := \left(\int_{\Omega_C} |\mathbf{Q}(\mathbf{x})|^2 \right)^{\frac{1}{2}}, \quad \|\mathbf{Q}\|_{\mu, \Omega_C} := \left(\int_{\Omega_C} \mu(\mathbf{x}) \mathbf{Q}(\mathbf{x}) \cdot \overline{\mathbf{Q}(\mathbf{x})} \right)^{\frac{1}{2}},$$

and, analogously, the norms $\|\mathbf{Q}\|_{\sigma, \Omega_C}$ and $\|\mathbf{Q}\|_{\mu, \Omega_I}$.

Let us introduce now the symmetric and positive definite matrix M by setting

$$M_{nj} = \int_{\Omega_I} \mu \boldsymbol{\rho}_n \cdot \boldsymbol{\rho}_j;$$

we will also use the vector norm $|\mathbf{q}|_M = (M\mathbf{q} \cdot \bar{\mathbf{q}})^{\frac{1}{2}}$, where $\mathbf{q} \in \mathbb{C}^g$.

Finally, we define an antilinear form $a : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{C}$ by

$$a[\mathbf{u}, \mathbf{v}] = \int_{\Omega_C} \sigma^{-1} \operatorname{curl} \mathbf{H} \cdot \operatorname{curl} \bar{\mathbf{W}} + \int_{\Omega_C} i\omega\mu \mathbf{H} \cdot \bar{\mathbf{W}} + \int_{\Omega_I} i\omega\mu \nabla\psi \cdot \nabla\bar{\eta} + i\omega M\boldsymbol{\alpha} \cdot \bar{\boldsymbol{\beta}}.$$

The form $a[\cdot, \cdot]$ is obviously continuous on $\mathbf{V} \times \mathbf{V}$ and it is also coercive (see, e.g., [2, p. 37]).

DEFINITION 2.5. *A triplet $\mathbf{u} = (\mathbf{H}, \psi, \boldsymbol{\alpha}) \in \mathbf{V}_0$ is said to be a weak solution of the eddy current model (2.11) below associated with $\mathbf{J}_e \in L^2(\Omega_C)^3$, if*

$$a[\mathbf{u}, \mathbf{v}] = \int_{\Omega_C} \sigma^{-1} \mathbf{J}_e \cdot \operatorname{curl} \bar{\mathbf{W}} \quad \forall \mathbf{v} := (\mathbf{W}, \eta, \boldsymbol{\beta}) \in \mathbf{V}_0. \quad (2.9)$$

We shall prove below that (2.9) is indeed the weak formulation of the system (2.11).

LEMMA 2.6 (Well posedness). *For all $\mathbf{J}_e \in L^2(\Omega_C)^3$, the equation (2.11) has a unique weak solution $(\mathbf{H}, \psi, \boldsymbol{\alpha})$. There is a constant $c > 0$ not depending on \mathbf{J}_e such that*

$$\|(\mathbf{H}, \psi, \boldsymbol{\alpha})\|_{\mathbf{V}} \leq c \|\mathbf{J}_e\|_{\Omega_C}. \quad (2.10)$$

Proof. The mapping $\Theta : H(\operatorname{curl}; \Omega_C) \rightarrow \mathbb{C}$ defined by

$$\Theta(\mathbf{W}, \eta, \boldsymbol{\beta}) := \int_{\Omega_C} \sigma^{-1} \bar{\mathbf{J}}_e \cdot \operatorname{curl} \mathbf{W}$$

(i.e., the conjugate complex value of the right hand side of (2.9)) is continuous and linear on \mathbf{V} , hence it belongs in particular to $(\mathbf{V}_0)'$. Moreover, the antilinear form a is coercive on \mathbf{V}_0 , hence the Lemma of Lax and Milgram ensures the existence of a unique solution of the variational equation (2.9) and of a constant $c_0 > 0$ such that

$$\|(\mathbf{H}, \psi, \boldsymbol{\alpha})\|_{\mathbf{V}} \leq c_0 \|\Theta\|_{(\mathbf{V}_0)'} \leq c \|\mathbf{J}_e\|_{\Omega_C}$$

holds. \square

Let us denote by \mathbf{n}_Ω the unit outward normal vector on $\partial\Omega$. For the sake of completeness, now we prove that the variational equation is the weak formulation of the following strong form of the eddy current problem (a similar procedure has been applied in [2, p. 42–43]):

THEOREM 2.7 (Strong eddy current problem). *If the solution $(\mathbf{H}, \psi, \boldsymbol{\alpha}) \in \mathbf{V}_0$ to the variational problem (2.9) is sufficiently smooth, then it satisfies the strong eddy current equations*

$$\begin{aligned} \operatorname{curl}(\sigma^{-1} \operatorname{curl} \mathbf{H}) + i\omega\mu \mathbf{H} &= \operatorname{curl}(\sigma^{-1} \mathbf{J}_e) && \text{in } \Omega_C \\ \mathbf{H} \times \mathbf{n} &= \nabla\psi \times \mathbf{n} + \sum_{j=1}^g \alpha_j \boldsymbol{\rho}_j \times \mathbf{n} && \text{on } \Gamma \\ \mu \mathbf{H} \cdot \mathbf{n} &= \mu \nabla\psi \cdot \mathbf{n} && \text{on } \Gamma \\ -\operatorname{div}(\mu \nabla\psi) &= 0 && \text{in } \Omega_I \\ \mu \nabla\psi \cdot \mathbf{n}_\Omega &= 0 && \text{on } \partial\Omega \end{aligned} \quad (2.11)$$

and the geometrical conditions

$$(M\boldsymbol{\alpha})_j = (i\omega)^{-1} \int_{\Gamma} \sigma^{-1}(\operatorname{curl} \mathbf{H} - \mathbf{J}_e) \cdot (\mathbf{n} \times \boldsymbol{\rho}_j) \quad \forall j \in \{1, \dots, g\}. \quad (2.12)$$

Proof. (i) First, we assume that in (2.9) $\eta = 0$ and $\boldsymbol{\beta} = \mathbf{0}$ holds. By the interface conditions (2.8), which are to be understood as equations in $H^{-1/2}(\operatorname{div}_{\tau}; \Gamma)$, we have then $\mathbf{W} \times \mathbf{n} = \mathbf{0}$ on Γ and also $\overline{\mathbf{W}} \times \mathbf{n} = \mathbf{0}$ on Γ , since \mathbf{n} is real. Integrating by parts in the weak formulation (2.9), we obtain (remember that \mathbf{n} is the unit outward normal vector to Ω_I)

$$\begin{aligned} & \int_{\Gamma} \sigma^{-1}(\overline{\mathbf{W}} \times \mathbf{n}) \cdot \operatorname{curl} \mathbf{H} + \int_{\Omega_C} \operatorname{curl}(\sigma^{-1} \operatorname{curl} \mathbf{H}) \cdot \overline{\mathbf{W}} + \int_{\Omega_C} i\omega \mu \mathbf{H} \cdot \overline{\mathbf{W}} \\ &= \int_{\Gamma} \sigma^{-1}(\overline{\mathbf{W}} \times \mathbf{n}) \cdot \mathbf{J}_e + \int_{\Omega_C} \operatorname{curl}(\sigma^{-1} \mathbf{J}_e) \cdot \overline{\mathbf{W}} \end{aligned} \quad (2.13)$$

for all $\mathbf{W} \in H(\operatorname{curl}; \Omega_C)$. Here, and in all what follows, the integrals on the interface Γ are defined in the duality sense on $H^{-1/2}(\operatorname{div}_{\tau}; \Gamma)$. Since both integrals on Γ vanish by $\overline{\mathbf{W}} \times \mathbf{n} = \mathbf{0}$, this implies that the first equation of (2.11) holds in the sense of distributions, because \mathbf{W} can be chosen arbitrarily out of $C_0^{\infty}(\Omega_C)$.

(ii) Next, we allow η to vary while still $\boldsymbol{\beta} = \mathbf{0}$ is required. Then, by the condition (2.8) that holds in \mathbf{V}_0 , in particular we have

$$\mathbf{W} \times \mathbf{n} = \nabla \eta \times \mathbf{n}. \quad (2.14)$$

Note also that (2.9) holds not only for each $\eta \in H^1(\Omega_I)/\mathbb{C}$ but also for each $\eta \in H^1(\Omega_I)$. We return to (2.9) but we can now use the first equation in (2.11), i.e., in (2.13) the integrals on Ω_C can be skipped. Adding the terms related to η and ψ , performing an integration by parts and using (2.14), we find for the remaining terms

$$\begin{aligned} & \int_{\Gamma} \sigma^{-1}(\nabla \overline{\eta} \times \mathbf{n}) \cdot \operatorname{curl} \mathbf{H} - \int_{\Gamma} \sigma^{-1}(\nabla \overline{\eta} \times \mathbf{n}) \cdot \mathbf{J}_e \\ &+ \int_{\partial\Omega} i\omega \overline{\eta} \mu \nabla \psi \cdot \mathbf{n}_{\Omega} + \int_{\Gamma} i\omega \overline{\eta} \mu \nabla \psi \cdot \mathbf{n} - \int_{\Omega_I} i\omega \overline{\eta} \operatorname{div}(\mu \nabla \psi) = 0. \end{aligned} \quad (2.15)$$

Selecting arbitrary $\eta \in C_0^{\infty}(\Omega_I)$, we deduce $-\operatorname{div}(\mu \nabla \psi) = 0$ in the sense of distributions. This yields the fourth equation of (2.11) so that the integral on Ω_I in (2.15) vanishes. Next, we vary η freely on $\partial\Omega$ subject to $\eta = 0$ in a neighborhood of Γ . Then we also find $\mu \nabla \psi \cdot \mathbf{n}_{\Omega} = 0$ on $\partial\Omega$, i.e., the last equation of (2.11).

(iii) In the next step, still assuming that $\boldsymbol{\beta} = \mathbf{0}$, we verify the two interface conditions on Γ in (2.11). The first one is included in the definition of the space \mathbf{V}_0 that underlies the definition of a weak solution. For the second one, let us start from equation (2.15), that, from what we have proved in (ii) and taking into account (2.14), can be re-written as

$$\int_{\Gamma} \sigma^{-1}(\overline{\mathbf{W}} \times \mathbf{n}) \cdot \operatorname{curl} \mathbf{H} - \int_{\Gamma} \sigma^{-1}(\overline{\mathbf{W}} \times \mathbf{n}) \cdot \mathbf{J}_e + \int_{\Gamma} i\omega \overline{\eta} \mu \nabla \psi \cdot \mathbf{n} = 0, \quad (2.16)$$

for each $\eta \in H^1(\Omega_I)$. We transform the first two terms of this equation as follows:

$$\begin{aligned} \int_{\Gamma} \sigma^{-1}(\overline{\mathbf{W}} \times \mathbf{n}) \cdot (\operatorname{curl} \mathbf{H} - \mathbf{J}_e) &= \int_{\Gamma} \sigma^{-1}(\nabla \bar{\eta} \times \mathbf{n}) \cdot (\operatorname{curl} \mathbf{H} - \mathbf{J}_e) \\ &= - \int_{\Gamma} \nabla \bar{\eta} \cdot [\sigma^{-1}(\operatorname{curl} \mathbf{H} - \mathbf{J}_e) \times \mathbf{n}] = \int_{\Gamma} \operatorname{div}_{\tau} [\sigma^{-1}(\operatorname{curl} \mathbf{H} - \mathbf{J}_e) \times \mathbf{n}] \bar{\eta} \quad (2.17) \\ &= \int_{\Gamma} \operatorname{curl} [\sigma^{-1}(\operatorname{curl} \mathbf{H} - \mathbf{J}_e)] \cdot \mathbf{n} \bar{\eta} = - \int_{\Gamma} i\omega \mu \mathbf{H} \cdot \mathbf{n} \bar{\eta}, \end{aligned}$$

where we have used in the third equation the surface divergence $\operatorname{div}_{\tau}$ that in particular satisfies for each $\mathbf{a} \in H(\operatorname{curl}; \Omega_C)$

$$- \int_{\Gamma} \nabla \bar{\eta} \cdot (\mathbf{a} \times \mathbf{n}) = \int_{\Gamma} \operatorname{div}_{\tau} (\mathbf{a} \times \mathbf{n}) \bar{\eta}$$

(see [23, p. 49]). Moreover, in the fourth equation above we have used the identity

$$\operatorname{div}_{\tau} (\mathbf{a} \times \mathbf{n}) = \operatorname{curl} \mathbf{a} \cdot \mathbf{n}$$

(see, e.g., [2, Appen. A.1]), while in the fifth equation we invoked the first equation of (2.11) that we had already obtained before.

Now, we insert (2.17) in (2.16) and arrive at

$$- \int_{\Gamma} i\omega \mu \mathbf{H} \cdot \mathbf{n} \bar{\eta} + \int_{\Gamma} i\omega \bar{\eta} \mu \nabla \psi \cdot \mathbf{n} = 0$$

for each $\eta \in H^1(\Omega_I)$, hence for each $\eta|_{\Gamma} \in H^{1/2}(\Gamma)$. Dividing by $i\omega$ yields the second interface condition of (2.11), that holds in $H^{-1/2}(\Gamma)$.

(iv) To verify the geometrical condition, we finally let also $\boldsymbol{\beta}$ vary in \mathbb{C}^g . Then, in view of (2.8), \mathbf{W} , η , $\boldsymbol{\beta}$ obey

$$\mathbf{W} \times \mathbf{n} = \nabla \eta \times \mathbf{n} + \sum_{j=1}^g \beta_j \boldsymbol{\rho}_j \times \mathbf{n} \quad \text{on } \Gamma. \quad (2.18)$$

Moreover, we have to take into account the term $i\omega M \boldsymbol{\alpha} \cdot \bar{\boldsymbol{\beta}}$ in the variational equation. We return to (2.16), add $i\omega M \boldsymbol{\alpha} \cdot \bar{\boldsymbol{\beta}}$, and insert (2.18) for $\mathbf{W} \times \mathbf{n}$. This yields

$$\begin{aligned} 0 &= \int_{\Gamma} (\overline{\mathbf{W}} \times \mathbf{n}) \cdot \sigma^{-1}(\operatorname{curl} \mathbf{H} - \mathbf{J}_e) + \int_{\Gamma} i\omega \bar{\eta} \mu \nabla \psi \cdot \mathbf{n} + i\omega M \boldsymbol{\alpha} \cdot \bar{\boldsymbol{\beta}} \\ &= \int_{\Gamma} (\nabla \bar{\eta} \times \mathbf{n} + \sum_{j=1}^g \bar{\beta}_j \boldsymbol{\rho}_j \times \mathbf{n}) \cdot \sigma^{-1}(\operatorname{curl} \mathbf{H} - \mathbf{J}_e) + \int_{\Gamma} i\omega \bar{\eta} \mu \nabla \psi \cdot \mathbf{n} \\ &\quad + i\omega M \boldsymbol{\alpha} \cdot \bar{\boldsymbol{\beta}} \\ &= \int_{\Gamma} (\nabla \bar{\eta} \times \mathbf{n}) \cdot \sigma^{-1}(\operatorname{curl} \mathbf{H} - \mathbf{J}_e) + \int_{\Gamma} i\omega \bar{\eta} \mu \nabla \psi \cdot \mathbf{n} \quad (2.19) \\ &\quad - \int_{\Gamma} \left(\sum_{j=1}^g \bar{\beta}_j \mathbf{n} \times \boldsymbol{\rho}_j \right) \cdot \sigma^{-1}(\operatorname{curl} \mathbf{H} - \mathbf{J}_e) + i\omega M \boldsymbol{\alpha} \cdot \bar{\boldsymbol{\beta}} \\ &= - \int_{\Gamma} \left(\sum_{j=1}^g \bar{\beta}_j \mathbf{n} \times \boldsymbol{\rho}_j \right) \cdot \sigma^{-1}(\operatorname{curl} \mathbf{H} - \mathbf{J}_e) + i\omega M \boldsymbol{\alpha} \cdot \bar{\boldsymbol{\beta}}, \end{aligned}$$

in view of (2.17) and the second interface condition in (2.11). Since this must hold for arbitrary $\boldsymbol{\beta} \in \mathbb{C}^g$, the last equation amounts to

$$i\omega(M\boldsymbol{\alpha})_j = \int_{\Gamma} \sigma^{-1}(\operatorname{curl} \mathbf{H} - \mathbf{J}_e) \cdot (\mathbf{n} \times \boldsymbol{\rho}_j) \quad \forall j \in \{1, \dots, g\},$$

i.e., (2.12) is verified. \square

3. The optimal control problem.

3.1. The optimal current problem and its well-posedness. In our paper, we will consider several optimal control problems. Here, we discuss the following steady state optimal control problem of elliptic type, where the impressed current \mathbf{J}_e is the control function.

As fixed data, vector functions $\mathbf{H}_d \in \mathbf{L}^2(\Omega)$, $\mathbf{E}_d \in L^2(\Omega_C)^3$ and constants $\nu_C \geq 0$, $\nu_A \geq 0$, $\nu_B \geq 0$, $\nu_E \geq 0$, $\nu \geq 0$ with $\nu_C + \nu_A + \nu_B + \nu_E + \nu > 0$ are given. In Ω_I the reference magnetic field \mathbf{H}_d is split as $\nabla\psi_d + \sum_{j=1}^g \alpha_{d,j} \boldsymbol{\rho}_j$. Moreover, a set of admissible controls $\mathbf{J}_{ad} \subset L^2(\Omega_C)^3$ is given and is assumed to be nonempty, bounded, convex and closed. We will specify later possible choices for \mathbf{J}_{ad} .

Thanks to Lemma 2.6, for each control $\mathbf{J}_e \in \mathbf{J}_{ad}$ there exists a unique weak solution of (2.11). To indicate the correspondence of this solution to the given control \mathbf{J}_e , we denote this solution by $(\mathbf{H}_{J_e}, \psi_{J_e}, \boldsymbol{\alpha}_{J_e})$. In what follows, we will skip the subscript e from the controls and denote them just by \mathbf{J} . Notice that \mathbf{J} stands now for the impressed current \mathbf{J}_e and is not equal to the total current that was formerly denoted by $\mathbf{J} = \sigma \mathbf{E} + \mathbf{J}_e$.

As optimization criterion, we use the following (reduced) objective functional F ,

$$\begin{aligned} F(\mathbf{J}) := & \frac{\nu_C}{2} \|\mathbf{H}_J - \mathbf{H}_d\|_{\mu, \Omega_C}^2 + \frac{\nu_A}{2} \|\nabla\psi_J - \nabla\psi_d\|_{\mu, \Omega_I}^2 + \frac{\nu_B}{2} |\boldsymbol{\alpha}_J - \boldsymbol{\alpha}_d|_M^2 \\ & + \frac{\nu_E}{2} \|\mathbf{E}_J - \mathbf{E}_d\|_{\sigma, \Omega_C}^2 + \frac{\nu}{2} \|\mathbf{J}\|_{\Omega_C}^2, \end{aligned} \quad (3.1)$$

where \mathbf{E}_J denotes the electric field associated with \mathbf{J} . In the terms of F , the magnetic energy and the electric energy (per unit time) of \mathbf{H} and \mathbf{E} , respectively, appear. These weighted norms are more natural than the standard L^2 -norms and will later lead to some simplifications in the adjoint equation. For the L^2 -norm, the theory is similar and can be covered by setting μ and σ to one in all the terms that are associated with the objective functional.

The electric field \mathbf{E} is equal to $\mathbf{E} = \sigma^{-1}(\operatorname{curl} \mathbf{H} - \mathbf{J})$, hence

$$\begin{aligned} F(\mathbf{J}) = & \frac{\nu_C}{2} \|\mathbf{H}_J - \mathbf{H}_d\|_{\mu, \Omega_C}^2 + \frac{\nu_A}{2} \|\nabla\psi_J - \nabla\psi_d\|_{\mu, \Omega_I}^2 + \frac{\nu_B}{2} |\boldsymbol{\alpha}_J - \boldsymbol{\alpha}_d|_M^2 \\ & + \frac{\nu_E}{2} \|\sigma^{-1}(\operatorname{curl} \mathbf{H}_J - \mathbf{J}) - \mathbf{E}_d\|_{\sigma, \Omega_C}^2 + \frac{\nu}{2} \|\mathbf{J}\|_{\Omega_C}^2. \end{aligned} \quad (3.2)$$

This objective functional F aims at minimizing the weighted distance to desired (or measured) magnetic and electric fields, while the norm of the control function \mathbf{J} is included as a Tikhonov regularization term weighted by ν .

The optimal control problem, written in short form, is

$$\min_{\mathbf{J} \in \mathbf{J}_{ad}} F(\mathbf{J}). \quad (3.3)$$

DEFINITION 3.1 (Optimality). *A control $\mathbf{J}^* \in \mathbf{J}_{ad}$ is said to be optimal, if*

$$F(\mathbf{J}^*) \leq F(\mathbf{J}) \quad \forall \mathbf{J} \in \mathbf{J}_{ad}.$$

THEOREM 3.2. *The optimal control problem (3.3) admits at least one optimal control denoted by \mathbf{J}^* . The optimal control is unique, if $\nu > 0$.*

Proof. Thanks to Lemma 2.6, the mappings $\mathbf{J} \mapsto \mathbf{H}_J$, $\mathbf{J} \mapsto \psi_J$ and $\mathbf{J} \mapsto \boldsymbol{\alpha}_J$ is well defined, linear and continuous from $L^2(\Omega_C)^3$ to $H(\text{curl}; \Omega_C)$, $H^1(\Omega_I)/\mathbb{C}$ and \mathbb{C}^g , respectively. Therefore, the reduced objective functional F is continuous and convex, hence also weakly lower semicontinuous. Moreover, the set \mathbf{J}_{ad} of admissible controls is weakly sequentially compact in $L^2(\Omega_C)^3$ so that the existence of an optimal control $\mathbf{J} \in \mathbf{J}_{ad}$ with

$$F(\mathbf{J}^*) = \inf_{\mathbf{J} \in \mathbf{J}_{ad}} F(\mathbf{J})$$

is an immediate consequence. Notice that F is bounded from below by zero so that the existence of a non-negative infimum is guaranteed. For $\nu > 0$, the functional F is strictly convex and that implies the uniqueness of the optimal control. \square

3.2. Necessary optimality conditions. The next step of our analysis is the derivation of first-order necessary optimality conditions for an optimal control \mathbf{J}^* . By convexity of F and \mathbf{J}_{ad} , they are also sufficient for optimality.

Prior to this, let us mention the following simple calculation concerning the directional derivative of the complex but real valued function $g : z \mapsto |z|^2$. For any fixed $z \in \mathbb{C}$ and varying $h \in \mathbb{C}$, we have

$$|z + h|^2 = |z|^2 + z\bar{h} + \bar{z}h + |h|^2 = |z|^2 + 2\text{Re}[z\bar{h}] + |h|^2.$$

Therefore, the complex function g has the directional derivative

$$g'(z)h := \lim_{t \rightarrow 0} \frac{|z + th|^2 - |z|^2}{t} = 2\text{Re}[z\bar{h}] = 2\text{Re}[\bar{z}h]$$

(here, $t \in \mathbb{R}$). Notice that the mapping $h \mapsto 2\text{Re}[\bar{z}h]$ is not complex linear. However, it is real linear, because $\text{Re}[\bar{z}\alpha h] = \alpha \text{Re}[\bar{z}h]$ for all real α . The function g is not holomorphic, i.e., not differentiable in the sense of complex analysis.

The directional derivative of the objective functional F can be determined analogously. The derivative in the direction \mathbf{J} at an arbitrary fixed (not necessarily optimal or admissible) control $\hat{\mathbf{J}}$ with associated magnetic field $\hat{\mathbf{H}} := \mathbf{H}_{\hat{\mathbf{J}}}$, $\hat{\psi} := \psi_{\hat{\mathbf{J}}}$ and $\hat{\boldsymbol{\alpha}} := \boldsymbol{\alpha}_{\hat{\mathbf{J}}}$ is given by

$$\begin{aligned} F'(\hat{\mathbf{J}})\mathbf{J} &= \nu_C \int_{\Omega_C} \text{Re}[\mu(\hat{\mathbf{H}} - \mathbf{H}_d) \cdot \overline{\mathbf{H}_J}] \\ &+ \nu_A \int_{\Omega_I} \text{Re}[\mu(\nabla\hat{\psi} - \nabla\psi_d) \cdot \nabla\overline{\psi_J}] + \nu_B \text{Re}[M(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_d) \cdot \overline{\boldsymbol{\alpha}_J}] \\ &+ \nu_E \int_{\Omega_C} \text{Re}[(\sigma^{-1}(\text{curl}\hat{\mathbf{H}} - \hat{\mathbf{J}}) - \mathbf{E}_d) \cdot (\text{curl}\overline{\mathbf{H}_J} - \overline{\mathbf{J}})] \\ &+ \nu \int_{\Omega_C} \text{Re}[\hat{\mathbf{J}} \cdot \overline{\mathbf{J}}]. \end{aligned}$$

Notice that $\sigma^{-1}(\text{curl } \widehat{\mathbf{H}} - \widehat{\mathbf{J}}) = \widehat{\mathbf{E}} := \mathbf{E}_{\widehat{\mathbf{J}}}$. Re-arranging the terms, we see

$$\begin{aligned}
F'(\widehat{\mathbf{J}}) \mathbf{J} &= \text{Re} \left\{ \int_{\Omega_C} \nu_C \mu (\widehat{\mathbf{H}} - \mathbf{H}_d) \cdot \overline{\mathbf{H}_J} \right. \\
&\quad + \int_{\Omega_I} \nu_A \mu (\nabla \widehat{\psi} - \nabla \psi_d) \cdot \nabla \overline{\psi_J} + \nu_B M (\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_d) \cdot \overline{\boldsymbol{\alpha}_J} \\
&\quad + \int_{\Omega_C} \nu_E (\widehat{\mathbf{E}} - \mathbf{E}_d) \cdot \text{curl } \overline{\mathbf{H}_J} \\
&\quad \left. - \int_{\Omega_C} \nu_E (\widehat{\mathbf{E}} - \mathbf{E}_d) \cdot \overline{\mathbf{J}} + \nu \int_{\Omega_C} \widehat{\mathbf{J}} \cdot \overline{\mathbf{J}} \right\}.
\end{aligned} \tag{3.4}$$

Note that taking the real part of a complex number is an additive operation. In (3.4), the variable direction \mathbf{J} is appearing explicitly in the last two integrals, while it occurs implicitly in the first four terms through the mappings $\mathbf{J} \mapsto \mathbf{H}_J$, $\mathbf{J} \mapsto \psi_J$ and $\mathbf{J} \mapsto \boldsymbol{\alpha}_J$. By introducing an adjoint state, this implicit dependence on \mathbf{J} can be transformed in a standard way to an explicit dependence.

DEFINITION 3.3 (Adjoint equation). *Let $\widehat{\mathbf{J}} \in L^2(\Omega_C)^3$ be a given control with associated states $\widehat{\mathbf{H}} := \mathbf{H}_{\widehat{\mathbf{J}}}$, $\widehat{\mathbf{E}} := \mathbf{E}_{\widehat{\mathbf{J}}}$, $\widehat{\psi} := \psi_{\widehat{\mathbf{J}}}$, $\widehat{\boldsymbol{\alpha}} := \boldsymbol{\alpha}_{\widehat{\mathbf{J}}}$, and let $\mathbf{H}_d \in L^2(\Omega_C)^3$, $\psi_d \in H^1(\Omega_I)/\mathbb{C}$, $\boldsymbol{\alpha}_d \in \mathbb{C}^g$, $\mathbf{E}_d \in L^2(\Omega_C)^3$ be given as above. The equation for $(\mathbf{W}, \eta, \boldsymbol{\beta})$,*

$$\begin{aligned}
&\int_{\Omega_C} \sigma^{-1} \text{curl } \mathbf{W} \cdot \text{curl } \overline{\mathbf{H}} - i\omega \int_{\Omega_C} \mu \mathbf{W} \cdot \overline{\mathbf{H}} - i\omega \int_{\Omega_I} \mu \nabla \eta \cdot \nabla \overline{\psi} - i\omega M \boldsymbol{\beta} \cdot \overline{\boldsymbol{\alpha}} \\
&= \int_{\Omega_C} \nu_C \mu (\widehat{\mathbf{H}} - \mathbf{H}_d) \cdot \overline{\mathbf{H}} \\
&\quad + \int_{\Omega_I} \nu_A \mu (\nabla \widehat{\psi} - \nabla \psi_d) \cdot \nabla \overline{\psi} + \nu_B M (\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_d) \cdot \overline{\boldsymbol{\alpha}} \\
&\quad + \int_{\Omega_C} \nu_E (\widehat{\mathbf{E}} - \mathbf{E}_d) \cdot \text{curl } \overline{\mathbf{H}} \quad \forall (\mathbf{H}, \psi, \boldsymbol{\alpha}) \in \mathbf{V}_0
\end{aligned} \tag{3.5}$$

is said to be the adjoint equation of equation (2.9). The solution $(\mathbf{W}_{\widehat{\mathbf{J}}}, \eta_{\widehat{\mathbf{J}}}, \boldsymbol{\beta}_{\widehat{\mathbf{J}}}) \in \mathbf{V}_0$ is called the adjoint state associated with $\widehat{\mathbf{J}}$.

COROLLARY 3.4. *For all given $\mathbf{H}_d \in L^2(\Omega_C)^3$, $\psi_d \in H^1(\Omega_I)/\mathbb{C}$, $\boldsymbol{\alpha}_d \in \mathbb{C}^g$, $\mathbf{E}_d \in L^2(\Omega_C)^3$, $\widehat{\mathbf{J}} \in L^2(\Omega_C)^3$, the adjoint equation (3.5) has a unique solution $(\mathbf{W}_{\widehat{\mathbf{J}}}, \eta_{\widehat{\mathbf{J}}}, \boldsymbol{\beta}_{\widehat{\mathbf{J}}})$.*

This result follows as Lemma 2.6 by the Lemma of Lax and Milgram. Notice that the mapping

$$\begin{aligned}
(\mathbf{H}, \psi, \boldsymbol{\alpha}) &\mapsto \int_{\Omega_C} \nu_C \mu \overline{(\widehat{\mathbf{H}} - \mathbf{H}_d)} \cdot \mathbf{H} \\
&\quad + \int_{\Omega_I} \nu_A \mu \overline{(\nabla \widehat{\psi} - \nabla \psi_d)} \cdot \nabla \psi + \nu_B M \overline{(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_d)} \cdot \boldsymbol{\alpha} \\
&\quad + \int_{\Omega_C} \nu_E \overline{(\widehat{\mathbf{E}} - \mathbf{E}_d)} \cdot \text{curl } \mathbf{H}
\end{aligned}$$

(i.e., the conjugate complex value of the right hand side of (3.5)) is linear and continuous from \mathbf{V}_0 to \mathbb{C} , hence it belongs to $(\mathbf{V}_0)'$.

To see that the adjoint state transforms the implicit appearance of the control J in (3.4) to an explicit one, we prove the following auxiliary result:

LEMMA 3.5. *We have that*

$$\begin{aligned} & \operatorname{Re} \left[\nu_C \int_{\Omega_C} \mu(\widehat{\mathbf{H}} - \mathbf{H}_d) \cdot \overline{\mathbf{H}}_J + \nu_A \int_{\Omega_I} \mu(\nabla\widehat{\psi} - \nabla\psi_d) \cdot \nabla\overline{\psi}_J \right. \\ & \quad \left. + \nu_B M(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_d) \cdot \overline{\boldsymbol{\alpha}}_J + \nu_E \int_{\Omega_C} (\widehat{\mathbf{E}} - \mathbf{E}_d) \cdot \operatorname{curl} \overline{\mathbf{H}}_J \right] \\ & = \operatorname{Re} \int_{\Omega_C} \sigma^{-1} \operatorname{curl} \mathbf{W}_{\widehat{J}} \cdot \overline{\mathbf{J}}, \end{aligned} \quad (3.6)$$

where the function $\mathbf{W}_{\widehat{J}}$ is the first component of the adjoint state $(\mathbf{W}_{\widehat{J}}, \eta_{\widehat{J}}, \boldsymbol{\beta}_{\widehat{J}})$ associated with $\widehat{\mathbf{J}}$.

Proof. We write down the variational equation defining the weak solution \mathbf{H}_J , ψ_J and $\boldsymbol{\alpha}_J$, and insert the solution $(\mathbf{W}_{\widehat{J}}, \eta_{\widehat{J}}, \boldsymbol{\beta}_{\widehat{J}})$ of the adjoint equation as test function; we obtain

$$\begin{aligned} & \int_{\Omega_C} \sigma^{-1} \operatorname{curl} \mathbf{H}_J \cdot \operatorname{curl} \overline{\mathbf{W}}_{\widehat{J}} + i\omega \int_{\Omega_C} \mu \mathbf{H}_J \cdot \overline{\mathbf{W}}_{\widehat{J}} \\ & \quad + i\omega \int_{\Omega_I} \mu \nabla \psi_J \cdot \nabla \overline{\eta}_{\widehat{J}} + i\omega M \boldsymbol{\alpha}_J \cdot \overline{\boldsymbol{\beta}}_{\widehat{J}} \\ & = \int_{\Omega_C} \sigma^{-1} \mathbf{J} \cdot \operatorname{curl} \overline{\mathbf{W}}_{\widehat{J}}. \end{aligned} \quad (3.7)$$

On the other hand, inserting $(\mathbf{H}_J, \psi_J, \boldsymbol{\alpha}_J)$ as test function in the adjoint equation (3.5), we find

$$\begin{aligned} & \int_{\Omega_C} \sigma^{-1} \operatorname{curl} \overline{\mathbf{H}}_J \cdot \operatorname{curl} \mathbf{W}_{\widehat{J}} - i\omega \int_{\Omega_C} \mu \overline{\mathbf{H}}_J \cdot \mathbf{W}_{\widehat{J}} \\ & \quad - i\omega \int_{\Omega_I} \mu \nabla \overline{\psi}_J \cdot \nabla \eta_{\widehat{J}} - i\omega M \overline{\boldsymbol{\alpha}}_J \cdot \boldsymbol{\beta}_{\widehat{J}} \\ & = \nu_C \int_{\Omega_C} \mu(\widehat{\mathbf{H}} - \mathbf{H}_d) \cdot \overline{\mathbf{H}}_J \\ & \quad + \nu_A \int_{\Omega_I} \mu(\nabla\widehat{\psi} - \nabla\psi_d) \cdot \nabla\overline{\psi}_J + \nu_B M(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_d) \cdot \overline{\boldsymbol{\alpha}}_J \\ & \quad + \nu_E \int_{\Omega_C} (\widehat{\mathbf{E}} - \mathbf{E}_d) \cdot \operatorname{curl} \overline{\mathbf{H}}_J. \end{aligned} \quad (3.8)$$

We see that the left hand side of (3.7) is the complex conjugate of the left-hand side of (3.8). Therefore, the conjugate complex value of the right-hand side of (3.7) is equal to the right-hand side of (3.8), i.e.,

$$\begin{aligned} & \int_{\Omega_C} \sigma^{-1} \operatorname{curl} \mathbf{W}_{\widehat{J}} \cdot \overline{\mathbf{J}} = \nu_C \int_{\Omega_C} \mu(\widehat{\mathbf{H}} - \mathbf{H}_d) \cdot \overline{\mathbf{H}}_J \\ & \quad + \nu_A \int_{\Omega_I} \mu(\nabla\widehat{\psi} - \nabla\psi_d) \cdot \nabla\overline{\psi}_J + \nu_B M(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_d) \cdot \overline{\boldsymbol{\alpha}}_J \\ & \quad + \nu_E \int_{\Omega_C} (\widehat{\mathbf{E}} - \mathbf{E}_d) \cdot \operatorname{curl} \overline{\mathbf{H}}_J. \end{aligned}$$

The claim of the theorem follows by taking the real part of each side above. \square

THEOREM 3.6 (Necessary optimality conditions). *Let \mathbf{J}^* be an optimal control of problem 3.3 and let \mathbf{H}_{J^*} and \mathbf{E}_{J^*} be the associated optimal magnetic and electric fields, respectively. Then there exists a unique solution $(\mathbf{W}_{J^*}, \eta_{J^*}, \boldsymbol{\beta}_{J^*})$ of the adjoint equation (3.5) such that the variational inequality*

$$\operatorname{Re} \int_{\Omega_C} \left(\sigma^{-1} \operatorname{curl} \mathbf{W}_{J^*} - \nu_E (\mathbf{E}_{J^*} - \mathbf{E}_d) + \nu \mathbf{J}^* \right) \cdot (\bar{\mathbf{J}} - \bar{\mathbf{J}}^*) \geq 0 \quad \forall \mathbf{J} \in \mathbf{J}_{ad} \quad (3.9)$$

is satisfied.

Proof. The optimal control \mathbf{J}^* must obey the standard variational inequality

$$F'(\mathbf{J}^*)(\mathbf{J} - \mathbf{J}^*) \geq 0 \quad \forall \mathbf{J} \in \mathbf{J}_{ad}. \quad (3.10)$$

We show that this is equivalent to the variational inequality (3.9). We first consider the expression (3.4) for $F'(\hat{\mathbf{J}})$ for the particular choice $\hat{\mathbf{J}} := \mathbf{J}^*$ and have

$$\begin{aligned} & F'(\mathbf{J}^*)(\mathbf{J} - \mathbf{J}^*) \\ &= \operatorname{Re} \left[\nu_C \int_{\Omega_C} \mu (\mathbf{H}_{J^*} - \mathbf{H}_d) \cdot \overline{\mathbf{H}_{J-J^*}} \right. \\ &\quad + \nu_A \int_{\Omega_I} \mu (\nabla \psi_{J^*} - \nabla \psi_d) \cdot \overline{\nabla \psi_{J-J^*}} + \nu_B M (\boldsymbol{\alpha}_{J^*} - \boldsymbol{\alpha}_d) \cdot \overline{\boldsymbol{\alpha}_{J-J^*}} \\ &\quad + \nu_E \int_{\Omega_C} (\mathbf{E}_{J^*} - \mathbf{E}_d) \cdot \operatorname{curl} \overline{\mathbf{H}_{J-J^*}} - \nu_E \int_{\Omega_C} (\mathbf{E}_{J^*} - \mathbf{E}_d) \cdot (\bar{\mathbf{J}} - \bar{\mathbf{J}}^*) \\ &\quad \left. + \nu \int_{\Omega_C} \mathbf{J}^* \cdot (\bar{\mathbf{J}} - \bar{\mathbf{J}}^*) \right]. \end{aligned}$$

Thanks to Lemma 3.5, we obtain

$$\begin{aligned} & F'(\mathbf{J}^*)(\mathbf{J} - \mathbf{J}^*) \\ &= \operatorname{Re} \left[\int_{\Omega_C} \sigma^{-1} \operatorname{curl} \mathbf{W}_{J^*} \cdot (\bar{\mathbf{J}} - \bar{\mathbf{J}}^*) \right. \\ &\quad \left. - \int_{\Omega_C} \nu_E (\mathbf{E}_{J^*} - \mathbf{E}_d) \cdot (\bar{\mathbf{J}} - \bar{\mathbf{J}}^*) + \int_{\Omega_C} \nu \mathbf{J}^* \cdot (\bar{\mathbf{J}} - \bar{\mathbf{J}}^*) \right] \quad (3.11) \\ &= \operatorname{Re} \int_{\Omega_C} \left(\sigma^{-1} \operatorname{curl} \mathbf{W}_{J^*} - \nu_E (\mathbf{E}_{J^*} - \mathbf{E}_d) + \nu \mathbf{J}^* \right) \cdot (\bar{\mathbf{J}} - \bar{\mathbf{J}}^*), \end{aligned}$$

where \mathbf{W}_{J^*} is the first component of the adjoint state associated with \mathbf{J}^* . \square

DEFINITION 3.7. *For convenience, we define*

$$\mathbf{D}_{J^*} := \sigma^{-1} \operatorname{curl} \mathbf{W}_{J^*} - \nu_E (\mathbf{E}_{J^*} - \mathbf{E}_d). \quad (3.12)$$

By this definition, the variational inequality (3.9) simplifies to

$$\operatorname{Re} \int_{\Omega_C} \left(\mathbf{D}_{J^*} + \nu \mathbf{J}^* \right) \cdot (\bar{\mathbf{J}} - \bar{\mathbf{J}}^*) \geq 0 \quad \forall \mathbf{J} \in \mathbf{J}_{ad}. \quad (3.13)$$

This is our main necessary condition that will be later used to handle various particular cases for \mathbf{J}_{ad} .

The next result is important for the choice of the descent direction in gradient type methods for the numerical solution of our optimal control problem.

COROLLARY 3.8. *At an arbitrarily given control $\widehat{\mathbf{J}} \in L^2(\Omega_C)^3$, the maximum*

$$\max_{\|\mathbf{J}\|_{\Omega_C}=1} F'(\widehat{\mathbf{J}}) \mathbf{J},$$

i.e., the direction of steepest ascent, is attained by

$$\mathbf{J} = \frac{\mathbf{D}_{\widehat{\mathbf{J}}} + \nu \widehat{\mathbf{J}}}{\|\mathbf{D}_{\widehat{\mathbf{J}}} + \nu \widehat{\mathbf{J}}\|_{\Omega_C}}.$$

Proof. The integral in (3.11) can be written in terms of the inner product $(\cdot, \cdot)_{\Omega_C}$ of the space $L^2(\Omega_C)^3$ by

$$F'(\widehat{\mathbf{J}}) \mathbf{J} = \operatorname{Re} \left(\mathbf{D}_{\widehat{\mathbf{J}}} + \nu \widehat{\mathbf{J}}, \mathbf{J} \right)_{\Omega_C}.$$

Invoking the Cauchy-Schwarz inequality, we estimate

$$\begin{aligned} F'(\widehat{\mathbf{J}}) \mathbf{J} &\leq \left| \operatorname{Re} \left(\mathbf{D}_{\widehat{\mathbf{J}}} + \nu \widehat{\mathbf{J}}, \mathbf{J} \right)_{\Omega_C} \right| \\ &\leq \left\| \mathbf{D}_{\widehat{\mathbf{J}}} + \nu \widehat{\mathbf{J}} \right\|_{\Omega_C} \end{aligned}$$

if $\|\mathbf{J}\|_{\Omega_C} = 1$. This maximal value at the end of this inequality is attained by the function \mathbf{J} defined in the Corollary. \square

DEFINITION 3.9. *The direction of steepest ascent of a differentiable function is given (after normalization) by its gradient. Though our objective functional F is only directionally differentiable and does not have a gradient, let us denote the direction of steepest ascent of $F'(\widehat{\mathbf{J}})$ as its reduced gradient:*

$$\nabla F(\widehat{\mathbf{J}}) := \mathbf{D}_{\widehat{\mathbf{J}}} + \nu \widehat{\mathbf{J}}. \quad (3.14)$$

This notion is introduced to shorten the arguments below.

3.3. The strong form of the adjoint equation. For completeness, and to compare the adjoint equation with the state equation, we also present its strong formulation. We obtain the following representation theorem, where, as before, we write $\widehat{\mathbf{E}} = \sigma^{-1}(\operatorname{curl} \widehat{\mathbf{H}} - \widehat{\mathbf{J}})$.

THEOREM 3.10 (Strong adjoint equation). *If the adjoint state $(\mathbf{W}, \eta, \boldsymbol{\alpha})$ is sufficiently smooth, then it satisfies the system*

$$\begin{aligned} \operatorname{curl}(\sigma^{-1} \operatorname{curl} \mathbf{W}) - i\omega \mu \mathbf{W} &= \nu_C \mu (\widehat{\mathbf{H}} - \mathbf{H}_d) \\ &\quad + \nu_E \operatorname{curl}(\widehat{\mathbf{E}} - \mathbf{E}_d) && \text{in } \Omega_C \\ \mathbf{W} \times \mathbf{n} &= \nabla \eta \times \mathbf{n} + \sum_{j=1}^g \beta_j \boldsymbol{\rho}_j \times \mathbf{n} && \text{on } \Gamma \\ \mu \mathbf{W} \cdot \mathbf{n} - \mu \nabla \eta \cdot \mathbf{n} &= -(i\omega)^{-1} \nu_C \mu (\widehat{\mathbf{H}} - \mathbf{H}_d) \cdot \mathbf{n} \\ &\quad + (i\omega)^{-1} \nu_A \mu (\nabla \widehat{\psi} - \nabla \psi_d) \cdot \mathbf{n} && \text{on } \Gamma \\ -\operatorname{div}(\mu \nabla \eta) &= (i\omega)^{-1} \nu_A \operatorname{div}(\mu (\nabla \widehat{\psi} - \nabla \psi_d)) && \text{in } \Omega_I \\ \mu \nabla \eta \cdot \mathbf{n}_\Omega &= -(i\omega)^{-1} \nu_A \mu (\nabla \widehat{\psi} - \nabla \psi_d) \cdot \mathbf{n}_\Omega && \text{on } \partial\Omega, \end{aligned} \quad (3.15)$$

with the geometrical conditions

$$(M\boldsymbol{\beta})_j = -(i\omega)^{-1} \int_{\Gamma} (\sigma^{-1} \operatorname{curl} \mathbf{W} - \nu_E (\widehat{\mathbf{E}} - \mathbf{E}_d)) \cdot (\mathbf{n} \times \boldsymbol{\rho}_j) - (i\omega)^{-1} \nu_B [M(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_d)]_j \quad \forall j \in \{1, \dots, g\}. \quad (3.16)$$

The result can be shown analogously to Theorem 2.7; therefore, we do not detail the proof.

At first view, the adjoint system (3.15) exhibits a different structure than the state equation. In particular, the vector field $\mu \nabla \eta$ is not divergence free. However, we can cover both equations by the following unified form:

$$\begin{aligned} \operatorname{curl}(\sigma^{-1} \operatorname{curl} \mathbf{Q}) \pm i\omega \mu \mathbf{Q} &= \mathbf{f}_C + \operatorname{curl} \mathbf{F}_C && \text{in } \Omega_C \\ \mathbf{Q} \times \mathbf{n} - \nabla \chi \times \mathbf{n} - \sum_{j=1}^g \zeta_j \boldsymbol{\rho}_j \times \mathbf{n} &= 0 && \text{on } \Gamma \\ \mu \mathbf{Q} \cdot \mathbf{n} - \mu \nabla \chi \cdot \mathbf{n} &= \pm (i\omega)^{-1} \mathbf{f}_C \cdot \mathbf{n} - \mathbf{G}_I \cdot \mathbf{n} && \text{on } \Gamma \\ -\operatorname{div}(\mu \nabla \chi) &= -\operatorname{div} \mathbf{G}_I && \text{in } \Omega_I \\ \mu \nabla \chi \cdot \mathbf{n}_\Omega &= \mathbf{G}_I \cdot \mathbf{n}_\Omega && \text{on } \partial\Omega \\ \int_{\Gamma} \sigma^{-1} \operatorname{curl} \mathbf{Q} \cdot (\mathbf{n} \times \boldsymbol{\rho}_j) \mp i\omega (M\boldsymbol{\zeta})_j &= r_j + \int_{\Gamma} \mathbf{F}_C \cdot (\mathbf{n} \times \boldsymbol{\rho}_j) && \forall j \in \{1, \dots, g\}. \end{aligned} \quad (3.17)$$

having sign + in the first and third equations and sign – in the last equation for the state problem, and the other way around for the adjoint problem.

Precisely, for the state equation we have

$$\mathbf{f}_C = \mathbf{0}, \quad \mathbf{F}_C = \sigma^{-1} \mathbf{J}_e, \quad \mathbf{G}_I = \mathbf{0}, \quad r_j = 0,$$

whereas for the adjoint equation we have

$$\begin{aligned} \mathbf{f}_C &= \nu_C \mu (\widehat{\mathbf{H}} - \mathbf{H}_d), \quad \mathbf{F}_C = \nu_E (\widehat{\mathbf{E}} - \mathbf{E}_d), \\ \mathbf{G}_I &= -(i\omega)^{-1} \nu_A \mu (\nabla \widehat{\psi} - \nabla \psi_d), \quad r_j = -\nu_B [M(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_d)]_j. \end{aligned}$$

In particular, this says that, for the solution of the adjoint equation, the quantity $\mu \mathbf{W} \cdot \mathbf{n} - \mu \nabla \eta \cdot \mathbf{n}$ has a jump on Γ ; hence the overall field

$$\mathbf{K}_\Omega = \begin{cases} \mu \mathbf{W} & \text{in } \Omega_C \\ \mu \nabla \eta + \sum_{j=1}^g \beta_j \mu \boldsymbol{\rho}_j & \text{in } \Omega_I \end{cases}$$

has not a square-summable divergence, even if the desired fields $\mu \mathbf{H}_d$ and $\mu \nabla \psi_d$ were divergence free in Ω_C and Ω_I , respectively (this property is true for the state variables, that satisfy $\operatorname{div}(\mu \widehat{\mathbf{H}}) = 0$ in Ω_C and $\operatorname{div}(\mu \nabla \widehat{\psi}) = 0$ in Ω_I). In contrast to this, the solution of the state equation is the magnetic field

$$\mathbf{H}_\Omega = \begin{cases} \mathbf{H} & \text{in } \Omega_C \\ \nabla \psi + \sum_{j=1}^g \alpha_j \boldsymbol{\rho}_j & \text{in } \Omega_I, \end{cases}$$

whose associated magnetic induction $\mathbf{B}_\Omega = \mu \mathbf{H}_\Omega$ is divergence free, as the magnetic Gauss law requires.

Example 3.11. Let us consider the particular choice $\nu_A = \nu_C$ and assume in addition that the desired fields \mathbf{H}_d and ψ_d are compatible on the interface, i.e.,

$$\mu \mathbf{H}_d \cdot \mathbf{n} = \mu \nabla \psi_d \cdot \mathbf{n} \quad \text{on } \Gamma. \quad (3.18)$$

Since also $\mu \widehat{\mathbf{H}} \cdot \mathbf{n} = \mu \nabla \widehat{\psi} \cdot \mathbf{n}$ holds on Γ , the second interface condition of the adjoint system then simplifies to

$$\mu \mathbf{W} \cdot \mathbf{n} = \mu \nabla \eta \cdot \mathbf{n} \quad \text{on } \Gamma.$$

Therefore, the jump between $\mu \mathbf{W} \cdot \mathbf{n}$ and $\mu \nabla \eta \cdot \mathbf{n}$ disappears and the field \mathbf{K}_Ω defined here above is divergence free in Ω , provided that the desired field \mathbf{H}_d and $\nabla \psi_d$ satisfy $\operatorname{div}(\mu \mathbf{H}_d) = 0$ in Ω_C and $\operatorname{div}(\mu \nabla \psi_d) = 0$ in Ω_I .

Remark 3.12. For the sake of completeness, let us also point out that the weak form of (3.17) reads

$$\begin{aligned} & \int_{\Omega_C} \sigma^{-1} \operatorname{curl} \mathbf{Q} \cdot \operatorname{curl} \bar{\mathbf{p}} \pm i\omega \int_{\Omega_C} \mu \mathbf{Q} \cdot \bar{\mathbf{p}} \pm i\omega \int_{\Omega_I} \mu \nabla \chi \cdot \nabla \bar{\vartheta} \pm i\omega M \boldsymbol{\zeta} \cdot \bar{\boldsymbol{\pi}} \\ & = \int_{\Omega_C} \mathbf{f}_C \cdot \bar{\mathbf{p}} + \int_{\Omega_C} \mathbf{F}_C \cdot \operatorname{curl} \bar{\mathbf{p}} \pm i\omega \int_{\Omega_I} \mathbf{G}_I \cdot \nabla \bar{\vartheta} - \mathbf{r} \cdot \bar{\boldsymbol{\pi}}, \end{aligned} \quad (3.19)$$

for each $(\mathbf{p}, \vartheta, \boldsymbol{\pi}) \in \mathbf{V}_0$, with sign + for the state problem and sign – for the adjoint problem. The antilinear form at the left hand side is continuous and coercive in $V \times V$, for both choices of the sign. The right hand side is an antilinear functional on V , provided that $\mathbf{f}_C \in L^2(\Omega_C)^3$, $\mathbf{F}_C \in L^2(\Omega_C)^3$, $\mathbf{G}_I \in L^2(\Omega_I)^3$ and $\mathbf{r} \in \mathbb{C}^g$.

3.4. Discussion of the optimality conditions for different cases of \mathbf{J}_{ad} .

Let us discuss the conclusions of the necessary optimality conditions for several particular cases of \mathbf{J}_{ad} .

3.4.1. Unbounded complex control vectors. If $\nu > 0$, the unbounded control set

$$\mathbf{J}_{ad} = L^2(\Omega_C)^3 \quad (3.20)$$

can be used. Notice that the choice $\nu = 0$ is not appropriate in this case, because we cannot prove the existence of an optimal control unless the desired fields \mathbf{H}_d and \mathbf{E}_d are in the range of the control-to-state mapping.

For unbounded controls, i.e., for $\mathbf{J}_{ad} = L^2(\Omega_C)^3$, the equation $\mathbf{D}_{J^*} + \nu \mathbf{J}^* = 0$ is necessary and sufficient for the optimality of \mathbf{J}^* , i.e.

$$\mathbf{J}^* = -\frac{1}{\nu} \mathbf{D}_{J^*}.$$

These conditions follow immediately from the variational inequality (3.9).

3.4.2. Complex control vectors bounded by box constraints. For all $\nu \geq 0$, the set

$$\mathbf{J}_{ad} = \left\{ \mathbf{J} \in L^2(\Omega_C)^3 : |\operatorname{Re} J_\ell(\mathbf{x})| \leq \operatorname{Re}_{\max}, |\operatorname{Im} J_\ell(\mathbf{x})| \leq \operatorname{Im}_{\max} \right. \\ \left. \text{for } \ell = 1, 2, 3, \text{ and for almost all } \mathbf{x} \in \Omega_C \right\} \quad (3.21)$$

might be taken, if positive bounds Re_{\max} and Im_{\max} must be imposed on the possible currents.

If \mathbf{J}_{ad} is defined by box constraints as in (3.21), then we perform the following standard discussion: Using the representation (3.14), the variational inequality (3.9) can be re-written as

$$\operatorname{Re} \int_{\Omega_C} \nabla F(\mathbf{J}^*) \cdot \bar{\mathbf{J}} \leq \operatorname{Re} \int_{\Omega_C} \nabla F(\mathbf{J}^*) \cdot \bar{\mathbf{J}} \quad \forall \mathbf{J} \in \mathbf{J}_{ad}.$$

Expanding the terms under the integral, we find

$$\begin{aligned} & \int_{\Omega_C} \operatorname{Re}(\nabla F(\mathbf{J}^*)) \cdot \operatorname{Re} \mathbf{J}^* + \int_{\Omega_C} \operatorname{Im}(\nabla F(\mathbf{J}^*)) \cdot \operatorname{Im} \mathbf{J}^* \\ & \leq \int_{\Omega_C} \operatorname{Re}(\nabla F(\mathbf{J}^*)) \cdot \operatorname{Re} \mathbf{J} + \int_{\Omega_C} \operatorname{Im}(\nabla F(\mathbf{J}^*)) \cdot \operatorname{Im} \mathbf{J} \quad \forall \mathbf{J} \in \mathbf{J}_{ad}. \end{aligned}$$

This inequality can now be evaluated by a sequence of standard steps. First of all, we observe that $\operatorname{Re} \mathbf{J}$ and $\operatorname{Im} \mathbf{J}$ can be chosen completely independent. Therefore, the inequality above is equivalent to the following two inequalities:

$$\begin{aligned} & \int_{\Omega_C} \operatorname{Re}(\nabla F(\mathbf{J}^*)) \cdot \operatorname{Re} \mathbf{J}^* \leq \int_{\Omega_C} \operatorname{Re}(\nabla F(\mathbf{J}^*)) \cdot \operatorname{Re} \mathbf{J} \quad \forall \mathbf{J} : |\operatorname{Re} \mathbf{J}(\cdot)| \leq \operatorname{Re}_{\max}, \\ & \int_{\Omega_C} \operatorname{Im}(\nabla F(\mathbf{J}^*)) \cdot \operatorname{Im} \mathbf{J}^* \leq \int_{\Omega_C} \operatorname{Im}(\nabla F(\mathbf{J}^*)) \cdot \operatorname{Im} \mathbf{J} \quad \forall \mathbf{J} : |\operatorname{Im} \mathbf{J}(\cdot)| \leq \operatorname{Im}_{\max}. \end{aligned}$$

Here, the inequalities $|\operatorname{Re} \mathbf{J}(\cdot)| \leq \operatorname{Re}_{\max}$ and $|\operatorname{Im} \mathbf{J}(\cdot)| \leq \operatorname{Im}_{\max}$ have to be understood in pointwise and componentwise sense. For instance, this means

$$|\operatorname{Re} J_\ell(\mathbf{x})| \leq \operatorname{Re}_{\max} \text{ for a.a. } \mathbf{x} \in \Omega_C \text{ and all } \ell \in \{1, 2, 3\}.$$

These inequalities with control functions appearing under the integral can be discussed further in a pointwise way (for this type of argument, see, e.g., [28, Sect. 2.8]). For instance, the first inequality is equivalent to the condition that

$$\operatorname{Re} \nabla F(\mathbf{J}^*)(\mathbf{x}) \cdot \operatorname{Re} \mathbf{J}^*(\mathbf{x}) \leq \operatorname{Re} \nabla F(\mathbf{J}^*)(\mathbf{x}) \cdot \mathbf{v} \quad \forall \mathbf{v} \in \mathbb{R}^3 : |v_\ell| \leq \operatorname{Re}_{\max}, \ell \in \{1, 2, 3\}$$

holds for almost all $\mathbf{x} \in \Omega_C$. All components of the vector $\mathbf{v} \in \mathbb{R}^3$ can be selected independently. Then the inequality above means for the ℓ th component that

$$\min_{v \in \mathbb{R} : |v| \leq \operatorname{Re}_{\max}} \operatorname{Re}(\nabla F(\mathbf{J}^*))_\ell(\mathbf{x}) v = \operatorname{Re}(\nabla F(\mathbf{J}^*))_\ell(\mathbf{x}) \operatorname{Re} J_\ell^*(\mathbf{x}),$$

i.e., that, for a.a. $\mathbf{x} \in \Omega_C$, the minimum at the left-hand side is attained by $\operatorname{Re} J_\ell^*(\mathbf{x})$.

Inserting the concrete expression for the reduced gradient ∇F (see (3.14)), we find

$$\operatorname{Re} J_\ell^*(\mathbf{x}) = \begin{cases} -\operatorname{Re}_{\max}, & \text{if } \operatorname{Re}(\mathbf{D}_{J^*} + \nu \mathbf{J}^*)_\ell(\mathbf{x}) > 0 \\ \operatorname{Re}_{\max}, & \text{if } \operatorname{Re}(\mathbf{D}_{J^*} + \nu \mathbf{J}^*)_\ell(\mathbf{x}) < 0 \end{cases} \quad (3.22)$$

for almost all $\mathbf{x} \in \Omega_C$ and all $\ell \in \{1, 2, 3\}$. The formula for the imaginary part is the same with Im substituted for Re .

If the Tikhonov regularization parameter is positive, then this is equivalent to the projection formula

$$\operatorname{Re} J_\ell^*(\mathbf{x}) = \mathbb{P}_{[-\operatorname{Re}_{\max}, \operatorname{Re}_{\max}]} \left\{ -\frac{1}{\nu} \operatorname{Re}(\mathbf{D}_{J^*})_\ell(\mathbf{x}) \right\} \quad (3.23)$$

for almost all $\mathbf{x} \in \Omega_C$ and all $\ell \in \{1, 2, 3\}$. Here, the projection function $\mathbb{P}_{[a,b]} : \mathbb{R} \rightarrow [a,b]$ is defined by

$$\mathbb{P}_{[a,b]}(s) := \max(a, \min(b, s)).$$

This projection formula (3.23) is a standard result of optimal control theory, but it is not completely trivial. It can be explained by the fact that the minimizer v^* of the real function $v \mapsto \frac{\nu}{2}v^2 + dv$ in the interval $[a, b]$ is

$$v^* = \arg \min_{v \in [a, b]} \left\{ \frac{\nu}{2}v^2 + dv \right\} = \mathbb{P}_{[a, b]} \left\{ -\frac{1}{\nu}d \right\}.$$

We refer to [28, Theor. 2.28].

Example 3.13 (Optimal control as inverse problem).

In electro-encephalography (EEG) or magneto-encephalography (MEG), magnetic or electric fields associated to the electrical activity of the human brain are measured. Then one looks for the electrical currents, located in certain regions of the brain, that generated these fields. Under certain assumptions, this problem can be cast into the form of our optimal control problem, where the desired fields \mathbf{H}_d and \mathbf{E}_d stand for the measurements. Normally, these measurements can be taken only at the boundary Γ of the conductor, say at the surface of the human head Ω_C (which can be assumed to be simply connected). Moreover, they are only given at certain points. Let us assume that these measurements can be interpolated to get a measurement of $\mu \mathbf{H}_d \cdot \mathbf{n}$ on the interface Γ . In view of the interface conditions on Γ , we have then also $\mu \nabla \psi_d \cdot \mathbf{n}$ on Γ . Together with the homogeneous boundary conditions on $\partial\Omega$, we then can determine the harmonic scalar potential ψ_d and hence also $\nabla \psi_d$ in Ω_I that can serve as measurement in Ω_I .

In the case of an inverse problem, one cannot prescribe any particular form or direction of the unknown electrical current \mathbf{J}_e . Here the general class \mathbf{J}_{ad} of arbitrary bounded \mathbf{L}^2 -controls is meaningful indeed, and the optimal control problem of minimizing the objective functional F is nothing more than the inverse problem to determine unknown electrical currents by given measurements. Here, we have $\nu_C = 0$, $\nu_E = 0$, ν_B is not considered (since we have $g = 0$), and finally ν is a Tikhonov regularization parameter that is standard for (ill posed) inverse problems.

Meaningful selections of \mathbf{J}_{ad} are the definitions (3.20) and (3.21).

3.4.3. Electrical current in an induction coil. Another typical application is the case where the electrical current is prescribed in an induction coil (see, e.g., [25]). A standard induction coil is composed by one wire that is twisted in many windings around the core. Here, the direction of the electrical current in one point is very precisely given by the direction of the wire in that point. The strength j of the current is the only unknown that is to be determined.

In that case, the control \mathbf{J}_e has the form

$$\mathbf{J}_e(\mathbf{x}) = \frac{N_\star}{Q_{\text{coil}}} \mathbf{e}(\mathbf{x}) j$$

where j is a complex number, the unit vector function \mathbf{e} is the direction of the wire in the point \mathbf{x} of the coil, N_\star is the number of windings and Q_{coil} is the area of the cross section of the coil that is perpendicular to the direction of the windings.

Let us assume for convenience that $N_\star/Q_{\text{coil}} = 1$ to simplify our notation. This does not cause any restriction, because this constant can be included by the choice of the constants Re_{max} and Im_{max} in (3.25) below.

Example 3.14. In [25] the following geometry was chosen for the induction coil, which is topologically equivalent to a torus:

$$\Omega_{\text{coil}} = \{\mathbf{x} \in \mathbb{R}^3 : 0 < r_1 < x_1^2 + x_2^2 < r_2, c_1 < x_3 < c_2\},$$

where $r_2 > r_1 > 0$ and $c_1 < c_2$ are given real numbers. Here the function \mathbf{e} is defined by

$$\mathbf{e}(x_1, x_2, x_3) = \begin{cases} \frac{1}{\sqrt{x_1^2 + x_2^2}} \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} & \text{in } \Omega_{\text{coil}} \\ 0 & \text{in } \Omega_C \setminus \text{cl } \Omega_{\text{coil}}. \end{cases} \quad (3.24)$$

In Ω_{coil} , \mathbf{e} is a unit vector.

Notice that in this case the control is just one complex number. Here, the analogue of (3.21) is

$$\mathbf{J}_{ad} = \{\mathbf{e}(\cdot)j : |\text{Re } j| \leq \text{Re}_{\max} \text{ and } |\text{Im } j| \leq \text{Im}_{\max}\}. \quad (3.25)$$

Next, we derive the necessary optimality conditions for this case. Let j^* be optimal and set $\mathbf{J}^* = \mathbf{e}j^*$. We begin with the general variational inequality (3.13) and obtain

$$\begin{aligned} 0 &\leq \text{Re} \int_{\Omega_C} (\mathbf{D}_{J^*} + \nu \mathbf{J}^*) \cdot (\bar{J} - \bar{J}^*) = \text{Re} \int_{\Omega_C} (\mathbf{D}_{J^*} + \nu \mathbf{e}j^*) \cdot \mathbf{e}(\bar{j} - \bar{j}^*) \\ &= \int_{\Omega_C} \text{Re}(\mathbf{D}_{J^*} + \nu \mathbf{e}j^*) \cdot \mathbf{e} \text{Re}(j - j^*) + \int_{\Omega_C} \text{Im}(\mathbf{D}_{J^*} + \nu \mathbf{e}j^*) \cdot \mathbf{e} \text{Im}(j - j^*) \\ &= \text{Re}(j - j^*) \int_{\Omega_C} \text{Re}(\mathbf{D}_{J^*} \cdot \mathbf{e} + \nu j^*) + \text{Im}(j - j^*) \int_{\Omega_C} \text{Im}(\mathbf{D}_{J^*} \cdot \mathbf{e} + \nu j^*) \quad \forall j \in j_{ad}. \end{aligned}$$

This inequality can be discussed as in Section (3.4.2). We deduce for $\nu \geq 0$

$$\text{Re } j^* = \begin{cases} -\text{Re}_{\max}, & \text{if } \int_{\Omega_C} \text{Re}(\mathbf{D}_{J^*} \cdot \mathbf{e} + \nu j^*) > 0 \\ \text{Re}_{\max}, & \text{if } \int_{\Omega_C} \text{Re}(\mathbf{D}_{J^*} \cdot \mathbf{e} + \nu j^*) < 0. \end{cases} \quad (3.26)$$

If $\nu > 0$, we have the projection formula

$$\text{Re } j^* = \mathbb{P}_{[-\text{Re}_{\max}, \text{Re}_{\max}]} \left\{ -\frac{1}{\nu} \int_{\Omega_C} \text{Re } \mathbf{D}_{J^*} \cdot \mathbf{e} \right\}. \quad (3.27)$$

Analogous conditions are satisfied by $\text{Im } j^*$ with Im substituted for Re .

3.4.4. Electrical currents in a package of wires. The following situation is somehow intermediate between the two cases mentioned above. Here, the induction coil is composed of a package of single wires. The electrical currents in each wire can be controlled separately. It is assumed that each one of these currents can be controlled independently from the others. The cross section of this package of wires can be viewed as a discrete approximation of a function $j : \Omega_{\text{coil}} \rightarrow \mathbb{C}$ that stands for the strength of the current while the direction is still given by a function such as \mathbf{e} above.

Let us consider the geometry of Example 3.14. Here, the strength j of the current depends only on the radius r and the coordinate x_3 , while the direction of the current is given again by \mathbf{e} . In terms of cylindrical coordinates, this reads

$$\mathbf{J}_e = \mathbf{e}(r, \varphi, z) j(r, z),$$

where $r_1 \leq r \leq r_2$, $0 \leq \varphi < 2\pi$, $c_1 \leq z \leq c_2$.

In this situation, a useful set of admissible control functions might be

$$\mathbf{J}_{ad} = \mathbf{e} j_{ad} \tag{3.28}$$

where we take controls out of the complex space $L^2((r_1, r_2) \times (c_1, c_2))$,

$$j_{ad} = \{j \in L^2((r_1, r_2) \times (c_1, c_2)) : |\operatorname{Re} j| \leq \operatorname{Re}_{\max} \text{ and } |\operatorname{Im} j| \leq \operatorname{Im}_{\max}\}$$

and the actual control function would be $j \in L^2((r_1, r_2) \times (c_1, c_2))$.

Given a partition of the square $[r_1, r_2] \times [c_1, c_2]$ by finitely many small rectangles, an associated finite-dimensional approximation of j by a step function would define the finitely many single currents in the rectangles. These rectangles can be interpreted as the cross sections of the wires of the package of wires. This view is certainly academic but it gives an interpretation on how a controlled distributed current might be generated.

The necessary optimality conditions are analogous to (3.22) and (3.23), but $(r_1, r_2) \times (c_1, c_2)$ must be substituted for Ω_C . For instance, the optimal solution obeys the projection formula

$$\operatorname{Re} j^*(r, z) = \mathbb{P}_{[-\operatorname{Re}_{\max}, \operatorname{Re}_{\max}]} \left\{ -\frac{1}{\nu} \int_0^{2\pi} \operatorname{Re} \mathbf{D}_{J^*}(r, z) \cdot \mathbf{e}(r, \varphi, z) d\varphi \right\} \tag{3.29}$$

for almost all $(r, z) \in [r_1, r_2] \times [c_1, c_2]$.

3.4.5. Real current vectors. A smaller but perhaps more realistic class of controls \mathbf{J} has the particular form

$$\mathbf{J}(\mathbf{x}) = e^{i\phi} J(\mathbf{x}), \tag{3.30}$$

where J is a real vector function and ϕ is fixed. Here, J varies in the admissible set

$$J_{ad} = \{J \in L^2_{\mathbb{R}}(\Omega_C)^3 : -j_{\max} \leq J_\ell(\mathbf{x}) \leq j_{\max} \text{ for a.a. } \mathbf{x} \in \Omega_C, \text{ all } \ell \in \{1, 2, 3\}\} \tag{3.31}$$

with a given bound $j_{\max} > 0$. To cover this ansatz by the optimal control problem (3.3), we define the functional

$$f(J) := F(e^{i\phi} J)$$

and consider the problem

$$\min_{J \in J_{ad}} f(J). \tag{3.32}$$

This is nothing more than a particular case of the optimal control problem (3.3) subject to the particular control set defined by (3.30) and (3.31).

The associated optimal control $\mathbf{J}^* = e^{i\phi} J^*$ has to obey the necessary optimality conditions of Theorem 3.6, in particular the variational inequality (3.9), or, using the notation (3.12),

$$\operatorname{Re} \int_{\Omega_C} (\mathbf{D}_{J^*} + \nu \mathbf{J}^*) \cdot (\bar{\mathbf{J}} - \bar{\mathbf{J}}^*) \geq 0 \quad \forall \mathbf{J} \in \mathbf{J}_{ad}.$$

In view of the particular ansatz (3.30), this variational inequality can be simplified: inserting the particular form of \mathbf{J} , we find

$$\operatorname{Re} \int_{\Omega_C} (\mathbf{D}_{J^*} + \nu e^{i\phi} J^*) \cdot e^{-i\phi} (J - J^*) \geq 0 \quad \forall J \in J_{ad},$$

or

$$\int_{\Omega_C} (D_{J^*} + \nu J^*) \cdot (J - J^*) dx \geq 0 \quad \forall J \in J_{ad}, \quad (3.33)$$

with

$$D_{J^*} := \operatorname{Re} (e^{-i\phi} \mathbf{D}_{J^*}). \quad (3.34)$$

The further pointwise discussion of (3.33) is analogous to (3.22) and (3.23), where “Re” can be omitted, since all quantities in (3.33) are real.

4. Sparse optimal controls.

4.1. Introduction to sparse controls. In the problem of controlling the current in a package of independent wires, the whole cross section of the induction coil is densely filled with wires. However, it might happen as the result of numerical calculations that only some part of the wires is really important while the optimal current in some others is negligible. In such cases, one might be interested to find out those wires that are most important for achieving the desired goal of optimization. The result would be a better geometry of the coil. This is an issue, where the method of sparse controls might be useful.

Sparsity techniques originated from the field of image processing, where L^1 distance functionals are used for some purpose. In the context of optimal control of partial differential equations, a first reference is [27], where this technique is applied to problems with linear elliptic equations. This opened an active research in this field. We refer to the contributions [5], [6], [7], [8], [14] to the application of sparsity methods for different types of elliptic or parabolic PDEs.

To our best knowledge, the method of sparse control was not yet applied in the control of electromagnetic fields. Though the underlying analysis does not essentially differ from that in the papers mentioned above, we think it is worth presenting it for our particular setting. Our analysis follows the steps outlined in [8]. However, since our electromagnetic fields might be unbounded, we obtain a slightly weaker result.

To start the discussion of sparse control, we recall the problem (3.32) that has a quadratic, hence smooth objective functional F . For sparse controls, we add to this functional a multiple of the L^1 -norm of J and consider the functional

$$J \mapsto f(J) + \kappa \sum_{\ell=1}^3 \int_{\Omega_C} |J_\ell(\mathbf{x})|, \quad (4.1)$$

where κ is the so-called sparse parameter. For convenience, we define

$$\begin{aligned}\gamma(j) &:= \int_{\Omega_C} |j(\mathbf{x})| \\ g(J) &:= \sum_{\ell=1}^3 \gamma(J_\ell) = \sum_{\ell=1}^3 \int_{\Omega_C} |J_\ell(\mathbf{x})|.\end{aligned}$$

This motivates the following optimal control problem with sparse parameter κ :

$$\min_{J \in J_{ad}} \{f(J) + \kappa g(J)\}. \quad (4.2)$$

Again, the existence of an optimal control $J^* \in J_{ad}$ follows by standard arguments. If $\nu > 0$, then the objective functional of (4.2) is strictly convex and hence the optimal control is unique.

We shall sketch below that the sparse parameter influences the size of the support of the optimal control of the problem (4.2). The larger κ is, the smaller is the support of the optimal control.

To understand this effect that is meanwhile well studied (see, for instance, [5], [6], [7], [8], [14], [27]), we first have to set up the associated system of necessary optimality conditions. To this aim, we need the subdifferential $\partial\gamma(j)$ of the convex but non-differentiable functional $\gamma : L_{\mathbb{R}}^1(\Omega_C) \rightarrow \mathbb{R}$ at an arbitrary but fixed $j \in L_{\mathbb{R}}^1(\Omega_C)$.

This subdifferential is the set of all elements $\lambda \in L_{\mathbb{R}}^\infty(\Omega_C)$ such that

$$\gamma(v) \geq \gamma(j) + \int_{\Omega_C} \lambda(\mathbf{x}) (v(\mathbf{x}) - j(\mathbf{x})) \quad \forall v \in L_{\mathbb{R}}^1(\Omega_C). \quad (4.3)$$

It is hence defined by

$$\partial\gamma(j) := \{\lambda \in L_{\mathbb{R}}^\infty(\Omega_C) : (4.3) \text{ is satisfied}\}.$$

The following representation is known for $\partial\gamma(j)$ (see, e.g., [20, Sect. 4.5.1]):

$$\partial\gamma(j) := \left\{ \lambda \in L_{\mathbb{R}}^\infty(\Omega_C) : \lambda \text{ satisfies (4.5) below} \right\}, \quad (4.4)$$

$$\lambda(\mathbf{x}) = \begin{cases} 1, & \text{if } j(\mathbf{x}) > 0 \\ [-1, 1], & \text{if } j(\mathbf{x}) = 0 \\ -1, & \text{if } j(\mathbf{x}) < 0. \end{cases} \quad (4.5)$$

After some easy computation, the subdifferential of g is obtained as

$$\partial g(J) = \{\Lambda = (\lambda_1, \lambda_2, \lambda_3) \in L_{\mathbb{R}}^\infty(\Omega_C)^3 : \lambda_\ell \in \partial\gamma(J_\ell), \ell = 1, 2, 3\}; \quad (4.6)$$

notice that $g(J) = \sum_{\ell=1}^3 \gamma(J_\ell)$.

4.2. Necessary optimality conditions. For the case $\kappa = 0$, where the functional (4.2) is differentiable, we derived the variational inequality (3.33) as necessary condition. In the case $\kappa > 0$, the variational inequality (3.33) has to be complemented by the subdifferential of g . The following result is obtained:

THEOREM 4.1 (Necessary conditions for sparse optimal controls). *Let J^* be the optimal control for the problem (4.2) and let $\mathbf{J}^* := e^{i\phi} J^*$. Then there exists a*

unique adjoint state $(\mathbf{W}_{J^*}, \eta_{J^*}, \boldsymbol{\alpha}_{J^*})$ solving the adjoint equation (3.5) and a function $\Lambda^* \in \partial g(J^*) \subset L_{\mathbb{R}}^\infty(\Omega_C)^3$ such that the variational inequality

$$\int_{\Omega_C} (D_{J^*} + \nu J^* + \kappa \Lambda^*) \cdot (J - J^*) \geq 0 \quad \forall J \in J_{ad} \quad (4.7)$$

is satisfied. Here, we have $D_{J^*} = \operatorname{Re}(e^{-i\phi} \mathbf{D}_{J^*})$, where

$$\mathbf{D}_{J^*} = \sigma^{-1} \operatorname{curl} \mathbf{W}_{J^*} - \nu_E (\mathbf{E}_{J^*} - \mathbf{E}_d)$$

is defined according to (3.12).

Proof. The main line of proof is more or less standard in convex optimization. However, it is not completely obvious how the associated ideas should be merged to derive our result in the case of optimal control. We therefore detail the proof for convenience of the reader.

Due to our notation, J^* minimizes the functional

$$\Phi := f + \kappa g$$

in the set J_{ad} . In a first step, we derive an auxiliary variational inequality by differentiating only the smooth part f of Φ . For all $0 \leq s \leq 1$ and arbitrary fixed $J \in J_{ad}$, we have

$$\begin{aligned} 0 &\leq \frac{\Phi(J^* + s(J - J^*)) - \Phi(J^*)}{s} \\ &\leq \frac{f(J^* + s(J - J^*)) - f(J^*)}{s} + \kappa(g(J) - g(J^*)) \end{aligned}$$

because g is convex. Passing to the limit $s \downarrow 0$, it follows

$$0 \leq f'(J^*)(J - J^*) + \kappa g(J) - \kappa g(J^*) \quad \forall J \in J_{ad}.$$

This variational inequality is a standard result for minimizing the sum of a convex and of a differentiable functional (see [11, Chap. II, Prop. 2.2]). It can be re-written as

$$f'(J^*)J^* + \kappa g(J^*) \leq f'(J^*)J + \kappa g(J) \quad \forall J \in J_{ad}.$$

In other words, we have

$$J^* \in \arg \min_{J \in J_{ad}} \{f'(J^*)J + \kappa g(J)\}. \quad (4.8)$$

Next, we include the constraint $J \in J_{ad}$ in the objective functional. To this aim, we introduce the indicator function

$$\Psi_{J_{ad}}(J) = \begin{cases} 0, & J \in J_{ad} \\ \infty, & \text{else.} \end{cases}$$

We also define the linear part of the functional above by

$$\varphi : J \mapsto f'(J^*)J = \int_{\Omega_C} (D_{J^*} + \nu J^*) \cdot J.$$

Thanks to (4.8), J^* is the minimizer of the convex optimization problem

$$J^* = \arg \min \{ \varphi(J) + \kappa g(J) + \Psi_{J_{ad}}(J) \},$$

and hence J^* must satisfy the associated necessary optimality condition

$$0 \in \partial(\varphi + \kappa g + \Psi_{J_{ad}})(J^*). \quad (4.9)$$

The subdifferential of $\partial\Psi_{J_{ad}}$ is equal to the normal cone $N_{J_{ad}}$ at J^* , where

$$N_{J_{ad}}(J^*) = \left\{ z \in L_{\mathbb{R}}^{\infty}(\Omega_C)^3 : \int_{\Omega_C} z \cdot (J - J^*) dx \leq 0 \quad \forall J \in J_{ad} \right\},$$

if $J^* \in J_{ad}$. For $J^* \notin J_{ad}$, we have $N_{J_{ad}}(J^*) = \emptyset$. Applying the theorem of Moreau–Rockafellar (see [20, Sect. 4.2.2]), we find

$$\begin{aligned} \partial(\varphi + \kappa g + \Psi_{J_{ad}})(J^*) &= \partial\varphi(J^*) + \kappa \partial g(J^*) + \partial\Psi_{J_{ad}}(J^*) \\ &= (D_{J^*} + \nu J^*) + \kappa \partial g(J^*) + N_{J_{ad}}(J^*); \end{aligned}$$

notice that the assumptions of the Moreau–Rockafellar theorem are satisfied, because the functional $\varphi + \kappa g$ is continuous on the whole space $L_{\mathbb{R}}^2(\Omega_C)$. By (4.9), we have

$$-(D_{J^*} + \nu J^*) \in \kappa \partial g(J^*) + N_{J_{ad}}(J^*),$$

i.e., there exist $\Lambda^* \in \partial g(J^*) \subset L_{\mathbb{R}}^{\infty}(\Omega_C)^3$ (notice that $g : L_{\mathbb{R}}^1(\Omega_C)^3 \rightarrow \mathbb{R}$, hence the properties of ∂g remain true, if the argument J even belongs to $L_{\mathbb{R}}^2(\Omega_C)^3$) and $Z^* \in N_{J_{ad}}(J^*)$ such that

$$-(D_{J^*} + \nu J^* + \kappa \Lambda^*) = Z^* \in N_{J_{ad}}(J^*).$$

By definition of $N_{J_{ad}}(J^*)$ this means

$$-\int_{\Omega_C} (D_{J^*} + \nu J^* + \kappa \Lambda^*) \cdot (J - J^*) \leq 0 \quad \forall J \in J_{ad},$$

the inequality being equivalent to (4.7). \square

Let us describe a few consequences of this theorem. The main one is the sparsity of the optimal control J^* .

COROLLARY 4.2 (Sparsity). *Assume $\nu > 0$ and $\kappa > 0$ and let J^* be optimal for the problem (4.2). Then, for $\ell = 1, 2, 3$,*

$$J_{\ell}^*(\mathbf{x}) = 0 \text{ if and only if } \kappa \geq |(D_{J^*})_{\ell}(\mathbf{x})| \quad (4.10)$$

holds for a.a. $\mathbf{x} \in \Omega_C$. For almost all $\mathbf{x} \in \Omega_C$, the element $\Lambda^ = (\lambda_1^*, \lambda_2^*, \lambda_3^*) \in \partial g(J^*)$ is given by the projection formula*

$$\lambda_{\ell}^*(\mathbf{x}) = \mathbb{P}_{[-1,1]} \left\{ -\frac{1}{\kappa} (D_{J^*})_{\ell}(\mathbf{x}) \right\}, \quad \ell \in \{1, 2, 3\}. \quad (4.11)$$

Proof. Let us fix $\ell \in \{1, 2, 3\}$. To avoid an extensive use of subscripts, let us write for short $D^* := D_{J^*}$. First, we show the implication $\kappa \geq |D_{\ell}^*(\mathbf{x})| \Rightarrow J_{\ell}^*(\mathbf{x}) = 0$. Assume the contrary, i.e., $J_{\ell}^*(\mathbf{x}) \neq 0$. It follows from (4.7) that

$$\begin{aligned} J_{\ell}^*(\mathbf{x}) > 0 &\Rightarrow (D_{\ell}^* + \nu J_{\ell}^* + \kappa \lambda_{\ell}^*)(\mathbf{x}) \leq 0 \\ J_{\ell}^*(\mathbf{x}) < 0 &\Rightarrow (D_{\ell}^* + \nu J_{\ell}^* + \kappa \lambda_{\ell}^*)(\mathbf{x}) \geq 0. \end{aligned}$$

If $J_\ell^*(\mathbf{x}) > 0$, then we have $\lambda_\ell^*(\mathbf{x}) = 1$ and the first case above implies

$$D_\ell^*(\mathbf{x}) + \nu J_\ell^*(\mathbf{x}) + \kappa \leq 0,$$

hence

$$0 < \nu J_\ell^*(\mathbf{x}) \leq -D_\ell^*(\mathbf{x}) - \kappa$$

must hold; this yields

$$\kappa < -D_\ell^*(\mathbf{x}), \quad (4.12)$$

a contradiction. Analogously, if $J_\ell^*(\mathbf{x}) < 0$, then $\lambda_\ell^*(\mathbf{x}) = -1$ and $D_\ell^*(\mathbf{x}) + \nu J_\ell^*(\mathbf{x}) - \kappa \geq 0$ must hold. This leads to

$$\kappa < D_\ell^*(\mathbf{x}), \quad (4.13)$$

a contradiction. Altogether, we have proved that

$$\kappa \geq |D_\ell^*(\mathbf{x})| \quad \Rightarrow \quad J_\ell^*(\mathbf{x}) = 0.$$

Next, we verify the converse implication $J_\ell^*(\mathbf{x}) = 0 \Rightarrow \kappa \geq |D_\ell^*(\mathbf{x})|$. From the variational inequality (4.7) we deduce for almost all $\mathbf{x} \in \Omega_C$

$$0 = (D_\ell^* + \nu J_\ell^* + \kappa \lambda_\ell^*)(\mathbf{x}) = (D_\ell^* + \kappa \lambda_\ell^*)(\mathbf{x}),$$

hence $|D_\ell^*(\mathbf{x})| = \kappa |\lambda_\ell^*(\mathbf{x})|$. By the definition of the subdifferential, we have $|\lambda_\ell^*(\mathbf{x})| \leq 1$. Therefore, $\kappa \geq |D_\ell^*(\mathbf{x})|$ must be satisfied.

Finally, let us confirm the projection formula (4.11). For $J_\ell^*(\mathbf{x}) = 0$, we found $D_\ell^*(\mathbf{x}) + \kappa \lambda_\ell^*(\mathbf{x}) = 0$, i.e. $\lambda_\ell^*(\mathbf{x}) = -\kappa^{-1} D_\ell^*(\mathbf{x})$. Since $|\lambda_\ell^*(\mathbf{x})| \leq 1$, this implies (4.11). For $J_\ell^*(\mathbf{x}) > 0$ we have derived the inequality (4.12) that yields

$$\lambda_\ell^*(\mathbf{x}) = 1 < -\frac{D_\ell^*(\mathbf{x})}{\kappa}.$$

Again, this complies with (4.11). Analogously, we invoke (4.13), if $J_\ell^*(\mathbf{x}) < 0$. \square

In view of this result, we can expect that for increasing κ the support of the optimal control functions J_ℓ^* becomes smaller. This is expressed by the following conclusion.

COROLLARY 4.3. *Assume $\nu > 0$ and denote by J_κ^* the optimal control of the problem (4.2) for given $\kappa > 0$. Then there holds*

$$\lim_{\kappa \rightarrow \infty} \text{meas}\{\mathbf{x} \in \Omega_C : |(J_\kappa^*)_\ell(\mathbf{x})| > 0\} = 0 \quad \forall \ell \in \{1, 2, 3\}. \quad (4.14)$$

Proof. By the definition (3.31) of J_{ad} , we have a bound $c_1 > 0$ such that

$$\|J\|_{L^2_{\mathbb{R}}(\Omega_C)^3} \leq c_1 \quad \forall J \in J_{ad}.$$

This bound remains valid for all associated $\mathbf{J} = e^{i\phi} J$, $J \in J_{ad}$, of the (complex) space $L^2(\Omega_C)^3$. The control-to-state mapping $\mathbf{J} \mapsto (\mathbf{H}_J, \psi_J, \alpha_J)$ defined by the state equation (2.9) is continuous from $L^2(\Omega_C)^3$ to $H(\text{curl}; \Omega_C) \times (H^1(\Omega_I)/\mathbb{C}) \times \mathbb{C}^g$. Therefore, the mapping $\mathbf{J} \mapsto \mathbf{E}_J = \sigma^{-1}(\text{curl } \mathbf{H}_J - \mathbf{J})$ is continuous in $L^2(\Omega_C)^3$.

In the adjoint equation (3.5), considered for $\widehat{\mathbf{J}} := \mathbf{J}$, the terms $\mathbf{H}_J - \mathbf{H}_d$, $\nabla\psi_J - \nabla\psi_d$, $\alpha_J - \alpha_d$, and $\sigma^{-1}(\text{curl}\mathbf{H}_J - \mathbf{J}) - \mathbf{E}_d$ appear. In view of the continuity properties stated above, these terms depend continuously on \mathbf{J} in the spaces $H(\text{curl}; \Omega_C)$, $L^2(\Omega_C)^3$, \mathbb{C}^g , and $L^2(\Omega_C)^3$, respectively.

Therefore, also the mapping $J \mapsto (\mathbf{W}_J, \eta_J, \beta_J)$ is continuous from $L^2(\Omega_C)^3$ to the associated spaces. Consequently, in this way, the boundedness of the admissible set J_{ad} implies the boundedness of the set of all adjoint states $(\mathbf{W}_J, \eta_J, \beta_J)$ in $H(\text{curl}; \Omega_C) \times (H^1(\Omega_I)/\mathbb{C}) \times \mathbb{C}^g$ that can be generated by $J \in J_{ad}$.

By the definition (3.12) of \mathbf{D}_J , the set of all possible functions \mathbf{D}_J that are generated by the controls $J \in J_{ad}$ is bounded in $L^2(\Omega_C)^3$. Taking the real part of $e^{-i\phi}\mathbf{D}_J$, this implies the existence of $c_3 > 0$ such that

$$\|D_J\|_{L^2_{\mathbb{R}}(\Omega_C)^3} \leq c_3 \quad \forall J \in J_{ad}. \quad (4.15)$$

In particular, this holds true for $D^* = D_{J^*}$, that is related to the optimal control J^* .

After having found this bound, we argue by contradiction and assume that (4.14) is not true. Then there exist $\ell_0 \in \{1, 2, 3\}$, $\delta > 0$, and a sequence $\kappa_n \rightarrow \infty$ as $n \rightarrow \infty$, such that

$$\text{meas}(S_n) \geq \delta \quad \forall n \in \mathbb{N},$$

where

$$S_n = \{\mathbf{x} \in \Omega_C : |(J_{\kappa_n}^*)_{\ell_0}(\mathbf{x})| > 0\}.$$

Let us write for short $D_n^* := D_{J_{\kappa_n}^*}$ and $D_{n, \ell_0}^* := (D_{J_{\kappa_n}^*})_{\ell_0}$. From Corollary 4.2, condition (4.10), we deduce that

$$\kappa_n < |D_{n, \ell_0}^*(\mathbf{x})| \quad \text{a.e. in } S_n.$$

Now we find

$$c_3 \geq \|D_n^*\|_{L^2_{\mathbb{R}}(\Omega_C)^3} \geq \left(\int_{S_n} |D_{n, \ell_0}^*(\mathbf{x})|^2 \right)^{1/2} \geq \kappa_n (\text{meas}(S_n))^{1/2} \geq \kappa_n \sqrt{\delta}.$$

This is a contradiction to (4.15), since $\kappa_n \rightarrow \infty$ as $n \rightarrow \infty$. \square

Remark 4.4. By the particular form of J_{ad} , we might expect a stronger result. The set of all possible control functions J is bounded in $L^{\infty}_{\mathbb{R}}(\Omega_C)^3$ by the constant j_{\max} . If we were able to deduce from this fact that all associated functions D_J are bounded in $L^{\infty}_{\mathbb{R}}(\Omega_C)^3$ by a joint constant as well, then we would obtain the existence of some κ_0 such that the optimal controls must vanish whenever $\kappa > \kappa_0$. However, to our knowledge a boundedness result for the state functions in the space $L^{\infty}_{\mathbb{R}}(\Omega_C)^3$ is not available.

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