

On the inverse eigenvalue problem for T -alternating and T -palindromic matrix polynomials

Leonhard Batzke* Christian Mehl*

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Abstract

The inverse eigenvalue problem for T -alternating matrix polynomials over arbitrary algebraically closed fields of characteristic different from two is considered. The main result shows that the necessary conditions obtained in [10] for a matrix polynomial to be the Smith form of a T -alternating matrix polynomial are under mild conditions also sufficient to be the Smith form of a T -alternating matrix polynomial with invertible leading coefficient which is additionally in anti-triangular form.. In particular, this result implies that any T -alternating matrix polynomial with invertible leading coefficient is equivalent to a T -alternating matrix polynomial in anti-triangular form that has the same finite and infinite elementary divisors as the original matrix polynomial. Finally, the inverse eigenvalue problem for T -palindromic matrix polynomials is considered excluding the case that both $+1$ and -1 are eigenvalues.

Key words. Matrix polynomial, matrix pencil, Smith form, alternating matrix polynomial, palindromic matrix polynomial, triangularization, anti-triangular form.

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1 Introduction

Matrix polynomials and polynomial eigenvalue problems have been studied intensively in the last few decades. Recently, two particular topics have gained considerable interest: *inverse polynomial eigenvalue problems* and *the triangularization of matrix polynomials*. The aim of this paper is to combine these two topics with special emphasis on T -alternating matrix polynomials..

A matrix polynomial of degree k , where k is a positive integer, is an expression $P(\lambda) = \sum_{j=0}^k \lambda^j A_j$ with coefficient matrices $A_j \in \mathbb{F}^{n \times n}$ over an arbitrary field \mathbb{F} . If the characteristic of \mathbb{F} is different from two, such a $P(\lambda)$ is called T -even if $P(-\lambda) = P(\lambda)^T$,

*Institut für Mathematik, MA 4-5, Technische Universität Berlin, 10623 Berlin, Germany, Emails: batzke@math.tu-berlin.de, mehl@math.tu-berlin.de.

and it is called *T-odd* if $P(-\lambda) = -P(\lambda)^T$. Observe that these definitions are equivalent to the fact that the sequence of coefficient matrices (A_0, A_1, \dots, A_k) alternates between symmetric and skew-symmetric matrices, starting with a symmetric A_0 in the *T-even* case and with a skew-symmetric A_0 in the *T-odd* case. Therefore, the term *T-alternating* has been introduced in [15] as a hypernym for *T-even* and *T-odd* matrix polynomials.

In [9] it was observed that there exist *T-alternating* matrix polynomials that do not allow a *T-alternating* strong linearization and a detailed explanation of this effect was presented in [10] by characterizing the possible Smith forms of *T-alternating* matrix polynomials in terms of pairing properties of elementary divisors (see Section 2 for details). In particular, it was shown that those pairing conditions for the elementary divisors of a given matrix polynomial $S(\lambda)$ in Smith form were necessary and sufficient for the existence of a *T-alternating* matrix polynomial $P(\lambda)$ having $S(\lambda)$ as its Smith form, but the constructed $P(\lambda)$ may have a rather high degree resulting in a high number of infinite elementary divisors. Thus, the following question cannot be answered based on the results obtained in [10]:

Problem 1.1 *Let $S(\lambda) = \text{diag}(d_1(\lambda), \dots, d_n(\lambda))$ be a matrix polynomial in Smith form that satisfies the necessary conditions for being a Smith form of a *T-even* matrix polynomial, and assume that $\sum_{j=1}^n \deg(d_j) = nk$. Does there exist a *T-even* $n \times n$ matrix polynomial $P(\lambda)$ of degree k with invertible leading coefficient that has the Smith form $S(\lambda)$?*

An important application for inverse quadratic eigenvalue problems is the design of feedback controllers for second order systems, see [1, 16]. Since in many cases the coefficient matrices of the second order system are symmetric, the inverse symmetric quadratic eigenvalue problem has been studied intensively. Important contributions to its solution have been made in [4, 5, 7] under the additional assumption that the eigenvalues of the designed systems are semisimple. While this limitation may not be of importance in applications, it turns out to be a restriction to the solution of Problem 1.1, where the eigenvalues of the given Smith form need not be semisimple. Other techniques to tackle inverse quadratic eigenvalue problems include solvents [6] and the construction of *quasi-canonical forms* [8], but it is not clear if these techniques can easily be generalized to inverse polynomial eigenvalue problems of higher degree.

A different approach to the solution of inverse polynomial eigenvalue problems involves the *triangularization of matrix polynomials*: in [19], the authors raised the question whether any regular, complex, quadratic matrix polynomial can be transformed to a quadratic matrix polynomial in triangular form with the same finite and infinite elementary divisors. Here, *triangular form* means that all coefficient matrices of the matrix polynomial are upper triangular. The question was motivated by the lack of existence of a generalized Schur form for matrix polynomials of degree greater than one: in general, it is not possible to transform a given matrix polynomial to triangular form under strict equivalence, let alone under strict unitary equivalence.

On the other hand, it was shown in the proof of [2, Theorem 1.7] that any complex matrix polynomial $P(\lambda)$ of degree k with nonsingular leading coefficient is unimodularly

equivalent to a monic upper triangular matrix polynomial $T(\lambda)$ of degree k . Thus, in particular $P(\lambda)$ and $T(\lambda)$ have the same elementary divisors. Recently, the result hidden in the proof of [2, Theorem 1.7] has been generalized in [19] in the case of quadratic matrix polynomials by relaxing the condition of nonsingularity of the leading coefficient and allowing regular matrix polynomials. Finally, it was shown in [17] that any rectangular $n \times m$ matrix polynomial over an algebraically closed field is unimodularly equivalent to a matrix polynomial in triangular form if $n \leq m$. In particular, those results solve an inverse eigenvalue problem as it was highlighted in [17, Lemma 3.2]: *If $d_1(\lambda), \dots, d_n(\lambda)$ are monic polynomials with entries in an algebraically closed field \mathbb{F} such that $d_j(\lambda)$ divides $d_{j+1}(\lambda)$ for $j = 1, \dots, n-1$, then there exists a monic, triangular $n \times n$ matrix polynomial $P(\lambda)$ of degree k over \mathbb{F} with $d_1(\lambda), \dots, d_n(\lambda)$ as invariant polynomials if and only if $\sum_{j=1}^n \deg(d_j) = nk$.*

In the case of T -alternating polynomials, triangular forms turn out to be too restrictive. Indeed, if $P(\lambda)$ is a T -alternating matrix polynomial in triangular form, then all symmetric coefficient matrices are diagonal and all skew-symmetric coefficient matrices are zero. Instead, we aim to construct *anti-triangular forms*. Recall that a matrix $A = [a_{ij}] \in \mathbb{F}^{n \times n}$ is called anti-triangular if $a_{ij} = 0$ for all (i, j) satisfying $i + j \leq n$, see, e.g., [14].

The remainder of the paper is organized as follows. In Section 2, we compile some preliminary results that will be needed in the following. In Section 3, we state and prove the main result Theorem 3.1 which gives an affirmative answer to Problem 1.1 by constructing a T -even, anti-triangular matrix polynomial whose Smith form is the given matrix polynomial $S(\lambda)$. This result is generalized in Section 4 to related structures of matrix polynomials.

Throughout the paper, \mathbb{F} denotes an arbitrary field of characteristic different from two. By $\mathbb{F}[\lambda]^{n \times n}$, we will denote the set of $n \times n$ matrix polynomials over \mathbb{F} . Finally, R_n denotes the $n \times n$ reverse identity

$$R_n = \begin{bmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{bmatrix}.$$

2 Preliminaries

Recall that $P(\lambda), Q(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ are called *unimodularly equivalent* (short: *equivalent*) if there exist $E(\lambda), F(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ that are unimodular (i.e., having constant nonzero determinant) such that

$$Q(\lambda) = E(\lambda)P(\lambda)F(\lambda).$$

We will also denote equivalence of matrix polynomials by $P(\lambda) \sim Q(\lambda)$. In some of our results, we will need a more restrictive equivalence relation than equivalence, the so-called *unimodular alternating-congruence*.

Definition 2.1 $P(\lambda), Q(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ are called *unimodularly alternatingly-congruent* (short: *congruent*) if there exists a unimodular $E(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ such that

$$Q(\lambda) = E(\lambda)P(\lambda)E^T(-\lambda).$$

It is straightforward to check that the T -alternating structure of matrix polynomials is preserved under unimodular alternating-congruence: If $P(\lambda)$ and $E(\lambda)$ are $n \times n$ matrix polynomials and $P(\lambda)$ is T -even or T -odd, then also $E(\lambda)P(\lambda)E^T(-\lambda)$ is T -even or T -odd, respectively.

Due to this feature, we added the prefix “alternating-” in order to distinguish it properly from *unimodular congruence* transformations of the form $P(\lambda) \mapsto E(\lambda)P(\lambda)E^T(\lambda)$ introduced in [13] for the sake of preserving the structure of skew-symmetric matrix polynomials. For simplicity, we will use the term *congruence* instead of *unimodular alternating-congruence* as there will be no ambiguity in this paper.

Unfortunately, a canonical form for matrix polynomials under congruence seems not to be available yet, and hence, we will mainly use (unimodular) equivalence transformations and the *Smith form* which is the corresponding canonical form, see, e.g., [2] for details.

Theorem 2.2 (Smith form) *Let $P(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$. Then there exists a nonnegative integer r and unimodular $E(\lambda), F(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ such that*

$$E(\lambda)P(\lambda)F(\lambda) = \begin{bmatrix} D(\lambda) & 0 \\ 0 & 0 \end{bmatrix}, \quad D(\lambda) = \text{diag}(d_1(\lambda), \dots, d_r(\lambda)),$$

where $d_1(\lambda), \dots, d_r(\lambda)$ are monic and $d_j(\lambda) \mid d_{j+1}(\lambda)$ for $j = 1, \dots, r-1$. Moreover, $D(\lambda)$ and r are unique.

The polynomials $d_1(\lambda), \dots, d_r(\lambda)$ are called the *invariant polynomials* of $P(\lambda)$. They can be characterized as ratios of greatest common divisors (short: gcd’s) of minors of $P(\lambda)$. We recall that a minor of order j of $P(\lambda)$ is defined to be the determinant of a $j \times j$ submatrix of $P(\lambda)$ that is obtained by extracting j rows and j columns of $P(\lambda)$.

Theorem 2.3 *Let $P(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ have the Smith form*

$$S(\lambda) = \text{diag}(p_1(\lambda), \dots, p_r(\lambda), 0, \dots, 0).$$

Set $p_0(\lambda) \equiv 1$ and define $p_j(\lambda) \equiv 0$ if all minors of $P(\lambda)$ of order j are zero, otherwise let $p_j(\lambda)$ be the greatest common divisor (gcd) of all minors of $P(\lambda)$ of order j . Then r is the largest integer j such that $p_j(\lambda) \not\equiv 0$ and the invariant polynomials of $P(\lambda)$ are given by

$$d_j(\lambda) = \frac{p_j(\lambda)}{p_{j-1}(\lambda)}, \quad j = 1, \dots, r.$$

In order to apply this result, we will need a few lemmas involving gcd’s. The first one is well-known; a proof can be found, e.g., in [3].

Lemma 2.4 (Lemma of Bézout) *Let $p(\lambda), q(\lambda) \in \mathbb{F}[\lambda]$, then there exist polynomials $z_1(\lambda), z_2(\lambda) \in \mathbb{F}[\lambda]$ such that*

$$z_1(\lambda)p(\lambda) + z_2(\lambda)q(\lambda) = \text{gcd}\{p(\lambda), q(\lambda)\}.$$

Since the matrix polynomials focused on in this paper are T -alternating, we will frequently make use of the following properties of alternating scalar polynomials. The *parity* $\epsilon(p)$ of an alternating $p(\lambda) \in \mathbb{F}[\lambda]$ is then defined to be $\epsilon(p) := +1$ if $p(\lambda) = p(-\lambda)$ is even, and $\epsilon(p) := -1$ if $p(\lambda) = -p(-\lambda)$ is odd. The proofs of the following two lemmas can be found in [10].

Lemma 2.5 *Let $p(\lambda) \in \mathbb{F}[\lambda]$, then $\gcd\{p(\lambda), p(-\lambda)\}$ is an alternating polynomial.*

Lemma 2.6 *Let $p(\lambda) \in \mathbb{F}[\lambda]$ be divided by $d(\lambda) \in \mathbb{F}[\lambda] \setminus \{0\}$ with $\deg(d) \leq \deg(p)$ to get*

$$p(\lambda) = d(\lambda)q(\lambda) + r(\lambda), \quad \deg(r) < \deg(d).$$

If $p(\lambda)$ and $d(\lambda)$ are alternating (not necessarily with the same parity), then $q(\lambda)$ and $r(\lambda)$ are alternating as well. Moreover, $p(\lambda), r(\lambda)$, and $d(\lambda)q(\lambda)$ all three have the same parity.

A key lemma in our constructions will be the following factorization result for even scalar polynomials.

Lemma 2.7 *Let $s(\lambda) \in \mathbb{F}[\lambda]$ be even and of degree $2k$, where k is a nonnegative integer, and let the field \mathbb{F} be algebraically closed. Then there exists a monic $x(\lambda) \in \mathbb{F}[\lambda]$ of degree k such that*

$$s(\lambda) = c x(\lambda)x(-\lambda),$$

where $c \in \mathbb{F} \setminus \{0\}$. If $s(\lambda)$ itself is monic, then $c = (-1)^k$ and if $s(0) \neq 0$, then $x(\lambda)$ can be chosen such that

$$\gcd\{x(\lambda), x(-\lambda)\} = 1.$$

Proof. By [10, Lemma 4.1], $s(\lambda)$ admits the factorization

$$s(\lambda) = c \lambda^{\alpha_0} [(\lambda - \lambda_1)(\lambda + \lambda_1)]^{\alpha_1} \cdots [(\lambda - \lambda_r)(\lambda + \lambda_r)]^{\alpha_r},$$

where $\alpha_0 \in \mathbb{N}$ is even, $c \in \mathbb{F} \setminus \{0\}$, $\alpha_1, \dots, \alpha_r \in \mathbb{N} \setminus \{0\}$, and $\lambda_1, \dots, \lambda_r, -\lambda_1, \dots, -\lambda_r$ are pairwise distinct. The result then follows easily by setting

$$x(\lambda) := \lambda^{\alpha_0/2} \prod_{j=1}^r (\lambda - \lambda_j)^{\alpha_j}. \quad \square$$

The following theorems are the main results from [10]; they completely characterize the possible Smith forms of T -alternating matrix polynomials.

Theorem 2.8 (E-Smith form) *Suppose that*

$$S(\lambda) = \text{diag}(\lambda^{\alpha_1} p_1(\lambda), \lambda^{\alpha_2} p_2(\lambda), \dots, \lambda^{\alpha_r} p_r(\lambda), 0, \dots, 0) \in \mathbb{F}[\lambda]^{n \times n},$$

*where $0 \leq \alpha_1 \leq \dots \leq \alpha_r$ are nonnegative integers, all $p_j(\lambda)$ are monic with $p_j(0) \neq 0$, and $p_j(\lambda) | p_{j+1}(\lambda)$ for $j = 1, \dots, r-1$. Then $S(\lambda)$ is the Smith form of some T -**even** $n \times n$ matrix polynomial if and only if:*

1) $p_j(\lambda)$ is even for $j = 1, \dots, r$.

2) If ν is the number of **odd** exponents among $\alpha_1, \dots, \alpha_r$, then ν is an even integer. Letting $k_1 < k_2 < \dots < k_\nu$ be the positions on the diagonal of $S(\lambda)$, where these **odd** exponents α_{k_j} occur, the following properties hold:

(a) adjacency-pairing of positions:

$$k_2 = k_1 + 1, \quad k_4 = k_3 + 1, \quad \dots, \quad k_\nu = k_{\nu-1} + 1,$$

(b) equality-pairing of odd exponents:

$$\alpha_{k_2} = \alpha_{k_1}, \quad \alpha_{k_4} = \alpha_{k_3}, \quad \dots, \quad \alpha_{k_\nu} = \alpha_{k_{\nu-1}}. \quad (2.1)$$

Theorem 2.9 (O-Smith form) Suppose that

$$S(\lambda) = \text{diag}(\lambda^{\alpha_1} p_1(\lambda), \lambda^{\alpha_2} p_2(\lambda), \dots, \lambda^{\alpha_r} p_r(\lambda), 0, \dots, 0) \in \mathbb{F}[\lambda]^{n \times n},$$

where $0 \leq \alpha_1 \leq \dots \leq \alpha_r$ are nonnegative integers, all $p_j(\lambda)$ are monic with $p_j(0) \neq 0$, and $p_j(\lambda) | p_{j+1}(\lambda)$ for $j = 1, \dots, r-1$. Then $S(\lambda)$ is the Smith form of some **T-odd** $n \times n$ matrix polynomial if and only if:

1) $p_j(\lambda)$ is even for $j = 1, \dots, r$.

2) If ν is the number of **even** exponents among $\alpha_1, \dots, \alpha_r$, then ν is an even integer. Letting $k_1 < k_2 < \dots < k_\nu$ be the positions on the diagonal of $S(\lambda)$ where these **even** exponents α_{k_j} occur, the following properties hold:

(a) adjacency-pairing of positions:

$$k_2 = k_1 + 1, \quad k_4 = k_3 + 1, \quad \dots, \quad k_\nu = k_{\nu-1} + 1,$$

(b) equality-pairing of even exponents:

$$\alpha_{k_2} = \alpha_{k_1}, \quad \alpha_{k_4} = \alpha_{k_3}, \quad \dots, \quad \alpha_{k_\nu} = \alpha_{k_{\nu-1}}. \quad (2.2)$$

We will further need the following lemma that was proved in [13].

Lemma 2.10 (Exchange lemma) Let $f(\lambda), g(\lambda), h(\lambda) \in \mathbb{F}[\lambda]$ have the property that $f(\lambda)$ is relatively prime to both $g(\lambda)$ and $h(\lambda)$. Then:

$$1) \begin{bmatrix} f(\lambda)g(\lambda) & 0 \\ 0 & h(\lambda) \end{bmatrix} \sim \begin{bmatrix} g(\lambda) & 0 \\ 0 & f(\lambda)h(\lambda) \end{bmatrix};$$

$$2) \text{ for any } \ell, m \in \mathbb{N}, \text{ it is: } \begin{bmatrix} f(\lambda)^\ell g(\lambda) & 0 \\ 0 & f(\lambda)^m h(\lambda) \end{bmatrix} \sim \begin{bmatrix} f(\lambda)^m g(\lambda) & 0 \\ 0 & f(\lambda)^\ell h(\lambda) \end{bmatrix}.$$

Finally, we review the result that was obtained in the proof of [2, Theorem 1.7] for the case $\mathbb{F} = \mathbb{C}$, but the proof easily generalizes to arbitrary algebraically closed fields. For this and for the remainder of the paper, let k always denote a positive integer.

Theorem 2.11 *Let $P(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ satisfy $\deg(\det P(\lambda)) = nk$ and assume the field \mathbb{F} to be algebraically closed. Then $P(\lambda)$ is equivalent to an upper triangular matrix polynomial, whose diagonal elements have degree k .*

We note that the matrix polynomial $P(\lambda)$ in Theorem 2.11 may have off-diagonal entries of arbitrary degree, which can by a simple procedure be reduced to degree $k - 1$ or less, which is shown in [17]. However, for our purpose the statement in Theorem 2.11 is sufficient.

3 The inverse T -even polynomial eigenvalue problem

In this section, we will prove our main theorem.

Theorem 3.1 *Let the field \mathbb{F} be algebraically closed and let $S(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ be in E -Smith form as in Theorem 2.8. If $\deg(\det S(\lambda)) = nk$, then $S(\lambda)$ is equivalent to a T -even, lower anti-triangular $n \times n$ matrix polynomial of degree k .*

The proof of Theorem 3.1 will be carried out in the following subsections and proceed in two main steps:

- I) Construct a T -even, lower anti-triangular matrix polynomial such that all entries on the anti-diagonal have degree k . (The entries below the main anti-diagonal may have arbitrary degree.)
- II) Reduce the degrees of the entries in the strict lower anti-triangular part so that the resulting matrix polynomial has degree k .

We will begin by describing the procedure that will be used to carry out Step II in Subsection 3.1. In the following subsections, we will then prove Theorem 3.1 by executing Step I in special cases: in the case $n = 2$ in Subsection 3.2, in the general case of even n in Subsection 3.3 and in the case of odd n in Subsection 3.4. This covers all possible cases for n and thus completes the proof of Theorem 3.1.

3.1 Reducing the degree

In the following result, the field \mathbb{F} need not be algebraically closed.

Theorem 3.2 *Let $P(\lambda) = [p_{ij}(\lambda)]_{i,j=1}^n \in \mathbb{F}[\lambda]^{n \times n}$ be T -even and lower anti-triangular, and let its anti-diagonal elements all have degree k . Then $P(\lambda)$ is congruent to a T -even,*

lower anti-triangular matrix polynomial of degree k with anti-diagonal leading coefficient. More precisely, there exists a unimodular $E(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ such that

$$E(\lambda)P(\lambda)E(-\lambda)^T =: \check{P}(\lambda) = [\check{p}_{ij}(\lambda)]_{i,j=1}^n$$

is lower anti-triangular and has the same anti-diagonal elements as $P(\lambda)$ and all other elements have degrees not exceeding $k - 1$, i.e.,

$$\check{p}_{i,n+1-i}(\lambda) = p_{i,n+1-i}(\lambda) \quad \text{and} \quad \deg(\check{p}_{ij}) \leq k - 1 \quad \text{for } i + j > n + 1, \quad i, j = 1, \dots, n.$$

Proof. Let

$$\kappa := \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n - 1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

We will construct $\check{P}(\lambda)$ in two steps. In the first step, the degree of the elements in the diagonal positions $(n - \kappa + 1, n - \kappa + 1), \dots, (n, n)$ will be reduced to $k - 1$ or less, in the second step, all other elements in the strict lower anti-triangular part will be considered.

Step 1: reducing the diagonal elements. For each $i = n - \kappa + 1, \dots, n$ we aim to reduce the degree of the element in the (i, i) position to $k - 1$ or less. We proceed by an induction argument and assume that the degrees of the diagonal elements in the positions $(n - \kappa + 1, n - \kappa + 1), \dots, (i - 1, i - 1)$ have already been reduced resulting in a matrix polynomial with anti-diagonal elements identical to those of $P(\lambda)$. For simplicity, we will again denote this matrix polynomial by $P(\lambda)$. Let $\tilde{p}(\lambda) := p_{i,n-i+1}(\lambda)$ be the anti-diagonal element in the same row as $p_{ii}(\lambda)$. Since $P(\lambda)$ is T -even, we obtain that

$$g(\lambda) := \gcd\{p_{i,n-i+1}(\lambda), p_{n-i+1,i}(\lambda)\} = \gcd\{\tilde{p}(\lambda), \tilde{p}(-\lambda)\}$$

is alternating by Lemma 2.5 and clearly $\deg(g) \leq k$. Assume $\deg(p_{ii}) \geq k$ (otherwise there is nothing to do) and let $p_{ii}(\lambda) = g(\lambda)q(\lambda) + r(\lambda)$, where $\deg(r) < \deg(g) \leq k$. By Lemma 2.6, the polynomial $q(\lambda)$ is alternating and of the same parity as $g(\lambda)$, and $r(\lambda)$ is even. By Lemma 2.4, there exist polynomials $z_1(\lambda), z_2(\lambda)$ such that

$$z_1(\lambda)\tilde{p}(\lambda) + z_2(\lambda)\tilde{p}(-\lambda) = g(\lambda).$$

Setting $z(\lambda) := -q(\lambda)(z_1(\lambda) + \epsilon(q)z_2(-\lambda))/2$, we obtain

$$z(\lambda)\tilde{p}(\lambda) + z(-\lambda)\tilde{p}(-\lambda) = -\frac{1}{2}q(\lambda)g(\lambda) - \frac{1}{2}q(-\lambda)g(-\lambda) = -q(\lambda)g(\lambda).$$

Thus, adding the $z(\lambda)$ -multiple of the $(n + i - 1)$ st column to the i th column and then the $z(-\lambda)$ -multiple of the $(n + i - 1)$ st row to the i th row (note that this is a congruence transformation), the element in the (i, i) position is changed to

$$p_{ii}(\lambda) + z(\lambda)\tilde{p}(\lambda) + z(-\lambda)\tilde{p}(-\lambda) = r(\lambda),$$

which is even and of degree $k - 1$ or less. Observe that no anti-diagonal elements and no diagonal elements other than $p_{ii}(\lambda)$ have been changed. Thus, after κ steps, we obtain a

matrix polynomial again denoted by $P(\lambda)$, which is T -even, lower anti-triangular, and its diagonal elements $p_{ii}(\lambda)$ for $i = n - \kappa + 1, \dots, n$ have degree $k - 1$ or less.

Step 2: reducing the off-diagonal elements. Similar to Step 1, we will use an induction argument and reduce the degree of the elements in the (i, j) and (j, i) positions simultaneously for $j = n - i + 2, \dots, i - 1$ and for $i = \kappa + 2, \dots, n$. Thus, assume that we are currently considering the elements in the (i, j) and (j, i) positions and denote the matrix polynomial resulting from the previous steps again by $P(\lambda)$. For simplicity, set $\hat{p}(\lambda) := p_{ij}(\lambda)$ and $\tilde{p}(\lambda) := p_{i, n-i+1}(\lambda)$. Since $P(\lambda)$ is T -even, we obtain

$$p_{ji}(\lambda) = \hat{p}(-\lambda) \quad \text{and} \quad p_{n-i+1, i}(\lambda) = \tilde{p}(-\lambda).$$

If $\deg(\hat{p}) \geq k$ (otherwise there is nothing to do), let $\hat{p}(\lambda) = \tilde{p}(\lambda)q(\lambda) + r(\lambda)$, where $\deg(r) < \deg(\tilde{p}) = k$. Adding the $(-q(-\lambda))$ -multiple of the $(n - i + 1)$ st row to the j th row and the $(-q(\lambda))$ -multiple of the $(n - i + 1)$ st column to the j th column (note that this is a congruence transformation), we obtain a matrix polynomial again denoted by $P(\lambda)$ with elements in the (i, j) and (j, i) positions now given by

$$\hat{p}(\lambda) - q(\lambda)\tilde{p}(\lambda) = r(\lambda) \quad \text{and} \quad \hat{p}(-\lambda) - q(-\lambda)\tilde{p}(-\lambda) = r(-\lambda),$$

respectively. These polynomials have degrees less than or equal to $k - 1$. Furthermore, $P(\lambda)$ is T -even and lower anti-triangular and all elements on the anti-diagonal, as well as those on the diagonal and those previously reduced are unchanged. Completing the induction, we finally obtain a matrix polynomial in the desired form that is congruent to the matrix polynomial we started with. \square

3.2 The 2×2 case

For the remainder of this paper we will assume the additional condition that the field \mathbb{F} is algebraically closed.

Let $S(\lambda) = \text{diag}(\lambda^\alpha p_1(\lambda), \lambda^\beta p_2(\lambda))$ be a possible Smith form of a T -even matrix polynomial, where $p_1(0), p_2(0) \neq 0$. Then by Theorem 2.8, both $p_1(\lambda)$ and $p_2(\lambda)$ are even polynomials, and the exponents α and β are either both even (including the case $\alpha = \beta = 0$) or they are both odd and equal. Furthermore, since $p_1(\lambda)$ divides $p_2(\lambda)$, the latter can be factorized as $p_2(\lambda) = p_1(\lambda)s(\lambda)$, where by Lemma 2.6 also $s(\lambda)$ is even. The following lemma therefore covers Theorem 3.1 in the case $n = 2$.

Lemma 3.3 *Let $S(\lambda) \in \mathbb{F}[\lambda]^{2 \times 2}$ be in Smith form*

$$S(\lambda) = \text{diag}(\lambda^\alpha p(\lambda), \lambda^\beta p(\lambda)s(\lambda)),$$

where $0 \leq \alpha \leq \beta$ and where $p(\lambda)$ and $s(\lambda)$ are monic, even polynomials with $p(0), s(0) \neq 0$, and assume $\deg(\det S(\lambda)) = 2k$.

- 1) If $\beta \neq \alpha$, but $\beta - \alpha$ is even, then $S(\lambda)$ is equivalent to a lower anti-triangular matrix polynomial of degree k of the form

$$\begin{bmatrix} 0 & (-\lambda)^{(\alpha+\beta)/2}p(\lambda)x(-\lambda) \\ \lambda^{(\alpha+\beta)/2}p(\lambda)x(\lambda) & \lambda^\alpha p(\lambda) \end{bmatrix}, \quad (3.1)$$

where $x(\lambda)x(-\lambda)(-1)^{\deg(x)} = s(\lambda)$. Moreover, the matrix polynomial (3.1) is T -even if α (and therefore also β) is even.

- 2) If $\beta = \alpha$, then $S(\lambda)$ is equivalent to a T -even, lower anti-diagonal matrix polynomial of degree k of the form

$$\begin{bmatrix} 0 & (-\lambda)^\alpha p(\lambda)x(-\lambda) \\ \lambda^\alpha p(\lambda)x(\lambda) & 0 \end{bmatrix}, \quad (3.2)$$

where $x(\lambda)x(-\lambda)(-1)^{\deg(x)} = s(\lambda)$.

Proof. Since $s(\lambda)$ is even and monic with $s(0) \neq 0$, it follows from Lemma 2.7 that it can be factorized as $s(\lambda) = x(\lambda)x(-\lambda)(-1)^{\deg(x)}$ for some polynomial $x(\lambda)$ satisfying $\gcd\{x(\lambda), x(-\lambda)\} = 1$.

We will first prove 1). Observe that

$$\begin{aligned} \begin{bmatrix} \lambda^\alpha p(\lambda) & 0 \\ 0 & \lambda^\beta p(\lambda)s(\lambda) \end{bmatrix} &\sim \begin{bmatrix} \lambda^\alpha p(\lambda) & 0 \\ \lambda^{(\alpha+\beta)/2}p(\lambda)x(-\lambda)(-1)^{\deg(x)+1} & \lambda^\beta p(\lambda)s(\lambda) \end{bmatrix} \\ &\sim \begin{bmatrix} \lambda^\alpha p(\lambda) & \lambda^{(\alpha+\beta)/2}p(\lambda)x(\lambda) \\ \lambda^{(\alpha+\beta)/2}p(\lambda)x(-\lambda)(-1)^{\deg(x)+1} & 0 \end{bmatrix}. \end{aligned}$$

Indeed, the first equivalence follows by adding the $(\lambda^{(\beta-\alpha)/2}x(-\lambda)(-1)^{\deg(x)+1})$ -multiple of the first row of $S(\lambda)$ to its second row, and the second equivalence follows by adding the $(\lambda^{(\beta-\alpha)/2}x(\lambda))$ -multiple of the first column of the resulting matrix polynomial to its second column. The latter matrix polynomial is easily seen to be equivalent to the desired shape (3.1) by multiplying the second row with a suitable power of -1 followed by a row and column permutation. It is now straightforward to check that the constructed matrix polynomial has degree k and that it is T -even if α is even.

Next let us prove 2), thus assuming that $\beta = \alpha$. As in 1), it follows that $S(\lambda)$ is equivalent to

$$\begin{bmatrix} 0 & (-\lambda)^\alpha p(\lambda)x(-\lambda) \\ \lambda^\alpha p(\lambda)x(\lambda) & \lambda^\alpha p(\lambda) \end{bmatrix}.$$

Since $\gcd\{x(\lambda), x(-\lambda)\} = 1$, it follows from Lemma 2.4 that there exist polynomials $z_1(\lambda), z_2(\lambda)$ with

$$z_1(\lambda) \lambda^\alpha p(\lambda)x(\lambda) + z_2(\lambda) (-\lambda)^\alpha p(\lambda)x(-\lambda) = \lambda^\alpha p(\lambda).$$

Subtracting the $z_1(\lambda)$ -multiple of the first column from the second column and the $z_2(\lambda)$ -multiple of the first row from the second row we obtain the desired form (3.2). \square

3.3 The case of even n

In this subsection, we prove Theorem 3.1 in the case that n is even but greater than two.

Lemma 3.4 *Let $n \in \mathbb{N}$ be even and let $S(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ be in E -Smith form as in Theorem 2.8 satisfying $\deg(\det S(\lambda)) = nk$. Then $S(\lambda)$ is equivalent to a T -even, lower anti-triangular matrix polynomial of degree k .*

Proof. We will proceed with the proof in two steps. In the first step we will transform $S(\lambda)$ to a lower anti-triangular matrix polynomial that is T -even, and in the second step we will reduce its degree to k .

Step 1: reduction to anti-triangular form. Let

$$S(\lambda) = \text{diag}(\lambda^{\alpha_1} p_1(\lambda), \lambda^{\alpha_2} p_2(\lambda), \dots, \lambda^{\alpha_n} p_n(\lambda)),$$

where $p_j(0) \neq 0$ for $j = 1, \dots, n$. Further, let $k_1 < \dots < k_\nu$ denote the indices j for which α_j is odd and let $l_1 < \dots < l_\mu$ denote the indices j for which α_j is even. Then by Theorem 2.8, the integer ν is even and thus with n also μ must be even. Set $\kappa := n/2$ and let (m_1, m_2, \dots, m_n) be a permutation of $\{1, 2, \dots, n\}$ with the following two properties:

- (a) either $(m_{2j-1}, m_{2j}) = (k_{2i-1}, k_{2i})$ for an $i \in \{1, \dots, \nu/2\}$, or $(m_{2j-1}, m_{2j}) = (l_{2i-1}, l_{2i})$ for an $i \in \{1, \dots, \mu/2\}$ for $j = 1, \dots, \kappa$;
- (b) $\alpha_{m_{2j-3}} + \alpha_{m_{2j-2}} \leq \alpha_{m_{2j-1}} + \alpha_{m_{2j}}$ for $j = 2, \dots, \kappa$.

Such a permutation exists, since it can be obtained by ordering the pairs (k_{2i-1}, k_{2i}) and (l_{2i-1}, l_{2i}) with respect to the arithmetic mean of the corresponding exponents. By Lemma 2.10, $S(\lambda)$ is equivalent to the matrix polynomial

$$\text{diag}(\lambda^{\alpha_{m_1}} p_1(\lambda), \lambda^{\alpha_{m_2}} p_2(\lambda), \dots, \lambda^{\alpha_{m_n}} p_n(\lambda)). \quad (3.3)$$

This step has permuted the α_j 's on the diagonal of $S(\lambda)$ in such a way that a partition of (3.3) into diagonal 2×2 blocks groups together adjacent pairs of odd exponents that are equal by Theorem 2.8. The ascending order of the α_j 's as in the Smith form is no longer in place, but we do have an ascending ordering of the 2×2 blocks on the diagonal with respect to their exponents' arithmetic mean $(\alpha_{m_{2j-1}} + \alpha_{m_{2j}})/2$. Thus, a 2×2 block from such a partitioning of (3.3) has the form

$$\text{diag}(\lambda^{\alpha_{m_{2j-1}}} p_{2j-1}(\lambda), \lambda^{\alpha_{m_{2j}}} p_{2j}(\lambda)), \quad (3.4)$$

where $\alpha_{m_{2j-1}}$ and $\alpha_{m_{2j}}$ are either both even or they are odd and equal. Moreover, $p_{2j-1}(\lambda)$ and $p_{2j}(\lambda)$ are monic, even polynomials with $p_{2j-1}(\lambda) \mid p_{2j}(\lambda)$ and $p_{2j-1}(0), p_{2j}(0) \neq 0$. Hence, applying Lemma 3.3 we find that each 2×2 block (3.4) is equivalent to a T -even matrix polynomial of the form (3.1) or (3.2), respectively, so that $S(\lambda)$ is equivalent to a T -even matrix polynomial of the form

$$\begin{bmatrix} 0 & q_1(-\lambda) \\ q_1(\lambda) & r_1(\lambda) \end{bmatrix} \oplus \begin{bmatrix} 0 & q_2(-\lambda) \\ q_2(\lambda) & r_2(\lambda) \end{bmatrix} \oplus \dots \oplus \begin{bmatrix} 0 & q_\kappa(-\lambda) \\ q_\kappa(\lambda) & r_\kappa(\lambda) \end{bmatrix}.$$

Applying a row and column permutation, we obtain the T -even matrix polynomial

$$\begin{bmatrix} 0 & T(-\lambda)^T \\ T(\lambda) & R(\lambda) \end{bmatrix}, \quad \text{where} \quad T(\lambda) = \begin{bmatrix} 0 & & q_\kappa(\lambda) \\ & \ddots & \\ q_1(\lambda) & & 0 \end{bmatrix} \quad (3.5)$$

is anti-diagonal and $R(\lambda) = \text{diag}(r_1(\lambda), \dots, r_\kappa(\lambda))$ is diagonal. Observe that by construction (property (b) of the permutation), we have $q_j(\lambda) \mid q_{j+1}(\lambda)$ for $j = 1, \dots, \kappa - 1$.

Step 2: reduction to degree k . Since the polynomials $q_j(\lambda)$ are monic and form a divisibility chain, we find that $R_\kappa T(\lambda)$ is in Smith form. Further, we have $\deg(\det S(\lambda)) = \deg(\det T(\lambda)) \cdot \deg(\det T(-\lambda)^T)$ and therefore $\deg(\det T(\lambda)) = \kappa k$. Then Theorem 2.11 applied to $R_\kappa T(\lambda)$ implies that there are unimodular $\kappa \times \kappa$ matrix polynomials $E(\lambda), F(\lambda)$ such that $E(\lambda) R_\kappa T(\lambda) F(\lambda)$ is an upper triangular matrix polynomial of degree k . Hence, setting $\tilde{E}(\lambda) := R_\kappa E(\lambda) R_\kappa$ and $M(\lambda) := \tilde{E}(\lambda) T(\lambda) F(\lambda)$, we obtain that

$$\begin{bmatrix} F(-\lambda)^T & 0 \\ 0 & \tilde{E}(\lambda) \end{bmatrix} \begin{bmatrix} 0 & T(-\lambda)^T \\ T(\lambda) & R(\lambda) \end{bmatrix} \begin{bmatrix} F(\lambda) & 0 \\ 0 & \tilde{E}(-\lambda)^T \end{bmatrix} = \begin{bmatrix} 0 & M(-\lambda)^T \\ M(\lambda) & \tilde{E}(\lambda) R(\lambda) \tilde{E}(-\lambda)^T \end{bmatrix}$$

is a T -even, lower anti-triangular matrix polynomial that is equivalent to $S(\lambda)$. Moreover, its anti-diagonal entries all have degree k . Thus, by Theorem 3.2, it is congruent to a T -even, lower anti-triangular matrix polynomial of degree k . \square

3.4 The case of odd n

The proof in the case that n is odd is more involved and needs the two technical Lemmas 6.1 and 6.2 as prerequisites. We will defer the statements and proofs of these lemmas to the appendix.

Lemma 3.5 *Let $n \in \mathbb{N}$ be odd and let $S(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ be in E -Smith form as in Theorem 2.8 satisfying $\deg(\det S(\lambda)) = nk$. Then $S(\lambda)$ is equivalent to a T -even, lower anti-triangular matrix polynomial of degree k .*

Proof. We first highlight that k is even because by [10, Lemma 3.4] the determinant of a T -even matrix polynomial is an even scalar polynomial, so it has even degree. Thus, as n is odd, k must be even. Similar to the proof of Lemma 3.4, we will then proceed in two steps.

Step 1: reduction to anti-triangular form. Again, let

$$S(\lambda) = \text{diag}(\lambda^{\alpha_1} p_1(\lambda), \lambda^{\alpha_2} p_2(\lambda), \dots, \lambda^{\alpha_n} p_n(\lambda)),$$

where $p_j(0) \neq 0$ for $j = 1, \dots, n$. Further, let $k_1 < \dots < k_\nu$ denote the indices j for which α_j is odd and let $\ell_1 < \dots < \ell_\mu$ denote the indices j for which α_j is even. Then, by Theorem 2.8 the integer ν is even and thus with n also μ is odd. Therefore, $\mu \geq 1$. By row and column permutations, $S(\lambda)$ is equivalent to

$$\text{diag}(\lambda^{\alpha_{\ell_1}} p_{\ell_1}(\lambda), \lambda^{\alpha_1} p_1(\lambda), \dots, \lambda^{\alpha_{\ell_1-1}} p_{\ell_1-1}(\lambda), \lambda^{\alpha_{\ell_1+1}} p_{\ell_1+1}(\lambda), \dots, \lambda^{\alpha_n} p_n(\lambda)).$$

Set $\kappa := (n-1)/2$ and let $(m_1, m_2, \dots, m_{n-1})$ be a permutation of $\{1, 2, \dots, n\} \setminus \{\ell_1\}$ with the following two properties:

- (a) either $(m_{2j-1}, m_{2j}) = (k_{2i-1}, k_{2i})$ for an $i \in \{1, \dots, \nu/2\}$, or $(m_{2j-1}, m_{2j}) = (\ell_{2i}, \ell_{2i+1})$ for an $i \in \{1, \dots, (\mu-1)/2\}$ for $j = 1, \dots, \kappa$;
- (b) $\alpha_{m_{2j-3}} + \alpha_{m_{2j-2}} \leq \alpha_{m_{2j-1}} + \alpha_{m_{2j}}$ for $j = 2, \dots, \kappa$.

Again, such a permutation can be obtained by ordering the pairs (k_{2i-1}, k_{2i}) and (ℓ_{2i}, ℓ_{2i+1}) with respect to the arithmetic mean of the corresponding exponents. By Lemma 2.10, $S(\lambda)$ is equivalent to the matrix polynomial

$$\text{diag}(q(\lambda), \lambda^{\alpha_{m_1}} \widehat{p}_1(\lambda), \lambda^{\alpha_{m_2}} \widehat{p}_2(\lambda), \dots, \lambda^{\alpha_{m_{n-1}}} \widehat{p}_{n-1}(\lambda)), \quad (3.6)$$

where we defined $q(\lambda) := \lambda^{\alpha_{\ell_1}} p_{\ell_1}(\lambda)$ and for each $j = 1, \dots, n-1$

$$\widehat{p}_j(\lambda) := \begin{cases} p_j(\lambda) & \text{if } j < \ell_1, \\ p_{j+1}(\lambda) & \text{if } j \geq \ell_1. \end{cases}$$

Thus, we can partition the lower $(n-1) \times (n-1)$ principal submatrix of (3.6) into diagonal 2×2 blocks of the form

$$\text{diag}(\lambda^{\alpha_{m_{2j-1}}} \widehat{p}_{2j-1}(\lambda), \lambda^{\alpha_{m_{2j}}} \widehat{p}_{2j}(\lambda)), \quad (3.7)$$

where $\alpha_{m_{2j-1}}$ and $\alpha_{m_{2j}}$ are either both even or they are odd and equal. Further, the 2×2 blocks are ordered ascendingly with respect to the arithmetic mean $(\alpha_{m_{2j-1}} + \alpha_{m_{2j}})/2$. Moreover, $\widehat{p}_{2j-1}(\lambda)$ and $\widehat{p}_{2j}(\lambda)$ are monic, even polynomials with $\widehat{p}_{2j-1}(\lambda) | \widehat{p}_{2j}(\lambda)$ and $\widehat{p}_{2j-1}(0), \widehat{p}_{2j}(0) \neq 0$. Thus, we can apply Lemma 3.3 and find that each 2×2 block (3.7) is equivalent to a T -even matrix polynomial of the form (3.1) or (3.2), respectively, so that $S(\lambda)$ is equivalent to the T -even matrix polynomial

$$\widehat{S}(\lambda) := [q(\lambda)] \oplus \begin{bmatrix} 0 & q_1(-\lambda) \\ q_1(\lambda) & r_1(\lambda) \end{bmatrix} \oplus \begin{bmatrix} 0 & q_2(-\lambda) \\ q_2(\lambda) & r_2(\lambda) \end{bmatrix} \oplus \dots \oplus \begin{bmatrix} 0 & q_\kappa(-\lambda) \\ q_\kappa(\lambda) & r_\kappa(\lambda) \end{bmatrix}.$$

We now claim that the matrix polynomial $\widehat{S}(\lambda)$ is equivalent to a T -even matrix polynomial of the form

$$\widehat{P}(\lambda) = \begin{bmatrix} 0 & 0 & T(-\lambda)^T \\ 0 & \widehat{q}(\lambda) & a(-\lambda)^T \\ T(\lambda) & a(\lambda) & X(\lambda) \end{bmatrix}, \quad (3.8)$$

where $\deg(\widehat{q}) = k$. If this is indeed true, then the proof can be finished similarly to the case of even n : clearly, the determinant of (3.8) has degree nk . Since \widehat{q} has degree k , it follows that $\det T(\lambda)$ has degree κk . Thus, Theorem 2.11 implies that there are unimodular $\kappa \times \kappa$ matrix polynomials $E(\lambda), F(\lambda)$ such that $E(\lambda) R_\kappa T(\lambda) F(\lambda)$ is an upper triangular matrix

polynomial of degree k . Hence, setting $\tilde{E}(\lambda) := R_\kappa E(\lambda) R_\kappa$ and $M(\lambda) := \tilde{E}(\lambda) T(\lambda) F(\lambda)$, we obtain that

$$\begin{aligned} & \begin{bmatrix} F(-\lambda)^T & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tilde{E}(\lambda) \end{bmatrix} \begin{bmatrix} 0 & 0 & T(-\lambda)^T \\ 0 & \hat{q}(\lambda) & a(-\lambda)^T \\ T(\lambda) & a(\lambda) & X(\lambda) \end{bmatrix} \begin{bmatrix} F(\lambda) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tilde{E}(-\lambda)^T \end{bmatrix} \\ = & \begin{bmatrix} 0 & 0 & M(-\lambda)^T \\ 0 & \hat{q}(\lambda) & a(-\lambda)^T \tilde{E}(-\lambda)^T \\ M(\lambda) & E(\lambda) a(\lambda) & \tilde{E}(\lambda) X(\lambda) \tilde{E}(-\lambda)^T \end{bmatrix} =: P(\lambda) \end{aligned}$$

is a lower anti-triangular matrix polynomial that is equivalent to $S(\lambda)$. Further, all its anti-diagonal entries have degree k and since $P(\lambda)$ is congruent to the T -even matrix polynomial $\hat{P}(\lambda)$, it is T -even itself. Hence, by Theorem 3.2, $S(\lambda)$ is equivalent to a T -even, lower anti-triangular matrix polynomial of degree k , whose anti-diagonal entries have degree k .

Thus, it remains to prove the claim that $\hat{S}(\lambda)$ is equivalent to a T -even matrix polynomial of the form (3.8). We distinguish three subcases depending on the degree of the first entry $q(\lambda)$.

Subcase 1a): $\deg(q) = k$.

By row and column permutations, $\hat{S}(\lambda)$ is equivalent to

$$\begin{bmatrix} 0 & 0 & T(-\lambda)^T \\ 0 & q(\lambda) & 0 \\ T(\lambda) & 0 & X(\lambda) \end{bmatrix},$$

where $R_\kappa T(\lambda) = \text{diag}(q_1(\lambda), \dots, q_\kappa(\lambda))$ and where $X(\lambda) = \text{diag}(r_\kappa(\lambda), \dots, r_1(\lambda))$ is T -even. This matrix polynomial is of the desired form (3.8).

Subcase 1b): $\deg(q) < k$.

By row and column permutations, $\hat{S}(\lambda)$ is equivalent to

$$Q(\lambda) := \begin{bmatrix} 0 & 0 & 0 & 0 & \hat{T}(-\lambda) \\ 0 & q(\lambda) & 0 & 0 & 0 \\ 0 & 0 & 0 & q_\kappa(-\lambda) & 0 \\ 0 & 0 & q_\kappa(\lambda) & r_\kappa(\lambda) & 0 \\ \hat{T}(\lambda) & 0 & 0 & 0 & \hat{X}(\lambda) \end{bmatrix}.$$

where $R_{\kappa-1} \hat{T}(\lambda) = \text{diag}(q_1(\lambda), \dots, q_{\kappa-1}(\lambda))$ and where $\hat{X}(\lambda) = \text{diag}(r_{\kappa-1}(\lambda), \dots, r_1(\lambda))$. Note that $Q(\lambda)$ is T -even and lower anti-triangular except for the middle 3×3 block. Undoing the transformations on the lower principal 2×2 submatrix of that block, we obtain

$$\left([q(\lambda)] \oplus \begin{bmatrix} 0 & q_\kappa(-\lambda) \\ q_\kappa(\lambda) & r_\kappa(\lambda) \end{bmatrix} \right) \sim \text{diag}(\lambda^{\alpha_{\ell_1}} p_{\ell_1}(\lambda), \lambda^{\alpha_{m_{n-2}}} \hat{p}_{n-2}(\lambda), \lambda^{\alpha_{m_{n-1}}} \hat{p}_{n-1}(\lambda)). \quad (3.9)$$

We claim that the latter matrix polynomial is in Smith form. Indeed, all its entries are monic and we observe that $\ell_1 < n$ because otherwise

$$k > \deg(q) = \alpha_{\ell_1} + \deg(p_{\ell_1}) = \alpha_n + \deg(p_{\alpha_n}) \geq \dots \geq \alpha_1 + \deg(p_1)$$

in contradiction to $\deg(\det Q(\lambda)) = nk$. Thus $\ell_1 < n$ and therefore $\alpha_{\ell_1} \leq \alpha_n$. If α_n is odd, then $\alpha_{\ell_1} < \alpha_{n-1} = \alpha_n$ and thus clearly $(m_{n-2}, m_{n-1}) = (n-1, n)$ and $\ell_1 < n-1$. But then $\widehat{p}_{n-2}(\lambda) = p_{n-1}(\lambda)$ and $\widehat{p}_{n-1}(\lambda) = p_n(\lambda)$ so that the diagonal entries in the right hand side of (3.9) form a divisibility chain. If on the other hand α_n is even, then $\mu \geq 3$ as μ is odd, so $\ell_1 < \dots < \ell_{\mu-1} < \ell_\mu = n$ implies $\ell_1 < n-1$. Consequently $\widehat{p}_{n-2}(\lambda) = p_{n-1}(\lambda)$ and $\widehat{p}_{n-1}(\lambda) = p_n(\lambda)$ and thus $p_{\ell_1}(\lambda) \mid \widehat{p}_{n-2}(\lambda) \mid \widehat{p}_{n-1}(\lambda)$ holds. Moreover, we have

$$\alpha_{\ell_1} \leq \frac{\alpha_{\ell_{\mu-1}} - \alpha_{\ell_\mu}}{2} = \frac{\alpha_{m_{n-2}} - \alpha_{m_{n-1}}}{2}$$

as $(m_{n-2}, m_{n-1}) = (\ell_{\mu-1}, \ell_\mu)$, so again the diagonal entries of the right hand side of (3.9) form a divisibility chain. Hence, writing

$$\lambda^{\alpha_{\ell_1}} p_{\ell_1}(\lambda) = \lambda^\alpha p(\lambda), \quad \lambda^{\alpha_{m_{n-2}}} \widehat{p}_{n-2}(\lambda) = \lambda^\beta p(\lambda) s(\lambda), \quad \lambda^{\alpha_{m_{n-1}}} \widehat{p}_{n-1}(\lambda) = \lambda^\gamma p(\lambda) s(\lambda) t(\lambda),$$

we can observe that $\alpha + \deg(p) = \alpha_{\ell_1} + \deg(p_{\ell_1}) = \deg(q) < k$ and

$$k < \deg(p) + \deg(s) + \frac{\beta + \gamma + \deg(t)}{2}$$

because otherwise

$$\begin{aligned} k &\geq \deg(p) + \deg(s) + \frac{\beta + \gamma + \deg(t)}{2} = \frac{\alpha_{m_{n-2}} + \alpha_{m_{n-1}} + \deg(\widehat{p}_{n-2}) + \deg(\widehat{p}_{n-1})}{2} \\ &= \deg(q_\kappa) \geq \deg(q_{\kappa-1}) \geq \dots \geq \deg(q_1) \end{aligned}$$

would with $k > \deg(q)$ contradict $\deg(\det Q(\lambda)) = nk$. Moreover, k and $\alpha = \alpha_{\ell_1}$ are both even. Therefore, we can apply Lemma 6.1 from the appendix, which implies that there are unimodular 3×3 matrix polynomials $\widehat{E}(\lambda), \widehat{F}(\lambda)$ such that

$$\widehat{E}(\lambda) \begin{bmatrix} 0 & 0 & q_\kappa(-\lambda) \\ 0 & q(\lambda) & 0 \\ q_\kappa(\lambda) & 0 & r_\kappa(\lambda) \end{bmatrix} \widehat{F}(\lambda) = \begin{bmatrix} 0 & 0 & \widetilde{q}(-\lambda) \\ 0 & \widehat{q}(\lambda) & a(-\lambda) \\ \widetilde{q}(\lambda) & a(\lambda) & \widehat{r}(\lambda) \end{bmatrix}$$

is a T -even matrix polynomial where $\widehat{q}(\lambda)$ has degree k . Thus,

$$(I_{\kappa-1} \oplus \widehat{E}(\lambda) \oplus I_{\kappa-1}) Q(\lambda) (I_{\kappa-1} \oplus \widehat{F}(\lambda) \oplus I_{\kappa-1})$$

has the desired form (3.8).

Subcase 1c): $\deg(q) > k$.

By row and column permutations, $\widehat{S}(\lambda)$ is equivalent to

$$Q(\lambda) := \begin{bmatrix} 0 & 0 & 0 & 0 & \widehat{T}(-\lambda) \\ 0 & 0 & q_1(-\lambda) & 0 & 0 \\ 0 & q_1(\lambda) & r_1(\lambda) & 0 & 0 \\ 0 & 0 & 0 & q(\lambda) & 0 \\ \widehat{T}(\lambda) & 0 & 0 & 0 & \widehat{X}(\lambda) \end{bmatrix}.$$

where $R_{\kappa-1}\widehat{T}(\lambda) = \text{diag}(q_2(\lambda), \dots, q_\kappa(\lambda))$ and where $\widehat{X}(\lambda) = \text{diag}(r_\kappa(\lambda), \dots, r_2(\lambda))$. Similarly to the previous case, $Q(\lambda)$ is T -even and lower anti-triangular except for the middle 3×3 block. Also

$$\left(\begin{bmatrix} 0 & q_1(-\lambda) \\ q_1(\lambda) & r_1(\lambda) \end{bmatrix} \oplus [q(\lambda)] \right) \sim \text{diag}(\lambda^{\alpha_{m_1}}\widehat{p}_1(\lambda), \lambda^{\alpha_{m_2}}\widehat{p}_2(\lambda), \lambda^{\alpha_{\ell_1}}p_{\ell_1}(\lambda)), \quad (3.10)$$

where the latter matrix polynomial is in Smith form. Indeed, $\deg q(\lambda) > k$ implies $\ell_1 > 1$ (otherwise contradicting $\deg(\det Q(\lambda)) = nk$). Hence, α_1 is odd and we obtain $\alpha_1 = \alpha_2$ which implies $m_1 = 1$, $m_2 = 2$, and $\ell_1 > 2$. Thus, the diagonal entries in the right hand side of (3.10) form a divisibility chain and they are clearly monic. Therefore, writing

$$\lambda^{\alpha_{m_1}}\widehat{p}_1(\lambda) = \lambda^\alpha p(\lambda), \quad \lambda^{\alpha_{m_2}}\widehat{p}_2(\lambda) = \lambda^\alpha p(\lambda)s(\lambda), \quad \lambda^{\alpha_{\ell_1}}p_{\ell_1}(\lambda) = \lambda^\beta p(\lambda)s(\lambda)t(\lambda),$$

where α is odd and β is even, we observe that

$$\beta + \deg(p) + \deg(s) + \deg(t) = \alpha_{\ell_1} + \deg(p_{\ell_1}) = \deg(q) > k.$$

Furthermore, we have

$$k > \alpha + \deg(p) + \frac{\deg(s)}{2}$$

because otherwise

$$k \leq \alpha + \deg(p) + \frac{\deg(s)}{2} = \alpha_{m_1} + \frac{\deg(\widehat{p}_1) + \deg(\widehat{p}_2)}{2} = \deg(q_1) \leq \deg(q_2) \leq \dots \leq \deg(q_\kappa)$$

would with $k < \deg(q)$ contradict $\deg(\det Q(\lambda)) = nk$. Thus, the hypotheses of Lemma 6.2 from the appendix are satisfied and so there exist unimodular 3×3 matrix polynomials $\widehat{E}(\lambda), \widehat{F}(\lambda)$ such that

$$\widehat{E}(\lambda) \begin{bmatrix} 0 & 0 & q_\kappa(-\lambda) \\ 0 & q(\lambda) & 0 \\ q_\kappa(\lambda) & 0 & r_\kappa(\lambda) \end{bmatrix} \widehat{F}(\lambda) = \begin{bmatrix} 0 & 0 & \widetilde{q}(-\lambda) \\ 0 & \widehat{q}(\lambda) & a(-\lambda) \\ \widetilde{q}(\lambda) & a(\lambda) & \widehat{r}(\lambda) \end{bmatrix}$$

is T -even, where $\widehat{q}(\lambda)$ has degree k . Hence,

$$(I_{\kappa-1} \oplus \widehat{E}(\lambda) \oplus I_{\kappa-1})Q(\lambda)(I_{\kappa-1} \oplus \widehat{F}(\lambda) \oplus I_{\kappa-1})$$

has the desired form (3.8), which concludes the remaining part of the proof. \square

4 Inverse polynomial eigenvalue problems for related structures

It is straightforward to generalize the results derived in the last section to T -odd matrix polynomials.

Theorem 4.1 *Let $S(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ be in O -Smith form as in Theorem 2.9 satisfying $\deg(\det S(\lambda)) = nk$. Then $S(\lambda)$ is equivalent to a T -odd, lower anti-triangular matrix polynomial of degree k .*

Proof. Comparing the conditions of Theorems 2.8 and 2.9, we observe that if $S(\lambda)$ is in Smith form satisfying the conditions of Theorem 2.9, then $\widehat{S}(\lambda) := \lambda S(\lambda)$ is also in Smith form and satisfies the conditions of Theorem 2.8. Moreover, $\deg(\widehat{S}(\lambda)) = n(k+1)$. Hence, by Theorem 3.1, there exist unimodular matrix polynomials $E(\lambda), F(\lambda)$ such that

$$E(\lambda)\widehat{S}(\lambda)F(\lambda) = \lambda E(\lambda)S(\lambda)F(\lambda)$$

is a T -even, lower anti-triangular matrix polynomial of degree $k+1$. Then $E(\lambda)S(\lambda)F(\lambda)$ is the desired T -odd, lower anti-triangular matrix polynomial of degree k . \square

It turns out that Theorem 3.1 and Theorem 4.1 do not only provide a solution to the inverse T -alternating polynomial eigenvalue problem, but they also show that any T -alternating matrix polynomial with invertible leading coefficient is equivalent to a T -alternating matrix polynomial in a particular form.

Corollary 4.2 *Let $P(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ be a T -alternating matrix polynomial of degree k with invertible leading coefficient. Then $P(\lambda)$ is equivalent to a T -alternating matrix polynomial of degree k that is in lower anti-triangular form and has either the leading coefficient R_n or*

$$\begin{bmatrix} 0 & R_{n/2} \\ -R_{n/2} & 0 \end{bmatrix}.$$

Proof. By [10], the Smith form of $P(\lambda)$ satisfies the hypotheses of Theorem 3.1 or Theorem 4.1 and it follows $P(\lambda)$ is equivalent to a T -alternating matrix polynomial $T(\lambda)$ of degree k (also with invertible leading coefficient), which is in lower anti-triangular form. From the statement of Theorem 3.2, we observe that we can assume its leading coefficient to be anti-diagonal. Then, applying an appropriate congruence transformation $T(\lambda) \mapsto M^T T(\lambda) M$, the leading coefficient can be reduced to either R_n or

$$\begin{bmatrix} 0 & R_{n/2} \\ -R_{n/2} & 0 \end{bmatrix},$$

depending on it being symmetric or skew-symmetric. \square

Next, let us turn to palindromic matrix polynomials. Recall that a matrix polynomial $P(\lambda) = \sum_{j=0}^k \lambda^j A_j$ (where leading coefficients A_k, A_{k-1}, \dots are allowed to be zero) is called *T -palindromic* (or *T -palindromic of type +1*) if $\text{rev } P(\lambda) = P(\lambda)^T$ and that it is called *T -anti-palindromic* (or *T -palindromic of type -1*) if $\text{rev } P(\lambda) = -P(\lambda)^T$, where

$$\text{rev } P(\lambda) = \sum_{j=0}^k \lambda^j A_{k-j}$$

is the *reversal* of $P(\lambda)$. We will further call $P(\lambda)$ *palindromic* if it is either T -palindromic or T -anti-palindromic. In [11, Theorem 7.6] necessary conditions for a Smith form to be that of a palindromic matrix polynomial were presented consisting of pairing conditions for the

elementary divisors associated with $+1$ and -1 that are parallel to the pairing conditions for the elementary divisors associated with zero in Theorem 2.8 and Theorem 2.9. However, these conditions were not sufficient. In particular, it was highlighted that the problem of finding conditions that are both necessary and sufficient for a matrix polynomial in Smith form to be the Smith form of a palindromic matrix polynomial remains an open problem. Based on Theorem 3.1, we are able to get one step closer to solving this open problem.

As a tool, we will consider the Cayley transformations of matrix polynomials that were used in [9] to relate palindromic and T -alternating matrix polynomials. Recall that for a matrix polynomial $P(\lambda)$ the two Cayley transformations with pole at -1 and $+1$ are given by

$$\mathcal{C}_{-1}(P)(\mu) := (\mu + 1)^k P\left(\frac{\mu - 1}{\mu + 1}\right) \quad \text{and} \quad \mathcal{C}_{+1}(P)(\mu) := (1 - \mu)^k P\left(\frac{1 + \mu}{1 - \mu}\right), \quad (4.1)$$

respectively. The following table is extracted from [9] and adapted to our focus. It displays the correspondence in the structures of the matrix polynomial $P(\lambda)$ and its Cayley transforms.

Table 4.1: Cayley transforms of structured matrix polynomials.

$P(\lambda)$	$\mathcal{C}_{-1}(P)(\mu)$		$\mathcal{C}_{+1}(P)(\mu)$	
	k even	k odd	k even	k odd
T -palindromic	T -even	T -odd	T -even	
T -anti-palindromic	T -odd	T -even	T -odd	
T -even	T -palindromic		T -palindromic	T -anti-palindromic
T -odd	T -anti-palindromic		T -anti-palindromic	T -palindromic

In order to keep the discussion short, we do not review in detail the necessary conditions for a matrix polynomial to be in P -Smith form, instead we refer the reader to [11, Theorem 7.6]. Then, we obtain the following result parallel to Theorem 3.1.

Theorem 4.3 *Let $S(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ be regular and in P -Smith form as in [11, Theorem 7.6], i.e., it is the possible Smith form of a T -palindromic or T -anti-palindromic matrix polynomial. Further, assume $\deg(\det S(\lambda)) = nk - m$, where m is the algebraic multiplicity of the eigenvalue $\lambda_0 = 0$.*

If $+1$ and -1 are not both eigenvalues of $S(\lambda)$, then $S(\lambda)$ is equivalent to a lower anti-triangular matrix polynomial of degree k which is T -palindromic or T -anti-palindromic, respectively.

Proof. We will only prove the theorem in the case that $S(\lambda)$ is the possible Smith form of a T -palindromic matrix polynomial and that -1 is not an eigenvalue of $S(\lambda)$. The other cases, i.e., $S(\lambda)$ is the possible Smith form of a T -anti-palindromic matrix polynomial and/or $+1$ is not an eigenvalue of $S(\lambda)$ can be shown analogously.

Then, by [11, Theorem 7.6] we have

$$S(\lambda) = \text{diag}(\lambda^{\alpha_1}(\lambda - 1)^{\beta_1}p_1(\lambda), \dots, \lambda^{\alpha_n}(\lambda - 1)^{\beta_n}p_n(\lambda)), \quad (4.2)$$

where $0 \leq \alpha_1 \leq \dots \leq \alpha_n$, $0 \leq \beta_1 \leq \dots \leq \beta_n$, and $p_j(-1), p_j(0), p_j(1) \neq 0$ for $j = 1, \dots, n$. Furthermore, all $p_j(\lambda)$ are palindromic (of type +1) and all odd exponents β_j occur in equal pairs in adjacent diagonal positions.

Now, consider the list of all elementary divisors of $S(\lambda)$ including infinite elementary divisors of degrees $\alpha_1, \dots, \alpha_n$. (More precisely, the degrees of the infinite elementary divisors in this list are exactly the nonzero entries of $(\alpha_1, \dots, \alpha_n)$ as elementary divisors have positive degree.) Then, because of $\sum_{j=1}^n \alpha_j = m$, the sum of the degrees of all (finite and infinite) elementary divisors in this list is nk . Hence, by [17, Section 5] this list of elementary divisors is realizable by an $n \times n$ matrix polynomial $Q(\lambda)$ of degree k , i.e., the Smith form of $Q(\lambda)$ is $S(\lambda)$ and its infinite elementary divisors have the degrees $\alpha_1, \dots, \alpha_n$.

Then, the Cayley transformation \mathcal{C}_{+1} transforms $Q(\lambda)$ to the matrix polynomial $\widehat{Q}(\lambda) := \mathcal{C}_{+1}(Q)(\lambda)$ of degree k that does not have infinity as an eigenvalue, in particular its leading coefficient is invertible. Let

$$\widehat{S}(\lambda) = \text{diag}(\lambda^{\widetilde{\beta}_1}\widetilde{p}_1(\lambda), \dots, \lambda^{\widetilde{\beta}_n}\widetilde{p}_n(\lambda))$$

be the Smith form of $\widehat{Q}(\lambda)$, where $0 \leq \widetilde{\beta}_1 \leq \dots \leq \widetilde{\beta}_n$ and $\widetilde{p}_j(0) \neq 0$ for $j = 1, \dots, n$. Then $\widehat{S}(\lambda)$ is in E-Smith form as in Theorem 2.8.

Indeed, since all $p_j(\lambda)$ from (4.2) are palindromic, it follows from [11, Corollary 5.9] that for any $\lambda_0 \in \mathbb{F} \setminus \{-1, 0, 1\}$, the elementary divisors of $Q(\lambda)$ associated with λ_0 and λ_0^{-1} occur in pairs, i.e., if $(\lambda - \lambda_0)^{\gamma_1}, \dots, (\lambda - \lambda_0)^{\gamma_\nu}$ are the elementary divisors associated with λ_0 , then $(\lambda - \lambda_0^{-1})^{\gamma_1}, \dots, (\lambda - \lambda_0^{-1})^{\gamma_\nu}$ are the elementary divisors associated with λ_0^{-1} . Further, by construction the elementary divisors of $Q(\lambda)$ at $\lambda_0 = 0$ are paired to those at ∞ both having the degrees $\alpha_1, \dots, \alpha_n$.

By [18, Theorem 3.4] (see also [12]), the Cayley transform \mathcal{C}_{+1} transports this pairing to pairs $(\mu_0, -\mu_0)$, where $\mu_0 = (\lambda_0 - 1)/(\lambda_0 + 1) \in \mathbb{F} \setminus \{0\}$, i.e., the degrees of the elementary divisors of $\widehat{S}(\lambda)$ associated with μ_0 and $-\mu_0$ are given by $\gamma_1, \dots, \gamma_\nu$ if $\lambda_0 \neq 0$ and by $\alpha_1, \dots, \alpha_n$ otherwise. Hence, it is straightforward that $\widetilde{p}_1(\lambda), \dots, \widetilde{p}_n(\lambda)$ are necessarily even. Finally, it follows from [18, Theorem 3.4] that $\widetilde{\beta}_j = \beta_j$ for $j = 1, \dots, n$, of which all odd exponents occur in equal pairs in adjacent diagonal positions. Therefore, $\widehat{S}(\lambda)$ is indeed in E-Smith form as in Theorem 2.8.

Moreover, $\deg(\det \widehat{S}(\lambda)) = \deg(\det \widehat{Q}(\lambda)) = nk$ as $\widehat{Q}(\lambda)$ is regular and does not have the eigenvalue ∞ . Thus, by Theorem 3.1 there exist unimodular matrix polynomials $E(\lambda), F(\lambda)$ such that $\widehat{P}(\lambda) := E(\lambda)\widehat{Q}(\lambda)F(\lambda)$ is a T -even, lower anti-triangular matrix polynomial of degree k . Now applying the inverse Cayley transform \mathcal{C}_{-1} , we obtain that $P(\lambda) := \mathcal{C}_{-1}(\widehat{P})(\lambda)$ is a T -palindromic matrix polynomial, which is in anti-triangular form. Clearly, the Smith form of $P(\lambda)$ is $S(\lambda)$ (using again [18, Theorem 3.4]) which concludes the proof. \square

5 Conclusion

We have studied the inverse T -alternating and T -palindromic polynomial eigenvalue problem over arbitrary algebraically closed fields of characteristic different from two. In particular, we have developed sufficient conditions under which an $n \times n$ matrix polynomial is the Smith form of a T -alternating $n \times n$ matrix polynomial with nonsingular leading coefficient of degree k . The analogous problem for T -palindromic matrix polynomials was considered in the case that not both $+1$ and -1 are eigenvalues. Additionally, the constructed matrix polynomials are in lower anti-triangular form. It remains an open problem to consider the inverse T -alternating eigenvalue problem for the case that not only finite elementary divisors, but also infinite elementary divisors are prescribed – then the techniques developed in this paper cannot be applied. Similarly, the inverse T -palindromic eigenvalue problem remains unsolved if both $+1$ and -1 are prescribed eigenvalues.

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Appendix

In the appendix, we will state and prove two technical lemmas that are needed in the proof of Theorem 3.5. We recall that \mathbb{F} is an algebraically closed field of characteristic different from two.

Lemma 6.1 *Let $S(\lambda) \in \mathbb{F}[\lambda]^{3 \times 3}$ be in Smith form*

$$S(\lambda) = \text{diag}(\lambda^\alpha p(\lambda), \lambda^\beta p(\lambda)s(\lambda), \lambda^\gamma p(\lambda)s(\lambda)t(\lambda)),$$

where $0 \leq \alpha \leq \beta \leq \gamma$ and where $p(\lambda), s(\lambda), t(\lambda)$ are monic, even polynomials with $p(0), s(0), t(0) \neq 0$. Assume that α is even and that either both β and γ are even or both are odd and equal. If $k \in \mathbb{N}$ is even such that

$$\alpha + \deg(p) < k < \deg(p) + \deg(s) + \frac{\beta + \gamma + \deg(t)}{2}, \quad (6.1)$$

then $S(\lambda)$ is equivalent to a T -even, lower anti-triangular matrix polynomial, whose middle anti-diagonal entry has degree k .

Proof. By hypothesis, $k - \alpha - \deg(p)$ is even and by (6.1) it is bounded by

$$k - \alpha - \deg(p) \leq \gamma - \alpha + \deg(s) + \deg(t).$$

Thus, let $\ell, \widehat{\ell}, \widetilde{\ell} \in \mathbb{N}$ satisfy

$$k - \alpha - \deg(p) = 2(\ell + \widehat{\ell} + \widetilde{\ell}) \quad \text{and} \quad 2\ell \leq \gamma - \alpha, \quad 2\widehat{\ell} \leq \deg(s), \quad 2\widetilde{\ell} \leq \deg(t). \quad (6.2)$$

Since \mathbb{F} is algebraically closed, we can choose two monic and even polynomials $\widehat{s}(\lambda)$ of degree $(\deg(s) - 2\widehat{\ell})$ and $\widehat{t}(\lambda)$ of degree $2\widetilde{\ell}$ such that $\widehat{s}(\lambda) \mid s(\lambda)$ and $\widehat{t}(\lambda) \mid t(\lambda)$. Then by Lemmas 2.6 and 2.7 there exist polynomials $x(\lambda), y(\lambda)$ and constants $c_x, c_y \in \mathbb{F} \setminus \{0\}$ such that

$$s(\lambda) = c_x \widehat{s}(\lambda)x(\lambda)x(-\lambda) \quad \text{and} \quad t(\lambda) = c_y \widehat{t}(\lambda)y(\lambda)y(-\lambda). \quad (6.3)$$

Note that $\deg(x) = \widehat{\ell}$ and that

$$n := \frac{\beta + \gamma}{2} - \alpha - \ell \geq \frac{\beta + \gamma}{2} - \alpha - \frac{\gamma - \alpha}{2} = \frac{\beta - \alpha}{2} \geq 0. \quad (6.4)$$

Moreover, let

$$\widehat{\beta} := 2\lceil \beta/2 \rceil, \quad m := \min\{2\ell, \widehat{\beta} - \alpha\}, \quad \text{and} \quad M := \max\{2\ell, \widehat{\beta} - \alpha\} - m.$$

Thus, $\widehat{\beta} = \beta + 1$ if β is odd and $\widehat{\beta} = \beta$ if β is even, additionally m and M are both even. We further define

$$w_1(\lambda) := \begin{cases} \widehat{t}(\lambda) & \text{if } m = \widehat{\beta} - \alpha, \\ -c_x \widehat{s}(\lambda) & \text{else,} \end{cases} \quad \text{and} \quad w_2(\lambda) := \begin{cases} -c_x \widehat{s}(\lambda) & \text{if } m = \widehat{\beta} - \alpha, \\ \widehat{t}(\lambda) & \text{else,} \end{cases}$$

which implies

$$\lambda^{2\ell} \widehat{t}(\lambda) - c_x \lambda^{\widehat{\beta} - \alpha} \widehat{s}(\lambda) = \lambda^m (\lambda^M w_1(\lambda) + w_2(\lambda)). \quad (6.5)$$

By construction $\lambda^M w_1(\lambda) + w_2(\lambda)$ is an even polynomial, and hence by Lemma 2.7 there exists a polynomial $b(\lambda)$ and a constant $c \in \mathbb{F} \setminus \{0\}$ such that

$$\lambda^M w_1(\lambda) + w_2(\lambda) = c b(\lambda) b(-\lambda). \quad (6.6)$$

We go on to show that $S(\lambda)$ is equivalent to the matrix polynomial

$$\begin{bmatrix} 0 & 0 & (-\lambda)^{\alpha+n} p(\lambda) \widehat{s}(\lambda) x(-\lambda) y(-\lambda) \\ 0 & \lambda^{\alpha+2\ell} p(\lambda) \widehat{t}(\lambda) x(\lambda) x(-\lambda) & (-\lambda)^{\alpha+m/2} p(\lambda) x(-\lambda) b(\lambda) \\ \lambda^{\alpha+n} p(\lambda) \widehat{s}(\lambda) x(\lambda) y(\lambda) & \lambda^{\alpha+m/2} p(\lambda) x(\lambda) b(-\lambda) & \lambda^\alpha c^{-1} (-1)^{m/2} p(\lambda) \end{bmatrix} \quad (6.7)$$

by proving that the Smith form of (6.7) is $S(\lambda)$, which is computed using Theorem 2.3. We immediately find $p_1(\lambda) = \lambda^\alpha p(\lambda)$; to compute $p_2(\lambda)$ in particular the minor corresponding

to the lower right 2×2 submatrix of (6.7) is needed. Using (6.6), (6.5), and (6.3), and we compute that it is given by

$$d(\lambda) := c^{-1}c_x (-1)^{m/2} \lambda^{\alpha+\widehat{\beta}} p(\lambda)^2 s(\lambda).$$

We can now obtain $p_2(\lambda)$ as follows, where the 2×2 minors of (6.7) are ordered lexicographically and powers of -1 are ignored:

$$\begin{aligned} p_2(\lambda) = \gcd \{ & 0, 0, \lambda^{2\alpha+2\ell+n} p(\lambda)^2 \widehat{t}(\lambda) x(\lambda) x(-\lambda)^2 \widehat{s}(\lambda) y(-\lambda), \\ & 0, \lambda^{2\alpha+2n} p(\lambda)^2 \widehat{s}(\lambda)^2 x(\lambda) y(\lambda) x(-\lambda) y(-\lambda), \\ & \lambda^{2\alpha+n+m/2} p(\lambda)^2 x(\lambda) b(-\lambda) \widehat{s}(\lambda) x(-\lambda) y(-\lambda), \\ & \lambda^{2\alpha+2\ell+n} p(\lambda)^2 \widehat{s}(\lambda) x(\lambda)^2 y(\lambda) \widehat{t}(\lambda) x(-\lambda), \\ & \lambda^{2\alpha+n+m/2} p(\lambda)^2 \widehat{s}(\lambda) x(\lambda) y(\lambda) x(-\lambda) b(\lambda), d(\lambda) \}. \end{aligned} \quad (6.8)$$

In order to show $p_2(\lambda) = \lambda^{\alpha+\beta} p(\lambda)^2 s(\lambda)$, observe that all polynomials in (6.8) are divisible by $p(\lambda)^2 s(\lambda)$ because of (6.3). Since $d(\lambda)$ is equal to $\lambda^{\alpha+\widehat{\beta}} p(\lambda)^2 s(\lambda)$ up to a scalar factor and powers of λ and the polynomial $p(\lambda)^2 s(\lambda)$ are relatively prime, it only remains to investigate the occurring exponents of λ . Hence, we show that $2\alpha + n + m/2$, $2\alpha + 2\ell + n$, and $2\alpha + 2n$ are greater than or equal to $\alpha + \beta$. By the choice of ℓ and n it follows that

$$2\alpha + 2\ell + n = \alpha + \ell + \frac{\beta + \gamma}{2} \geq \alpha + \frac{\beta + \gamma}{2} \geq \alpha + \beta.$$

Next, we observe

$$\ell \leq \frac{\gamma - \alpha}{2} \quad \Rightarrow \quad \frac{\beta + \gamma}{2} - \alpha - n \leq \frac{\gamma - \alpha}{2} \quad \Rightarrow \quad \frac{\beta - \alpha}{2} \leq n,$$

which implies $2\alpha + 2n \geq \alpha + \beta$. To achieve the estimate $2\alpha + n + m/2 \geq \alpha + \beta$, we consider both possible cases for m . If $m = \widehat{\beta} - \alpha$, then

$$2\alpha + n + \frac{m}{2} \geq 2\alpha + \frac{\beta - \alpha}{2} + \frac{\widehat{\beta} - \alpha}{2} \geq \alpha + \beta,$$

else we have $m = 2\ell$ and:

$$2\alpha + n + \frac{m}{2} = 2\alpha + n + \ell = 2\alpha + \frac{\beta + \gamma}{2} - \alpha \geq \alpha + \beta.$$

In the case that β is even, this proves the desired form of $p_2(\lambda)$ since then $d(\lambda)$ contains the factor $\lambda^{\alpha+\widehat{\beta}} = \lambda^{\alpha+\beta}$. Otherwise, we still have to show that $\lambda^{\alpha+\beta+1}$ does not divide all 2×2 minors in (6.8). But if β is odd, by hypothesis $\beta = \gamma$ holds and we obtain

$$\ell \leq \frac{\gamma - \alpha}{2} = \frac{\beta - \alpha}{2} < \frac{\widehat{\beta} - \alpha}{2}.$$

Hence, in this case $m = 2\ell$ always holds true and it is

$$2\alpha + n + \frac{m}{2} = 2\alpha + n + \ell = 2\alpha + \frac{\gamma + \beta}{2} - \alpha = \alpha + \beta,$$

from which the desired form of $p_2(\lambda)$ follows as well. Finally, $p_3(\lambda)$ is the normalized determinant of (6.7):

$$p_3(\lambda) = \gcd \left\{ \lambda^{3\alpha+2n+2\ell} p(\lambda)^3 \widehat{s}(\lambda)^2 \widehat{t}(\lambda) x(\lambda)^2 x(-\lambda)^2 y(\lambda) y(-\lambda) \right\} = \lambda^{\alpha+\beta+\gamma} p(\lambda)^3 s(\lambda)^2 t(\lambda),$$

where we used (6.4) and (6.3) to achieve the last equality. The invariant polynomials of (6.7) are thus given by:

$$d_1(\lambda) = \lambda^\alpha p(\lambda), \quad d_2(\lambda) = \lambda^\beta p(\lambda) s(\lambda), \quad d_3(\lambda) = \lambda^\gamma p(\lambda) s(\lambda) t(\lambda).$$

and hence $S(\lambda)$ is the Smith form of (6.7). Furthermore, (6.7) is T -even and its middle anti-diagonal element is of the degree

$$\alpha + 2\ell + \deg(p) + \deg(\widehat{t}) + 2\deg(x) = \alpha + \deg(p) + 2(\ell + \widehat{\ell} + \widetilde{\ell}) = k,$$

where we used $\deg(\widehat{t}) = 2\widetilde{\ell}$, $\deg(x) = \widehat{\ell}$, and (6.2). \square

Lemma 6.2 *Let $S(\lambda) \in \mathbb{F}[\lambda]^{3 \times 3}$ be in Smith form*

$$S(\lambda) = \text{diag} \left(\lambda^\alpha p(\lambda), \lambda^\alpha p(\lambda) s(\lambda), \lambda^\beta p(\lambda) s(\lambda) t(\lambda) \right),$$

with $0 < \alpha < \beta$, where α is odd and β is even, and where $p(\lambda), s(\lambda), t(\lambda)$ are monic, even polynomials with $p(0), s(0), t(0) \neq 0$. If $k \in \mathbb{N}$ is even such that

$$\alpha + \deg(p) + \frac{\deg(s)}{2} < k < \beta + \deg(p) + \deg(s) + \deg(t), \quad (6.9)$$

then $S(\lambda)$ is equivalent to a T -even, lower anti-triangular matrix polynomial, whose middle anti-diagonal entry has degree k .

Proof. The proof proceeds similar to the one of Lemma 6.1. By hypothesis, the integer $\beta + \deg(p) + \deg(s) + \deg(t) - k$ is even and by (6.9) it is bounded by

$$\begin{aligned} & \beta + \deg(p) + \deg(s) + \deg(t) - k \\ & \leq \beta + \deg(p) + \deg(s) + \deg(t) - \alpha - \deg(p) - \frac{\deg(s)}{2} - 1 \\ & \leq \beta - \alpha - 1 + \deg(s) + \deg(t). \end{aligned}$$

Thus, let $\ell, \widehat{\ell}, \widetilde{\ell} \in \mathbb{N}$ satisfy

$$\beta + \deg(p) + \deg(s) + \deg(t) - k = 2(\ell + \widehat{\ell} + \widetilde{\ell}) \quad (6.10)$$

and $2\ell \leq \beta - \alpha - 1$, $2\widehat{\ell} \leq \deg(s)$, $2\widetilde{\ell} \leq \deg(t)$. Since \mathbb{F} is algebraically closed, we can choose two monic and even polynomials $\widehat{s}(\lambda)$ of degree $2\widehat{\ell}$ and $\widehat{t}(\lambda)$ of degree $(\deg(t) - 2\widetilde{\ell})$, such that $\widehat{s}(\lambda) \mid s(\lambda)$ and $\widehat{t}(\lambda) \mid t(\lambda)$. Then by Lemmas 2.6 and 2.7 there exist polynomials $x(\lambda), y(\lambda)$ and constants $c_x, c_y \in \mathbb{F} \setminus \{0\}$ such that

$$s(\lambda) = c_x \widehat{s}(\lambda) x(\lambda) x(-\lambda) \quad \text{and} \quad t(\lambda) = c_y \widehat{t}(\lambda) y(\lambda) y(-\lambda). \quad (6.11)$$

Note that $\deg(y) = \widetilde{\ell}$ and that

$$n := \beta - \alpha - 2\ell \geq 0 \quad (6.12)$$

is odd. We now consider the even polynomial $-\lambda^{n+1}\widehat{t}(\lambda) + c_x \widehat{s}(\lambda)$, by Lemma 2.7 there exists a polynomial $b(\lambda)$ and a constant $c \in \mathbb{F} \setminus \{0\}$ such that

$$-\lambda^{n+1}\widehat{t}(\lambda) + c_x \widehat{s}(\lambda) = c b(\lambda) b(-\lambda). \quad (6.13)$$

We will now show that $S(\lambda)$ is equivalent to the matrix polynomial

$$\begin{bmatrix} 0 & 0 & (-\lambda)^{\alpha+\ell} p(\lambda) \widehat{s}(\lambda) x(-\lambda) y(-\lambda) \\ 0 & \lambda^{\alpha+n} p(\lambda) \widehat{t}(\lambda) x(\lambda) x(-\lambda) & -\lambda^\alpha p(\lambda) x(-\lambda) b(\lambda) \\ \lambda^{\alpha+\ell} p(\lambda) \widehat{s}(\lambda) x(\lambda) y(\lambda) & \lambda^\alpha p(\lambda) x(\lambda) b(-\lambda) & c^{-1} \lambda^{\alpha+1} p(\lambda) \end{bmatrix}, \quad (6.14)$$

and again applying Theorem 2.3, we immediately find $p_1(\lambda) = \lambda^\alpha p(\lambda)$. Moreover,

$$\begin{aligned} p_2(\lambda) = \gcd \{ & 0, 0, \lambda^{2\alpha+\ell+n} p(\lambda)^2 \widehat{t}(\lambda) x(\lambda) x(-\lambda)^2 \widehat{s}(\lambda) y(-\lambda), \\ & 0, \lambda^{2\alpha+2\ell} p(\lambda)^2 \widehat{s}(\lambda)^2 x(\lambda) y(\lambda) x(-\lambda) y(-\lambda), \\ & \lambda^{2\alpha+\ell} p(\lambda)^2 x(\lambda) b(-\lambda) \widehat{s}(\lambda) x(-\lambda) y(-\lambda), \\ & \lambda^{2\alpha+\ell+n} p(\lambda)^2 \widehat{s}(\lambda) x(\lambda)^2 y(\lambda) \widehat{t}(\lambda) x(-\lambda), \\ & \lambda^{2\alpha+\ell} p(\lambda)^2 \widehat{s}(\lambda) x(\lambda) y(\lambda) x(-\lambda) b(\lambda), c^{-1} \lambda^{2\alpha} p(\lambda)^2 s(\lambda) \}, \end{aligned}$$

where again the 2×2 minors of (6.14) are given in lexicographical order, powers of -1 are ignored, and the latter minor (the determinant of the lower principal 2×2 submatrix) has been simplified using (6.13) and (6.11). Because of (6.11), clearly all above minors are divisible by $\lambda^{2\alpha} p(\lambda)^2 s(\lambda)$, hence we obtain $p_2(\lambda) = \lambda^{2\alpha} p(\lambda)^2 s(\lambda)$. Finally, $p_3(\lambda)$ is the normalized determinant of (6.14):

$$p_3(\lambda) = \gcd \{ \lambda^{3\alpha+2\ell+n} p(\lambda)^3 \widehat{s}(\lambda)^2 \widehat{t}(\lambda) x(\lambda)^2 x(-\lambda)^2 y(\lambda) y(-\lambda) \} = \lambda^{2\alpha+\beta} p(\lambda)^3 s(\lambda)^2 t(\lambda),$$

where the last equality follows from (6.12) and (6.11). The invariant polynomials of the matrix polynomial (6.14) are thus given by:

$$d_1(\lambda) = \lambda^\alpha p(\lambda), \quad d_2(\lambda) = \lambda^\alpha p(\lambda) s(\lambda), \quad d_3(\lambda) = \lambda^\beta p(\lambda) s(\lambda) t(\lambda),$$

i.e., $S(\lambda)$ is the Smith form of (6.14). Furthermore, (6.14) is T -even and its middle anti-diagonal element has the degree

$$\alpha + n + \deg(p) + \deg(\widehat{t}) + 2 \deg(x) = \beta - 2\ell + \deg(p) + \deg(t) - 2\widetilde{\ell} + \deg(s) - 2\widehat{\ell} = k,$$

where we inserted the degrees of $\widehat{t}(\lambda)$ and $x(\lambda)$, as well as (6.12) and (6.10). \square