STRUCTURED EIGENVALUE BACKWARD ERRORS OF MATRIX PENCILS AND POLYNOMIALS WITH PALINDROMIC STRUCTURES*

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Abstract. We derive formulas for the backward error of an approximate eigenvalue of a $*$-palindromic matrix polynomial with respect to $*$-palindromic perturbations. Such formulas are also obtained for complex T-palindromic pencils and quadratic polynomials. When the T-palindromic polynomial is real, then we derive the backward error of a real number considered as an approximate eigenvalue of the matrix polynomial with respect to real T-palindromic perturbations. In all cases the corresponding minimal structure preserving perturbations are obtained as well. The results are illustrated by numerical experiments. These show that there is significant difference between the backward errors with respect to structure preserving and arbitrary perturbations in many cases.

Key words. palindromic matrix pencil, palindromic matrix polynomial, perturbation theory, eigenvalue backward error, structured eigenvalue backward error

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1. Introduction. Given $n \times n$ matrices $A_0, \ldots, A_m$, the corresponding matrix polynomial $P(z) := A_0 + zA_1 + \cdots + z^m A_m$ is said to have an eigenvalue $\lambda \in \mathbb{C}$, with corresponding eigenvector $v \in \mathbb{C}^n \setminus \{0\}$ if $P(\lambda)v = 0$. We consider such eigenvalue problems for the special case that the coefficient matrices of $P(z)$ satisfy certain symmetries. This is indicated by stating that the ordered tuple $(A_0, \ldots, A_m)$ belongs to $\mathbb{S}$ where $\mathbb{S} \subset (\mathbb{C}^{n\times n})^{m+1}$. In particular, given $\lambda \in \mathbb{C}$, we are interested in perturbations $(\Delta_0, \ldots, \Delta_m) \in \mathbb{S}$, to $(A_0, \ldots, A_m) \in \mathbb{S}$ that are minimal with respect to a specified norm such that λ is an eigenvalue of the perturbed polynomial $\widetilde{P}(z) := (A_0 - \Delta_0) + z(A_1 - \Delta_1) \cdots + z^m(A_m - \Delta_m)$. The norm of such a minimal structure preserving perturbation is called the structured backward error of λ as an approximate eigenvalue of $P(z)$. We refer to this also as the structured eigenvalue backward error of λ with respect to $P(z)$ and $\mathbb{S}$. In contrast, we refer to the norm of a minimal but not necessarily structure preserving perturbation to $P(z)$ such that $\lambda \in \mathbb{C}$ is an eigenvalue of the perturbed polynomial simply as the eigenvalue backward error of λ with respect to $P(z)$.

Matrix polynomials with symmetries in their coefficients are referred to as structured matrix polynomials. For example the coefficients of Hermitian matrix polynomials are all Hermitian matrices, i.e., they satisfy, $A_j^* = A_j, j = 0, 1, \ldots, m$ where A^* denotes the complex conjugate transpose of a matrix A. Other structured matrix polynomials closely related to the Hermitian matrix polynomials are the skew-Hermitian matrix polynomials where the coefficients are all skew-Hermitian matrices and $*$-even and $*$-odd matrix polynomials where the coefficient matrices alternate between Hermitian and skew-Hermitian structure due to which they are also referred to as $*$-alternating polynomials [20]. Yet another interesting class of structured matrix polynomials are the $*$-palindromic polynomials where the coefficient matrices satisfy

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$A_j^* = A_{m-j}, j = 0, \ldots, m$. The $*$-palindromic polynomials $P(z)$ have the property that $\operatorname{rev} P^*(z) = P(z)$ where $P^*(z) := \Sigma_{j=0}^m z^j A_j^*$, and $\operatorname{rev} P(z) := z^m P(1/z)$ represents the reversal of $P(z)$ obtained by reversing the order of the coefficient matrices in $P(z)$. Replacing the complex conjugate transpose $*$ by the transpose T in the above definition of $*$-palindromic matrix polynomials results in the T-palindromic matrix polynomials which satisfy $\operatorname{rev} P^T(z) = P(z)$. Each of the above mentioned structured matrix polynomials display a symmetry in their eigenvalue distributions as an immediate consequence of the symmetry in their coefficient matrices. For example, the eigenvalues of Hermitian and skew-Hermitian matrix polynomials occur in pairs $(\lambda, \bar{\lambda})$. Eigenvalues of $*$-even and $*$-odd matrix polynomials occur in pairs $(\lambda, -\bar{\lambda})$ when the polynomial is complex and in quadruples $(\lambda, -\bar{\lambda}, \bar{\lambda}, -\lambda)$ when the polynomial is real. On the other hand, for complex $*$-palindromic and T-palindromic matrix polynomials, these pairings are $(\lambda, 1/\bar{\lambda})$ and $(\lambda, 1/\lambda)$, respectively. If these matrix polynomials are real, then the eigenvalues occur in quadruples $(\lambda, 1/\bar{\lambda}, \bar{\lambda}, 1/\lambda)$.

This paper is a follow up of [6] which considered the similar problem of finding structured backward errors of approximate eigenvalues and associated minimal structure preserving perturbations for the particular case of the Hermitian matrix polynomial $P(z)$ and related structures like skew-Hermitian and $*$-alternating matrix polynomials. In the same work, it was stated that when the problem concerns $*$-palindromic and T-palindromic structures, the solutions require a different treatment from that of the Hermitian and related structures and the aim of this work is to focus on such structures.

As already mentioned in [6], structured matrix polynomials occur widely in various applications. For example, Hermitian matrix polynomials occur in structural mechanics, fluid flows and signal processing to name a few (for details, see [24] and references therein) while $*$-even matrix polynomials arise in linear quadratic optimal control problems [19, 22] and in gyroscopic systems [18]. On the other hand $*$-palindromic matrix polynomials occur in discrete time optimal control theory [20] and T-palindromic matrix polynomials arise in the mathematical modelling and numerical simulation of Surface Acoustic Wave (SAW) filters [25] and in the vibration analysis of railway tracks excited by high speed trains [12, 13]. It is already well established [24] that it is important to use algorithms that preserve the structure of the matrix polynomials when computing their eigenvalues so that the eigenvalue pairing associated with the structure remains intact under roundoff errors and any analysis or application based on them is physically meaningful and useful. Therefore, structure preserving perturbation and backward error analysis of eigenvalues and eigenvectors of structured matrix polynomials play a crucial role in all applications involving such polynomials. Significant contributions towards structured backward error and perturbation analysis have been made in [2, 3, 4, 10, 16, 17, 23].

The problem of finding minimal structure preserving perturbations to a given structured polynomial $P(z)$ such that the perturbed polynomial has a prescribed eigenvalue λ and a corresponding eigenvector v gives the structured backward error for the approximate *eigenpair* (λ, v) of $P(z)$. This problem has been solved for several classes of structured matrix pencils and polynomials in [2, 3]. However, except for the work done in [6], the problem of computing structured *eigenvalue* backward errors has not been undertaken so far.

The structured eigenvalue backward errors are important for the stability analysis of structure preserving algorithms that compute only eigenvalues. They are also important for finding solutions to distance problems involving structured matrices as

in the case of the distance to bounded realness for Hamiltonian matrices [5] and in the passivation of linear time invariant control systems. The backward error for an approximate eigenvalue λ of $P(z)$ may be obtained by minimizing the corresponding expression for the backward error of the approximate eigenpair (λ, v) over all nonzero vectors v. But, the expressions for the structured backward errors for approximate eigenpairs in [2, 3] indicate that this approach may not be feasible for structured eigenvalue backward errors. There are certain special situations when the backward errors of approximate eigenvalues are equal with respect to structured and arbitrary perturbations. For instance, this is the case for approximate real eigenvalues for Hermitian and skew-Hermitian matrix polynomials, purely imaginary approximate eigenvalues of $*$-alternating matrix polynomials and approximate eigenvalues on the unit circle for $*$-palindromic matrix polynomials [1, 2, 3]. Apart from these cases, in all other situations, the computation of the structured eigenvalue backward error is a challenging and important problem.

This paper is organized as follows. In section 2, we formulate our problems, provide definitions and give the preliminary results necessary for the computation of the structured eigenvalue backward errors. In section 3, we reformulate the original problem of computing the structured eigenvalue backward error as an equivalent problem of maximizing the Rayleigh quotient of a Hermitian matrix with respect to certain constraints. We derive formulas for the structured eigenvalue backward error of $\lambda \in \mathbb{C}$ for $*$-palindromic polynomials in section 4 and the T-palindromic pencils and quadratic polynomials in section 5. We also find formulas for the structured backward error of $\lambda \in \mathbb{R}$ for real T-palindromic polynomials of any degree with respect to real T-palindromic perturbations.

Notations: We use the notations $\mathrm{Herm}(n)$ and $\mathrm{Sym}(n)$ to denote the sets of Hermitian and symmetric matrices of size $n \times n$, respectively. The notations $\lambda_{\max}(H)$ and $\lambda_2(H)$ denote the largest and second largest eigenvalues of a Hermitian matrix H. Also, $\sigma_2(S)$ denotes the second largest singular value of a matrix S. The symbol $\otimes$ denotes the Kronecker product of matrices (or vectors).

2. Preliminaries. We introduce a norm on $(\mathbb{C}^{n\times n})^{m+1}$ associated with a weight vector $w \in \mathbb{R}^{m+1}$ to be able to measure perturbations of matrix polynomials in a flexible way.

DEFINITION 2.1. *Let $\|\cdot\|$ be the spectral norm and let $w = (w_0, \ldots, w_m) \in \mathbb{R}^{m+1}$, where $w_0, \ldots, w_m > 0$.*

1) *w is called a* weight vector *and its entries w_j are called* weights.
2) *The* reciprocal weight vector *of w is defined as $w^{-1} := (w_0^{-1}, \ldots, w_m^{-1})$.*
3) *A weight vector $w = (w_0, \ldots, w_m)$ is said to be a* palindromic weight vector *if $w_j = w_{m-j}$ for $j = 0, \ldots, m$.*
4) *For a tuple of matrices $\Delta_0, \ldots, \Delta_m \in \mathbb{C}^{n\times n}$, we define*

$$\|(\Delta_0, \ldots, \Delta_m)\|_w := \sqrt{w_0^2\|\Delta_0\|^2 + \cdots + w_m^2\|\Delta_m\|^2}. \tag{2.1}$$

DEFINITION 2.2. *Let $P(z) = z^m A_m + \cdots + zA_1 + A_0$ be a matrix polynomial, where $A_0, \ldots, A_m \in \mathbb{C}^{n\times n}$ and let $\lambda \in \mathbb{C}$. Furthermore, let $w = (w_0, \ldots, w_m) \in \mathbb{R}^{m+1}$ be a weight vector and let $\mathbb{S} \subseteq (\mathbb{C}^{n\times n})^{m+1}$. Then we call*

$$\eta_w^{\mathbb{S}}(P, \lambda) := \inf\left\{ \|(\Delta_0, \ldots, \Delta_m)\|_w \;\middle|\; \det\Big(\sum_{j=0}^{m} \lambda^j (A_j - \Delta_j)\Big) = 0,\ (\Delta_0, \ldots, \Delta_m) \in \mathbb{S} \right\}$$

the backward error of λ with respect to $P(z)$, $\mathbb{S}$ and w.

Thus, $\eta_w^{\mathbb{S}}(P,\lambda)$ is the norm of the smallest perturbation from $\mathbb{S}$ so that λ becomes an eigenvalue of the perturbed matrix polynomial $\widetilde{P}(z) = \sum_{j=0}^{m} z^j(A_j - \Delta_j)$. Clearly, we have $\eta_w^{\mathbb{S}}(P,\lambda) = 0$ if the matrix $P(\lambda) \in \mathbb{C}^{n\times n}$ is singular, i.e., if λ is already an eigenvalue of $P(z)$ (including the case that the matrix polynomial $P(z)$ is singular). Throughout this paper we assume that $P(z)$ is regular and that $P(\lambda)$ is nonsingular. Also it is important to note that if $(A_0,\dots,A_m) \in \mathbb{S}$ then

$$\eta_w^{\mathbb{S}}(P,\lambda) \le \|(A_0,\dots,A_m)\|_w < \infty,$$

because the perturbation with the tuple $(A_0,\dots,A_m)$ results in the zero polynomial. The case $\mathbb{S} = \mathbb{C}^{n\times n}$ in the definition of $\eta_w^{\mathbb{S}}(P,\lambda)$ corresponds to the backward error with respect to arbitrary perturbations and we denote this by $\eta_w(P,\lambda)$. It is well known (see for example [4, Proposition 4.6], or [6, Theorem 4.2]) that

$$\eta_w(P,\lambda) = \frac{\sigma_{\min}\big(P(\lambda)\big)}{\|(1,\lambda,\dots,\lambda^m)\|_{w^{-1}}},$$

where $\sigma_{\min}(A)$ stands for the smallest singular value of a matrix A.

For the sake of brevity, whenever we make statements that are valid for both $*$-palindromic and T-palindromic structures, we use the term $\star$-palindromic where $\star = *$ or $\star = T$. Thus, denoting the $\star$-palindromic structure by $\mathrm{pal}_\star$, we have

$$\mathrm{pal}_\star = \begin{cases} \{(A_0,\dots,A_m) \in (\mathbb{C}^{n\times n})^{m+1} : A_j^* = A_{m-j}\} & \text{if } \star = *, \\ \{(A_0,\dots,A_m) \in (\mathbb{C}^{n\times n})^{m+1} : A_j^T = A_{m-j}\} & \text{if } \star = T. \end{cases}$$

The flexibility to perturb a polynomial with coefficients in $\mathrm{pal}_\star$ in a structure preserving way is restricted by the fact that equal weights must be given to coefficients in position j and position $m-j$. Therefore unless otherwise stated, we assume that the weight vector w is a palindromic weight vector as defined in Definition 2.1. Our aim will be to solve the following problem.

PROBLEM 2.3. *Let $P(z) = \sum_{i=0}^{m} z^j A_j$ be $\star$-palindromic and $\lambda \in \mathbb{C}\setminus\{0\}$. Suppose that $P(\lambda)$ is nonsingular. Find the smallest structured perturbation from $\mathrm{pal}_\star$ that makes λ an eigenvalue of the perturbed $\star$-palindromic polynomial. More precisely: calculate*

$$\eta_w^{\mathrm{pal}_\star}(P,\lambda) := \inf\Big\{ \|(\Delta_0,\dots,\Delta_m)\|_w \,\Big|\, \det\Big(\sum_{j=0}^{m} \lambda^j(A_j - \Delta_j)\Big) = 0,\ (\Delta_0,\dots,\Delta_m) \in \mathrm{pal}_\star \Big\}$$

and construct the corresponding perturbation $\Delta P(z) = \sum_{j=0}^{m} \lambda^j \Delta_j$ that attains the infimum.

Note that the assumption $\lambda \in \mathbb{C}\setminus\{0\}$ is justified because

$$\eta_w^{\mathrm{pal}_\star}(P,0) = \sqrt{2} w_0 \sigma_{\min}(A_0),$$

where $\sigma_{\min}(A_0)$ is the minimum singular value of A_0.

The problem of computing $\eta_w^{\mathbb{S}}(P,\lambda)$ for the Hermitian and related structures was reformulated in terms of structured mapping problems in [6]. The result [6, Lemma 2.4] was crucial to this reformulation. The result is restated below together with a proof for the sake of completeness.

LEMMA 2.4. *Let $P(z) := \sum_{j=0}^{m} z^j A_j$ be a complex $n \times n$ matrix polynomial. Also, let $\Delta_0, \dots, \Delta_m \in \mathbb{C}^{n\times n}$ and $\lambda \in \mathbb{C}$ be such that $P(\lambda)$ is nonsingular and define $M := \big(P(\lambda)\big)^{-1}$. Then the following statements are equivalent:*

(a) $\det\Big(\sum_{j=0}^{m} \lambda^j (A_j - \Delta_j)\Big) = 0.$

(b) *There exist vectors $v_0, \dots, v_m \in \mathbb{C}^n$ satisfying $\sum_{j=0}^{m} \lambda^j v_j \neq 0$ such that*

$$v_j = \Delta_j M\big(\lambda^m v_m + \cdots + \lambda v_1 + v_0\big), \quad \text{for } j = 0, \dots, m.$$

Proof. Denote $\widetilde{P}(\lambda) = \sum_{j=0}^{m} \lambda^j (A_j - \Delta_j)$.
(a) $\Rightarrow$ (b): If (a) holds then there exists $x \neq 0$ such that $\widetilde{P}(\lambda)x = 0$. Let $v_j = \Delta_j\, x$ for $j = 0, \dots, m$. Then we have

$$P(\lambda)x = P(\lambda)x - \widetilde{P}(\lambda)x = \sum_{j=0}^{m} \lambda^j \Delta_j\, x = \sum_{j=0}^{m} \lambda^j v_j =: v_\lambda. \tag{2.2}$$

We have $v_\lambda \neq 0$ because $P(\lambda) = M^{-1}$ is nonsingular by assumption. On multiplying (2.2) from the left with $\Delta_j M$ we obtain the identities $v_j = \Delta_j M v_\lambda$ for $j = 0, \dots, m$.

(b) $\Rightarrow$ (a): Suppose that (b) holds and set $v_\lambda := \sum_{j=0}^{m} \lambda^j v_j$. Then

$$\widetilde{P}(\lambda) M v_\lambda = \Big(P(\lambda) - \sum_{j=0}^{m} \lambda^j \Delta_j\Big) M v_\lambda = v_\lambda - \sum_{j=0}^{m} \lambda^j \Delta_j\, M v_\lambda = 0,$$

because $\Delta_j\, M v_\lambda = v_j$ for $j = 0, \dots, m$. Since $M v_\lambda \neq 0$, this implies (a). ∎

Lemma 2.4 yields the following alternative characterization of $\eta_w^{\mathrm{pal}_\star}(P, \lambda)$ in terms of mapping problems.

COROLLARY 2.5. *Let $P(z) = \sum_{j=0}^{m} z^j A_j$ be $\star$-palindromic and $\lambda \in \mathbb{C} \setminus \{0\}$. Also let $k := \lfloor \frac{m-1}{2} \rfloor$ and $v_\lambda := \sum_{j=0}^{m} \lambda^j v_j$ where $v_0, \dots, v_m \in \mathbb{C}^n$. Assume that $P(\lambda)$ is nonsingular and let $M = \big(P(\lambda)\big)^{-1}$. If m is odd,*

$$\eta_w^{\mathrm{pal}_\star}(P, \lambda) = \inf\Big\{ \|(\Delta_0, \dots, \Delta_m)\|_w \,\Big|\, \exists\, v_0, \dots, v_m \in \mathbb{C}^n,\ v_\lambda \neq 0, (\Delta_0, \dots, \Delta_m) \in \mathrm{pal}_\star,\\ \Delta_j M v_\lambda = v_j,\ \Delta_j^\star M v_\lambda = v_{m-j},\ \ j = 0, \dots, k\Big\},$$

and if m is even,

$$\eta_w^{\mathrm{pal}_\star}(P, \lambda) = \inf\Big\{ \|(\Delta_0, \dots, \Delta_m)\|_w \,\Big|\, \exists\, v_0, \dots, v_m \in \mathbb{C}^n,\ v_\lambda \neq 0,\ (\Delta_0, \dots, \Delta_m) \in \mathrm{pal}_\star,\\ \Delta_{\frac{m}{2}} M v_\lambda = v_{\frac{m}{2}},\ \Delta_j M v_\lambda = v_j,\ \Delta_j^\star M v_\lambda = v_{m-j},\ \ j = 0, \dots, k\Big\}.$$

Necessary and sufficient conditions for the mapping problems

$$\Delta_j M v_\lambda = v_j,\ \Delta_j^\star M v_\lambda = v_{m-j}$$

in Corollary 2.5 to be solvable as well as minimal norm solutions to such problems have been obtained in [10] and [14, Theorem 2]. For convenience, we restate the result and include a formula for the desired minimal norm solution and also give a proof.

THEOREM 2.6. *Let $x, y, z \in \mathbb{C}^n$ with $x \neq 0$. Then there exists a matrix $\Delta \in \mathbb{C}^{n\times n}$ such that $\Delta x = y$ and $\Delta^\star x = z$ if and only if $x^\star y = z^\star x$. If the latter condition is satisfied then*

$$\min\left\{\|\Delta\| \,\middle|\, \Delta \in \mathbb{C}^{n\times n},\, \Delta x = y, \Delta^\star x = z\right\} = \max\left\{\frac{\|y\|}{\|x\|}, \frac{\|z\|}{\|x\|}\right\}. \tag{2.3}$$

Furthermore, let $\hat{x} = x$ when $\star = *$ and $\hat{x} = \bar{x}$ when $\star = T$, and let y_1 and z_1 denote the orthogonal projections of y and z, respectively, onto the orthogonal complement of $\hat{x}$. If $\|z_1\| \leq \|y_1\|$, then the minimum in (2.3) is attained for

$$\widetilde{\Delta} = \frac{1}{\|x\|}\begin{bmatrix}\frac{\hat{x}}{\|x\|} & \frac{y_1}{\|y_1\|}\end{bmatrix}\begin{bmatrix}\frac{\hat{x}^* y}{\|x\|} & \|z_1\| \\ \|y_1\| & -\hat{x}^* y \frac{\|z_1\|}{\|x\|\,\|y_1\|}\end{bmatrix}\begin{bmatrix}\frac{\hat{x}}{\|x\|} & \frac{z_1}{\|z_1\|}\end{bmatrix}^\star$$

if $z_1 \neq 0$ and for $\widetilde{\Delta} = \frac{1}{\|x\|^2} y x^*$ if $z_1 = 0$. If $\|y_1\| \leq \|z_1\|$ then these formulas can be used to construct $\widetilde{\Delta}^\star$.

Proof. Clearly, the identities $\Delta x = y$ and $\Delta^\star x = z$ imply $x^\star y = z^\star x$ and

$$\|\Delta\| \geq \max\left\{\frac{\|y\|}{\|x\|}, \frac{\|z\|}{\|x\|}\right\}.$$

Suppose now that $x^\star y = z^\star x$ holds true. Denote $x_0 = \hat{x}/\|x\|$. Then

$$y = (x_0^* y)x_0 + y_1,\ x_0^* y_1 = 0,\ \|y\|^2 = |x_0^* y|^2 + \|y_1\|^2,$$
$$z = (x_0^* z)x_0 + z_1,\ x_0^* z_1 = 0,\ \|z\|^2 = |x_0^* z|^2 + \|z_1\|^2.$$

Notice that $x_0^* y = z^* x_0$ and hence $\|y\|^2 - \|z\|^2 = \|y_1\|^2 - \|z_1\|^2$. For every $\alpha \in \mathbb{C}$ the matrix

$$\Delta_\alpha = \|x\|^{-1}\big((x_0^* y)x_0 x_0^\star + y_1 x_0^\star + x_0 z_1^\star + \alpha\, y_1 z_1^\star\big)$$

satisfies $\Delta_\alpha x = y$ and $\Delta_\alpha^\star x = z$. Hence,

$$\|\Delta_\alpha\| \geq \max\left\{\frac{\|y\|}{\|x\|}, \frac{\|z\|}{\|x\|}\right\}. \tag{2.4}$$

We show that equality holds in (2.4) for appropriate α. Without loss of generality we may assume $\|z\| \leq \|y\|$, or, equivalently, $\|z_1\| \leq \|y_1\|$. Otherwise, we may interchange the roles of z and y and Δ and $\Delta^\star$, respectively. We consider two cases.

Case 1: $z_1 \neq 0$. Let $\alpha = -x_0^* y/\|y_1\|^2$. Then $\Delta_\alpha = \widetilde{\Delta}$ and $\|\Delta_\alpha\| = \|y\|/\|x\|$. In order to see that let $y_0 = y_1/\|y_1\|$ and $z_0 = z_1/\|z_1\|$. Then

$$\Delta_\alpha = \|x\|^{-1}\begin{bmatrix}x_0 & y_0\end{bmatrix}\begin{bmatrix}x_0^* y & \|z_1\| \\ \|y_1\| & \alpha\|y_1\|\,\|z_1\|\end{bmatrix}\begin{bmatrix}x_0 & z_0\end{bmatrix}^\star$$

$$= \frac{\|y\|}{\|x\|}\begin{bmatrix}x_0 & y_0\end{bmatrix}\underbrace{\frac{1}{\|y\|}\begin{bmatrix}x_0^* y & \|y_1\| \\ \|y_1\| & -x_0^* y\end{bmatrix}}_{=:C}\underbrace{\begin{bmatrix}1 & 0 \\ 0 & \|z_1\|/\|y_1\|\end{bmatrix}}_{=:D}\begin{bmatrix}x_0 & z_0\end{bmatrix}^\star$$

Since $\begin{bmatrix}x_0 & y_0\end{bmatrix}^*\begin{bmatrix}x_0 & y_0\end{bmatrix} = \begin{bmatrix}x_0 & z_0\end{bmatrix}^*\begin{bmatrix}x_0 & z_0\end{bmatrix} = I$ we have

$$\|\begin{bmatrix}x_0 & y_0\end{bmatrix}\| = \|\begin{bmatrix}x_0 & z_0\end{bmatrix}^\star\| = 1.$$

The matrix C is easily seen to be unitary. Moreover, $\|D\| = \max\{1, \|z_1\|/\|y_1\|\} = 1$. Consequently, $\|\Delta_\alpha\| \leq \|y\|/\|x\|$. This inequality is actually an equality because of (2.4).

Case 2: $z_1 = 0$. Then for any α, $\Delta_\alpha = \|x\|^{-1} y x_0^*$, whence $\|\Delta_\alpha\| = \|y\|/\|x\|$. ∎

3. Reformulation of the problem. We first reformulate the original Problem 2.3 of finding the backward error for $\star$-palindromic polynomial into an equivalent problem of maximizing the Rayleigh quotient of a Hermitian matrix with respect to some constraints. These constraints involve Hermitian matrices when $\star = *$, and symmetric matrices when $\star = T$. Let $\lambda \in \mathbb{C} \setminus \{0\}$, let $P(z) = \sum_{j=0}^{m} z^j A_j$ be a $\star$-palindromic matrix polynomial such that $M = P(\lambda)^{-1}$ exists and define $v_\lambda = \sum_{j=0}^{m} \lambda^j v_j$ where $v_0, \ldots, v_m \in \mathbb{C}^n$. Also let $k = \lfloor \frac{m-1}{2} \rfloor$.

By Theorem 2.6 for any $v_0, \ldots, v_m \in \mathbb{C}^n$ that satisfy $v_\lambda \neq 0$, there exists a $\mathbf{\Delta} = (\Delta_0, \ldots, \Delta_m) \in \mathrm{pal}_\star$ such that

$$\Delta_j M v_\lambda = v_j \text{ and } \Delta_j^\star M v_\lambda = v_{m-j},\ j = 0, \ldots, k \tag{3.1}$$

if and only if $(Mv_\lambda)^\star v_j = v_{m-j}^\star (Mv_\lambda)$. For any $\Delta_j \in \mathbb{C}^{n\times n}$ satisfying (3.1) which is minimal with respect to the 2-norm, we have $\|\Delta_j\| = \max\left\{ \frac{\|v_j\|}{\|Mv_\lambda\|}, \frac{\|v_{m-j}\|}{\|Mv_\lambda\|} \right\}$.

If m is even, the matrix $\Delta_{\frac{m}{2}}$ of the tuple $\mathbf{\Delta} = (\Delta_0, \ldots \Delta_m)$ is Hermitian when $\star = *$ and symmetric when $\star = T$. In the case $\star = *$, the Hermitian matrix $\Delta_{\frac{m}{2}}$ may be chosen to satisfy

$$\Delta_{\frac{m}{2}} M v_\lambda = v_{\frac{m}{2}},$$

if and only if $(Mv_\lambda)^* v_{\frac{m}{2}} \in \mathbb{R}$, (for details, see [21] and [6, Theorem 2.6]). On the other hand, when $\star = T$ the symmetric matrix $\Delta_{\frac{m}{2}}$ may be chosen to satisfy $\Delta M v_\lambda = v_{\frac{m}{2}}$ without any restrictions on Mv_λ and $v_{\frac{m}{2}}$ and in either case, any minimal 2-norm solution of this mapping problem satisfies $\|\Delta_{\frac{m}{2}}\|_2 = \frac{\|v_{m/2}\|}{\|Mv_\lambda\|}$, (see [21]). Therefore, if all the constraints are fulfilled, the minimal norm of $\mathbf{\Delta}$ is given by

$$\|\mathbf{\Delta}\|_w^2 = f(v_0, \ldots, v_m),$$

where

$$f(v_0, \ldots, v_m) := \begin{cases} \sum_{j=0}^{k} 2w_j^2 \max\left\{ \frac{\|v_j\|^2}{\|Mv_\lambda\|^2}, \frac{\|v_{m-j}\|^2}{\|Mv_\lambda\|^2} \right\} & \text{if } m \text{ is odd,} \\ \sum_{j=0}^{k} 2w_j^2 \max\left\{ \frac{\|v_j\|^2}{\|Mv_\lambda\|^2}, \frac{\|v_{m-j}\|^2}{\|Mv_\lambda\|^2} \right\} + w_{\frac{m}{2}}^2 \frac{\|v_{\frac{m}{2}}\|^2}{\|Mv_\lambda\|^2} & \text{if } m \text{ is even.} \end{cases}$$

Thus, Corollary 2.5 yields

$$\eta_w^{\mathrm{pal}_\star}(P, \lambda)^2 = \inf\left\{ f(v_0, \ldots, v_m) \middle| (v_0, \ldots, v_m) \in \mathcal{K} \right\}, \tag{3.2}$$

where $\mathcal{K} \subseteq (\mathbb{C}^n)^{m+1}$ is given by

$$\mathcal{K} := \left\{ (v_0, \ldots, v_m) \,\middle|\, v_\lambda \neq 0,\ (Mv_\lambda)^\star v_j = v_{m-j}^\star M v_\lambda,\ j = 0 \ldots, k \right\} \tag{3.3}$$

if $\star = T$, or if m is odd and $\star = *$ and by

$$\mathcal{K} := \left\{ (v_0, \ldots, v_m) \,\middle|\, v_\lambda \neq 0,\ (Mv_\lambda)^* v_{\frac{m}{2}} \in \mathbb{R}, (Mv_\lambda)^* v_j = v_{m-j}^* M v_\lambda,\ j = 0, \ldots, k \right\} \tag{3.4}$$

otherwise (i.e., when $\star = *$ and m is even). Observe that $(Mv_\lambda)^\star v_j = v_{m-j}^\star (Mv_\lambda)$ for $j = 0, \ldots, k$, if and only if

$$0 = \Big(M(v_0 + \ldots + \lambda^m v_m) \Big)^\star v_j - v_{m-j}^\star \Big(M(v_0 + \ldots + \lambda^m v_m) \Big) = v^\star \widetilde{C}_j v,$$

where $v := [v_0^T, \dots, v_m^T]^T$ and

$$\widetilde{C}_j := (\Lambda_m^\star e_{j+1}^\star) \otimes M^\star - (e_{m-j+1}\Lambda_m) \otimes M, \tag{3.5}$$

with $\Lambda_m := [1, \lambda, \dots, \lambda^m] \in \mathbb{C}^{1\times(m+1)}$. Similarly $(Mv_\lambda)^* v_{\frac{m}{2}} \in \mathbb{R}$ if and only if

$$0 = -2\,\mathrm{Im}\Big((Mv_\lambda)^* v_{\frac{m}{2}}\Big) = i\Big(v_{\frac{m}{2}}^*(Mv_\lambda) - (Mv_\lambda)^* v_{\frac{m}{2}}\Big) = v^*\widetilde{C}_{\frac{m}{2}} v,$$

where

$$\widetilde{C}_{\frac{m}{2}} := i\Big((\Lambda_m^* e_{\frac{m}{2}+1}^*) \otimes M^* - (e_{\frac{m}{2}+1}\Lambda_m) \otimes M\Big). \tag{3.6}$$

Note that $\widetilde{C}_{\frac{m}{2}}$ is a Hermitian matrix but the matrices $\widetilde{C}_j, j = 0, \dots, k$ are not Hermitian. Thus, from (3.3) if $\star = T$, or if m is odd and $\star = *$

$$\mathcal{K} = \Big\{(v_0, \dots, v_m) \,\Big|\, v_\lambda \neq 0,\, v^\star \widetilde{C}_j v = 0,\ j = 0, \dots, k\Big\}, \tag{3.7}$$

and from (3.4)

$$\mathcal{K} = \Big\{(v_0, \dots, v_m) \,\Big|\, v_\lambda \neq 0,\, v^*\widetilde{C}_{\frac{m}{2}} v = 0,\, v^*\widetilde{C}_j v = 0,\ j = 0, \dots, k\Big\}, \tag{3.8}$$

otherwise.

As already stated in the beginning of this section, our aim is to reformulate the computation of the structured eigenvalue backward error as an equivalent problem of maximizing the Rayleigh quotient of a Hermitian matrix subject to some constraints. The same strategy was applied in [6] to find the structured eigenvalue backward error $\eta_w^{\mathbb{S}}(P, \lambda)$ for Hermitian and related structures. But the reformulation was aided by the fact $\eta_w^{\mathbb{S}}(P, \lambda)$ satisfied

$$\eta_w^{\mathbb{S}}(P, \lambda) = \left(\sup\left\{\frac{\|Mv_\lambda\|^2}{\sum_{j=0}^m w_j^2 \|v_j\|^2} \,\middle|\, v_\lambda \neq 0, v_j^* M v_\lambda \in \mathbb{R}\right\}\right)^{-\frac{1}{2}}. \tag{3.9}$$

for those structures, because the quotient in the right hand side of (3.9) could easily be seen to be a Rayleigh quotient of a particular Hermitian matrix. (For details, we refer to the proof of [6, Theorem 4.4].) However, as (3.2) suggests, this is not the case for the structured eigenvalue backward error $\eta_w^{\mathrm{pal}_\star}(P, \lambda)$ for the $\star$-palindromic structures, because the function f in the right hand side of (3.2) involves taking a maximum. The following lemma is a key step towards establishing a relationship similar to (3.9) for $\eta_w^{\mathrm{pal}_\star}(P, \lambda)$, because it shows that minimizers of the function f in (3.2) also minimize a related function g that can in the following be interpreted as a Rayleigh quotient of a certain Hermitian matrix.

LEMMA 3.1. *Let $P(z) = \sum_{j=0}^m z^j A_j$ be $\star$-palindromic and $\lambda \in \mathbb{C} \setminus \{0\}$. Assume further that $M = \big(P(\lambda)\big)^{-1}$ exists and $k = \lfloor \frac{m-1}{2} \rfloor$. Then*

$$\eta_w^{\mathrm{pal}_\star}(P, \lambda)^2 = \inf\Big\{g(v_0, \dots, v_m) \,\Big|\, (v_0, \dots, v_m) \in \mathcal{K}\Big\}, \tag{3.10}$$

where

$$g(v_0, \dots, v_m) := \begin{cases} \sum\limits_{j=0}^{k} \frac{2\, w_j^2 (\|v_j\|^2 + |\lambda|^{m-2j} \|v_{m-j}\|^2)}{(1+|\lambda|^{m-2j})\, \|Mv_\lambda\|^2} & \text{if } m \text{ is odd,} \\ \sum\limits_{j=0}^{k} \frac{2\, w_j^2 (\|v_j\|^2 + |\lambda|^{m-2j} \|v_{m-j}\|^2)}{(1+|\lambda|^{m-2j})\, \|Mv_\lambda\|^2} + \frac{w_{\frac{m}{2}}^2 \|v_{\frac{m}{2}}\|^2}{\|Mv_\lambda\|^2} & \text{if } m \text{ is even,} \end{cases}$$

and $\mathcal{K}$ is as defined in (3.7) *and* (3.8)*, respectively.*

Proof. Set $\nu := \inf\big\{g(v_0,\dots,v_m)\,\big|\,(v_0,\dots,v_m)\in\mathcal{K}\big\}$. It is easily verified that $g(v_0,\dots,v_m)\le f(v_0,\dots,v_m)$ for all $(v_0,\dots,v_m)\in(\mathbb{C}^n)^{m+1}$ with $v_\lambda\neq 0$. This together with (3.2) implies $\nu\le\eta_w^{\mathrm{pal}_\star}(P,\lambda)^2$. The opposite inequality is an immediate consequence of the following facts:

(a) *The infimum of g in the definition of ν is attained for some $(\hat v_0,\dots,\hat v_m)\in\mathcal{K}$.*

(b) *For every minimizer $(\hat v_0,\dots,\hat v_m)\in\mathcal{K}$ of g we have*

$$g(\hat v_0,\dots,\hat v_m)=f(\hat v_0,\dots,\hat v_m).$$

Proof of (a): Since $\mathcal{K}$ is closed under scalar multiplication and since for all $t\in\mathbb{R}\setminus\{0\}$ and all $(v_0,\dots,v_m)\in\mathcal{K}$ we have $g(v_0,\dots,v_m)=g(t\,v_0,\dots,t\,v_m)$, we obtain that $g(\mathcal{K})=g(\mathcal{K}\cap\mathcal{S})$, where $\mathcal{S}$ is defined as

$$\mathcal{S}=\left\{(v_0,\dots,v_m)\in(\mathbb{C}^n)^{m+1}\;\middle|\;\sum_{j=0}^{k}\|v_j\|^2=1\right\}.$$

Let $(v_0^{(\ell)},\dots,v_m^{(\ell)})$, $\ell\in\mathbb{N}$, be a sequence in $\mathcal{K}\cap\mathcal{S}$ for which

$$\lim_{\ell\to\infty}g\big(v_0^{(\ell)},\dots,v_m^{(\ell)}\big)=\nu.$$

Since $\mathcal{S}$ is compact we may assume without loss of generality that the sequence $(v_0^{(\ell)},\dots,v_m^{(\ell)})$ has a limit $(\hat v_0,\dots,\hat v_m)\in\mathcal{S}$. Suppose that $(\hat v_0,\dots,\hat v_m)\notin\mathcal{K}$. Then we have $\hat v_0+\lambda v_1+\cdots+\lambda^m\hat v_m=0$, as $(\hat v_0,\dots,\hat v_m)$ belongs to the closure of $\mathcal{K}$. This implies

$$\lim_{\ell\to\infty}\left\|M\left(v_0^{(\ell)}+\lambda v_1^{(\ell)}+\cdots+\lambda^m v_m^{(\ell)}\right)\right\|^{-1}=\infty$$

and hence

$$\lim_{\ell\to\infty}g(v_0^{(\ell)},\dots,v_m^{(\ell)})=\infty\neq\nu$$

which is a contradiction. Thus, $(\hat v_0,\dots,\hat v_m)\in\mathcal{K}$ and $g(\hat v_0,\dots,\hat v_m)=\nu$.

Proof of (b): Let $(\hat v_0,\dots,\hat v_m)\in\mathcal{K}$ be such that $g(\hat v_0,\dots,\hat v_m)=\nu$. Observe that, to show that $g(\hat v_0,\dots,\hat v_m)=f(\hat v_0,\dots,\hat v_m)$, it is sufficient to show that $\|\hat v_j\|=\|\hat v_{m-j}\|$ for all $j=0,\dots,k$. Let

$$x_0=\begin{cases}M(\hat v_\lambda)/\|M(\hat v_\lambda)\| & \text{if } \star=*,\\ \overline{M(\hat v_\lambda)}/\|M(\hat v_\lambda)\| & \text{if } \star=T.\end{cases}$$

and y_j, y_{m-j} be the projections of $\hat v_j$ and $\hat v_{m-j}$, respectively, onto the orthogonal complement of x_0, for $0\le j\le k$. Then

$$\hat v_j=y_j+c_jx_0\quad\text{and}\quad\hat v_{m-j}=y_{m-j}+c_{m-j}x_0$$

for some $c_j,c_{m-j}\in\mathbb{C}$. Since $(\hat v_0,\dots,\hat v_m)\in\mathcal{K}$ we have $\bar c_j=c_{m-j}$ when $\star=*$ and $c_j=c_{m-j}$ when $\star=T$. Hence

$$\|\hat v_j\|^2=\|y_j\|^2+|c_j|^2\quad\text{and}\quad\|\hat v_{m-j}\|^2=\|y_{m-j}\|^2+|c_{m-j}|^2.\tag{3.11}$$

Let $y = \bar\lambda^{m-2j}\, y_j + |\lambda|^{m-2j}\, y_{m-j}$. Observe that

$$(\hat v_0, \dots, \hat v_j + t\,\lambda^{m-2j}\, y, \dots, \hat v_{m-j} - t\, y, \dots, \hat v_m) \in \mathcal{K}$$

for all $t \in \mathbb{R}$. Thus as $(\hat v_0, \dots, \hat v_m)$ is a minimizer of g over $\mathcal{K}$,, we have

$$\begin{aligned}
0 \;&=\; \frac{d}{dt} g\big(\hat v_0, \dots, \hat v_j + t\,\lambda^{m-2j}\, y, \dots, \hat v_{m-j} - t\, y, \dots, \hat v_m\big)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\frac{2w_j^2(\|\hat v_j + t\,\lambda^{m-2j}\, y\|^2 + |\lambda|^{m-2j}\|\hat v_{m-j} - t\, y\|^2)}{(1+|\lambda|^{m-2j})\,\|M\hat v_\lambda\|^2} \right)\Big|_{t=0} \\
&= \frac{2w_j^2 \Re\big(\hat v_j^*(\lambda^{m-2j}\, y) - |\lambda|^{m-2j}\, \hat v_{m-j}^* y\big)}{(1+|\lambda|^{m-2j})\,\|M\hat v_\lambda\|^2} \qquad \Big(\text{since, } \frac{d}{dt}\|v+ty\|^2\Big|_{t=0} = 2\Re(v^*y)\Big) \\
&= \frac{2w_j^2 \Re\big(y_j^*(\lambda^{m-2j}\, y) - |\lambda|^{m-2j}\, y_{m-j}^* y\big)}{(1+|\lambda|^{m-2j})\,\|M\hat v_\lambda\|^2} \\
&= \frac{2w_j^2 |\lambda|^{m-2j}\, \Re\big(|\lambda|^{m-2j}\,\|y_j\|^2 + \lambda^{m-2j}\, y_j^* y_{m-j} - \bar\lambda^{m-2j}\, y_{m-j}^* y_j - |\lambda|^{m-2j}\,\|y_{m-j}\|^2\big)}{(1+|\lambda|^{m-2j})\,\|M\hat v_\lambda\|^2} \\
&= \frac{2w_j^2\, |\lambda|^{2(m-2j)}\big(\|y_j\|^2 - \|y_{m-j}\|^2\big)}{(1+|\lambda|^{m-2j})\,\|M\hat v_\lambda\|^2},
\end{aligned}$$

which implies $\|y_j\| = \|y_{m-j}\|$. This together with (3.11) yields $\|\hat v_j\| = \|\hat v_{m-j}\|$. Hence $\|\hat v_j\| = \|\hat v_{m-j}\|$ for all j, and the latter implies $g(\hat v_0, \dots, \hat v_m) = f(\hat v_0, \dots, \hat v_m)$. This completes the proof. □

Recalling that $k = \lfloor \frac{m-1}{2} \rfloor$, define $\gamma_{j1} = w_j \sqrt{\frac{2}{1+|\lambda|^{m-2j}}}$, $\gamma_{j2} = w_j \sqrt{\frac{2|\lambda|^{m-2j}}{1+|\lambda|^{m-2j}}}$, $j = 0, \dots, k$ and

$$\Gamma := \begin{cases} \operatorname{diag}(\gamma_{01}, \dots, \gamma_{k1}, \gamma_{k2} \dots, \gamma_{02}) \otimes I_n, & \text{if } m \text{ is odd,} \\ \operatorname{diag}(\gamma_{01}, \dots, \gamma_{k1}, w_{\frac{m}{2}}, \gamma_{k2} \dots, \gamma_{02}) \otimes I_n & \text{if } m \text{ is even.} \end{cases}$$

Also recall that $\Lambda_m = [1, \lambda, \dots, \lambda^m] \in \mathbb{C}^{1\times(m+1)}$. Then we have

$$g(v_0, \dots, v_m) = \frac{v^*\Gamma^2 v}{v^*\widetilde G v}, \quad \text{where } \widetilde G := (\Lambda_m^*\Lambda_m) \otimes (M^*M), \quad v = [v_0^T, \dots, v_m^T]^T \tag{3.12}$$

and where $v^*\widetilde G v = \|Mv_\lambda\| \neq 0$, or, equivalently, $v_\lambda \neq 0$. It follows that

$$\begin{aligned}
\eta_w^{\mathrm{pal}_\star}(P, \lambda) &= \Big(\inf\Big\{ f(v_0, \dots, v_m) \;\Big|\; (v_0, \dots, v_m) \in \mathcal{K} \Big\}\Big)^{1/2} \\
&= \Big(\inf\Big\{ g(v_0, \dots, v_m) \;\Big|\; (v_0, \dots, v_m) \in \mathcal{K} \Big\}\Big)^{1/2} \qquad \text{(by Lemma 3.1)} \\
&= \Big(\sup\Big\{ g(v_0, \dots, v_m)^{-1} \;\Big|\; (v_0, \dots, v_m) \in \mathcal{K} \Big\}\Big)^{-1/2}.
\end{aligned}$$

Set $u := \Gamma v$ and

$$G := \Gamma^{-1}\widetilde G\Gamma^{-1}, \tag{3.13}$$

$$C_j := \Gamma^{-1}\widetilde C_j\Gamma^{-1}, \tag{3.14}$$

$$C_{\frac{m}{2}} := \Gamma^{-1}\widetilde C_{\frac{m}{2}}\Gamma^{-1}, \tag{3.15}$$

where $\widetilde{G}, \widetilde{C}_j$ and $\widetilde{C}_{\frac{m}{2}}$ are as defined in (3.12), (3.5) and (3.6) respectively.

By Lemma 3.1 and (3.12), for $k = \lfloor \frac{m-1}{2} \rfloor$, we have

$$\begin{aligned}\left(\eta_w^{\mathrm{pal}_\star}(P,\lambda)\right)^{-2} &= \sup\Big\{ \frac{v^*\widetilde{G}v}{v^*\Gamma^2 v} \,\Big|\, v \in \mathbb{C}^{n(m+1)} \setminus \{0\},\, v^\star \widetilde{C}_j v = 0,\, j = 0,\dots,k \Big\} \\ &= \sup\Big\{ \frac{u^*Gu}{u^*u} \Big| u \in \mathbb{C}^{n(m+1)} \setminus \{0\},\, u^\star C_j u = 0,\, j = 0,\dots,k \Big\},\end{aligned}$$

if $\star = T$, or if m is odd and $\star = *$, and

$$\left(\eta_w^{\mathrm{pal}_*}(P,\lambda)\right)^{-2} = \sup\Big\{ \frac{u^*Gu}{u^*u} \Big| u \in \mathbb{C}^{n(m+1)} \setminus \{0\},\, u^* C_{\frac{m}{2}} u = 0,\, u^* C_j u = 0,\, j = 0,\dots,k \Big\}$$

otherwise. Note that the condition $v_\lambda \neq 0$ from the definition of $\mathcal{K}$ in (3.7) or (3.8), respectively, or, equivalently, the conditions $v^*\widetilde{G}v \neq 0$ and $u^*Gu \neq 0$ can be dropped in the two expressions for $\left(\eta_w^{\mathrm{pal}_*}(P,\lambda)\right)^{-2}$, because $\widetilde{G}$ and G are semidefinite. This implies $\frac{u^*Gu}{u^*u} \geq 0$ and hence the supremum of this Rayley quotient over all nonzero vectors u satisfying some constraints will be the same with or without the additional condition $u^*Gu \neq 0$.

In order to state the main result of this section, for each $j = 0, \dots, k$ we define

$$H_j := C_j + C_j^*, \tag{3.16}$$
$$H_{m-j} := i(C_j - C_j^*), \tag{3.17}$$
$$H_{\frac{m}{2}} := C_{\frac{m}{2}}, \tag{3.18}$$
$$S_j := C_j + C_j^T. \tag{3.19}$$

Observe that for $j = 0, \dots, k$,

$$\begin{aligned} v^*\widetilde{C}_j v = 0 &\iff u^* H_j u = 0 \text{ and } u^* H_{m-j} u = 0, \\ v^T \widetilde{C}_j v = 0 &\iff u^T S_j u = 0, \\ v^* \widetilde{C}_{\frac{m}{2}} v = 0 &\iff u^* H_{\frac{m}{2}} u = 0. \end{aligned}$$

Therefore we have proved the following theorem which gives the desired reformulation.

THEOREM 3.2. *Let $P(z) = \sum_{j=0}^m z^j A_j$ be $\star$-palindromic and $\lambda \in \mathbb{C}\setminus\{0\}$. Suppose that $P(\lambda)$ is nonsingular and $M = (P(\lambda))^{-1}$. Furthermore, let $k = \lfloor \frac{m-1}{2} \rfloor$, G be defined in* (3.13)*, $H_j, j = 0, \dots, m$ be defined by* (3.16)–(3.18) *and $S_j, j = 0, \dots, k$ be defined by* (3.19)*. Then*

$$\eta_w^{\mathrm{pal_T}}(P,\lambda) = \left(\sup\Big\{ \frac{u^*Gu}{u^*u} \;\Big|\; u \in \mathbb{C}^{n(m+1)} \setminus \{0\},\, u^T S_j u = 0,\, j = 0,\dots,k \Big\} \right)^{-\frac{1}{2}} \tag{3.20}$$

and

$$\eta_w^{\mathrm{pal}_*}(P,\lambda) = \left(\sup\Big\{ \frac{u^*Gu}{u^*u} \;\Big|\; u \in \mathbb{C}^{n(m+1)} \setminus \{0\},\, u^* H_j u = 0,\, j = 0,\dots,m \Big\} \right)^{-\frac{1}{2}}. \tag{3.21}$$

4. Backward errors for approximate eigenvalues of $*$-palindromic matrix polynomials. In this section, we obtain structured eigenvalue backward errors $\eta_w^{\mathrm{pal}_*}(\lambda, P)$ for matrix polynomials $P(z)$ with $*$-palindromic structure. As already mentioned in section 1, if $\lambda \in \mathbb{C} \setminus \{0\}$ is such that $|\lambda| = 1$, then there is no difference between the eigenvalue backward errors with respect to structure preserving and arbitrary perturbations. This fact was shown in [1] for the weight vector $w = (1, \ldots, 1)$ and easily generalizes to arbitrary choices of palindromic weight vectors. The situation is completely different if $|\lambda| \neq 1$. In this case, we obtain the structured backward error via minimization of the maximal eigenvalue of a parameter-depending Hermitian matrix. The following theorem which is a combination of [6, Theorem 3.2 and Theorem 3.5] is crucial to this process.

THEOREM 4.1. *Let $G, H_0, \ldots, H_p \in \mathbb{C}^{n\times n}$ be Hermitian. Assume that any nonzero linear combination $\alpha_0 H_0 + \cdots + \alpha_p H_p$, $(\alpha_0, \ldots, \alpha_p) \in \mathbb{R}^{p+1} \setminus \{0\}$ is indefinite. (Here, "indefinite" is used in the sense "strictly not semi-definite" as opposed to "not necessarily definite" as it is used in [7].) Then the following statements hold:*

(1) *The function $L : \mathbb{R}^{p+1} \to \mathbb{R}$, $(t_0, \ldots, t_p) \mapsto \lambda_{\max}(G + t_0 H_0 + \cdots + t_p H_p)$ is convex and has a global minimum*

$$\lambda_{\max}^* = \min_{t_0,\ldots,t_p \in \mathbb{R}} L(t_0, \ldots, t_p).$$

(2) *If either $p = 1$, or the minimum $\lambda_{\max}^*$ of L is attained at $(t_0^*, \ldots, t_p^*) \in \mathbb{R}^{p+1}$ and is a simple eigenvalue of $H_* := G + t_0^* H_0 + \cdots + t_p^* H_p$, then there exists an eigenvector $u \in \mathbb{C}^n \setminus \{0\}$ of H_* associated with $\lambda_{\max}^*$ satisfying*

$$u^* H_j u = 0 \quad \text{for } j = 0, \ldots, p. \tag{4.1}$$

(3) *Under the assumptions of (2) we have*

$$\sup \left\{ \frac{u^* G u}{u^* u} \,\middle|\, u \neq 0,\ u^* H_j u = 0,\ j = 0, \ldots, p \right\} = \lambda_{\max}^*. \tag{4.2}$$

In particular, the supremum of the left hand side of (4.2) is a maximum and attained for the eigenvector u from (4.1).

As seen in [6, Theorem 3.5], the assumption that $\lambda_{\max}^*$ is simple is not necessary when $p = 1$. This is due to the fact that the joint numerical range

$$\mathcal{W}_0(F_0, F_1) := \left\{ (x^* F_0 x, x^* F_1 x) \in \mathbb{R}^2 \,\middle|\, x \in \mathbb{C}^\ell, \|x\| = 1 \right\}$$

of two Hermitian matrices $F_0, F_1 \in \mathbb{C}^{\ell\times\ell}$ is a convex set [11]. But as [6, Example 3.8] shows, the joint numerical range of three or more complex Hermitian matrices need not be convex. Hence the assumption of simplicity of $\lambda_{\max}^*$ is essential when $p > 1$. We have the following result which gives a formula for $\eta_w^{\mathrm{pal}_*}(P, \lambda)$ when $|\lambda| \neq 1$.

THEOREM 4.2. *Let $P(z) = \sum_{j=0}^m z^j A_j$ be $*$-palindromic and $\lambda \in \mathbb{C} \setminus \{0\}$ such that $|\lambda| \neq 1$. Suppose that $P(\lambda)$ is nonsingular and $M = (P(\lambda))^{-1}$. Then for G as defined in (3.13) and H_j, $j = 0, \ldots, m$ as defined in (3.16)–(3.18), we have that*

$$\lambda_{\max}^* := \min_{t_0,\ldots,t_m \in \mathbb{R}} \lambda_{\max}(G + t_0\, H_0 + \cdots + t_m\, H_m)$$

is attained for some $(t_0^*, \ldots, t_m^*) \in \mathbb{R}^{m+1}$. If $m = 1$ or $\lambda_{\max}^*$ is a simple eigenvalue of $G + t_0^*\, H_0 + \cdots + t_m^*\, H_m$, then

$$\eta_w^{\mathrm{pal}_*}(P, \lambda) = \frac{1}{\sqrt{\lambda_{\max}^*}} = \left(\min_{t_0,\ldots,t_m \in \mathbb{R}} \lambda_{\max}(G + t_0\, H_0 + \cdots + t_m\, H_m) \right)^{-1/2}.$$

Proof. Setting $k = \lfloor \frac{m-1}{2} \rfloor$ let $\widetilde{H}_j = \widetilde{C}_j + \widetilde{C}_j^*$ and $\widetilde{H}_{m-j} = i\big(\widetilde{C}_j - \widetilde{C}_j^*\big)$, where $\widetilde{C}_j$ for $j = 0, \ldots, k$ are as defined in (3.5) and (3.6). In view of Theorem 3.2, we aim to apply Theorem 4.1 in the proof. Thus we check whether each nontrivial linear combination of $H_0, \ldots, H_m$, or, equivalently, of $\widetilde{H}_0, \ldots, \widetilde{H}_m$ is indefinite. Let $H := \sum_{j=0}^{m} \alpha_j \widetilde{H}_j$. Recalling that $\Lambda_m := [1, \lambda, \ldots, \lambda^m] \in \mathbb{C}^{1\times(m+1)}$, we have

$$H = \sum_{j=0}^{k} \Big(\alpha_j \Big(\big(\Lambda_m^*(e_{j+1}^* - e_{m+1-j}^*)\big) \otimes M^* + \big((e_{j+1} - e_{m+1-j})\Lambda_m\big) \otimes M \Big) \\ + i\,\alpha_{m-j} \Big(\big(\Lambda_m^*(e_{j+1}^* + e_{m+1-j}^*)\big) \otimes M^* - \big((e_{j+1} + e_{m+1-j})\Lambda_m\big) \otimes M \Big) \Big)$$

when m is odd and

$$H = \sum_{j=0}^{k} \Big(\alpha_j \Big(\big(\Lambda_m^*(e_{j+1}^* - e_{m+1-j}^*)\big) \otimes M^* + \big((e_{j+1} - e_{m+1-j})\Lambda_m\big) \otimes M \Big) \\ + i\,\alpha_{m-j} \Big(\big(\Lambda_m^*(e_{j+1}^* + e_{m+1-j}^*)\big) \otimes M^* - \big((e_{j+1} + e_{m+1-j})\Lambda_m\big) \otimes M \Big) \Big) \\ + i\,\alpha_{\frac{m}{2}} \Big((\Lambda_m^* e_{\frac{m}{2}+1}^*) \otimes M^* - (e_{\frac{m}{2}+1}\Lambda_m) \otimes M \Big)$$

when m is even. Thus we have,

$$H = (\Lambda_m^* \alpha^*) \otimes M^* + (\alpha \Lambda_m) \otimes M,$$

where $\alpha := [\alpha_0 - i\alpha_m, \ldots, \alpha_k - i\alpha_{m-k},\ -(\alpha_k + i\alpha_{m-k}), \ldots, -(\alpha_0 + i\alpha_m)]^T$ if m is odd and

$$\alpha = \big[\alpha_0 - i\alpha_m, \ldots, \alpha_k - i\alpha_{m-k},\ -i\alpha_{\frac{m}{2}},\ -(\alpha_k + i\alpha_{m-k}), \ldots, -(\alpha_0 + i\alpha_m)\big]^T$$

if m is even. To complete the proof, we show that if H is semidefinite then $\alpha = 0$ and hence $\alpha_0 = \cdots = \alpha_m = 0$. Let

$$Q := \begin{bmatrix} 1 & -\lambda & 0 & \ldots & 0 \\ 0 & 1 & -\lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -\lambda \\ 0 & \ldots & \ldots & 0 & 1 \end{bmatrix} \in \mathbb{C}^{(m+1)\times(m+1)} \tag{4.3}$$

and $a = [a_0\ \ldots,\ a_m]^T := Q^*\alpha$. Since $\Lambda_m Q = e_1^*$ we have

$$\begin{aligned} (Q \otimes I_n)^* H (Q \otimes I_n) &= (Q^* \Lambda_m^* \alpha^* Q) \otimes M^* + (Q^* \alpha \Lambda_m Q) \otimes M \\ &= (e_1 a^*) \otimes M^* + (a e_1^*) \otimes M \\ &= \begin{bmatrix} a_0 M + \bar{a}_0 M^* & \bar{a}_1 M^* & \cdots & \bar{a}_m M^* \\ a_1 M & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ a_m M & 0 & \ldots & 0 \end{bmatrix} \end{aligned}$$

If H is semidefinite, then $a_1 = \cdots = a_m = 0$ and hence $Q^*\alpha = a = a_0 e_1$. When $m \geq 3$, observing that

$$a_1 = 0 \Rightarrow \alpha_1 - i\alpha_{m-1} = \bar{\lambda}(\alpha_0 - i\alpha_m),$$
$$\bar{a}_m = 0 \Rightarrow \alpha_0 - i\alpha_m = \lambda(\alpha_1 - i\alpha_{m-1}),$$

we have $a_0 = \alpha_0 - i\alpha_m = \lambda(\alpha_1 - i\alpha_{m-1}) = \lambda\bar{\lambda}(\alpha_0 - i\alpha_m) = \lambda\bar{\lambda}a_0$.

Similarly, when $m = 1$,

$$a_1 = 0 \Rightarrow \alpha_0 + i\alpha_1 = -\bar{\lambda}(\alpha_0 - i\alpha_1) \text{ and } \alpha_0 - i\alpha_1 = -\lambda(\alpha_0 + i\alpha_1)$$

so that $a_0 = \alpha_0 - i\alpha_1 = -\lambda(\alpha_0 + i\alpha_1) = \lambda\bar{\lambda}(\alpha_0 - i\alpha_1) = \lambda\bar{\lambda}a_0$.

Finally when $m = 2$,

$$a_1 = 0 \Rightarrow i\alpha_1 = -\bar{\lambda}(\alpha_0 - i\alpha_2)$$
$$\bar{a}_2 = 0 \Rightarrow \alpha_0 - i\alpha_2 = -i\lambda\alpha_1$$

so that $a_0 = \alpha_0 - i\alpha_2 = -i\lambda\alpha_1 = \lambda\bar{\lambda}(\alpha_0 - i\alpha_2) = \lambda\bar{\lambda}a_0$.

In all cases we have $a_0 = \lambda\bar{\lambda}a_0$ and hence $a_0 = 0$ as $\lambda\bar{\lambda} \neq 1$. Therefore, $\alpha = (Q^*)^{-1}a = 0$ which implies that $\alpha_0 = \cdots = \alpha_m = 0$. □

REMARK 4.3. Although it cannot be established that $\lambda^*_{\max}$ is always simple for the particular matrices $G, H_0, \ldots, H_m$ in Theorem 4.2, numerical experiments suggests that this holds generically.

REMARK 4.4. There may be situations when it would be necessary to find the backward error $\eta_w^{\mathrm{pal}_*}(P, \lambda)$ under the restriction that T-palindromic perturbations can affect only some of the coefficient matrices. This is equivalent to setting some of the entries in the palindromic weight vector w to zero. Let $\mathcal{I} := \{j_0, j_1, \ldots, j_\ell\}$ be a subset of $\{0, 1, \ldots, \lfloor m-1 \rfloor/2\}$ if m is odd and of $\{0, 1, \ldots, m/2\}$ if m is even and assume that $j_0 < j_1 < \cdots < j_\ell$. Suppose that $\mathcal{I}$ is the set of indices such that only the coefficients A_j and A_{m-j}, $j \in \mathcal{I}$ of $P(z)$ are affected by perturbations. Let $\widehat{w}$ be a palindromic weight vector extracted from w by retaining only its nonzero entries. Then $\widehat{w}$ belongs to $\mathbb{R}^{2\ell+3}$ if m is even and $j_\ell = \frac{m}{2}$ and to $\mathbb{R}^{2\ell+2}$ otherwise. For any $(\Delta_0, \ldots, \Delta_m) \in \mathrm{pal}_*$ with $\Delta_j = \Delta_{m-j} = 0$ if $j \notin \mathcal{I}$, we then have

$$\begin{aligned}
&\|(\Delta_0, \ldots, \Delta_m)\|_w^2 \\
&= \|(\Delta_{j_0}, \Delta_{j_1}, \ldots, \Delta_{m-j_1}, \Delta_{m-j_0})\|_{\widehat{w}}^2 \\
&= \begin{cases} \displaystyle\sum_{j\in\mathcal{I}} \left(w_j^2 \|\Delta_j\|^2 + w_{m-j}^2 \|\Delta_{m-j}\|^2 \right) + w_{\frac{m}{2}}^2 \|\Delta_{\frac{m}{2}}\|^2 & \text{if } m \text{ is even and } j_l = \frac{m}{2}, \\ \displaystyle\sum_{j\in\mathcal{I}} \left(w_j^2 \|\Delta_j\|^2 + w_{m-j}^2 \|\Delta_{m-j}\|^2 \right) & \text{otherwise.} \end{cases}
\end{aligned}$$

Then $\|\cdot\|_{\widehat{w}}$ defines a norm on $(\mathbb{C}^{n\times n})^{2\ell+3}$ in the first case and on $(\mathbb{C}^{n\times n})^{2\ell+2}$ in the second case. By using this, and the weight vector $\widehat{w}$, the strategy of reformulation proposed in section 3 may be used to compute the structured backward error $\eta_w^{\mathrm{pal}_*}(P, \lambda)$ with fewer constraints and smaller Hermitian matrices involved in each constraint. The details of this process are similar to those in [6, Section 6] for Hermitian matrix polynomials and are therefore omitted.

REMARK 4.5. To obtain an optimal $*$-palindromic perturbation to $P(z)$ with norm equal to the structured backward error $\eta_w^{\mathrm{pal}_*}(P, \lambda)$ such that the perturbed polynomial has an eigenvalue at λ, we first compute the eigenvector u corresponding to

the eigenvalue $\lambda^*_{\max}$ of $G+t_0^*H_0+\cdots+t_m^*H_m$ that satisfies the constraints $u^*H_ju=0$. Setting, $v:=\Gamma u$, the coefficient matrices Δ_j of the $*$-palindromic perturbation may be obtained from Theorem 2.6 and [6, Theorem 2.6], the second result being necessary only to construct $\Delta_{\frac{m}{2}}$ when m is even.

To highlight the fact that the assumption of simplicity of $\lambda^*_{\max}$ is not required when $m=1$, we state the result for the case of the $*$-palindromic pencils separately.

THEOREM 4.6. *Let $A\in\mathbb{C}^{n\times n}$ and $\lambda\in\mathbb{C}\setminus\{0\}$ with $|\lambda|\neq 1$. Suppose that the pencil $P(\lambda)=A+\lambda A^*$ is nonsingular and let $M:=(A+\lambda A^*)^{-1}$. Furthermore define $\gamma_1:=\sqrt{\frac{2}{1+|\lambda|}}$, $\gamma_2:=\sqrt{\frac{2|\lambda|}{1+|\lambda|}}$,*

$$\widetilde{G}:=\begin{bmatrix} M^*M & \lambda M^*M \\ \bar{\lambda}M^*M & |\lambda|^2M^*M\end{bmatrix},\ C:=\begin{bmatrix} M^* & 0\\ \bar{\lambda}M^*-M & -\lambda M\end{bmatrix},\ \Gamma:=\begin{bmatrix} w_0\gamma_1 I_n & 0\\ 0 & w_0\gamma_2 I_n\end{bmatrix},$$

$$G:=\Gamma^{-1}\widetilde{G}\Gamma^{-1},\quad H_0=\Gamma^{-1}(C+C^*)\Gamma^{-1},\quad \textit{and}\quad H_1:=i\,\Gamma^{-1}(C-C^*)\Gamma^{-1}.$$

Then

$$\eta_w^{\mathrm{pal}_*}(P,\lambda)=\left(\min_{t_0,t_1\in\mathbb{R}}\lambda_{\max}(G+t_0\,H_0+t_1\,H_1)\right)^{-1/2}.$$

5. Structured eigenvalue backward errors for T-palindromic matrix polynomials. In this section, we obtain the structured eigenvalue backward error $\eta_w^{\mathrm{pal_T}}(P,\lambda)$ for a matrix pencil or quadratic matrix polynomial $P(z)$ with T-palindromic structure. We also obtain the structured backward error for approximate real eigenvalues of real T-palindromic polynomials without restrictions on the degree of such polynomials.

Due to the results in [1], if $\lambda=\pm1$ then there is no difference between the eigenvalue backward errors with respect to arbitrary perturbations and with respect to complex T-palindromic perturbations (when $P(z)$ is complex) and real T-palindromic perturbations (when $P(z)$ is real). This is proved for the weight vector $w=(1,\ldots,1)$, but it may be easily generalized to arbitrary choices of palindromic weight vectors.

However, the situation is different, if $\lambda\neq\pm1$. Due to Theorem 3.2 the original Problem 2.3 of finding the structured backward error $\eta_w^{\mathrm{pal_T}}(P,\lambda)$ for T-palindromic polynomials is equivalent to an optimization problem which requires maximizing the Rayleigh quotient of a Hermitian matrix subject to a number of constraints involving symmetric matrices. In these cases, the structured backward error may be obtained by using a theorem from [15]. To state this theorem and other results that follow from it, we recall that $\lambda_2(B)$ denotes the second largest eigenvalue of a Hermitian matrix B and $\sigma_2(S)$ denotes the second largest singular value of a matrix S.

THEOREM 5.1. [15] *Let $H\in\mathrm{Herm(n)}$ and $S\in\mathrm{Sym(n)}$ with $\mathrm{rank}(S)\geq 2$. Then*

$$\sup\left\{v^*Hv\;\middle|\; v\in\mathbb{C}^n, v^TSv=0, \|v\|=1\right\}=\min_{0\leq t\leq t_1}\lambda_2\left(\begin{bmatrix} H & t\bar{S}\\ tS & \bar{H}\end{bmatrix}\right),$$

where $t_1=\frac{2\|H_0\|}{\sigma_2(S)}$.

The following theorem gives a formula for the structured eigenvalue backward error $\eta_w^{\mathrm{pal_T}}(P,\lambda)$ when $P(z)$ is a T-palindromic pencil and $\lambda\in\mathbb{C}\setminus\{0,1,-1\}$.

THEOREM 5.2. *Let $A \in \mathbb{C}^{n\times n}$ and suppose that $P(\lambda) := A + \lambda A^T$ is nonsingular for $\lambda \in \mathbb{C}\setminus\{0,1,-1\}$. Let $M := (A+\lambda A^T)^{-1}$ and define $\gamma_1 := \sqrt{\frac{2}{1+|\lambda|}}$, $\gamma_2 := \sqrt{\frac{2|\lambda|}{1+|\lambda|}}$,*

$$\widetilde{G} := \begin{bmatrix} M^*M & \lambda M^*M \\ \bar{\lambda}M^*M & |\lambda|^2 M^*M \end{bmatrix},\ C := \begin{bmatrix} M^T & 0 \\ \lambda M^T - M & -\lambda M \end{bmatrix},\ \Gamma := \begin{bmatrix} w_0\gamma_1 I_n & 0 \\ 0 & w_0\gamma_2 I_n \end{bmatrix},$$

$$G := \Gamma^{-1}\widetilde{G}\Gamma^{-1} \quad \textit{and} \quad S := \Gamma^{-1}(C + C^T)\Gamma^{-1}.$$

Then

$$\eta_w^{\mathrm{pal_T}}(P,\lambda) = \left(\min_{0\le t\le t_1} \lambda_2\Big(\begin{bmatrix} G & t\bar{S} \\ tS & \bar{G} \end{bmatrix} \Big) \right)^{-1/2},$$

where $t_1 = \frac{2\|G\|}{\sigma_2(S)}$.

Proof. Since $P(z) = A + zA^T$, (3.20) implies that

$$\eta_w^{\mathrm{pal_T}}(P,\lambda) = \left(\sup\Big\{ \frac{u^*Gu}{u^*u} \;\Big|\; u \in \mathbb{C}^{2n}\setminus\{0\},\ u^TS_0u = 0, \Big\} \right)^{-\frac{1}{2}}$$

where $S_0 = S$. The proof then follows by applying Theorem 5.1. ∎

Similarly, the following theorem gives $\eta_w^{\mathrm{pal_T}}(P,\lambda)$ when $P(z)$ is a T-palindromic quadratic matrix polynomial and $\lambda \in \mathbb{C}\setminus\{0,1,-1\}$.

THEOREM 5.3. *Let $P(z) = A_0 + zA_1 + z^2A_0^T$ be a T-palindromic quadratic polynomial and $\lambda \in \mathbb{C}\setminus\{0,1,-1\}$. Suppose that $\det(P(\lambda)) \neq 0$, and let $M := (P(\lambda))^{-1}$. Furthermore, let $\gamma_1 := \sqrt{\frac{2}{1+|\lambda|^2}}$ and $\gamma_2 := \sqrt{\frac{2|\lambda|^2}{1+|\lambda|^2}}$, and define*

$$\widetilde{G} := \begin{bmatrix} 1 & \lambda & \lambda^2 \\ \bar{\lambda} & |\lambda|^2 & \lambda|\lambda|^2 \\ \bar{\lambda}^2 & \bar{\lambda}|\lambda|^2 & |\lambda|^4 \end{bmatrix} \otimes M^*M, \quad C := \begin{bmatrix} M^T & 0 & 0 \\ \lambda M^T & 0 & 0 \\ \lambda^2 M^T - M & -\lambda M & -\lambda^2 M \end{bmatrix},$$

$$\Gamma := \mathrm{diag}(w_0\gamma_1, w_1, w_0\gamma_2) \otimes I_n, \quad G := \Gamma^{-1}\widetilde{G}\Gamma^{-1} \quad \textit{and} \quad S = \Gamma^{-1}(C + C^T)\Gamma^{-1}.$$

Then

$$\eta_w^{\mathrm{pal_T}}(P,\lambda) = \left(\min_{0\le t\le t_1} \lambda_2\Big(\begin{bmatrix} G & t\bar{S} \\ tS & \bar{G} \end{bmatrix} \Big) \right)^{-1/2},$$

where $t_1 = \frac{2\|G\|}{\sigma_2(S)}$.

Proof. Since $P(z) = A_0 + zA_1 + z^2A_0^T$, from (3.20) we have,

$$\eta_w^{\mathrm{pal_T}}(P,\lambda) = \left(\sup\Big\{ \frac{u^*Gu}{u^*u} \;\Big|\; w \in \mathbb{C}^{3n}\setminus\{0\},\ u^TS_0u = 0 \Big\} \right)^{-\frac{1}{2}}$$

where $S_0 = S$. The proof then follows by applying Theorem 5.1. ∎

REMARK 5.4. Due to (3.20), computing $\eta_w^{\mathrm{pal_T}}(P,\lambda)$ for polynomials of degree greater than 2 involves maximizing the Rayleigh quotient $\frac{v^*Gv}{v^*v}$ with respect non zero vectors v that satisfies more than one constraint each involving a symmetric matrix. Generalizing our approach to compute $\eta_w^{\mathrm{pal_T}}(P,\lambda)$ in these cases may involve obtaining

appropriate extensions of Theorem 5.1. This does not seem to be straightforward and will be the subject of future research.

REMARK 5.5. A strategy similar to the one outlined in Remark 4.4 may be used to compute $\eta_w^{\mathrm{pal_T}}(P,\lambda)$ for the case that the T-palindromic perturbations affect only some of the coefficients of $P(z)$.

It may be noted that if $P(z)$ is a T-palindromic polynomials of even degree and only the coefficient $A_{\frac{m}{2}}$ is affected by perturbation, then there are no constraints in the computation of $\eta_w^{\mathrm{pal_T}}(P,\lambda)$ and therefore it is equal to the backward error $\eta_w(P,\lambda)$ with respect to arbitrary perturbations. Moreover, $\eta_w^{\mathrm{pal_T}}(P,\lambda)$ may be computed for any degree if the restrictions are such that the computation of the backward error involves only one constraint.

REMARK 5.6. Note that in Theorem 3.2 if $\lambda \in \mathbb{R} \setminus \{0,1,-1\}$ and $P(z)$ is a real T-palindromic polynomial, then G and S_i are *real* Hermitian matrices. Therefore, denoting the *real* $\star$-palindromic structure by $\mathrm{pal}_{\star,\mathbb{R}}$, that is,

$$\mathrm{pal}_{\star,\mathbb{R}} := \left\{(\Delta_0,\ldots,\Delta_m) \in (\mathbb{R}^{n\times n})^{m+1} \middle| \Delta_j^\star = \Delta_{m-j}, j = 0,\ldots,m\right\} \tag{5.1}$$

we have,

$$\eta_w^{\mathrm{pal_{T,\mathbb{R}}}}(P,\lambda) = \eta_w^{\mathrm{pal_{*,\mathbb{R}}}}(P,\lambda) \tag{5.2}$$

in such cases.

The final result of this section gives the structured backward error $\eta_w^{\mathrm{pal_{T,\mathbb{R}}}}(P,\lambda)$ when $\lambda \in \mathbb{R} \setminus \{0,1,-1\}$ and $P(z)$ is a real T-palindromic polynomial of any degree. It may be noted that despite the equality (5.2), this result is not a corollary of Theorem 4.2. However the proof is based on similar arguments that make use of the real version of Theorem 4.1 with $\mathbb{C}$ replaced by $\mathbb{R}$ as given below.

THEOREM 5.7. *Let $G, H_0, \ldots, H_p \in \mathbb{R}^{n\times n}$ be Hermitian matrices. Assume that any nonzero linear combination $\alpha_0 H_0 + \cdots + \alpha_p H_p$, $(\alpha_0,\ldots,\alpha_p) \in \mathbb{R}^{p+1} \setminus \{0\}$ is indefinite. (Here, "indefinite" is used in the same sense as in Theorem 4.1.) Then the following statements hold:*

(1) *The function $L : \mathbb{R}^{p+1} \to \mathbb{R}$, $(t_0,\ldots,t_p) \mapsto \lambda_{\max}(G + t_0H_0 + \cdots + t_pH_p)$ is convex and has a global minimum*

$$\lambda_{\max}^* = \min_{t_0,\ldots,t_p\in\mathbb{R}} L(t_0,\ldots,t_p).$$

(2) *If $p = 0$ or the minimum $\lambda_{\max}^*$ of L is attained at $(t_0^*,\ldots,t_p^*) \in \mathbb{R}^{p+1}$ and is a simple eigenvalue of $H_* := G + t_0^*H_0 + \cdots + t_p^*H_p$, then there exists an eigenvector $u \in \mathbb{R}^n \setminus \{0\}$ of H_* associated with $\lambda_{\max}^*$ satisfying*

$$u^T H_j u = 0 \quad \text{for } j = 0,\ldots,p. \tag{5.3}$$

(3) *Under the assumptions of (2) we have*

$$\sup\left\{ \frac{u^TGu}{u^Tu} \,\middle|\, u \neq 0,\ u^TH_ju = 0,\ j = 0,\ldots,p \right\} = \lambda_{\max}^*. \tag{5.4}$$

In particular, the supremum of the left hand side of (5.4) is a maximum and attained for the eigenvector u from (5.3).

Proof. The proofs of (1) and (3) follow from the proofs of the corresponding parts of [6, Theorem 3.2].

If $p = 0$, then the proof of part (2) follows by arguing as in the proof of [6, Theorem 3.5] due to the fact that the set

$$\{x^T H_0 x \,\Big|\, x \in \mathbb{R}^n, x^T x = 1\}$$

is convex. If $p > 0$ and $\lambda^*_{\max}$ is a simple eigenvalue of H_* then, part (2) of Theorem 4.1 implies that there exists a non zero (possibly complex) eigenvector u corresponding to $\lambda^*_{\max}$ of H_* that satisfies (5.3). However as H_* is real and $\lambda^*_{\max}$ is real and simple, the eigenvector u can be chosen to be real. This proves part (2) and completes the proof of the theorem. □

It is important to note that the assumption of simplicity of $\lambda^*_{\max}$ made in the hypothesis of Theorem 5.7 is necessary even when $p = 1$. This is evident from the following example which is a slight modification of [6, Example 3.8].

EXAMPLE 5.8. Let $G = \mathrm{diag}(\alpha, \alpha, \beta)$ where $\alpha > \beta \geq 0$. Also let

$$H_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } H_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then any non zero real linear combination of H_0 and H_1 is indefinite and for $t_0, t_1 \in \mathbb{R}$, the matrix

$$H(t_0, t_1) = G + t_0 H_0 + t_1 H_1 = \begin{bmatrix} \alpha + t_0 & t_1 & 0 \\ t_1 & \alpha - t_0 & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

has eigenvalues $\alpha \pm \sqrt{t_0^2 + t_1^2}$ and β. Clearly $L : \mathbb{R}^2 \to \mathbb{R}$ defined by $L(t_0, t_1) = \lambda_{\max}(H(t_0, t_1)) = \alpha + \sqrt{t_0^2 + t_1^2}$ has its minimum $\lambda^*_{\max}$ at $(t_0, t_1) = (0, 0)$, i.e., when $H(0, 0) = G$. But the maximal eigenvalue α of G is a double eigenvalue with corresponding eigenvectors e_1 and e_2 which are the first two basis vectors of $\mathbb{R}^3$. Therefore, the matrix whose columns form an orthonormal basis of the eigenspace of G corresponding to α is $U = [\ e_1 \quad e_2\ ] \in \mathbb{R}^{3\times 2}$. There exists a real non zero vector x in the eigenspace of G corresponding to α satisfying $x^* H_0 x = x^* H_1 x = 0$ if and only if the real joint numerical range of the matrices

$$U_1 := U^T H_0 U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } U_2 := U^T H_1 U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

defined by

$$\{(x^T U_1 x, x^T U_2 x) \,\Big|\, x \in \mathbb{R}^2, \|x\|_2 = 1\}$$

contains 0. Clearly this is not true in this case as this set represents the unit circle.

Note that this example does not contradict Theorem 4.1 as the eigenvector of G with respect to α that satisfies $u^* H_0 u = u^* H_1 u = 0$ is the complex vector $u = [1 \quad i]^T$.

The following theorem gives formulae for the structured eigenvalue backward error $\eta_w^{\mathrm{pal_T}}(P, \lambda)$ when $P(z)$ is a real T-palindromic polynomial and $\lambda \in \mathbb{R} \setminus \{0, \pm 1\}$.

THEOREM 5.9. *Let $P(z) = \sum_{j=0}^m z^j A_j$ be a real T-palindromic polynomial and $\lambda \in \mathbb{R} \setminus \{0, \pm 1\}$. Suppose that $\det(P(\lambda)) \neq 0$ so that $M = (P(\lambda))^{-1}$ exists and set $k = \lfloor \frac{m-1}{2} \rfloor$. Then*

$$\lambda^*_{\max} := \min_{t_0, \ldots, t_k \in \mathbb{R}} \lambda_{\max}(G + t_0\, S_0 + \cdots + t_k\, S_k)$$

is attained for some $(t_0^*, \ldots, t_k^*) \in \mathbb{R}^{k+1}$ *where* G *is defined by* (3.13) *and* $S_j, j = 0, \ldots, k$ *are defined by* (3.19)*, respectively. If* $m = 1$ *or* $m = 2$*, or if* $\lambda_{\max}^*$ *is a simple eigenvalue of* $G + t_0^* S_0 + \cdots + t_k^* S_k$*, then*

$$\eta_w^{\mathrm{pal_{T,\mathbb{R}}}}(P, \lambda) = \frac{1}{\sqrt{\lambda_{\max}^*}} = \left(\min_{t_0, \ldots, t_k \in \mathbb{R}} \lambda_{\max}(G + t_0 S_0 + \cdots + t_k S_k) \right)^{-1/2}.$$

Proof. Consider the Hermitian matrices $\widetilde{S}_j := \widetilde{C}_j + \widetilde{C}_j^T$, $j = 0, \ldots, k$, where $\widetilde{C}_j$ are as defined in (3.5). Observe that $k = 0$ whenever $m = 1$ or $m = 2$. Therefore, the proof follows from Theorem 5.7 if it is established that each nontrivial linear combination of $S_0, \ldots, S_k$ given by (3.19), or equivalently, of $\widetilde{S}_0, \ldots, \widetilde{S}_k$ is indefinite. Suppose there exists $\begin{bmatrix} \alpha_0, & \ldots, & \alpha_k \end{bmatrix}^T \in \mathbb{R}^{k+1}$ such that $S := \sum_{j=0}^k \alpha_j \widetilde{S}_j$ is semidefinite. Then recalling that $\Lambda_m = [1, \lambda, \ldots, \lambda^m] \in \mathbb{C}^{1 \times (m+1)}$, we have

$$\begin{aligned} S &= \sum_{j=0}^{k} \alpha_j \left(\left(\Lambda_m^T (e_{j+1}^T - e_{m+1-j}^T) \right) \otimes M^T + \left((e_{j+1} - e_{m+1-j}) \Lambda_m \right) \otimes M \right) \\ &= (\Lambda_m^T \alpha^T) \otimes M^T + (\alpha \Lambda_m) \otimes M \end{aligned}$$

where the vector α is given by $\alpha := [\alpha_0, \ldots, \alpha_k, \ -\alpha_k, \ldots, -\alpha_0]^T$ when m is odd and by $\alpha := [\alpha_0, \ldots, \alpha_k, 0, -\alpha_k, \ldots, -\alpha_0]^T$ when m is even.

Setting Q as in (4.3) and $a = [a_0, \ldots, a_m] := Q^T \alpha$, since $\Lambda_m Q = e_1^T$, we have

$$\begin{aligned} (Q \otimes I_n)^T S (Q \otimes I_n) &= (Q^T \Lambda_m^T \alpha^T Q) \otimes M^T + (Q^T \alpha \Lambda_m Q_m) \otimes M \\ &= (e_1 a^T) \otimes M^T + (a e_1^T) \otimes M \\ &= \begin{bmatrix} a_0(M + M^T) & a_1 M^T & \cdots & a_m M^T \\ a_1 M & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_m M & 0 & \cdots & 0 \end{bmatrix}. \end{aligned}$$

If S is semidefinite, then $a_1 = \cdots = a_m = 0$ which implies that $Q^T \alpha = a = a_0 e_1$. Therefore, $a_0 = \alpha_0$ and

$$-\lambda \alpha_{j-1} + \alpha_j = 0, \ j = 1, \ldots, k. \tag{5.5}$$

Also,

$$(\lambda + 1)\alpha_k = 0 \text{ when } m \text{ is odd and} \tag{5.6}$$

$$\lambda \alpha_k = 0 \text{ when } m \text{ is even.} \tag{5.7}$$

The identities (5.5) imply that

$$\alpha_k = \lambda^k \alpha_0. \tag{5.8}$$

When m is odd, we have $\alpha_k = 0$ from (5.6) since $\lambda \neq -1$. Similarly when m is even, (5.7) gives $\alpha_k = 0$ as $\lambda \neq 0$. In either case, (5.8) implies that $a_0 = \alpha_0 = 0$ as $\lambda \neq 0$ and completes the proof. ∎

REMARK 5.10. A strategy identical to the one suggested in Remark 4.5 gives an optimal T-palindromic perturbation to $P(z)$ corresponding to $\eta_w^{\mathrm{pal_T}}(P, \lambda)$ in Theorems 5.2 and 5.3 and an optimal real T-palindromic perturbation to $P(z)$ corresponding to $\eta_w^{\mathrm{pal_{T,\mathbb{R}}}}(P, \lambda)$ in Theorem 5.9.

6. Numerical Experiments. In this section we present some numerical examples to illustrate the proposed method for computing the structured backward error $\eta_w^{\mathbb{S}}(P,\lambda)$ of some $\lambda \in \mathbb{C} \setminus \{0\}$ for the structures $\mathbb{S} = \text{pal}_\star$ and for $w = (1,1,\ldots,1)$. In all cases we have used the software package CVX [8, 9] in MATLAB to solve the associated optimization problems.

EXAMPLE 6.1. $L(z) = A + zA^*$ is a $*$-palindromic pencil of size 4 with eigenvalues $0.4624 - 0.8867i, -0.5697 + 1.7298i, -0.1718 + 0.5215i, -0.9765 + 0.2155i$. For $\lambda = 0.4853 - 0.5955i$ the backward error with respect to arbitrary perturbations is $\eta_w(L,\lambda) = 0.0912$ while the structured backward error satisfies $\eta_w^{\text{pal}_*}(L,\lambda) = 0.3320$. The plot on the left of Figure 6.1 illustrates the movement of the eigenvalues of the pencil $L(z)$ under the homotopic perturbations $L(z) + t\Delta L(z)$ as t varies from 0 to 1 and $\Delta L(z)$ is an optimal $*$-palindromic perturbation corresponding to $\eta_w^{\text{pal}_*}(L,\lambda)$ that induces eigenvalues at $(\lambda, 1/\bar{\lambda})$. The eigenvalue curves starting from $0.4624 - 0.8867i$ and $-0.9765 + 0.2155i$ (each marked by a star surrounded by a circle) come together on the unit circle and split out to form the pair of eigenvalues $(\lambda, 1/\bar{\lambda})$ (where λ is marked by a star surrounded by a diamond) of the pencil $L(z) + \Delta L(z)$.

On the other hand on the plot on the right hand side of Figure 6.1 gives the movement of the eigenvalues under the homotopic perturbations $L(z) + t\widetilde{\Delta L}(z)$ when t moves from 0 to 1 and $\widetilde{\Delta L}(z)$ is an optimal perturbation corresponding to $\eta_w(L,\lambda)$ that induces an eigenvalue at λ without preserving $*$-palindromic structure. In this case the perturbations move the nearest eigenvalue of the pencil to λ.

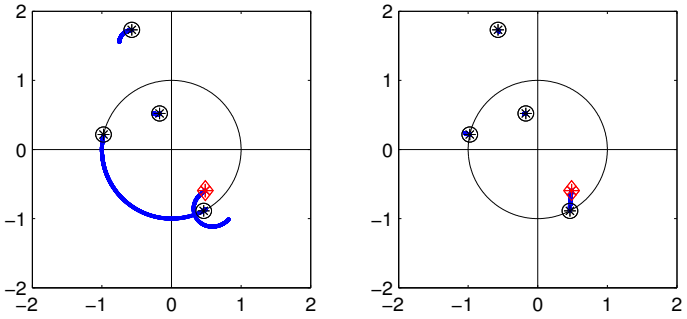

FIG. 6.1. Eigenvalue perturbation curves for the $*$-palindromic pencil $L(z)$ of Example 6.1 with respect $*$-palindromic perturbations (left) and arbitrary perturbations (right).

EXAMPLE 6.2. $Q(z) = A + zB^* + z^2A^*$ is a $*$-palindromic polynomial of size 3 with eigenvalues $-3.2746 - 0.4165i, -0.3597 + 1.82221i, 0.9896 - 0.1437i, -0.0961 - 0.9954i, -0.3005 - 0.0382i, -0.1043 + 0.5282i$. For $\lambda = 0.88 + 0.15i$, the eigenvalue backward error with respect to arbitrary perturbations is 0.609 while with respect to $*$-palindromic perturbations this is 1.7059. The plot on the left of Figure 6.2 illustrates the effect of perturbations $Q(z) + t\Delta Q(z)$ on the eigenvalues of $Q(z)$ as t varies from 0 to 1, $\Delta Q(z)$ being an optimal $*$-palindromic perturbation to $Q(z)$ corresponding to $\eta_w^{\text{pal}_*}(Q,\lambda)$ that induces eigenvalues at $(\lambda, 1/\bar{\lambda})$. It shows eigenvalue curves starting from the eigenvalues $-0.1043 + 0.528i$ and $-0.3597 + 1.82221i$ (each marked by a star surrounded by circle) of $Q(z)$ coalescing on the unit circle and moving along the circle till they next coalesce with the eigenvalue curve starting from the eigenvalue $0.9896 - 0.1437i$ on the unit circle. After the second coalescence, the eigenvalues curves split out of the unit circle to form the pair of eigenvalues $(\lambda, 1/\bar{\lambda})$ (where λ is

marked by a star surrounded by a diamond) of $Q(z) + \Delta Q(z)$.

The plot on the right of Figure 6.2 shows the movement of the eigenvalues of $Q(z)$ under perturbations $Q(z) + t\widetilde{\Delta Q}(z)$ where $\widetilde{\Delta Q}(z)$ is an optimal perturbation to $Q(z)$ corresponding to $\eta_w(Q, \lambda)$ that induces an eigenvalue at λ and is not $*$-palindromic. In this case the nearest eigenvalue of $Q(z)$ on the unit circle moves to form the eigenvalue λ of $Q(z) + \widetilde{\Delta Q}(z)$.

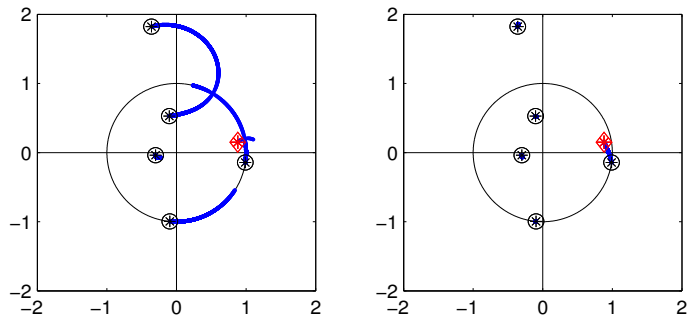

FIG. 6.2. Eigenvalue perturbation curves for the $*$-palindromic polynomial $Q(z)$ of Example 6.2 with respect to $*$-palindromic perturbations (left) and arbitrary perturbations (right).

EXAMPLE 6.3. $L(z) = A + zA^T$ is a real T-palindromic pencil of size 3 with eigenvalues -1 and $-0.5954 \pm 0.8034i$ all on the unit circle. For $\lambda = -1.6656$, the eigenvalue backward error is 0.6563 with respect to real T-palindromic perturbations and 0.5614 with respect to complex T-palindromic perturbations. With respect to arbitrary perturbations, the eigenvalue backward error is 0.3177. Figure 6.3 illustrates the effect of real T-palindromic, complex T-palindromic and arbitrary perturbations on the eigenvalues of $L(z)$ so that they move to form an eigenvalue at λ for the respective perturbed pencils.

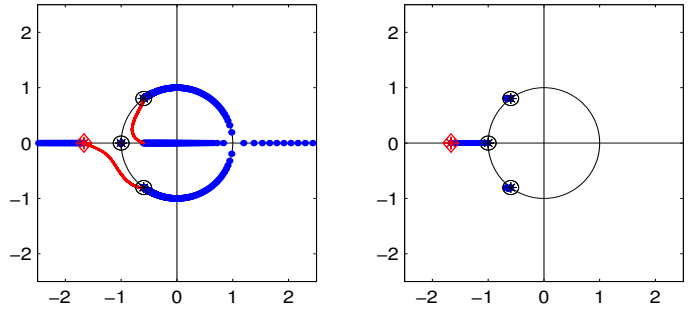

FIG. 6.3. Eigenvalue perturbation curves for the real T-palindromic pencil $L(z)$ of Example 6.3 with respect to real and complex T-palindromic perturbations (left) and arbitrary perturbations (right).

The plot on the left of Figure 6.3 shows the effect of perturbations $L(z) + t\Delta L(z)$ on the eigenvalues of $L(z)$ (in thick curves) as t moves from 0 to 1, $\Delta L(z)$ being the minimal real T-palindromic perturbation to $L(z)$ corresponding to $\eta_w^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}(L, \lambda)$

that induces eigenvalues at $(\lambda, 1/\lambda)$. In this case eigenvalue curves starting from the eigenvalues $-0.5954 \pm 0.8034i$ (each marked by a star surrounded by a circle) on the unit circle coalesce on the unit circle and split out with one of the branches moving over ∞ to form the eigenvalue λ (marked by a star surrounded by a diamond) and the other moving out to form the eigenvalue $1/\lambda$ (inside the unit circle) as t moves from 0 to 1.

The movement of the perturbed eigenvalues under the given real T-palindromic perturbations may partly be attributed to two facts. The first is that -1 is always an eigenvalue of a T-palindromic polynomial of odd degree and odd size (since $P(-1)$ is then a skew symmetric matrix of odd size and thus singular) of which the given pencil $L(z)$ is a particular case. The second fact is that eigenvalues of real T-palindromic polynomials occur in quadruples $(\mu, \bar{\mu}, 1/\mu, 1/\bar{\mu})$. This symmetry breaks down only on the unit circle and on the real line where it reduces to the pairing $(\mu, 1/\mu)$. Since the only eigenvalues of $L(z)$ other than -1 are also on the unit circle, in order to maintain the eigenvalue symmetry, they have to pass through the intersection of the unit circle and the real line to form the eigenvalues at λ and $1/\lambda$.

The plot on the left of Figure 6.3 also shows the effect of perturbations $L(z) + t\widehat{\Delta L}(z)$ on the eigenvalues of $L(z)$ (in thin curves) as t moves from 0 to 1, $\widehat{\Delta L}(z)$ being the minimal complex T-palindromic perturbation to $L(z)$ corresponding to $\eta_w^{\mathrm{pal_{T,\mathbb{C}}}}(L, \lambda)$ that induces eigenvalues at $(\lambda, 1/\lambda)$. Also in this case -1 cannot be moved to λ under T-palindromic perturbations. Instead, an eigenvalue curve starting from $-0.5954 + 0.8034i$ moves to λ while another starting from $-0.5954 - 0.8034i$ moves to $1/\lambda$ (inside the unit circle) as t moves from 0 to 1.

Finally, the plot on the right of Figure 6.3 shows the effect of perturbations $L(z) + t\widetilde{\Delta L}(z)$ on the eigenvalues of $L(z)$ as t moves from 0 to 1, $\widetilde{\Delta L}(z)$ being an optimal perturbation corresponding to $\eta_w(L, \lambda)$ that induces an eigenvalue at λ and is not T-palindromic. In this case, the nearest eigenvalue -1 of $L(z)$ moves to form the eigenvalue λ of $L(z) + \widetilde{\Delta L}(z)$.

We also compare $\eta_w(P, \lambda)$ with $\eta_w^{\mathrm{pal}_\star}(P, \lambda)$ for the cases that the values of λ converge to an eigenvalue of $P(z)$ as well as for arbitrary values of λ.

Table 6.1 illustrates these comparisons for the $*$-palindromic pencil $L(z)$ of Example 6.1 as λ values converge to the eigenvalue $0.4624 - 0.8867i$ on the unit circle. Observe that while $\eta_w(L, \lambda)$ decreases to 0, this is not the case for $\eta_w^{\mathrm{pal}_*}(L, \lambda)$ leading to large differences in the values of the two backward errors.

Table 6.2 does the same comparison for values of λ that converge to the eigenvalue $-0.5697 + 1.7298i$ not on the unit circle as well as for arbitrary values of λ. In the first case, both $\eta_w(L, \lambda)$ and $\eta_w^{\mathrm{pal}_*}(L, \lambda)$ decrease to 0. However in the second case, there is significant difference between the two backward errors even when the values of λ are away from the unit circle.

Table 6.3 compares the backward errors $\eta_w(L, \lambda)$, $\eta_w^{\mathrm{pal_{T,\mathbb{R}}}}(L, \lambda)$ and $\eta_w^{\mathrm{pal_T}}(L, \lambda)$ for the T-palindromic pencil $L(z)$ in Example 6.3 as λ converges to -1 along the real line. Observe that while both $\eta_w(L, \lambda)$, and $\eta_w^{\mathrm{pal_{T,\mathbb{C}}}}(L, \lambda)$ decrease, $\eta_w^{\mathrm{pal_{T,\mathbb{R}}}}(L, \lambda)$ increases as λ approaches -1. This leads to large differences between $\eta_w^{\mathrm{pal_{T,\mathbb{R}}}}(L, \lambda)$ and the other backward errors at values of λ close to -1.

Conclusions. We have obtained formulas for the backward error of approximate eigenvalues of $*$-palindromic matrix polynomials and T-palindromic pencils and quadratic polynomials with respect to structure preserving perturbations. When the T-palindromic polynomial is real, we have also obtained the backward error of a real number considered as an approximate eigenvalue of the matrix polynomial with re-

TABLE 6.1

Values of $\eta_w(L,\lambda)$ and $\eta_w^{\mathrm{pal}_*}(L,\lambda)$ for the $*-$palindromic pencil $L(z)$ in Example 6.1 where $\lambda \to 0.4624 - 0.8867i$.

λ	$\eta_w(L,\lambda)$	$\eta_w^{\mathrm{pal}_*}(L,\lambda)$
0.600 - 1.200i	0.0710	0.2755
0.550 - 1.100i	0.0510	0.2772
0.520 - 1.000i	0.0300	0.2824
0.500 - 0.980i	0.0240	0.2807
0.480 - 0.950i	0.0159	0.2799
0.475 - 0.930i	0.0111	0.2811
0.470 - 0.900i	0.0038	0.2835
0.465 - 0.895i	0.0022	0.2830

TABLE 6.2

Values of $\eta_w(L,\lambda)$ and $\eta_w^{\mathrm{pal}_*}(L,\lambda)$ for the $*-$palindromic pencil $L(z)$ in Example 6.1 where $\lambda \to -0.5697 + 1.7298i$ (left) and for arbitrary λ (right).

λ	$\eta_w(L,\lambda)$	$\eta_w^{\mathrm{pal}_*}(L,\lambda)$
$-2.50 + 0.50i$	0.1134	0.1605
$-2.00 + 1.00i$	0.1006	0.1344
$-1.50 + 1.40i$	0.0882	0.1012
$-0.90 + 1.50i$	0.0569	0.0593
$-0.60 + 1.62i$	0.0196	0.0203
$-0.58 + 1.70i$	0.0053	0.0055

λ	$\eta_w(L,\lambda)$	$\eta_w^{\mathrm{pal}_*}(L,\lambda)$
$1.1890 + 0.0376i$	0.4726	0.5937
$0.2940 - 1.3362i$	0.0850	0.2190
$1.1910 - 1.2025i$	0.1571	0.3415
$0.9410 - 0.9921i$	0.1173	0.3494
$0.4850 - 0.5955i$	0.0912	0.3320
$0.6680 - 0.0783i$	0.3688	0.5149

TABLE 6.3

Values of $\eta_w(L,\lambda)$, $\eta_w^{\mathrm{pal_T}}(L,\lambda)$ and $\eta_w^{\mathrm{pal_{T,\mathbb{R}}}}(L,\lambda)$ for the T-palindromic pencil $L(z)$ in Example 6.3 as $\lambda \to -1$.

λ	$\eta_w(P,\lambda)$	$\eta_w^{\mathrm{pal_T}}(P,\lambda)$	$\eta_w^{\mathrm{pal_{T,\mathbb{R}}}}(P,\lambda)$
-1.6656	0.1692	0.3177	0.5614
-1.5500	0.1501	0.3076	0.5623
-1.4500	0.1308	0.2992	0.5631
-1.3500	0.1086	0.2912	0.5638
-1.2500	0.0827	0.2842	0.5644
-1.1500	0.0528	0.2788	0.5649

spect to real T-palindromic perturbations. For each case, a procedure for constructing an optimal perturbation that corresponds to the structured backward error is provided. Numerical experiments suggest that there is significant difference between the backward errors with respect to structure preserving and arbitrary perturbations in many cases.

REFERENCES

[1] B. Adhikari. *Backward perturbation and sensitivity analysis of structured polynomial eigenvalue problems*. PhD thesis, Department of Mathematics, Indian Institute of Technology Guwahati, December, 2008.

[2] B. Adhikari and R. Alam. Structured backward errors and pseudospectra of structured matrix pencils. *SIAM J. Matrix Anal. Appl.*, 31:331–359, 2009.

[3] B. Adhikari and R. Alam. On backward errors of structured polynomial eigenproblems solved by structure preserving linearizations. *Linear Algebra Appl.*, 434:1989–2017, 2011.

[4] Sk.S. Ahmad and R. Alam. Pseudospectra, critical points and multiple eigenvalues of matrix polynomials. *Linear Algebra Appl.*, 430:1171–1195, 2009.

[5] R. Alam, S. Bora, M Karow, V. Mehrmann, and J. Moro. Perturbation theory for Hamiltonian matrices and the distance to bounded-realness. *SIAM J. Matrix Anal. Appl.* 32: 484–514, 2011.

[6] S. Bora, M. Karow, C. Mehl, and P. Sharma. Structured Eigenvalue Backward Errors of Matrix Pencils and Polynomials with Hermitian and Related Structures. *SIAM J. Matrix Anal. Appl.*, 35:2, 453-475,2014.

[7] I. Gohberg, P. Lancaster, and L. Rodman. *Indefinite Linear Algebra.* Birkhäuser, Basel, 2005.

[8] M. Grant and S. Boyd. *Graph implementations for nonsmooth convex programs, Recent Advances in Learning and Control (a tribute to M. Vidyasagar),* V. Blondel, S. Boyd, and H. Kimura, editors, pages 95-110. Lecture Notes in Control and Information Sciences, Springer, 2008. `http://stanford.edu/boyd/graph_dcp.html`.

[9] M. Grant and S. Boyd. CVX: Matlab software for disciplined convex programming, version 2.0 beta. http://cvxr.com/cvx, September 2012.

[10] D.J. Higham and N.J. Higham. Structured backward error and condition of generalized eigenvalue problems. *SIAM J. Matrix Anal. Appl.*, 20:493–512, 1998.

[11] R.A. Horn and C.R. Johnson. *Topics in Matrix Analysis.* Cambridge University press, Cambridge, 1991.

[12] A. Hilliges. *Numerische Lösung von quadratischen Eigenwertproblemen mit Anwendung in der Schienendynamik.* Diplomarbeit, TU Berlin, Inst. f. Mathematik, 2004.

[13] A. Hilliges, C. Mehl and V. Mehrmann. *On the solution of palindromic eigenvalue problems.* In Proceedings of the 4th European Congress on Computational Methods in applied Sciences and Engineering (ECCOMAS). Jyväskylä, Finland, 2004. CD-ROM.

[14] W. Kahan, B.N. Parlett, and E. Jiang. Residual bounds on approximate eigensystems of nonnormal matrices. *SIAM J. Numer. Anal.*, 19:470–484, 1982.

[15] M. Karow. μ-values and spectral value sets for linear perturbation classes defined by a scalar product. *SIAM J. Matrix Anal. Appl.*, 32:845–865, 2011.

[16] M. Karow, D. Kressner, and F. Tisseur. Structured eigenvalue condition numbers. *SIAM J. Matrix Anal. Appl.*, 28:1052–1068, 2006.

[17] D. Kressner, M.J. Peláez, and J. Moro. Structured Hölder condition numbers for multiple eigenvalues. *SIAM J. Matrix Anal. Appl.*, 31:175–201, 2009.

[18] P. Lancaster. Lambda-Matrices and Vibrating Systems. Pergamon Press, Oxford, 1966. Reprinted by Dover, New York, 2002.

[19] P. Lancaster and L. Rodman. *Algebraic Riccati Equations.* Clarendon Press, Oxford, UK, 1995. *SIAM J. Matrix Anal. Appl.*, 28:971–1004, 2006.

[20] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Palindromic polynomial eigenvalue problems: Good vibrations from good linearizations. Technical Report No. 239, DFG Research Center MATHEON, Berlin, 2006.

[21] D.S. Mackey, N. Mackey, and F. Tisseur. Structured mapping problems for matrices associated with scalar products Part I: Lie and Jordan algebras. *SIAM J. Matrix Anal. Appl.*, 29:1389–1410, 2008.

[22] V. Mehrmann. *The Autonomous Linear Quadratic Control Problem. Theory and Numerical Solution.* Lecture Notes in Control and Information Sciences, 163. Springer-Verlag, Berlin, Germany, 1991.

[23] F. Tisseur. A chart of backward errors and condition numbers for singly and doubly structured eigenvalue problems. *SIAM J. Matrix Anal. Appl.*, 24:877–897, 2003.

[24] F. Tisseur and K. Meerbergen. The quadratic eigenvalue problem. *SIAM Review*, 43:235–286, 2001.

[25] S. Zaglmayr. *Eigenvalue problems in SAW-filter simulations.* Diplomarbeit, Inst. of Comp. Mathematics, J. Kepler Univ. Linz, Austria, 2002.