Robust transmission conditions of high order for thin conducting sheets in two dimensions

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Resolving thin conducting sheets for shielding or even skin layers inside by the mesh of numerical methods like the finite element method (FEM) can be avoided by using impedance transmission conditions (ITCs). Those ITCs shall provide an accurate approximation for small sheet thicknesses d, where the accuracy is best possible independent of the conductivity or the frequency being small or large – this we will call *robustness*. We investigate the accuracy and robustness of popular and recently developed ITCs, and propose robust ITCs which are accurate up to $O(d^2)$.

***Index Terms*—Electromagnetic shielding, Eddy currents, Impedance Transmission Conditions.**

I. Introduction

Thin conducting sheets for the protection of electronic devices exhibit large ratios of characteristic lengths which require a small mesh size when using finite difference or finite element schemes. Besides this issue of computational cost due to the small geometry detail, many commercial mesh generators get difficulties with anisotropic geometrical features.

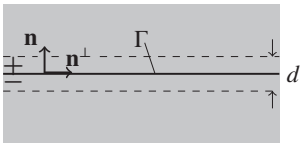


Figure 1. Impedance transmission conditions are set on the mid-line Γ of the sheet and shall approximate the exact field outside the area the sheet was originally located.

We consider the time-harmonic eddy current model [3] (convention $\exp(-\mathrm{i}\omega t)$, $\omega > 0$) in two dimensions

$$\mathbf{curl}_{2D}\, e(x,y) = \mathrm{i}\omega\mu(x,y)\,\mathbf{h}(x,y), \tag{1a}$$
$$\mathrm{curl}_{2D}\,\mathbf{h}(x,y) = \sigma(x,y)\,e(x,y) + j_0(x,y) \tag{1b}$$

where e and $\mathbf{h}$ are the out-of-plane electric and in-plane magnetic fields, σ is the conductivity of the thin sheet of thickness d and zero elsewhere, μ is the permeability, and j_0 is the out-of-plane imposed current which is outside the conductor. We have used the 2D rotation operators $\mathbf{curl}_{2D} = (\partial_y, -\partial_x)^\top$ and $\mathrm{curl}_{2D} = (-\partial_y, \partial_x)$. Denoting the skin depth by $\delta = \sqrt{2/\omega\mu\sigma}$, we may distinguish the "low frequency case" for which the sheet thickness is much smaller than the skin depth from the "high frequency case", for which the skin depth is much smaller than the sheet thickness.

The shielding behaviour can be modelled alternatively by replacing the thin sheet by an interface on which *impedance transmission conditions* are set. Popular ITCs for the time-harmonic eddy current model are the perfectly electric conductor (PEC) boundary condition, the shielding elements [4], and probably most used the thin sheet conditions [5], [6], [1]. With $\beta = i\omega\mu$ and $\gamma = \sqrt{-i\omega\mu\sigma}$ they are given by

$$\begin{aligned} e^+_{\mathrm{MB}} - e^-_{\mathrm{MB}} &= \tfrac{\beta}{\gamma}\tanh\left(\tfrac{\gamma d}{2}\right)(\mathbf{h}^+_{\mathrm{MB}}\cdot\mathbf{n}^\perp + \mathbf{h}^-_{\mathrm{MB}}\cdot\mathbf{n}^\perp),\\ \mathbf{h}^+_{\mathrm{MB}}\cdot\mathbf{n}^\perp - \mathbf{h}^-_{\mathrm{MB}}\cdot\mathbf{n}^\perp &= \tfrac{\gamma}{\beta}\tanh\left(\tfrac{\gamma d}{2}\right)(e^+_{\mathrm{MB}} + e^-_{\mathrm{MB}}) \end{aligned} \tag{2}$$

which are conditions on the mid-line Γ of the thin sheet. In their derivation it is assumed that the fields vary more in thickness direction than along the sheet. Here, the subscript MB denotes the approximative electric and magnetic field, the superscript ± denotes the values on the two sides of the sheet, and $\mathbf{n} = (n_1, n_2)^\top$ and $\mathbf{n}^\perp = (n_2, -n_1)^\top$ are the normalised normal and tangential vectors on Γ like shown in Fig. 1. Note, the relation of (2) to the conditons in [6].

The purpose of this article is the introduction of higher order transmission conditions using the technique of asymptotic expansions. In Sec. II we compare different asymptotic limits for vanishing thickness and derive in Sec. III the higher order asymptotic expansion resulting in higher order ITCs. Finally, in Sec. IV we discuss and compare the properties of the ITCs.

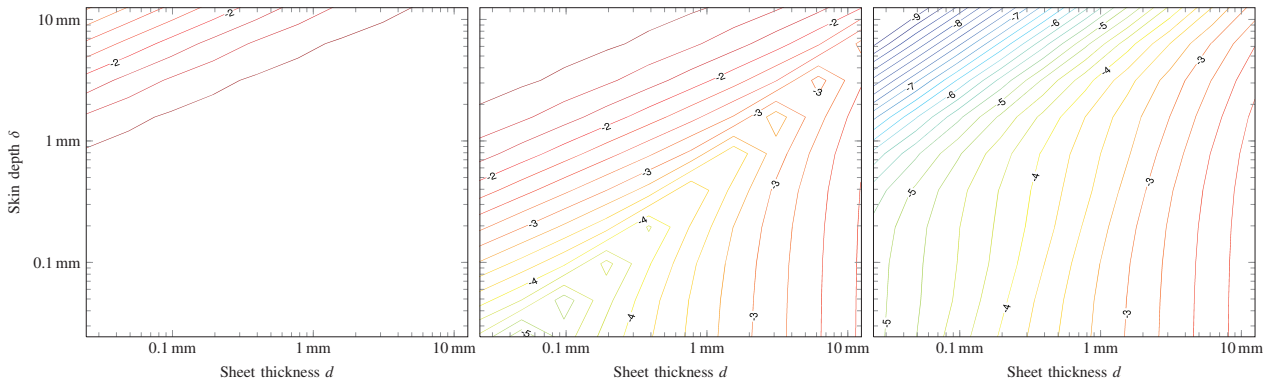
II. Asymptotic limits of vanishing thickness

The asymptotic analysis is a systematic tool which answers the questions which terms are smaller than others and which terms should be included first. In this section we present the asymptotic limits of sheets with vanishing thicknesses while the conductivity is changed with the thickness [10]. We denote the varying sheet thickness by ε to distinguish it from the thickness d of the original problem. In the asymptotic process $\varepsilon \to 0$ we replace the conductivity σ by $\sigma_\alpha(\varepsilon) = \sigma\cdot(d/\varepsilon)^\alpha$, *i. e.*, the conductivity is proportional to $\varepsilon^{-\alpha}$. With this choice the original problem is met for $\varepsilon = d$. We observe three different limits for vanishing sheet thickness ($\varepsilon \to 0$):

1) The conductivity is remained or is increased less than $1/\varepsilon$ (*i. e.*, $0 \le \alpha < 1$). Then, we have twofold continuity

$$\begin{aligned} e^+_0 - e^-_0 &= 0,\\ \mathbf{h}^+_0\cdot\mathbf{n}^\perp - \mathbf{h}^-_0\cdot\mathbf{n}^\perp &= 0, \end{aligned} \tag{3a}$$

the sheet vanishes like for $\delta = \sqrt{2/\omega\mu\sigma} \to \infty$.



(a) Low-frequency limit. (b) High-frequency limit. (c) Non-trivial limit (ITC-1-0).

Figure 2. The modelling error in the magnetic field outside the thin sheet when using (a) the low-frequency limit, (b) the high-frequency limit, (c) the non-trivial limit to approximate the shielding of conducting sheets of different thicknesses d and different frequencies or skin depth $\delta = \sqrt{2/\omega\mu\sigma}$.

2) The conductivity increases like $1/\varepsilon$ (*i. e.*, $\alpha = 1$), where we get the non-trivial limit conditions [7],

$$\begin{aligned} e_1^+ \quad - \quad e_1^- \quad &= \quad 0, \\ \mathbf{h}_1^+ \cdot \mathbf{n}^\perp \quad - \quad \mathbf{h}_1^- \cdot \mathbf{n}^\perp \quad &= \quad -\tfrac{\sigma d}{2}(e_1^+ + e_1^-), \end{aligned} \tag{3b}$$

which has been the first time introduced by Levi-Civita [8].

3) The conductivity increases more than $1/\varepsilon$ ($1 < \alpha \leq 2$). Then, the electric field on both sides get zero

$$e_2^+ = e_2^- = 0. \tag{3c}$$

in the limit $\varepsilon \to 0$, which corresponds to PEC.

The electric field is in all limit cases continuous as the permeability is not assumed to be large and, hence, does not scale with ε. The limit conditions (3a), (3b) and (3c) can be used as ITCs whose properties we will discuss in Sec. IV.

III. Higher order transmission conditions

To derive higher order transmission conditions we use an asymptotic expansion of the solution in the interior and exterior of the sheet with respect to the varying sheet thickness ε. This asymptotic expansion of the solution fulfills the differential equation inside and outside the sheet and the coupling conditions asymptotically, *i. e.*, the better the smaller ε. Note, that the original thickness d is through the scaling of $\sigma_\alpha(\varepsilon)$ present as a parameter.

For simplicity of the presentation we assume the sheet to be straight in this section where the sheet mid-line is at $y = 0$, x is the tangential direction, *i. e.*, $\mathbf{n}^\perp = (1, 0)^\top$.

A. Asymptotic expansion inside the sheet

Inside the thin sheet we can rewrite (1) in terms of the electric field as

$$\operatorname{curl}_{2D} \mathbf{curl}_{2D}\, e(\mathbf{x}) + \gamma^2 e(\mathbf{x}) = 0.$$

Due to $\gamma_\alpha^2 = \gamma^2 (d/\varepsilon)^\alpha$ this equation is identical to the equation

$$\operatorname{curl}_{2D} \mathbf{curl}_{2D}\, e_{\text{int}}^\varepsilon(\mathbf{x}) + \gamma_\alpha^2(\varepsilon)\, e_{\text{int}}^\varepsilon(\mathbf{x}) = 0 \tag{4}$$

for the ε-dependent field $e_{\text{int}}^\varepsilon$ (the subscript "int" stands for the interior of the sheet). Here, we will consider two cases $\alpha = 1$ and $\alpha = 2$ which lead to the families of transmission conditions ITC-1-N and ITC-2-N, where N stands for the order, respectively. Inside the sheet we use the following ansatz for the electric field

$$e_{\text{int}}^\varepsilon(x, y) = E_{\text{int}}^0(x, y/\varepsilon) + \frac{\varepsilon}{d} E_{\text{int}}^1(x, y/\varepsilon) + \ldots,$$

where $E_{\text{int}}^0(x, Y)$ is the limit field in the stretched coordinate $Y = y/\varepsilon \in [-\frac{1}{2}, \frac{1}{2}]$ and $E_{\text{int}}^j(x, Y)$, $j > 0$, are corrector fields of first or higher order. Inserting this ansatz into (4) and using the identity $\operatorname{curl}_{2D} \mathbf{curl}_{2D} = -\Delta$ we find

$$\left(-\partial_x^2 - \frac{1}{\varepsilon^2}\partial_Y^2 + \gamma^2 \frac{d^\alpha}{\varepsilon^\alpha}\right)\left(E_{\text{int}}^0(x, Y) + \frac{\varepsilon}{d} E_{\text{int}}^1(x, Y) + \ldots\right) = 0.$$

Now, we assume that the equality holds for all powers in ε independently. This means for $\alpha = 1$ that

$$\partial_Y^2 E_{\text{int}}^0(x, Y) = 0, \tag{5a}$$

$$\partial_Y^2 E_{\text{int}}^1(x, Y) = \gamma^2 d^2\, E_{\text{int}}^0(x, Y), \tag{5b}$$

and for $\alpha = 2$ for $n = 0, 1$ that

$$\left(\partial_Y^2 - \gamma^2 d^2\right) E_{\text{int}}^n(x, Y) = 0. \tag{6}$$

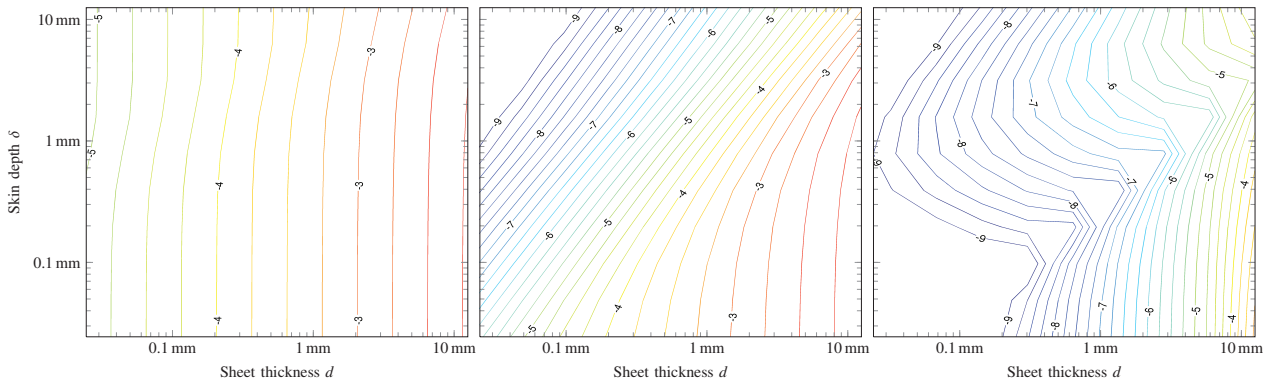
We see that for the first terms the derivatives ∂_x^2 along the sheet play a minor role, which is the usual assumption for deriving the thin sheet conditions. The behaviour in thickness direction is different. For $\alpha = 1$ we have a possible linear behaviour in Y for E_{int}^0 and, assuming that σ is constant in y, a possible cubic behaviour for E_{int}^1. Using the notation $\{V\}(\cdot) := \frac{1}{2}(V(\cdot, +\frac{1}{2}) + V(\cdot, -\frac{1}{2}))$, $[V](\cdot) := V(\cdot, +\frac{1}{2}) - V(\cdot, -\frac{1}{2})$ we write for $\alpha = 2$ and $n = 0, 1$, if σ is again constant in y,

$$E_{\text{int}}^n(x, Y) := \{E_{\text{int}}^n\}(x) \tfrac{\cosh(\gamma d Y)}{\cosh(\gamma \frac{d}{2})} + [E_{\text{int}}^n](x) \tfrac{\sinh(\gamma d Y)}{2 \sinh(\gamma \frac{d}{2})}. \tag{7}$$

With these hyberbolic functions the internal field adapts to the skin depth to thickness ratio d/δ. Taking the normal derivatives of E_{int}^n at $Y = \pm\frac{1}{2}$ we find then

$$[\partial_Y E_{\text{int}}^n](x) := 2\gamma d \tanh(\gamma \tfrac{d}{2}) \{E_{\text{int}}^n\}(x), \tag{8a}$$

$$\{\partial_Y E_{\text{int}}^n\}(x) := \tfrac{1}{2}\gamma d \coth(\gamma \tfrac{d}{2}) [E_{\text{int}}^n](x). \tag{8b}$$



(a) Thin sheet conditions. (b) ITC-1-1 (c) ITC-2-1

Figure 3. Error of (a) the thin sheet conditions (2) and the impedance transmission conditions ITC-1-1 and ITC-2-1.

B. Asymptotic expansion outside the sheet

As the electric and magnetic fields $e^{\varepsilon}_{\text{ext}}$ and $\mathbf{h}^{\varepsilon}_{\text{ext}}$ outside the thin sheet are coupled to those inside, the external fields depend on the parameter ε as well, and we use the ansatz

$$e^{\varepsilon}_{\text{ext}}(x,y) = e^{0}_{\text{ext}}(x,y) + \frac{\varepsilon}{d} e^{1}_{\text{ext}}(x,y) + \dots , \tag{9}$$

$$\mathbf{h}^{\varepsilon}_{\text{ext}}(x,y) = \mathbf{h}^{0}_{\text{ext}}(x,y) + \frac{\varepsilon}{d} \mathbf{h}^{1}_{\text{ext}}(x,y) + \dots . \tag{10}$$

Here e^{0}_{ext} and $\mathbf{h}^{0}_{\text{ext}}$ are the limit fields and first and higher corrector fields are e^{n}_{ext}, $\mathbf{h}^{n}_{\text{ext}}$ $n > 0$, which all are assumed to be independent of ε and which shall be defined for any ε and so up to $y = 0$. However, they may take two different values on both sides of Γ. Once again we claim that (1) is fulfilled for each order in ε separately, which implies that each pair e^{n}_{ext}, $\mathbf{h}^{n}_{\text{ext}}$ fulfills (1), where for $n > 0$ no impressed current is present.

The fields outside the sheet couple at $y = \pm\frac{\varepsilon}{2}$ to those inside the sheet. To obtain at the end conditions on Γ we have to relate the outer electric field at $y = \pm\frac{\varepsilon}{2}$ with its value at $y = \pm 0$. If the sheet is smooth enough, which we have assumed here, we can use the Taylor expansion. For $n = 0$ we have for example

$$\begin{aligned} e^{0}_{\text{ext}}(x, \pm\tfrac{\varepsilon}{2}) &= e^{0}_{\text{ext}}(x, \pm 0) \pm \tfrac{\varepsilon}{2} \partial_y e^{0}_{\text{ext}}(x, \pm 0) + \dots , \\ &= e^{0}_{\text{ext}}(x, \pm 0) \pm \tfrac{1}{2} i\omega\mu\varepsilon\, \mathbf{h}^{0}_{\text{ext}}(x, \pm 0) \cdot \mathbf{n}^{\perp} + \dots \end{aligned}$$

and

$$\mathbf{h}^{0}_{\text{ext}}(x, \pm\tfrac{\varepsilon}{2}) = \mathbf{h}^{0}_{\text{ext}}(x, \pm 0) \pm \tfrac{\varepsilon}{2} \partial_y \mathbf{h}^{0}_{\text{ext}}(x, \pm 0) + \dots .$$

Here, we used that e^{0}_{ext}, $\mathbf{h}^{0}_{\text{ext}}$ shall satisfy (1) up to $y = \pm 0$ with no impressed current for $|y| < \varepsilon/2$.

Using this Taylor expansion is a key feature of this derivation to obtain higher order ITCs. In all previously classical transmission conditions for thin conducting sheets the fields at $\pm d/2$ and at ± 0 are directly identified which leads to a maximal linear decay of the modelling error with d.

C. Coupling the internal and external fields

On the surface of the conductor the tangential electric and magnetic field components are continuous. So, we have for the electric field

$$E^{0}_{\text{int}}(x, \pm\tfrac{1}{2}) + \tfrac{\varepsilon}{d} E^{1}_{\text{int}}(x, \pm\tfrac{1}{2}) + \dots = e^{0}_{\text{ext}}(x, \pm\tfrac{\varepsilon}{2}) + \tfrac{\varepsilon}{d} e^{1}_{\text{ext}}(x, \pm\tfrac{\varepsilon}{2}) + \dots$$

and inserting the above Taylor expansion and the rule that the equality should hold for all powers in ε

$$E^{0}_{\text{int}}(x, \pm\tfrac{1}{2}) = e^{0}_{\text{ext}}(x, \pm 0)\,, \tag{11}$$

$$E^{1}_{\text{int}}(x, \pm\tfrac{1}{2}) = e^{1}_{\text{ext}}(x, \pm 0) \pm \tfrac{1}{2} i\omega\mu d\, \mathbf{h}^{0}_{\text{ext}}(x, \pm 0) \cdot \mathbf{n}^{\perp}. \tag{12}$$

For the magnetic field we have

$$\begin{aligned} \frac{1}{i\omega\mu} \left(\frac{1}{\varepsilon} \partial_Y E^{0}_{\text{int}}(x, \pm\tfrac{1}{2}) + \frac{1}{d} \partial_Y E^{1}_{\text{int}}(x, \pm\tfrac{1}{2}) + \dots \right) \\ = \mathbf{h}^{0}_{\text{ext}}(x, \pm\tfrac{\varepsilon}{2}) \cdot \mathbf{n}^{\perp} + \frac{\varepsilon}{d} \mathbf{h}^{1}_{\text{ext}}(x, \pm\tfrac{\varepsilon}{2}) \cdot \mathbf{n}^{\perp} + \dots , \end{aligned} \tag{13}$$

and inserting the above Taylor expansion we obtain the following coupling conditions

$$\partial_Y E^{0}_{\text{int}}(x, \pm\tfrac{1}{2}) = 0, \tag{14a}$$

$$\partial_Y E^{1}_{\text{int}}(x, \pm\tfrac{1}{2}) = i\omega\mu d\, \mathbf{h}^{0}_{\text{ext}}(x, \pm 0) \cdot \mathbf{n}^{\perp}. \tag{14b}$$

D. Conditions for the external fields with $1/\varepsilon^2$ scaling

The only solution to (6) for $n = 0$ and (14a) is $E^{0}_{\text{int}} \equiv 0$, and so (11) implies perfectly electric conductor boundary

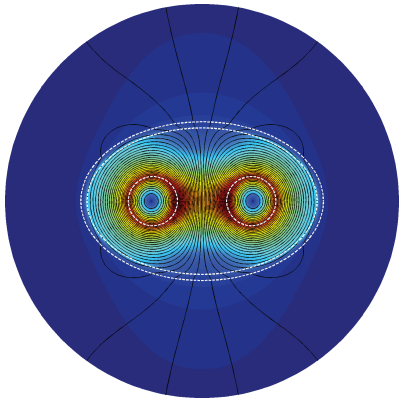


Figure 4. Flux lines and magnitude of magnetic field for an elliptic thin sheet, which is highlighted by white dashed lines.

conditions in the limit $\varepsilon \to 0$

$$e^0_{\text{ext}}(x, \pm 0) = 0.$$

Nevertheless, the limit of the external magnetic field $\mathbf{h}^0_{\text{ext}}(x, \pm 0) \cdot \mathbf{n}^\perp$ is not necessarily zero. Inserting the electric field coupling condition (12) into (8a) and comparing with the magnetic field coupling condition (14b) we conclude

$$\gamma \tanh(\gamma \tfrac{d}{2})\,(e^{1,+}_{\text{ext}} + e^{1,-}_{\text{ext}}) = \beta\,(1 - \tfrac{\gamma d}{2} \tanh(\gamma \tfrac{d}{2}))(\mathbf{h}^{0,+}_{\text{ext}} - \mathbf{h}^{0,-}_{\text{ext}}) \cdot \mathbf{n}^\perp,$$
$$\gamma \coth(\gamma \tfrac{d}{2})\,(e^{1,+}_{\text{ext}} - e^{1,-}_{\text{ext}}) = \beta\,(1 - \tfrac{\gamma d}{2} \coth(\gamma \tfrac{d}{2}))(\mathbf{h}^{0,+}_{\text{ext}} + \mathbf{h}^{0,-}_{\text{ext}}) \cdot \mathbf{n}^\perp,$$

where $e^{1,\pm}_{\text{ext}} := e^1(\cdot, \pm 0)$ and $\mathbf{h}^{0,\pm}_{\text{ext}} := \mathbf{h}^0(\cdot, \pm 0)$.

Taking the first order electric field $e_{2,1} := e^0_{\text{ext}} + e^1_{\text{ext}}$ and the zeroth order magnetic field $\mathbf{h}_{2,1} := \mathbf{h}^0_{\text{ext}}$ as approximation to the fields in the exterior of a thin sheet of thickness $\varepsilon = d$ we obtain the impedance transmission conditions ITC-2-1

$$e^+_{2,1} - e^-_{2,1} = \frac{\beta}{\gamma}\Big(\tanh\Big(\frac{\gamma d}{2}\Big) - \frac{\gamma d}{2}\Big)(\mathbf{h}^+_{2,1} \cdot \mathbf{n} + \mathbf{h}^-_{2,1} \cdot \mathbf{n}),$$
$$\mathbf{h}^+_{2,1} \cdot \mathbf{n}^\perp - \mathbf{h}^-_{2,1} \cdot \mathbf{n}^\perp = \frac{\gamma}{\beta}\,\frac{\tanh\Big(\frac{\gamma d}{2}\Big)}{1 - \frac{\gamma d}{2}\tanh\Big(\frac{\gamma d}{2}\Big)}\,(e^+_{2,1} + e^-_{2,1}),$$

which have the same form as the ITC-2-1 with different coefficients. Additional terms will be present for curved sheets [10].

E. Conditions for the external fields with $1/\varepsilon$ scaling

The first order ITCs related to $\alpha = 1$, which we call ITC-1-1, are given by [2]

$$e^+_{1,1} - e^-_{1,1} = 0,$$
$$\mathbf{h}^+_{1,1} \cdot \mathbf{n}^\perp - \mathbf{h}^-_{1,1} \cdot \mathbf{n}^\perp = -\tfrac{\sigma d}{2}(1 - \tfrac{1}{6}\mathrm{i}\omega\mu\sigma d^2)(e^+_{1,1} + e^-_{1,1}).$$

The second and third ITCs involve curvature terms and second order tangential derivatives, see [2].

IV. Discussion

We investigated the thin sheet conditions (2), the limit conditions (3) and the higher order conditions ITC-2-1 and ITC-1-1 for a thin sheet whose mid-line is an ellipse with semi-axes 60 mm and 38.7 mm and two circular live wires with opposite current direction, see Fig. 4 for an example field plot. We computed the electric and magnetic fields for the original model resolving the thin sheet as well as the impedance transmission conditions by means of finite elements of higher order. The computations performed using the numerical C++ library Concepts [9], see also `www.concepts.math.ethz.ch`, provide so accurate results that the discretisation error is negligible over the modelling error. We computed for elliptic thin sheets of thickness from 0.0244 mm up to 12.5 mm and varied the frequency to obtain skin depths in the same range. This study includes all practical combinations of very small thicknesses for very large skin depths up to very large thicknesses for very small skin depths. We use contour plots for the illustration of the modelling, see Figs. 2 and 3, in which specified error levels for the L^2-norm of the magnetic field in the exterior of the sheet allow to compare the accuracy of the different ITCs. In those contour plots we observe robustness with respect to the frequency, if for fixed sheet thickness the error decreases or reaches a saturation level for very small or very large skin depths.

The original thin sheet conditions (2) turn out to be robust with respect to the skin depth or frequency, see Fig. 3(a), which is obvious as they transform into (3a) for low frequencies and into (3c) for high frequencies, cf. [6].

The low-frequency limit conditions (3a) achieve only some accuracy if the sheet thickness is more than one or two orders smaller than the skin depth. The high-frequency limit conditions (3c) entail some accuracy if the skin depth is at least at the order of the sheet thickness.

The non-trivial limit conditions ITC-1-0, see (3b), are again robust. Their accuracy is comparable to the one of the original thin sheet conditions for large skin depths / thickness ratios, a parameter domain in which both conditions tend to PEC. The ITC-1-0 meet much higher accuracies than the thin sheet conditions if the skin depth gets relatively small, see Fig. 2(c). This obvervation is remarkable as the expression of (3b) is much simpler than the one of (2).

The proposed ITC-1-1 and ITC-2-1 are robust and get improved accuracy in comparison to the non-trivial limit and the original thin sheet conditions (see Figs. 3(b),(c)). We used for the curved sheets the conditions ITC-2-1 with curvature terms. We observe that the accuracy of ITC-2-1 is asymptotically like $O(d^2)$ independent of the skin depth and achieves accurate results even for larger sheet thicknesses. The reason that an application of ITC-2-1 results in lower errors than of the thin sheet conditions is that in the derivation of the latter the external fields at the sheet midline are matched to the internal fields on the sheet surface, whereas for ITC-2-1 the matching is based on Taylor expansions. Since their expression has the same form as the original thin sheet conditions (2) they are preferable – for low and for high frequencies. As the family ITC-1-N is based on a scaling of the skin depth like $O(\sqrt{\varepsilon})$, which exceed the thickness ε for $\varepsilon \to 0$, we cannot expect high accuracy for small skin depths-to-thickness ratios. This is visible as ITC-2-1 outperforms ITC-1-1 for small skin depths.

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