

An incremental algorithm for the uncapacitated facility location problem

Ashwin Arulsevan^{1,2} Olaf Maurer^{1,3} Martin Skutella^{1,4}

*Institut für Mathematik
Technische Universität Berlin
Straße des 17. Juni 136, 10623 Berlin, Germany*

Abstract

We study the incremental facility location problem, wherein we are given an instance of the uncapacitated facility location problem. We seek an incremental sequence of opening facilities and an incremental sequence of serving customers along with their fixed assignments to facilities open in the partial sequence. Our aim is to have the solution obtained for serving the first ℓ customers in the sequence be competitive with the optimal solution to serve any ℓ customers. We provide an incremental framework that provides an overall competitive factor of 8 and a worst case instance that provides the lower bound of 3. The problem has applications in multi-stage network planning.

Keywords: incremental algorithm, facility location problem, network design

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² Email: arulse@math.tu-berlin.de

³ Email: maurer@math.tu-berlin.de

⁴ Email: martin.skutella@tu-berlin.de

1 Introduction

The problem studied is motivated by the need to deploy telecommunication networks in phases due to budget, time or resource restrictions. In practical situations, problems arising in network planning require incremental solutions that help planners to build the network in stages in the most efficient way. The uncapacitated facility location problem (UFLP) with metric assignment cost has a long line of research [1,3,8]. The best approximation guarantee known for this problem is 1.48 [5]. The problem has also been studied in incremental settings [7,6] but prior work involve in different settings compared to one considered in our work. The authors in previous work were interested in a nested sequence of facilities and a threshold sequence for the scaling of the assignment costs. A solution in the sequence is a solution corresponding to a specific scaling factor, which could be inferred from the threshold sequence. Each solution in the sequence serves all customers and we differ from them in our problem definition.

The incremental facility location problem (IncFLP) is based on the robust facility location problem (RFLP) [2]. In that problem, we are given a set F of potential facility locations and a set R of customers. We are also given a service cost $c_S : R \times F \rightarrow \mathbb{R}_+$, a facility opening cost $c_F : F \rightarrow \mathbb{R}_+$ and a number ℓ of customers to be connected. We seek a set of facilities F^* and a set customers R^* incurring minimal cost with $|R^*| \geq \ell$. We denote the robust facility location problem with given number ℓ by ℓ -RFLP.

An instance of IncFLP comprises of the same input as the RFLP without the number ℓ . Let n the number of customers in the instance. We seek nested sets of facilities, customers and assignment edges between them such that for all $\ell = 1, \dots, n$, the induced solutions of the ℓ -RFLPs are k -approximative for some k .

We need more notation to make this precise: For a subset of facilities $L \subseteq F$, we denote the total facility cost of this set by $c_F(L)$. We denote the service cost obtained by serving a set of customers $M \subseteq R$ by a set of facilities $L \subseteq F$ by $c_S(M, L) = \sum_{j \in M} c_S(j, L)$, where $c_S(j, L)$ is the cost of the cheapest assignment of customer j to a facility in L . For an edge subset E , let $c_S(M, L, E) = \sum_{j \in M} c_S(j, L, E)$, where $c_S(j, L, E)$ is the cost of the cheapest assignment of customer j to a facility in L existing in E . Let $m = |F|$ be the number of facilities and $n = |R|$ the number of customers.

In the incremental version, the goal is to give a sequence to serve customers

$$R_1 \subsetneq R_2 \subsetneq \dots \subsetneq R_n = R$$

where $|R_\ell| = \ell$ and a sequence to open facilities

$$F_1 \subseteq F_2 \subseteq \dots \subseteq F_n$$

such that the customers in $R_\ell \setminus R_{\ell-1}$ could be assigned only to facilities in F_ℓ , assuming $R_0 = \emptyset$. The objective is to minimize the competitive ratio of the sequence. The competitive ratio of a sequence is defined as

$$\max_{\ell=1\dots n} \frac{c_F(F_\ell) + \sum_{\ell=1}^n c_S(F_\ell, R_\ell \setminus R_{\ell-1})}{OPT_\ell},$$

where OPT_ℓ is the optimal cost of the ℓ -RFLP.

Let A be an approximation algorithm for the RFLP. In our approximation algorithm, we will treat A as a black-box. By writing $(Z, M) = A(F, R, \ell)$ we mean that A takes as input the set of potential facilities locations F , the set of customers R and an integer, $\ell \leq n$, number of customers to be served. It produces the output (Z, M) , where $Z \subseteq F$ is the set of facilities opened and $M \subseteq R, |M| = \ell$ is the set of customers served by the facilities in Z . The algorithm does not need to return the actual assignment as it is easily computable. We are now ready to provide our incremental framework $FacInc(F, R)$.

2 Algorithm $FacInc(F, R)$

We start with a solution to the uncapacitated facility location problem obtained from any approximation algorithm. Let $F' \subseteq F$ be the set of facilities opened in this solution. The algorithm runs in two phases. The first phase constructs good partial solutions and in the second phase these partial solutions are glued together to construct an incremental solution. Let $D(F, R)$ denote the solution obtained from the solution (F, R) by removing exactly one customer with the highest service cost (ties broken arbitrarily) and removing any unused facilities (if any) from the solution.

Incremental Phase:

- 1: Initialize: $k = 1, F_n = F', R_n = R$
- 2: **for** $\ell = (n - 1)$ to 1 **do**
- 3: $(F_\ell, R_\ell) = D(F_{\ell+1}, R_{\ell+1})$ and $(L_O, M_O) = A(F, R, \ell)$
- 4: **if** $c_F(F_\ell) + c_S(F_\ell, R_\ell) \geq 2c_F(L_O) + c_S(M_O, L_O)$ **then**
- 5: $F_\ell = L_O$
- 6: $R_\ell = M_O$

7: $k = k + 1$
8: **end if**
9: **end for**
10: Let $K = k$

We call the point when the “If” condition in step 4 passes as a refinement point. We have a total of K such refinement points. We shall reverse the indexing of the solutions from 1 to K , in the increasing fashion (smallest to largest), in order to help us present the analysis with clarity. Let us denote these solutions as $(F_1, R_1), (F_2, R_2), \dots, (F_K, R_K)$. Now (F_1, R_1) and (F_K, R_K) have the least and most number of customers respectively. Let r_k be the number of customers served in the k^{th} refinement point solution. Note that we can construct good solutions (within some constant factor of the optimal) from (F_k, R_k) to serve $\ell = (r_{k-1} + 1) \dots r_k$ customers. In the second phase, we glue these partial solutions together to construct an incremental solution. We denote by $J(F, R, k)$ the set of k customers from the set R with the cheapest service cost when assigned to the facilities in F and we denote such a service cost by $c_S(F, R, k)$.

Refinement Phase:

1: Initialize: $\tilde{R}_0 = R_0, \tilde{F}_0 = F_0$
2: **for** $k = 0$ to $K - 1$ **do**
3: $\tilde{F}_{k+1} = \tilde{F}_k \cup F_{k+1}$
4: Let $J_{k+1} = J(\tilde{F}_{k+1}, R_{k+1} \setminus \tilde{R}_k, r_{k+1} - r_k)$
5: $\tilde{R}_{k+1} = \tilde{R}_k \cup J_{k+1}$
6: **end for**

We now have our incremental sequence of customers and facilities:

$$\tilde{R}_0 \subseteq \tilde{R}_1 \cdots \subseteq \tilde{R}_{K-1} \subseteq \tilde{R}_K = R \quad (1)$$

$$\tilde{F}_0 \subseteq \tilde{F}_1 \cdots \subseteq \tilde{F}_{K-1} \subseteq \tilde{F}_K \quad (2)$$

We are yet to explain why this is an incremental sequence at the non-refinement or intermediate points. We will do this later.

3 Analysis

We would first like to show that for all $k = 1, \dots, K$, the following inequality holds:

$$c_F(\tilde{F}_k) + c_S(\tilde{F}_k, \tilde{R}_k) \leq 2c_F(A(F, R, r_k)) + c_S(A(F, R, r_k)) \quad (3)$$

where A is the approximation algorithm used as our black-box. Our refinement condition at step 4 of the refinement phase implies

$$2c_F(F_k) + c_S(F_k, R_k) \leq c_F(F_{k+1}) + c_S(F_{k+1}, R_{k+1}, r_k), \forall k \in \{1 \dots K - 1\} \quad (4)$$

Lemma 3.1

$$\begin{aligned} 2c_F(F_k) + c_S(F_k, R_k) + c_F(F_{k+1}) + c_S(F_{k+1}, R_{k+1} \setminus \tilde{R}_k, r_{k+1} - r_k) \\ \leq 2c_F(F_{k+1}) + c_S(F_{k+1}, R_{k+1}), \text{ for all } k \in \{0, \dots, K - 1\} \end{aligned}$$

Proof. Add $c_F(F_{k+1}) + c_S(F_{k+1}, R_{k+1} \setminus \tilde{R}_k, r_{k+1} - r_k)$ to both sides of the refinement condition 4. The right hand side is then equal to

$$\begin{aligned} 2c_F(F_{k+1}) + c_S(F_{k+1}, R_{k+1}, r_k) + c_S(F_{k+1}, R_{k+1} \setminus \tilde{R}_k, r_{k+1} - r_k) \\ \leq 2c_F(F_{k+1}) + c_S(F_{k+1}, R_{k+1}, r_{k+1}) = 2c_F(F_{k+1}) + c_S(F_{k+1}, R_{k+1}). \end{aligned}$$

The inequality is due to the observation that the cost of choosing the cheapest r_k customers and some $r_{k+1} - r_k$ customers from the set R_{k+1} must be less than the cost of choosing all r_{k+1} customers. \square

Lemma 3.2

$$\begin{aligned} 2c_F(F_1) + c_S(F_1, R_1) + \sum_{j=2}^k \left[c_F(F_j) + c_S(F_j, R_j \setminus \tilde{R}_{j-1}, r_j - r_{j-1}) \right] \\ \leq 2c_F(F_k) + c_S(F_k, R_k), \text{ for all } k \in \{0, \dots, K\} \quad (5) \end{aligned}$$

The left hand side is obviously an upper bound to the cost of the incremental solution.

Proof. For $k = 1$, the claim is true from Lemma 3.1. Let the claim be proven for k . We prove it for $k + 1$ and we know that (5) is true for k . We have to

show

$$\begin{aligned} 2c_F(F_1) + c_S(F_1, R_1) + \sum_{j=2}^{k+1} \left[c_F(F_j) + c_S(F_j, R_j \setminus \tilde{R}_{j-1}, r_j - r_{j-1}) \right] \\ \leq 2c_F(F_{k+1}) + c_S(F_{k+1}, R_{k+1}) \end{aligned}$$

By Lemma 3.1, we know that

$$\begin{aligned} 2c_F(F_k) + c_S(F_k, R_k) + c_F(F_{k+1}) + c_S(F_{k+1}, R_{k+1} \setminus \tilde{R}_k, r_{k+1} - r_k) \\ \leq 2c_F(F_{k+1}) + c_S(F_{k+1}, R_{k+1}) \end{aligned}$$

By applying the induction hypothesis, we get

$$\begin{aligned} 2c_F(F_1) + c_S(F_1, R_1) + \sum_{j=2}^{k+1} \left[c_F(F_j) + c_S(F_j, R_j \setminus \tilde{R}_{j-1}, r_j - r_{j-1}) \right] \\ \leq 2c_F(F_k) + c_S(F_k, R_k) + c_F(F_{k+1}) + c_S(F_{k+1}, R_{k+1} \setminus \tilde{R}_k, r_{k+1} - r_k) \\ \leq 2c_F(F_{k+1}) + c_S(F_{k+1}, R_{k+1}) \end{aligned}$$

□

We need to show that we can upper bound the incremental solution at intermediate points (non-refinement points) as well. In order to do this, we first need to explain how to retrieve an incremental solution at these points from the incremental solution we constructed. If we notice the incremental solution (1) and (2), we only specified the nested subsets of customers and facilities at the refinement points. Let us construct the customer set J_{k+1} in step 4 of the refinement phase in the following equivalent way: iteratively augment \tilde{R}_k by one customer until the size of the set reaches r_{k+1} . We denote this set after p augmentations by \tilde{R}_{k+1}^p . Let $R_{k+1}^p \subset R_{k+1}$, be the subset of customers of size $p = r_k + 1, \dots, r_{k+1}$, that we accepted in the refinement phase (If condition fails) and $F_{k+1}^p \subset F_{k+1}$ be the set of facilities that are serving these customers in that accepted solution. In each iteration, the new customer is picked from the set $R_{k+1}^p \setminus \tilde{R}_k^{p-1}$ and note that at least one such customer exist, since the cardinalities of these two sets differ exactly by 1. Now, we are ready to prove the following Lemma (which is similar to Lemma 3.1).

Lemma 3.3 *For all $p = |R_k| + 1, \dots, |R_{k+1}|$, we have the following true,*

$$\begin{aligned} 2c_F(F_k) + c_S(F_k, R_i) + c_F(F_{k+1}^p) + c_S(F_{k+1}^p, R_{k+1}^p \setminus \tilde{R}_k^{p-1}, 1) \\ \leq 2c_F(F_{k+1}) + c_S(F_{k+1}^p, R_{k+1}^p). \end{aligned}$$

Proof. The proof is identical to the proof of Lemma 3.1. The additional thing to observe is the fact that the sets $F_{k+1}^{r_k+1} \subseteq \dots \subseteq F_{k+1}^p \subseteq \dots \subseteq F_{k+1}^{r_{k+1}}$ are nested, since we only close down a facility if it is not serving any more customers. Hence, when we picked a customer from the set $R_{k+1}^p \setminus \tilde{R}_k^{p-1}$, we can pick the facility from the set F_{k+1}^p that was serving this customer into our incremental solution. The rest of the arguments are similar to proof of Lemma 3.1. \square

Now, with Lemma 3.3 and Lemma 3.2, we have established the following theorem.

Theorem 3.4 *For all $p = r_k + 1, \dots, r_{k+1}$ and for all $k \in \{0, \dots, K - 1\}$, we have the following true,*

$$\begin{aligned} 2c_F(F_1) + c_S(F_1, R_1) + \sum_{j=2}^k \left[c_F(F_j) + c_S(F_j, R_j \setminus \tilde{R}_{j-1}, r_j - r_{j-1}) \right] \\ + c_F(F_{k+1}^p) + c_S(F_{k+1}^p, R_{k+1}^p \setminus \tilde{R}_k^{p-1}, 1) \\ \leq 2c_F(F_{k+1}) + c_S(F_{k+1}^p, R_{k+1}^p). \end{aligned}$$

4 Lower bound

The lower bound here imply the price we need to pay for seeking an incremental solution. The instances we present cannot have any sequence that can have a competitive factor less than 3. We give a family of instances, which yield a lower bound of at least 2.99 for $m \geq 200$, as can be evaluated computationally. The ratio for this construction does not seem to exceed a value of 3.

Let there be n facilities. Each facility has zero-cost-connections to a set R_i with 2^i customers. All sets R_i are mutually disjoint. The connection cost of all other connections is some constant $M \gg 0$. We refer to the facility together with its customers of zero service cost as a cluster.

Let x_0, \dots, x_{m-1} be rest facility opening costs. Consider the following system of linear inequalities:

$$x_{i+1} + \sum_{j=0}^{i-1} x_j \geq \alpha \cdot x_i \quad \forall i = 1, \dots, m - 1.$$

α is the achieved minimal competitive ratio by these inequalities. As an example, for $m = 4$ this system is feasible for $\alpha = 2.246$, but infeasible for

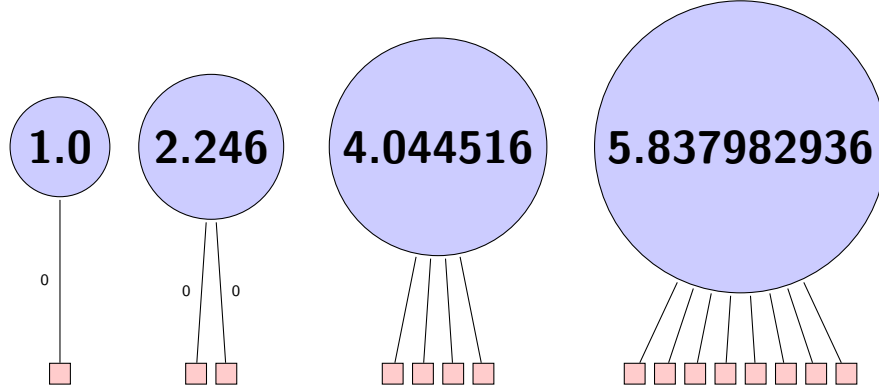


Fig. 1. Lower bound example for $m = 4$, $\alpha = 2.246$. All assignment costs are zero.

$\alpha = 2.247$. So by setting the facility costs to a solution of this system, we get a lower bound of 2.246.

The intuitive explanation behind these inequalities is the following. Suppose one looks for a solution of the RFLP with $\ell = 2^k$ customers. To keep the competitive ratio down below, one needs to open the first cluster, as the second cluster is α times as expensive. The following inequalities give an incentive to opening the clusters one by one in the order of their size, as the competitive ratio for the sequence is immediately as bad as α as soon as one skips a cluster.

An example for $m = 4$, $\alpha = 2.246$ can be seen in Figure 1.

5 Robust facility location problem

The best known approximation guarantee known for the robust facility location problem is 2 [4]. The LP has an integrality gap of 2 even after parametric pruning. This worst case example works on an instance with zero service cost, so we cannot hope to exploit the unequal guarantees that we achieved from our framework for the facility and service cost. This is the best one could hope to achieve with the known techniques to solve the facility location problem. If we plug in the algorithm of Jain et.al. [4] in our framework above, we can solve the incremental version that guarantees a competitive factor 8. For the worst case instance presented in section 4, the ℓ -RFLP could be solved exactly as it is a knapsack problem with the budget being ℓ and our guarantee could be improved to 4 for these instances.

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