

# Shape Optimization for a Sharp Interface Model of Distortion Compensation

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## Abstract

We study a mechanical equilibrium problem for a material consisting of two components with different densities, which allows to change the outer shape by changing the interface between the subdomains. We formulate the shape design problem of compensating unwanted workpiece changes by controlling the interface, employ regularity results for transmission problems for a rigorous derivation of optimality conditions based on the speed method, and conclude with some numerical results based on a spline approximation of the interface.

## 1 Introduction

By definition, distortion means undesired alterations in workpiece size and shape, which may happen as a side effect at some stage in the manufacturing chain. In that case an additional final step has to be added to the manufacturing chain, i.e., the elimination of distortion by mechanical surface finishing. It has been shown that this leads to severe economic losses within machine, automotive, and transmission industry [Thoben et al., 2002]. Recently, a new strategy has been developed, which allows the elimination of distortion already during heat treatment [Schüttenberg et al., 2005], the last step in many manufacturing chains anyway.

Without going into details the cause why such a heat treatment might lead to a change in geometry are the occurring solid-solid phase transitions during such a heat treatment. They may lead to a microstructure consisting of phases with different densities, resulting in internal stresses along phase boundaries and also to macroscopic changes in geometry. Distortion compensation then means finding a desired phase mixture such that the resulting internal stresses and accompanying changes in geometry compensate the distortion and hence lead to the desired workpiece size and shape.

Assuming that no rate effects occur during cooling, i.e., neglecting transformation-induced plasticity [Chelminski et al., 2008], one can tackle this problem mathematically in a hybrid

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approach. In the first step the optimal microstructure for distortion compensation is computed solving a shape design problem subject to a stationary mechanical equilibrium problem. In the second step an optimal cooling strategy is computed to realize this microstructure. While the latter has been studied extensively, see, e.g., [Hömborg and Volkwein, 2003, Hömborg and Kern, 2009] the goal of this paper is to develop a novel approach to compute an optimal microstructure or phase mixture to compensate for distortion.

In contrast to [Chelminski et al., 2013], where a phasefield approach to distortion compensation is taken, here we assume that the workpiece domain  $D \subset \mathbf{R}^d$  consists of a microstructure with two phases in the domains  $\Omega \subset D$  and  $D \setminus \bar{\Omega}$ , separated by a sharp interface. For instance one might think of these two phases as having been created from one parent phase during a heat treatment. To distinguish between the subdomains we introduce the characteristic function  $\chi = \chi_\Omega$  of the set  $\Omega$ , which is equal to 1 for  $x \in \Omega$  and 0 otherwise.

Now assume the workpiece to be in equilibrium. Then the stress tensor  $\sigma$  satisfies

$$-\operatorname{div} \sigma = 0, \quad \text{in } D \tag{1.1}$$

$$\sigma n = 0, \quad \text{in } \Gamma^N \tag{1.2}$$

$$\mathbf{u} = 0, \quad \text{in } \Gamma_0 \tag{1.3}$$

with  $\bar{\Gamma}^N \cup \bar{\Gamma}_0 = \partial D$ . According to Hooke's law only elastic strains contribute to the stress, so in the case of small deformations we obtain

$$\sigma = A(\varepsilon(\mathbf{u}) - \tilde{\varepsilon}), \tag{1.4}$$

with the stiffness tensor  $A$ , the internal strain  $\tilde{\varepsilon}$  and the linearized overall strain

$$\varepsilon(\mathbf{u}) = \frac{1}{2}(D\mathbf{u} + (D\mathbf{u})^T).$$

In general, the stiffness might be different in both subdomains, hence we make the ansatz

$$A = A_\chi(x) := \chi(x)A_1 + (1 - \chi(x))A_2. \tag{1.5}$$

The main reason for the occurrence of internal stresses lies in the different densities of the two subdomains. Thus we make an analogous mixture ansatz, i.e.

$$\tilde{\varepsilon} = \tilde{\varepsilon}_\chi(x) := \chi(x)\tilde{\varepsilon}_1 + (1 - \chi(x))\tilde{\varepsilon}_2,$$

and assume in addition isotropy, i.e.

$$A_i \tilde{\varepsilon}_i = \beta_i(x)I,$$

where  $I$  is the identity matrix. Then the constitutive relation reads as

$$\sigma_\chi(x) = A_\chi \varepsilon(\mathbf{u}) - \beta_\chi I, \tag{1.6}$$

with

$$\beta_\chi(x) := \chi(x)\beta_1 + (1 - \chi(x))\beta_2. \tag{1.7}$$

To motivate this modelling assumption, one might think that (1.1) describes the steady state of an isotropic homogeneous linear thermoelastic body after cooling from a reference

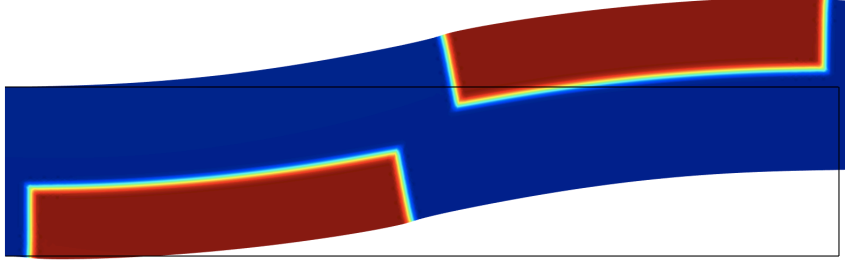


Figure 1: Deformation of a rectangular reference domain caused by subdomains with different densities.

temperature  $\theta_{ref}$  to the asymptotic one,  $\theta_\infty$ . In that case the internal stress corresponds to the asymptotic linear thermoelastic stress, which can be described as

$$\varepsilon^{th} = \delta(\theta_\infty - \theta_{ref})I,$$

where  $\delta$  denotes the thermal expansion.

Figure 1 demonstrates the effect of subdomains with different densities for the mechanical equilibrium shape. For details about the chosen data we refer to Section 4. The goal of this paper is to utilize this effect by finding an optimal mixture of subdomains  $\Omega$  and its complement in  $D$ , such that the workpiece attains a desired equilibrium shape. To this end we consider the cost functional

$$J(\chi, \mathbf{y}) = \int_{\Sigma} \|\mathbf{y} - \mathbf{u}_d\|^2 ds + \alpha \mathcal{P}_D(\chi), \quad (1.8)$$

where  $\Sigma \subset \Gamma^N$  and  $\mathbf{u}_d \in H^1(\mathbf{R}^d, \mathbf{R}^d)$ . The geometric part of the cost functional, i.e., the first two terms will drag the workpiece towards the desired equilibrium shape. Details how to choose  $L_{1,2}$  in practice will be described in Section 4. It is known that minimizing this geometric part alone would lead to homogenized or laminated microstructures [Allaire, 2002]. To avoid this scenario, we have added the last term in (1.8) term which penalizes subdomains  $\Omega$  with large perimeter  $\hat{\mathcal{P}}_D(\chi)$ . Details about its definition will be given in Section 2.1. Note that if the boundary is  $C^2$  the perimeter corresponds to the total surface area of the boundary in three-dimensional problems, and to the total arc length of the boundary in two-dimensional problems.

The optimal shape design problem to be studied in this paper then amounts to finding an optimal subdomain  $\hat{\Omega}$  and optimal displacement  $\hat{\mathbf{u}} = \mathbf{u}(\hat{\Omega}) = \mathbf{u}(\hat{\chi})$ , which solves the equilibrium problem (1.1).

The main contribution of this paper is the rigorous derivation of the shape derivative of our cost functional using a saddle point formulation and recent regularity results for transmission problems [Costabel et al., 2010]. Similar results for scalar transmission problems can be found in [Pantz, 2005, Afraites et al., 2007].

Note that this problem is related to topology optimization, where the subdomain  $\Omega$  represents the final structure or topology of the workpiece and its complement  $D \setminus \bar{\Omega}$  is interpreted as void. The goal is to find the stiffest structure corresponding to the minimal compliance. In [Haber et al., 1996] this problem is studied with a perimeter penalization. Recently, a number of authors investigated topology optimization, where

the perimeter term is replaced by a Ginzburg Landau type phasefield relaxation [Bourdin and Chambolle, 2003, Burger and Stainko, 2006].

In the next Section we will detail the optimal design problem, analyse the state system and proof the existence of an optimal design. In Section 3 we derive optimality conditions which we will utilize for the numerical computation of optimal phase mixtures in Section 4.

## 2 The shape design problem

### 2.1 Assumptions, notations and problem definition

Throughout this paper  $D \subset \mathbf{R}^d$  is open, bounded and with Lipschitz boundary. In the following we write  $\chi := \chi_\Omega$ , for  $\Omega \subset D$  open and with boundary  $\Gamma := \partial\Omega$ . Henceforth we will call  $\Gamma$  also interface and assume it locally to be the graph of a Lipschitz function. We assume that  $\Omega$  has at least  $\varepsilon > 0$  distance to the boundary  $\partial D$ , i.e.  $d_{\partial D}(x) := \inf_{y \in \partial D} |x - y| > \varepsilon$  for  $x \in \Omega$ . Moreover, we set  $\Omega^+ := \Omega$  and  $\Omega^- := D \setminus \overline{\Omega}$ . Thus we have  $\Gamma = \overline{\Omega}^- \cap \overline{\Omega}^+$ . The set of characteristic function is defined by

$$X(D) := \{\chi_\Omega : \Omega \text{ Lebesgue measurable and } \Omega \subset D\}. \quad (2.1)$$

The considerations of the first chapter, namely equations (1.1),(1.5), (1.6), (1.7), lead to following interface model

$$\begin{aligned} -\operatorname{div}(A_1 \varepsilon(\mathbf{u}^+)) &= 0 & \text{in } & \Omega^+, \\ -\operatorname{div}(A_2 \varepsilon(\mathbf{u}^-)) &= 0 & \text{in } & \Omega^-, \\ -A_2 \varepsilon(\mathbf{u}^-) n_D &= 0 & \text{on } & \Gamma^N, \\ \mathbf{u}^- &= 0 & \text{on } & \Gamma_0, \end{aligned} \quad (2.2)$$

complemented by the transmission conditions

$$(A_1 \varepsilon(\mathbf{u}^+) - A_2 \varepsilon(\mathbf{u}^-)) n = (\beta_1 - \beta_2) n \quad \text{on } \Gamma, \quad (2.3)$$

where the *displacement field*  $\mathbf{u} : \overline{D} \rightarrow \mathbf{R}^d$  is the unknown function and  $n$  and  $n_D$  are unit normal fields along  $\partial\Omega$  and  $\partial D$ , respectively, see [Antman, 2005]. Here, we assume that  $\Gamma^N$  and  $\Gamma_0$  are disjoint parts of the boundary  $\Gamma$  with positive surface measure  $|\Gamma_0| > 0$ . The material is assumed to be *isotropic* and *homogeneous* in each phase, such that the stiffness take the form

$$A_i(\Theta) := 2\mu_i \Theta + \lambda \operatorname{tr}(\Theta) I, \quad \Theta \in \mathbf{R}^{n,n}, \quad \mu_i, \lambda_i > 0,$$

$i = 1, 2$ . Mathematically, we control  $\chi$  which describes the distribution of martensite. As described in the introduction, we choose the cost functional

$$\hat{J}(\chi) := \frac{1}{2} \int_{\Sigma} \|\mathbf{u}(\chi) - \mathbf{u}_d\|^2 ds + \alpha \hat{\mathcal{P}}_D(\chi), \quad (\alpha > 0), \quad (2.4)$$

where  $\Sigma \subset \Gamma \setminus \Gamma_0$ . The function  $\mathbf{u}(\chi)$  is a solution to (2.2),  $\mathbf{u}_d \in H^1(D, \mathbf{R}^3)$  describes the desired shape of the body.

The total variation of a function  $\chi \in X(D)$  is defined by

$$\hat{\mathcal{P}}_D(\chi) := \text{Var}(\chi, D) := \sup_{\substack{\varphi \in C_c^1(D; \mathbf{R}^d) \\ \|\varphi\|_{L^\infty(D)} < \infty}} \int_D \text{div}(\varphi) \chi dx.$$

A subset  $\Omega \subset \mathbf{R}^d$  has finite perimeter relative to  $D \subset \mathbf{R}^d$  if  $\mathcal{P}_D(\Omega) := \hat{\mathcal{P}}_D(\chi_\Omega) < \infty$ . If  $D = \mathbf{R}^d$  then we define  $\hat{\mathcal{P}}(\chi) := \text{Var}(\chi, \mathbf{R}^d)$  and  $\mathcal{P}(\Omega) := \hat{\mathcal{P}}(\chi_\Omega)$ . In other words, a subset  $\Omega \subset D$  has finite perimeter if the characteristic function  $\chi = \chi_\Omega \in X(D)$  belongs to the space

$$BV(D) := \{f \in L_1(D) \mid \text{Var}(f, D) < \infty\}.$$

Since  $\Omega \subset D$ , we have  $\mathcal{P}_D(\Omega) = \mathcal{P}(\Omega)$ . One should keep in mind that a finite perimeter set  $\Omega \subset \mathbf{R}^d$ ,  $\mathcal{P}_D(\Omega) < \infty$ , can have non zero  $d$ -dimensional Lebesgue measure, i.e.  $\mu^d(\partial\Omega) > 0$ . This is even true for the relative boundary  $\partial\Omega \cap D$ , see [Giusti, 1984, p. 7]. A reference rich of results concerning spaces of bounded variation is [Ambrosio et al., 2000]. We will search for optimal solutions in the set

$$BV_\chi(D) := \{\chi \in X(D) \mid \chi \in BV(D)\}.$$

In the following we will study the problem

$$\min_{\chi \in BV_\chi(D)} \hat{J}(\chi), \quad (2.5)$$

which we will see has at least one solution.

## 2.2 Analysis of state system

For each  $\chi \in X(D)$  let us first introduce a bilinear form  $a_\chi : H^1(D; \mathbf{R}^d) \times H^1(D; \mathbf{R}^d) \rightarrow \mathbf{R}$  by

$$a_\chi(\varphi, \psi) := a_\chi^+(\varphi^+, \psi^+) + a_\chi^-(\varphi^-, \psi^-),$$

where

$$a_\chi^i(\varphi^i, \psi^i) = \int_D \chi_{\Omega^i} A_i \varepsilon(\varphi) : \varepsilon(\psi) dx \quad (i \in \{+, -\}).$$

Then the weak form of the interface problem reads: Seek  $\mathbf{u}_\chi = \mathbf{u} \in H^1(D; \mathbf{R}^d)$

$$a_\chi(\mathbf{u}, \psi) = \int_D \beta_\chi \text{div}(\varphi) dx, \text{ for all } \varphi \in \mathcal{W} \quad (2.6)$$

where

$$\mathcal{W} := \{v \in H^1(D; \mathbf{R}^d) : v|_{\Gamma_0} = 0 \text{ in the trace sense}\}.$$

Using the characteristic function  $\chi = \chi_\Omega$  (note that  $\mu^d(\Gamma) = 0$ ) the last equation can be rewritten as

$$\int_\Omega A_1 \varepsilon(\mathbf{u}) : \varepsilon(\varphi) dx + \int_{D \setminus \bar{\Omega}} A_2 \varepsilon(\mathbf{u}) : \varepsilon(\varphi) dx = \beta_1 \int_\Omega \text{div}(\varphi) dx + \beta_2 \int_{D \setminus \bar{\Omega}} \text{div}(\varphi) dx.$$

We have the following result concerning existence and uniqueness of the state equation.

**Theorem 2.1** For given  $\chi \in X(D)$  the equation (2.6) has exactly one weak solution  $\mathbf{u}(\chi)$  and we have the following apriori bound

$$\|\mathbf{u}(\chi)\|_{H^1(D;\mathbf{R}^d)} \leq C, \quad (2.7)$$

where  $C := \max\{\beta_1, \beta_2\} \sqrt{3|D|}/\alpha$  depends only on  $\beta_1, \beta_2, |D|$  and  $\alpha$ . Additionally, if the interface  $\Gamma$  is  $C^2$ , we have

$$\mathbf{u}(\chi)|_{\Omega^+} \in H^2(\Omega^+; \mathbf{R}^d), \quad \mathbf{u}(\chi)|_{\hat{\Omega}^-} \in H^2(\hat{\Omega}^-; \mathbf{R}^d),$$

for each  $\hat{\Omega}^- \subset \Omega^-$  such that  $\sup_{y \in \partial D} |x - y| > 0$  for all  $x \in \hat{\Omega}^-$ .

**Proof:** The higher regularity result is a direct consequence of [Costabel et al., 2010, Theorem 5.3.8]. Here, we show that the equation (2.6) has indeed for every  $\chi \in X(D)$  a unique solution. This can be seen as follows:<sup>1</sup> Since  $A_1, A_2$  are positive definite with  $k_1 > k_2 > 0$  and from Korn's inequality (with  $\alpha_K$ ) it follows, that there exist constants  $C, \alpha > 0$ , independent of  $\chi$  such that for all  $\varphi \in H^1(D; \mathbf{R}^d)$

$$\begin{aligned} a_\chi(\varphi, \varphi) &= \int_D \chi A_1 \varepsilon(\varphi) : \varepsilon(\varphi) dx + \int_D (1 - \chi) A_2 \varepsilon(\varphi) : \varepsilon(\varphi) dx \\ &\geq \underbrace{\int_D \chi (k_1 - k_2) \varepsilon(\varphi) : \varepsilon(\varphi) dx}_{\geq 0} + \int_D k_2 \varepsilon(\varphi) : \varepsilon(\varphi) dx \\ &\geq \alpha \|\varphi\|_{H^1(D;\mathbf{R}^d)}^2 \quad (\alpha_a := k_2 \alpha_K) \end{aligned} \quad (2.9)$$

and

$$a_\chi(\varphi, \psi) \leq C \|\varphi\|_{H^1(D;\mathbf{R}^d)} \|\psi\|_{H^1(D;\mathbf{R}^d)}. \quad (2.10)$$

Thus the Lemma of Lax Milgram (see [Evans, 1998, p. 297-299, Theorem 1]) guarantees the unique solvability of the variational problem:

$$\text{Seek } \mathbf{u} \in \mathcal{W} : \quad a_\chi(\mathbf{u}, \varphi) = \int_D \beta_\chi \operatorname{div}(\varphi) dx \quad \text{for all } \varphi \in \mathcal{W}, \quad (2.11)$$

since  $\varphi \mapsto \int_D \beta_\chi \operatorname{div}(\varphi) dx \in \mathcal{W}^{-1}$  according to the estimate

$$\int_D \beta_\chi \operatorname{div}(\varphi) dx \leq \max\{\beta_2, \beta_1\} \sqrt{|D|} \|\varphi\|_{H^1(D;\mathbf{R}^d)}. \quad (2.12)$$

Notice that the constants are independent of  $\Omega$ . To see the apriori bound, let  $\chi \in X(D)$  be any characteristic functions and  $\mathbf{u}_\chi$  the corresponding solutions to (2.6). Using (2.9) we compute

$$\begin{aligned} \alpha \|\mathbf{u}_n\|_{H^1(D;\mathbf{R}^d)}^2 &\leq a_{\chi_n}(\mathbf{u}_n, \mathbf{u}_n) \\ &= \int_D \beta_{\chi_n}(x) \operatorname{div}(\mathbf{u}_n) dx \\ &\leq C \|\mathbf{u}_n\|_{H^1(D;\mathbf{R}^d)}. \end{aligned} \quad (2.13)$$

where  $C := \max\{\beta_1, \beta_2\} \sqrt{3|D|}/\alpha$  depends only on  $\beta_1, \beta_2, |D|$  and  $\alpha$ .  $\square$

<sup>1</sup>Note that at this stage of investigation it is by no means necessary to assume that  $A_1, A_2$  are constant tensors. Indeed, assume  $A_1, A_2 : D \rightarrow L(\mathbf{R}^{3,3}; \mathbf{R}^{3,3})$  and assuming

$$\text{For a.e. } x \in D : \quad A_i(x)G : G \geq k_i \|G\|^2, \quad \text{for all } G \in \operatorname{Mat}_{\operatorname{sym}}(\mathbf{R}^{d,d}), \quad (2.8)$$

all calculations remain valid. Similarly, we could assume that  $\beta_1, \beta_2 \in L_\infty(D)$  instead of  $\beta_1, \beta_2 \in \mathbf{R}^+$ .

**Remark 2.2** *The space  $X(D)$  is not closed for the norms of  $L_p(D)$  ( $1 \leq p < \infty$ ) and the weak convergence of elements of  $X(D)$  coincides with the strong convergence if the limit element is in  $X(D)$ , cf. [Delfour and Zolésio, 2011].*

Next we proof that the function  $X(D) \ni \chi \mapsto \mathbf{u}(\chi) \in \mathcal{W}$ , considered as function from  $L_q(D) \rightarrow \mathcal{W}$  for some large  $q > 2$  Lipschitz continuous.

**Lemma 2.3** *There is a constant  $C > 0$  and  $q > 2$  such that for all  $\chi_1, \chi_2 \in X(D)$*

$$\|\mathbf{u}(\chi_1) - \mathbf{u}(\chi_2)\|_{H^1(D; \mathbf{R}^d)} \leq C \|\chi_1 - \chi_2\|_{L_q(D)}, \quad (2.14)$$

where  $\mathbf{u}(\chi_1), \mathbf{u}(\chi_2)$  is a solution of (2.6).

**Proof:** Let  $\chi_1, \chi_2 \in X(D)$  be two characteristic functions. Set  $\mathbf{u}_i := \mathbf{u}(\chi_i)$  ( $i = 1, 2$ ) and  $\mathbf{u} := \mathbf{u}(\chi)$ , then we estimate

$$\begin{aligned} c \|\mathbf{u}_1 - \mathbf{u}_2\|_{H^1(D; \mathbf{R}^d)}^2 &\leq \int_D A_{\chi_1} \varepsilon(\mathbf{u}_1 - \mathbf{u}_2) : \varepsilon(\mathbf{u}_1 - \mathbf{u}_2) dx \\ &= \int_D (\beta_{\chi_1} - \beta_{\chi_2}) \operatorname{div}(\mathbf{u}_1 - \mathbf{u}_2) dx + \int_D (A_{\chi_2} - A_{\chi_1}) \varepsilon(\mathbf{u}_2) : \varepsilon(\mathbf{u}_1 - \mathbf{u}_2) dx \\ &\leq \|\chi_1 - \chi_2\|_{L_2(D)} \|\mathbf{u}_1 - \mathbf{u}_2\|_{H^1(D; \mathbf{R}^d)} \\ &\quad + \| |A_{\chi_2} - A_{\chi_1}| |\varepsilon(\mathbf{u}_2)| \|_{L_2(D)} \|\mathbf{u}_1 - \mathbf{u}_2\|_{H^1(D; \mathbf{R}^d)} \dots \end{aligned} \quad (2.15)$$

Now from [Herzog et al., 2011] we know that  $\varepsilon(\mathbf{u}) \in L_{2+\gamma}(D)$  for some  $\gamma > 0$ . Therefore dividing (2.15) by  $\|\mathbf{u}_1 - \mathbf{u}_2\|_{H^1(D; \mathbf{R}^d)}$  and estimating the right hand site with the Hölder inequality with  $q = \frac{2+\gamma}{2}$  and  $q' := \frac{q}{q-1} = \frac{2}{\gamma} + 1$ , we obtain

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{H^1(D; \mathbf{R}^d)} \leq C (\|\beta_1 - \beta_2\|_{L_2(D)} + \|A_{\chi_2} - A_{\chi_1}\|_{L_{q'}(D)} \|\varepsilon(\mathbf{u}_2)\|_{L_q(D)}) \quad (2.16)$$

Finally noting that

$$\begin{aligned} |A_{\chi_1} - A_{\chi_2}| &= |\chi_1 A_1 + (1 - \chi_1) A_2 - (\chi_2 A_1 + (1 - \chi_2) A_2)| \\ &\leq |\chi_1 - \chi_2| (|A_1| + |A_2|) \end{aligned}$$

and

$$\begin{aligned} |\beta_{\chi_1} - \beta_{\chi_2}| &= |\chi_1 \beta_1 + (1 - \chi_1) \beta_2 - (\chi_2 \beta_1 + (1 - \chi_2) \beta_2)| \\ &\leq |\chi_1 - \chi_2| (|\beta_1| + |\beta_2|) \end{aligned}$$

we finish the proof.  $\square$

### 2.3 Existence of an optimal shape

After the preparations of the last section, we can study the constraint optimization problem

$$(\mathcal{P}_\chi) \quad \inf_{\chi \in BV_\chi(D)} \hat{J}(\chi),$$

**Theorem 2.4** *For each  $\alpha > 0$  the problem  $(\mathcal{P}_\chi)$  with the cost functional (2.4) admits at least one solution.*

**Proof:** Let  $\chi_{\Omega_n} = \chi_n \in BV_\chi(D)$  be a minimizing sequence,  $\Omega_n$  the corresponding sets of finite perimeter and  $\mathbf{u}_n = \mathbf{u}(\chi_n)$  the corresponding solutions of (2.6). Let us denote by  $j \geq 0$  the infimum of  $(\mathcal{P}_\chi)$ . We have

$$j \leq \int_{\Gamma_d} \|\mathbf{u}_d\|^2 dx < \infty$$

because  $\hat{\mathcal{P}}_D(\chi_\emptyset) = 0$  and  $\mathbf{u}(\chi_\emptyset) = 0$ , which implies  $j < \infty$  and therefore there is a  $c > 0$  such that

$$\forall n \in \mathbf{N} : \hat{\mathcal{P}}_D(\chi_{\Omega_n}) \leq c,$$

and using Theorem 6.3 p. 247 in [Delfour and Zolésio, 2011] yields that there exists a subsequence  $\{\Omega_{n_k}\}$  of  $\{\Omega_n\}$  still indexed by  $n$  and a measurable subset  $\Omega \subset D$  with

$$\hat{\mathcal{P}}_D(\chi_\Omega) \leq \underline{\lim}_{n \rightarrow \infty} \hat{\mathcal{P}}_D(\chi_{\Omega_n}) \text{ and } \chi_{\Omega_n} \rightarrow \chi_\Omega \text{ in } L_1(D; \mathbf{R}^d), \quad (2.17)$$

that means in particular  $\chi_\Omega \in BV_\chi(D)$ . Now Theorem 2.3 and (2.17) give us the lower-semi continuity of the cost  $\hat{J}$ .  $\square$

### 3 Necessary optimality condition

#### 3.1 Main result

In this section we are going to make the sensivity analysis for our cost functional. Instead of introducing the material derivative to derive the shape derivative we are going to use the saddle point formulation first introduced by Correa-Seeger in [Correa and Seeger, 1985] and then applied to shape optimization and further developed by Morgan, Zolesio and Delfour, see [Delfour and Zolésio, 2011] and the references therein. The main contribution of this work is the derivation of nessacary optimality conditions for our interface problem.

**Theorem 3.1** *Let  $\Omega \subset D$  be an open set with Lipschitz boundary, which solves the minimization problem (2.17). Then the following nessacary optimality condition for the problem  $(\mathcal{P}_\chi)$  with  $J(\Omega) := \hat{J}(\chi_\Omega)$  holds*

$$dJ(\Omega)[V] \geq 0 \quad \text{for all admissible } V \in \mathcal{V}_D, \quad (3.1)$$

and the shape derivative of (3.7) exists for all  $V \in \mathcal{V}_D$  and is given by

$$\begin{aligned} dJ(\Omega)[V] &= \int_D \operatorname{div}(V) A_\chi \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{p}) dx - \int_D A_\chi \mathcal{S}(D\mathbf{u}\partial V) : \varepsilon(\mathbf{p}) dx \\ &\quad - \int_D A_\chi \varepsilon(\mathbf{u}) : \mathcal{S}(D\mathbf{p}\partial V) dx \\ &\quad + \int_D \beta_\chi \operatorname{div}(V) \operatorname{div}(\mathbf{p}) dx + \int_D \beta_\chi (\partial V)^T : D\mathbf{p} dx + \alpha \int_\Gamma \operatorname{div}_\Gamma(V) dH^{d-1}, \end{aligned} \quad (3.2)$$

where  $\mathcal{S}(A) := \frac{1}{2}(A + A^T)$ . If the interface  $\Gamma$  is  $C^2$ , we obtain the following formula



$$(V_n := V \cdot n)$$

$$\begin{aligned}
dJ(\Omega)[V] &= \int_{\Gamma} (A_1(\varepsilon(\mathbf{u}^+) - \beta_1 I) : \varepsilon(\mathbf{p}^+) - A_2(\varepsilon(\mathbf{u}^-) - \beta_2 I) : \varepsilon(\mathbf{p}^-)) V_n ds \\
&\quad - \int_{\Gamma} (A_1(\varepsilon(\mathbf{u}^+) - \beta_1 I) n) \cdot \partial_n \mathbf{p}^+ V_n ds - \int_{\Gamma} A_1 \varepsilon(\mathbf{p}^+) n \cdot \partial_n \mathbf{u}^+ V_n ds \\
&\quad + \int_{\Gamma} (A_2(\varepsilon(\mathbf{u}^-) - \beta_2 I) n) \cdot \partial_n \mathbf{p}^- V_n ds + \int_{\Gamma} A_2 \varepsilon(\mathbf{p}^-) n \cdot \partial_n \mathbf{u}^- V_n ds \\
&\quad + \alpha \int_{\Gamma} \kappa V_n dH^{n-1}.
\end{aligned} \tag{3.3}$$

The state  $\mathbf{u}$  and adjoint  $\mathbf{p}$  are given as the solution of

$$\begin{aligned}
\mathbf{u} \in \mathcal{W}, \quad a_{\chi\Omega}(\mathbf{u}, \boldsymbol{\psi}) &= \beta_1 \int_{\Omega} \operatorname{div}(\boldsymbol{\psi}^+) dx + \beta_2 \int_{D \setminus \bar{\Omega}} \operatorname{div}(\boldsymbol{\psi}^-) dx, \quad \forall \boldsymbol{\psi} \in \mathcal{W}, \\
\mathbf{p} \in \mathcal{W}, \quad a_{\chi}(\boldsymbol{\varphi}, \mathbf{p}) &+ \int_{\Sigma} (\mathbf{u}^+ - \mathbf{u}_d) \boldsymbol{\varphi}^+ ds = 0, \quad \forall \boldsymbol{\varphi} \in \mathcal{W}.
\end{aligned}$$

### 3.2 Speed method and Shape derivatives

Let  $D \subset \mathbf{R}^d$  be open, bounded with Lipschitz boundary. For  $k \geq 1$  we consider the space

$$\mathcal{V}_D^k := \{V \in \mathcal{C}^k(\mathbf{R}^d; \mathbf{R}^d) : \operatorname{supp}(V) \subset D\} \tag{3.4}$$

and for  $k = \infty$ , we set  $\mathcal{V}_D^\infty := \mathcal{V}_D$ . The *flow* of the vector field  $V \in \mathcal{V}_D^k$  is defined for each  $x_0 \in D$  by  $\Phi_t^V(x_0) := x(t)$ , where

$$\begin{aligned}
\dot{x}(t) &= V(x(t)) \quad \text{in } (0, \tau) \\
x(0) &= x_0.
\end{aligned} \tag{3.5}$$

**Definition 3.2 (Eulerian semi-derivative)** *Suppose we are given a shape functional  $J : \mathcal{A} \rightarrow \mathbf{R}$  and a flow  $\Phi_t^V : \bar{D} \times \mathbf{R} \rightarrow \mathbf{R}^d$  generated by a vector field  $V \in \mathcal{V}_D$ . Set  $\Omega_t := \Phi_t(\Omega)$ . Then the Eulerian semi-derivative of  $J$  at  $\Omega \subset D$  in the direction  $V$  is defined as the limit (if it exists)*

$$dJ(\Omega)[V] \stackrel{\text{def}}{=} \lim_{t \rightarrow 0} \frac{1}{t} (J(\Omega_t) - J(\Omega)).$$

The derivative  $dJ(\Omega)[V]$  may be non-linear in  $V$  in general.

**Definition 3.3** *Let  $\Omega \subset D$  and  $D \subset \mathbf{R}^d$  open. The functional  $J$  is said to be shape differentiable at  $\Omega$  if the Eulerian semi-derivative  $dJ(\Omega)[V]$  exists for all  $\mathcal{V}_D$  and the map*

$$V \mapsto dJ(\Omega)[V] : \mathcal{V}_D \rightarrow \mathbf{R}, \tag{3.6}$$

*is linear and continuous.*

We cite the following special case of a theorem from [Lamboley and Pierre, 2007, p. 3].

**Theorem 3.4** *Let  $A \subset D \subset \mathbf{R}^d$  be a domain, whose boundary  $\partial A$  locally is the graph of a Lipschitz function. Then the shape derivative of the perimeter is given by*

$$\forall V \in \mathcal{V}_D : d\mathcal{P}(A)[V] = \int_{\partial A} \operatorname{div}_{\partial A} V \, dH^{d-1},$$

$H^{d-1}$  being the  $d - 1$  dimensional Hausdorff measure in  $\mathbf{R}^d$  and

$$\operatorname{div}_{\partial A} V := \operatorname{div}(V)|_{\partial A} - DV\nu_A \cdot \nu_A$$

the tangential divergence. Moreover, if  $\partial A$  is  $C^2$  we have

$$\forall V \in \mathcal{V}_D : d\mathcal{P}(A)[V] = \int_{\partial A} \kappa V \cdot n \, dH^{d-1},$$

$n$  being the unit outwarding normal along  $\partial A$  and  $\kappa$  the curvature of  $\partial A$ .

### 3.3 Saddle points formulation

We now consider the objective function (2.4), but now we choose explicitly the subset  $\Omega$  as variable instead of  $\chi$ . Again we explicitly write

$$J(\Omega, \varphi) = \frac{1}{2} \int_{\Sigma} \|\varphi - \mathbf{u}_d\|^2 ds + \alpha \mathcal{P}_D(\Omega), \quad (3.7)$$

where we have  $\hat{J}(\Omega) = J(\Omega, \mathbf{u}(\Omega))$  when  $\mathbf{u} = \mathbf{u}(\Omega)$  solves (2.6). Let us rewrite the minimization problem (2.5) in control type form by introducing the Lagrangian function:

$$\mathcal{L}(\Omega, \varphi, \psi) := J(\Omega, \varphi) + a_{\chi}(\varphi, \psi) - \int_D \beta_{\chi} \operatorname{div}(\psi) dx,$$

which can be written as

$$\begin{aligned} \mathcal{L}(\Omega, \varphi, \psi) &:= \frac{1}{2} \int_{\Sigma} \|\varphi^- - \mathbf{u}_d\|^2 ds + \int_{\Omega} A_1 \varepsilon(\varphi^+) : \varepsilon(\psi^+) dx + \int_{D \setminus \bar{\Omega}} A_2 \varepsilon(\varphi^-) : \varepsilon(\psi^-) dx \\ &\quad - \beta_1 \int_{\Omega} \operatorname{div}(\psi^+) dx - \beta_2 \int_{D \setminus \bar{\Omega}} \operatorname{div}(\psi^-) dx + \alpha \mathcal{P}_D(\Omega) \end{aligned} \quad (3.8)$$

One easily verifies

$$\forall \varphi \in \mathcal{W} : \sup_{\psi \in \mathcal{W}} \mathcal{L}(\Omega, \varphi, \psi) = \begin{cases} J(\Omega, \mathbf{u}(\Omega)) & \text{if } \varphi = \mathbf{u}(\Omega) \\ +\infty & \text{if } \varphi \neq \mathbf{u}(\Omega) \end{cases}$$

and therefore

$$\boxed{\min_{\varphi \in \mathcal{W}} \sup_{\psi \in \mathcal{W}} \mathcal{L}(\Omega, \varphi, \psi) = J(\Omega, \mathbf{u}(\Omega)).}$$

Thus for  $\Omega \subset D$ , we can write the function  $\hat{J}(\Omega)$  as a min-max of a Lagrangian function  $\mathcal{L}(\Omega, \varphi, \psi)$ . Next we will characterize the saddle points of  $\mathcal{L}(\Omega, \varphi, \psi)$ .

**Definition 3.5** *Let  $A, B$  be sets and  $G : A \times B \rightarrow \mathbf{R}$  a map. Then we say a pair  $(\bar{u}, \bar{p}) \in A \times B$  is a saddle point on  $A \times B$  if*

$$G(\bar{u}, p) \leq G(\bar{u}, \bar{p}) \leq G(u, \bar{p}) \quad \forall u \in A \quad \forall p \in B.$$

By convention  $p \mapsto G(\bar{u}, p)$  has a maximum and  $u \mapsto G(u, \bar{p})$  a minimum at  $\bar{p}$  respectively  $\bar{u}$ . The saddle points are characterized by the next Lemmata.

**Lemma 3.6** *A function  $G$  on  $A \times B$  has a saddle point  $(\bar{u}, \bar{p})$  on  $A \times B$  if and only if*

$$\sup_{p \in B} \inf_{u \in A} G(u, p) = \inf_{u \in A} \sup_{p \in B} G(u, p),$$

and in this case it is equal to  $G(\bar{u}, \bar{p})$ .

**Proof:** See Temam, Ekeland [Ekeland and Temam, 1976]. □

**Lemma 3.7** *Let  $X, Y$  be two Banach spaces. Let us suppose that  $A \subset X$  and  $B \subset Y$ ,  $A, B$  are closed, convex and non-empty. Moreover, let  $G : A \times B \rightarrow \mathbf{R}$  such that*

$$\begin{aligned} \forall u \in A, p \mapsto G(u, p) \text{ is u.s.c., convex and Gateaux differentiable,} \\ \forall p \in B, u \mapsto G(u, p) \text{ is l.s.c., concave and Gateaux differentiable.} \end{aligned}$$

Then  $(\bar{u}, \bar{p}) \in A \times B$  is a saddle point if and only if

$$\begin{aligned} \left\langle \frac{\partial G}{\partial u}(\bar{u}, \bar{p}), u - \bar{u} \right\rangle &\geq 0, \quad \forall u \in A, \\ \left\langle \frac{\partial G}{\partial p}(\bar{u}, \bar{p}), p - \bar{p} \right\rangle &\leq 0, \quad \forall p \in B. \end{aligned} \tag{3.9}$$

**Proof:** See Temam, Ekeland [Ekeland and Temam, 1976]. □

**Remark 3.8** *When  $A$  and  $B$  in the previous Lemma are linear spaces, then we have indeed equality in (3.9).*

The Lagrangean enjoys nice properties

- $\mathcal{L}(\Omega, \varphi, \psi)$  is *convex* and *continuous* with respect to the variable  $\varphi$  and
- *continuous* and *affine-linear* with respect to  $\psi$ :

Let  $(\mathbf{u}, \mathbf{p}) \in \mathcal{W} \times \mathcal{W}$  be a point saddle point, that is

$$\inf_{\varphi \in \mathcal{W}} \sup_{\psi \in \mathcal{W}} \mathcal{L}(\Omega, \varphi, \psi) = \sup_{\psi \in \mathcal{W}} \inf_{\varphi \in \mathcal{W}} \mathcal{L}(\Omega, \varphi, \psi) = \mathcal{L}(\Omega, \mathbf{u}, \mathbf{p}).$$

Then by Lemma 3.7  $(\mathbf{u}, \mathbf{p})$  satisfies

$$\begin{aligned} \mathbf{u} \in \mathcal{W}, \quad \partial \mathcal{L}_\psi(\Omega, \mathbf{u}, \hat{\psi}) = 0, \quad \forall \hat{\psi} \in \mathcal{W}, \\ \mathbf{p} \in \mathcal{W}, \quad \partial \mathcal{L}_\varphi(\Omega, \hat{\varphi}, \mathbf{p}) = 0, \quad \forall \hat{\varphi} \in \mathcal{W}, \end{aligned}$$

or equivalently

$$\begin{aligned} \mathbf{u} \in \mathcal{W}, \quad a_{\chi_\Omega}(\mathbf{u}, \psi) = \beta_1 \int_\Omega \operatorname{div}(\psi^+) dx + \beta_2 \int_{D \setminus \bar{\Omega}} \operatorname{div}(\psi^-) dx, \quad \forall \psi \in \mathcal{W}, \\ \mathbf{p} \in \mathcal{W}, \quad a_\chi(\varphi, \mathbf{p}) + \int_\Sigma (\mathbf{u}^+ - \mathbf{u}_d) \varphi^+ ds = 0, \quad \forall \varphi \in \mathcal{W}. \end{aligned} \tag{3.10}$$

The last equations is called adjoint equation. The strong formulation of the adjoint equation is as follows

$$\begin{aligned}
-\operatorname{div}(A_1\varepsilon(\mathbf{p}^+)) &= 0 & \text{in } \Omega^+, \\
-\operatorname{div}(A_2\varepsilon(\mathbf{p}^-)) &= 0 & \text{in } \Omega^-, \\
-A_2\varepsilon(\mathbf{p}^-)n_D &= -(\mathbf{u}^+ - \mathbf{u}_d) & \text{on } \Sigma, \\
\mathbf{p}^- &= 0 & \text{on } \Gamma_0, \\
-A_2\varepsilon(\mathbf{p}^-)n_D &= 0 & \text{on } \partial D \setminus (\Sigma \cup \Gamma_0),
\end{aligned} \tag{3.11}$$

complemented by the transmission conditions

$$A_1\varepsilon(\mathbf{p}^+)n = A_2\varepsilon(\mathbf{p}^-)n \quad \text{on } \Gamma. \tag{3.12}$$

### 3.4 Perturbation of the domain $\Omega$

Notice that by construction  $\Phi_t^V : D \rightarrow D$  are bi-Lipschitzian for all  $t \geq 0$ , where  $V \in \mathcal{V}_D$ . Moreover,  $\Phi_t^V =: \Phi_t$  are homeomorphism and  $\Phi_t(\operatorname{int}(D)) = \operatorname{int}(D)$ ,  $\Phi_t(\partial D) = \partial D$ . Thus given  $\mathbf{v} \in W^{1,p}(D; \mathbf{R}^d)$ ,  $p \geq 1$ , we conclude (see Theorem 2.2.2, p. 52 [Ziemer, 1989]) that

$$\mathbf{v} \circ \Phi_t \in W^{1,p}(D; \mathbf{R}^d) \text{ if and only if } \mathbf{v} \in W^{1,p}(D; \mathbf{R}^d)$$

and therefore

$$\mathbf{v} \circ \Phi_t \in \mathcal{W} \text{ if and only if } \mathbf{v} \in \mathcal{W}.$$

In the same way we characterized the saddle points of the function  $\mathcal{L}(\Omega, \boldsymbol{\varphi}, \boldsymbol{\psi})$ , we can characterize the saddle points of the perturbed reparametrized Lagrangean  $\mathcal{L}(\Omega_t, \boldsymbol{\varphi}, \boldsymbol{\psi})$ . As we have seen the objective function  $J(\Omega, \mathbf{u}(\Omega))$  can be written as the min-max of the Lagrangean  $\mathcal{L}(\Omega, \boldsymbol{\varphi}, \boldsymbol{\psi})$ , which has a saddle point  $(\mathbf{u}, \mathbf{p})$  completely characterized by (3.10). We define the new domain  $\Omega_t := \Phi_t(\Omega)$ . Then a saddle point  $(\mathbf{u}_t, \mathbf{p}_t)$  of

$$J(\Omega_t, \mathbf{u}(\Omega_t)) = \min_{\boldsymbol{\varphi} \in \mathcal{W}} \sup_{\boldsymbol{\psi} \in \mathcal{W}} \mathcal{L}(\Omega_t, \boldsymbol{\varphi}, \boldsymbol{\psi}),$$

is characterized again by Lemma 3.7 through

$$\begin{aligned}
\mathbf{u}_t \in \mathcal{W}, \quad a_{\chi_{\Omega_t}}(\mathbf{u}_t, \boldsymbol{\psi}) &= \beta_1 \int_{\Omega_t} \operatorname{div}(\boldsymbol{\psi}^+) dx + \beta_2 \int_{D \setminus \overline{\Omega_t}} \operatorname{div}(\boldsymbol{\psi}^-) dx, \quad \forall \boldsymbol{\psi} \in \mathcal{W}, \\
\mathbf{p}_t \in \mathcal{W}, \quad a_{\chi_{\Omega_t}}(\boldsymbol{\varphi}, \mathbf{p}_t) + \int_{\Sigma} (\mathbf{u}_t - \mathbf{u}_d) \boldsymbol{\varphi}^- ds &= 0, \quad \forall \boldsymbol{\varphi} \in \mathcal{W}.
\end{aligned} \tag{3.13}$$

The system (3.13) has exactly one solution  $(\mathbf{u}_t, \mathbf{p}_t) \in \mathcal{W} \times \mathcal{W}$  since  $a_{\chi_{\Omega_t}}(\cdot, \cdot)$  is  $\mathcal{W}$ -coercive and continuous. Note that the space  $\mathcal{W}$  is independent of  $\Omega$ . The following identity holds

$$\varepsilon(\boldsymbol{\varphi}) \circ \Phi_t : \varepsilon(\boldsymbol{\psi}) \circ \Phi_t = \mathcal{S}(D(\boldsymbol{\varphi} \circ \Phi_t) \cdot (D\Phi_t)^{-1}) : \mathcal{S}(D(\boldsymbol{\psi} \circ \Phi_t) \cdot (D\Phi_t)^{-1}),$$

where  $\mathcal{S} : \mathbf{R}^{d,d} \rightarrow \mathbf{R}^{d,d}$  is the symmetrization operator defined as  $S(A) := \frac{1}{2}(A + A^T)$ , note that  $S(D\varphi) = \varepsilon(\varphi)$ . Thus a change of variables leads to

$$\begin{aligned}
& \int_{\Omega} J_t A_1 \mathcal{S}(D(\mathbf{u}_t^+ \circ \Phi_t)(D\Phi_t)^{-1}) : \mathcal{S}(D(\boldsymbol{\psi}^+ \circ \Phi_t)(D\Phi_t)^{-1}) dx \\
& + \int_{D \setminus \bar{\Omega}} J_t A_2 \mathcal{S}(D(\mathbf{u}_t^- \circ \Phi_t)(D\Phi_t)^{-1}) : \mathcal{S}(D(\boldsymbol{\psi}^- \circ \Phi_t)(D\Phi_t)^{-1}) dx \\
& = \beta_1 \int_{\Omega} D\Phi_t^{-T} : D(\boldsymbol{\psi}^+ \circ \Phi_t^{-1}) dx + \beta_2 \int_{D \setminus \bar{\Omega}} D\Phi_t^{-T} : D(\boldsymbol{\psi}^- \circ \Phi_t^{-1}) dx \\
& \qquad \qquad \qquad \forall \boldsymbol{\psi} \in \mathcal{W}, \\
& \int_{\Omega} J_t A_1 \mathcal{S}(D(\boldsymbol{\varphi}^+ \circ \Phi_t)(D\Phi_t)^{-1}) : \mathcal{S}(D(\mathbf{p}_t^+ \circ \Phi_t)(D\Phi_t)^{-1}) dx \\
& + \int_{D \setminus \bar{\Omega}} J_t A_2 \mathcal{S}(D(\boldsymbol{\varphi}^- \circ \Phi_t)(D\Phi_t)^{-1}) : \mathcal{S}(D(\mathbf{p}_t^- \circ \Phi_t)(D\Phi_t)^{-1}) dx \\
& \qquad \qquad \qquad + \int_{\Sigma} (\mathbf{u}_t^- \circ \Phi_t - \mathbf{u}_d \circ \Phi_t) \boldsymbol{\varphi}^- \circ \Phi_t ds = 0 \\
& \qquad \qquad \qquad \forall \boldsymbol{\varphi} \in \mathcal{W}.
\end{aligned} \tag{3.14}$$

The pull backs  $\mathbf{u}^t := \Phi_t^*(\mathbf{u}_t) := \mathbf{u}_t \circ \Phi_t$  and  $\mathbf{p}^t := \Phi_t^*(\mathbf{p}_t) := \mathbf{p}_t \circ \Phi_t$  satisfy therefore

$$\begin{aligned}
& \int_{\Omega} J_t A_1 \mathcal{S}(D\mathbf{u}^{t,+}(D\Phi_t)^{-1}) : \mathcal{S}(D\boldsymbol{\psi}^+(D\Phi_t)^{-1}) dx \\
& + \int_{D \setminus \bar{\Omega}} J_t A_2 \mathcal{S}(D\mathbf{u}^{t,-}(D\Phi_t)^{-1}) : \mathcal{S}(D\boldsymbol{\psi}^-(D\Phi_t)^{-1}) dx \\
& = \beta_1 \int_{\Omega} D\Phi_t^{-T} : D\boldsymbol{\psi}^+ dx + \beta_2 \int_{D \setminus \bar{\Omega}} D\Phi_t^{-T} : D\boldsymbol{\psi}^- dx \\
& \qquad \qquad \qquad \forall \boldsymbol{\psi} \in \mathcal{W}, \\
& \int_{\Omega} J_t A_1 \mathcal{S}(D\boldsymbol{\varphi}^+(D\Phi_t)^{-1}) : \mathcal{S}(D\mathbf{p}^{t,+}(D\Phi_t)^{-1}) dx \\
& + \int_{D \setminus \bar{\Omega}} J_t A_2 \mathcal{S}(D\boldsymbol{\varphi}^-(D\Phi_t)^{-1}) : \mathcal{S}(D\mathbf{p}^{t,-}(D\Phi_t)^{-1}) dx \\
& \qquad \qquad \qquad + \int_{\Sigma} (\mathbf{u}^{t,-} - \mathbf{u}_d \circ \Phi_t) \boldsymbol{\varphi}^- ds = 0 \\
& \qquad \qquad \qquad \forall \boldsymbol{\varphi} \in \mathcal{W}.
\end{aligned} \tag{3.15}$$

**Remark 3.9** Note that since  $\text{supp}(V) \subset D$  is compactly contained in  $D$  we have that  $\Phi_t$  equals the identity near the boundary and therefore the integral  $\int_{\Sigma} (\boldsymbol{\varphi} - \mathbf{u}_d \circ \Phi_t) \boldsymbol{\varphi}^- ds$  is independent of  $t$ .

This system has also exactly one solution due to the one-to-one correspondence to the equations (3.14). It is convenient to reparametrize the Lagrange function (3.8)

$$\tilde{\mathcal{L}}(\Omega_t, \boldsymbol{\varphi}, \boldsymbol{\psi}) := \mathcal{L}(\Omega_t, \boldsymbol{\varphi} \circ \Phi_t^{-1}, \boldsymbol{\psi} \circ \Phi_t^{-1}) \quad (\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{W}). \tag{3.16}$$

A change of variables gives us

$$\begin{aligned}
\tilde{\mathcal{L}}(\Omega_t, \varphi, \psi) &= \frac{1}{2} \int_{\Sigma} \|\varphi^- - \mathbf{u}_d\|^2 ds + \int_{\Omega} J_t A_1 \mathcal{S}(D\varphi^+(D\Phi_t)^{-1}) : \mathcal{S}(D\psi^+(D\Phi_t)^{-1}) dx \\
&\quad + \int_{D \setminus \bar{\Omega}} J_t A_2 \mathcal{S}(D\varphi^-(D\Phi_t)^{-1}) : \mathcal{S}(D\psi^-(D\Phi_t)^{-1}) dx \\
&\quad \beta_1 \int_{\Omega} D\Phi_t^{-T} : D\psi^+ dx + \beta_2 \int_{D \setminus \bar{\Omega}} D\Phi_t^{-T} : D\psi^- dx + \alpha \mathcal{P}_D(\Omega_t).
\end{aligned} \tag{3.17}$$

It can easily be seen that the saddle point of  $\tilde{\mathcal{L}}$  coincides with the solutions of the equations (3.15). Thus we have

$$\min_{\varphi \in \mathcal{W}} \sup_{\psi \in \mathcal{W}} \mathcal{L}(\Omega_t, \varphi, \psi) = \min_{\tilde{\varphi} \in \mathcal{W}} \sup_{\tilde{\psi} \in \mathcal{W}} \tilde{\mathcal{L}}(\Omega_t, \tilde{\varphi}, \tilde{\psi})$$

and the saddle points of both Lagrangeans are connected via  $\Phi_t$ , i.e.  $\mathbf{u}_t = \mathbf{u}^t \circ \Phi_t^{-1}$  and  $\mathbf{p}_t = \mathbf{p}^t \circ \Phi_t^{-1}$ . In order to show that our function is shape differentiable we have to investigate if the min-max of the function  $G(t, \tilde{\varphi}, \tilde{\psi}) := \tilde{\mathcal{L}}(\Omega_t, \tilde{\varphi}, \tilde{\psi})$  is differentiable with respect to  $t$ . This problem is devoted to the next section.

### 3.5 Differentiability of the min-max

In view of the formal relation

$$dJ(\Omega)[V] = \frac{d}{dt} \left( \min_{\varphi \in \mathcal{W}} \sup_{\psi \in \mathcal{W}} \tilde{\mathcal{L}}(\Omega_t, \varphi, \psi) \right) \Big|_{t=0}$$

we can obtain the *Eulerian derivative*, if we can answer the question under which conditions we are allowed to differentiate

$$\min_{\varphi \in \mathcal{W}} \sup_{\psi \in \mathcal{W}} \tilde{\mathcal{L}}(\Omega_t, \varphi, \psi).$$

Generally, we consider a map

$$G : [0, \tau] \times E \times E \rightarrow \mathbf{R},$$

for  $\tau > 0$  and Banach spaces  $E$  and  $E$ . For each  $t \in [0, \tau]$  we define

$$g(t) := \inf_{x \in E} \sup_{y \in F} G(t, x, y), \quad h(t) := \sup_{y \in F} \inf_{x \in E} G(t, x, y)$$

and the sets associated solution sets

$$\begin{aligned}
E(t) &= \left\{ x^t \in E : \sup_{y \in F} G(t, x^t, y) = g(t) \right\} \\
F(t) &= \left\{ y^t \in F : \inf_{x \in E} G(t, x, y^t) = h(t) \right\}.
\end{aligned}$$

Finally we introduce, according to Lemma 3.6, the set of *saddle points*

$$S(t) := \{(x, y) \in E \times F : g(t) = G(t, x, y) = h(t)\}, \tag{3.18}$$

$S(t) \subset E \times F$  by definition. Now we state a result from [Delfour and Zolésio, 2011] which gives a condition on  $G$  that allows us to differentiate  $g(t)$  at  $t = 0$ . It amounts in asserting that the set valued maps  $E(\cdot)$  and  $F(\cdot)$  are continuous in some sense.

**Theorem 3.10 (R. Correa and A. Seeger)** *Suppose we are given two Banach spaces  $E$  and  $F$  with topologies  $\mathcal{T}_E$  and  $\mathcal{T}_F$  on them. Let a real number  $\tau > 0$ , and the functional*

$$G : [0, \tau] \times E \times F \rightarrow \mathbf{R},$$

*be given. Suppose the following conditions*

(H1) *For all  $(t, x) \in [0, \tau] \times E$ ,  $y \mapsto G(t, x, y)$  is concave and for all  $(t, y) \in [0, \tau] \times F$ ,  $x \mapsto G(t, x, y)$  is convex .*

(H2)  *$\partial_t G(t, x, y)$  exists for all  $(t, x, y) \in [0, \tau] \times E \times F$ .*

(H3) *For any sequence  $t_n \rightarrow 0$  there exists a subsequence  $t_{n_k}$  and an element  $x_0 \in E$ ,  $x_{n_k} \in E(t_{n_k})$  such that*

(i)  *$x_{n_k} \rightarrow x_0$  in the  $\mathcal{T}_E$  topology, and*

(ii) *for all  $y \in F(0)$ ,*

$$\lim_{\substack{k \rightarrow \infty \\ t \rightarrow 0}} \partial_t G(t, x_{n_k}, y) = \partial_t G(0, x, y).$$

(H4) *For any sequence  $t_n \rightarrow 0$  there exists a subsequence  $t_{n_k}$  and an element  $y_0 \in F$ ,  $y_{n_k} \in F(t_{n_k})$  such that*

(i)  *$y_{n_k} \rightarrow y_0$  in the  $\mathcal{T}_F$  topology, and*

(ii) *for all  $x \in E(0)$ ,*

$$\lim_{\substack{k \rightarrow \infty \\ t \rightarrow 0}} \partial_t G(t, x, y_{n_k}) = \partial_t G(0, x, y).$$

*Then for each limit  $(x_0, y_0) \in E(0) \times F(0)$  we have*

$$\frac{d}{dt} g(t)|_{t=0} = \partial_t G(0, x_0, y_0).$$

**Remark 3.11** *This version of the theorem is a special case of the one in [Delfour and Zolésio, 2011, p. 556, Theorem 5.1]. The version we present is more appropriate for our application. In the forthcoming paper [Sturm, 2013], we will show how one can use a similar result to the above one for non-linear problems.*

### 3.6 Application of Correa-Seeger

We set

$$G(t, \varphi, \psi) := \tilde{\mathcal{L}}(\Omega_t, \varphi, \psi) \quad (\varphi, \psi \in \mathcal{W}).$$

and consider vector fields  $V \in \mathcal{V}_D$ . We set  $E := F := H^1(D; \mathbf{R}^d)$  and choose for  $\mathcal{T}_E$  the topology generated by the norm  $\|\cdot\|_{H^1(D; \mathbf{R}^d)}$  and for  $\mathcal{T}_F$  the weak topology. First note that (H1) is clearly verified and it follows by Lemma 3.7 and [Delfour and Zolésio, 2011, p. 556, Lemma 5.1]

$$X(t) \times Y(t) = S(t)$$

and

$$E(t) = \{\mathbf{u}^t\}, \quad F(t) = \{\mathbf{p}^t\},$$

for all  $t \in [0, \tau]$ , where  $\mathbf{u}^t, \mathbf{p}^t$  are saddle points of  $G(t, \cdot, \cdot)$  and characterized by (3.15). Next we verify (H2). Let us turn back to the reparametrized Lagrangean (3.16) and differentiate this expression

$$\begin{aligned}
\partial_t G(t, \boldsymbol{\varphi}, \boldsymbol{\psi}) &= \alpha \int_{\partial^* \Omega} \operatorname{div}_{\partial^* \Omega} V dH^{d-1} + \int_D J_t \operatorname{div}(V(t)) A_\chi \mathcal{S}(D\boldsymbol{\varphi}(D\Phi_t)^{-1}) : \mathcal{S}(D\boldsymbol{\psi}(D\Phi_t)^{-1}) dx \\
&\quad - \int_D J_t A_\chi \mathcal{S}(D\boldsymbol{\varphi}(D\Phi_t)^{-1} \partial V(t)) : \mathcal{S}(D\boldsymbol{\psi}(D\Phi_t)^{-1}) dx \\
&\quad - \int_D J_t A_\chi \mathcal{S}(D\boldsymbol{\varphi}(D\Phi_t)^{-1}) : \mathcal{S}(D\boldsymbol{\psi}(D\Phi_t)^{-1} \partial V(t)) dx \\
&\quad + \int_D \operatorname{div}(V(t)) J_t \beta_\chi (D\Phi_t^{-1})^T : D\boldsymbol{\psi} dx + \int_D J_t \beta_\chi (\partial V(t))^T (D\Phi_t^{-1})^T : D\boldsymbol{\psi} dx
\end{aligned} \tag{3.19}$$

where we use the notation  $V(t) := V(\Phi_t(x))$ . By the choice of  $V \in \mathcal{V}_D$  we have  $t \mapsto V(\Phi_t(x))$ ,  $t \mapsto DV(t)$ ,  $t \mapsto D\Phi_t(x)$  and  $t \mapsto (D\Phi_t(x))^{-1}$  are continuous on the interval  $[0, \tau]$ . Thus (H2) is verified. Now we are going to verify (H3)(i) and (H4)(i). From Theorem 2.1 we infer that the solutions  $\mathbf{u}_t$  and  $\mathbf{p}_t$  of (3.14) are uniformly bounded in  $t$ , i.e. there are constants  $C_1, C_2 > 0$  such that

$$\forall t \in [0, \tau] : \quad \|\mathbf{u}_t\|_{H^1(D; \mathbf{R}^d)} \leq C_1, \quad \|\mathbf{p}_t\|_{H^1(D; \mathbf{R}^d)} \leq C_2. \tag{3.20}$$

**Lemma 3.12** *The pull backs  $\Phi_t^*(\mathbf{u}_t) = \mathbf{u}^t$  and  $\Phi_t^*(\mathbf{p}_t) = \mathbf{p}^t$  are uniformly bounded, i.e. there are constants  $C_1, C_2 > 0$  such that*

$$\forall t \in [0, \tau] \quad \|\mathbf{u}^t\|_{H^1(D; \mathbf{R}^d)} \leq C_1 \quad \|\mathbf{p}^t\|_{H^1(D; \mathbf{R}^d)} \leq C_2.$$

**Proof:** From 3.20, we have

$$\|\mathbf{u}_t\|_{H^1(D; \mathbf{R}^d)}^2 \leq C_1^2 \tag{3.21}$$

but after the change of variables  $\Phi_t(x) = y$  we get

$$\begin{aligned}
\int_D \mathbf{u}_t \cdot \mathbf{u}_t + D\mathbf{u}_t : D\mathbf{u}_t dx &= \int_D J_t ((\mathbf{u}_t \circ \Phi_t)^2 + (D\mathbf{u}_t) \circ \Phi_t : (D\mathbf{u}_t) \circ \Phi_t) dy \\
&= \int_D J_t (\mathbf{u}^t \cdot \mathbf{u}^t + (D\mathbf{u}^t) : D\mathbf{u}^t (D\Phi_t)^{-1} (D\Phi_t)^{-T}) dy,
\end{aligned} \tag{3.22}$$

where  $J_t = \det(D\Phi_t) > 0$  for small  $t$ . Setting  $B(t) := J_t (D\Phi_t)^{-1} (D\Phi_t)^{-T}$  we conclude there are constants  $\gamma_1, \gamma_2$  such that

$$\begin{aligned}
\forall \zeta \in \mathbf{R}^d, \forall t \in [0, t^*] : \quad \gamma_1 \zeta^2 \leq \zeta \cdot B(t) \cdot \zeta \leq \gamma_2 \zeta^2, \\
\gamma_1 \leq J_t \leq \gamma_2,
\end{aligned}$$

for some  $t^* > 0$  small. Thus particularly for  $t \in [0, t^*]$

$$\begin{aligned}
(D\mathbf{u}^t) : (D\mathbf{u}^t) B(t) &= B(t) (D\mathbf{u}^t)^T : (D\mathbf{u}^t)^T \\
&\geq \gamma_1 (D\mathbf{u}^t)^T : (D\mathbf{u}^t)^T \\
&= \gamma_1 D\mathbf{u}^t : D\mathbf{u}^t.
\end{aligned} \tag{3.23}$$

From (3.21)-(3.23) we conclude

$$\forall t \in [0, t^*] : \quad \|\mathbf{u}^t\|_{H^1(D; \mathbf{R}^d)} \leq C_1,$$



for some constant  $C_1$ . □

Using the previous Lemma, we conclude that for any sequence  $t_n \rightarrow 0$  for  $n \rightarrow \infty$  there exists a subsequence, still denoted by  $t_n$  and elements  $\mathbf{z}, \mathbf{q} \in H^1(D; \mathbf{R}^d)$  such that

$$\mathbf{u}^{t_n} \rightharpoonup \mathbf{z} \quad \text{and} \quad \mathbf{p}^{t_n} \rightharpoonup \mathbf{q} \quad \text{as } n \rightarrow \infty.$$

Going to the limit in the variational formulation, we conclude by uniqueness  $\mathbf{z} = \mathbf{u}$  and  $\mathbf{q} = \mathbf{p}$ .

Indeed we can show the strong convergence of this sequence and thus we verify (H3) and (H4).

**Lemma 3.13** *Suppose we are given solutions  $\mathbf{u}^t, \mathbf{p}^t$  of (3.13) for  $t > 0$  and  $\mathbf{u}, \mathbf{p} \in H^1(D; \mathbf{R}^d)$  a solution of (3.13) for  $t = 0$ . Then there is a constant  $c > 0$  such that*

$$\|\mathbf{u}^t - \mathbf{u}\|_{H^1(D; \mathbf{R}^d)} \leq ct \text{ for small } t. \quad (3.24)$$

Moreover, we have  $\mathbf{p}^t \rightharpoonup \mathbf{p}$  in  $H^1(D; \mathbf{R}^d)$  as  $t \rightarrow 0$ .

**Proof:** The functions  $\mathbf{u}^t, \mathbf{u}$  satisfy

$$\begin{aligned} \int_D A_{\chi\Omega} \varepsilon(\mathbf{u}^t) : \varepsilon(\boldsymbol{\psi}) dx + \int_D J_t A_{\chi\Omega} \mathcal{S}(D\mathbf{u}^t(D\Phi_t(x))^{-1}) : D\boldsymbol{\psi}(D\Phi_t(x))^{-1} \\ - A_{\chi\Omega} \varepsilon(\mathbf{u}^t) : \varepsilon(\boldsymbol{\psi}) dx \\ = \int_D \beta_{\chi\Omega} \operatorname{div}(\boldsymbol{\psi}) dx + \int_D (\beta_{\chi\Omega} (J_t(D\Phi_t(x))^{-T} - I) : D\boldsymbol{\psi} dx \end{aligned}$$

and

$$\mathbf{u} \in \mathcal{W}, \quad \int_D A_{\chi\Omega} \varepsilon(\mathbf{u}) : D\boldsymbol{\psi} dx = \int_D \beta_{\chi\Omega} \operatorname{div}(\boldsymbol{\psi}) dx \\ \forall \boldsymbol{\psi} \in \mathcal{W},$$

and subtracting both equations yield

$$\begin{aligned} \int_D A_{\chi\Omega} \varepsilon(\mathbf{u}^t - \mathbf{u}) : \varepsilon(\boldsymbol{\psi}) dx = - \int_D (J_t(A_{\chi\Omega} \mathcal{S}(D\mathbf{u}^t(D\Phi_t(x))^{-1}))(D\Phi_t(x))^{-T} - \varepsilon(\mathbf{u}^t)) : D\boldsymbol{\psi} \\ + \int_D (\beta_{\chi\Omega} (J_t(D\Phi_t(x))^{-T} - I) : D\boldsymbol{\psi} dx \end{aligned} \quad (3.25)$$

Now we can estimate

$$\begin{aligned} \|(J_t A_{\chi\Omega} \mathcal{S}(D\mathbf{u}^t(D\Phi_t)^{-1})(D\Phi_t)^{-T} - \varepsilon(\mathbf{u}^t))\|_{L_2(D; \mathbf{R}^{d,d})} \leq \\ + \|(D\Phi_t)^{-T} - I\|_{L_\infty(D; \mathbf{R}^{d,d})} \|(J_t A_{\chi\Omega} \mathcal{S}(D\mathbf{u}^t(D\Phi_t)^{-1}))\|_{L_2(D; \mathbf{R}^{d,d})} \\ + |J_t - 1|_\infty \|A_{\chi\Omega} \mathcal{S}(D\mathbf{u}^t(D\Phi_t)^{-1})\|_{L_2(D; \mathbf{R}^{d,d})} \\ + \|A_{\chi\Omega} (D\mathbf{u}^t((D\Phi_t)^{-1} - I))\|_{L_2(D; \mathbf{R}^{d,d})} \\ + \|A_{\chi\Omega} ((D\Phi_t)^{-T} - I)(D\mathbf{u}^t)^T\|_{L_2(D; \mathbf{R}^{d,d})} \end{aligned} \quad (3.26)$$

and

$$\begin{aligned} \|(\beta_{\chi_\Omega}(J_t(D\Phi_t)^{-T} - I))\|_{L_2(D; \mathbf{R}^d)} &\leq \|(\beta_{\chi_\Omega}((J_t - 1)(D\Phi_t)^{-T}))\|_{L_2(D; \mathbf{R}^d)} \\ &\quad + \|\beta_{\chi_\Omega}(D\Phi_t^{-T} - I)\|_{L_2(D; \mathbf{R}^d)} \\ &\leq c_1|B(t) - I|_\infty + c_2|J_t - 1|_\infty|B(t)|_\infty \end{aligned} \quad (3.27)$$

thus inserting  $\psi = \mathbf{u}^t - \mathbf{u}$  as test function in (3.25) and using (3.26) and (3.27), we see that  $\|\mathbf{u}^t - \mathbf{u}\|_{H^1(D; \mathbf{R}^d)} = o(t)$ .  $\square$

**Remark 3.14** *Note that we could directly conclude from Theorem 2.3 that  $\mathbf{u}_t = \mathbf{u}(\chi_{\Omega_t}) \rightarrow \mathbf{u}(\chi_\Omega) = \mathbf{u}$  in  $H^1(D; \mathbf{R}^d)$ , since for  $f \in L_q(D)$   $q > 1$*

$$\lim_{t \rightarrow 0} (f \circ \Phi_t - f) = 0 \quad \text{in } L_q(D)$$

and thus

$$\lim_{t \rightarrow 0} (\chi_{\Omega_t} - \chi_\Omega) = \lim_{t \rightarrow 0} (\chi_\Omega \circ \Phi_t - \chi_\Omega) = 0 \quad \text{in } L_q(D).$$

Thus it follows immediately  $\mathbf{u}^t = \mathbf{u}_t \circ \Phi_t \rightarrow \mathbf{u}$  in  $H^1(D; \mathbf{R}^d)$ .

### 3.7 Boundary Expression

Now we aim for to differentiate the function

$$j(t) \stackrel{\text{def}}{=} \mathcal{L}(\Omega_t, \boldsymbol{\varphi}^t, \boldsymbol{\psi}^t),$$

where  $\boldsymbol{\varphi}^t = \boldsymbol{\varphi} \circ \Phi_t^{-1}$  and  $\boldsymbol{\psi}^t = \boldsymbol{\psi} \circ \Phi_t^{-1}$  and  $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{W}$ . For this, we will apply the following Corollary from [Henrot and Pierre, 2005, p. 173, Corollaire 5.2.5].

**Lemma 3.15** *Let  $\Omega \subset \mathbf{R}^d$  be open and  $\Phi_t : \Omega \rightarrow \mathbf{R}^d$  be a transformation with  $\Phi_0(\Omega) = \Omega$ , such that  $t \rightarrow f(t, \cdot) \in L_1(\Omega_t)$  for  $t \in (0, \varepsilon)$ . Assume*

$$t \mapsto f(t, \Phi_t(\cdot)) \text{ is differentiable in } 0,$$

then  $I(t) = \int_{\Omega_t} f(t, \Phi_t(x)) dx$  is differentiable and when the boundary  $\Gamma = \partial\Omega$  is Lipschitz then we have

$$I'(0) = \int_{\Omega} f'(0) dx + \int_{\Gamma} f(0)(V \cdot n) ds,$$

with  $n$  denoting the unit normal field along  $\Gamma$ .

We have

$$\begin{aligned} j(t) &:= \frac{1}{2} \int_{\Sigma} \|\boldsymbol{\varphi}^{t,-} - \mathbf{u}_d\|^2 ds + \int_{\Omega_t} A_1 \varepsilon(\boldsymbol{\varphi}^{t,+}) : \varepsilon(\boldsymbol{\psi}^{t,+}) dx + \int_{\Phi_t(D \setminus \overline{\Omega})} A_2 \varepsilon(\boldsymbol{\varphi}^{t,-}) : \varepsilon(\boldsymbol{\psi}^{t,-}) dx \\ &\quad - \beta_1 \int_{\Omega_t} \operatorname{div}(\boldsymbol{\psi}^{t,+}) dx - \beta_2 \int_{\Phi_t(D \setminus \overline{\Omega})} \operatorname{div}(\boldsymbol{\psi}^{t,-}) dx + \alpha \mathcal{P}_D(\Omega_t). \end{aligned}$$

Due to the mixed boundary conditions we have just  $\mathbf{u}^{t,-}, \mathbf{p}^{t,-} \in \mathcal{H}_{loc}^2(\Phi_t(D \setminus \overline{\Omega}); \mathbf{R}^d)$  and  $\mathbf{u}^{t,+}, \mathbf{p}^{t,+} \in \mathcal{H}^2(\Phi_t(\Omega); \mathbf{R}^d)$ . Thus the only problematic terms could be the integrals over  $\Phi_t(D \setminus \overline{\Omega})$ . However, since  $\operatorname{supp}(V) \subset D$ ,  $\Phi_t$  is the identity in the vicinity of  $\partial D$ , hence

these terms give no contribution to the derivative. We define  $\hat{\Omega} = \text{supp}(V) \cap (D \setminus \Omega)$  and apply the theorem to obtain  $j'(0) = dJ(\Omega)[V]$  with

$$\begin{aligned} dJ(\Omega)[V] &= \int_{\Gamma} \kappa V \cdot n ds + \left\{ \int_{\hat{\Omega}} A_2 \varepsilon(\dot{\mathbf{u}}^-) : \varepsilon(\mathbf{p}^-) dx + \int_{\Omega} A_1 \varepsilon(\dot{\mathbf{u}}^+) : \varepsilon(\mathbf{p}^+) dx \right\} \\ &+ \left\{ \int_{\hat{\Omega}} A_2 (\varepsilon(\mathbf{u}^-) - \beta_2 I) : \varepsilon(\dot{\mathbf{p}}^+) dx + \int_{\Omega} A_1 (\varepsilon(\mathbf{u}^+) - \beta_1 I) : \varepsilon(\dot{\mathbf{p}}^+) dx \right\} \quad (3.28) \\ &+ \left\{ \int_{\Gamma} (A_1 (\varepsilon(\mathbf{u}^+) - \beta_1 I) : \varepsilon(\mathbf{p}^+) - A_2 (\varepsilon(\mathbf{u}^-) - \beta_2 I) : \varepsilon(\mathbf{p}^-)) (V \cdot n^-) ds \right\}, \end{aligned}$$

where we used the definitions

$$\begin{aligned} \dot{\mathbf{u}}^i &= \frac{d}{dt} (\mathbf{u}^i \circ \Phi_t^{-1})_{t=0} = -D\mathbf{u}^i \cdot V, \\ \dot{\mathbf{p}}^i &= \frac{d}{dt} (\mathbf{p}^i \circ \Phi_t^{-1})_{t=0} = -D\mathbf{p}^i \cdot V, \end{aligned}$$

( $i \in \{+, -\}$ ). In the following we use the tangential gradient  $\nabla_{\Gamma} f$  of a function  $f \in C^1(\Gamma)$  which is defined as

$$\nabla_{\Gamma} f := \nabla \tilde{f}|_{\Gamma} - (\nabla \tilde{f} \cdot n)n,$$

where  $\tilde{f}$  is an arbitrary extension of  $f$ . This definition is independent of the extension, cf. [Delfour and Zolésio, 2011, p. 496]. With this definition we introduce the tangential gradient  $D_{\Gamma} v$  of a function  $v \in C^1(\Gamma; \mathbf{R}^3)$  by

$$(D_{\Gamma} v)^T = (\nabla_{\Gamma} v_1, \nabla_{\Gamma} v_2, \nabla_{\Gamma} v_3)$$

The last line in (3.28) has already the right form but the other lines are still volume integrals. From now on we use the fact that  $\mathbf{u}^i, \mathbf{p}^i \in \mathcal{H}_{loc}^2(\Omega_i; \mathbf{R}^d)$  ( $i \in \{+, -\}$ ) at least and they satisfy the equations in the strong sense. We start with the first and second line in (3.28) by applying Gauss and using that  $\mathbf{u}^i, \mathbf{p}^i$  are strong solutions in the respectively domains

$$\begin{aligned} &\int_{\hat{\Omega}} A_2 (\varepsilon(\mathbf{u}^-) - \beta_2 I) : \varepsilon(\dot{\mathbf{p}}^-) dx + \int_{\Omega^+} A_1 (\varepsilon(\mathbf{u}^+) - \beta_1 I) : \varepsilon(\dot{\mathbf{p}}^+) dx \\ &= \int_{\hat{\Omega}} \text{div} (A_2 (\varepsilon(\mathbf{u}^-) - \beta_2 I)) \cdot \dot{\mathbf{p}}^- dx + \int_{\Omega} \text{div} (A_1 (\varepsilon(\mathbf{u}^+) - \beta_1 I)) \cdot \dot{\mathbf{p}}^+ dx \\ &- \int_{\Gamma} A_2 (\varepsilon(\mathbf{u}^-) - \beta_2 I) \cdot \dot{\mathbf{p}}^- \cdot n dx + \int_{\Gamma} A_1 (\varepsilon(\mathbf{u}^+) - \beta_1 I) \dot{\mathbf{p}}^+ \cdot n dx \\ &= - \int_{\Gamma} A_2 (\varepsilon(\mathbf{u}^-) - \beta_2 I) \cdot \dot{\mathbf{p}}^- \cdot n ds + \int_{\Gamma} A_1 (\varepsilon(\mathbf{u}^+) - \beta_1 I) \dot{\mathbf{p}}^+ \cdot n ds \end{aligned} \quad (3.29)$$

and similarly

$$\begin{aligned} &\int_{\hat{\Omega}} A_2 \varepsilon(\dot{\mathbf{u}}^-) : \varepsilon(\mathbf{p}^-) dx + \int_{\Omega} A_1 \varepsilon(\dot{\mathbf{u}}^+) : \varepsilon(\mathbf{p}^+) dx \\ &= - \int_{\Omega} \text{div} (A_1 \varepsilon(\mathbf{p}^+)) \cdot \dot{\mathbf{u}}^+ dx - \int_{\hat{\Omega}} \text{div} (A_2 \varepsilon(\mathbf{p}^-)) \cdot \dot{\mathbf{u}}^- dx \\ &+ \int_{\Gamma} (A_1 \varepsilon(\mathbf{p}^+) \dot{\mathbf{u}}^+) \cdot n ds - \int_{\Gamma} (A_2 \varepsilon(\mathbf{p}^-) \dot{\mathbf{u}}^-) \cdot n ds \\ &= \int_{\Gamma} (A_1 \varepsilon(\mathbf{p}^+) \dot{\mathbf{u}}^+) \cdot n ds - \int_{\Gamma} (A_2 \varepsilon(\mathbf{p}^-) \dot{\mathbf{u}}^-) \cdot n ds. \end{aligned} \quad (3.30)$$

Therefore using (3.29) and (3.30) in (3.28) we get

$$\begin{aligned}
dJ(\Omega)[V] &= \int_{\Gamma} (A_1(\varepsilon(\mathbf{u}^+) - \beta_1 I) : \varepsilon(\mathbf{p}^+) - A_2(\varepsilon(\mathbf{u}^-) - \beta_2 I) : \varepsilon(\mathbf{p}^-))(V \cdot n) ds \\
&\quad - \int_{\Gamma} A_2(\varepsilon(\mathbf{u}^-) - \beta_2 I) \dot{\mathbf{p}}^- \cdot n ds + \int_{\Gamma} A_1(\varepsilon(\mathbf{u}^+) - \beta_1 I) \dot{\mathbf{p}}^+ \cdot n ds \\
&\quad + \int_{\Gamma} (A_1 \varepsilon(\mathbf{p}^+) \dot{\mathbf{u}}^+) \cdot n ds - \int_{\Gamma} (A_2 \varepsilon(\mathbf{p}^-) \dot{\mathbf{u}}^-) \cdot n ds + \int_{\Gamma} \kappa (V \cdot n) ds.
\end{aligned} \tag{3.31}$$

The tensor product  $\otimes$  between two vectors  $\mathbf{v}, \mathbf{w} \in \mathbf{R}^d$  is given by  $(\mathbf{v} \otimes \mathbf{w})\mathbf{u} := (\mathbf{w} \cdot \mathbf{u})\mathbf{v}$ , where  $\cdot$  is the scalar product in  $\mathbf{R}^d$ . The last two lines are not following the structure theorem, but we can rewrite this by decomposing  $D\mathbf{u}|_{\Gamma} = D_{\Gamma}\mathbf{u} + (\partial_n \mathbf{u}) \otimes n$  into normal and tangential part. We have  $((\partial_n \mathbf{u}) \otimes n)n = \partial_n \mathbf{u}$  and  $((\partial_n \mathbf{u}) \otimes n)T = 0$ . Here  $n$  is the normal vector along  $\Gamma$  and  $T$  such that  $n \cdot T = 0$ . Similarly, we define  $V_{\Gamma} := V - V_n n$ , where  $V_n = V \cdot n$ . We have

$$\begin{aligned}
\dot{\mathbf{u}}_i &= -D\mathbf{u}^i V = -D_{\Gamma}\mathbf{u}^i V_{\Gamma} - V_n \partial_n \mathbf{u}^i && \text{on } \Gamma \\
\dot{\mathbf{p}}_i &= -D\mathbf{p}^i V = -D_{\Gamma}\mathbf{p}^i V_{\Gamma} - V_n \partial_n \mathbf{p}^i && \text{on } \Gamma,
\end{aligned}$$

and thus we conclude

$$\begin{aligned}
& - \int_{\Gamma} A_2(\varepsilon(\mathbf{u}^-) - \beta_2 I) \dot{\mathbf{p}}^- \cdot n ds + \int_{\Gamma} (A_1(\varepsilon(\mathbf{u}^+) - \beta_1 I) \dot{\mathbf{p}}^+ \cdot n ds \\
&= \int_{\Gamma} A_2(\varepsilon(\mathbf{u}^-) - \beta_2 I) n \cdot \partial_n \mathbf{p}^- V_n ds - \int_{\Gamma} (A_1(\varepsilon(\mathbf{u}^+) - \beta_1 I) n \cdot \partial_n \mathbf{p}^+) V_n ds \\
&+ \int_{\Gamma} \underbrace{[(A_2(\varepsilon(\mathbf{u}^-) - \beta_2 I) n) - (A_1(\varepsilon(\mathbf{u}^+) - \beta_1 I) n)]}_{=0, \text{transmission condition } \mathbf{u}, (2.3)} \cdot (D_{\Gamma}\mathbf{p} V_{\Gamma}) ds \\
&= \int_{\Gamma} A_2(\varepsilon(\mathbf{u}^-) - \beta_2 I) n \cdot \partial_n \mathbf{p}^- V_n ds - \int_{\Gamma} (A_1(\varepsilon(\mathbf{u}^+) - \beta_1 I) n \cdot (\partial_n \mathbf{p}^+) V_n ds,
\end{aligned}$$

and similarly

$$\begin{aligned}
& \int_{\Gamma} (A_1 \varepsilon(\mathbf{p}^+) \dot{\mathbf{u}}^+) \cdot n ds - \int_{\Gamma} A_2 \varepsilon(\mathbf{p}^-) \dot{\mathbf{u}}^- \cdot n ds \\
&= - \int_{\Gamma} (A_1 \varepsilon(\mathbf{p}^+) n \cdot \partial_n \mathbf{u}^+) V_n ds + \int_{\Gamma} A_2 \varepsilon(\mathbf{p}^-) n \cdot (\partial_n \mathbf{u}^-) V_n ds \\
&+ \int_{\Gamma} \underbrace{[(A_2 \varepsilon(\mathbf{p}^-) n) - A_1 \varepsilon(\mathbf{p}^+) n]}_{=0, \text{transmission condition } \mathbf{p}, (3.12)} \cdot (D_{\Gamma}\mathbf{u} V_{\Gamma}) V_n ds \\
&= - \int_{\Gamma} (A_1 \varepsilon(\mathbf{p}^+) n \cdot \partial_n \mathbf{u}^+) V_n ds + \int_{\Gamma} A_2 \varepsilon(\mathbf{p}^-) n \cdot \partial_n \mathbf{u}^- V_n ds.
\end{aligned}$$

Thus

$$\begin{aligned}
dJ(\Omega)[V] &= \int_{\Gamma} (A_1(\varepsilon(\mathbf{u}^+) - \beta_1 I) : \varepsilon(\mathbf{p}^+) - A_2(\varepsilon(\mathbf{u}^-) - \beta_2 I) : \varepsilon(\mathbf{p}^-)) V_n ds \\
&\quad - \int_{\Gamma} (A_1(\varepsilon(\mathbf{u}^+) - \beta_1 I)n) \cdot \partial_n \mathbf{p}^+ V_n ds - \int_{\Gamma} A_1 \varepsilon(\mathbf{p}^+) n \cdot \partial_n \mathbf{u}^+ V_n ds \\
&\quad + \int_{\Gamma} (A_2(\varepsilon(\mathbf{u}^-) - \beta_2 I)n) \cdot \partial_n \mathbf{p}^- V_n ds + \int_{\Gamma} A_2 \varepsilon(\mathbf{p}^-) n \cdot \partial_n \mathbf{u}^- V_n ds \\
&\quad + \int_{\Gamma} \kappa V_n ds.
\end{aligned} \tag{3.32}$$

Now we define for a matrix function  $A \in \mathcal{H}^1(D; \mathbf{R}^{n,n})$ , the tangential part as  $A_{\Gamma} := A|_{\Gamma} - An \otimes n$ , where  $\Gamma \subset D$ . Note that for all  $\mathbf{v}, \mathbf{w}$  and  $C \in \mathbf{R}^{n,n}$ , we have  $C : \mathbf{v} \otimes \mathbf{w} = \mathbf{v} \cdot C \mathbf{w}$ . Then we obtain

$$\begin{aligned}
dJ(\Omega)[V] &= \int_{\Gamma} [(A_1(\varepsilon(\mathbf{u}^+) - \beta_1 I))_{\Gamma} : \varepsilon_{\Gamma}(\mathbf{p}^+) - (A_2(\varepsilon(\mathbf{u}^-) - \beta_2 I))_{\Gamma} : \varepsilon_{\Gamma}(\mathbf{p}^-)] V_n ds \\
&\quad + \int_{\Gamma} (A_2 \varepsilon(\mathbf{p}^-) n) \cdot \partial_n \mathbf{u}^- V_n ds - \int_{\Gamma} A_1 \varepsilon(\mathbf{p}^+) n \cdot \partial_n \mathbf{u}^+ V_n ds + \int_{\Gamma} \kappa V_n ds.
\end{aligned} \tag{3.33}$$

Due to the asymmetry of the transmission conditions for  $\mathbf{u}$  and  $\mathbf{p}$ , we obtain a formula that is not intrinsic in the sense that the quantities  $\partial_n \mathbf{u}^+$  and  $\partial_n \mathbf{u}^-$  in the last line make only sense as restriction of functions of a bigger domain.

**Remark 3.16** *A close inspection of the boundary expression shows that the linear elliptic transmission problem from [Afraites et al., 2007] is contained in our model if  $\beta_1 = \beta_2 = 0$  and the coefficients  $A_1, A_2$  are only scalars.*

**Lemma 3.17** *In the case  $A := A_1 = A_2$  we have  $\mathbf{p} \in \mathcal{H}^2(K; \mathbf{R}^d)$  for each  $K \subset\subset D$  and the shape derivative (3.32) reduces to*

$$dJ(\Omega)[V] = (\beta_2 - \beta_1) \int_{\Gamma} \operatorname{div}(\mathbf{p}) V_n ds.$$

**Proof:** Since  $A := A_1 = A_2$  the adjoint  $\mathbf{p}$  is more regular across the interface, i.e.  $D\mathbf{p}^+ = D\mathbf{p}^-$  on  $\Gamma$ . Therefore in particular  $\partial_n \mathbf{p}^+ = \partial_n \mathbf{p}^-$  and thus the second and third line in (3.32) cancel out if we use the transmission conditions for  $\mathbf{u}$ . Let's have a close look at the first line in (3.32)

$$\begin{aligned}
&\int_{\Gamma} (A\varepsilon(\mathbf{u}^+) : \varepsilon(\mathbf{p}^+) - A\varepsilon(\mathbf{u}^-) : \varepsilon(\mathbf{p}^-)) V_n ds \\
&= \int_{\Gamma} (\partial_t \mathbf{u}^+ : A\varepsilon(\mathbf{p}) - \partial_t \mathbf{u}^- : A\varepsilon(\mathbf{p})) V_n ds \\
&= \int_{\Gamma} (\partial_t \mathbf{u}^+ : A\varepsilon(\mathbf{p}) - \partial_t \mathbf{u}^- : A\varepsilon(\mathbf{p})) V_n ds \\
&+ \int_{\Gamma} (\partial_n \mathbf{u}^+ \otimes n : A\varepsilon(\mathbf{p}) - \partial_n \mathbf{u}^- \otimes n : A\varepsilon(\mathbf{p})) V_n ds \\
&= \int_{\Gamma} (\partial_n \mathbf{u}^+ \otimes n : A\varepsilon(\mathbf{p}) - \partial_n \mathbf{u}^- \otimes n : A\varepsilon(\mathbf{p})) V_n ds
\end{aligned}$$

since  $D_\Gamma \mathbf{u}^+ = D_\Gamma \mathbf{u}^-$  on  $\Gamma$ . Note we have for all  $v, w \in \mathbf{R}^d$  and  $B \in \mathbf{R}^{d,d}$

$$v \otimes w : B = Bw \cdot v,$$

and thus

$$\begin{aligned} & \int_\Gamma (A\varepsilon(\mathbf{u}^+) : \varepsilon(\mathbf{p}^+) - A\varepsilon(\mathbf{u}^-) : \varepsilon(\mathbf{p}^-)) V_n ds \\ &= \int_\Gamma (A\varepsilon(\mathbf{p})n \cdot \partial_n \mathbf{u}^+ - A\varepsilon(\mathbf{p})n \cdot \partial_n \mathbf{u}^-) V_n ds. \end{aligned}$$

Using this identity together with  $\partial_n \mathbf{p}^+ = \partial_n \mathbf{p}^-$  in (3.32) we get the assertion.  $\square$

## 4 Numerics

In the first part of this section we assume  $D \subset \mathbf{R}^2$ , thus we are dealing with planar elasticity. We follow the ideas of [Laurain and Privat, 2010]. If the interface  $\Gamma = \partial\Omega$  is smooth it can be approximated by a (smooth) curve  $\gamma : [0, 1] \rightarrow \mathbf{R}^2$  such that  $\gamma([0, 1]) \approx \Gamma$  in a certain sense.

### 4.1 Clamped and closed B-Splines and B-Spline surfaces

Let  $k, N \in \mathbf{N}$  be fixed integers, set  $p = k - 1$  and define  $m = p + N + 1 = N + k$ . The  $N + 1$  vectors  $U_0, \dots, U_N \in \mathbf{R}^2$  will be called **control points**. Furthermore, we define recursively so called basis functions  $N_k^i : [t_0, t_m] \rightarrow \mathbf{R}$  by

$$N_i^0(t) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } t_i < t_{i+1} \text{ and } t_i \leq t \leq t_{i+1} \\ 0 & \text{else} \end{cases}$$

where  $i = 0, 1, \dots, N$  and

$$N_i^r(t) = \frac{t - t_i}{t_{i+r-1} - t_i} N_i^{r-1}(t) + \frac{t_{i+r} - t}{t_{i+r} - t_{i+1}} N_{i+1}^{r-1}(t)$$

for  $r > 0$  and  $i > 1$  and the knots  $(t_0, \dots, t_{N+k}) \in \mathbf{R}^{m+1}$ .  $N_i^r(t)$  are polynomials of degree  $r - 1$ . Then the basis spline (B-Spline) curve  $\gamma : [t_0, t_m] \rightarrow \mathbf{R}^2$  is defined by

$$\gamma(t) = \sum_{i=0}^N N_i^k(t) U_i. \quad (4.1)$$

Remark that since  $N_i^k(t) = 0$  for  $t \in \mathbf{R} \setminus [t_i, t_{i+k}]$  the curve is local in the sense that if we move the point  $U_i$  it affects maximal  $k$  curve segments, which makes those curves attractive for shape optimization problems. We will refer to  $k$  as the **order** of the B-Spline curve. For an **clamped curve**, that is a curve where the start and end points are not necessarily the same, we choose the knot vector

$$t_j = \begin{cases} 0 & \text{if } j < k \\ j - k + 1 & \text{if } k \leq j \leq N \\ N - k + 2 & \text{if } j > N \end{cases} \quad (4.2)$$

then  $t$  runs in  $[0, n - k + 2]$ . Then we define the curve  $\hat{\gamma} : [0, 1] \rightarrow \mathbf{R}^2$  by  $\hat{\gamma}(t) = \gamma(t(n - k + 2))$ . This curve is again denoted by  $\gamma$ . Let us assume that  $\gamma$  has no self intersections, i.e.

$$\text{For all } t_1, t_2 \in [0, 1] : \gamma(t_1) = \gamma(t_2) \implies t_1 = t_2.$$

This curve satisfies  $\gamma(0) = U_0$  and  $\gamma(1) = U_N$ . A closed curve satisfies by definition  $\gamma(0) = \gamma(1)$ . To close the B-Spline, we choose the knotvector instead of (4.2) as

$$t_j = j/m \quad \text{for } j = 0, 1, \dots, m,$$

such that  $t_{j+1} - t_j = 1/m$ , therefore the knot vector is uniform. Additionally, we have to overlap  $k$  control points as follows

$$U_i = U_{N-(k-1)+i} \quad i = 0, 1, \dots, k - 1.$$

This curve is defined on  $[t_k, t_{m-k}]$  and we have the formula

$$\gamma(t) = \sum_{i=k-1}^{N-k} N_i^k(t)U_i + \sum_{i=N-k}^N (N_{i-(N-k)}^k(t) + N_i^k(t))U_i \quad (4.3)$$

#### 4.1.1 Clamped B-Spline surface

Let  $N_1, N_2, k_1, k_2 \in \mathbf{N}^+$  be fixed and set  $m_1 = p_1 + 1 + N_1$  and  $m_2 = p_2 + 1 + N_2$ . Moreover, let  $N_1 \times N_2$  control points  $U_{ij} \in \mathbf{R}^3$  be given. Suppose we have given two knot vectors  $T_1 = (u_0, \dots, u_{N_1}), T_2 = (v_0, \dots, v_{N_2}) \in \mathbf{R}^m$ , where assume that the knot vectors are defined by (4.2). Then a B-Spline surface  $\varphi : [u_0, u_{N_1}] \times [v_0, v_{N_2}] \rightarrow \mathbf{R}^3$  is defined as follows

$$\varphi(u, v) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} U_{ij} N_i^{k_1}(u) N_j^{k_2}(v).$$

We define then  $\hat{\varphi} : [0, 1] \times [0, 1] \rightarrow \mathbf{R}^3$  by setting  $\hat{\varphi}(u, v) = \varphi((N_1 - k_1 + 2)u, (N_2 - k_2 + 2)v)$ . We will denote  $\hat{\varphi}$  again  $\varphi$ .

## 4.2 Algorithm

In the planar case, i.e.  $D \in \mathbf{R}^2$ , we define a vector field  $V : \Gamma \rightarrow \mathbf{R}^2$  as follows

$$V(x, y) \stackrel{\text{def}}{=} c \sum_{i=0}^N c_i N_i^k(\gamma^{-1}(x, y)) \tilde{U}_i,$$

where  $1/c_i = \int_{\Gamma} N_i^k(\gamma^{-1}(x)) ds > 0$ ,  $c = \sum_{i=1}^N c_i$  and the  $\tilde{U}_i$ 's are to be determined. We denote by  $\hat{V}(t) = V(\gamma(t)) : [0, 1] \rightarrow \mathbf{R}^2$  the reduced vector field. For the case  $D \in \mathbf{R}^3$  we define  $V : \Gamma \rightarrow \mathbf{R}^3$  by

$$V(x, y, z) \stackrel{\text{def}}{=} \hat{c} \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \hat{c}_{ij} N_{i,k_1}(\varphi_1^{-1}(x, y, z)) N_{j,k_2}(\varphi_2^{-1}(x, y, z)) \tilde{U}_{ij},$$

where  $1/\hat{c}_{ij} = \int_{\Gamma} N_{i,k_1}(\varphi_1^{-1}(x, y, z))N_{j,k_2}(\varphi_2^{-1}(x, y, z))ds > 0$  and  $\hat{c} = \sum_{i,j} \hat{c}_{ij}$ . Note, that  $\varphi^{-1}(x, y, z) = (\varphi_1^{-1}(x, y, z), \varphi_2^{-1}(x, y, z))$ . Let us denote

$$g(\Gamma) = (A_1(\varepsilon(\mathbf{u}^+) - \beta_1 I) : \varepsilon(\mathbf{p}^+) - A_2(\varepsilon(\mathbf{u}^-) - \beta_2 I) : \varepsilon(\mathbf{p}^-)) \\ - A_1(\varepsilon(\mathbf{u}^+) - \beta_1 I) \cdot n \cdot (\partial_n \mathbf{p}^+ - \partial_n \mathbf{p}^-) - (A_1 \varepsilon(\mathbf{p}^+) \cdot n)(\partial_n \mathbf{u}^+ - \partial_n \mathbf{u}^-) + \alpha \kappa,$$

plugging this Ansatz into the shape derivative (3.32), we obtain for the clamped curve the formula in 2D

$$dJ(\Omega)[V] = \sum_{i=0}^N \tilde{U}_i \int_0^1 g(\Gamma)(\gamma(s))N_i^k(s)J\dot{\gamma}(s)ds,$$

since  $n(s) = J \frac{\dot{\gamma}(s)}{|J\dot{\gamma}(s)|}$ ,  $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . For the closed curve, we have the formula

$$dJ(\Omega)[V] = \sum_{i=k-1}^{N-k} \tilde{U}_i \int_0^1 g(\Gamma)(\gamma(s))(N_i^k(s)J\dot{\gamma}(s)ds \\ + \sum_{i=N-k+1}^N \tilde{U}_i \int_0^1 g(\Gamma)(\gamma(s))(N_i^k(s) + N_{i-(N-k),k}(s))J\dot{\gamma}(s)ds.$$

For a clamped B-Spline surface, we have

$$dJ(\Omega)[V] = \sum_{i=0}^N \tilde{U}_{ij} \int_0^1 \int_0^1 g(\Gamma)(\varphi(u, v))N_i^{k_1}(u)N_{j,k_2}(v) \frac{\partial \varphi(u, v)}{\partial u} \times \frac{\partial \varphi(u, v)}{\partial v} dudv$$

since  $n(u, v) = \frac{\partial \varphi(u, v)}{\partial u} \times \frac{\partial \varphi(u, v)}{\partial v} / |\frac{\partial \varphi(u, v)}{\partial u} \times \frac{\partial \varphi(u, v)}{\partial v}|$ .

To find a decent direction, that is  $dJ(\Omega)[V] \leq 0$ , we achieve

$$\tilde{U}_i = - \int_0^1 g(\Gamma)(\gamma(s))N_i^k(s)J\dot{\gamma}(s)ds, \quad (i = 0, \dots, N) \quad (4.4)$$

for the clamped curve. For the closed curve, we choose

$$\tilde{U}_i = - \int_{\Gamma} g(\Gamma)(\gamma(s))N_i^k(s)J\dot{\gamma}(s)ds, \quad (i = k-1, \dots, N-k) \\ \tilde{U}_i = - \int_{\Gamma} g(\Gamma)(\gamma(s))(N_i^k(s) + N_{i-(N-k),k}(s))\dot{\gamma}(s)ds, \quad (i = N-k+1, \dots, N) \quad (4.5)$$

and for the clamped surface we choose

$$\tilde{U}_{ij} = - \int_0^1 \int_0^1 g(\Gamma)(\varphi(u, v))N_i^{k_1}(u)N_{j,k_2}(v) \frac{\partial \varphi(u, v)}{\partial u} \times \frac{\partial \varphi(u, v)}{\partial v} dudv, \quad (4.6)$$

for  $i = 0, \dots, N_1$  and  $j = 0, \dots, N_2$ .

**Remark 4.1** *Since for our numerical examples we need not closed surfaces we won't work them out in detail here, but mention that they exist and one gets similar formula like (4.5).*



Note that we have the following relation between the moved curve  $\Gamma_t = \gamma([0, 1]) + \alpha V(\gamma([0, 1]))$  and the moved control points

$$\gamma(t) + \alpha V(\gamma(t)) = \sum_{i=0}^N N_i^k(t)(U_i + \alpha \tilde{U}_i). \quad (\alpha > 0)$$

We have a similar formula for the B-Spline surface  $\varphi(u, v) : [0, 1] \times [0, 1] \rightarrow \mathbf{R}^3$

$$\varphi(u, v) + \alpha V(\varphi(u, v)) = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} N_i^k(u) N_j^k(v) (U_{ij} + \alpha \tilde{U}_{ij}). \quad (\alpha > 0)$$

Thus by moving the control points by means of (4.4), (4.5) or (4.6) depending which dimension and curve type we have at hand, we may move the interface  $\Gamma$ . Here the (mean) curvature of the curve  $\gamma = (\gamma_1, \gamma_2)$  at  $t \in (0, 1)$  is computed using the well known formula

$$\kappa(t) = \frac{\dot{\gamma}_1(t)\ddot{\gamma}_2(t) - \dot{\gamma}_2(t)\ddot{\gamma}_1(t)}{|\dot{\gamma}(t)|^3}.$$

for the surface  $\varphi : [0, 1] \times [0, 1] \rightarrow \mathbf{R}^3$  the (mean) curvature  $H = \frac{1}{2}(K_1 + K_2)$ , where  $K_1, K_2$  are the principal curvatures, can be computed by

$$H = \frac{Gl - 2Fm + En}{EG - F^2},$$

where  $E = |\varphi_u|^2$ ,  $G = |\varphi_v|^2$ ,  $F = \varphi_u \cdot \varphi_v$  and  $l = N \cdot \varphi_{uu}$ ,  $n = N \cdot \varphi_{vv}$ ,  $m = N \cdot \varphi_{uv}$ . Since  $\Gamma$  depends on the control points  $U_0, \dots, U_N$  we write

$$J(\Omega, U_0, \dots, U_N) = J(\Omega, \Gamma).$$

To summarise, we have developed the following algorithm:

- 1 Initialize the control points  $U_0, \dots, U_N$  which give us  $\gamma(t)$  and set  $J_0 \stackrel{\text{def}}{=} J(\Omega_0, U_0, \dots, U_N)$ .
- 2 Calculate  $\tilde{U}_0, \dots, \tilde{U}_N$  using the shape derivative.
- 3 Set  $\hat{U}_i = U_i + \eta \tilde{U}_i$  and decrease  $\eta$  until  $J_1 - J_0 \leq -\lambda \eta \sum_{k=1}^N |U_k - \hat{U}_k|^2$  where  $J_1 \stackrel{\text{def}}{=} J(\Omega, \hat{U}_0, \dots, \hat{U}_N)$ . (Usually  $\eta = \zeta \beta^a$ ,  $a$  an integer,  $\beta \in [0, 1]$ ,  $\zeta$  positive.)

$\varrho_1$	$\varrho_2$	$\lambda$	$\mu$
7850 kg	7770 kg	$1.5 \cdot 10^{11}$ Pa	$7.5 \cdot 10^{11}$ Pa

Table 1: Material data for a plain carbon steel.

### 4.3 Numerical results

In this section, we will give two numerical results computed by the algorithm presented in the last section. We will use cubic B-Spline curves to model the interface, i.e. we choose  $k = 4$  in the last subsection. Moreover we choose  $A_2 = A_1 = A$  and  $\beta_2 = 0$  and  $\beta_1 = (1 + \nu)\alpha\frac{1}{2}$ , where  $\nu$  is the shear contraction number and  $\alpha = \frac{\rho_1}{\rho_2} - 1$ , i.e.

$$\begin{aligned}\sigma_\chi &= A\varepsilon(\mathbf{u}) - (1 + \nu)\alpha\frac{1}{2}\chi I \\ &= \lambda \operatorname{div}(\mathbf{u})I + 2\mu\varepsilon(\mathbf{u}) - (1 + \nu)\alpha\frac{1}{2}\chi I.\end{aligned}$$

In this way, we achieve that no stresses occur if there is only one phase present, i.e. if  $\Omega = \emptyset$  then  $\chi = 0$  a.e. on  $D$  and thus  $\sigma_\chi = 0$ . For the discretization, we use the Finite Element Method with linear elements as implemented in the FE/FV toolbox PDELib. The material data correspond to a plain carbon steel, see Table 1.

#### 4.3.1 Spherification of an ellipse

In the first example, we consider a work piece, whose reference configuration is a quarter ellipsoid with periodic boundary conditions. i.e. we set  $u_y = 0$  on the  $x$ -axes and  $u_x = 0$  on the  $y$ -axes. On the curved part of the boundary we impose homogenous Neumann boundary conditions. Our goal is to change the ellipse to a quarter circle. For this example we take the following cost functional

$$J(\Omega) \stackrel{\text{def}}{=} \int_{\Sigma} (\|\mathbf{u}(x) + x\| - R)^2 dx,$$

where  $R > 0$  denotes the desired radius of the circle and  $\mathbf{u}(x) + x$  is the actual deformation of the material point  $x \in D$  and  $\Sigma$  denotes the curved part of the boundary. Unfortunately, since the densities in different steel phases only differ by less than 1%, the ellipticity is hardly visible by eye. The major axis is in the  $x$ - and the minor axis in the  $y$ -direction. Figure 2 shows the  $y$ - component of the adjoint  $p$  for several iterations of the optimisation algorithm. Since the derivative of the cost functional acts as a force in the adjoint equation and the  $y$ - component of the ellipse has to be pushed upwards to obtain a circle this quantity is especially relevant.

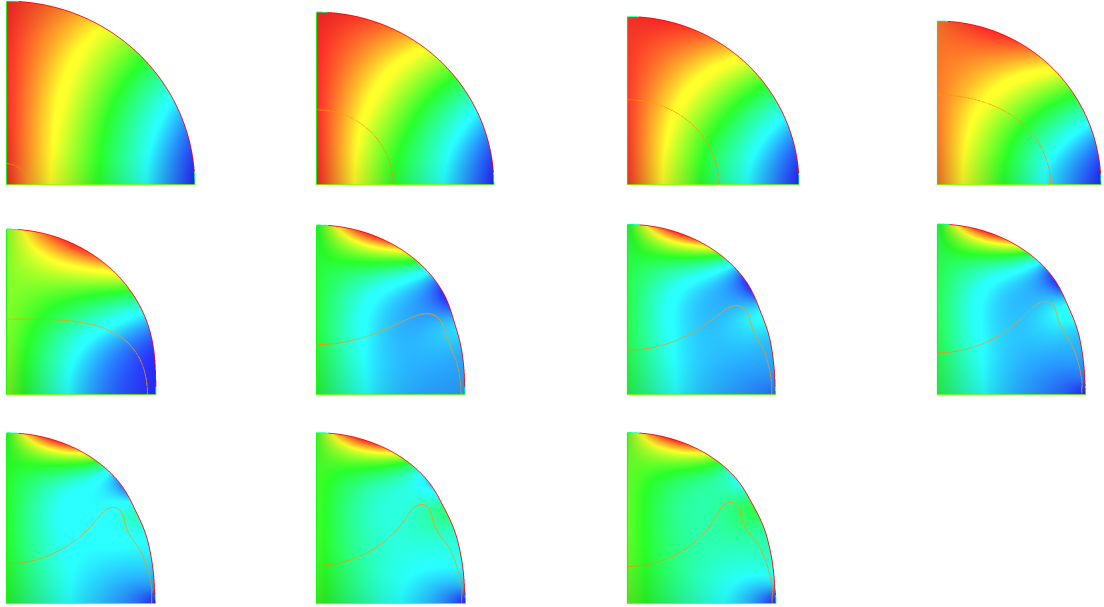


Figure 2: Several iterations for  $p_y$  with  $\mathbf{p} = (p_x, p_y)$ .

Figure 4.3.1 shows the convergence history and Figure 4 a comparison between initial and final shape. As said before, since we have chosen to magnify the displacement, the final shape looks rather like an ellipse with minor on the  $x$ - axis. But a closer look reveals that the interface which initially was a small circle looks like a quarter cross section of a barbell, which is due to the fact that around the  $y$ - axis more material with a lower density is necessary to achieve a circular shape.

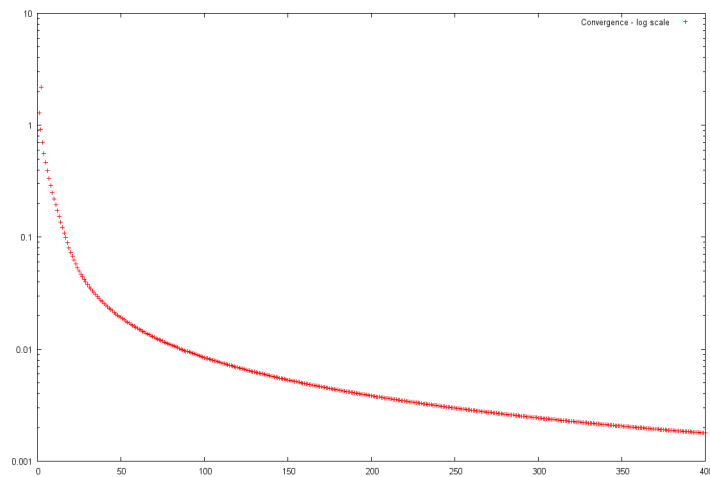


Figure 3: Convergence history for the spherification of an ellipsoid.

### 4.3.2 Straightening of a wavy block

As the second example we consider a rectangular domain  $D$  with a wavy upper surface. We assume Dirichlet boundary conditions on the bottom and Neuman conditions on the

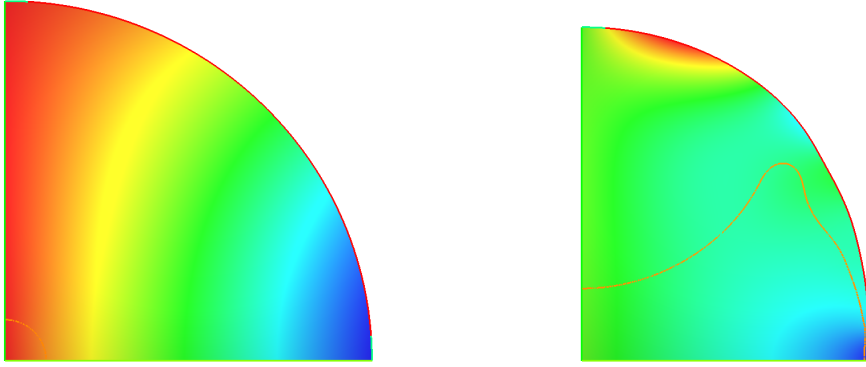


Figure 4: Initial and optimal shape.

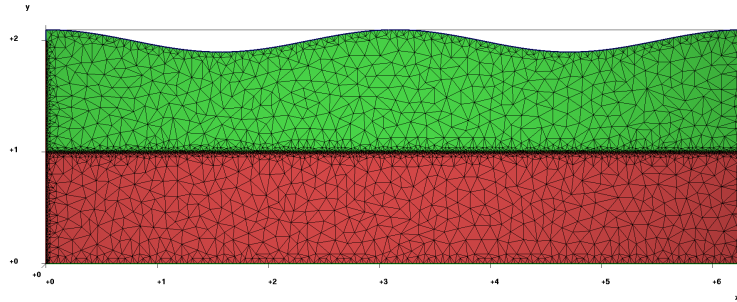


Figure 5: Triangulation of the wavy block.

top and on the sides and use the cost functional

$$J(\Omega) = \int_{\Sigma} |u_y - R|^2 ds$$

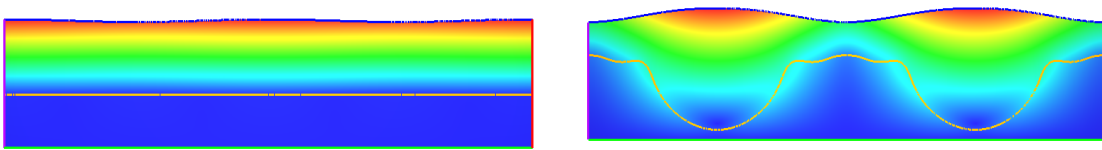


Figure 6: Initial (left) and optimal shape (right).

The goal is to straighten the upper surface. The initial and final block shape are depicted in Figure 6. Unfortunately, since the densities in different steel phases only differ by less than 1%, the waviness of the upper surface is hardly visible. In the optimal shape on the right-hand side the displacement has been magnified to show the effect, hence the surface looks even more wavy. However, this is an artefact due to the magnification. Figure 8 shows the magnified shape of the upper boundary for several iterations of the optimisation algorithm and one can indeed observe, how the surface gradually straightens.

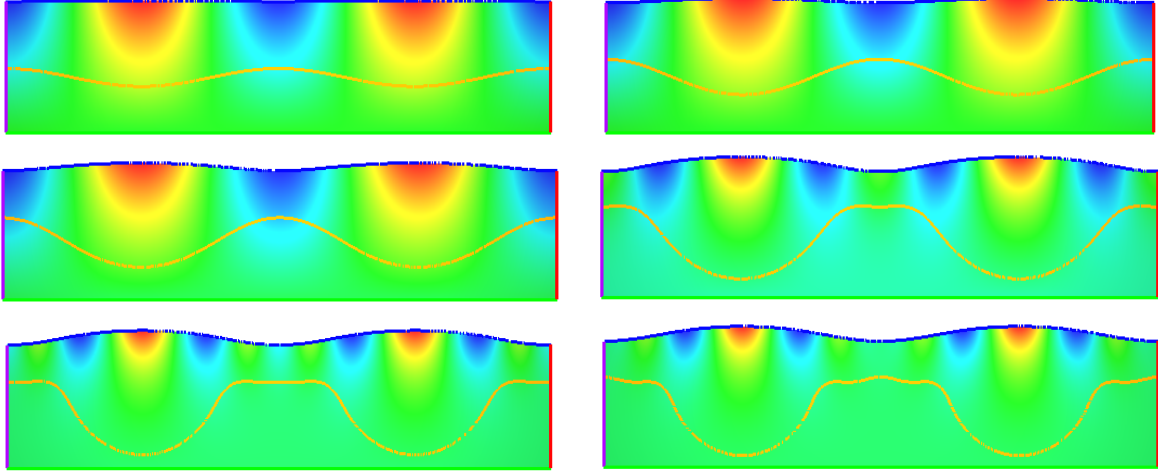


Figure 7: Several iterations of  $p_y$  with  $\mathbf{p} = (p_x, p_y)$ .

Finally, Figure 7 shows several iterations for the  $y$ - component of the adjoint variable, where the gradient acts as a force term on the upper boundary,

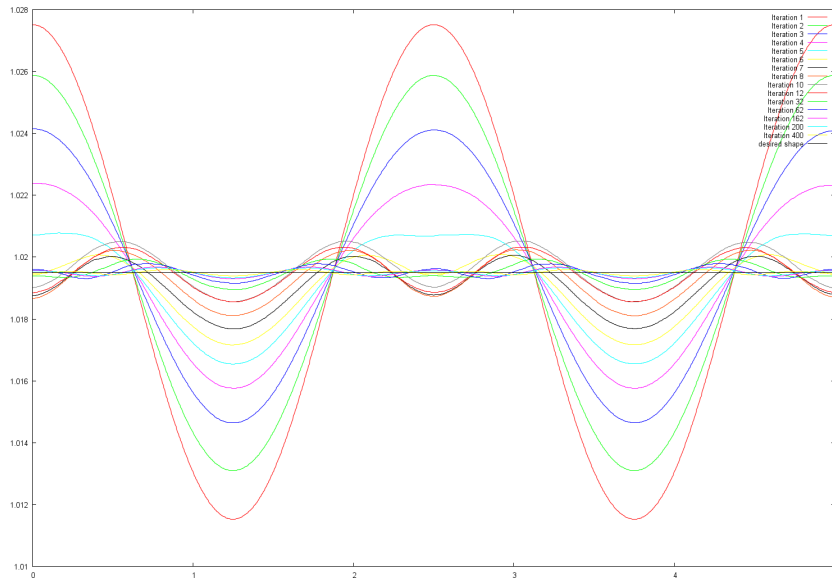


Figure 8: Surface shape of the wavy block for different iteration steps.

## 5 Conclusion

In this paper we have discussed a transmission problem for a mechanical equilibrium problem for subdomains with different densities. A future challenge will be the study of interface problems for nonlinear elasticity, which do not allow for a direct application of the Correa-Seeger theorem.

For other scalar nonlinear problems it has been shown recently that the application of the Correa-Seeger theorem can be justified by the material derivative method [Sturm, 2013].

In less regular situations, which do not allow for a rigorous derivation of boundary supported shape gradients it is of interest to use the distributed representation of the shape derivative. In a forthcoming paper [Sturm and Laurain, 2013] it will be shown that this approach allows for a straight-forward numerical realisation with level-set methods.

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