An Optimal Adaptive FEM for Eigenvalue Clusters *

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Abstract

The analysis of adaptive finite element methods in practice immediately leads to eigenvalue clusters which requires the simultaneous marking in adaptive finite element methods. A first analysis for multiple eigenvalues of the recent work [Dai, He, Zhou, arXiv Preprint 1210.1846v2] introduces an adaptive method whose marking strategy is based on the element-wise sum of local error estimator contributions for multiple eigenvalues. This paper proves optimality of a practical adaptive algorithm for eigenvalue clusters for the eigenvalues of the Laplace operator in terms of nonlinear approximation classes. All estimates are explicit in the initial mesh-size, the eigenvalues and the cluster width to clarify the dependence of the involved constants.

Keywords eigenvalue problem, eigenvalue cluster, adaptive finite element method, optimality
AMS subject classifications 65M12, 65M60, 65N25

1 Introduction

Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain with polyhedral boundary. The adaptive finite element approximation of multiple eigenvalues of the model problem

$$-\triangle u = \lambda u$$

leads to the situation of an eigenvalue cluster as the eigenvalues of interest and their multiplicities may not be resolved by the initial mesh. The optimality analysis of adaptive finite element methods for eigenvalue problems [DXZ08, GM11, CG12, CGS13] is based on the comparison of the finite element solutions on different meshes. In the case of multiple eigenvalues, this leads to the difficulty that the discrete orthonormal systems of eigenfunctions produced by the adaptive algorithm may change in each step of the adaptive loop. The works [GMZ09, GG09, CG11] study the plain convergence of adaptive algorithms for eigenvalue problems. The proofs of [DXZ08, GM11, CG12, CGS13] on optimal convergence rates are restricted to simple eigenvalues. The first optimal convergence result of adaptive finite element schemes with a multiple eigenvalue [DHZ13] introduces the innovative methodology to use one bulk criterion for all discrete eigenfunctions in the algorithm for automatic mesh refinement and proves equivalence to the simultaneous error of the discrete eigenvalue approximation to the fixed orthonormal basis of the exact eigenspace. In practice, little perturbations in coefficients or in the geometry immediately lead to an eigenvalue cluster of finite length.

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In case that the multiplicities are resolved by the initial mesh in the sense that the multiplicities are known and each of the corresponding discrete eigenvalues is well-separated from the remaining spectrum, the algorithm of [DHZ13], which is based on the multiplicity of the exact eigenvalues, can still be employed. In the case that the cluster is narrow, this separation condition is unrealistic and a modification is required. A posteriori error estimates and adaptive marking strategies for eigenvalue clusters were discussed in [GO09, BGO13]. This article extends the approach of [DHZ13] based on explicit residual-based error estimators to the more practical case of eigenvalue clusters. A first-glance generalisation of the analysis of [DHZ13] to clustered eigenvalues would lead to an additive term in the reliability estimate that describes the cluster-width and thus is not efficient. This paper presents a different analysis that leads to "cluster-robust" error estimates.

The energy error for an L^2-normalised eigenfunction u is measured by the quantity $|||u-\Lambda_\ell u|||$. Here, Λ_ℓ is the L^2 projection of the Galerkin projection $G_\ell u$ onto the space W_ℓ spanned by the discrete eigenfunctions within the cluster (more details follow in Section 2). A theoretical non-computable error estimator is employed which allows a proof of equivalence to the refinement indicator of the adaptive algorithm. In contrast to the case of one multiple eigenvalue, care has to be taken that the reliability and equivalence constants of the error estimator do not depend on the cluster or its length. This restricts the analysis to the lowest-order conforming finite element method based on piecewise affine functions.

The non-computable error estimator allows reliable and efficient error estimates for $|||u-\Lambda_\ell u|||$ and is locally equivalent to the computable explicit residual-based error estimator. The proof of this property requires a careful analysis and further conditions on the initial mesh-size. This is due to the fact that Λ_ℓ does not map orthonormal sets onto orthonormal sets.

The equivalence of error estimators allows to consider the theoretical error estimator in the analysis with some modified bulk parameter. Since this error estimator does not depend on the choice of discrete eigenfunctions, it allows for the error estimator reduction and contraction properties. These ingredients lead to the proof of optimal convergence rates for the error of the simultaneous approximation of the eigenfunctions within the cluster. Let $(\lambda_j \mid j\in J)$ denote the cluster under consideration with (possibly different) eigenspaces $E(\lambda_j)$. Then the error quantities

$$\left(\frac{|\lambda_k-\lambda_{\ell,k}|}{\lambda_{\ell,k}}\right)^{1/2} \quad\text{and}\quad \sup_{j\in J}\sup_{\substack{w\in E(\lambda_j)\\ \|w\|=1}}\inf_{v_\ell\in W_\ell}|||w-v_\ell|||$$

decay as $(\mathrm{card}(\mathscr{T}_\ell)-\mathrm{card}(\mathscr{T}_0))^{-\sigma}$, provided all eigenfunctions are in the approximation class $\mathscr{A}_\sigma$, see Section 3 for more details.

One subtle aspect are the requirements on the fineness of the initial mesh for the optimality proof and the dependence of the bulk parameter on the cluster and its length. This paper gives a detailed explicit analysis of all those sufficient conditions for optimal convergence in Table 1.

Although the analysis in this paper is concerned with the eigenvalues of the Laplace operator for the sake of exposition, it can also be applied to more general symmetric second-order elliptic eigenvalue problems of the form $\operatorname{div} KDu + c\,u = \lambda u$ with a symmetric uniformly positive definite diffusion matrix field K and a nonnegative coefficient function c, provided the coefficients are essentially bounded and resolved by the initial mesh.

(H1)	$M_J := \sup_{\mathscr{T}_\ell \in \mathbb{T}} \max_{j \in \{1,\dots,\dim(V_\ell)\}\setminus J} \max_{k\in J} \frac{\lambda_k}{\lvert\lambda_{\ell,j}-\lambda_k\rvert} < \infty$	Section 2.3
(H2)	$H_0^{2s} B^2 C_{\mathrm{drel}}^2 (1+M_J)^2 \leq 1$	Section 4
(H3)	$\varepsilon := \max_{j\in J} \lVert u_j - \Lambda_\ell u_j\rVert \leq \sqrt{1+(2N)^{-1}} - 1$	Lemma 5.1
(H4)	$(1+M_J)^2 (BC_{\mathrm{qo}} H_0^{2s} + 2B^2 C_{\mathrm{reg}}^2 H_0^{2+2s}) < \min\left\{1, \frac{1-\rho_1}{KC_{\mathrm{drel}}^2}\right\}/4$	Prop. 6.3

Table 1: Overview of assumptions on the initial mesh-size and their first occurrence in this article. B acts as upper bound for all $(\lambda_j \mid j \in J)$.

The remaining parts of this article are organised as follows. Section 2 introduces the eigenvalue problem and its finite element discretisation. The adaptive algorithm and the optimality result based on the concept of approximation classes is introduced in Section 3. Sections 4–5 introduce the theoretical error estimator and prove equivalence to the computable error estimator. Section 6 establishes the contraction property; the proof of optimal convergence rates is given in Section 7. The numerical tests of Section 8 and the comments of Section 9 conclude the paper.

Throughout this article, standard notation on Lebesgue and Sobolev spaces and their norms is employed; the L^2 scalar product is denoted by $(\cdot,\cdot)_{L^2(\Omega)}$. The d-dimensional Lebesgue measure of a measurable set $\omega \subseteq \mathbb{R}^d$ is denoted by $\mathrm{meas}(\omega)$. All estimates are explicit with respect to the initial mesh-size H_0, the eigenvalues $(\lambda_k)_{k\in J}$ and the separation constant M_J, which depends on the cluster J. All other constants may depend on the domain Ω, the space dimension d or the angles of the initial triangulation $\mathscr{T}_0$; they do neither depend on the mesh-size nor on the eigenvalue cluster.

2 Preliminaries

This section introduces the finite element discretisation of the eigenvalue problem along with the necessary notation for simplicial triangulations and discrete spaces.

2.1 Triangulations and Finite Element Spaces

Let $\mathscr{T}_0$ be a regular simplicial triangulation of Ω in the sense of [Ste08], i.e., $\cup\mathscr{T}_0 = \overline{\Omega}$ and any two elements of $\mathscr{T}_0$ are either disjoint or share exactly one k-dimensional face for $k \leq d$ (e.g., a vertex or an edge). Throughout this paper, any regular triangulation of Ω is assumed to be admissible in the sense that it is regular and a refinement of $\mathscr{T}_0$ created by the refinement rules of [Ste08]. The set of all admissible refinements is denoted by $\mathbb{T}$. Given a triangulation $\mathscr{T}_\ell \in \mathbb{T}$, the piecewise constant mesh-size function $h_\ell := h_{\mathscr{T}_\ell}$ is defined by $h_\ell|_T := h_T := \mathrm{meas}(T)^{1/d}$ for any simplex $T \in \mathscr{T}_\ell$.

The set of $(d-1)$-dimensional hyper-faces (e.g., edges for $d=2$ or faces for $d=3$) of $\mathscr{T}_\ell$ is denoted by $\mathscr{F}_\ell$ while the interior $(d-1)$-dimensional hyper-faces are denoted by $\mathscr{F}_\ell(\Omega)$. Let every $F \in \mathscr{F}_\ell$ be equipped with a fixed normal vector ν_F. Given $F \in \mathscr{F}_\ell(\Omega)$, $F = \partial T_+ \cap \partial T_-$ shared by two simplices $(T_+, T_-) \in \mathscr{T}_\ell^2$, and a piecewise smooth function v, define the jump of v across F by

$$[v]_F := v|_{T_+} - v|_{T_-}.$$

For a simplex T, the set of $(d-1)$-dimensional hyper-faces belonging to T is denoted by $\mathscr{F}(T)$.

The set of piecewise polynomial functions of degree $\leq k$ with respect to $\mathscr{T}_\ell$ is denoted by $\mathfrak{P}_k(\mathscr{T}_\ell)$. The L^2 projection onto $\mathfrak{P}_k(\mathscr{T}_\ell)$ is denoted by $\Pi^k_{\mathscr{T}_\ell} \equiv \Pi^k_\ell$. The conforming finite element space of piecewise affine functions is defined as

$$V_\ell := V(\mathscr{T}_\ell) := \mathfrak{P}_1(\mathscr{T}_\ell) \cap H^1_0(\Omega).$$

2.2 The Eigenvalue Problem and its Discretisation

Let $V := H^1_0(\Omega)$ be equipped with the scalar products

$$a(v,w) := (Dv, Dw)_{L^2(\Omega)} \quad \text{and} \quad b(v,w) := (v,w)_{L^2(\Omega)}$$

and induced norms $|||v||| := a(v,v)^{1/2}$ and $\|v\| := b(v,v)^{1/2}$. The Laplace eigenvalue problem seeks eigenpairs $(\lambda, u) \in \mathbb{R} \times V$ with $\|u\| = 1$ such that

$$a(u,v) = \lambda b(u,v) \quad \text{for all } v \in V. \tag{2.1}$$

The finite element discretisation based on a regular triangulation $\mathscr{T}_\ell$ seeks discrete eigenpairs $(\lambda_\ell, u_\ell) \in \mathbb{R} \times V_\ell$ with $\|u_\ell\| = 1$ and

$$a(u_\ell, v_\ell) = \lambda_\ell b(u_\ell, v_\ell) \quad \text{for all } v_\ell \in V_\ell. \tag{2.2}$$

It is well known [Eva10] that (2.1) has countably many eigenvalues, which are real and positive with $+\infty$ as only accumulation point. Suppose that the eigenvalues and the discrete eigenvalues are enumerated

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \quad \text{and} \quad 0 < \lambda_{\ell,1} \leq \dots \leq \lambda_{\ell,\dim(V_\ell)}.$$

Let $(u_1, u_2, u_3, \dots)$ and $(u_{\ell,1}, u_{\ell,2}, \dots, u_{\ell,\dim(V_\ell)})$ denote some b-orthonormal systems of corresponding eigenfunctions. For a cluster of eigenvalues $\lambda_{n+1}, \dots, \lambda_{n+N}$ of length $N \in \mathbb{N}$ define the index set $J := \{n+1, \dots, n+N\}$ and the spaces $W := \operatorname{span}\{u_j\}_{j\in J}$ and $W_\ell := \operatorname{span}\{u_{\ell,j}\}_{j\in J}$. In the situation of a true eigenvalue cluster, the eigenspaces $E(\lambda_j)$ may differ for different $j \in J$. It is assumed that in the eigenvalue cluster multiple eigenvalues are not split, in other words it holds that

$$\lambda_n < \lambda_{n+1} \leq \dots \leq \lambda_{n+N} < \lambda_{n+N+1}$$

(with the convention $\lambda_0 := 0$). The cluster is contained in a bounded interval $[A,B]$. In the present case of nested conforming finite element spaces, A and B can be chosen as $A := \lambda_{n+1}$, $B := \lambda_{0,n+N}$ with respect to the coarse initial triangulation $\mathscr{T}_0$. The min-max principle [SF73] assures that the discrete eigenvalues of the cluster will be contained in $[A,B]$ for all $\mathscr{T}_\ell$ which are refined from $\mathscr{T}_0$.

2.3 Approximation of Eigenvalue Clusters and L^2 Control

The a priori analysis of [SF73, BO91, Bof10] shows that the eigenvalues converge on a sequence of uniformly refined triangulations. In particular, for a sufficiently fine initial mesh-size $H_0 \ll 1$, the quantities $\lambda_{n+1} - \lambda_{\ell,n}$ and $\lambda_{\ell,n+N+1} - \lambda_{n+N}$ are positive and uniformly bounded from below by some positive real number. Hence, there exists a separation bound

$$M_J := \sup_{\mathscr{T}_\ell \in \mathbb{T}} \max_{j \in \{1,\dots,\dim(V_\ell)\}\setminus J} \max_{k\in J} \frac{\lambda_k}{|\lambda_{\ell,j} - \lambda_k|} < \infty. \tag{H1}$$

Note that upper bounds for M_J can be computed by employing the recent results of [CG14, LO13] on guaranteed lower eigenvalue bounds which allow to compute a lower bound for A. Let $G_\ell : V \to V_\ell$ denote the Galerkin projection, i.e., for any $w \in V$ the function $G_\ell w$ satisfies

$$a(w - G_\ell w, v_\ell) = 0 \quad \text{for all } v_\ell \in V_\ell.$$

Let P_ℓ denote the L^2 projection onto W_ℓ and define

$$\Lambda_\ell := \Lambda_{\mathscr{T}_\ell} := P_\ell \circ G_\ell. \tag{2.3}$$

For any eigenfunction $u \in W$, the function $\Lambda_\ell u \in W_\ell$ is regarded as its finite element approximation. This approximation does not depend on the choice of the discrete eigenfunctions in the computation. Notice that $\Lambda_\ell u$ is neither computable nor necessarily an eigenfunction.

The following lemma provides an L^2 error estimate for $\Lambda_\ell u$. The proof follows immediately from Proposition 3.1 of [CG11] and the Aubin-Nitsche duality technique [Bra08] for the boundary value problem.

Lemma 2.1. *Provided the condition* (H1) *holds true, any eigenfunction $u \in W$ with $\|u\| = 1$ satisfies*

$$\|\Lambda_\ell u - G_\ell u\| \le M_J \|u - G_\ell u\| \tag{2.4}$$

and

$$\|u - P_\ell u\| \le \|u - \Lambda_\ell u\| \le (1 + M_J) C_{\mathrm{reg}} H_0^s |||u - G_\ell u||| \le (1 + M_J) C_{\mathrm{reg}} H_0^s |||u - \Lambda_\ell u||| \tag{2.5}$$

for a constant C_{reg} and a parameter $0 < s \le 1$ that describes the elliptic regularity of the function u.

Proof. The first stated inequality (2.4) is proven in [CG11, Proposition 3.1]. The fact that $P_\ell u$ is the L^2 best-approximation of u in W_ℓ proves $\|u - P_\ell u\| \le \|u - \Lambda_\ell u\|$. The triangle inequality and (2.4) prove $\|u - \Lambda_\ell u\| \le (1 + M_J)\|u - G_\ell u\|$. The Aubin-Nitsche duality technique therefore leads to some constant C_{reg} such that

$$\|u - \Lambda_\ell u\| \le (1 + M_J) C_{\mathrm{reg}} H_0^s |||u - G_\ell u|||.$$

The last inequality in (2.5) follows from the fact that $G_\ell u$ is the best-approximation of u in V_ℓ with respect to the energy norm $|||\cdot|||$. ∎

The following algebraic identity will be frequently used in the analysis.

Lemma 2.2. *Any exact eigenpair $(\lambda, u) \in \mathbb{R} \times V$ satisfies*

$$a(\Lambda_\ell u, v_\ell) = \lambda b(P_\ell u, v_\ell) \quad \textit{for all } v_\ell \in V_\ell.$$

Proof. The representation of $\Lambda_\ell u$ in terms of the orthonormal basis $(u_{\ell,j})_{j \in J}$ reads as

$$\Lambda_\ell u = \sum_{j \in J} \alpha_j u_{\ell,j} \quad \text{with } \alpha_j = b(G_\ell u, u_{\ell,j}) \quad \text{for all } j \in J.$$

The symmetry of a and b proves for any $j \in J$ that

$$\alpha_j = b(G_\ell u, u_{\ell,j}) = \lambda_{\ell,j}^{-1} a(G_\ell u, u_{\ell,j}) = \lambda_{\ell,j}^{-1} \lambda b(u, u_{\ell,j}).$$

Therefore, the discrete eigenvalue problem reveals

$$a(\Lambda_\ell u, v_\ell) = \sum_{j\in J} \alpha_j \lambda_{\ell,j} b(u_{\ell,j}, v_\ell) = \lambda \sum_{j\in J} b(b(u,u_{\ell,j})u_{\ell,j}, v_\ell) = \lambda b(P_\ell u, v_\ell). \qquad \blacksquare$$

The analysis of this paper is merely concerned with an approximation of the eigenfunctions. The following consequence of [KO06, Corollary 3.4] gives a bound on the eigenvalue error. Recall that the eigenvalue cluster is contained in the interval $[A,B]$.

Proposition 2.3. *There exists some constant C such that for any $j \in J$ the eigenvalue error is controlled as*

$$|\lambda_{\ell,j} - \lambda_j| \leq \lambda_{\ell,j}(1 + M_J^2 B^2 C) \sup_{\substack{w\in E(\lambda_j)\\ |||w|||=1}} \inf_{v_\ell \in W_\ell} |||w - v_\ell|||^2. \qquad \blacksquare$$

3 Adaptive Algorithm and Optimality Result

This section introduces the adaptive algorithm AFEM and states the optimality result, which is based on the concept of approximation classes.

3.1 Adaptive Algorithm

For any simplex $T \in \mathscr{T}_\ell$, the explicit residual-based error estimator from [DPR03] consists of the sum of the residuals of the computed discrete eigenfunctions $(u_{\ell,j})_{j\in J}$,

$$\eta_\ell^2(T) := \sum_{j\in J} \left(h_T^2 \|\lambda_{\ell,j} u_{\ell,j}\|_{L^2(T)}^2 + \sum_{F\in \mathscr{F}(T)\cap \mathscr{F}_\ell(\Omega)} h_T \|[Du_{\ell,j}]_F \cdot \nu_F\|_{L^2(F)}^2 \right).$$

Let, for any subset $\mathscr{K} \subseteq \mathscr{T}$,

$$\eta_\ell^2(\mathscr{K}) := \sum_{T\in\mathscr{K}} \eta_\ell^2(T).$$

The adaptive algorithm is driven by this computable error estimator and runs the following loop

Input. Initial triangulation $\mathscr{T}_0$, bulk parameter $0 < \theta \leq 1$.
for $\ell = 0,1,2,\ldots$ **do**
Solve. Compute discrete solutions $(\lambda_{\ell,j}, u_{\ell,j})_{j\in J}$ of (2.2) with respect to $\mathscr{T}_\ell$.
Estimate. Compute local contributions of the error estimator $\left(\eta_\ell^2(T)\right)_{T\in\mathscr{T}_\ell}$.
Mark. The Dörfler marking chooses a minimal subset $\mathscr{M}_\ell \subseteq \mathscr{T}_\ell$ such that

$$\theta \eta_\ell^2(\mathscr{T}_\ell) \leq \eta_\ell^2(\mathscr{M}_\ell).$$

Refine. Generate a new triangulation $\mathscr{T}_{\ell+1}$ as the smallest admissible refinement of $\mathscr{T}_\ell$ satisfying $\mathscr{M}_\ell \cap \mathscr{T}_{\ell+1} = \emptyset$ by using the refinement rules of [Ste08] (see Figure 1 for $d = 2$). **end do**
Output. Sequences of triangulations $(\mathscr{T}_\ell)_\ell$ and discrete solutions $\left((\lambda_{\ell,j}, u_{\ell,j})_{j\in J}\right)_\ell$.

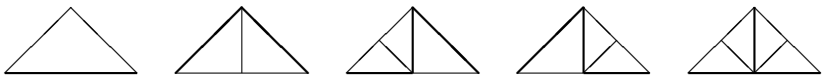

Figure 1: Possible refinements of a triangle T in one level in 2D. The thick lines indicate the refinement edges of the sub-triangles as in [BDD04, Ste08].

3.2 Approximation Classes

Let, for any $m \in \mathbb{N}$, the set of triangulations in $\mathbb{T}$ whose cardinality differs from that of $\mathscr{T}_0$ by m or less be denoted by

$$\mathbb{T}(m) := \{\mathscr{T} \in \mathbb{T} \mid \operatorname{card}(\mathscr{T}) - \operatorname{card}(\mathscr{T}_0) \leq m\}.$$

Define the seminorm

$$|u|_{\mathscr{A}_\sigma} := \sup_{m\in\mathbb{N}} m^\sigma \inf_{\mathscr{T}\in\mathbb{T}(m)} \|(1-\Pi^0_{\mathscr{T}})Du\|$$

and the approximation class

$$\mathscr{A}_\sigma := \left\{ v \in V \;\middle|\; |v|_{\mathscr{A}_\sigma} < \infty \right\}.$$

The set $\mathscr{A}_\sigma$ is a true approximation class which does not depend on the finite element method and instead concerns the approximability of the gradient by piecewise constant functions. The following alternative set, also referred to as approximation class, will turn out to be more suitable for the analysis

$$\mathscr{A}'_\sigma := \left\{ u \in V \;\middle|\; u \text{ is eigenfunction and } |u|_{\mathscr{A}'_\sigma} < \infty \right\}$$

for

$$|u|_{\mathscr{A}'_\sigma} := \sup_{m\in\mathbb{N}} m^\sigma \inf_{\mathscr{T}\in\mathbb{T}(m)} |||u - \Lambda_{\mathscr{T}} u|||.$$

Subsection 3.3 will establish the equivalence of those two approximation classes in the sense that any eigenfunction $u \in W$ satisfies $u \in \mathscr{A}_\sigma$ if and only if $u \in \mathscr{A}'_\sigma$.

3.3 Equivalence of Approximation Classes

The following best-approximation result generalises [CPS12, Gud10] to eigenvalue problems.

Proposition 3.1 (best-approximation result)**.** *There exists a constant C_{ba} such that, provided the condition* (H1) *holds, any eigenpair $(\lambda, u) \in \mathbb{R} \times W$ with $\|u\| = 1$ satisfies*

$$|||u - \Lambda_\ell u||| \leq C_{\mathrm{ba}}(1 + (1+M_J)\lambda H_0^s)(1+\lambda H_0^2)\|(1-\Pi^0_\ell)Du\|.$$

Proof. Let $v_\ell := G_\ell u - \Lambda_\ell u$. Lemma 2.2 and the eigenvalue problem for u lead to

$$a(G_\ell u - \Lambda_\ell u, v_\ell) = \lambda b(u - P_\ell u, v_\ell) \leq \lambda \|u - P_\ell\| \|v_\ell\|.$$

Hence, the Friedrichs inequality with constant C_{F} proves $|||G_\ell u - \Lambda_\ell u||| \leq \lambda C_{\mathrm{F}} \|u - P_\ell u\|$. This and the triangle inequality lead to

$$|||u - \Lambda_\ell u||| \leq |||u - G_\ell u||| + \lambda C_{\mathrm{F}} \|u - P_\ell u\|. \tag{3.1}$$

Since P_ℓ is the L^2 projection, it holds that $\|u-P_\ell u\| \le \|u-\Lambda_\ell u\|$. Therefore, (3.1) and Lemma 2.1 prove

$$|||u-\Lambda_\ell u||| \le (1+H_0^s\lambda(1+M_J)C_{\mathrm{reg}}C_{\mathrm{F}})|||u-G_\ell u|||.$$

The comparison results of [CPS12, Gud10] prove the existence of a constant C_{comp}, which only depends on the shape-regularity in $\mathbb{T}$, such that

$$|||u-G_\ell u||| \le C_{\mathrm{comp}}(\|(1-\Pi_\ell^0)Du\| + \|h_\ell(1-\Pi_\ell^0)\lambda u\|).$$

(Note that the analysis of [CPS12, Gud10] is carried out for $d=2$. The generalisation to $d\ge 3$, however, is immediate.) The remaining part of the proof bounds the oscillation term $\|h_\ell(1-\Pi_\ell^0)\lambda u\|$. Let $T\in\mathscr{T}_\ell$ and let $\flat_T \in H_0^1(\mathrm{int}(T))\cap\mathfrak{P}_{d+1}(T)$ denote the bubble function on T with $\|\flat_T\|_{L^\infty(T)}=1$. Let $\Pi_\ell^1 : L^2(\Omega)\to\mathfrak{P}_1(\mathscr{T}_\ell)$ denote the L^2 projection onto piecewise affine functions. Define $\psi_T := \flat_T\Pi_\ell^1(\lambda u)$. The arguments of [Ver96] with the equivalence constant C_{equiv} yield

$$\|h_T\lambda u\|_{L^2(T)}^2 \le C_{\mathrm{equiv}}\big((h_T^2 b(\lambda u,\psi_T) + \|h_T(1-\Pi_\ell^1)(\lambda u)\|_{L^2(T)}^2\big). \tag{3.2}$$

Since $\Pi_\ell^0 D\psi = 0$ by the divergence theorem, the eigenvalue problem implies

$$b(\lambda u,\psi_T) = (Du,D\psi_T)_{L^2(\Omega)} = ((1-\Pi_\ell^0)Du,D\psi_T)_{L^2(\Omega)}.$$

This and an inverse estimate $\|D\psi_T\|_{L^2(T)} \le C_{\mathrm{inv}}h_T^{-1}\|\psi_T\|_{L^2(T)}$ and $\|\flat_T\|_{L^\infty(T)}=1$ prove

$$b(\lambda u,\psi_T) \le C_{\mathrm{inv}}\|(1-\Pi_\ell^0)Du\|_{L^2(T)}\|h_T^{-1}\Pi_\ell^1(\lambda u)\|_{L^2(T)}.$$

The second term of (3.2) can be bounded as

$$\|h_T(1-\Pi_\ell^1)(\lambda u)\|_{L^2(T)} \le \kappa h_T^2\lambda\|(1-\Pi_\ell^0)Du\|_{L^2(T)}$$

for the constant κ from the error estimate for the nonconforming $\mathfrak{P}_1$-interpolation [CG14]. ∎

3.4 Optimal Convergence Rates

This subsection presents the optimality of the adaptive algorithm. The proof will be given in Section 7.

Theorem 3.2 (optimal convergence rates)**.** *Provided the bulk parameter $\theta \ll 1$ is sufficiently small and the initial mesh size H_0 satisfies the conditions* (H1)–(H4) *of Table 1, the adaptive algorithm computes triangulations $(\mathscr{T}_\ell)_\ell$ and discrete eigenpairs $\big((\lambda_{\ell,j},u_{\ell,j})_{j\in J}\big)_\ell$ with optimal rate of convergence in the sense that, for some constant C_{opt},*

$$\left(\sum_{j\in J}|||u_j-\Lambda_\ell u_j|||^2\right)^{1/2} \le C_{\mathrm{opt}}\big(\mathrm{card}(\mathscr{T}_\ell)-\mathrm{card}(\mathscr{T}_0)\big)^{-\sigma}\left(\sum_{j\in J}|u_j|_{\mathscr{A}_\sigma'}^2\right)^{1/2}.$$

Propositions 2.3 and 3.1 imply the following immediate consequence.

Corollary 3.3. *Under the conditions of Theorem 3.2, the adaptive algorithm computes triangulations $(\mathscr{T}_\ell)_\ell$ and discrete eigenpairs $\big((\lambda_{\ell,j},u_{\ell,j})_{j\in J}\big)_\ell$ with optimal rate of convergence in the sense that, for all $k\in J$,*

$$\lambda_1^{1/2}(1+M_J^2B^2C)^{-1/2}\left(\frac{|\lambda_k-\lambda_{\ell,k}|}{\lambda_{\ell,k}}\right)^{1/2}+\sup_{j\in J}\sup_{\substack{w\in E(\lambda_j)\\ \|w\|=1}}\inf_{v_\ell\in W_\ell}|||w-v_\ell|||$$

$$\leq 2C_{\mathrm{ba}}(1+(1+M_J)BH_0^s)(1+BH_0^2)C_{\mathrm{opt}}(\mathrm{card}(\mathscr{T}_\ell)-\mathrm{card}(\mathscr{T}_0))^{-\sigma}\left(\sum_{j\in J}|u_j|^2_{\mathscr{A}_\sigma}\right)^{1/2}.$$

Proof. The proof follows from Theorem 3.2 together with Propositions 2.3 and 3.1 and the observation that

$$\sup_{j\in J}\sup_{\substack{w\in E(\lambda_j)\\ |||w|||=1}}\inf_{v_\ell\in W_\ell}|||w-v_\ell|||^2\leq\lambda_1^{-1}\sup_{j\in J}\sup_{\substack{w\in E(\lambda_j)\\ |||w|||=1}}\inf_{v_\ell\in W_\ell}|||w-v_\ell|||^2$$

where $\lambda_1^{-1}=\sup_{v\in V}\big(\|v\|^2/|||v|||^2\big)$ acts as the square of the Friedrichs constant. ∎

4 Theoretical Error Estimator

In order to compare two finite element solutions on different meshes, the analysis relies on a theoretical, non-computable error estimator that does not depend on the choice of the discrete eigenfunctions. Given an eigenpair (λ,u), the error estimator includes the elementwise residuals in terms of $P_\ell u$ and $\Lambda_\ell u$. More precisely, define, for any $T\in\mathscr{T}_\ell$,

$$\mu_\ell^2(T,\lambda,u):=h_T^2\|\lambda P_\ell u\|^2_{L^2(T)}+\sum_{F\in\mathscr{F}(T)\cap\mathscr{F}_\ell(\Omega)}h_T\|[D\Lambda_\ell u]_F\cdot\nu_F\|^2_{L^2(F)}$$

and, for any subset $\mathscr{K}\subseteq\mathscr{T}_\ell$,

$$\mu_\ell^2(\mathscr{K},\lambda_j,u_j):=\sum_{T\in\mathscr{K}}\mu_\ell^2(T,\lambda_j,u_j)\quad\text{and}\quad\mu_\ell^2(\mathscr{K}):=\sum_{j\in J}\mu_\ell^2(\mathscr{K},\lambda_j,u_j).$$

The theoretical error estimator satisfies the following discrete reliability.

Proposition 4.1 (discrete reliability)**.** *Under the assumption* (H1) *there exists C_{drel} solely dependent on $\mathscr{T}_0$ such that any discrete eigenpair $(\lambda,u)\in\mathbb{R}\times W$ with $\|u\|=1$ satisfies*

$$2|||\Lambda_{\ell+m}u-\Lambda_\ell u|||^2\leq C_{\mathrm{drel}}^2\Big(\mu_\ell^2(\mathscr{T}_\ell\setminus\mathscr{T}_{\ell+m},\lambda,u)+H_0^{2s}\lambda^2(1+M_J)^2|||u-\Lambda_\ell u|||^2\Big).$$

Proof. Let $\varphi_{\ell+m}:=\Lambda_{\ell+m}u-\Lambda_\ell u\in V_{\ell+m}$. It is well-established [SZ90] that there exists a quasi-interpolant $\varphi_\ell\in V_\ell$ with quasi-local approximation and stability properties

$$h_T^{-1}\|\varphi_{\ell+m}-\varphi_\ell\|_{L^2(T)}+\|D\varphi_\ell\|_{L^2(T)}\leq C_{\mathrm{stab}}\|D\varphi_{\ell+m}\|_{L^2(\omega_T)}$$

for any $T\in\mathscr{T}_\ell$ and its patch ω_T, which is the interior of the set $\cup\{K\in\mathscr{T}_\ell\mid K\cap T\neq\emptyset\}$ consisting of those simplices that meet T. Moreover, following [SZ90], the function φ_ℓ can be chosen in such a way that $\varphi_\ell=\varphi_{\ell+m}$ holds along all $(d-1)$-dimensional hyper-faces in the set $\mathscr{F}_\ell\cap\mathscr{F}_{\ell+m}$. Elementary algebraic manipulations lead to

$$a((\Lambda_{\ell+m}-\Lambda_\ell)u,\varphi_{\ell+m})=a(\Lambda_{\ell+m}u,\varphi_{\ell+m})-a(\Lambda_\ell u,\varphi_\ell)+a(\Lambda_\ell u,\varphi_\ell-\varphi_{\ell+m}).$$

The arguments of [Ste07, Theorem 4.1] and the aforementioned properties of φ_ℓ lead to some mesh-size independent constant C_1 such that

$$a(\Lambda_\ell u, \varphi_\ell - \varphi_{\ell+m}) \leq C_1 \left(\sum_{T\in\mathscr{T}_\ell\setminus\mathscr{T}_{\ell+m}} \sum_{F\in\mathscr{F}(T)\cap\mathscr{F}_\ell(\Omega)} h_T \|[D\Lambda_\ell u]_F \cdot \nu_F\|^2_{L^2(F)} \right) |||\varphi_{\ell+m}|||.$$

Here, the fact that $\varphi_\ell = \varphi_{\ell+m}$ holds along all hyper-faces of $\mathscr{F}_\ell \cap \mathscr{F}_{\ell+m}$ was used to obtain only the error estimator contributions of the set $\mathscr{T}_\ell \setminus \mathscr{T}_{\ell+m}$ on the right-hand side. Lemma 2.2 and the approximation and stability properties of the quasi interpolation imply for the constant C_{F} of the Friedrichs inequality that

$$\begin{aligned}
&a(\Lambda_{\ell+m}u, \varphi_{\ell+m}) - a(\Lambda_\ell u, \varphi_\ell) \\
&\quad = \lambda\,(b(P_{\ell+m}u, \varphi_{\ell+m}) - b(P_\ell u, \varphi_\ell)) \\
&\quad = \lambda\,\big(b(P_{\ell+m}u - P_\ell u, \varphi_{\ell+m}) + b(P_\ell u, \varphi_{\ell+m} - \varphi_\ell)\big) \\
&\quad \leq \max\{C_{\mathrm{F}}, C_{\mathrm{stab}}\} \left(\lambda \|P_{\ell+m}u - P_\ell u\| + \|h_\ell \lambda P_\ell u\|_{L^2(\cup\mathscr{T}_\ell\setminus\mathscr{T}_{\ell+m})}\right) |||\varphi_{\ell+m}|||.
\end{aligned}$$

The triangle inequality reveals

$$\|P_{\ell+m}u - P_\ell u\| \leq \|u - P_{\ell+m}u\| + \|u - P_\ell u\|. \tag{4.1}$$

Lemma 2.1 proves that

$$\|u - P_{\ell+m}u\| \leq \|u - \Lambda_{\ell+m}u\| \leq (1+M_J) C_{\mathrm{reg}} H_0^s |||u - G_{\ell+m}u|||.$$

An analogous argument for the second term of (4.1) and the relation $V_\ell \subseteq V_{\ell+m}$ conclude the proof of the proposition. ∎

Proposition 3.1 shows that $\Lambda_{\ell+m}u \to u$ with respect to $|||\cdot|||$ on a sequence of triangulation such that $\|h_{\ell+m}\|_{L^\infty(\Omega)} \to 0$ for $m \to \infty$. Hence, Proposition 4.1 proves the reliability estimate

$$2|||u - \Lambda_\ell u|||^2 \leq C^2_{\mathrm{drel}} \left(\mu^2_\ell(\mathscr{T}_\ell, \lambda, u) + H_0^{2s}\lambda^2(1+M_J)^2 |||u - \Lambda_\ell u|||^2\right).$$

Provided the initial mesh is sufficiently fine in the sense that

$$H_0^{2s}\lambda^2 C^2_{\mathrm{drel}}(1+M_J)^2 \leq 1, \tag{H2}$$

the reliability reads as

$$|||u - \Lambda_\ell u|||^2 \leq C^2_{\mathrm{drel}} \mu^2_\ell(\mathscr{T}_\ell, \lambda, u). \tag{4.2}$$

The efficiency

$$2\mu_\ell(\mathscr{T}_\ell, \lambda, u) \leq C_{\mathrm{eff}} \left(1 + \lambda H_0^{1+s}(1+M_J) C_{\mathrm{reg}}\right) |||u - \Lambda_\ell u|||$$

follows from the standard arguments of [Ver96], cf. the proof of Proposition 3.1. The assumption (H4) from Table 1 implies

$$\mu_\ell(\mathscr{T}_\ell, \lambda, u) \leq C_{\mathrm{eff}} |||u - \Lambda_\ell u|||. \tag{4.3}$$

Remark 4.2. In the analysis of this paper, the cluster-width $(B-A)$ does *not* enter the estimates of Proposition 4.1 and inequality (4.2) as an additive constant. Lemma 2.2 is the technical tool that allows the 'cluster-robust' (discrete) reliability. Corresponding error estimates in the present literature on multiple eigenvalues, e.g., [DHZ13, eqn. (3.17)], are only valid in the case of one multiple eigenvalue, that is in the case that $\lambda_{n+1} = \cdots = \lambda_{n+N}$.

5 Equivalence of Error Estimators

This section establishes the equivalence of the local error estimator contributions of the practical and the theoretical error estimator.

Lemma 5.1. *Suppose $H_0 \ll 1$ is small enough such that*

$$\varepsilon := \max_{j\in J} \|u_j - \Lambda_\ell u_j\| \leq \sqrt{1+(2N)^{-1}} - 1 \tag{H3}$$

holds. Then, both $(P_\ell u_j)_{j\in J}$ and $(\Lambda_\ell u_j)_{j\in J}$ form a basis of W_ℓ. For any $w_\ell \in W_\ell$ with $\|w_\ell\| = 1$, the coefficients of the representation $w_\ell = \sum_{j\in J} \beta_j P_\ell u_j$ and $w_\ell = \sum_{j\in J} \gamma_j \Lambda_\ell u_j$ are controlled as

$$\max\left\{ \sum_{j\in J} |\beta_j|^2, \sum_{j\in J} |\gamma_j|^2 \right\} \quad \leq 2+4N \tag{5.1}$$

for $N = \mathrm{card}(J)$.

Proof. The proof is carried out for Λ_ℓ only. Analogous arguments and $\max\{\|u_j - P_\ell u_j\| \mid j \in J\} \leq \varepsilon$ yield the result for P_ℓ.

For any $(j,k) \in J^2$ the triangle inequality plus $\|u_j\| = 1$ and the definition of ε reveal (δ_{jk} denotes the Kronecker δ)

$$\begin{aligned} |b(\Lambda_\ell u_j, \Lambda_\ell u_k) - \delta_{jk}| &= |b(\Lambda_\ell u_j - u_j, \Lambda_\ell u_k) + b(u_j, \Lambda_\ell u_k - u_k)| \\ &\leq \varepsilon(1 + \|\Lambda_\ell u_k\|) \\ &\leq \varepsilon(2 + \|u_k - \Lambda_\ell u_k\|) \leq \varepsilon(2+\varepsilon). \end{aligned} \tag{5.2}$$

For any $j \in J$ it follows from (H3) and (5.2) that

$$b(\Lambda_\ell u_j, \Lambda_\ell u_j) \geq 1 - \varepsilon(2+\varepsilon) \geq \frac{2N-1}{2N}$$

and

$$\sum_{k\in J\setminus\{j\}} |b(\Lambda_\ell u_j, \Lambda_\ell u_k)| \leq (N-1)\varepsilon(2+\varepsilon) \leq \frac{N-1}{2N}.$$

Thus, the Gershgorin theorem implies that all eigenvalues of the matrix

$$[b(\Lambda_\ell u_j, \Lambda_\ell u_k)]_{(j,k)\in J^2}$$

are positive and, hence, $(\Lambda_\ell u_j)_{j\in J}$ is a basis of W_ℓ. Let $w_\ell \in W_\ell$ with $\|w_\ell\| = 1$ and $w_\ell = \sum_{j\in J} \gamma_j \Lambda_\ell u_j$ for real coefficients $(\gamma_j \mid j \in J)$.

For any $k \in J$ it holds that

$$b(\Lambda_\ell u_k, w_\ell) = \sum_{j\in J} \gamma_j b(\Lambda_\ell u_k, \Lambda_\ell u_j) = \gamma_k + \sum_{j\in J} \gamma_j (b(\Lambda_\ell u_k, \Lambda_\ell u_j) - \delta_{jk}).$$

Hence, the triangle and Young inequalities together with (5.2) and $\|\Lambda_\ell u_k\| \leq 1+\varepsilon$ prove

$$\begin{aligned} |\gamma_k|^2 &\leq \left(|b(\Lambda_\ell u_k, w_\ell)| + \sum_{j\in J} |\gamma_j| |b(\Lambda_\ell u_k, \Lambda_\ell u_j) - \delta_{jk}| \right)^2 \\ &\leq 2|b(\Lambda_\ell u_k, w_\ell)|^2 + 2N \sum_{j\in J} |\gamma_j|^2 |b(\Lambda_\ell u_k, \Lambda_\ell u_j) - \delta_{jk}|^2 \\ &\leq 2(1+\varepsilon)^2 + 2N(\varepsilon(2+\varepsilon))^2 \sum_{j\in J} |\gamma_j|^2. \end{aligned}$$

Summation over $k \in J$ yields

$$\sum_{k\in J}|\gamma_k|^2 \le 2N(1+\varepsilon)^2 + 2N^2(\varepsilon(2+\varepsilon))^2 \sum_{j\in J}|\gamma_j|^2.$$

Since $\varepsilon(2+\varepsilon) \le (2N)^{-1}$ by assumption (H3), it follows that

$$\sum_{j\in J}|\gamma_j|^2 \le 4N(1+\varepsilon)^2 \le 2+4N. \qquad \blacksquare$$

Recall that all eigenvalues in the cluster as well as their approximations are contained in the interval $[A,B]$ and therefore $\sup_{(j,k)\in J^2} \lambda_k^{-1}\lambda_{\ell,j} \le B/A$.

Proposition 5.2 (bulk criterion)**.** *Suppose* (H1) *and* (H3) *are satisfied. Then, for any $T \in \mathscr{T}_\ell$, the error estimator contributions can be compared as follows*

$$N^{-1}\sum_{j\in J}\mu_\ell^2(T,\lambda_j,u_j) \le (B/A)^2\eta_\ell^2(T) \le (B/A)^4(2N+4N^2)\sum_{j\in J}\mu_\ell^2(T,\lambda_j,u_j). \quad (5.3)$$

Therefore, $\mu_\ell^2(\mathscr{M}_\ell)$ satisfies the bulk criterion

$$\tilde\theta\,\mu_\ell^2(\mathscr{T}_\ell) \le \mu_\ell^2(\mathscr{M}_\ell)$$

for the modified bulk parameter

$$\tilde\theta := \left((B/A)^4(2N^2+4N^3)\right)^{-1}\theta < 1. \quad (5.4)$$

Proof of the first inequality in (5.3)*.* Let $k \in J$. For the volume term the expansion with respect to the orthonormal basis $(u_{\ell,j} \mid j \in J)$ reads $P_\ell u_k = \sum_{j\in J}\alpha_j u_{\ell,j}$ with $\sum_{j\in J}\alpha_j^2 = \|P_\ell u_k\|^2 \le 1$. Thus, the triangle inequality followed by the Cauchy inequality proves

$$\begin{aligned}\|\lambda_k P_\ell u_k\|_{L^2(T)}^2 &\le \left(\sum_{j\in J}\lambda_k|\alpha_j|\|u_{\ell,j}\|_{L^2(T)}\right)^2 \\ &\le \lambda_k^2\left(\sum_{j\in J}\alpha_j^2\right)\sum_{j\in J}\|u_{\ell,j}\|_{L^2(T)}^2 \le (B/A)^2\sum_{j\in J}\|\lambda_{\ell,j}u_{\ell,j}\|_{L^2(T)}^2.\end{aligned}$$

An analogous expansion of the jump terms of $\Lambda_\ell u_k = \sum_{j\in J}\beta_j u_{\ell,j}$ with real coefficients $\beta_j = b(G_\ell u_k, u_{\ell,j}) = \lambda_{\ell,j}^{-1}\lambda_k b(u_k,u_{\ell,j})$ results in $\sum_{j\in J}\beta_j^2 \le (B/A)^2$ and, thus, for any hyper-face $F \in \mathscr{F}(T)$,

$$\|[D\Lambda_\ell u_k]_F\cdot\nu_F\|_{L^2(F)}^2 \le (B/A)^2\sum_{j\in J}\|[Du_{\ell,j}]_F\cdot\nu_F\|_{L^2(F)}^2.$$

Proof of the second inequality in (5.3)*.* Let $k \in J$. According to Lemma 5.1 $(P_\ell u_j)_{j\in J}$ is a basis of W_ℓ and $u_{\ell,k} = \sum_{j\in J}\gamma_j P_\ell u_j$ for coefficients $(\gamma_j \mid j \in J)$. The triangle inequality proves for the volume term

$$\|\lambda_{\ell,k}u_{\ell,k}\|_{L^2(T)}^2 = \|\lambda_{\ell,k}\sum_{j\in J}\gamma_j P_\ell u_j\|_{L^2(T)}^2 \le (B/A)^2\left(\sum_{j\in J}\gamma_j^2\right)\sum_{j\in J}\|\lambda_j P_\ell u_j\|_{L^2(T)}^2.$$

As proven in Lemma 5.1 it holds that $\sum_{j\in J}\gamma_j^2 \le (2+4N)$. Analogously, Lemma 5.1 shows that there exist real coefficients $(\delta_j \mid j\in J)$ such that $u_{\ell,k}=\sum_{j\in J}\delta_j\Lambda_\ell u_j$. For any interior hyper-face $F\in\mathscr{F}_\ell(\Omega)$, the triangle inequality leads to

$$\|[Du_{\ell,k}]_F\cdot\nu_F\|^2_{L^2(F)} \le \left(\sum_{j\in J}\delta_j^2\right)\sum_{j\in J}\|[D\Lambda_\ell u_j]_F\cdot\nu_F\|^2_{L^2(F)}$$

Lemma 5.1 therefore proves

$$\|[Du_{\ell,k}]_F\cdot\nu_F\|^2_{L^2(F)} \le (2+4N)\sum_{j\in J}\|[D\Lambda_\ell u_j]_F\cdot\nu_F\|^2_{L^2(F)}. \qquad \blacksquare$$

6 Contraction Property

This section presents the contraction property for a linear combination of error and error estimator under the conditions (H1)–(H4) of Table 1.

Proposition 6.1 (quasi-orthogonality)**.** *Let $(\lambda,u)\in\mathbb{R}\times W$ with $\|u\|=1$ be an eigenpair of* (2.1)*. Under hypothesis* (H1) *there exists a constant $C_{\rm qo}$ such that any $\mathscr{T}_\ell\in\mathbb{T}$ and any admissible refinement $\mathscr{T}_{\ell+m}\in\mathbb{T}$ satisfy*

$$|2a(u-\Lambda_{\ell+m}u,\Lambda_{\ell+m}u-\Lambda_\ell u)| \le \lambda C_{\rm qo}(1+M_J)^2H_0^{2s}|||u-\Lambda_\ell u|||^2.$$

Proof. The eigenvalue problem (2.1) and Lemma 2.2 followed by the Cauchy and Young inequalities show

$$\begin{aligned} a(u-\Lambda_{\ell+m}u,\Lambda_{\ell+m}u-\Lambda_\ell u) &= \lambda b(u-P_{\ell+m}u,\Lambda_{\ell+m}u-\Lambda_\ell u)\\ &\le \lambda(\|u-P_{\ell+m}u\|^2+\|\Lambda_{\ell+m}u-\Lambda_\ell u\|^2)/2. \end{aligned}$$

The triangle inequality and Lemma 2.1 prove the result. $\blacksquare$

Proposition 6.2 (error estimator reduction for μ_ℓ)**.** *Provided the initial mesh-size H_0 satisfies* (H1) *and* (H3)*, there exist constants $0<\rho_1<1$ and $0<K<\infty$ such that $\mathscr{T}_\ell$ and its one-level refinement $\mathscr{T}_{\ell+1}$ generated by AFEM and any eigenfunction $u\in W$ with $\|u\|=1$ and eigenvalue λ satisfy*

$$\begin{aligned} &\mu^2_{\ell+1}(\mathscr{T}_{\ell+1},\lambda,u)\\ &\le \rho_1\mu^2_\ell(\mathscr{T}_\ell,\lambda,u)+K\left(|||\Lambda_{\ell+1}u-\Lambda_\ell u|||^2+\|h_{\ell+1}\lambda(P_{\ell+1}u-P_\ell u)\|^2\right). \end{aligned}$$

Proof. The design of the error estimator $\mu^2_\ell(\mathscr{T}_\ell,\lambda,u)$ allows the use of the standard arguments of [CKNS08, Ste07] to prove the result. $\blacksquare$

Proposition 6.3 (contraction property)**.** *Under the conditions* (H1)–(H4) *of Table 1 there exist $0<\rho_2<1$ and $0<\beta<\infty$ such that, for any eigenpair $(\lambda,u)\in\mathbb{R}\times W$ with $\|u\|=1$, the term $\xi^2_\ell:=\mu^2_\ell(\mathscr{T}_\ell,\lambda,u)+\beta|||u-\Lambda_\ell u|||^2$ satisfies*

$$\xi^2_{\ell+1}\le\rho_2\xi^2_\ell \quad \textit{for all } \ell\in\mathbb{N}.$$

Proof. Throughout the proof, the following shorthand notation is employed

$$\begin{aligned} e_\ell &:= u-\Lambda_\ell u, & e_{\ell+1} &:= u-\Lambda_{\ell+1}u,\\ \mu^2_\ell &:= \mu^2_\ell(\mathscr{T}_\ell,\lambda,u), & \mu^2_{\ell+1} &:= \mu^2_{\ell+1}(\mathscr{T}_{\ell+1},\lambda,u). \end{aligned}$$

The error estimator reduction from Proposition 6.2 and elementary algebraic manipulations lead to

$$\begin{aligned}&\mu_{\ell+1}^2+K|||e_{\ell+1}|||^2\\&\quad\le\rho_1\mu_\ell^2+K\Big(|||e_\ell|||^2+2a(e_{\ell+1},(\Lambda_\ell-\Lambda_{\ell+1})u)+H_0^2\|\lambda(P_{\ell+1}-P_\ell)u\|^2\Big).\end{aligned}\tag{6.1}$$

The quasi-orthogonality of Proposition 6.1 reads as

$$|2a(e_{\ell+1},(\Lambda_{\ell+1}-\Lambda_\ell)u)|\le\lambda C_{\mathrm{qo}}(1+M_J)^2H_0^{2s}|||e_\ell|||^2.$$

The triangle inequality and Lemma 2.1 lead to

$$H_0^2\|\lambda(P_{\ell+1}-P_\ell)u\|^2\le 2(1+M_J)^2\lambda^2C_{\mathrm{reg}}^2H_0^{2+2s}(|||e_{\ell+1}|||^2+|||e_\ell|||^2).$$

The combination of the preceding two displayed formulas with (6.1) leads to

$$\begin{aligned}&\mu_{\ell+1}^2+K\Big(1-(1+M_J)^2{2\lambda}^2C_{\mathrm{reg}}^2H_0^{2+2s}\Big)|||e_{\ell+1}|||^2\\&\quad\le\rho_1\mu_\ell^2+K\Big(1+(1+M_J)^2(\lambda C_{\mathrm{qo}}H_0^{2s}+2\lambda^2C_{\mathrm{reg}}^2H_0^{2+2s})\Big)|||e_\ell|||^2.\end{aligned}\tag{6.2}$$

For any $0<\delta<1$, the reliability (4.2) bounds the right-hand side of (6.2) by

$$\begin{aligned}&\Big(\rho_1+\delta C_{\mathrm{drel}}^2K\big(1+(1+M_J)^2(\lambda C_{\mathrm{qo}}H_0^{2s}+2\lambda^2C_{\mathrm{reg}}^2H_0^{2+2s})\big)\Big)\mu_\ell^2\\&\qquad+K\big(1+(1+M_J)^2(\lambda C_{\mathrm{qo}}H_0^{2s}+2\lambda^2C_{\mathrm{reg}}^2H_0^{2+2s})\big)(1-\delta)|||e_\ell|||^2\\&\le\rho_2\left(\mu_\ell^2(\lambda,u,\mathscr{T}_\ell)+\beta|||u-\Lambda_\ell u|||^2\right)\end{aligned}$$

for $\beta:=K(1-(1+M_J)^2{2\lambda}^2C_{\mathrm{reg}}^2H_0^{2+2s})$ and

$$\begin{aligned}\rho_2:=\max\Big\{&\rho_1+\delta C_{\mathrm{drel}}^2K\big(1+(1+M_J)^2(\lambda C_{\mathrm{qo}}H_0^{2s}+2\lambda^2C_{\mathrm{reg}}^2H_0^{2+2s})\big),\\&\frac{1+(1+M_J)^2(\lambda C_{\mathrm{qo}}H_0^{2s}+2\lambda^2C_{\mathrm{reg}}^2H_0^{2+2s})}{1-(1+M_J)^2{2\lambda}^2C_{\mathrm{reg}}^2H_0^{2+2s}}(1-\delta)\Big\}.\end{aligned}$$

The choice of a sufficiently small H_0 such that

$$(1+M_J)^2(\lambda C_{\mathrm{qo}}H_0^{2s}+2\lambda^2C_{\mathrm{reg}}^2H_0^{2+2s})<\min\left\{1,\frac{1-\rho_1}{KC_{\mathrm{drel}}^2}\right\}/4\tag{H4}$$

and $\delta:=2(1+M_J)^2(\lambda C_{\mathrm{qo}}H_0^{2s}+2\lambda^2C_{\mathrm{reg}}^2H_0^{2+2s})$ leads to $\rho_2<1$ and $0<\beta<K$. ∎

7 Proof of Optimal Convergence Rates

This section is devoted to the proof of Theorem 3.2. While the results of the preceding sections were stated for each eigenfunction $u_k\in W$ separately, the optimality proof of this section is concerned with the simultaneous error of all eigenfunction approximations. Consider

$$\Xi_\ell^2:=\mu_\ell^2(\mathscr{T}_\ell)+\beta\sum_{j\in J}|||u_j-\Lambda_\ell u_j|||^2\quad\text{for all }\ell=0,1,2,\dots$$

for the parameter β from Proposition 6.3. The proof excludes the pathological case $\Xi_0 = 0$. Choose $0 < \tau \leq \sum_{j\in J} |u_j|^2_{\mathscr{A}'_\sigma} / \Xi_0^2$, and set $\varepsilon(\ell) := \sqrt{\tau}\,\Xi_\ell$. Let $N(\ell) \in \mathbb{N}$ be minimal with the property

$$\sum_{j\in J} |u_j|^2_{\mathscr{A}'_\sigma} \leq \varepsilon(\ell)^2 N(\ell)^{2\sigma}.$$

The minimality of $N(\ell)$ (and the definition of $\varepsilon(\ell)$ plus the contraction property from Proposition 6.3 in the case of $N(\ell) = 1$) proves

$$N(\ell) \leq 2\bigg(\sum_{j\in J} |u_j|^2_{\mathscr{A}'_\sigma}\bigg)^{1/(2\sigma)} \varepsilon(\ell)^{-1/\sigma} \quad \text{for all } \ell \in \mathbb{N}_0. \tag{7.1}$$

Let $\widetilde{\mathscr{T}_\ell} \in \mathbb{T}$ denote the optimal triangulation of cardinality

$$\operatorname{card}(\widetilde{\mathscr{T}_\ell}) \leq \operatorname{card}(\mathscr{T}_0) + N(\ell)$$

in the sense that the projection $\widetilde{\Lambda} := \Lambda_{\widetilde{\mathscr{T}_\ell}}$ from (2.3) with respect to $\widetilde{\mathscr{T}_\ell}$ satisfies

$$\sum_{j\in J} |||u_j - \widetilde{\Lambda} u_j|||^2 \leq N(\ell)^{-2\sigma} \sum_{j\in J} |u_j|^2_{\mathscr{A}'_\sigma} \leq \varepsilon(\ell)^2. \tag{7.2}$$

The overlay $\widehat{\mathscr{T}_\ell}$ is the smallest common refinement of $\mathscr{T}_\ell$ and $\widetilde{\mathscr{T}_\ell}$ and is known [CKNS08] to satisfy

$$\operatorname{card}(\widehat{\mathscr{T}_\ell}) - \operatorname{card}(\mathscr{T}_\ell) \leq \operatorname{card}(\widetilde{\mathscr{T}_\ell}) - \operatorname{card}(\mathscr{T}_0) \leq N(\ell). \tag{7.3}$$

It is known that $\operatorname{card}(\mathscr{T}_\ell \setminus \widehat{\mathscr{T}_\ell}) \leq \operatorname{card}(\widehat{\mathscr{T}_\ell}) - \operatorname{card}(\mathscr{T}_\ell)$. This and (7.1)–(7.3) lead to

$$\operatorname{card}(\mathscr{T}_\ell \setminus \widehat{\mathscr{T}_\ell}) \leq N(\ell) \leq 2\bigg(\sum_{j\in J} |u_j|^2_{\mathscr{A}'_\sigma}\bigg)^{1/(2\sigma)} \varepsilon(\ell)^{-1/\sigma}. \tag{7.4}$$

Let $\widehat{\Lambda} := \Lambda_{\widehat{\mathscr{T}_\ell}}$ denote the projection from (2.3) with respect to $\widehat{\mathscr{T}_\ell}$.

Lemma 7.1. *Under the assumptions of Table 1 there holds*

$$\sum_{j\in J} |||u_j - \widehat{\Lambda} u_j|||^2 \leq 2\varepsilon(\ell)^2. \tag{7.5}$$

Proof. Elementary manipulations and the quasi-orthogonality from Proposition 6.1 reveal for any $j \in J$ that

$$\begin{aligned} |||u_j - \widehat{\Lambda} u_j|||^2 &= |||u_j - \widetilde{\Lambda} u_j|||^2 - |||\widetilde{\Lambda}_\ell u_j - \widehat{\Lambda} u_j|||^2 - 2a(u_j - \widehat{\Lambda} u_j, \widehat{\Lambda} u_j - \widetilde{\Lambda} u_j) \\ &\leq (1 + \lambda_j C_{\mathrm{qo}}(1 + M_J)^2 H_0^{2s}) |||u_j - \widetilde{\Lambda} u_j|||^2 - |||\widetilde{\Lambda}_\ell u_j - \widehat{\Lambda} u_j|||^2. \end{aligned}$$

This, the assumption (H4) and (7.2) lead to

$$\sum_{j\in J} |||u_j - \widehat{\Lambda} u_j|||^2 \leq 2 \sum_{j\in J} |||u_j - \widetilde{\Lambda} u_j|||^2 \leq 2\varepsilon(\ell)^2. \qquad \blacksquare$$

Lemma 7.2 (key argument). *There exists C_2 with*

$$\mu_\ell^2(\mathscr{T}_\ell) \le C_2 \mu_\ell^2(\mathscr{T}_\ell \setminus \widehat{\mathscr{T}_\ell})$$

and therefore, by Proposition 5.2,

$$\eta_\ell^2(\mathscr{T}_\ell) \le C_2 (B/A)^4 (2N^2+N^3) \eta_\ell^2(\mathscr{T}_\ell \setminus \widehat{\mathscr{T}_\ell}) \tag{7.6}$$

Proof. The quasi-orthogonality from Proposition 6.1 and the discrete reliability from Proposition 4.1 plus (H2) and (H4) yield, for any $j \in J$, that

$$\begin{aligned}
|||u_j - \Lambda_\ell u_j|||^2 &= 2a(u_j - \widehat{\Lambda}_\ell u_j, \widehat{\Lambda}_\ell u_j - \Lambda_\ell u_j) + |||u_j - \widehat{\Lambda}_\ell u_j|||^2 + |||\widehat{\Lambda}_\ell u_j - \Lambda_\ell u_j|||^2 \\
&\le |||u_j - \widehat{\Lambda}_\ell u_j|||^2 + \lambda_j^2 (1+M_J)^2 H_0^{2s} (C_{\mathrm{qo}} + 2^{-1} C_{\mathrm{drel}}^2) |||u_j - \Lambda_\ell u_j|||^2 \\
&\qquad + 2^{-1} C_{\mathrm{drel}}^2 \mu_\ell^2(\mathscr{T}_\ell \setminus \widehat{\mathscr{T}_\ell}, \lambda_j, u_j) \\
&\le |||u_j - \widehat{\Lambda}_\ell u_j|||^2 + \frac{1}{2} |||u_j - \Lambda_\ell u_j|||^2 + 2^{-1} C_{\mathrm{drel}}^2 \mu_\ell^2(\mathscr{T}_\ell \setminus \widehat{\mathscr{T}_\ell}, \lambda_j, u_j).
\end{aligned}$$

Therefore, (7.5) implies

$$\frac{1}{2} \sum_{j \in J} |||u_j - \Lambda_\ell u_j|||^2 \le 2\varepsilon(\ell)^2 + 2^{-1} C_{\mathrm{drel}}^2 \mu_\ell^2(\mathscr{T}_\ell \setminus \widehat{\mathscr{T}_\ell}).$$

Let C_{eq} denote the constant of $2\Xi_\ell^2 \le C_{\mathrm{eq}} \mu_\ell^2(\mathscr{T}_\ell)$ (which exists by reliability). The efficiency (4.3), the definition of $\varepsilon(\ell)$ and the preceding estimates prove

$$\begin{aligned}
\frac{1}{2} C_{\mathrm{eff}}^{-2} \mu_\ell^2(\mathscr{T}_\ell) &\le 2\varepsilon(\ell)^2 + 2^{-1} C_{\mathrm{drel}}^2 \mu_\ell^2(\mathscr{T}_\ell \setminus \widehat{\mathscr{T}_\ell}) \\
&\le \tau C_{\mathrm{eq}} \mu_\ell^2(\mathscr{T}_\ell) + 2^{-1} C_{\mathrm{drel}}^2 \mu_\ell^2(\mathscr{T}_\ell \setminus \widehat{\mathscr{T}_\ell}).
\end{aligned}$$

Define $C_2 := (2^{-1} C_{\mathrm{eff}}^{-2} - \tau C_{\mathrm{eq}})^{-1} C_{\mathrm{drel}}^2/2$, which is positive for sufficiently small choice of τ. ∎

The finish of the optimality proof follows the arguments of [CKNS08, Ste07].

Lemma 7.3 (finish of the optimality proof). *The choice*

$$0 < \theta \le 1 \big/ \big(C_2 (B/A)^4 (2N^2 + 4N^3) \big)$$

implies

$$\big(\mathrm{card}(\mathscr{T}_\ell) - \mathrm{card}(\mathscr{T}_0)\big)^\sigma \left(\sum_{j \in J} |||u_j - \Lambda_\ell u_j|||^2 \right)^{1/2} \le C(\sigma) \left(\sum_{j \in J} |u_j|_{\mathscr{A}_\sigma'}^2 \right)^{1/2}$$

for a constant $C(\sigma)$.

Proof. The marking step in the adaptive algorithm selects $\mathscr{M}_\ell \subseteq \mathscr{T}_\ell$ with minimal cardinality such that $\theta \eta_\ell^2(\mathscr{T}_\ell) \le \eta_\ell^2(\mathscr{M}_\ell)$. Lemma 7.2 and

$$\theta \le 1 \big/ \big(C_2 (B/A)^4 (2N^2 + 4N^3) \big) \tag{7.7}$$

imply that also $\mathscr{T}_\ell \setminus \widehat{\mathscr{T}_\ell}$ satisfies the bulk criterion. The minimality of $\mathscr{M}_\ell$ and (7.4) prove that

$$\mathrm{card}(\mathscr{M}_\ell) \le \mathrm{card}(\mathscr{T}_\ell \setminus \widehat{\mathscr{T}_\ell}) \le 2 \left(\sum_{j \in J} |u_j|_{\mathscr{A}_\sigma'}^2 \right)^{1/(2\sigma)} \tau^{-1/(2\sigma)} \Xi_\ell^{-1/\sigma}.$$

It is proven in [BDD04, Ste08] that there exists a constant C_{BDV} such that

$$\begin{aligned}\operatorname{card}(\mathscr{T}_\ell)-\operatorname{card}(\mathscr{T}_0) &\le C_{\mathrm{BDV}}\sum_{k=0}^{\ell-1}\operatorname{card}(\mathscr{M}_k)\\ &\le 2C_{\mathrm{BDV}}\Big(\sum_{j\in J}|u_j|^2_{\mathscr{A}'_\sigma}\Big)^{1/(2\sigma)}\tau^{-1/(2\sigma)}\sum_{k=0}^{\ell-1}\Xi_k^{-1/\sigma}.\end{aligned}$$

The contraction property from Proposition 6.3 implies $\Xi_\ell^2\le\rho_2^{\ell-m}\Xi_m^2$ for $0\le m\le\ell$. Since $\rho_2<1$, a geometric series argument leads to

$$\sum_{j=0}^{\ell-1}\Xi_j^{-1/\sigma}\le\Xi_\ell^{-1/\sigma}\sum_{j=0}^{\ell-1}\rho_2^{(\ell-j)/(2\sigma)}\le\Xi_\ell^{-1/\sigma}\rho_2^{1/(2\sigma)}\Big/\Big(1-\rho_2^{1/(2\sigma)}\Big).$$

The combination of the above estimates results in

$$\begin{aligned}&\operatorname{card}(\mathscr{T}_\ell)-\operatorname{card}(\mathscr{T}_0)\\ &\qquad\le 2C_{\mathrm{BDV}}\Big(\sum_{j\in J}|u_j|^2_{\mathscr{A}'_\sigma}\Big)^{1/(2\sigma)}\tau^{-1/(2\sigma)}\Xi_\ell^{-1/\sigma}\rho_2^{1/(2\sigma)}\Big/\Big(1-\rho_2^{1/(2\sigma)}\Big).\end{aligned}$$

The equivalence of Ξ_ℓ^2 with the error $\sum_{j\in J}|||u_j-\Lambda_\ell u_j|||^2$ concludes the proof. ∎

8 Numerical Examples

This section presents three numerical tests on non-convex domains where not all eigenfunctions have full H^2-regularity. All adaptive computations are based on the bulk parameter $\theta=0.1$. All convergence graphs are logarithmically scaled, the error quantities are plotted against the degrees of freedom.

8.1 Simultaneous Approximation on a Symmetric Geometry

Let the square domain with four symmetric slits be defined as

$$\Omega=(-1,1)^2\setminus\left(\begin{array}{l}\operatorname{conv}\{(0.5,0),(1,0)\}\cup\operatorname{conv}\{(0,0.5),(0,1)\}\\ \cup\operatorname{conv}\{(-0.5,0),(-1,0)\}\cup\operatorname{conv}\{(0,-0.5),(0,-1)\}\end{array}\right).$$

The symmetry of the domain shows that there are simple as well as multiple eigenvalues. The following close approximations of the first 12 eigenvalues were obtained by a $\mathfrak{P}_2$ finite element method on uniform meshes and Aitken extrapolation

$$\begin{array}{llll}\lambda_1=10.1474 & \lambda_2=17.6625 & \lambda_3=17.6625 & \lambda_4=19.7392\\ \lambda_5=26.1018 & \lambda_6=38.3491 & \lambda_7=38.3491 & \lambda_8=46.5539\\ \lambda_9=49.1496 & \lambda_{10}=49.1496 & \lambda_{11}=49.3480 & \lambda_{12}=49.3480.\end{array}$$

Figure 2 displays the convergence history of the eigenvalue errors under uniform mesh-refinement. The sub-optimal convergence rate of several eigenvalues suggests that some of the eigenfunctions are singular. The simultaneous adaptive approximation with $J=\{1,\dots,12\}$ and the bulk parameter $\theta=0.1$ leads to the optimal convergence rates displayed in Figure 3. Figure 4 displays the efficiency index, which describes the ratio of the squared error estimator and the sum of the relative eigenvalue errors via

$$\textit{effind}:=\eta_\ell^2\Big/\sum_{j\in J}\frac{\lambda_{\ell,j}-\lambda_j}{\lambda_{\ell,j}}.$$

The values for the efficiency index displayed in Figure 4 range from 30 to 60.

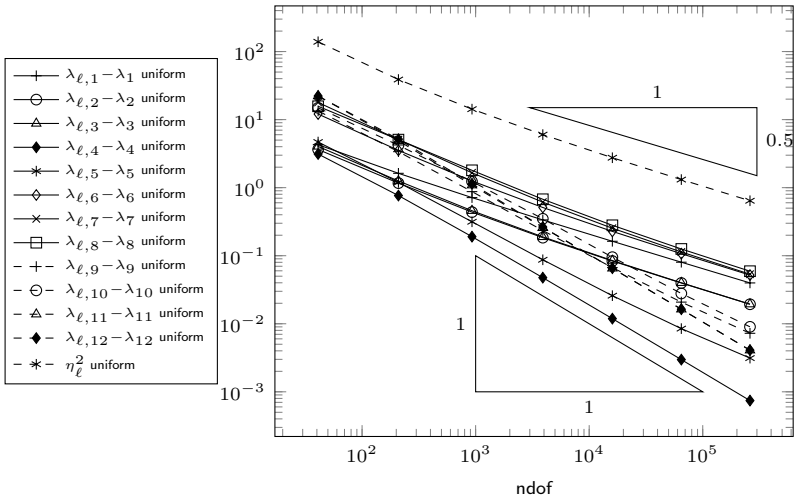

Figure 2: Convergence history on the square with symmetric slits under uniform mesh-refinement.

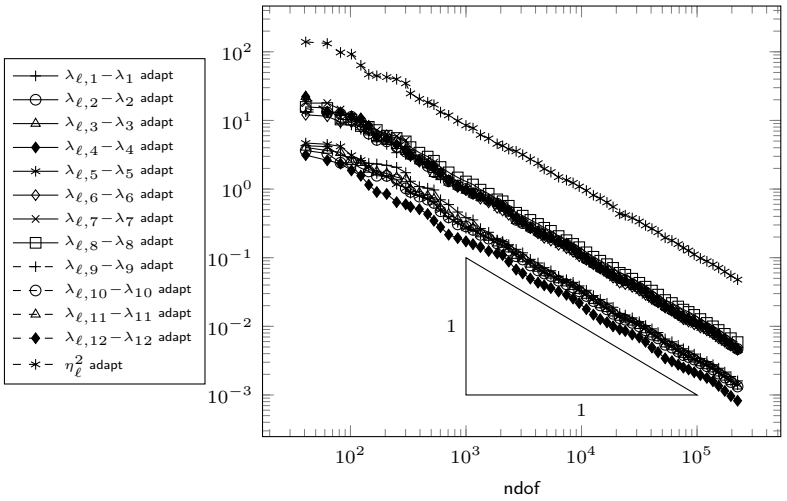

Figure 3: Convergence history on the square with symmetric slits under adaptive mesh-refinement based on $J = \{1,\dots,12\}$ and $\theta = 0.1$.

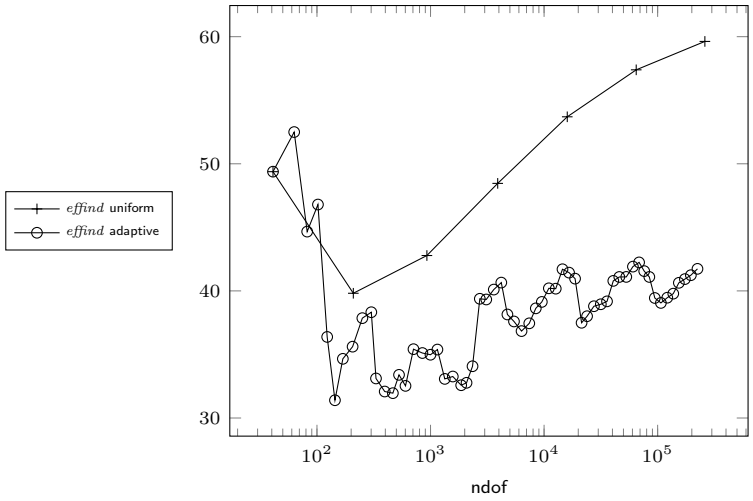

Figure 4: Efficiency index for the symmetric slitted square domain.

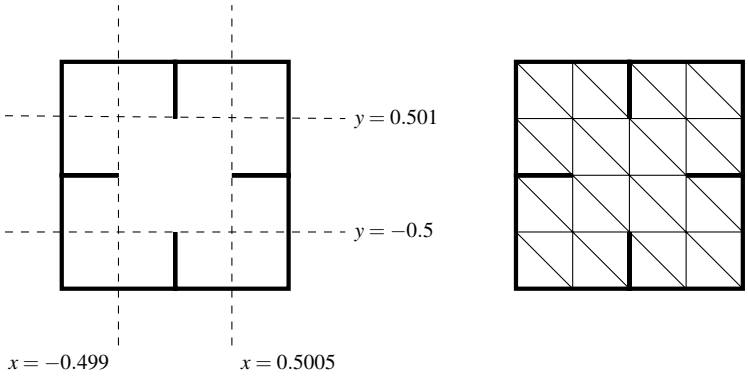

Figure 5: Square domain $(-1,1)^2$ with perturbed slits and coarse initial partition with 5 interior vertices.

8.2 Narrow Cluster on a Perturbed Symmetric Geometry

This example studies a perturbation of the geometry of Section 8.1 displayed in Figure 5. The second and third eigenvalues of the Laplacian on this domain are approximated by a $\mathfrak{P}_2$ FEM and the Aitken extrapolation as

$$\lambda_2 = 17.6557 \qquad \lambda_3 = 17.6660.$$

The convergence history based on the coarse initial triangulation of Figure 5 is displayed in Figure 6. Uniform mesh refinement leads to a suboptimal convergence rate. The simultaneous adaptive approximation based on $J = \{2,3\}$ leads to the optimal convergence rate. The choice of $J = \{2\}$, that is marking only with respect to the second computed eigenfunction u_2, yields a large pre-asymptotic effect up to 4×10^5 degrees of freedom where the adaptive algorithm is not significantly better than uniform refinement. This behaviour can –at least on a heuristic level– be explained from the plots of

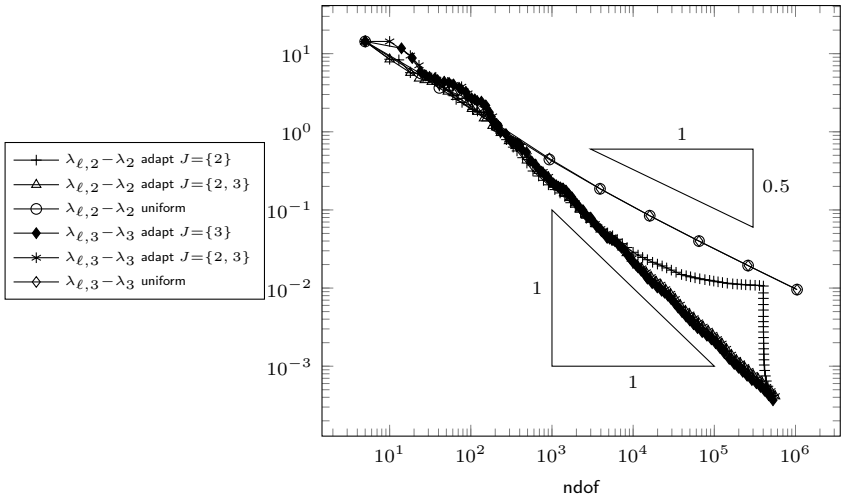

Figure 6: Convergence history on the square with perturbed slits. Adaptive mesh-refinement is based on $J \subseteq \{2,3\}$. The initial triangulation $\mathscr{T}_0$ has 5 degrees of freedom.

discrete eigenmodes and the adaptive meshes. The second eigenfunction u_2 shows its significant singularities at different slit tips than the third eigenfunction u_3 (this follows from the axial symmetry in the non-perturbed case). Figure 7 displays (close approximations to) the eigenmodes u_2 and u_3. Figure 8 displays the eigenmodes computed by the adaptive algorithm for $J = \{2\}$. The initial triangulation does not resolve the eigenvalue cluster in that the computed eigenfunction $u_{\ell,2}$ does not capture the shape of u_2. Accordingly, the AFEM refines near those reentrant corners where the error estimator contributions of $u_{\ell,2}$ are large, but where u_2 is smooth. This can be seen in the adaptive mesh of Figure 9 for $J = \{2\}$. This yields only little improvement for the approximation of u_2. The eigenvalue cluster is only resolved when the global mesh-size is sufficiently small, which requires a large amount of iterations in an adaptive algorithm. In contrast, the adaptive algorithm for $J = \{2,3\}$ refines at all reentrant corners, even if the eigenvalues in the cluster are not well-separated on the initial mesh. This explains why this algorithm is more robust in the sense that it only requires separation of the cluster from the remaining spectrum, but no resolution within the cluster.

On a finer initial mesh (after one red-refinement) with 41 degrees of freedom, this effect is no more present and the adaptive algorithm appears optimal for any choice of $J \subseteq \{2,3\}$ as displayed in the convergence history of Figure 10.

8.3 Multiple Eigenvalue on a Symmetric Geometry

The first five eigenvalues on the slitted squared domain

$$\Omega = (-1,1)^2 \setminus \left(\begin{array}{l} \operatorname{conv}\{(0.15,0),(1,0)\} \cup \operatorname{conv}\{(0,0.15),(0,1)\} \\ \cup \operatorname{conv}\{(-0.15,0),(-1,0)\} \cup \operatorname{conv}\{(0,-0.15),(0,-1)\} \end{array} \right)$$

displayed in Figure 11 are approximated by a $\mathfrak{P}_2$ FEM and Aitken extrapolation as

$$\lambda_1 = 19.6518 \quad \lambda_2 = 19.7198 \quad \lambda_3 = 19.7198 \quad \lambda_4 = 19.7392 \quad \lambda_5 = 48.4497.$$

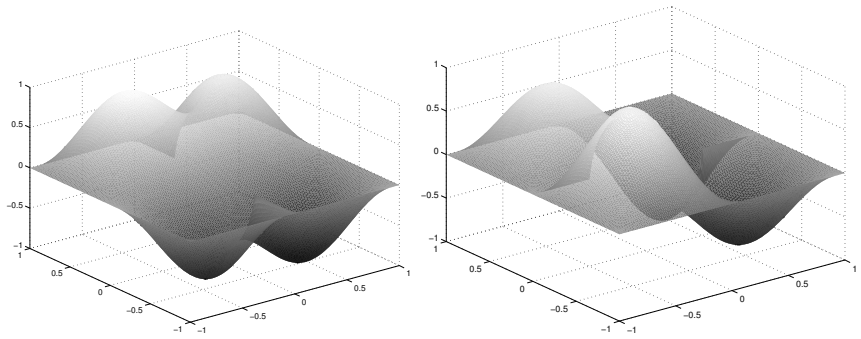

Figure 7: Approximation of the eigenmodes u_2 and u_3 on the perturbed slit domain on a uniform mesh with 16 001 degrees of freedom.

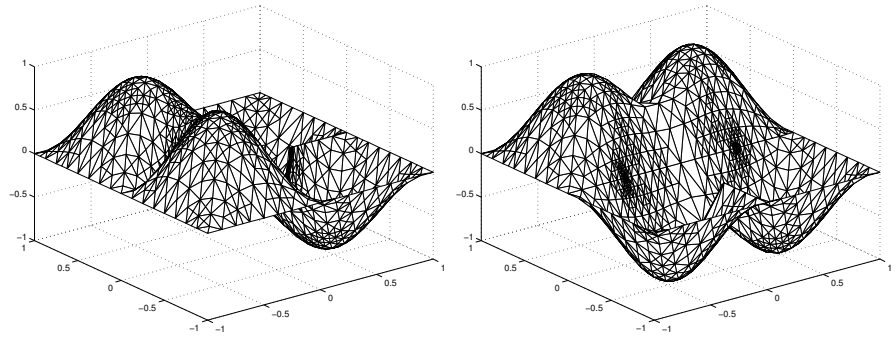

Figure 8: Adaptive approximation of the eigenmodes u_2 and u_3 on the perturbed slit domain, $J = \{2\}$, 1 085 degrees of freedom, level 30, from initial triangulation $\mathscr{T}_0$ with 5 degrees of freedom.

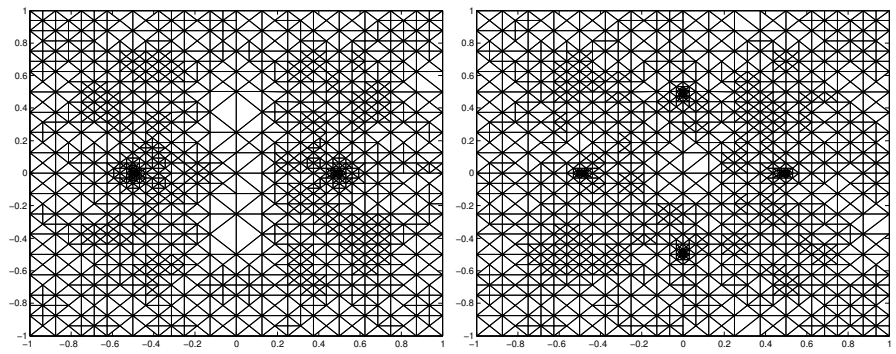

Figure 9: Adaptive meshes for the perturbed slit domain from initial triangulation $\mathscr{T}_0$ with 5 degrees of freedom. Left: $J = \{2\}$, 1 085 degrees of freedem, level 30. Right: $J = \{2,3\}$, 1 056 degrees of freedom, level 27.

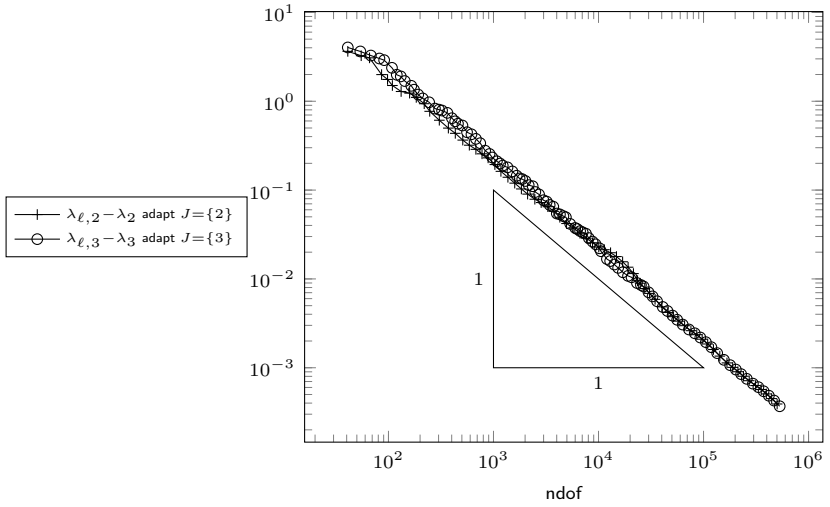

Figure 10: Convergence history on the square with perturbed slits for adaptive mesh-refinement based on $J \subseteq \{2,3\}$. The initial triangulation $\mathscr{T}_0$ has 41 degrees of freedom.

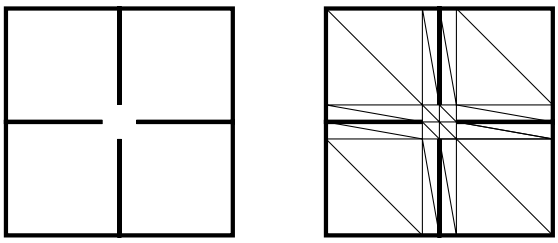

Figure 11: Square domain $(-1,1)^2$ with symmetric slits of length 0.85 and coarse initial partition with 5 interior vertices.

The symmetry of the domain shows that the second eigenvalue has indeed multiplicity 2. The first four eigenvalues are close to $2\pi^2$ which would be the first eigenvalue on each of the four sub-squares in the case of full decoupling. Figure 12 displays the convergence history of the first four eigenvalues under uniform mesh-refinement. The FEM approximation of the multiple eigenvalue $\lambda_2 = \lambda_3$ for uniform and adaptive mesh-refinement results in the convergence history of Figure 13. One observes that marking with respect to $J = \{2,3\}$ leads to a kink in the convergence history of $\lambda_{\ell,3}$ up to 3×10^4 degrees of freedom. An explanation for this behaviour can be given from the adaptive mesh in Figure 14 (left): The discrete eigenvalues $\lambda_{\ell,2}$, $\lambda_{\ell,3}$ are not well-separated from $\lambda_{\ell,4}$ and the discrete eigenfunctions $u_{\ell,2}$ and $u_{\ell,3}$ lead, similar to the example in Subsection 8.2, to an inappropriate adaptive mesh. The simultaneous marking with respect to the cluster $J = \{2,3,4\}$ leads to the optimal convergence rate which can be observed starting from 100 degrees of freedom. A typical adaptive mesh is displayed in Figure 14 (right).

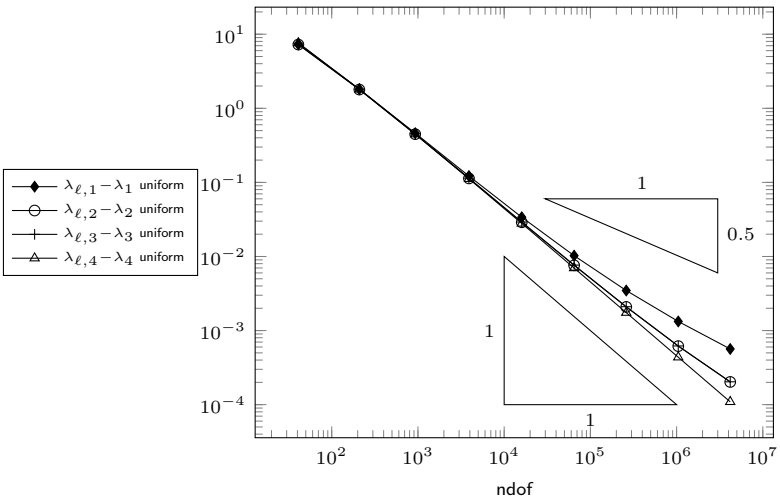

Figure 12: Convergence history on the square with symmetric slits of length 0.85 for uniform mesh-refinement.

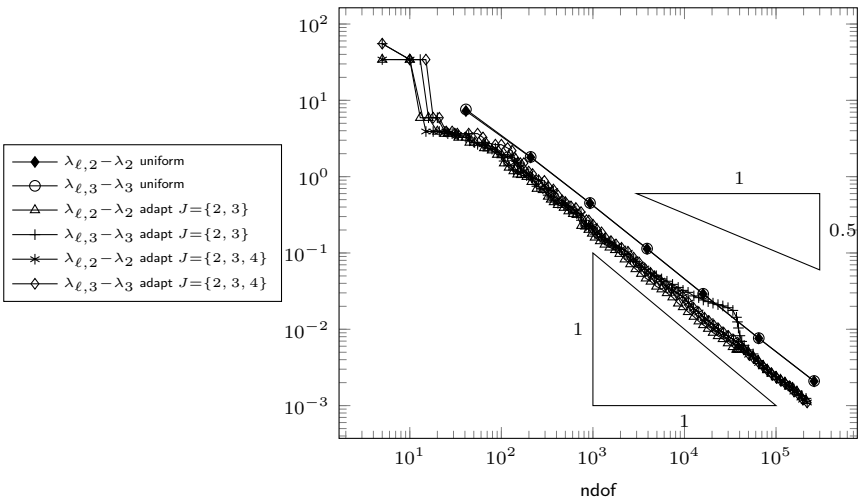

Figure 13: Convergence history on the square with symmetric slits of length 0.85 for adaptive mesh-refinement based on $J \subseteq \{2,3,4\}$. The initial triangulation $\mathscr{T}_0$ has 5 degrees of freedom.

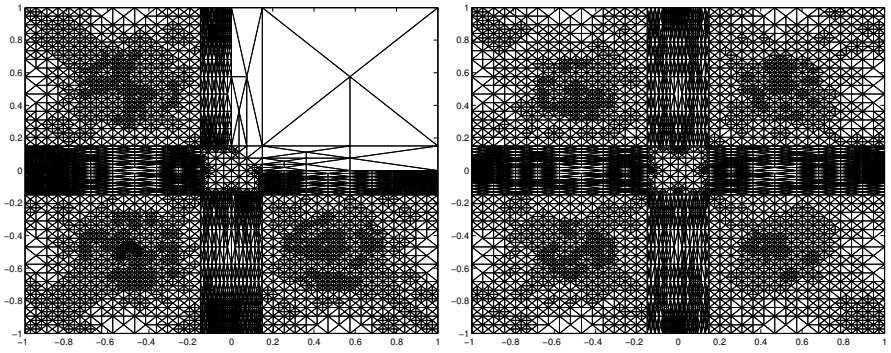

Figure 14: Adaptive meshes for the symmetric domain with slits of length 0.85 from initial triangulation $\mathscr{T}_0$ with 5 degrees of freedom. Left: $J = \{2,3\}$, 4 543 degrees of freedem, level 49. Right: $J = \{2,3,4\}$, 4 496 degrees of freedom, level 47.

9 Conclusions

(a) This paper reduces the usual assumption of sufficiently small H_0 to the four conditions displayed in Table 1. While the constants $C_{\mathrm{drel}}, C_{\mathrm{qo}}, C_{\mathrm{reg}}$ can (in principle) be computed at least on simple domains with structured initial meshes, and condition (H1) can be verified a posteriori with the methodology of [CG14, LO13], sufficient criteria to verify condition (H3) are not known.

(b) In particular for coarse initial triangulations, the simultaneous approximation by the algorithm of Section 3 seems to be superior compared to the use of an adaptive scheme for each eigenvalue separately, even if all eigenvalues on the continuous level are simple: While the use of a coarse initial mesh in the example of Subsection 8.2 with the latter strategy leads to a wide pre-asymptotic regime, the simultaneous approximation produces optimal rates which are observed in Figure 6 even for less than 100 degrees of freedom.

(c) The choice of the bulk parameter $\theta = 0.1$ in the experiments leads to optimal results. From the numerical tests one cannot observe a different behaviour than in the case of a linear PDE although (7.7) appears to be a severe (theoretical) restriction on the bulk parameter which scales badly with the size of the cluster. The pre-asymptotic regime, however, may be large as seen in the experiment of Subsection 8.2. That experiment and the heuristic explanation suggest that an improvement of this pre-asymptotic behaviour is rather achieved by a finer initial mesh-size than by a smaller bulk parameter.

(d) It is evident from the convergence history plot in Figure 6 that the initial separation of the eigenvalue cluster has a strong influence on the quality of the adaptive approximation. The critical quantity is the separation of the cluster from the remaining spectrum and not a full resolution of the multiplicities within the cluster. The separation of the cluster is a more relaxed condition than the separation of a single eigenvalue. Accordingly, the simultaneous marking strategy yields better results. The further assumptions on sufficiently small initial mesh-sizes seemingly indicate that a uniform refinement might be competitive with an adaptive mesh-refinement at least in a large pre-asymptotic regime. This is indeed untrue

as seen in the numerical examples for simultaneous marking. While the remaining theoretical conditions on the initial mesh-size are vital for the analysis, the numerical tests for the simultaneous marking strategy yield optimal results in all examples.

(e) The optimal convergence rates proven in this paper are of asymptotic nature and do not quantify the pre-asymptotic range. The results of [GMZ09] state that for any arbitrarily coarse initial mesh $\mathcal{T}_0$ there is convergence towards *some* eigenpair and the global mesh-size $\|h_\ell\|_{L^\infty(\Omega)} \to 0$ tends to zero. The combination with the results of this paper implies that at some point the global mesh-size is small enough to allow for the optimal convergence rate. The quasi-optimality results in this paper, however, cannot dispense with the restrictions on the initial mesh-size because the involved constant (e.g., C_{opt} in Theorem 3.2) has a universal character in that it only depends on the domain and the shape-regularity of the triangulations. Without any restrictions on the mesh-size one also eventually obtains the optimal convergence rate, but with an unquantified constant that may be sensitive with respect to the eigenvalue of interest or the angles in the initial triangulation. Furthermore, it is clear that the separation constant M_J heavily depends on the definition of the cluster J and the numerical experiments show that the separation has influence on the width of the pre-asymptotic range.

(f) The extension of the present analysis to higher-order methods based on $\mathfrak{P}_k$ polynomials with $k \geq 2$ remains as an open question. The volume contribution of the error estimator on a simplex T reads as $\|\lambda_{\ell,j} u_{\ell,j} + \triangle u_{\ell,j}\|_{L^2(T)}$. The proof of equivalence of this term to theoretical quantities of the form

$$\|\lambda P_\ell u + \triangle \Lambda_\ell u\|_{L^2(T)} \quad \text{or} \quad \|\lambda \Lambda_\ell u + \triangle \Lambda_\ell u\|_{L^2(T)}$$

with the methodology of Proposition 5.2 seems problematic unless only one multiple eigenvalue is considered. Let $k \in J$ and let, in analogy to the proof of Proposition 5.2, $\Lambda_\ell u_k = \sum_{j\in J} \alpha_j u_{\ell,j}$. The Cauchy and triangle inequalities show

$$\begin{aligned}
&\|\lambda_k \Lambda_\ell u_k + \triangle \Lambda_\ell u_k\|^2_{L^2(T)} \\
&\quad \leq \|\Lambda_\ell u_k\|_{L^2(T)} \sum_{j\in J} \|\lambda_k u_{\ell,j} + \triangle u_{\ell,j}\|^2_{L^2(T)} \\
&\quad \leq \|\Lambda_\ell u_k\|_{L^2(T)} \sum_{j\in J} \Big(\|\lambda_{\ell,j} u_{\ell,j} + \triangle u_{\ell,j}\|_{L^2(T)} + \|u_{\ell,j}\|_{L^2(T)} |\lambda_{\ell,j} - \lambda_k| \Big)^2 .
\end{aligned}$$

Only in the case that all $(\lambda_{\ell,j} \mid j \in J)$ converge to one multiple eigenvalue λ the additional term $|\lambda_{\ell,j} - \lambda_k|$ is an appropriate error measure. In case of an eigenvalue cluster, this term describes the cluster width $(B-A)$ and, thus, is not efficient.

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