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Structure preserving deflation of infinite eigenvalues in structured pencils *

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Abstract

The long standing problem is discussed of how to deflate the part associated with the eigenvalue infinity in a structured matrix pencil using structure preserving unitary transformations. We derive such a deflation procedure and apply this new technique to symmetric, Hermitian or alternating pencils and in a modified form to (anti)-palindromic pencils. We present a detailed error and perturbation analysis of this and other deflation procedures and demonstrate the properties of the new algorithm with several numerical examples.

Keywords: structured staircase form, structured Kronecker canonical form, symmetric pencil, Hermitian pencil, alternating pencil, palindromic pencil, linear quadratic control, H_∞ control
AMS subject classification.: 65F15, 15A21, 93B40.

1 Introduction

In this paper we develop numerical methods to calculate a structure preserving deflation under unitary (or in the real case real orthogonal) equivalence transformations for the infinite eigenvalue part in eigenvalue problems for structured matrix pencils

$$(\lambda N - M)x = 0, \tag{1}$$

where $N = \sigma_N N^\star,\ M = \sigma_M M^\star \in \mathbb{K}^{n,n}$, $\sigma_N, \sigma_M \in \{\pm 1\}$, and $\star$ is either T, the transpose or $*$, the conjugate transpose. Here $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$.

Eigenvalue problems of this form arise in many applications, in particular in the context of linear quadratic optimal control problems, see e.g. [9, 13, 16, 18, 20], H_∞ control problems, see e.g. [1, 7, 15, 21], and other applications, see e.g. [10, 14].

In most applications one needs structural information about the Kronecker structure of these pencils, see [19]. To compute this information in a numerically backward stable way, a

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structured staircase form under unitary congruence transformations has been derived in [5] and implemented as production software in [4]. In the real case the unitary transformations can be chosen to be real orthogonal but to simplify, in the following when we speak of unitary transformations we implicitly mean real orthogonal transformations in the case of real matrices.

The following result summarizes the staircase form.

Theorem 1 *[**Structured staircase form.**] For a pencil $\lambda N - M$ with $N = \sigma_N N^\star, M = \sigma_M M^\star \in \mathbb{K}^{n,n}$, there exists a real orthogonal, or unitary matrix $U \in \mathbb{K}^{n,n}$, respectively, such that*

$U^\star N U =$

$$\left[\begin{array}{cccc|c|ccc|c|c}
N_{11} & \cdots & \cdots & N_{1,m} & N_{1,m+1} & N_{1,m+2} & \cdots & N_{1,2m} & 0 & n_1 \\
\vdots & \ddots & & \vdots & \vdots & \vdots & \iddots & \iddots & & \vdots \\
\vdots & & \ddots & \vdots & \vdots & N_{m-1,m+2} & \iddots & & & \vdots \\
\sigma_N N_{1,m}^\star & \cdots & \cdots & N_{m,m} & N_{m,m+1} & 0 & & & & n_m \\
\hline
\sigma_N N_{1,m+1}^\star & \cdots & \cdots & \sigma_N N_{m,m+1}^\star & N_{m+1,m+1} & & & & & l \\
\hline
\sigma_N N_{1,m+2}^\star & \cdots & \sigma_N N_{m-1,m+2}^\star & 0 & & & & & & q_m \\
\vdots & \iddots & \iddots & & & & & & & \vdots \\
\sigma_N N_{1,2m}^\star & \iddots & & & & & & & & q_2 \\
0 & & & & & & & & & q_1
\end{array}\right]$$

$U^\star M U =$ (2)

$$\left[\begin{array}{cccc|c|ccc|c|c}
M_{11} & \cdots & \cdots & M_{1,m} & M_{1,m+1} & M_{1,m+2} & \cdots & \cdots & M_{1,2m+1} & n_1 \\
\vdots & \ddots & & \vdots & \vdots & \vdots & & \iddots & & \vdots \\
\vdots & & \ddots & \vdots & \vdots & \vdots & \iddots & & & \vdots \\
\sigma_M M_{1,m}^\star & \cdots & \cdots & M_{m,m} & M_{m,m+1} & M_{m,m+2} & & & & n_m \\
\hline
\sigma_M M_{1,m+1}^\star & \cdots & \cdots & \sigma_M M_{m,m+1}^\star & M_{m+1,m+1} & & & & & l \\
\hline
\sigma_M M_{1,m+2}^\star & \cdots & \cdots & \sigma_M M_{m,m+2}^\star & & & & & & q_m \\
\vdots & & \iddots & & & & & & & \vdots \\
\vdots & \iddots & & & & & & & & \vdots \\
\sigma_M M_{1,2m+1}^\star & & & & & & & & & q_1
\end{array}\right],$$

where $q_1 \geq n_1 \geq q_2 \geq n_2 \geq \ldots \geq q_m \geq n_m$,

$$\begin{aligned}
N_{j,2m+1-j} &\in \mathbb{K}^{n_j,q_{j+1}}, \qquad 1 \leq j \leq m-1, \\
N_{m+1,m+1} &= \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}, \quad \Delta = \sigma_N \Delta^\star \in \mathbb{K}^{2p,2p}, \\
M_{j,2m+2-j} &= \begin{bmatrix} \Gamma_j & 0 \end{bmatrix} \in \mathbb{K}^{n_j,q_j}, \quad \Gamma_j \in \mathbb{R}^{n_j,n_j}, \qquad 1 \leq j \leq m, \\
M_{m+1,m+1} &= \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \sigma_M \Sigma_{12}^\star & \Sigma_{22} \end{bmatrix}, \quad \Sigma_{11} = \sigma_M \Sigma_{11}^\star \in \mathbb{K}^{2p,2p}, \quad \Sigma_{22} = \sigma_M \Sigma_{22}^\star \in \mathbb{K}^{l-2p,l-2p},
\end{aligned}$$

and the blocks Σ_{22} and Δ and Γ_j, $j = 1, \ldots, m$ are nonsingular.

Proof. The proof for the real case, $\sigma_N = -1$, and $\sigma_M = 1$ has been given in [5], the other cases follow analogously. □

It is clear that the transformation introduced in Theorem 1 preserves the structure of the pencil, but note that in the real case or in the complex case with $\star$ being the complex

conjugate this transformation is a congruence transformation, while in the complex case with $\star$ being the transpose this is just a structure preserving equivalence transformation but not a congruence transformation.

This staircase form allows to deflate the singular part and some of the infinite eigenvalue parts of the pencil in a structure preserving way so that for eigenvalue, eigenvector and invariant subspace computation only the central block $\lambda N_{m+1,m+1} - M_{m+1,m+1}$ of the form (1) has to be considered, which is regular and of index at most one, i.e., it has only finite eigenvalue eigenvalues and infinite eigenvalues with Kronecker blocks of size one, due to the fact that Δ and Σ_{22} are nonsingular.

The staircase form has recently been extended to parameter dependent matrix pencils arising in the control of differential-algebraic systems with variable coefficients, [9], where however the transformation is more complex.

While it is clear how to deflate all the parts associated with higher (than one) indices and singular parts using unitary transformations (despite the usual difficulties with rank decisions), it remained for a long time an open problem how to deflate the remaining index one part associated with the eigenvalue infinity in a structure preserving way using unitary structure preserving transformations, i.e., to reduce the central sub-pencil to a sub-pencil $\lambda N_f - M_f$ of the same structure (with a nonsingular N_f) that contains all the information about the finite eigenvalues. In this paper we solve this problem and develop a new technique for structure preserving deflation of infinite eigenvalues via unitary transformations. We present an error and perturbation analysis and also several numerical examples that demonstrate the properties of the new method, and also compare it to non-unitary structure preserving transformations.

2 Deflation of index one part

A simple way to achieve a structure preserving but non-unitary deflation is to use the Schur complement. Compute a factorization

$$U^\star N = \begin{bmatrix} N_1 \\ 0 \end{bmatrix} \tag{3}$$

of N with N_1 of full row rank. This can be done via the rank-revealing QR decomposition or the singular value decomposition, see [8]. The critical part in this factorization is the decision about the rank of N, which is done in the usual way by setting the part associated to those singular values which are smaller than the machine precision times the norm of N to zero, see [6] for a detailed discussion of this topic.

Using the structure of the pencil, we have

$$\widetilde{N} = U^\star N U = \begin{bmatrix} \widetilde{N}_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad \widetilde{M} = U^\star M U = \begin{bmatrix} \widetilde{M}_{11} & \widetilde{M}_{12} \\ \sigma_M \widetilde{M}_{12}^\star & \widetilde{M}_{22} \end{bmatrix}, \tag{4}$$

where $\widetilde{N}_{11} = \sigma_N \widetilde{N}_{11}^\star$, $\widetilde{M}_{ii} = \sigma_M \widetilde{M}_{ii}^\star$, $i = 1, 2$. By construction, $\widetilde{N}_{11}$ is invertible, and since $\lambda N - M$ is regular and of index at most one, $\widetilde{M}_{22}$ is also invertible.

Then, with

$$\widetilde{L} = \begin{bmatrix} I & 0 \\ -\sigma_M \widetilde{M}_{22}^{-1} \widetilde{M}_{12}^\star & I \end{bmatrix}, \tag{5}$$

one has

$$\begin{aligned}(U\widetilde{L})^\star N(U\widetilde{L}) &= \widetilde{L}^\star\widetilde{N}\widetilde{L} = \begin{bmatrix} \widetilde{N}_{11} & 0 \\ 0 & 0 \end{bmatrix}, \\ (U\widetilde{L})^\star M(U\widetilde{L}) &= \widetilde{L}^\star\widetilde{M}\widetilde{L} = \begin{bmatrix} S & 0 \\ 0 & \widetilde{M}_{22} \end{bmatrix}, \end{aligned} \tag{6}$$

with the *Schur complement*

$$S = \widetilde{M}_{11} - \sigma_M \widetilde{M}_{12}\widetilde{M}_{22}^{-1}\widetilde{M}_{12}^\star.$$

Hence, $\lambda N - M$ is equivalent to the decoupled block diagonal pencil

$$(\lambda\widetilde{N}_{11} - S) \oplus (\lambda 0 - \widetilde{M}_{22}).$$

This deflation procedure preserves the structure, but if $\widetilde{M}_{22}$ is ill-conditioned with respect to inversion, then the eigenvalues of the pencil $\lambda\widetilde{N}_{11} - S$ may be corrupted due the ill-conditioning in the computation of the Schur complement S. We analyze the properties of the Schur complement approach in Section 4 and present some numerical examples in section 7.

Another possibility to deflate the part associated with the infinite eigenvalues is to use equivalence transformations that are unitary but not structure preserving. Starting again with the factorization (3), we form $U^\star M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ partitioned analogously and let V be a unitary matrix such that

$$M_2 V = \begin{bmatrix} 0 & \widehat{M}_{22} \end{bmatrix} \tag{7}$$

with $\widehat{M}_{22}$ square nonsingular. Setting

$$\begin{aligned}\widehat{N} &= U^\star N V = \begin{bmatrix} \widehat{N}_{11} & \widehat{N}_{12} \\ 0 & 0 \end{bmatrix}, \\ \widehat{M} &= U^\star M V = \begin{bmatrix} \widehat{M}_{11} & \widehat{M}_{12} \\ 0 & \widehat{M}_{22} \end{bmatrix}, \end{aligned} \tag{8}$$

then, since $\widehat{N}_{11}$ and $\widehat{M}_{22}$ in (8) are nonsingular, we can eliminate $\widehat{N}_{12}$ and $\widehat{M}_{12}$ simultaneously by block Gaussian eliminations. It follows that $\lambda N - M$ is equivalent to the decoupled block diagonal pencil

$$(\lambda\widehat{N}_{11} - \widehat{M}_{11}) \oplus (\lambda 0 - \widehat{M}_{22}).$$

The computation of the sub-pencil $\lambda\widehat{N}_{11} - \widehat{M}_{11}$ can be performed in a backward stable way, but it has lost its symmetric structures and as a consequence it is hard to obtain the resulting symmetric structure in the finite spectrum of $\lambda N - M$ in further computations. Thus this method is inadequate for many of the tasks where the exact eigenvalue symmetry is needed. We present some numerical examples to demonstrate this deficiency in section 7.

3 Deflation under unitary structure preserving transformations

To derive a structure preserving deflation under unitary structure preserving transformations, consider the unitary matrices U, V obtained in (3) and (7). Form

$$V^\star N V =: \begin{bmatrix} N_{11} & N_{12} \\ \sigma_N N_{12}^\star & N_{22} \end{bmatrix}, \; V^\star M V =: \begin{bmatrix} M_{11} & M_{12} \\ \sigma_M M_{12}^\star & M_{22} \end{bmatrix}. \tag{9}$$

Then we have the following surprising result.

Theorem 2 *Consider a structured pencil of the form (1), which is regular and of index at most one. Then the finite eigenvalues of $\lambda N - M$ are exactly the finite eigenvalues of the structured sub-pencil $\lambda N_{11} - M_{11}$ in (9).*

Proof. We show that the structured sub-pencil $\lambda N_{11} - M_{11}$ of (9) is equivalent to the pencils $\lambda\widehat{N}_{11} - \widehat{M}_{11}$ and $\lambda\widetilde{N}_{11} - S$ obtained in the previous section, and thus contains the finite eigenvalue part.

Let the unitary matrix

$$Q = U^*V = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

be partitioned conformably to the block structure in (9). Then

$$\lambda V^\star NV - V^\star MV = Q^\star[\lambda\widehat{N} - \widehat{M}], \tag{10}$$

and

$$\lambda V^\star NV - V^\star MV = Q^\star[\lambda\widetilde{N} - \widetilde{M}]Q. \tag{11}$$

Based on (4), one has

$$\begin{bmatrix} N_1 \\ 0 \end{bmatrix} = U^\star N = \begin{bmatrix} \widetilde{N}_{11} & 0 \\ 0 & 0 \end{bmatrix} U^*, \quad \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = U^\star M = \begin{bmatrix} \widetilde{M}_{11} & \widetilde{M}_{12} \\ \sigma_M\widetilde{M}_{12}^\star & \widetilde{M}_{22} \end{bmatrix} U^*$$

and then $M_2V = \begin{bmatrix} 0 & \widehat{M}_{22} \end{bmatrix}$ becomes

$$\begin{bmatrix} \sigma_M\widetilde{M}_{12}^\star & \widetilde{M}_{22} \end{bmatrix} Q = \begin{bmatrix} 0 & \widehat{M}_{22} \end{bmatrix}.$$

From this relation we obtain that

$$\sigma_M\widetilde{M}_{12}^\star Q_{11} + \widetilde{M}_{22}Q_{21} = 0 \tag{12}$$

and

$$\sigma_M\widetilde{M}_{12}^\star = \widehat{M}_{22}Q_{12}^*, \quad \widetilde{M}_{22} = \widehat{M}_{22}Q_{22}^*. \tag{13}$$

Since $\widetilde{M}_{22}$ is invertible, so are $\widehat{M}_{22}$ and Q_{22}. For Q, there exists a CS decomposition, see [8]. If the size n_1 of Q_{11} is larger or equal than the size n_2 of Q_{22}, then this has the form

$$\begin{bmatrix} W_1^* & 0 \\ 0 & W_2^* \end{bmatrix} \left[\begin{array}{c|c} Q_{11} & Q_{12} \\ \hline Q_{21} & Q_{22} \end{array}\right] \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} = \left[\begin{array}{cc|c} C & 0 & S \\ 0 & I_{n_1-n_2} & 0 \\ \hline -S & 0 & C \end{array}\right], \tag{14}$$

where W_1, W_2, Z_1, Z_2 are unitary matrices, $C = \mathrm{diag}(c_1, \ldots, c_{n_2})$ and $S = \mathrm{diag}(s_1, \ldots, s_{n_2})$ are both diagonal with nonnegative diagonal entries satisfying $C^2 + S^2 = I_{n_2}$. If $n_1 \le n_2$ then one has

$$\begin{bmatrix} W_1^* & 0 \\ 0 & W_2^* \end{bmatrix} \left[\begin{array}{c|c} Q_{11} & Q_{12} \\ \hline Q_{21} & Q_{22} \end{array}\right] \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} = \left[\begin{array}{c|cc} C & S & 0 \\ \hline -S & C & 0 \\ 0 & 0 & I_{n_2-n_1} \end{array}\right]. \tag{15}$$

Furthermore, since Q_{22} is invertible, it follows that C and thus also Q_{11} are invertible. Hence, from (12) it follows that

$$Q_{21}Q_{11}^{-1} = -\sigma_M \widetilde{M}_{22}^{-1}\widetilde{M}_{12}^{\star}, \tag{16}$$

and from (10) and (8), one has the equivalence relation

$$\lambda N_{11} - M_{11} = Q_{11}^{\star}[\lambda \widehat{N}_{11} - \widehat{M}_{11}].$$

Using (16), the matrix $\widetilde{L}$ defined in (5) becomes

$$\widetilde{L} = \begin{bmatrix} I & 0 \\ Q_{21}Q_{11}^{-1} & I \end{bmatrix},$$

and, defining

$$\widetilde{R} = \begin{bmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22}^{-*} \end{bmatrix},$$

and using the fact that Q is unitary, it follows that

$$Q = \widetilde{L}\widetilde{R}.$$

Then from (11) it follows that

$$\lambda V^{\star}NV - V^{\star}MV = \widetilde{R}^{\star}(\lambda \widetilde{L}^{\star}\widetilde{N}\widetilde{L} - \widetilde{L}^{\star}\widetilde{M}\widetilde{L})\widetilde{R}, \tag{17}$$

and comparing the $(1,1)$ block on both sides leads to

$$\lambda N_{11} - M_{11} = Q_{11}^{\star}(\lambda \widetilde{N}_{11} - S)Q_{11},$$

and then, since

$$\lambda \widehat{N}_{11} - \widehat{M}_{11} = (\lambda \widetilde{N}_{11} - S)Q_{11},$$

it follows that all three sub-pencils are equivalent. □

The relation (17) implies

$$\widetilde{R}^{-\star}(\lambda V^{\star}NV - V^{\star}MV)\widetilde{R}^{-1} = \widetilde{L}^{\star}(\lambda \widetilde{N} - \widetilde{M})\widetilde{L} = (\lambda \widetilde{N}_{11} - S) \oplus (\lambda 0 - \widetilde{M}_{22}).$$

Setting

$$L = \begin{bmatrix} Q_{11} & 0 \\ Q_{21} & I \end{bmatrix} = \widetilde{L}\begin{bmatrix} Q_{11} & 0 \\ 0 & I \end{bmatrix}, \quad R = \begin{bmatrix} I & Q_{11}^{-1}Q_{12} \\ 0 & Q_{22}^{-*} \end{bmatrix} = \begin{bmatrix} Q_{11} & 0 \\ 0 & I \end{bmatrix}^{-1}\widetilde{R},$$

we have $Q = LR$ and also

$$R^{-\star}(\lambda V^{\star}NV - V^{\star}MV)R^{-1} = L^{\star}(\lambda \widetilde{N} - \widetilde{M})L = (\lambda N_{11} - M_{11}) \oplus (\lambda 0 - \widetilde{M}_{22}), \tag{18}$$

which has the pencil $(\lambda N_{11} - M_{11})$ in the $(1,1)$ position.

The pencil $\lambda V^{\star}NV - V^{\star}MV$, however, is unitarily congruent or equivalent to the original pencil, and the finite eigenvalue part can be simply extracted from the $(1,1)$ block.

It follows from (18), that the columns of the matrices

$$\begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ I \end{bmatrix}$$

form orthonormal bases for the right deflating subspaces corresponding to the finite and infinite eigenvalues of $\lambda\widetilde{N} - \widetilde{M}$, respectively. In contrast to this, (6) shows that the Schur complement method simply uses the non-orthonormal basis with the columns of

$$\begin{bmatrix} I \\ Q_{21}Q_{11}^{-1} \end{bmatrix},$$

i.e., the first block column of $\widetilde{L}$ for a structure preserving transformation to block diagonalize $\lambda\widetilde{N} - \widetilde{M}$.

For the original pencil $\lambda N - M$, with $V = [V_1, V_2]$ and $U = [U_1, U_2]$, the columns of

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = U \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} \quad \text{and} \quad U_2 = \begin{bmatrix} U_{12} \\ U_{22} \end{bmatrix}$$

span the right deflating subspaces corresponding to the finite and infinite eigenvalues, respectively.

The *minimal angle* between the two right deflating subspaces is

$$\Theta_{\min} = \min_{0 \neq x \in \operatorname{range} V_1,\, 0 \neq y \in \operatorname{range} U_2} \cos^{-1} \frac{|x^* y|}{\|x\|\|y\|} = \cos^{-1} \|V_1^* U_2\| = \cos^{-1} \|Q_{21}\|,$$

where $\|\cdot\|$ denotes the Euclidean norm for vectors and spectral norm for matrices. The minimal angle measures the closeness of the two deflating subspaces. Based on (14) or (15), suppose that the singular values of Q_{21} are $\cos\theta_1, \ldots, \cos\theta_{\min\{n_1,n_2\}}$, then except for possibly some extra singular values at 1, the singular values of Q_{11} are $\sin\theta_1, \ldots, \sin\theta_{\min\{n_1,n_2\}}$, where $\theta_1, \ldots, \theta_{\min\{n_1,n_2\}} \in [0, \pi/2]$. Thus, we have

$$\Theta_{\min} = \min \theta_j.$$

Introducing

$$\rho := \|\widetilde{M}_{22}^{-1}\widetilde{M}_{12}^{\star}\| = \|\widetilde{M}_{12}\widetilde{M}_{22}^{-1}\|, \tag{19}$$

and using (16), one has

$$\cot\Theta_{\min} = \max_j \cot\theta_j = \rho,$$

and hence,

$$\Theta_{\min} = \cot^{-1}\rho.$$

Thus the smallest singular values of Q_{11} and Q_{22} are given by

$$\sigma_{\min}(Q_{11}) = \sigma_{\min}(Q_{22}) = \sin\Theta_{\min} = \frac{1}{\sqrt{1+\rho^2}} \tag{20}$$

and $\|Q_{12}\| = \|Q_{21}\| = \frac{\rho}{\sqrt{1+\rho^2}}$. If ρ is large, which happens commonly when M_{22} is nearly singular, then the numerical computation of S may be subject to large errors. On the other hand, in this case the block $N_{11} = Q_{11}^{\star}\tilde{N}_{11}Q_{11}$ may be close to a singular matrix as well, which may also lead to big numerical problems, in particular, when deciding about the rank of N.

Let us summarize the methods for computing the sub-pencils $\lambda N_{11} - M_{11}$ in the following algorithms.

Algorithm 1 *Consider a regular and of index at most one structured pencil $\lambda N - M$ with $N^\star = \sigma_N N$ and $M^\star = \sigma_M M$.*

1. *Use a rank revealing QR or singular value decomposition to compute a unitary matrix U such that*
$$U^\star N = \begin{bmatrix} N_1 \\ 0 \end{bmatrix},$$
where N_1 is of full row rank. Partition $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ accordingly and set
$$M_2 = U_2^\star M$$

2. *Use a rank revealing QR or singular value decomposition to determine a unitary matrix V such that*
$$M_2 V = \begin{bmatrix} 0 & \widehat{M}_{22} \end{bmatrix}, \quad \det \widehat{M}_{22} \neq 0.$$

3. *Compute*
$$V^\star N V = \begin{bmatrix} N_{11} & N_{12} \\ \sigma_N N_{12}^\star & N_{22} \end{bmatrix}, \quad V^\star M V = \begin{bmatrix} M_{11} & M_{12} \\ \sigma_M M_{12}^\star & M_{22} \end{bmatrix},$$
and return $\lambda N_{11} - M_{11}$ as the pencil associated with the finite spectrum of $\lambda N - M$.

Algorithm 1 requires two singular value decomposition or rank revealing QR factorizations, as well as the associated matrix-matrix multiplications. The most difficult part is the rank decision for the matrix N in Step 1, which will affect the number of finite eigenvalues and the size and therefore also the conditioning of M_2.

An alternative version of Algorithm 1 is given in the following algorithm.

Algorithm 2 *Consider a regular and of index at most one structured pencil $\lambda N - M$ with $N^\star = \sigma_N N$ and $M^\star = \sigma_M M$.*

1. *Use a Schur-like decomposition of N to compute a unitary matrix U such that*
$$U^\star N U = \begin{bmatrix} \widetilde{N}_{11} & 0 \\ 0 & 0 \end{bmatrix} =: \widetilde{N}, \quad \det \widetilde{N}_{11} \neq 0,$$
and compute
$$\widetilde{M} = U^\star M U = \begin{bmatrix} \widetilde{M}_{11} & \widetilde{M}_{12} \\ \sigma_M \widetilde{M}_{12}^\star & \widetilde{M}_{22} \end{bmatrix}.$$

2. *Use a rank revealing QR or singular value decomposition to compute a unitary matrix $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$ partitioned accordingly, such that*
$$\begin{bmatrix} \sigma_M \widetilde{M}_{12}^\star & \widetilde{M}_{22} \end{bmatrix} Q = \begin{bmatrix} 0 & \widehat{\widetilde{M}_{22}} \end{bmatrix}.$$

3. *Compute*
$$Q^\star \widetilde{N} Q = \begin{bmatrix} N_{11} & N_{12} \\ \sigma_N N_{12}^\star & N_{22} \end{bmatrix}, \quad Q^\star \widetilde{M} Q = \begin{bmatrix} M_{11} & M_{12} \\ \sigma_M M_{12}^\star & M_{22} \end{bmatrix}.$$
or, if only $\lambda N_{11} - M_{11}$ is needed, then
$$N_{11} = Q_{11}^\star \widetilde{N}_{11} Q_{11}, \quad M_{11} = Q_{11}^\star (\widetilde{M}_{11} Q_{11} + \widetilde{M}_{12} Q_{21}).$$

Algorithm 2 uses in contrast to Algorithm 1 a structured Schur-like decomposition in the first step, and performs two unitary transformations on the pencil. The other costs are comparable to those of Algorithm 1.

We will perform the error and perturbation analysis for these algorithms as well as that of the Schur complement approach in the next section.

4 Error and perturbation analysis

In this section we present a detailed error and perturbation analysis for the different deflation procedures presented in the last sections.

Our error analysis for the structure preserving method under unitary transformations will be based on Algorithm 2. The error analysis results for Algorithm 1 are essentially the same.

Let $\mathbf{u}$ be the unit roundoff and denote for each of the matrices defined in the previous section the computed counterpart by the corresponding calligraphic letter, e.g., let $\mathcal{U}$, $\widetilde{\mathcal{N}}_{11}$ be the computed U and $\widetilde{N}_{11}$ in Step 1 of Algorithm 2. Following standard backward error analysis ([8]) there exists a unitary matrix $\widetilde{U}$ such that

$$\widetilde{U}^\star(N+\delta N_u)\widetilde{U} = \begin{bmatrix} \widetilde{\mathcal{N}}_{11} & 0 \\ 0 & 0 \end{bmatrix} = \widetilde{\mathcal{N}},$$

where $\|\widetilde{U}-\mathcal{U}\| = O(\mathbf{u})$, $\delta N_u^\star = \sigma_N \delta N_u$, and $\|\delta N_u\| = O(\|N\|\mathbf{u})$. For the computed $\widetilde{\mathcal{M}}$ we have

$$\widetilde{\mathcal{M}} = \widetilde{U}^\star(M+\delta M_u)\widetilde{U} = \begin{bmatrix} \widetilde{\mathcal{M}}_{11} & \widetilde{\mathcal{M}}_{12} \\ \sigma_M \widetilde{\mathcal{M}}_{12}^\star & \widetilde{\mathcal{M}}_{22} \end{bmatrix},$$

with $\delta M_u^\star = \sigma_M \delta M_u$, $\|\delta M_u\| = O(\|M\|\mathbf{u})$.

Setting $X = U^*\widetilde{U}$ and, without loss of generality, expressing it as

$$X = I + \delta X = \begin{bmatrix} I+\delta X_{11} & \delta X_{12} \\ \delta X_{21} & I+\delta X_{22} \end{bmatrix},$$

we obtain from the exact factorization (4) that

$$\widetilde{\mathcal{N}} = \begin{bmatrix} \widetilde{\mathcal{N}}_{11} & 0 \\ 0 & 0 \end{bmatrix} = X^\star(\widetilde{N} + U^\star \delta N_u U)X = X^\star \begin{bmatrix} \widetilde{N}_{11}+\delta\widetilde{N}_{11} & \delta\widetilde{N}_{12} \\ \sigma_N \delta\widetilde{N}_{12}^\star & \delta\widetilde{N}_{22} \end{bmatrix} X. \tag{21}$$

Introduce the condition number

$$\kappa_N = \|N\|\|\widetilde{N}_{11}^{-1}\| = \|\widetilde{N}_{11}\|\|\widetilde{N}_{11}^{-1}\|,$$

and assume that $\|\delta\widetilde{N}_{11}\| < \sigma_{\min}(\widetilde{N}_{11})$. Then $\widetilde{N}_{11}+\delta\widetilde{N}_{11}$ is nonsingular and by comparing the blocks on both sides of (21), we obtain

$$\delta X_{12}(I+\delta X_{22})^{-1} = -(\widetilde{N}_{11}+\delta\widetilde{N}_{11})^{-1}\delta\widetilde{N}_{12} = O(\kappa_N \mathbf{u}).$$

Using the CS decomposition of X (which is unitary), one has

$$\|\delta X_{12}\|, \|\delta X_{21}\| = O(\kappa_N \mathbf{u}), \quad \|\delta X_{11}\|, \|\delta X_{22}\| = O(\kappa_N^2 \mathbf{u}^2).$$

Clearly, here κ_N measures the sensitivity of the null space of N. Hence,

$$\widetilde{\mathcal{N}}_{11} = (I + \delta X_{11})^\star \widetilde{N}_{11}(I + \delta X_{11}) + O(\|N\|\mathbf{u}) = \widetilde{N}_{11} + \delta\widetilde{\mathcal{N}}_{11}, \quad \|\delta\widetilde{\mathcal{N}}_{11}\| = O((\kappa_N^2\mathbf{u} + 1)\|N\|\mathbf{u}),$$

and from $\widetilde{\mathcal{M}} = X^\star(\widetilde{M} + U^\star \delta M_u U)X$, it follows that

$$\widetilde{\mathcal{M}} = \widetilde{M} + \delta\widetilde{\mathcal{M}}, \quad \|\delta\widetilde{\mathcal{M}}\| = O(\kappa_N\|M\|\mathbf{u}). \tag{22}$$

Partition $\widetilde{\mathcal{M}}$ conformable to $\widetilde{M}$ and assume that

$$\|\delta\widetilde{\mathcal{M}}_{22}\| < \sigma_{\min}(\widetilde{M}_{22}),$$

so that $\widetilde{\mathcal{M}}_{22}$ is invertible, and let $\mathcal{S}$ be the computed Schur complement based on $\widetilde{\mathcal{M}}$. Then, under the assumption that a backward stable linear system solver is used for computing $\widetilde{\mathcal{M}}_{22}^{-1}\widetilde{\mathcal{M}}_{12}^\star$ column-wise, one has

$$\mathcal{S} = \widetilde{\mathcal{M}}_{11} - \sigma_M \widetilde{\mathcal{M}}_{12}\widetilde{\mathcal{M}}_{22}^{-1}\widetilde{\mathcal{M}}_{12}^\star + E_s,$$

with

$$\|E_s\| = O((\|\widetilde{\mathcal{M}}_{11}\| + \rho\|\widetilde{\mathcal{M}}_{12}\| + \rho^2\|\widetilde{\mathcal{M}}_{22}\|)\mathbf{u}) = O((1+\rho)^2\|M\|\mathbf{u}),$$

where ρ is defined in (19). Some elementary calculations and (22) lead to

$$\mathcal{S} = S + E_s + F_s =: S + \delta\mathcal{S},$$

with

$$\|F_s\| = O((1+\rho)^2\kappa_N\|M\|\mathbf{u}),$$

where F_s results from the errors introduced in $\widetilde{M}$ by performing the transformation with $\widetilde{U}$. Note that due to the structure of the error we could absorb E_s into F_s. Altogether, we have

$$\lambda\widetilde{\mathcal{N}}_{11} - \mathcal{S} = \lambda(\widetilde{N}_{11} + \delta\widetilde{\mathcal{N}}_{11}) + (S + \delta\mathcal{S}),$$

with $\|\delta\widetilde{\mathcal{N}}_{11}\| = O((\kappa_N^2\mathbf{u} + 1)\|N\|\mathbf{u})$, and $\|\delta\mathcal{S}\| = O((1+\rho)^2\kappa_N\|M\|\mathbf{u})$.

It then follows that the columns of

$$\begin{bmatrix} I \\ -\sigma_M\widetilde{\mathcal{M}}_{22}^{-1}\widetilde{\mathcal{M}}_{12}^\star \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ I \end{bmatrix},$$

span the right deflating subspaces corresponding to the finite and infinite eigenvalues, respectively, of the pencil

$$\lambda\widetilde{\mathcal{N}} - \left(\widetilde{\mathcal{M}} + \begin{bmatrix} E_s & 0 \\ 0 & 0 \end{bmatrix}\right),$$

and $\lambda\widetilde{\mathcal{N}}_{11} - \mathcal{S}$, $\lambda 0 - \widetilde{\mathcal{M}}_{22}$ are the corresponding computed sub-pencils associated to these bases.

Let $\widetilde{\mathcal{Q}}$ be unitary such that

$$\begin{bmatrix} \sigma_M\widetilde{\mathcal{M}}_{12}^\star & \widetilde{\mathcal{M}}_{22} \end{bmatrix}\widetilde{\mathcal{Q}} = \begin{bmatrix} 0 & \widehat{\widetilde{\mathcal{M}}}_{22} \end{bmatrix}.$$

Then

$$-\sigma_M\widetilde{\mathcal{M}}_{22}^{-1}\widetilde{\mathcal{M}}_{12}^\star = \widetilde{\mathcal{Q}}_{21}\widetilde{\mathcal{Q}}_{11}^{-1},$$

and therefore the right deflating subspace associated with the finite eigenvalues is spanned by the orthonormal columns of the matrix

$$\begin{bmatrix} \widetilde{\mathcal{Q}}_{11} \\ \widetilde{\mathcal{Q}}_{21} \end{bmatrix}.$$

Introducing

$$\mathcal{Y} = Q^* \widetilde{\mathcal{Q}} = \begin{bmatrix} I + \delta\mathcal{Y}_{11} & \delta\mathcal{Y}_{12} \\ \delta\mathcal{Y}_{21} & I + \delta\mathcal{Y}_{22} \end{bmatrix},$$

it follows from (22) that

$$\begin{bmatrix} 0 & \widehat{\widetilde{\mathcal{M}}}_{22} \end{bmatrix} = \begin{bmatrix} \sigma_M \widetilde{\mathcal{M}}_{12}^{\star} & \widetilde{\mathcal{M}}_{22} \end{bmatrix} Q\mathcal{Y} = (\begin{bmatrix} 0 & \widehat{\widetilde{M}_{22}} \end{bmatrix} + \mathcal{E})\mathcal{Y}, \tag{23}$$

with

$$\mathcal{E} = \begin{bmatrix} \mathcal{E}_1 & \mathcal{E}_2 \end{bmatrix} = \begin{bmatrix} \sigma_M \delta\widetilde{\mathcal{M}}_{12}^{\star} & \delta\widetilde{\mathcal{M}}_{22} \end{bmatrix} Q.$$

Using

$$-(\widehat{M}_{22} + \mathcal{E}_2)^{-1}\mathcal{E}_1 = \delta\mathcal{Y}_{21}(I + \delta\mathcal{Y}_{11})^{-1},$$

it follows that

$$\|\delta\mathcal{Y}_{12}\|, \|\delta\mathcal{Y}_{21}\| = O(\kappa_N \widehat{\kappa}_M \mathbf{u}), \quad \|\delta\mathcal{Y}_{11}\|, \|\delta\mathcal{Y}_{22}\| = O(\kappa_N^2 \widehat{\kappa}_M^2 \mathbf{u}^2),$$

where $\widehat{\kappa}_M = \|\widehat{M}_{22}^{-1}\|\|M\| \geq 1$.

Since $\widetilde{U}^{\star}(\lambda(N + \delta N_u) - (M + \delta M_u))\widetilde{U}\widetilde{\mathcal{Q}}$ is block upper triangular, and because of the structure of the pencil, the columns of

$$\widetilde{U}\begin{bmatrix} \widetilde{\mathcal{Q}}_{11} \\ \widetilde{\mathcal{Q}}_{21} \end{bmatrix}, \quad \widetilde{U}\begin{bmatrix} 0 \\ I \end{bmatrix}$$

are the right deflating subspaces of $\lambda(N + \delta N_u) - (M + \delta M_u)$ corresponding to the finite and infinite eigenvalues, respectively. Hence, the minimal angle between these two subspaces is given by

$$\widetilde{\Theta}_{\min} = \cos^{-1}\|\widetilde{\mathcal{Q}}_{21}\|,$$

and, since $\widetilde{\mathcal{Q}}_{21} = Q_{21}(I + \delta\mathcal{Y}_{11}) + Q_{22}\delta\mathcal{Y}_{21}$, it follows that

$$\|\widetilde{\mathcal{Q}}_{21}\| = \|Q_{21}\| + O(\kappa_N \widehat{\kappa}_M \mathbf{u}),$$

and hence

$$\widetilde{\Theta}_{\min} = \Theta_{\min} + O(\csc\Theta_{\min}\kappa_N \widehat{\kappa}_M \mathbf{u}).$$

The perturbation in the deflating subspace associated to the finite eigenvalues can then be measured by $\delta_f := \sin^{-1}\|\Delta_f\|$, with

$$\Delta_f = \left(U\begin{bmatrix} Q_{12} \\ Q_{22} \end{bmatrix}\right)^* \widetilde{U}\begin{bmatrix} \widetilde{\mathcal{Q}}_{11} \\ \widetilde{\mathcal{Q}}_{21} \end{bmatrix} = \begin{bmatrix} Q_{12} \\ Q_{22} \end{bmatrix}^* X \begin{bmatrix} \widetilde{\mathcal{Q}}_{11} \\ \widetilde{\mathcal{Q}}_{21} \end{bmatrix},$$

and the perturbation in the deflating subspace of the eigenvalue infinity can be measured by $\delta_\infty := \sin^{-1}\|\Delta_\infty\|$, with

$$\Delta_\infty = \left(U\begin{bmatrix} I \\ 0 \end{bmatrix}\right)^* \widetilde{U}\begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Inserting the obtained norm bounds, we have

$$\delta_f = O(\|\delta\mathcal{Y}_{21}\| + \|\delta X\|) = O(\kappa_N \widehat{\kappa}_M \mathbf{u}), \quad \delta_\infty = O(\|\delta X_{12}\|) = O(\kappa_N \mathbf{u}).$$

It should be noted that in the case that N is already in block diagonal form as $\widetilde{N}$, which is for example the case when N arises from the middle block of the staircase form (2), then $U = \widetilde{U} = I$, and the computed sub-pencil corresponding to the finite eigenvalues is given by

$$\lambda\widetilde{N}_{11} - \mathcal{S}, \quad \mathcal{S} = S + E_s, \quad \|E_s\| = O((1+\rho)^2\|M\|\varepsilon),$$

where E_s is arising just from the computations of $\mathcal{S}$.

In Step 2 of Algorithm 2, for the computed $\mathcal{Q}$ version of Q, there exists a unitary $\widetilde{Q}$ such that $\|\mathcal{Q} - \widetilde{Q}\| = O(\mathbf{u})$ and

$$\left(\begin{bmatrix} \sigma_M\widetilde{\mathcal{M}}_{12}^\star & \widetilde{\mathcal{M}}_{22} \end{bmatrix} + E\right)\widetilde{Q} = \begin{bmatrix} 0 & \widehat{\mathcal{M}}_{22} \end{bmatrix}, \tag{24}$$

with $\|E\| = O(\|\widehat{\mathcal{M}}_{22}\|\mathbf{u}) = O(\|\widehat{M}_{22}\|\mathbf{u})$.

Introducing

$$Y = Q^*\widetilde{Q} = \begin{bmatrix} I + \delta Y_{11} & \delta Y_{12} \\ \delta Y_{21} & I + \delta Y_{22} \end{bmatrix},$$

then similar to (23) we obtain

$$\begin{bmatrix} \sigma_M\widetilde{\mathcal{M}}_{12}^\star & \widetilde{\mathcal{M}}_{22} \end{bmatrix} Q = \begin{bmatrix} 0 & \widehat{M}_{22} \end{bmatrix} + \begin{bmatrix} \sigma_M\delta\widetilde{\mathcal{M}}_{12}^\star & \delta\widetilde{\mathcal{M}}_{22} \end{bmatrix} Q =: \begin{bmatrix} 0 & \widehat{M}_{22} \end{bmatrix} + \mathcal{E},$$

with $\|\mathcal{E}\| = O(\kappa_N\|M\|\mathbf{u})$. Then from

$$(\begin{bmatrix} 0 & \widehat{M}_{22} \end{bmatrix} + EQ + \mathcal{E})Y = \begin{bmatrix} 0 & \widehat{\mathcal{M}}_{22} \end{bmatrix}$$

and by using the partitioning $EQ + \mathcal{E} = \begin{bmatrix} E_{11} & E_{12} \end{bmatrix}$, we obtain

$$\delta Y_{21}(I + \delta Y_{11})^{-1} = -(\widehat{M}_{22} + E_{12})^{-1}E_{11}, \tag{25}$$

with $\|E_{11}\| = O((\|\widehat{M}_{22}\| + \kappa_N\|M\|)\mathbf{u})$, and hence using the structure we obtain the estimates

$$\|\delta Y_{21}\|, \ \|\delta Y_{12}\| = O(\|E_{11}\|\|\widehat{M}_{22}^{-1}\|) = O((\kappa_{\widehat{M}_{22}} + \kappa_N\widehat{\kappa}_M)\mathbf{u})$$

and

$$\|\delta Y_{11}\|, \ \|\delta Y_{22}\| = O((\kappa_{\widehat{M}_{22}} + \kappa_N\widehat{\kappa}_M)^2\mathbf{u}^2),$$

where $\kappa_{\widehat{M}_{22}} := \|\widehat{M}_{22}\|\|\widehat{M}_{22}^{-1}\|$.

Using these estimates, the computed $\mathcal{N}_{11}$ and $\mathcal{M}_{11}$ can be expressed as

$$\mathcal{N}_{11} = \widetilde{Q}_{11}^\star\widetilde{\mathcal{N}}_{11}\widetilde{Q}_{11} + \delta\mathcal{N}_{11},$$

with $\|\delta\mathcal{N}_{11}\| = O(\|Q_{11}\|^2\|N\|\mathbf{u})$, and

$$\mathcal{M}_{11} = \widetilde{Q}_{11}^\star(\widetilde{\mathcal{M}}_{11}\widetilde{Q}_{11} + \widetilde{\mathcal{M}}_{12}\widetilde{Q}_{21}) + \delta\mathcal{M}_{11},$$

with $\|\delta\mathcal{M}_{11}\| = O((\|\widetilde{M}_{11}\|\|Q_{11}\| + \|\widetilde{M}_{12}\|\|Q_{21}\|)\|Q_{11}\|\mathbf{u})$.

Using the relation between $\lambda\widetilde{\mathcal{N}} - \widetilde{\mathcal{M}}$ and $\lambda\widetilde{N} - \widetilde{M}$, one has

$$\mathcal{N}_{11} = \widetilde{Q}_{11}^{\star}\widetilde{N}_{11}\widetilde{Q}_{11} + \Delta_N,$$

with $\|\Delta_N\| = O((\kappa_N^2\mathbf{u} + 1)\|N\|\|Q_{11}\|^2\mathbf{u})$, and

$$\mathcal{M}_{11} = \widetilde{Q}_{11}^{\star}(\widetilde{M}_{11}\widetilde{Q}_{11} + \widetilde{M}_{12}\widetilde{Q}_{21}) + \Delta_{M,1},$$

with $\|\Delta_{M,1}\| = O((\kappa_N\|M\| + \|\widetilde{M}_{11}\|\|Q_{11}\| + \|\widetilde{M}_{12}\|\|Q_{21}\|)\|Q_{11}\|\mathbf{u})$. Since

$$\begin{aligned}
\widetilde{Q}_{21}\widetilde{Q}_{11}^{-1} &= (Q_{21}(I + \delta Y_{11}) + Q_{22}\delta Y_{21})(Q_{11}(I + \delta Y_{11}) + Q_{12}\delta Y_{21})^{-1} \\
&= (Q_{21}Q_{11}^{-1} + Q_{22}\delta Y_{21}(I + \delta Y_{11})^{-1}Q_{11}^{-1})(I + Q_{12}\delta Y_{21}(I + \delta Y_{11})^{-1}Q_{11}^{-1})^{-1} \\
&= Q_{21}Q_{11}^{-1} + Q_{22}^{-*}\delta Y_{21}(I + \delta Y_{11})^{-1}Q_{11}^{-1}(I + Q_{12}\delta Y_{21}(I + \delta Y_{11})^{-1}Q_{11}^{-1})^{-1},
\end{aligned}$$

using (16), one obtains

$$\mathcal{M}_{11} = \widetilde{Q}_{11}^{\star}S\widetilde{Q}_{11} + \Delta_{M,1} + \Delta_{M,2},$$

where, by using (13),

$$\begin{aligned}
\Delta_{M,2} &= -Q_{11}^{\star}(Q_{21}Q_{11}^{-1})^{\star}\widehat{M}_{22}\delta Y_{21}(I + \delta Y_{11})^{-1}Q_{11}^{-1}(I + Q_{12}\delta Y_{21}(I + \delta Y_{11})^{-1}Q_{11}^{-1})^{-1}\widetilde{Q}_{11} \\
&= -Q_{21}^{\star}\widehat{M}_{22}\delta Y_{21}(I + \delta Y_{11})^{-1}(I + Q_{11}^{-1}Q_{12}\delta Y_{21}(I + \delta Y_{11})^{-1})^{-1}Q_{11}^{-1}\widetilde{Q}_{11}.
\end{aligned}$$

Using (25), it follows that

$$\Delta_{M,2} = Q_{21}^{\star}E_{11} + o(\mathbf{u}),$$

and hence,

$$\|\Delta_{M,2}\| = O(\|Q_{21}\|(\|\widehat{M}_{22}\| + \kappa_N\|M\|)\mathbf{u}),$$

and

$$\mathcal{M}_{11} = \widetilde{Q}_{11}^{\star}S\widetilde{Q}_{11} + \Delta_M,$$

with $\|\Delta_M\| = O((\kappa_N\|M\| + \|\widehat{M}_{22}\|\|Q_{21}\| + \widetilde{M}_{11}\|Q_{11}\|^2)\mathbf{u})$. Here we have used the fact that

$$\|M_{12}\| \leq \|M_{12}\widehat{M}_{22}^{-1}\|\|\widehat{M}_{22}\| = \|Q_{21}\|\|\widehat{M}_{22}\|.$$

In the situation that only the finite eigenvalues are considered, one can restrict the effort to compute $\lambda\mathcal{N}_{11} - \mathcal{M}_{11}$ only. So if we set

$$\sigma_N\mathcal{N}_{12}^{\star} = \widetilde{Q}_{12}^{\star}\widetilde{\mathcal{N}}_{11}\widetilde{Q}_{11}, \quad \sigma_M\mathcal{M}_{12}^{\star} = \widetilde{Q}_{12}^{\star}(\widetilde{\mathcal{M}}_{11}\widetilde{Q}_{11} + \widetilde{\mathcal{M}}_{12}\widetilde{Q}_{21}),$$

then the columns of

$$\begin{bmatrix} \widetilde{Q}_{11} \\ \widetilde{Q}_{21} \end{bmatrix}$$

form an orthonormal basis for the right deflating subspace of $\lambda\widetilde{\mathcal{N}} - \widetilde{\mathcal{M}}_1$ corresponding to the finite eigenvalues, where $\widetilde{\mathcal{M}}_1$ is a perturbed $\widetilde{\mathcal{M}}$ of order $O(\widetilde{M}_{11}\mathbf{u})$ because of the inexact QR factorization (24). The resulting sub-pencil is $\lambda\widetilde{Q}_{11}^{\star}\widetilde{\mathcal{N}}_{11}\widetilde{Q}_{11} - \widetilde{Q}_{11}^{\star}(\widetilde{\mathcal{M}}_{11}\widetilde{Q}_{11} + \widetilde{\mathcal{M}}_{12}\widetilde{Q}_{21})$, and the pencil $\lambda\mathcal{N}_{11} - \mathcal{M}_{11}$ is obtained by adding the extra error pencil $\lambda\Delta_N - \Delta_M$ introduced by the numerical computation of $\mathcal{N}_{11}$ and $\mathcal{M}_{11}$. Similarly, the columns of

$$\begin{bmatrix} 0 \\ I \end{bmatrix}$$

span the right deflating subspace of $\lambda\widetilde{\mathcal{N}} - \widetilde{\mathcal{M}}_1$ corresponding to the eigenvalue infinity and the columns of the matrices

$$\widetilde{U}\begin{bmatrix} \widetilde{Q}_{11} \\ \widetilde{Q}_{21} \end{bmatrix}, \quad \widetilde{U}\begin{bmatrix} 0 \\ I \end{bmatrix}$$

span the corresponding deflating subspaces of a pencil slightly perturbed from $\lambda N - M$.

Setting

$$\widetilde{\Delta}_f = \left(U\begin{bmatrix} Q_{12} \\ Q_{22} \end{bmatrix}\right)^* \widetilde{U}\begin{bmatrix} \widetilde{Q}_{11} \\ \widetilde{Q}_{21} \end{bmatrix} = \begin{bmatrix} Q_{12} \\ Q_{22} \end{bmatrix}^* X \begin{bmatrix} \widetilde{Q}_{11} \\ \widetilde{Q}_{21} \end{bmatrix}, \quad \epsilon_f := \sin^{-1}\|\widetilde{\Delta}_f\|,$$

and

$$\widetilde{\Delta}_\infty = \left(U\begin{bmatrix} I \\ 0 \end{bmatrix}\right)^* \widetilde{U}\begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \epsilon_\infty := \sin^{-1}\|\widetilde{\Delta}_\infty\|,$$

we have $\epsilon_\infty = \delta_\infty = O(\kappa_N \mathbf{u})$, and similarly,

$$\epsilon_f = O(\|\delta Y_{21}\| + \|\delta X\|) = O((\kappa_{\widehat{M}_{22}} + \kappa_N \widehat{\kappa}_M)\mathbf{u}),$$

and the minimal angle between the two perturbed subspaces is given by

$$\widehat{\Theta}_{\min} = \Theta_{\min} + O(\csc\Theta_{\min}(\kappa_{\widehat{M}_{22}} + \kappa_N \widehat{\kappa}_M)\mathbf{u}).$$

If $\lambda N - M$ is already in the form of $\lambda\widetilde{N} - \widetilde{M}$, then $U = \widetilde{U} = I$, and in this case,

$$\|\Delta_N\| = O(\|N\|\|Q_{11}\|^2\mathbf{u}), \quad \|\Delta_M\| = O((\|\widehat{M}_{22}\|\|Q_{21}\| + \|\widetilde{M}_{11}\|\|Q_{11}\|^2)\mathbf{u}),$$

and it follows that

$$\epsilon_\infty = 0, \quad \epsilon_f = O(\kappa_{\widehat{M}_{22}}\mathbf{u}), \quad \widehat{\Theta}_{\min} = \Theta_{\min} + O(\csc\Theta_{\min}\kappa_{\widehat{M}_{22}}\mathbf{u}).$$

We may express $\mathcal{N}_{11}$ and $\mathcal{M}_{11}$ as

$$\mathcal{N}_{11} = \widetilde{Q}_{11}^\star(\widetilde{N}_{11} + \widetilde{\Delta}_N)\widetilde{Q}_{11}, \quad \mathcal{M}_{11} = \widetilde{Q}_{11}^\star(S + \widetilde{\Delta}_M)\widetilde{Q}_{11}$$

with

$$\widetilde{\Delta}_N = \widetilde{Q}_{11}^{-\star}\Delta_N\widetilde{Q}_{11}^{-1}, \quad \widetilde{\Delta}_M = \widetilde{Q}_{11}^{-\star}\Delta_M\widetilde{Q}_{11}^{-1}.$$

and by using (20), we have

$$\begin{aligned} \|\widetilde{\Delta}_N\| &= O((\kappa_N^2\varepsilon + 1)\kappa_{Q_{11}}^2\|N\|\mathbf{u}), \\ \|\widetilde{\Delta}_M\| &= O((\kappa_N\|Q_{11}^{-1}\|^2\|M\| + \kappa_{Q_{11}}^2\|\widetilde{M}_{11}\| + \rho\|Q_{11}^{-1}\|\|\widehat{M}_{22}\|)\mathbf{u}), \end{aligned}$$

where $\kappa_{Q_{11}} = \|Q_{11}\|\|Q_{11}^{-1}\|$.

Recalling from (20) that $\|Q_{11}^{-1}\| = \sqrt{1+\rho^2}$, and comparing the errors in the sub-pencils $\lambda\widetilde{\mathcal{N}}_{11} - \mathcal{S}$ and $\lambda\mathcal{N}_{11} - \mathcal{M}_{11}$, it follows that $\|\widetilde{\Delta}_M\|$ and $\|\delta\mathcal{S}\|$ have the same order, while $\|\widetilde{\Delta}_N\|$ can be larger than $\|\delta\widetilde{\mathcal{N}}_{11}\|$ by a factor $\kappa_{Q_{11}}^2$. The latter will be of equal order only when $\kappa_{Q_{11}}^2$ is not too large.

On the other hand, if we transform $\lambda\widetilde{\mathcal{N}}_{11} - \mathcal{S}$ to

$$\widetilde{Q}_{11}^\star(\lambda\widetilde{\mathcal{N}}_{11} - \mathcal{S})\widetilde{Q}_{11} = \widetilde{Q}_{11}^\star(\lambda\widetilde{N}_{11} - S)\widetilde{Q}_{11} + \widetilde{Q}_{11}^\star(\lambda\delta\widetilde{\mathcal{N}}_{11} - \delta\mathcal{S})\widetilde{Q}_{11},$$

then we have

$$\begin{aligned}\|\widetilde{Q}_{11}^{\star}\delta\widetilde{\mathcal{N}}_{11}\widetilde{Q}_{11}\| &= O((\kappa_N^2\mathbf{u}+1)\|Q_{11}\|^2\|N\|\mathbf{u}),\\ \|\widetilde{Q}_{11}^{\star}\delta\mathcal{S}\widetilde{Q}_{11}\| &= O((1+\rho)^2\|Q_{11}\|^2\kappa_N\|M\|\mathbf{u}).\end{aligned}$$

The first quantity has the same order as $\|\Delta_N\|$, while the second one can be larger than $\|\Delta_M\|$ by a factor $\kappa_{Q_{11}}^2$ unless $\kappa_{Q_{11}}$ is not too large. So in terms of numerical stability, the error analysis results do not show which method has an advantage over the other. This is an unusual circumstance in numerical analysis, where usually the methods based on unitary transformations in the worst case situation have smaller errors than the ones based on non-unitary transformations. We will demonstrate this effect in the numerical examples in section 7.

5 An alternative point of view

Another way to understand the new unitary structure preserving method of deflating the infinite eigenvalue part is to consider the extension trick introduced in [9]. Partition the matrix U in (3) as $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ with U_1 of the same size as N_1. For any eigenvector x of $\lambda N - M$ corresponding to a finite eigenvalue λ, it then follows from $U^{\star}(\lambda N - M)x = 0$ that $U_2^{\star}Mx = 0$. So the original eigenvalue problem is equivalent to the non-square, non-symmetric eigenvalue problem

$$\left(\lambda\begin{bmatrix} N \\ 0 \end{bmatrix} - \begin{bmatrix} M \\ U_2^{\star}M \end{bmatrix}\right)x = 0, \tag{26}$$

and we are only looking for eigenvectors in the nullspace of $U_2^{\star}M$. Let

$$z := V^*x =: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

be partitioned according to the block structure in (9). Then the eigenvalue problem (26) is equivalent to the extended eigenvalue problem

$$\begin{aligned}&\begin{bmatrix} V^{\star} & 0 \\ 0 & I \end{bmatrix}\left(\lambda\begin{bmatrix} N \\ 0 \end{bmatrix} - \begin{bmatrix} M \\ U_2^{\star}M \end{bmatrix}\right)Vz\\ &= \left(\lambda\begin{bmatrix} N_{11} & N_{12} \\ \sigma_N N_{12}^{\star} & N_{22} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ \sigma_M M_{12}^{\star} & M_{22} \\ 0 & \widehat{M}_{22} \end{bmatrix}\right)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0.\end{aligned}$$

Clearly then it follows that $z_2 = 0$ and

$$(\lambda N_{11} - M_{11})z_1 = 0.$$

Hence, an eigenvector of $\lambda N - M$ corresponding to a finite eigenvalue has the form $x = V\begin{bmatrix} z_1 \\ 0 \end{bmatrix}$, where z_1 is an eigenvector of the sub-pencil $\lambda N_{11} - M_{11}$. More generally, the columns of $V\begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}$ form an orthonormal basis for the deflating subspace of $\lambda N - M$ corresponding to the finite eigenvalues.

6 Palindromic and anti-palindromic pencils

Structured pencils that are closely related to those discussed before are the palindromic/anti-palindromic pencils of the form

$$\lambda A^\star - \sigma A$$

where A is an arbitrary real or complex square matrix, and $\sigma = \pm 1$, see [10, 17] for a detailed discussion of the relationship. Assume further that the eigenvalue σ (if it occurs) is semi-simple, which corresponds to the regular and index at most one property for the structured pencils discussed before. Applying a Cayley transformation, see [10, 11], the pencil $\lambda N - M$ with

$$N := A^\star - \sigma A = -\sigma N^\star, \quad M := A^\star + \sigma A = \sigma M^\star$$

is a structured pencil of the form (1) with $\sigma_N = -\sigma$ and $\sigma_M = \sigma$, and the semi-simple eigenvalue σ becomes the eigenvalue ∞ and the pencil has index at most one. To this pencil we can apply the discussed transformations V, U as defined in (3) and (7). Then

$$U^\star(A^\star - \sigma A) = \begin{bmatrix} N_1 \\ 0 \end{bmatrix}, \quad U^\star(A^\star + \sigma A) = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}.$$

Partitioning

$$AU = \begin{bmatrix} A_1 & A_2 \end{bmatrix}, \quad U^\star A = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

conformably, one has

$$\begin{bmatrix} N_1 \\ 0 \end{bmatrix} = \begin{bmatrix} A_1^\star - \sigma B_1 \\ A_2^\star - \sigma B_2 \end{bmatrix}, \quad \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} A_1^\star + \sigma B_1 \\ A_2^\star + \sigma B_2 \end{bmatrix},$$

and hence

$$A_2^\star = \sigma B_2, \quad M_2 = 2\sigma B_2,$$

Furthermore, applying V to M_2, we obtain $M_2 V = \begin{bmatrix} 0 & \widehat{M}_{22} \end{bmatrix}$, and hence $B_2 V = \begin{bmatrix} 0 & \widehat{B}_{22} \end{bmatrix}$ with $\widehat{B}_{22} = \sigma \widehat{M}_{22}/2$.

Setting

$$V^\star A V = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

then

$$V^\star(\lambda N - M)V = \lambda \begin{bmatrix} A_{11}^\star - \sigma A_{11} & A_{21}^\star - \sigma A_{12} \\ A_{12}^\star - \sigma A_{21} & A_{22}^\star - \sigma A_{22} \end{bmatrix} - \begin{bmatrix} A_{11}^\star + \sigma A_{11} & A_{21}^\star + \sigma A_{12} \\ A_{12}^\star + \sigma A_{21} & A_{22}^\star + \sigma A_{22} \end{bmatrix},$$

and thus

$$\lambda N_{11} - M_{11} = \lambda(A_{11}^\star - \sigma A_{11}) - (A_{11}^\star + \sigma A_{11})$$

is the sub-pencil of $\lambda N - M$ with the finite eigenvalues only. By taking the inverse Cayley transformation, this is equivalent to the property that the palindromic sub-pencil

$$\lambda A_{11}^\star - \sigma A_{11}$$

contains all the eigenvalues of $\lambda A^\star - \sigma A$ except the eigenvalue σ, which has been deflated.

It should be noted though that the Cayley transformation or its inverse may lead to cancelation errors if there are eigenvalues close to σ but not equal to σ.

7 Numerical Examples

In this section we present several numerical tests to compare the computed finite eigenvalues of the sub-pencils generated by the three methods: structured unitary equivalence transformation, Schur complement transformation, and the non-structured equivalence transformation. All the tests were performed in `Matlab` ([12]) with double precision. All the tested pencils were real, and we use relative errors to measure the accuracy of the computed finite eigenvalues, where the "exact" eigenvalues are computed with the Matlab code `eig` in extended precision `vpa`. The tested pencils are all skew-symmetric/symmetric. The finite eigenvalues of the sub-pencil extracted with the two structure preserving methods are computed with the MEX code `skewHamileig` ([2, 3]). If the pencil associated with the finite eigenvalues was extracted via non-structured equivalence transformations, we use the `matlab` code `qz`. We also display the quantities $\Theta_{\min}$, $\delta_f, \epsilon_f, \epsilon_\infty$, as well as ρ defined in (19). Since $\widetilde{\Theta}_{\min}$ and $\widehat{\Theta}_{\min}$ are always very close to $\Theta_{\min}$, we did not display them. We do not show δ_∞ either, since it is the same as ϵ_∞.

The following quantities were produced from the numerical tests.

- $\mathrm{Eu}_{\max}$, $\mathrm{Es}_{\max}$, $\mathrm{Eh}_{\max}$: The maximum relative error of the finite eigenvalues for a given pencil with the structured unitary equivalence transformation method, the Schur complement method, and the nonstructured equivalence transformation method, respectively.

- $\mathrm{Eu}_{\min}$, $\mathrm{Es}_{\min}$, $\mathrm{Eh}_{\min}$: The minimum relative error of the finite eigenvalues for a given pencil with the structured unitary equivalence transformation method, the Schur complement method, and the nonstructured equivalence transformation method, respectively.

- Eu_{GM}, Es_{GM}, Eh_{GM}: The geometric mean of the relative errors of the finite eigenvalues for a given pencil with the structured unitary equivalence transformation method, the Schur complement method, and the nonstructured equivalence transformation method, respectively.

- Re_{hat}: The maximum real part of the eigenvalues computed by the nonstructured equivalence transformation method, in the case when all the exact finite eigenvalues are purely imaginary.

Example 1 We tested a set of skew-symmetric/symmetric pencils $X^T(\lambda N - M)X$, where

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{bmatrix} \oplus 0, \quad M = \begin{bmatrix} 2 & & & \\ & 3 & & \\ & & 3 & \\ & & & 2 \end{bmatrix} \oplus \begin{bmatrix} 100 & & \\ & \sqrt{\alpha} & \\ & & \alpha \end{bmatrix},$$

and X is a randomly generated nonsingular matrix. The finite eigenvalues of such a pencil are always $\pm i\sqrt{6}$ and $\pm i\sqrt{6}/\beta$, independent of X and α. If β is small in modulus, then N is close to a singular matrix, and when α is small in modulus, then it can be expected that the eigenvalue infinity part will affect the finite eigenvalues. Note that when $\beta = 1$, then the pencil has two pairs of double eigenvalues.

We have performed the computations 10 times for each pair of (α, β) where each time a different random X is generated. Tables 1a, 1b, 1c, and 1d show the results for each pair

of (α, β). The three methods determine the finite eigenvalues with about the same order of relative errors. The non-structured method produces slightly less accurate eigenvalues. Because it forms non-structured sub-pencils, the computed eigenvalues are no longer purely imaginary. The real parts of the computed finite eigenvalues grow as $|\beta|$ is getting smaller. For all three methods, in this example the errors seem insensitive to the value of α, while they increase when $|\beta|$ decreases. However, α does affect the deflating subspace associated with the finite eigenvalues.

	1	2	3	4	5	6	7	8	9	10
$\mathrm{Eu}_{\max}$	$4e-15$	$9e-15$	$2e-14$	$4e-14$	$6e-14$	$6e-14$	$7e-14$	$8e-14$	$2e-13$	$4e-13$
$\mathrm{Es}_{\max}$	$7e-15$	$4e-14$	$1e-14$	$6e-14$	$6e-14$	$6e-14$	$7e-14$	$1e-13$	$6e-14$	$5e-13$
$\mathrm{Eh}_{\max}$	$1e-14$	$8e-14$	$2e-14$	$1e-13$	$6e-14$	$8e-14$	$6e-14$	$2e-13$	$1e-12$	$5e-13$
$\mathrm{Eu}_{\min}$	$2e-15$	$9e-15$	$3e-15$	$1e-14$	$5e-15$	$6e-15$	$2e-14$	$1e-14$	$9e-16$	$3e-15$
$\mathrm{Es}_{\min}$	0	$1e-14$	$5e-15$	$9e-15$	$4e-15$	$4e-15$	$2e-14$	$6e-15$	$4e-15$	$4e-15$
$\mathrm{Eh}_{\min}$	$1e-15$	$4e-14$	$1e-14$	$1e-14$	$6e-15$	$3e-15$	$5e-15$	$9e-14$	$6e-15$	$2e-15$
Eu_{GM}	$2e-15$	$9e-15$	$8e-15$	$2e-14$	$2e-14$	$2e-14$	$3e-14$	$3e-14$	$1e-14$	$3e-14$
Es_{GM}	0	$2e-14$	$8e-15$	$2e-14$	$2e-14$	$2e-14$	$4e-14$	$3e-14$	$2e-14$	$4e-14$
Eh_{GM}	$4e-15$	$6e-14$	$2e-14$	$4e-14$	$2e-14$	$2e-14$	$2e-14$	$1e-13$	$8e-14$	$4e-14$
Re_{hat}	$2e-14$	$2e-13$	$2e-14$	$2e-13$	$1e-14$	$4e-14$	$9e-15$	$2e-13$	$4e-13$	$1e-13$
ϵ_f	$3e-11$	$3e-11$	$2e-12$	$2e-11$	$4e-11$	$4e-11$	$1e-11$	$2e-11$	$1e-10$	$7e-11$
δ_f	$3e-11$	$2e-11$	$2e-12$	$7e-12$	$2e-11$	$2e-11$	$1e-11$	$2e-11$	$1e-10$	$6e-11$
ϵ_∞	$6e-16$	$6e-16$	$5e-16$	$7e-16$	$5e-16$	$5e-16$	$3e-16$	$3e-16$	$2e-16$	$5e-16$
Θ	$6e-01$	$2e-01$	$2e-01$	$2e-01$	$8e-02$	$3e-01$	$2e-01$	$8e-02$	$8e-02$	$1e-01$
ρ	1.4	4.7	4.0	6.4	13	3.5	6.5	13	13	7.6

Table 1a $(\alpha, \beta) = (10^{-3}, 1)$: Eigenvalue errors, errors in deflating subspaces, Θ, ρ

	1	2	3	4	5	6	7	8	9	10
$\mathrm{Eu}_{\max}$	$8e-13$	$3e-12$	$6e-12$	$7e-12$	$9e-12$	$2e-11$	$8e-11$	$8e-11$	$3e-10$	$2e-09$
$\mathrm{Es}_{\max}$	$3e-12$	$3e-12$	$7e-12$	$9e-12$	$4e-11$	$3e-11$	$4e-11$	$4e-11$	$6e-10$	$2e-09$
$\mathrm{Eh}_{\max}$	$8e-12$	$7e-11$	$3e-10$	$8e-12$	$1e-10$	$3e-11$	$1e-10$	$4e-11$	$1e-09$	$3e-09$
$\mathrm{Eu}_{\min}$	$6e-14$	$4e-16$	$8e-14$	$4e-13$	$2e-14$	$9e-15$	$9e-13$	$3e-15$	$5e-16$	$1e-11$
$\mathrm{Es}_{\min}$	$6e-14$	$7e-16$	$3e-14$	$2e-13$	$2e-15$	$9e-15$	$1e-12$	$6e-15$	$7e-16$	$1e-11$
$\mathrm{Eh}_{\min}$	$5e-14$	$4e-14$	$2e-12$	$4e-13$	$4e-14$	$8e-15$	$9e-13$	$9e-15$	$2e-15$	$1e-11$
Eu_{GM}	$2e-13$	$3e-14$	$7e-13$	$2e-12$	$4e-13$	$4e-13$	$9e-12$	$5e-13$	$4e-13$	$2e-10$
Es_{GM}	$4e-13$	$5e-14$	$4e-13$	$1e-12$	$3e-13$	$5e-13$	$8e-12$	$5e-13$	$7e-13$	$1e-10$
Eh_{GM}	$6e-13$	$2e-12$	$2e-11$	$2e-12$	$2e-12$	$5e-13$	$1e-11$	$6e-13$	$2e-12$	$2e-10$
Re_{hat}	$2e-06$	$3e-06$	$7e-05$	$2e-06$	$2e-05$	$7e-06$	$1e-05$	$1e-05$	$3e-04$	$6e-04$
ϵ_f	$1e-08$	$8e-08$	$5e-08$	$8e-10$	$2e-07$	$1e-07$	$1e-07$	$1e-07$	$4e-07$	$6e-06$
δ_f	$1e-08$	$8e-08$	$5e-08$	$8e-10$	$2e-07$	$1e-07$	$1e-07$	$1e-07$	$4e-07$	$6e-06$
ϵ_∞	$2e-12$	$2e-11$	$6e-12$	$6e-12$	$4e-11$	$1e-11$	$7e-12$	$6e-12$	$1e-10$	$3e-11$
Θ	$2e-01$	$3e-01$	$2e-01$	$1e-01$	$1e-01$	$1e-01$	$5e-02$	$1e-01$	$8e-02$	$6e-03$
ρ	4.2	3.1	6.1	9.8	8.6	7.2	19	8.3	12	165

Table 1b $(\alpha, \beta) = (10^{-3}, 10^{-5})$: Eigenvalue errors, errors of deflating subspaces, and Θ, ρ

	1	2	3	4	5	6	7	8	9	10
$\mathrm{Eu}_{\max}$	$3e-15$	$4e-15$	$4e-15$	$6e-15$	$7e-15$	$9e-15$	$9e-15$	$9e-15$	$4e-14$	$6e-14$
$\mathrm{Es}_{\max}$	$1e-15$	$4e-15$	$8e-15$	$5e-15$	$4e-15$	$8e-15$	$4e-15$	$2e-14$	$2e-14$	$4e-14$
$\mathrm{Eh}_{\max}$	$3e-14$	$8e-15$	$3e-14$	$4e-15$	$5e-15$	$2e-14$	$3e-14$	$3e-14$	$4e-14$	$3e-14$
$\mathrm{Eu}_{\min}$	$7e-16$	$9e-16$	$2e-16$	$7e-16$	0	$2e-15$	$5e-15$	$4e-15$	$5e-15$	$2e-15$
$\mathrm{Es}_{\min}$	$2e-16$	$4e-16$	$1e-15$	$9e-16$	$1e-15$	$1e-15$	$2e-15$	$3e-15$	$2e-14$	$3e-15$
$\mathrm{Eh}_{\min}$	$4e-16$	$4e-15$	$3e-15$	$2e-15$	$4e-15$	$4e-15$	$4e-15$	$1e-15$	$8e-15$	$1e-14$
Eu_{GM}	$2e-15$	$2e-15$	$9e-16$	$2e-15$	0	$5e-15$	$6e-15$	$6e-15$	$1e-14$	$1e-14$
Es_{GM}	$5e-16$	$1e-15$	$3e-15$	$2e-15$	$2e-15$	$3e-15$	$3e-15$	$8e-15$	$2e-14$	$1e-14$
Eh_{GM}	$3e-15$	$5e-15$	$1e-14$	$3e-15$	$4e-15$	$9e-15$	$1e-14$	$7e-15$	$2e-14$	$2e-14$
Re_{hat}	$3e-14$	$2e-14$	$3e-14$	$3e-15$	$1e-14$	$1e-14$	$3e-14$	$7e-15$	$3e-14$	$3e-14$
ϵ_f	$5e-08$	$4e-08$	$1e-07$	$2e-07$	$2e-08$	$2e-07$	$4e-07$	$1e-07$	$1e-06$	$4e-07$
δ_f	$4e-08$	$3e-08$	$1e-07$	$9e-08$	$2e-08$	$1e-07$	$3e-07$	$1e-07$	$6e-07$	$3e-07$
ϵ_∞	$2e-16$	$5e-16$	$3e-16$	$3e-16$	$3e-16$	$7e-16$	$3e-16$	$2e-16$	$2e-15$	$5e-16$
Θ	$2e-01$	$7e-01$	$3e-01$	$2e-01$	$4e-01$	$3e-01$	$2e-01$	$7e-02$	$2e-01$	$2e-01$
ρ	4.6	1.2	3.7	5.8	2.4	3.8	4.9	14	4.6	5.0

Table 1c $(\alpha, \beta) = (10^{-7}, 1)$: Eigenvalue errors, errors of deflating subspaces, and Θ, ρ

	1	2	3	4	5	6	7	8	9	10
$\mathrm{Eu}_{\max}$	$1e-12$	$2e-12$	$9e-12$	$3e-11$	$3e-11$	$3e-11$	$52e-11$	$9e-11$	$1e-10$	$2e-10$
$\mathrm{Es}_{\max}$	$2e-11$	$2e-12$	$7e-12$	$5e-12$	$2e-11$	$3e-11$	$7e-11$	$1e-10$	$8e-11$	$1e-10$
$\mathrm{Eh}_{\max}$	$2e-11$	$3e-11$	$3e-11$	$7e-11$	$3e-10$	$1e-10$	$6e-11$	$1e-10$	$9e-11$	$1e-10$
$\mathrm{Eu}_{\min}$	$5e-16$	$1e-15$	$2e-15$	$1e-13$	$4e-15$	$2e-15$	$9e-14$	$1e-14$	$3e-13$	$1e-15$
$\mathrm{Es}_{\min}$	$2e-15$	$1e-14$	$9e-16$	$2e-13$	$5e-15$	$3e-15$	$4e-14$	$4e-15$	$2e-13$	$6e-15$
$\mathrm{Eh}_{\min}$	$6e-15$	$9e-14$	$7e-15$	$3e-13$	$3e-14$	$1e-14$	$7e-14$	$5e-15$	$7e-13$	$1e-14$
Eu_{GM}	$3e-14$	$4e-14$	$1e-13$	$2e-12$	$4e-13$	$3e-13$	$2e-12$	$1e-12$	$5e-12$	$4e-13$
Es_{GM}	$2e-13$	$1e-13$	$8e-14$	$9e-13$	$3e-13$	$3e-13$	$2e-12$	$7e-13$	$4e-12$	$8e-13$
Eh_{GM}	$3e-13$	$2e-12$	$8e-13$	$5e-12$	$2e-12$	$1e-12$	$2e-12$	$8e-13$	$8e-12$	$1e-12$
Re_{hat}	$4e-06$	$6e-06$	$2e-05$	$1e-05$	$6e-05$	$2e-05$	$4e-06$	$2e-05$	$2e-05$	$2e-05$
ϵ_f	$2e-04$	$4e-04$	$1e-03$	$4e-04$	$2e-03$	$5e-04$	$2e-03$	$2e-04$	$1e-04$	$2e-04$
δ_f	$2e-04$	$4e-04$	$1e-03$	$4e-04$	$2e-03$	$5e-04$	$2e-03$	$2e-04$	$1e-04$	$2e-04$
ϵ_∞	$6e-12$	$2e-12$	$2e-11$	$9e-12$	$3e-11$	$2e-11$	$1e-11$	$3e-11$	$4e-11$	$3e-11$
Θ	$3e-01$	$2e-01$	$5e-01$	$5e-02$	$1e-01$	$4e-01$	$7e-02$	$5e-02$	$2e-01$	$7e-02$
ρ	3.8	5.6	1.7	21	10	2.5	14	20	4.0	14

Table 1d $(\alpha, \beta) = (10^{-7}, 10^{-5})$: Eigenvalue errors, errors of deflating subspaces, and Θ, ρ

Example 2 We tested a set of skew-symmetric/symmetric pencils $X^T(\lambda N - M)X$, where

$$N = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \oplus 0, \quad M = \begin{bmatrix} 2 & 0 & \beta & 0 \\ 0 & -2 & 0 & \beta \\ \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 100 & & & \\ & \sqrt{\alpha} & & \\ & & \alpha & \end{bmatrix},$$

and X is a randomly generated matrix. The finite eigenvalues of such a pencil are always $\pm\beta$ with both algebraic and geometric multiplicities of 2.

When $|\beta|$ is tiny, then all the finite eigenvalues are close to zero and the eigenvalue condition number is expected to increase. Tables 2a, 2b, 2c, and 2d display the results for 4 different sets of (α, β) (with 10 pencils for each set as in Example 1), illustrating that the three methods have essentially the same accuracy. The results also show that the relative errors do increase when $|\beta|$ is getting tiny, and that again, the choice of α does not affect the accuracy very much. For this example, no method yields finite eigenvalues that are exactly real.

	1	2	3	4	5	6	7	8	9	10
$\mathrm{Eu}_{\max}$	$2e-15$	$2e-14$	$2e-14$	$2e-14$	$2e-14$	$2e-14$	$3e-14$	$5e-14$	$6e-14$	$7e-12$
$\mathrm{Es}_{\max}$	$2e-15$	$3e-14$	$3e-14$	$9e-16$	$2e-14$	$2e-14$	$4e-14$	$9e-14$	$7e-14$	$3e-12$
$\mathrm{Eh}_{\max}$	$2e-14$	$6e-14$	$6e-14$	$5e-14$	$2e-13$	$2e-14$	$9e-14$	$6e-14$	$2e-13$	$6e-12$
$\mathrm{Eu}_{\min}$	$2e-15$	$2e-14$	$2e-14$	$2e-14$	$2e-14$	$2e-15$	$9e-15$	$2e-14$	$2e-14$	$2e-14$
$\mathrm{Es}_{\min}$	$9e-16$	$3e-14$	$6e-15$	$9e-16$	$2e-14$	$7e-15$	$2e-14$	$2e-14$	$2e-15$	$5e-14$
$\mathrm{Eh}_{\min}$	$2e-15$	$8e-15$	$2e-14$	$5e-16$	$1e-14$	$5e-16$	$7e-15$	$6e-14$	$1e-14$	0
Eu_{GM}	$2e-15$	$2e-14$	$2e-14$	$2e-14$	$2e-14$	$6e-15$	$2e-14$	$3e-14$	$3e-14$	$4e-13$
Es_{GM}	$2e-15$	$3e-14$	$2e-14$	$9e-16$	$2e-14$	$9e-15$	$3e-14$	$4e-14$	$41e-14$	$4e-13$
Eh_{GM}	$5e-15$	$3e-14$	$3e-14$	$8e-15$	$4e-14$	$5e-15$	$3e-14$	$6e-14$	$4e-14$	0
ϵ_f	$8e-12$	$9e-11$	$9e-12$	$2e-11$	$4e-11$	$4e-12$	$5e-12$	$3e-11$	$9e-12$	$2e-10$
δ_f	$4e-12$	$4e-11$	$9e-12$	$1e-11$	$2e-11$	$4e-12$	$4e-12$	$2e-11$	$6e-12$	$2e-10$
ϵ_∞	$4e-16$	$5e-16$	$6e-16$	$3e-16$	$4e-16$	$4e-16$	$2e-16$	$5e-16$	$3e-16$	$5e-16$
Θ	$3e-01$	$3e-01$	$5e-01$	$2e-01$	$3e-01$	$3e-01$	$3e-01$	$5e-01$	$2e-01$	$3e-02$
ρ	4	4	3	8	5	5	4	3	6	40

Table 2a $(\alpha, \beta) = (10^{-3}, 1)$: Eigenvalue errors, errors of deflating subspaces, and Θ, ρ

	1	2	3	4	5	6	7	8	9	10
$\mathrm{Eu}_{\max}$	$9e-05$	$2e-04$	$2e-04$	$3e-04$	$4e-04$	$4e-04$	$5e-04$	$7e-04$	$3e-03$	$2e-02$
$\mathrm{Es}_{\max}$	$5e-05$	$5e-04$	$5e-05$	$2e-04$	$3e-04$	$3e-04$	$6e-04$	$2e-04$	$2e-03$	$6e-03$
$\mathrm{Eh}_{\max}$	$2e-04$	$3e-04$	$2e-04$	$6e-03$	$4e-04$	$2e-03$	$2e-03$	$2e-03$	$4e-03$	$3e-02$
$\mathrm{Eu}_{\min}$	$9e-05$	$2e-04$	$5e-06$	$7e-05$	$4e-04$	$5e-05$	$5e-04$	$5e-05$	$2e-05$	$2e-04$
$\mathrm{Es}_{\min}$	$5e-05$	$4e-05$	$5e-05$	$2e-04$	$3e-04$	$5e-05$	$6e-04$	$2e-04$	$5e-05$	$2e-04$
$\mathrm{Eh}_{\min}$	$2e-04$	$3e-04$	$9e-06$	$2e-04$	$4e-04$	$2e-04$	$3e-05$	$4e-05$	$7e-04$	$9e-04$
Eu_{GM}	$9e-05$	$2e-04$	$3e-05$	$2e-04$	$4e-04$	$2e-04$	$5e-04$	$2e-04$	$2e-04$	$2e-03$
Es_{GM}	$5e-05$	$2e-04$	$5e-05$	$2e-04$	$3e-04$	$2e-04$	$6e-04$	$2e-04$	$3e-04$	$8e-04$
Eh_{GM}	$2e-04$	$3e-04$	$5e-05$	$9e-04$	$4e-04$	$4e-04$	$3e-04$	$2e-04$	$2e-03$	$5e-03$
ϵ_f	$2e-12$	$3e-11$	$5e-12$	$4e-11$	$2e-11$	$2e-11$	$5e-11$	$6e-12$	$2e-11$	$2e-10$
δ_f	$2e-13$	$2e-11$	$4e-12$	$2e-11$	$2e-11$	$2e-11$	$4e-11$	$5e-12$	$1e-11$	$7e-11$
ϵ_∞	$3e-16$	$4e-16$	$3e-16$	$4e-16$	$4e-16$	$3e-16$	$3e-16$	$5e-16$	$4e-16$	$7e-16$
Θ	$5e-01$	$5e-01$	$3e-01$	$4e-01$	$2e-01$	$3e-01$	$7e-02$	$3e-01$	$2e-01$	$2e-02$
ρ	3	3	5	4	9	5	20	4	6	80

Table 2b $(\alpha, \beta) = (10^{-3}, 10^{-5})$: Eigenvalue errors, errors of deflating subspaces, and Θ, ρ

	1	2	3	4	5	6	7	8	9	10
$\mathrm{Eu}_{\max}$	$9e-15$	$1e-14$	$1e-14$	$2e-14$	$3e-14$	$3e-14$	$5e-14$	$5e-14$	$2e-13$	$4e-13$
$\mathrm{Es}_{\max}$	$2e-14$	$9e-15$	$3e-14$	$2e-14$	$3e-14$	$5e-14$	$3e-14$	$2e-13$	$2e-14$	$4e-13$
$\mathrm{Eh}_{\max}$	$2e-14$	$2e-14$	$3e-13$	$3e-13$	$2e-13$	$1e-14$	$6e-14$	$2e-13$	$3e-13$	$4e-13$
$\mathrm{Eu}_{\min}$	$6e-15$	$1e-14$	$7e-15$	$2e-14$	$3e-14$	$3e-14$	$4e-15$	$5e-14$	$4e-15$	$8e-15$
$\mathrm{Es}_{\min}$	$2e-15$	$3e-15$	$6e-15$	$2e-14$	$3e-14$	$3e-15$	$2e-15$	$4e-15$	$2e-14$	$7e-15$
$\mathrm{Eh}_{\min}$	$9e-15$	$7e-15$	$3e-14$	$6e-15$	$3e-14$	$7e-15$	$3e-15$	$2e-13$	$2e-15$	$7e-15$
Eu_{GM}	$7e-15$	$1e-14$	$9e-15$	$2e-14$	$3e-14$	$3e-14$	$2e-14$	$5e-14$	$3e-14$	$6e-14$
Es_{GM}	$5e-15$	$5e-15$	$2e-14$	$2e-14$	$3e-14$	$2e-14$	$8e-15$	$3e-14$	$2e-14$	$5e-14$
Eh_{GM}	$2e-14$	$2e-14$	$6e-14$	$5e-14$	$7e-14$	$8e-15$	$2e-14$	$2e-13$	$3e-14$	$5e-14$
ϵ_f	$9e-08$	$5e-07$	$9e-08$	$2e-07$	$9e-07$	$2e-07$	$5e-07$	$2e-08$	$2e-07$	$7e-07$
δ_f	$7e-08$	$2e-07$	$9e-08$	$9e-08$	$3e-07$	$1e-07$	$5e-07$	$8e-09$	$5e-08$	$7e-07$
ϵ_∞	$5e-16$	$3e-16$	$4e-16$	$7e-16$	$4e-16$	$3e-16$	$3e-16$	$4e-16$	$3e-16$	$6e-16$
Θ	$3e-01$	$5e-01$	$2e-01$	$7e-01$	$3e-01$	$7e-02$	$3e-01$	$7e-02$	$5e-01$	$2e-01$
ρ	$5e$	3	9	2	4	20	5	20	2	8

Table 2c $(\alpha, \beta) = (10^{-7}, 1)$: Eigenvalue errors, errors of deflating subspaces, and Θ, ρ

	1	2	3	4	5	6	7	8	9	10
$\mathrm{Eu}_{\max}$	$3e-05$	$3e-05$	$5e-05$	$2e-04$	$3e-04$	$4e-04$	$7e-04$	$3e-03$	$1e-01$	$2e-01$
$\mathrm{Es}_{\max}$	$8e-06$	$3e-05$	$6e-05$	$2e-04$	$6e-04$	$4e-04$	$8e-04$	$2e-03$	$3e-02$	$3e-01$
$\mathrm{Eh}_{\max}$	$5e-04$	$4e-05$	$4e-03$	$8e-04$	$2e-03$	$4e-04$	$3e-03$	$2e-03$	$8e-02$	$2e+00$
$\mathrm{Eu}_{\min}$	$2e-06$	$3e-05$	$5e-05$	$2e-04$	$1e-04$	$4e-04$	$7e-04$	$3e-05$	$5e-05$	$9e-05$
$\mathrm{Es}_{\min}$	$8e-06$	$3e-05$	$6e-05$	$2e-04$	$7e-05$	$4e-04$	$8e-04$	$6e-05$	$4e-05$	$2e-03$
$\mathrm{Eh}_{\min}$	$2e-05$	$4e-05$	$2e-04$	$2e-04$	$3e-04$	$4e-04$	$7e-04$	$4e-04$	$9e-05$	$2e-04$
Eu_{GM}	$6e-06$	$3e-05$	$5e-05$	$2e-04$	$2e-04$	$4e-04$	$7e-04$	$3e-04$	$3e-03$	$4e-03$
Es_{GM}	$8e-06$	$3e-05$	$6e-05$	$2e-04$	$2e-04$	$4e-04$	$8e-04$	$3e-04$	$2e-03$	$2e-02$
Eh_{GM}	$9e-05$	$4e-05$	$8e-04$	$4e-04$	$7e-04$	$4e-04$	$2e-03$	$9e-04$	$3e-03$	$2e-02$
ϵ_f	$4e-08$	$2e-08$	$5e-07$	$5e-07$	$2e-07$	$7e-07$	$2e-07$	$6e-07$	$9e-07$	$2e-06$
δ_f	$3e-08$	$3e-08$	$4e-07$	$5e-07$	$3e-07$	$8e-07$	$2e-07$	$3e-07$	$4e-07$	$2e-06$
ϵ_∞	$2e-16$	$1e-15$	$3e-16$	$7e-16$	$4e-16$	$7e-16$	$3e-16$	$5e-16$	$6e-16$	$3e-16$
Θ	$2e-01$	$5e-01$	$6e-01$	$2e-01$	$1e-01$	$8e-02$	$7e-02$	$6e-02$	$2e-02$	$5e-03$
ρ	6	3	2	5	20	20	20	20	90	300

Table 2d $(\alpha, \beta) = (10^{-7}, 10^{-5})$: Eigenvalue errors, errors of deflating subspaces, and Θ, ρ

Example 3 For skew-symmetric/symmetric pencils of the form $X^T(\lambda N - M)X$, where

$$N = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \oplus 0, \quad M = \begin{bmatrix} 2 & 0 & \beta & 0 \\ 0 & -2 & 0 & \beta \\ \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 100 & & & \\ & \sqrt{\alpha} & & \\ & & \alpha & \end{bmatrix},$$

and X being a randomly generated nonsingular matrix, the finite eigenvalues of such are always $\pm\beta$ with both algebraic and geometric multiplicities of 2.

The numerical results are similar to those of Example 2 and are not presented here.

Example 4 For skew-symmetric/symmetric pencils of the form $X^T(\lambda N - M)X$, where

$$N = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \oplus 0, \quad M = \begin{bmatrix} 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & \beta \\ \beta & 0 & 0 & 0 \\ 1 & \beta & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 100 & & \\ & \sqrt{\alpha} & \\ & & \alpha \end{bmatrix},$$

and X a randomly generated nonsingular matrix, there exist two 2×2 Jordan blocks, one for eigenvalue β and another for $-\beta$.

In this case it is expected that the computed finite eigenvalues have big errors due the Jordan blocks, and when $|\beta|$ becomes smaller, then the errors will get larger. Our numerical results presented in Tables 3a and 3b confirm these expectations for two sets of (α, β).

	1	2	3	4	5	6	7	8	9	10
$\mathrm{Eu}_{\max}$	$8e-08$	$8e-08$	$9e-08$	$1e-07$	$2e-07$	$2e-07$	$2e-07$	$2e-07$	$2e-07$	$1e-06$
$\mathrm{Es}_{\max}$	$2e-07$	$4e-08$	$7e-08$	$6e-08$	$1e-07$	$2e-07$	$5e-08$	$2e-07$	$2e-07$	$8e-07$
$\mathrm{Eh}_{\max}$	$2e-07$	$8e-08$	$2e-07$	$2e-07$	$2e-07$	$4e-07$	$3e-07$	$4e-07$	$6e-07$	$2e-06$
$\mathrm{Eu}_{\min}$	$8e-08$	$8e-08$	$9e-08$	$1e-07$	$2e-07$	$2e-07$	$2e-07$	$2e-07$	$2e-07$	$1e-06$
$\mathrm{Es}_{\min}$	$2e-07$	$4e-08$	$7e-08$	$6e-08$	$1e-07$	$2e-07$	$5e-08$	$2e-07$	$2e-07$	$8e-07$
$\mathrm{Eh}_{\min}$	$8e-08$	$7e-08$	$2e-07$	$4e-08$	$2e-07$	$3e-07$	$3e-07$	$2e-07$	$5e-07$	$1e-06$
Eu_{GM}	$8e-08$	$8e-08$	$9e-08$	$1e-07$	$2e-07$	$2e-07$	$2e-07$	$2e-07$	$2e-07$	$1e-06$
Es_{GM}	$2e-07$	$4e-08$	$7e-08$	$6e-08$	$1e-07$	$2e-07$	$5e-08$	$2e-07$	$2e-07$	$8e-07$
Eh_{GM}	$2e-07$	$7e-08$	$2e-07$	$7e-08$	$2e-07$	$3e-07$	$3e-07$	$3e-07$	$5e-07$	$2e-06$
ϵ_f	$2e-09$	$2e-09$	$3e-09$	$8e-10$	$3e-10$	$8e-11$	$7e-09$	$3e-10$	$4e-09$	$3e-09$
δ_f	$1e-09$	$3e-09$	$3e-09$	$9e-10$	$4e-10$	$2e-10$	$2e-09$	$3e-10$	$2e-09$	$3e-09$
ϵ_∞	$5e-16$	$4e-16$	$4e-16$	$2e-15$	$1e-15$	$3e-16$	$5e-16$	$6e-16$	$3e-16$	$8e-16$
Θ	$3e-01$	$4e-01$	$4e-01$	$6e-01$	$2e-01$	$3e-01$	$2e-01$	$5e-01$	$1e-01$	$2e-01$
ρ	5	3	3	2	7	5	7	2	20	10

Table 3a $(\alpha, \beta) = (10^{-5}, 1)$: Eigenvalue errors, errors of deflating subspaces, and Θ, ρ

	1	2	3	4	5	6	7	8	9	10
$\mathrm{Eu}_{\max}$	$5e-02$	$6e-02$	$7e-02$	$8e-02$	$1e-01$	$1e-01$	$2e-01$	$3e-01$	$4e-01$	$5e-01$
$\mathrm{Es}_{\max}$	$5e-02$	$5e-02$	$8e-02$	$8e-02$	$8e-02$	$2e-01$	$7e-02$	$3e-01$	$2e-01$	$6e-01$
$\mathrm{Eh}_{\max}$	$5e-02$	$6e-02$	$3e-01$	$3e-01$	$2e-01$	$9e-02$	$2e-01$	$6e-01$	$3e-01$	$1e+00$
$\mathrm{Eu}_{\min}$	$5e-02$	$6e-02$	$7e-02$	$8e-02$	$1e-01$	$1e-01$	$2e-01$	$3e-01$	$4e-01$	$5e-01$
$\mathrm{Es}_{\min}$	$5e-02$	$5e-02$	$8e-02$	$8e-02$	$8e-02$	$2e-01$	$7e-02$	$3e-01$	$2e-01$	$5e-01$
$\mathrm{Eh}_{\min}$	$2e-02$	$3e-02$	$2e-01$	$9e-02$	$5e-02$	$8e-02$	$2e-01$	$3e-01$	$3e-02$	$2e-01$
Eu_{GM}	$5e-02$	$6e-02$	$7e-02$	$8e-02$	$1e-01$	$1e-01$	$2e-01$	$3e-01$	$4e-01$	$5e-01$
Es_{GM}	$5e-02$	$5e-02$	$8e-02$	$8e-02$	$8e-02$	$2e-01$	$7e-02$	$3e-01$	$2e-01$	$6e-01$
Eh_{GM}	$3e-02$	$5e-02$	$3e-01$	$2e-01$	$8e-02$	$9e-02$	$2e-01$	$5e-01$	$9e-02$	$4e-01$
ϵ_f	$3e-10$	$9e-10$	$1e-09$	$2e-09$	$3e-09$	$2e-10$	$1e-09$	$3e-09$	$5e-09$	$3e-08$
δ_f	$3e-10$	$8e-10$	$4e-10$	$2e-09$	$2e-09$	$3e-10$	$6e-10$	$3e-09$	$2e-09$	$3e-08$
ϵ_∞	$4e-16$	$7e-16$	$5e-16$	$6e-16$	$3e-16$	$9e-16$	$2e-16$	$4e-16$	$7e-16$	$5e-16$
Θ	$4e-01$	$5e-01$	$2e-01$	$2e-01$	$4e-01$	$3e-01$	$6e-01$	$1e-01$	$6e-02$	$3e-02$
ρ	3	3	6	6	4	5	2	20	20	50

Table 3b $(\alpha, \beta) = (10^{-5}, 10^{-6})$: Eigenvalue errors, errors of deflating subspaces, and Θ, ρ

All the numerical tests confirm our expectations obtained from the error analysis and illustrate that, surprisingly, the Schur complement approach and the approach via orthogonal structure preserving transformations deliver the same accuracy, despite possible ill-conditioning. The unstructured orthogonal transformation, however, shows the expected problems with purely imaginary eigenvalues.

8 Conclusions

We have presented and analyzed several methods to deflate the infinite eigenvalue part in a structured regular pencil of index at most one. We have shown via a careful error analysis

that it is possible to do this deflation via a structure preserving real orthogonal or unitary transformation as well as with a Schur complement approach. Both methods yield similar results in the perturbation and error analysis and this is confirmed in the numerical tests. This is surprising, and counterintuitive to the general wisdom, that unitary transformations typically perform better than transformations with potentially unbounded transformations. Nonetheless, we suggest to use the unitary structure preserving transformations, since they are very much in line with the staircase algorithm. Both methods are definitely preferable to the non-structured unitary transformation.

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