IP modeling of the survivable hop constrained connected facility location problem

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Abstract

We consider a generalized version of the rooted connected facility location problem which occurs in planning of telecommunication networks with both survivability and hop-length constraints. Given a set of client nodes, a set of potential facility nodes including one predetermined root facility, a set of optional Steiner nodes, and the set of the potential connections among these nodes, that task is to decide which facilities to open, how to assign the clients to the open facilities, and how to interconnect the open facilities in such a way, that the resulting network contains at least λ edge-disjoint paths, each containing at most H edges, between the root and each open facility and that the total cost for opening facilities and installing connections is minimal. We study two IP models for this problem and present a branch-and-cut algorithm based on Benders decomposition for finding its solution. Finally, we report computational results.

Keywords: Connected facility location, survivability, integer programming.

1 Introduction

A typical metropolitan telecommunication network consists of several local access networks, that are connected by a (regional) core network to a central hub node, that provides connectivity to the national or international backbone.

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The traffic originating at the clients is sent through the access networks to the (regional) core nodes. From there, it traverses the core network(s) to reach the national core or the access network of its destination. Routing functionalities are typically only available at the regional or central core nodes. As the core networks play a primary role for the service availability and the service quality in such networks, it is common to require that these networks are fault-tolerant and lead to short routing paths. To model such a network planning scenario, we introduce a generalized version of the rooted connected facility location problem considering both survivability and hop-length constraints.

In the *Survivable Hop-constrained Connected Facility Location problem (SHConFL)*, we are given an undirected graph $G = (V, E)$ with $V = R\dot{\cup}S$, where R is the set of clients and S is the set of (potential) core nodes. The set $F \subseteq S$ is the set of potential facilities and $r \in S \setminus F$ is the root facility opened in advance, which corresponds to the central hub node. Only edges in $E(S) := \{uv \in E : u, v \in S\}$ may be used in the (regional) core. Furthermore, we are given the costs $f_i \in \mathbb{Z}_{\geq 0}$ for opening facility $i \in F$, the costs $a_{ij} \in \mathbb{Z}_{\geq 0}$ for assigning customer $j \in R$ to facility $i \in F \cup \{r\}$, and the costs $c_e \in \mathbb{Z}_{\geq 0}$ for installing edge $e \in E(S)$ in the (regional) core. Finally, we have an integer hop limit $H \geq 1$, and an integer connectivity requirement $\lambda \geq 1$. We seek for a subset $I \subseteq F$ of facilities to open, a function $\sigma(j) : R \to I \cup \{r\}$ assigning the clients to the open facilities, and an edge set $E' \subseteq E(S)$ such that (S, E') contains, for each facility $i \in I$, at least λ edge-disjoint (r, i)-paths of length at most H. The objective is to minimize the total cost $\sum_{i\in I} f_i + \sum_{j\in R} a_{\sigma(j)j} + \sum_{e\in E'} c_e$.

The SHConFL has not yet been studied in the literature. However, it is related to two other problems, namely the *Hop-Constrained Connected Facility Location (HConFL)* [4] and the *Hop-Constrained Survivable Network Design Problem (HSND)* [3,5]. The HConFL problem is a special case of the SHConFL with $\lambda = 1$. An overview of formulations and polyhedral results for HConFL can be found in [4]. The HSND problem also can be viewed as a special case of the problem where the set of facilities given in advance. Models for HSND can be found in [3,5].

In this paper we present two strong extended formulations for the SHConFL, inspired by the known formulations for HSND, and present computational results obtained with our implementation of a branch-and-cut algorithm. To make the stronger model computationally tractable, we applied Benders decomposition approach, projecting out the extended flow variables. To speed up the algorithm, we primarily separate connectivity inequalities stemming from (and valid for) the corresponding problem with connectivity constraints but without hop constraints. We only solve the computationally expensive separation problem for Benders cuts if the (computationally much easier) separation of connectivity inequalities fails.

2 IP Formulations

2.1 Multi-Commodity Flow Based Formulation.

For each facility $i \in F$, we create a directed layered graph $G^i = (S^i, A^i)$ with

$$S^i = \{r_0\} \cup \{u_h : u \in S,\, 1 \leq h < H\, u \neq r\} \cup \{i_H\} \text{ and}$$
$$A^i = \{(u_{h-1}, v_h) : u_{h-1}, v_h \in S^i,\, uv \in E(S),\, 1 \leq h \leq H\}\ .$$

The copy of node u in layer h of G^i is denoted by u_h and each edge uv has two directed copies (u_{h-1}, v_h) and (v_{h-1}, u_h) between layers $h-1$ and h of G^i.

Clearly, any (r, i) path of length $h \leq H$ in the (core) graph corresponds to a directed path of the same length from r_0 to i_h in G^i.

We introduce binary variables x_{ij}, y_i, and z_{uv} indicating whether customer j is assigned to facility i, facility i is opened, and edge $uv \in E(S)$ is selected, respectively. Additional binary variables f^{ih}_{uv} indicate if a path from r to facility i contains arc (u, v) in position h. Using these variables, a compact flow-based formulation for the problem is obtained as follows:

$$(MCF) \quad \min \sum_{i\in F} f_i y_i + \sum_{j\in R} \sum_{i\in F\cup\{r\}} a_{ij} x_{ij} + \sum_{e\in E(S)} c_e z_e$$

$$\sum_{i\in F\cup\{r\}} x_{ij} = 1 \qquad \forall j \in R \tag{1}$$

$$y_i - x_{ij} \geq 0 \qquad \forall i \in F, j \in R \tag{2}$$

$$\sum_{(v_{h-1},u_h)\in A^i} f^{i(h-1)}_{vu} - \sum_{(u_h,v_{h+1})\in A^i} f^{ih}_{uv} = 0 \qquad \forall i \in F, u_h \in S^i; u \notin \{r, i\} \tag{3}$$

$$\sum_{h=0}^{H-1} \sum_{(u_h,i_{h+1})\in A^i} f^{ih}_{ui} = \lambda y_i \qquad \forall i \in F \tag{4}$$

$$f^{i0}_{rv} \leq z_{rv} \qquad \forall i \in F, rv \in E(S) \tag{5}$$

$$\sum_{h=1}^{H-2} (f^{ih}_{uv} + f^{ih}_{vu}) \leq z_{uv} \qquad \forall i \in F, uv \in E(S), u, v \notin \{r, i\} \tag{6}$$

$$\sum_{h=1}^{H-1} f^{ih}_{ui} \leq z_{ui} \qquad \forall i \in F, ui \in E(S), u \neq r \tag{7}$$

$$x, y, z, f \in \{0, 1\} \tag{8}$$

Constraints (1) and (2) state that any customer has to be assigned to an open facility. Constraints (3) and (4) ensure flow conservation in each layered graph G^i, which guarantees λ length bounded flow paths between r and each open facility. Constraints (5)-(7) finally guarantee the edge-disjointness of these paths and the correct setting of the edge variables z_e.

2.2 Hop-Level Multi-Commodity Flow Based Formulation.

Recently, Mahjoub et al. [5] introduced a new idea to extend the general hop constrained formulation using additional variables indicating the distance of each core node from r in the solution. The formulation presented above can be improved by applying this idea. Given a solution (I, σ, E'), one can partition S into $H+2$ levels according to their distance from r: Level 0 only contains r; level l, $1 \leq l \leq H$, contains nodes with distance l form r; and level $H+1$ contains the nodes that are not connected to r in E'. We introduce binary variables w_u^l indicating if vertex u is in level l (according to its distance from r in the solution) and a_{uv}^{lk} indicating if edge uv belongs to the solution with u in level l and v in level k, respectively. Remark that $|k-l| \leq 1$. Together with the variables x, y, and z, we obtain a new formulation HL-MCF as follows.

Let $E_S' = E(S) \setminus \delta(r)$. With the following constraints, any binary vector (y,z) defines binary values (w, a) where each node is assigned to a single level.

$$\sum_{l=1}^{H+1} w_u^l = 1 \qquad \forall u \in S \setminus \{r\} \tag{9}$$

$$w_i^{H+1} \leq 1 - y_i \qquad \forall i \in F \setminus \{r\} \tag{10}$$

$$w_v^1 = a_{rv}^{01} = z_{rv} \qquad \forall rv \in E_S \tag{11}$$

$$\sum_{l=1}^{H-1} a_{uv}^{ll} + \sum_{l=1}^{H-1} (a_{uv}^{l(l+1)} + a_{vu}^{l(l+1)}) = z_{uv} \qquad \forall uv \in E_S' \tag{12}$$

$$a_{uv}^{11} + a_{uv}^{12} \leq w_u^1 \qquad \forall uv \in E_S' \tag{13}$$

$$a_{uv}^{11} + a_{vu}^{12} \leq w_v^1 \qquad \forall uv \in E_S' \tag{14}$$

$$a_{uv}^{ll} + a_{uv}^{l(l+1)} + a_{vu}^{(l-1)l} \leq w_u^l \qquad \forall uv \in E_S', 2 \leq l \leq H-1 \tag{15}$$

$$a_{uv}^{ll} + a_{vu}^{l(l+1)} + a_{uv}^{(l-1)l} \leq w_v^l \qquad \forall uv \in E_S', 2 \leq l \leq H-1 \tag{16}$$

$$a_{vu}^{(H-1)H} \leq w_u^H \qquad \forall uv \in E_S' \tag{17}$$

$$a_{uv}^{(H-1)H} \leq w_v^H \qquad \forall uv \in E_S' \tag{18}$$

$$w_v^l - \sum_{(u,v) \in \delta(v), u \neq r} a_{uv}^{(l-1)l} \leq 0 \qquad \forall v \in S, 2 \leq l \leq H \tag{19}$$

$$w, a, y, z \in \{0, 1\} \tag{20}$$

Constraints (9)-(10) state that each node should be in exactly one of the possible levels and chosen facilities must be reachable. Constraints (11) and (12) make the connection between (w, a) variables and variables z. Constraints (13)-(18) ensure that a variable a_{uv}^{lk} can only be 1 if both w_u^l and w_v^k are one. Constraints (19) state that a node can only be in level l if it is reached by at least one edge from level $l-1$.

Like in [5], it can be shown that a fractional (w, a) splits nodes and edges into different levels, which might increase the length of paths between r and open facilities in the level-expanded network induced by (w, a). Enforcing the existence of hop-limited arc-disjoint paths between r and open facilities in the level-expanded network, we can thus strengthen the formulation as follows.

For each facility $i \in F$, we create a directed layered graph $G^i_H = (S^i_H, A^i_H)$, where $S^i_H = \{r^0_0\} \cup \{u^l_h : u \in S,\ 1 \le l \le h \le H-1,\ u \neq r\} \cup \{i^l_H : 1 \le l \le H\}$ and $A^i_H = \{(u^l_{h-1}, v^k_h) : u^l_{h-1}, v^k_h \in S^i_H,\ uv \in E(S),\ 1 \le h \le H,\ |l-k| \le 1\} \cup \{(i^l_h, i^l_{h+1}) : 1 \le l \le h \le H-1\}$. The copy of node u in layer h and level l is denoted by u^l_h, while (u^l_{h-1}, v^k_h) denotes the directed arc corresponding to the copy of edge uv with node u in layer $h-1$ and level l and node v in layer h and level k. For each such arc, we have a binary flow variable g^{ihlk}_{uv} indicating that a path from r to facility i goes from u at level l to v at level k in hop h.

The following constraints ensure the existence of the λ arc-disjoint paths.

$$\sum_{(v^k_{h-1}, u^l_h) \in A^i_H} g^{i(h-1)kl}_{vu} - \sum_{(u^l_h, v^k_{h+1}) \in A^i_H} g^{ihlk}_{uv} = 0 \qquad \forall i \in F, u^l_h \in S^i_H, u \neq r, h \le H-1 \quad (21)$$

$$\sum_{l=1}^{H} \sum_{(v^k_{H-1}, i^l_H) \in A^i_H} g^{i(H-1)l}_{vi} = \lambda y_i \qquad \forall i \in F \quad (22)$$

$$g^{i001}_{rv} \le a^{01}_{rv} \qquad \forall i \in F, rv \in E(S) \quad (23)$$

$$\sum_{h=l}^{H-2} (g^{ihll}_{uv} + g^{ihll}_{vu}) \le a^{ll}_{uv} \qquad \forall i \in F, uv \in E'_S \setminus \delta(i), 1 \le l \le H-2 \quad (24)$$

$$\sum_{h=l}^{H-2} g^{ihl(l+1)}_{uv} + \sum_{h=l+1}^{H-2} g^{ih(l+1)l}_{vu} \le a^{l(l+1)}_{uv} \qquad \forall i \in F, uv \in E'_S \setminus \delta(i), 1 \le l \le H-2 \quad (25)$$

$$\sum_{h=l}^{H-1} g^{ihll}_{ui} \le a^{ll}_{ui} \qquad \forall i \in F, ui \in \delta(i) \setminus \delta(r), 1 \le l \le H-1 \quad (26)$$

$$\sum_{h=l}^{H-1} g^{ihl(l+1)}_{ui} \le a^{l(l+1)}_{ui} \qquad \forall i \in F, ui \in \delta(i) \setminus \delta(r), 1 \le l \le H-1 \quad (27)$$

Constraints (21)-(22) are the flow conservation constraints at every node of the layer- and level-extended graphs, guaranteeing λ units of flow from r to each open facility. Constraints (23)-(27) link the flow to the a variables.

The complete HL-MCF formulation is given by the objective function (MCF) subject to constraints (1)-(2) and (8)-(27).

Lemma 2.1 *Formulation HL-MCF is at least as strong as formulation MCF.*

3 Benders Decomposition for HL-MCF

In our implementation, we use a Benders decomposition approach to efficiently handle the huge number of variables and constraints of HL-MCF. However, we cannot directly apply Benders decomposition method, because all variables of

HL-MCF are integer and classical duality theory does not allow to project out integer variables. Similar to Botton et al. [3], we therefor relax the integrality restrictions of the flow variables in HL-MCF and apply Benders decomposition to this relaxation, called R-HL-MCF. We shall discuss whether R-HL-MCF provides the same optimal design variables x, y, and z as HL-MCF.

The master problem is given by the objective function (HL-MCF) subject to constraints (1)-(2), and (8)-(20). A solution $\bar{y}, \bar{x}, \bar{z}, \bar{w}, \bar{a}$ of the master problem defines a feasible solution for the R-HL-MCF if and only if for each $i \in F$ there exist fractional flow values satisfying (21)-(27) with $y_i = \bar{y}_i$ and $a = \bar{a}$.

To apply Farkas lemma to this linear system called SUB^i, we define dual variables $\pi^h_{u,l}$, π, $\sigma^{0(1)}_{rv}$, $\sigma^{l(l)}_{uv}$, $\sigma^{l(l+1)}_{uv}$, $\sigma^{l(l)}_{ui}$, $\sigma^{l(l+1)}_{ui}$ associated to the constraints (21)-(27), respectively. Let $\Pi(i)$ be the set of extreme rays of the corresponding dual system. It can be shown that the Benders reformulation of R-HL-MCF is given by adding the following Benders cuts to the master problem:

$$\lambda y_i \bar{\pi} - \sum_{(u,v)\in E(S), 0\le l,k\le H-1} a^{lk}_{uv} \bar{\sigma}^{l(k)}_{uv} \le 0 \quad \forall (\bar{\pi}, \bar{\sigma}) \in \cup_{i\in F} \Pi(i) \ . \tag{28}$$

Let ξ_{int} and ξ_{frac} denote the set of all binary vectors $(\bar{x}, \bar{y}, \bar{z})$ for which there exists an integral or a fractional solution for (SUB^i), $\forall i \in F$, respectively. Since $\xi_{int} \subseteq \xi_{frac}$, any Benders cut (28) is a valid inequality for conv(ξ_{int}), as well. Following Lemma shows that these Benders cuts are sufficient to describe ξ_{int} in some special cases, but not in general.

Lemma 3.1 *ξ_{int} and ξ_{frac} are equal for $H=2,3$ with any λ, and $H=4$ with $\lambda=2$. For $H \ge 4$, there exist a $\lambda \ge 2$ for which $\xi_{int} \subsetneq \xi_{frac}$, unless P=NP.*

The results follow immediately from the corresponding results for the HSND problem given by Botton et al. [3] (see also [2]).

For general λ and H, we retreat to the generation of *Benders feasibility cuts* to cut off infeasible integer solutions $(\bar{x}, \bar{y}, \bar{z}) \in \xi_{frac} \setminus \xi_{int}$. For this, we solve an integer programming formulation to check if there exist integral flows satisfying (21)-(27) with $y = \bar{y}$ and $a = \bar{a}$; see [3] for more details.

4 Branch-and-cut algorithm

We implemented a branch-and-cut algorithm based on the Benders decomposition discussed above. In every node of the branch-and-bound tree, we first separate (non-hop-restricted) connectivity inequalities of the form $z(\delta(U)) \ge \lambda y_i$ with $U \subseteq S$, $r \notin U$, $i \in U$ for some facility $i \in F$ via max-flow min-cut computations. Only if no violated connectivity cuts are found, we check for violated Benders cuts, which is computationally far more expensive. Formally, in every node of the (B&B) tree the algorithm works as follows:

(i) Solve the master problem; take the optimal (fractional) solution

(ii) Check if the current solution satisfies the connectivity requirements between root and facilities. As long as there are violated connectivity cuts, add them to the master and goto (i).

(iii) Solve the linear Benders subsystems (SUB^i) for the current solution. If this results in a violated Benders cut, add it to the master and goto (i).

(iv) Only at integer nodes of tree (and if needed; see Lemma 3.1): Solve the integer Benders subsystems (SUB^i) for the current solution. If infeasible, add corresponding feasibility cut to the master and goto (i).

Our computational experiences (see Section 5) show that connectivity cuts generated early in step (ii) are very important to avoid the exploration of too many infeasible nodes and to reduce the time spent in the computationally expensive generation of Benders cuts.

5 Computational Study

The algorithm has been implemented in C++, using SCIP [1] as a framework and CPLEX 12.4 as a LP solver, and run on a machine with a AMD Phenom(tm) II X6 1090T 3GHz and 8 GiB RAM. To generate our instances, we combine benchmarks from the HSND problem and benchmarks from uncapacitated facility location problem (UFL) for the core and access part of our instances, respectively. We follow Botton et al. [3] and generate the core graph as follows. Instances are complete graphs whose node set is of size 60 randomly placed among integer points of a grid 100×100. The first $|F| \in \{20, 30, 40\}$ nodes are selected to be potential facilities and the node with index 1 is selected as the root node. The edge costs are set to be the Euclidean distance between any two points. Now given an instance of UFL, $|F| \in \{20, 30, 40\}$ facilities are randomly selected. The number of customers, facility opening costs and assignment costs are provided in UFL instance. We consider a set of 6 instances obtained by combining two UFL instances $mp1$ and $mq1$ [4] (of size 200×200 and 300×300, respectively) with the three generated core instances.

Table 1 reports number of cuts of types "connectivity cut"and "Benders cut" generated throughout the execution of the algorithm (no feasibility cuts have been generated in any instance), denoted by "C(C)", and "C(B)", respectively, the number of branch and bound nodes visited, denoted by "B&B", the total time to solve the instances in seconds, denoted by "T(s)". Results in Table 1 show that easily computable connectivity cuts are very important to avoid generating many expensive Benders cuts.

For the algorithm without Step (ii), Table 2 reports the number of generated Benders cuts, the number of "B&B" nodes, and the running times. The results in Table 2 (compare to same results in Table 1) show the practical

[4] Available at http://www.mpi-inf.mpg.de/departments/d1/projects/benchmarks/UflLib.

Instances	H	$\lambda = 3$ C(C)	C(B)	B&B	T(s)	$\lambda = 5$ C(C)	C(B)	B&B	T(s)
MP1-20	3	148	7	54	84	286	9	56	209
MP1-20	5	199	19	101	478	231	2	32	448
MP1-30	3	67	0	5	25	40	1	1	17
MP1-30	5	69	0	7	78	55	4	1	39
MP1-40	3	575	14	136	457	1483	374	620	950
MP1-40	5	301	69	215	819	379	186	528	1092
MQ1-20	3	73	2	29	83	99	0	14	57
MQ1-20	5	91	0	33	163	95	2	8	198
MQ1-30	3	231	16	87	210	159	31	54	601
MQ1-30	5	146	50	92	476	120	27	91	969
MQ1-40	3	986	458	679	568	813	5	352	683
MQ1-40	5	440	268	935	1426	560	16	241	980

Table 1
Results of our algorithm on the set of instances

improvement of the algorithm due to the addition of cuts stemming from the plain (non-length-restricted) connectivity problem.

Instances	$H=3, \lambda=3$ Cut	B&B	T(s)	$H=3, \lambda=5$ Cut	B&B	T(s)
MP1-20	654	216	283	642	107	366
MP1-30	144	9	93	58	12	47
MQ1-20	178	6	163	185	4	194
MQ1-30	466	111	385	454	67	872

Table 2
Results of our algorithm without Step (ii) on a subset of the instances

References

[1] Achterberg, T. Constraint Integer Programming. PhD Thesis, TU Berlin, 2007.

[2] Bley, A., J. Neto. Approximability of 3- and 4-Hop Bounded Disjoint Paths Problems. In Proc. of IPCO 2010, pages 205-218.

[3] Botton,Q., B. Fortz, L. Gouveia, M. Poss. Benders Decomposition for the Hop-Constrained Survivable Network Design Problem. INFORMS Journal on Computing 25, pages 13-26, 2013.

[4] Ljubic, I., S. Gollowitzer. Layered Graph Approaches to the Hop Constrained Connected Facility Location Problem, INFORMS Journal on Computing, DOI: 10.1287/ijoc.1120.0500, 2012

[5] Mahjoub, R., L. Simonetti, E. Uchoa. Hop-Level Flow Formulation for the Hop Constrained Survivable Network Design Problem. Networks 61, pages 171-179, 2013.