Periodic Solutions to Dissipative Hyperbolic Systems. II: Hopf Bifurcation for Semilinear Problems

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Abstract

We consider boundary value problems for semilinear hyperbolic systems of the type

$$\partial_t u_j + a_j(x,\lambda)\partial_x u_j + b_j(x,\lambda,u) = 0,\ x \in (0,1),\ j = 1,\dots,n$$

with smooth coefficient functions a_j and b_j such that $b_j(x,\lambda,0) = 0$ for all $x \in [0,1]$, $\lambda \in \mathbb{R}$, and $j = 1,\dots,n$. We state conditions for Hopf bifurcation, i.e., for existence, local uniqueness (up to phase shifts), smoothness and smooth dependence on λ of time-periodic solutions bifurcating from the zero stationary solution. Furthermore, we derive a formula which determines the bifurcation direction.

The proof is done by means of a Liapunov-Schmidt reduction procedure. For this purpose, Fredholm properties of the linearized system and implicit function theorem techniques are used.

There are at least two distinguishing features of Hopf bifurcation theorems for hyperbolic PDEs in comparison with those for parabolic PDEs or for ODEs: First, the question if a non-degenerate time-periodic solution depends smoothly on the system parameters is much more delicate. And second, a sufficient amount of dissipativity is needed in the system, and a priori it is not clear how to verify this in terms of the data of the PDEs and of the boundary conditions.

Key words: first-order hyperbolic systems, time-periodic solutions, reflection boundary conditions, Liapunov-Schmidt procedure, implicit function theorem, fiber contraction principle.
Mathematics Subject Classification: 35B10, 35B32, 35L50, 35L60

1 Introduction

1.1 Problem and Main Result

This paper concerns hyperbolic systems of semilinear first-order PDEs in one space dimension of the type

$$\omega\partial_t u_j + a_j(x,\lambda)\partial_x u_j + b_j(x,\lambda,u) = 0, \quad x \in (0,1),\ j = 1,\ldots,n \tag{1.1}$$

with reflection boundary conditions

$$\begin{aligned} u_j(0,t) &= \sum_{k=m+1}^{n} r_{jk}u_k(0,t), \quad j = 1,\ldots,m, \\ u_j(1,t) &= \sum_{k=1}^{m} r_{jk}u_k(1,t), \quad j = m+1,\ldots,n \end{aligned} \tag{1.2}$$

and time-periodicity conditions

$$u_j(x,t+2\pi) = u_j(x,t), \quad x \in [0,1],\ j = 1,\ldots,n. \tag{1.3}$$

Here $m \le n$ are positive integers, and $r_{jk} \in \mathbb{R}$ are reflection coefficients. We suppose that

$$a_j : [0,1]\times\mathbb{R} \to \mathbb{R} \text{ and } b_j : [0,1]\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R} \text{ are } C^\infty\text{-smooth for all } j = 1,\ldots,n, \tag{1.4}$$

$$a_j(x,0) \ne 0 \text{ for all } x \in [0,1] \text{ and } j = 1,\ldots,n, \tag{1.5}$$

$$a_j(x,0) \ne a_k(x,0) \text{ for all } x \in [0,1] \text{ and } 1 \le j \ne k \le n. \tag{1.6}$$

The number $\omega > 0$ and the function $u = (u_1,\ldots,u_n) : [0,1]\times\mathbb{R} \to \mathbb{R}^n$ are the state parameters to be determined, and $\lambda \in \mathbb{R}$ is the bifurcation parameter. Speaking about solutions to (1.1)–(1.3), throughout the paper we mean classical solutions, i.e., C^1-smooth maps $u : [0,1]\times\mathbb{R} \to \mathbb{R}^n$ which satisfy (1.1)–(1.3) pointwise.

If, for given λ and ω, u is a solution to (1.1)–(1.3) and $U : [0,1]\times\mathbb{R} \to \mathbb{R}^n$ is defined as $U(x,t) := u(x,\omega t)$, then

$$\partial_t U_j + a_j(x,\lambda)\partial_x U_j + b_j(x,\lambda,U) = 0, \quad x \in (0,1),\ j = 1,\ldots,n, \tag{1.7}$$

and U is $2\pi/\omega$-periodic with respect to time; and vice versa. In other words: Solutions to (1.1)–(1.3) correspond to $2\pi/\omega$-periodic solutions to (1.7) (which satisfy the boundary conditions (1.2)).

We suppose that for all λ and ω the function $u = 0$ is a solution (the so-called trivial solution) to (1.1)–(1.3), i.e.,

$$b_j(x, \lambda, 0) = 0 \text{ for all } x \in [0, 1], \lambda \in \mathbb{R}, \text{ and } j = 1, \dots, n. \tag{1.8}$$

We are going to describe families of non-stationary solutions to (1.1)–(1.3) bifurcating from the family of trivial solutions. With this aim we consider the following eigenvalue problem for the linearization of (1.1)–(1.3) at the trivial solution

$$\left.\begin{aligned} a_j(x, \lambda)\frac{d}{dx}v_j + \sum_{k=1}^{n} \partial_{u_k} b_j(x, \lambda, 0)v_k = \mu v_j, \quad x \in (0, 1),\ j = 1, \dots, n \\ v_j(0) = \sum_{k=m+1}^{n} r_{jk}v_k(0), \quad j = 1, \dots, m, \\ v_j(1) = \sum_{k=1}^{m} r_{jk}v_k(1), \quad j = m + 1, \dots, n \end{aligned}\right. \tag{1.9}$$

and the corresponding adjoint eigenvalue problem

$$-\frac{d}{dx}\left(a_j(x, \lambda)w_j\right) + \sum_{k=1}^{n} \partial_{u_j} b_k(x, \lambda, 0)w_k = \nu w_j, \quad x \in (0, 1),\ j = 1, \dots, n, \tag{1.10}$$

$$\begin{aligned} a_j(0, \lambda)w_j(0) &= -\sum_{k=1}^{m} r_{kj}a_k(0, \lambda)w_k(0),\ j = m + 1, \dots, n, \\ a_j(1, \lambda)w_j(1) &= -\sum_{k=m+1}^{n} r_{kj}a_k(1, \lambda)w_k(1),\ j = 1, \dots, m. \end{aligned} \tag{1.11}$$

Here $\mu, \nu \in \mathbb{C}$ are the eigenvalues and $v = (v_1, \dots, v_n), w = (w_1, \dots, w_n) : [0, 1] \to \mathbb{C}^n$ are the corresponding eigenfunctions.

Let us formulate our assumptions which are analogous to corresponding assumptions in Hopf bifurcation for ODEs (see, e.g., [4, 9, 20]) and for parabolic PDEs (see, e.g., [10, 17, 18, 22]).

The first assumption states that there exists a pure imaginary pair of geometrically simple eigenvalues to (1.9) with $\lambda = 0$:

For $\lambda = 0$ and $\mu = i$ there exists exactly one (up to linear dependence) solution $v \neq 0$ to (1.9). $\Big\}$ (1.12)

The second assumption states that the eigenvalues $\mu = \pm i$ to (1.9) with $\lambda = 0$ are algebraically simple:

$$\left.\begin{array}{l} \text{For any solution } v \neq 0 \text{ to (1.9) with } \lambda = 0 \text{ and } \mu = i \text{ and} \\ \text{for any solution } w \neq 0 \text{ to (1.10) – (1.11) with } \lambda = 0 \text{ and } \nu = -i \\ \text{we have } \displaystyle\sum_{j=1}^{n} \int_0^1 v_j(x)\overline{w_j(x)}\, dx \neq 0. \end{array}\right\} \tag{1.13}$$

In what follows, we fix a solution $v = v^0$ to (1.9) with $\lambda = 0$ and $\mu = i$ and a solution $w = w^0$ to (1.10)–(1.11) with $\lambda = 0$ and $\mu = -i$ such that

$$\sum_{j=1}^{n} \int_0^1 v_j^0(x)\overline{w_j^0(x)}\, dx = 2. \tag{1.14}$$

The third assumption is the so-called transversality condition. It states that the eigenvalue $\mu = \mu(\lambda) \approx i$ to (1.9) with $\lambda \approx 0$ crosses the imaginary axis transversally if λ crosses zero:

$$\alpha := \frac{1}{2} \operatorname{Re} \sum_{j=1}^{n} \int_0^1 \left(\partial_\lambda a_j(x,0) \frac{d}{dx} v_j^0(x) + \sum_{k=1}^{n} \partial_\lambda \partial_{u_k} b_j(x,0,0) v_k^0(x) \right) \overline{w_j^0(x)}\, dx \neq 0 \tag{1.15}$$

(in fact it holds $\operatorname{Re} \mu'(0) = \alpha$, cf. (4.6)).

The fourth assumption is the so-called nonresonance condition:

$$\text{If } (\mu, v) \text{ is a solution to (1.9) with } \lambda = 0,\ \mu = ik,\ k \in \mathbb{Z}, \text{ and } v \neq 0, \text{ then } k = \pm 1. \tag{1.16}$$

In order to formulate our main result we need some more notation:

$$b_{jk}(x) := \partial_{u_k} b_j(x,0,0),\ b_{jkl}(x) := \partial^2_{u_k u_l} b_j(x,0,0),\ b_{jklr}(x) := \partial^3_{u_k u_l u_r} b_j(x,0,0), \tag{1.17}$$

$$R_0 := \max_{1 \le j \le m} \max_{0 \le x \le 1} \sum_{k=m+1}^{n} |r_{jk}| \exp\left(- \int_0^x \frac{b_{jj}(\xi)}{a_j(\xi,0)} d\xi \right), \tag{1.18}$$

$$R_1 := \max_{m+1 \le j \le n} \max_{0 \le x \le 1} \sum_{k=1}^{m} |r_{jk}| \exp \int_x^1 \frac{b_{jj}(\xi)}{a_j(\xi,0)} d\xi, \tag{1.19}$$

and

$$\beta := -\operatorname{Re} \int_0^1 \left(\frac{1}{8} \sum_{j,k,l,r=1}^{n} b_{jklr} v_k^0 v_l^0 \overline{v_r^0}\, \overline{w_j^0} + \sum_{j,k,l=1}^{n} b_{jkl} \left(y_k \overline{v_l^0} + z_k v_l^0 \right) \overline{w_j^0} \right) dx. \tag{1.20}$$

Here $y : [0,1] \to \mathbb{C}^n$ and $z : [0,1] \to \mathbb{R}^n$ are the solutions to the boundary value problems

$$\begin{aligned}
a_j(x,0)\frac{d}{dx}y_j - 2iy_j + \sum_{k=1}^n b_{jk}(x)y_k &= -\frac{1}{4}\sum_{k,l=1}^n b_{jkl}(x)v_k^0(x)v_l^0(x), \quad j = 1,\dots,n\\
y_j(0) &= \sum_{k=m+1}^n r_{jk}y_k(0), \qquad j = 1,\dots,m,\\
y_j(1) &= \sum_{k=1}^m r_{jk}y_k(1), \qquad j = m+1,\dots,n
\end{aligned}$$

and

$$\begin{aligned}
a_j(x,0)\frac{d}{dx}z_j + \sum_{k=1}^n b_{jk}(x)z_k &= -\frac{1}{2}\sum_{k,l=1}^n b_{jkl}(x)v_k^0(x)\overline{v_l^0(x)}, \quad j = 1,\dots,n\\
z_j(0) &= \sum_{k=m+1}^n r_{jk}z_k(0), \qquad j = 1,\dots,m,\\
z_j(1) &= \sum_{k=1}^m r_{jk}z_k(1), \qquad j = m+1,\dots,n.
\end{aligned}$$

Remark that the two boundary value problems above are uniquely solvable due to assumption (1.16).

Definition 1.1 *We denote by $\mathcal{C}_n$ the space of all continuous maps $u : [0,1]\times\mathbb{R} \to \mathbb{R}^n$, which satisfy the time-periodicity condition (1.3), with the norm*

$$\|u\|_\infty := \max_{1\le j\le n}\max_{0\le x\le 1}\max_{t\in\mathbb{R}}|u_j(x,t)|.$$

Now we are prepared to formulate our main result:

Theorem 1.2 *Suppose (1.4)–(1.6), (1.8), (1.12)–(1.16), and*

$$R_0R_1 < 1. \tag{1.21}$$

Then there exist $\varepsilon_0 > 0$ and C^2-maps $\hat\lambda : [0,\varepsilon_0] \to \mathbb{R}$, $\hat\omega : [0,\varepsilon_0] \to \mathbb{R}$, and $\hat u : [0,\varepsilon_0] \to \mathcal{C}_n$ such that the following is true:

(i) Existence of nontrivial solutions: *For all $\varepsilon \in (0,\varepsilon_0]$ the function $\hat u(\varepsilon)$ is a C^∞-smooth nontrivial solution to (1.1)–(1.3) with $\lambda = \hat\lambda(\varepsilon)$ and $\omega = \hat\omega(\varepsilon)$.*

(ii) Asymptotic expansion and bifurcation direction: *We have*

$$\hat\lambda(0) = \hat\lambda'(0) = 0,\ \hat\lambda''(0) = \frac{\beta}{\alpha},\ \omega(0) = 1,$$

and $\hat{u}(0)(x,t) = 0$, $\hat{u}'(0)(x,t) = \operatorname{Re} v^0(x) \cos t - \operatorname{Im} v^0(x) \sin t$ *for all* $x \in [0,1]$ *and* $t \in \mathbb{R}$.

(iii) Local uniqueness: *There exists* $\delta > 0$ *such that for all nontrivial solutions to (1.1)–(1.3) with* $|\lambda| + |\omega - 1| + \|u\|_\infty < \delta$ *there exist* $\varepsilon \in (0, \varepsilon_0]$ *and* $\varphi \in \mathbb{R}$ *such that* $\lambda = \hat{\lambda}(\varepsilon)$, $\omega = \hat{\omega}(\varepsilon)$, *and* $u(x,t) = \hat{u}(\varepsilon)(x, t + \varphi)$ *for all* $x \in [0,1]$ *and* $t \in \mathbb{R}$.

Our paper is organized as follows:

In Section 1.2 we comment about mathematical models which contain dissipative hyperbolic PDEs and which are used for describing destabilization of stationary states and/or for describing stable time-periodic processes.

In Section 1.3 we comment about some publications which are related to ours.

In Section 2 we derive the weak formulation (2.1), (2.2) for the PDE problem (1.1)–(1.3) via integration along characteristics, and we introduce operators in order to write this weak formulation as the operator equation (2.11).

In Section 3 we do a Liapunov-Schmidt procedure (as it is known for Hopf bifurcation for parabolic PDEs or for ODEs) in order to reduce (for $\lambda \approx 0, \omega \approx 1$ and $u \approx 0$) the problem (1.1)–(1.3) with infinite-dimensional state parameter $(\omega, u) \in \mathbb{R} \times \mathcal{C}_n$ to a problem with two-dimensional state parameter. Here the main technical results are Lemmas 3.3 and 3.4 about local unique solvability of the infinite dimensional part of the Liapunov-Schmidt system and Lemma 3.6 about smooth dependence of the solution on parameters. The proofs of these Lemmas are much more complicated as the corresponding proofs for parabolic PDEs or for ODEs. The point is that in the case of dissipative hyperbolic PDEs the question of Fredholm solvability of linear periodic problems as well as the question of smooth dependence of solutions on parameters are much more difficult. The difficulty with the Fredholmness is solved in [26].

In Section 4 we put the solution of the infinite dimensional part of the Liapunov-Schmidt system into the finite dimensional part and discuss the behavior of the resulting equation. This is completely analogous to what is known from Hopf bifurcation for ODEs and parabolic PDEs.

In Section 5 we present an example of a problem of the type (1.1)–(1.3) such that all assumptions (1.4)–(1.6), (1.8), (1.12)–(1.16), and (1.21) of Theorem 1.2 are satisfied.

Finally, in the appendix we present a simple linear version of the so-called fiber contraction principle, which is used in the proof of the key technical Lemma 3.6.

1.2 Applications

Almost all features of the phenomenon Hopf bifurcation for dissipative hyperbolic PDEs are the same as those for parabolic PDEs. Hence, one could ask why Hopf bifurcation is well established since 40 years for parabolic PDEs, but for dissipative hyperbolic PDEs almost nothing is known.

One reason is that the proofs for dissipative hyperbolic PDEs are more difficult. But a second reason is that in the past there existed only a few real world mathematical models containing dissipative hyperbolic PDEs and which were used for describing, optimizing and stabilizing time-periodic processes.

One field of appearance of those models is modeling of semiconductor laser devices and their applications in communication systems (see, e.g. [29, 38, 39, 40, 45]). Remark that the above cited semiconductor laser models have some specific features: There the hyperbolic PDEs (balance equations for the complex amplitudes of the light field) have complex coefficients, and they are coupled with ODEs (balance equations for the electron densities). Moreover, the models possess a nonlinear-Schrödinger-equation-like $SO(2)$-equivariance, therefore the Hopf bifurcations are bifurcations from relative equilibria (rotating waves) into relative periodic orbits (modulated waves). Anyway, for proving Hopf bifurcation there one has to overcome the same problems as in the present paper.

In [6] (with applications to population dynamics), [19] (with applications to correlated random walks), [21] (with applications to Brownian motion) and [35] (with applications to Rayleigh-Bénard convection) the authors considered semilinear hyperbolic systems of the type (1.1) with boundary conditions of the type (1.2) and with certain additional structures (determined by the applications) in the PDEs as well as in the boundary conditions. A linear stability analysis for the stationary solutions is done, and the bifurcation hypersurfaces in the space of control parameters are described, in particular those where the conditions (1.12), (1.13), and (1.15) are satisfied and, hence, where the authors expect Hopf bifurcation to appear.

1.3 Some remarks on related work

The main methods for proving Hopf bifurcation theorems are, roughly speaking, center manifold reduction and Liapunov-Schmidt reduction. In order to apply them to abstract evolution equations one needs to have a smooth center manifold for the corresponding semiflow (for center manifold reduction) or a Fredholm property of the linearized equation on spaces of periodic functions (for Liapunov-Schmidt reduction). It seems that hyperbolic partial differential operators do not satisfy conditions like [17, Hypothesis 2.7 in Chapter 2.2], [32, Chapter 2.4.2], or [46, Hypothesis (H)], which are usually assumed to be fulfilled by the operators in the abstract evolution equations and which imply the existence of center manifolds, and they also do not satisfy the conditions like [10, Hypothesis (HL)] or [22, Hypothesis I.8.8], which are assumed in order to apply Liapunov-Schmidt reduction.

The celebrated counter example of M. Renardy [44] shows that a reasonable linear hyperbolic differential operator in two space dimensions with periodic boundary conditions may not satisfy the spectral mapping property (see also [8, 13]), and, hence, its spectral decomposition does not create a corresponding spectral decomposition of the corresponding linear semiflow. Therefore, the question of existence of center manifolds for small nonlinear perturbations of this linear semiflow is completely open.

But hyperbolic PDEs in one space dimension are better: They satisfy, under reasonable assumptions, the spectral mapping property in L^p-spaces (see [37]) as well as in C-spaces (see [27]). The spectral mapping property in L^p-spaces is used in [38, 45] to show the existence of smooth center manifolds for linear first-order hyperbolic PDE systems which are coupled with nonlinear ODEs. Here the linearity of the problem with respect to the infinite dimensional part of the phase space is essential, because it implies the well-posedness and smoothness in L^p-spaces of the Nemyckii operators. The spectral mapping property in C-spaces is used in [28] to show the way how to prove the existence of smooth center manifolds in C-spaces for general semilinear first-order hyperbolic systems. It seems that going this way one could prove the Hopf bifurcation theorem of the the present paper as well.

The eigenvalue problem (1.9) is well-understood. The set of the real parts of all eigenvalues is bounded from above. All eigenvalues have finite multiplicity. The eigenvalues are asymptotically (for large imaginary parts) close to eigenvalues of the corresponding "diagonal" eigenvalue problem (i.e., if the non-diagonal terms $\partial_{u_k} b_j(x, \lambda, 0)v_k$ with $j \neq k$ in (1.9) are neglected). If (1.21) is satisfied, then the supremum of the real parts of the eigenvalues of the "diagonal" eigenvalue problem is negative, and, hence, only finitely many eigenvalues of the "full" eigenvalue problem (1.9) can be close to the imaginary axis. For related rigorous statements see, e.g., [27], [31, Chapter 6.1] and [36, 37, 42].

In [46] a 1D semilinear damped wave equation of the type $\partial_t^2 u = \partial_x^2 u - \gamma \partial_t u + f(u)$ with $f(0) = 0$ is considered. This equation is subjected to zero Dirichlet boundary conditions, which imply the exceptional property that for u satisfying the boundary conditions the nonlinearity $f(u)$ satisfies the boundary conditions also. This property is essentially used to prove the existence of a smooth center manifold.

In [2] Hopf bifurcation for (1.1)–(1.3) with $a_j(x, \lambda)$ not depending on λ is considered. It is assumed that many reflection coefficients r_{jk} vanish, which allows to use some smoothing property for the solutions to the corresponding linearized initial-boundary value problem [12]. But, unfortunately, there is an essential gap in the realization of the Liapunov-Schmidt procedure: Using our notation, the finite-dimensional part (3.14) of the Liapunov-Schmidt system is first locally solved with respect to λ and μ (in terms of v and w):

$$\lambda = \hat{\lambda}(v, w),\ \omega = \hat{\omega}(v, w).$$

Then these solutions are inserted into the infinite-dimensional part (3.15) of the Liapunov-Schmidt system. Finally the intention is to solve the resulting equation by means of the implicit function theorem with respect to w. But in the resulting equation there appear terms of the type

$$w_j(\xi, \tau_j(\xi, x, t, \hat{\omega}(v, w))).$$

In other words: The unknown function appears in the argument of the unknown function (like, for example, in ODEs with state depending delay). If one formally differentiates this expression with respect to w, then there appears $\partial_t w_j(\xi, \tau_j(\xi, x, t, \hat{\omega}(v, w)))$, which has less smoothness with respect to t than w_j. Roughly speaking, this loss of smoothness property is the reason that the nonlinear operator corresponding to the resulting equation is not differentiable in a neighborhood of zero, neither in the sense of C-spaces nor of C^1-spaces. Hence, the implicit function theorem (at least the classical one) is not applicable.

In [33] the authors considered scalar linear first-order PDEs of the type $(\partial_t+\partial_x+\mu)u=0$ on $(0,\infty)$ with a nonlinear integral boundary condition at $x=0$:

$$u(0,t)=h\left(\int_0^\infty \gamma(x)u(x,t)\,dx\right).$$

The nonlocality of the boundary condition is essential for the applied techniques in [33] (integrated semigroup theory, see also [30]) to get existence of center manifolds and Hopf bifurcation. It is easy to realize that we could also consider (linear or nonlinear) integral boundary conditions, i.e., if we would replace (1.2) by boundary conditions of the type

$$\begin{aligned} u_j(0,t)&=h_j\left(\int_0^1 \gamma_j(x)u(x,t)\,dx\right), \quad j=1,\dots,m,\\ u_j(1,t)&=h_j\left(\int_0^1 \gamma_j(x)u(x,t)\,dx\right), \quad j=m+1,\dots,n, \end{aligned}$$

then we would get essentially the same result as that described in Theorem 1.2, even without any assumption of the type (1.21). Roughly speaking, the reason is that the weak formulation of the problem will be of the type (2.11) again, but now with $C(\lambda,\omega)$ being compact (due to a smoothing property proved in [23]).

To the best of our knowledge, almost no results exist concerning smooth dependence on parameters of non-degenerate time-periodic solutions of dissipative hyperbolic PDEs. The paper [16, Chapter 3.5.1] shows the difficulty of this problem. There conditions are formulated such that a non-degenerate time-periodic solution to a system of semilinear damped wave equations survives under small parameter perturbations, but nothing is known if the perturbed solution depends smoothly on the perturbation parameters. Results about smooth dependence on data for time-periodic solutions to linear first-order hyperbolic systems with reflection boundary conditions are given in [25].

2 Abstract formulation of (1.1)–(1.3)

In this section we derive (for $\lambda\approx 0$) a weak formulation for the PDE problem (1.1)–(1.3) via integration along characteristics, and we introduce operators in order to write this weak formulation as an operator equation.

Let $\delta_0>0$ be sufficiently small such that (cf. assumption (1.5))

$$a_j(x,\lambda)\neq 0 \text{ for all } j=1,\dots,n,\ x\in[0,1] \text{ and } \lambda\in[-\delta_0,\delta_0].$$

Straightforward calculations (cf. [26, Section 2]) show that a C^1-map $u : [0,1] \times \mathbb{R} \to \mathbb{R}^n$ is a solution to the PDE problem (1.1)–(1.3) with $\lambda \in [-\delta_0, \delta_0]$ if and only if it is a solution to the following system of integral equations:

$$
\begin{aligned}
u_j(x,t) &= c_j(0,x,\lambda) \sum_{k=m+1}^{n} r_{jk} u_k(0, \tau_j(0,x,t,\lambda,\omega)) \\
&\quad + \int_0^x \frac{c_j(\xi,x,\lambda)}{a_j(\xi,\lambda)} f_j(\xi,\lambda,u(\xi,\tau_j(\xi,x,t,\lambda,\omega)))d\xi, \quad j = 1,\dots,m,
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
u_j(x,t) &= c_j(1,x,\lambda) \sum_{k=1}^{m} r_{jk} u_k(1, \tau_j(1,x,t,\lambda,\omega)) \\
&\quad - \int_x^1 \frac{c_j(\xi,x,\lambda)}{a_j(\xi,\lambda)} f_j(\xi,\lambda,u(\xi,\tau_j(\xi,x,t,\lambda,\omega)))d\xi, \quad j = m+1,\dots,n.
\end{aligned}
\tag{2.2}
$$

Here

$$
\tau_j(\xi,x,t,\lambda,\omega) := \omega \int_x^\xi \frac{d\eta}{a_j(\eta,\lambda)} + t \tag{2.3}
$$

is the j-th characteristic of the hyperbolic system (1.1) and

$$
c_j(\xi,x,\lambda) := \exp \int_x^\xi \frac{\partial_{u_j} b_j(\eta,\lambda,0)}{a_j(\eta,\lambda)} d\eta, \tag{2.4}
$$

$$
f_j(x,\lambda,u) := \partial_{u_j} b_j(x,\lambda,0) u_j - b_j(x,\lambda,u). \tag{2.5}
$$

If we set $b(x,\lambda,u) := (b_1(x,\lambda,u),\dots,b_n(x,\lambda,u))$ and $f(x,\lambda,u) := (f_1(x,\lambda,u),\dots,f_n(x,\lambda,u))$, then the nonlinear map $-f(x,\lambda,\cdot)$ is the difference of the nonlinear map $b(x,\lambda,\cdot)$ and of the diagonal part of the linear map $\partial_u b(x,\lambda,0)$. Hence, the diagonal part of $\partial_u f(x,\lambda,0)$ vanishes. This will be used later on (see (2.21) and the text there), because this implies that the linear operators $I - C(\lambda,\omega) - \partial_u F(\lambda,\omega,0)$ (see (2.9) and (2.10)) are Fredholm of index zero from $\mathcal{C}_n$ into $\mathcal{C}_n$.

Let us show that any solution to (1.1)–(1.3) satisfies (2.1)–(2.2): If u is a C^1-solution to (1.1)–(1.3), then

$$
\begin{aligned}
&\frac{d}{d\xi} u_j(\xi,\tau_j(\xi,x,t,\lambda,\omega)) + \frac{\partial_{u_j} b_j(\eta,\lambda,0)}{a_j(\xi,\tau_j(\xi,x,t,\lambda,\omega))} u_j(\xi,\omega_j(\xi,x,t,\lambda,\omega)) \\
&= \frac{f_j(\xi,\lambda,u(\xi,\tau_j(\xi,x,t,\lambda,\omega)))}{a_j(\xi,\tau_j(\xi,x,t,\lambda,\omega))}.
\end{aligned}
\tag{2.6}
$$

Now, applying the variation of constants formula and using the boundary conditions (1.3), one gets (2.1)–(2.2).

And vice versa: For any C^1-smooth map $u : [0,1] \times \mathbb{R} \to \mathbb{R}$ it holds

$$
\begin{aligned}
&(\omega\partial_t + a_j(x,\lambda)\partial_x)\left(c_j(\xi,x,\lambda)u_k(\xi,\tau_j(\xi,x,t,\lambda,\omega))\right)\\
&\quad = -\partial_{u_j} b_j(x,\lambda,0)c_j(\xi,x,\lambda)u_k(\xi,\tau_j(\xi,x,t,\lambda,\omega))
\end{aligned}
\tag{2.7}
$$

and

$$
\begin{aligned}
&(\omega\partial_t + a_j(x,\lambda)\partial_x)\int_\xi^x \frac{c_j(\eta,x,\lambda)}{a_j(\eta,\lambda)} f_j(\eta,\lambda,u(\eta,\tau_j(\eta,x,t,\lambda,\omega)))d\eta\\
&= f_j(x,\lambda,u(x,t)) - \partial_{u_j} b_j(x,\lambda,0)\int_\xi^x \frac{c_j(\eta,x,\lambda)}{a_j(\eta,\lambda)} f_j(\eta,\lambda,u(\eta,\tau_j(\eta,x,t,\lambda,\omega)))d\eta.
\end{aligned}
\tag{2.8}
$$

Therefore, if u satisfies (2.1)–(2.2), then it satisfies (1.1).

Now, for $\lambda \in [-\delta_0,\delta_0]$ and $\omega \in \mathbb{R}$ we define linear bounded operators $C(\lambda,\omega) : \mathcal{C}_n \to \mathcal{C}_n$ by

$$
\left(C(\lambda,\omega)u\right)_j(x,t) := \left\{
\begin{array}{l}
c_j(0,x,\lambda)\displaystyle\sum_{k=m+1}^{n} r_{jk}u_k(0,\tau_j(0,x,t,\lambda,\omega)),\ j = 1,\dots,m,\\
c_j(1,x,\lambda)\displaystyle\sum_{k=1}^{m} r_{jk}u_k(1,\tau_j(1,x,t,\lambda,\omega)),\ j = m+1,\dots,n
\end{array}
\right.
\tag{2.9}
$$

and nonlinear operators $F(\lambda,\omega,\cdot) : \mathcal{C}_n \to \mathcal{C}_n$ by

$$
F(\lambda,\omega,u)_j(x,t) := \int_{x_j}^x \frac{c_j(\xi,x,\lambda)}{a_j(\xi,\lambda)} f_j(\xi,\lambda,u(\xi,\tau_j(\xi,x,t,\lambda,\omega)))d\xi. \tag{2.10}
$$

Here and in what follows we denote

$$
x_j := \left\{
\begin{array}{ll}
0 & \text{for} \quad j = 1,\dots,m,\\
1 & \text{for} \quad j = m+1,\dots,n.
\end{array}
\right.
$$

Using this notation, system (2.1)–(2.2) is equivalent to the operator equation

$$
u = C(\lambda,\omega)u + F(\lambda,\omega,u). \tag{2.11}
$$

Because of assumption (1.4) and well-known differentiability properties of Nemytskii operators (see, e.g., [1, Lemma 2.4.18], [3, Lemma 6.1]) the superposition operator $u \in \mathcal{C}_n \mapsto f(\cdot,\lambda,u(\cdot)) \in \mathcal{C}_n$ (cf. (2.5)) is C^∞-smooth. Hence, $F(\lambda,\omega,\cdot)$ is C^∞-smooth. It is easy to verify that

$$
(\lambda,\omega,u) \in [-\delta_0,\delta_0] \times \mathbb{R} \times \mathcal{C}_n \mapsto C(\lambda,\omega)u \in \mathcal{C}_n \text{ is continuous} \tag{2.12}
$$

and that for all $k = 0, 1, 2, \ldots$

$$\begin{array}{l}(\lambda, \omega, u, v_1, \ldots, v_k) \in [-\delta_0, \delta_0] \times \mathbb{R} \times \mathcal{C}_n \times \ldots \times \mathcal{C}_n \\ \mapsto \partial_u^k F(\lambda, \omega, u)(v_1, \ldots, v_k) \in \mathcal{C}_n \text{ is continuous.}\end{array} \tag{2.13}$$

Moreover, for all $k = 0, 1, 2, \ldots$ and $\rho > 0$ there exists $c_k(\rho) > 0$ such that for all $\lambda \in [-\delta_0, \delta_0]$, $\omega \in \mathbb{R}$, $u \in \mathcal{C}_n$ with $\|u\|_\infty \le \rho$ and $v_1, \ldots, v_k \in \mathcal{C}_n$ it holds

$$\|\partial_u^k F(\lambda, \omega, u)(v_1, \ldots, v_k)\|_\infty \le c_k(\rho)\|v_1\|_\infty \ldots \|v_k\|_\infty. \tag{2.14}$$

Unfortunately, the maps $(\lambda, \omega) \in [-\delta_0, \delta_0] \times \mathbb{R} \mapsto C(\lambda, \omega)u \in \mathcal{C}_n$ as well as $(\lambda, \omega) \in [-\delta_0, \delta_0] \times \mathbb{R} \mapsto F(\lambda, \omega, u) \in \mathcal{C}_n$ are not smooth, in general, if u is only continuous and not smooth, cf. (3.54). This makes the question, if the data-to-solution map corresponding to (2.11) is smooth, very delicate.

Definition 2.1 *We denote by $\mathcal{C}_n^1$ the Banach space of all $u \in \mathcal{C}_n$ such that the partial derivatives $\partial_x u$ and $\partial_t u$ exist and are continuous with the norm $\|u\|_\infty + \|\partial_x u\|_\infty + \|\partial_t u\|_\infty$.*

Directly from the definitions (2.9) and (2.10) it follows that for all $u \in \mathcal{C}_n^1$ we have $C(\lambda, \omega)u \in \mathcal{C}_n^1$, $F(\lambda, \omega, u) \in \mathcal{C}_n^1$, and

$$\begin{array}{rcl}\partial_x C(\lambda, \omega)u &=& C_0^x(\lambda, \omega)u + C_1^x(\lambda, \omega)\partial_t u, \\ \partial_t C(\lambda, \omega)u &=& C(\lambda, \omega)\partial_t u, \\ \partial_x F(\lambda, \omega, u) &=& F_0^x(\lambda, \omega, u) + F_1^x(\lambda, \omega, u)\partial_t u, \\ \partial_t F(\lambda, \omega, u) &=& \partial_u F(\lambda, \omega, u)\partial_t u,\end{array} \tag{2.15}$$

where $C_0^x(\lambda, \omega)$, $C_1^x(\lambda, \omega)$, and $F_1^x(\lambda, \omega, u)$ are linear bounded operators from $\mathcal{C}_n$ into $\mathcal{C}_n$ defined by

$$\left(C_0^x(\lambda, \omega)u\right)_j(x, t) := \left\{\begin{array}{l}\partial_x c_j(0, x, \lambda) \displaystyle\sum_{k=m+1}^{n} r_{jk} u_k(0, \tau_j(0, x, t, \lambda, \omega)),\ j = 1, \ldots, m, \\ \partial_x c_j(1, x, \lambda) \displaystyle\sum_{k=1}^{m} r_{jk} u_k(1, \tau_j(1, x, t, \lambda, \omega)),\ j = m+1, \ldots, n,\end{array}\right.$$

$$\left(C_1^x(\lambda, \omega)v\right)_j(x, t) := \left\{\begin{array}{l}-\dfrac{\omega c_j(0, x, \lambda)}{a_j(x, \lambda)} \displaystyle\sum_{k=m+1}^{n} r_{jk} v_k(0, \tau_j(0, x, t, \lambda, \omega)),\ j = 1, \ldots, m, \\ -\dfrac{\omega c_j(1, x, \lambda)}{a_j(x, \lambda)} \displaystyle\sum_{k=1}^{m} r_{jk} v_k(1, \tau_j(1, x, t, \lambda, \omega)),\ j = m+1, \ldots, n,\end{array}\right.$$

and

$$\begin{array}{l}(F_1^x(\lambda, \omega, u)v)_j(x, t) \\ := -\displaystyle\int_{x_j}^{x} \frac{\omega c_j(\xi, x, \lambda)}{a_j(x, \lambda)a_j(\xi, \lambda)} \sum_{k=1}^{n} \partial_{u_k} f_j(\xi, \lambda, u(\xi, \tau_j(\xi, x, t, \lambda, \omega)))v_k(\xi, \tau_j(\xi, x, t, \lambda, \omega))d\xi,\end{array}$$

and $F_0^x(\lambda,\omega,u) \in \mathcal{C}_n$ is defined by

$$(F_0^x(\lambda,\omega,u))_j(x,t) := \frac{f_j(x,\lambda,u(x,t))}{a_j(x,t)} + \int_{x_j}^x \frac{\partial_x c_j(\xi,x,\lambda)}{a_j(\xi,\lambda)} f_j(\xi,\lambda,u(\xi,\tau_j(\xi,x,t,\lambda,\omega)))d\xi.$$

Lemma 2.2 *Suppose (1.21). Then for any $\gamma_1 > 0$ there exist $\delta_1 \in (0,\delta_0]$ and $c_1 > 0$ such that for all $\lambda \in [-\delta_1,\delta_1]$ and all $\omega \in [-\gamma_1,\gamma_1]$ the following is true:*

(i) The operators $I - C(\lambda,\omega)$ are isomorphisms on $\mathcal{C}_n$ as well as on $\mathcal{C}_n^1$, and

$$\|(I - C(\lambda,\omega))^{-1}\|_{\mathcal{L}(\mathcal{C}_n)} + \|(I - C(\lambda,\omega))^{-1}\|_{\mathcal{L}(\mathcal{C}_n^1)} \le c_1.$$

(ii) The operators $I - C(\lambda,\omega) - \partial_u F(\lambda,\omega,0)$ are Fredholm operators of index zero from $\mathcal{C}_n$ into $\mathcal{C}_n$.

Proof. To prove assertion (i), denote by $\mathcal{C}_m$ the space of all continuous maps $v : [0,1] \times \mathbb{R} \to \mathbb{R}^m$ with $v(x,t+2\pi) = v(x,t)$ for all $x \in [0,1]$ and $t \in \mathbb{R}$, with the norm

$$\|v\|_\infty := \max_{1\le j\le m} \max_{0\le x\le 1} \max_{t\in\mathbb{R}} |v_j(x,t)|.$$

Similarly we define the space $\mathcal{C}_{n-m}$. The spaces $\mathcal{C}_n$ and $\mathcal{C}_m \times \mathcal{C}_{n-m}$ will be identified, i.e., elements $u \in \mathcal{C}_n$ will be written as $u = (v,w)$ with $v \in \mathcal{C}_m$ and $w \in \mathcal{C}_{n-m}$. Then the operators $C(\lambda,\omega)$ work as

$$C(\lambda,\omega)u = (K(\lambda,\omega)w, L(\lambda,\omega)v) \quad \text{for } u = (v,w), \tag{2.16}$$

where the linear bounded operators $K(\lambda,\omega) : \mathcal{C}_{n-m} \to \mathcal{C}_m$ and $L(\lambda,\omega) : \mathcal{C}_m \to \mathcal{C}_{n-m}$ are defined by the right hand side of (2.9).

Let $f = (g,h) \in \mathcal{C}_n$ with $g \in \mathcal{C}_m$ and $h \in \mathcal{C}_{n-m}$ be arbitrarily given. We have $u = C(\lambda,\omega)u + f$ if and only if $v = K(\lambda,\omega)w + g,\ w = L(\lambda,\omega)v + h$, i.e., if and only if

$$v = K(\lambda,\omega)(L(\lambda,\omega)v + h) + g, \quad w = L(\lambda,\omega)v + h. \tag{2.17}$$

Moreover, it holds

$$\|K(\lambda,\omega)w\|_\infty = \max_{1\le j\le m} \max_{0\le x\le 1} \max_{t\in\mathbb{R}} \left| c_j(0,x,\lambda) \sum_{k=m+1}^n r_{jk} w_k(0,\tau_j(0,x,t,\lambda,\omega)) \right| \le R_0(\lambda)\|w\|_\infty$$

with

$$R_0(\lambda) := \max_{1\le j\le m} \max_{0\le x\le 1} \sum_{k=m+1}^n |r_{jk}| \exp\left(-\int_0^x \frac{\partial_{u_j} b_j(\eta,\lambda,0)}{a_j(\eta,\lambda)} d\eta \right).$$

Similarly one shows that $\|L(\lambda,\omega)v\|_\infty \le R_1(\lambda)\|v\|_\infty$ with

$$R_1(\lambda) := \max_{m+1\le j\le n} \max_{0\le x\le 1} \sum_{k=m+1}^n |r_{jk}| \exp \int_x^1 \frac{\partial_{u_j} b_j(\eta,\lambda,0)}{a_j(\eta,\lambda)} d\eta.$$

Since $R_0(\lambda)$ and $R_1(\lambda)$ depend continuously on λ, notation (1.18) and (1.19) and assumption (1.21) yield that there exist $\delta_1 \in (0, \delta_0]$ and $c \in (0,1)$ such that for all $\lambda \in [-\delta_1, \delta_1]$ and all $\omega \in \mathbb{R}$ we have

$$\|K(\lambda,\omega)\|_{\mathcal{L}(\mathcal{C}_{n-m};\mathcal{C}_m)}\|L(\lambda,\omega)\|_{\mathcal{L}(\mathcal{C}_m;\mathcal{C}_{n-m})} \le c.$$

Therefore, for those λ and ω it holds

$$\|(I - K(\lambda,\omega)L(\lambda,\omega))^{-1}\|_{\mathcal{L}(\mathcal{C}_{n-m})} \le \frac{1}{1-c},$$

hence, the system (2.17) is uniquely solvable with respect to v and w. Moreover, we have the following a priori estimates:

$$\begin{aligned} \|v\|_\infty &\le \frac{1}{1-c}\|K(\lambda,\omega)h + g\|_\infty \le \frac{1}{1-c}\left(R_0(\lambda)\|h\|_\infty + \|g\|_\infty\right), \\ \|w\|_\infty &= \|L(\lambda,\omega)v + h\|_\infty \le \text{const } \left(\|g\|_\infty + \|h\|_\infty\right), \end{aligned}$$

i.e., $\|u\|_\infty \le \text{const}\,\|f\|_\infty$. Here the constants do not depend on $\lambda \in [-\delta_1, \delta_1]$ and $\omega \in \mathbb{R}$.

Now fix an arbitrary $\gamma_1 > 0$. Suppose $u = C(\lambda,\omega)u + f$ with $f \in \mathcal{C}_n^1$, $\lambda \in [-\delta_1, \delta_1]$, and $\omega \in [-\gamma_1, \gamma_1]$. Then [25, Theorem 1.2 (iv)] implies that $u \in \mathcal{C}_n^1$. Hence, it remains to show that

$$\|u\|_\infty + \|\partial_x u\|_\infty + \|\partial_t u\|_\infty \le \text{const}\left(\|f\|_\infty + \|\partial_x f\|_\infty + \|\partial_t f\|_\infty\right), \tag{2.18}$$

where the constant can be chosen independently of u, f, λ, and ω.

Because of (2.15) we have

$$\partial_x u = C_0^x(\lambda,\omega)u + C_1^x(\lambda,\omega)\partial_t u + \partial_x f, \tag{2.19}$$
$$\partial_t u = C(\lambda,\omega)\partial_t u + \partial_t f. \tag{2.20}$$

The equation (2.20) now yields $\|\partial_t u\|_\infty \le \text{const } \|\partial_t f\|_\infty$, and this together with (2.19) gives (2.18). Here the constants do not depend on $\lambda \in [-\delta_1, \delta_1]$ and $\omega \in [-\gamma_1, \gamma_1]$, but on γ_1, because the norm of the operator $C_1^x(\lambda,\omega)$ grows if ω grows, in general.

To prove assertion (ii), we take into account (2.5) and (2.10), hence

$$\left(\partial_u F(\lambda,\omega,0)u\right)_j(x,t) = -\int_{x_j}^x \frac{c_j(\xi,x,\lambda)}{a_j(\xi,\lambda)} \sum_{\substack{k=1\\k\neq j}}^n \partial_{u_k} b_j(\xi,\lambda,0)u_k(\xi,\tau_j(\xi,x,t,\lambda,\omega)))d\xi. \tag{2.21}$$

Since the right-hand side of (2.21) does not depend on u_j, we can use assertions (i)–(iii) of [26, Theorem 1.2] and state that the operators $I - C(\lambda,\omega) - \partial_u F(\lambda,\omega,0)$ are Fredholm of index zero from $\mathcal{C}_n$ into $\mathcal{C}_n$ if $R_0(\lambda)R_1(\lambda) < 1$. Because of assumption (1.21) this is the case if λ is sufficiently close to zero. $\square$

3 Liapunov-Schmidt procedure

In this section we do a Liapunov-Schmidt procedure in order to reduce (locally for $\lambda \approx 0, \omega \approx 1$ and $u \approx 0$) the problem (2.11) with infinite-dimensional state parameter $(\omega, u) \in \mathbb{R} \times \mathcal{C}_n$ to a problem with two-dimensional state parameter.

For $\lambda \in [-\delta_0, \delta_0]$ and $\omega \in \mathbb{R}$ we introduce linear bounded operators $A(\lambda, \omega), \tilde{A}(\lambda, \omega) : \mathcal{C}_n^1 \to \mathcal{C}_n$ by

$$
\begin{aligned}
\left[A(\lambda,\omega)u\right](x,t) &:= \left[\omega\partial_t u_j(x,t) + a_j(x,\lambda)\partial_x u_j(x,t) + \partial_{u_j} b_j(x,\lambda,0)u_j(x,t)\right]_{j=1}^n, \\
\left[\tilde{A}(\lambda,\omega)u\right](x,t) &:= \left[-\omega\partial_t u_j(x,t) - \partial_x(a_j(x,\lambda)u_j(x,t)) + \partial_{u_j} b_j(x,\lambda,0)u_j(x,t)\right]_{j=1}^n
\end{aligned}
$$

and linear bounded operators $B(\lambda), \tilde{B}(\lambda), D(\lambda) : \mathcal{C}_n \to \mathcal{C}_n$ by

$$
\begin{aligned}
\left[B(\lambda)u\right](x,t) &:= \left[\sum_{\substack{k=1\\k\neq j}}^n \partial_{u_k} b_j(x,\lambda,0)u_k(x,t)\right]_{j=1}^n, \\
\left[\tilde{B}(\lambda)u\right](x,t) &:= \left[\sum_{\substack{k=1\\k\neq j}}^n \partial_{u_j} b_k(x,\lambda,0)u_k(x,t)\right]_{j=1}^n, \\
\left[D(\lambda,\omega)u\right](x,t) &:= \left[\int_{x_j}^x \frac{c_j(\xi,x,\lambda)}{a_j(\xi,0)} u_j(\xi,\tau_j(\xi,x,t,\lambda,\omega))d\xi\right]_{j=1}^n.
\end{aligned}
$$

Finally, we denote by

$$
\langle u,v\rangle := \frac{1}{2\pi}\sum_{j=1}^n \int_0^{2\pi}\int_0^1 u_j(x,t)v_j(x,t)dxdt \tag{3.1}
$$

the L^2 scalar product in $\mathcal{C}_n$. Obviously, for all $u, v \in \mathcal{C}_n^1$ it holds

$$
\langle A(\lambda,\omega)u,v\rangle - \langle u,\tilde{A}(\lambda,\omega)v\rangle = \left[\sum_{j=1}^n a_j(x,\lambda)\int_0^{2\pi} u_j(x,t)v_j(x,t)dt\right]_{x=0}^{x=1}, \tag{3.2}
$$

and for all $u, v \in \mathcal{C}_n$ we have

$$
\langle B(\lambda)u,v\rangle - \langle u,\tilde{B}(\lambda)v\rangle = 0. \tag{3.3}
$$

In particular, if u satisfies the boundary conditions (1.2) and v satisfies the adjoint boundary conditions

$$\begin{aligned} a_j(0,\lambda)v_j(0,t) &= -\sum_{k=1}^{m} r_{kj}a_k(0,\lambda)v_k(0,t), \quad j=m+1,\ldots,n, \\ a_j(1,\lambda)v_j(1,t) &= -\sum_{k=m+1}^{n} r_{kj}a_k(1,\lambda)v_k(1,t), \quad j=m+1,\ldots,n, \end{aligned} \tag{3.4}$$

then $\langle A(\lambda,\omega)u, v\rangle = \langle u, \tilde{A}(\lambda,\omega)v\rangle$.

Lemma 3.1 *(i) For all $u \in \mathcal{C}_n^1$ it holds $A(\lambda,\omega)C(\lambda,\omega)u = 0$, $A(\lambda,\omega)D(\lambda,\omega)u = u$, $A(\lambda,\omega)\partial_u F(\lambda,\omega,0)u = -B(\lambda)u$, and*

$$[A(\lambda,\omega)F(\lambda,\omega,u)]\,(x,t) = f(x,\lambda,u(x,t)). \tag{3.5}$$

(ii) For all $u \in \mathcal{C}_n^1$ satisfying the boundary conditions (1.2) it holds $D(\lambda,\omega)A(\lambda,\omega)u = (I - C(\lambda,\omega))u$.

Proof. (i) Take $u \in \mathcal{C}_n^1$. Then $A(\lambda,\omega)C(\lambda,\omega)u = 0$ follows from (2.7) (taking there $\xi = x_j$), $A(\lambda,\omega)D(\lambda,\omega)u = u$ follows from (2.8) (taking there $\xi = x_j$ and $f_j(x,\lambda,u) = u_j$). Similarly, (3.5) follows from (2.8) (taking there $\xi = x_j$). Finally, from (2.21) we have $\partial_u F(\lambda,\omega,0)u = -D(\lambda,\omega)B(\lambda)u$, hence $B(\lambda)u = -A(\lambda,\omega)\partial_u F(\lambda,\omega,0)u$.

(ii) In Section 2 we showed the following (cf. (2.6)): If we have $A(\lambda,\omega)u = f$ for some $f \in \mathcal{C}_n$ and some $u \in \mathcal{C}_n^1$ which satisfies the boundary conditions (1.2), then $(I - C(\lambda,\omega))u = D(\lambda,\omega)f$. □

3.1 Kernel and image of the linearization

Using the functions $v^0, w^0 : [0,1] \to \mathbb{C}^n$, introduced in Section 1.1 (see (1.14)), we define functions $\mathbf{v}, \mathbf{w} : [0,1] \times \mathbb{R} \to \mathbb{C}^n$ and $\mathbf{v_1}, \mathbf{v_2}, \mathbf{w_1}, \mathbf{w_2} : [0,1] \times \mathbb{R} \to \mathbb{R}^n$ by

$$\begin{aligned} &\mathbf{v}(x,t) := v^0(x)e^{-it},\ \mathbf{w}(x,t) := w^0(x)e^{-it}, \\ &\mathbf{v_1} := \operatorname{Re}\mathbf{v},\ \mathbf{v_2} := \operatorname{Im}\mathbf{v},\ \mathbf{w_1} := \operatorname{Re}\mathbf{w},\ \mathbf{w_2} := \operatorname{Im}\mathbf{w}. \end{aligned} \tag{3.6}$$

It follows from (1.14) and (3.1) that

$$\langle \mathbf{v_j}, \mathbf{w_k} \rangle = \delta_{jk}. \tag{3.7}$$

Remark that here we used the number two in the normalization condition (1.14). Further, we define a linear bounded operator $L_0 : \mathcal{C}_n \to \mathcal{C}_n$ by

$$L_0 := I - C(0,1) - \partial_u F(0,1,0).$$

Due to Lemma 2.2 (ii), L_0 is a Fredholm operator of index zero from $\mathcal{C}_n$ into $\mathcal{C}_n$. To simplify further notation we will write

$$A := A(0,1),\ \tilde{A} := \tilde{A}(0,1),\ B := B(0),\ \tilde{B} := \tilde{B}(0). \tag{3.8}$$

Lemma 3.2 *We have*

$$\ker L_0 = \operatorname{span}\{\mathbf{v_1}, \mathbf{v_2}\},\quad \operatorname{im} L_0 = \left\{f \in \mathcal{C}_n :\ \langle f, \tilde{A}\mathbf{w_1}\rangle = \langle f, \tilde{A}\mathbf{w_2}\rangle = 0\right\}.$$

Proof. Take $u \in \ker L_0$. Then, by [25, Theorem 1.2(iv)] we have $u \in \mathcal{C}_n^1$. Hence, because of Lemma 3.1 it holds $AL_0u = A(I - C(0,1) - \partial_u F(0,0))u = (A+B)u = 0$, i.e.,

$$\partial_t u_j(x,t) + a_j(x,0)\partial_x u_j(x,t) + \partial_{u_j} b_j(x,0,0)u_j(x,t) = 0.$$

Moreover, u satisfies the boundary conditions (1.2) because any function of the type $C(\lambda,\omega)v+\partial_u F(\lambda,\omega,0)v$ with arbitrary $v \in \mathcal{C}_n$ satisfies those boundary conditions. Doing the Fourier ansatz

$$u(x,t) = \sum_{s\in\mathbb{Z}} u^s(x)e^{ist},$$

we get the following boundary value problem for the coefficient u^s:

$$\begin{aligned}
a_j(x,0)\frac{d}{dx}u_j^s(x) + \sum_{k=1}^n \partial_{u_k} b_j(x,0,0)u_k^s(x) &= -isu_j^s(x), && j = 1,\dots,n,\\
u_j^s(0) &= \sum_{k=m+1}^n r_{jk}u_k^s(0), && j = 1,\dots,m,\\
u_j^s(1) &= \sum_{k=1}^m r_{jk}u_k^s(1), && j = m+1,\dots,n.
\end{aligned}$$

By assumptions (1.12) and (1.16), this is equivalent to

$$u^s = 0 \text{ for all } s \in \mathbb{Z}\setminus\{-1,1\} \text{ and } u^{\pm 1} \in \operatorname{span}\{\operatorname{Re} v^0, \operatorname{Im} v^0\},$$

i.e., to $u \in \operatorname{span}\{\operatorname{Re}\mathbf{v}, \operatorname{Im}\mathbf{v}\}$. In particular, we have $(A+B)\mathbf{v} = 0$. Similarly one shows that

$$(\tilde{A}+\tilde{B})\mathbf{w} = 0 \tag{3.9}$$

and that $\mathbf{w}$ satisfies the adjoint boundary conditions (3.4).

Now we show that $\operatorname{im} L_0 \subseteq \{f \in \mathcal{C}_n :\ \langle f, \tilde{A}\mathbf{w_1}\rangle = \langle f, \tilde{A}\mathbf{w_2}\rangle = 0\}$: Take $f \in \operatorname{im} L_0$, i.e., $f = (I - C(0,1) - \partial_u F(0,0))u$ with arbitrary $u \in \mathcal{C}_n$. Using Lemma 3.1, (3.2), the fact that any function of the type $C(\lambda,\omega)v + \partial_u F(\lambda,\omega,0)v$ with a certain $v \in \mathcal{C}_n$ satisfies (1.2) and that $\mathbf{w}$ satisfies (3.4), we get

$$\langle (C(0,1)+\partial_u F(0,1,0))u, \tilde{A}\mathbf{w_j}\rangle = \langle A(C(0,1)+\partial_u F(0,1,0))u, \mathbf{w_j}\rangle = -\langle Bu, \mathbf{w_j}\rangle = -\langle u, \tilde{B}\mathbf{w_j}\rangle.$$

Hence, (3.9) yields $\langle f, \tilde{A}\mathbf{w_j}\rangle = \langle (I - C(0,1) - \partial_u F(0,1,0))u, \tilde{A}\mathbf{w_j}\rangle = \langle u, (\tilde{A}+\tilde{B})\mathbf{w_j}\rangle = 0$.

Finally we show that $\{f \in \mathcal{C}_n : \langle f, \tilde{A}\mathbf{w_1}\rangle = \langle f, \tilde{A}\mathbf{w_2}\rangle = 0\} \subseteq \operatorname{im} L_0$: Because of Lemma 2.2 (i) there exist uniquely defined functions $\tilde{\mathbf{v}}_\mathbf{1}, \tilde{\mathbf{v}}_\mathbf{2} \in \mathcal{C}_n^1$ such that

$$(I - C(0,1))\tilde{\mathbf{v}}_\mathbf{k} = D\mathbf{v_k},\ k = 1, 2. \tag{3.10}$$

Moreover, these functions satisfy the boundary conditions (3.4). Therefore,

$$\langle \tilde{\mathbf{v}}_\mathbf{k}, \tilde{A}\mathbf{w_l}\rangle = \langle A\tilde{\mathbf{v}}_\mathbf{k}, \mathbf{w_l}\rangle = \langle \mathbf{v_k}, \mathbf{w_l}\rangle = \delta_{kl} \tag{3.11}$$

and, hence, $\dim\{f \in \mathcal{C}_n : \langle f, \tilde{A}\mathbf{w_1}\rangle = \langle f, \tilde{A}\mathbf{w_2}\rangle = 0\} \geq 2$. But $\operatorname{im} L_0$ is a closed subspace of codimension two in $\mathcal{C}_n$ because of Lemma 2.2, therefore the claim follows. □

3.2 Projectors and splitting of (2.11)

Lemma 3.2 and (3.11) imply that the linear bounded operator $P : \mathcal{C}_n \to \mathcal{C}_n$, which is defined by

$$Pu := \sum_{k=1}^{2} \langle u, \tilde{A}\mathbf{w_k}\rangle \tilde{\mathbf{v}}_\mathbf{k} \tag{3.12}$$

is a projection with $\ker P = \operatorname{im} L_0$, where the functions $\tilde{\mathbf{v}}_\mathbf{k}$ are implicitly defined in (3.10). Similarly, the linear bounded operator $Q : \mathcal{C}_n \to \mathcal{C}_n$, which is defined by

$$Qu := \sum_{k=1}^{2} \langle u, \mathbf{w_k}\rangle \mathbf{v_k} \tag{3.13}$$

is a projection with $\operatorname{im} Q = \ker L_0$.

Now we are going to solve the equation (2.11) by means of the ansatz

$$u = v + w,\ v \in \ker L_0 = \operatorname{im} Q,\ w \in \ker Q.$$

In other words, we have to solve a coupled system consisting of the finite dimensional equation

$$P((I - C(\lambda,\omega))(v + w) - F(\lambda,\omega,v + w)) = 0 \tag{3.14}$$

and the infinite dimensional equation

$$(I - P)((I - C(\lambda,\omega))(v + w) - F(\lambda,\omega,v + w)) = 0. \tag{3.15}$$

3.3 Local solution of (3.15)

In this subsection we will solve (3.15) locally with respect to $w \approx 0$ for parameters $\omega \approx 1$, $\lambda \approx 0$, and $v \approx 0$. Unfortunately, the classical implicit function theorem cannot be used for that purpose because the left-hand side of (3.15) is not C^1-smooth. More exactly, the map $(\lambda, \omega) \in \mathbb{R}^2 \mapsto (I-P)(I-C(\lambda,\omega)-\partial_u F(\lambda,\omega,0)) \in \mathcal{L}(\mathcal{C}_n)$ is not continuous. The reason is that for operators of "shift type" like $(S_\tau u)(x,t) := u(x,t+\tau)$ the map $\tau \in \mathbb{R} \mapsto S_\tau \in \mathcal{L}(\mathcal{C}_n)$ is not continuous (with respect to the operator norm in $\mathcal{L}(\mathcal{C}_n)$).

There exist several generalizations of the implicit function theorem in which the map "control parameter $\mapsto$ linearization with respect to the state parameter" is allowed to be discontinuous with respect to the operator norm, see, e.g., [5, Theorem 7], [43, Theorem 2.1]. However, it turns out that they do not fit to our problem, so we are going to adapt ideas of [34] and [41, Theorem 2.1].

Lemma 3.3 *There exist $\delta_2 \in (0,\delta_1)$ and $c_2 > 0$ such that for all $\lambda \in [-\delta_2, \delta_2]$, $\omega \in [1-\delta_2, 1+\delta_2]$ and $u \in \mathcal{C}_n$ with $\|u\|_\infty \le \delta_2$ it holds*

$$\|(I-P)(I-C(\lambda,\omega)-\partial_u F(\lambda,\omega,u))w\|_\infty \ge c_2\|w\|_\infty \text{ for all } w \in \ker Q.$$

Proof. Suppose the contrary. Then there exist sequences $\lambda^1, \lambda^2, \ldots \in \mathbb{R}$ with $\lambda^r \to 0$; $\omega^1, \omega^2, \ldots \in \mathbb{R}$ with $\omega^r \to 1$; $u^1, u^2, \ldots \in \mathcal{C}_n$ with $\|u^r\|_\infty \to 0$ and $w^1, w^2, \ldots \in \ker Q$ with

$$\|w^r\|_\infty = 1 \tag{3.16}$$

and

$$\|(I-P)(I-C(\lambda^r,\omega^r)-\partial_u F(\lambda^r,\omega^r,u^r))w^r\|_\infty \to 0. \tag{3.17}$$

In a first step we show that

$$\|(I-P)(I-C(\lambda^r,\omega^r)-\partial_u F(\lambda^r,\omega^r,0))w^r\|_\infty \to 0, \tag{3.18}$$

i.e., without loss of generality we can assume $u^1 = u^2 = \ldots = 0$. Indeed, because of (2.14) we have $\|(\partial_u F(\lambda^r,\omega^r,u^r)-\partial_u F(\lambda^r,\omega^r,0))w^r\|_\infty \le \text{const}\|u^r\|_\infty$, where the constant does not depend on r. Hence, the assumptions $\|u^r\|_\infty \to 0$ and (3.17) yield (3.18).

In a second step we show that

$$(C(\lambda^r,\omega^r)-C(0,1))\,w^r + (\partial_u F(\lambda^r,\omega^r,0)-\partial_u F(0,1,0))\,w^r \rightharpoonup 0 \text{ in } \mathcal{C}_n. \tag{3.19}$$

Indeed, denote by $t_j(\xi,x,\cdot,\lambda,\omega)$ the inverse function to the function $\tau_j(\xi,x,\cdot,\lambda,\omega)$, i.e.,

$$t_j(\xi,x,\tau,\lambda,\omega) = \tau - \omega\int_x^\xi \frac{d\eta}{a_j(\eta,\lambda)}$$

(cf. (2.3)). Take a test function $\varphi \in \mathcal{C}_n^1$. Then we have

$$
\begin{aligned}
&\int_0^1 \int_0^{2\pi} c_j(x_j,x,\lambda^r) w_k^r(x_j,\tau_j(x_j,x,t,\lambda^r,\omega^r)))\varphi_j(x,t)dtdx \\
&\quad - \int_0^1 \int_0^{2\pi} c_j(x_j,x,0) w_k^r(x_j,\tau_j(x_j,x,t,0,1))\varphi_j(x,t)dtdx \\
&\quad = \int_0^1 \int_0^{2\pi} c_j(x_j,x,\lambda^r)\varphi_j(x,t_j(x_j,x,\tau,\lambda^r,\omega^r)) w_k^r(x_j,\tau)d\tau dx \\
&\quad - \int_0^1 \int_0^{2\pi} c_j(x_j,x,0)\varphi_j(x,t_j(x_j,x,\tau,0,1)) w_k^r(x_j,\tau)d\tau dx,
\end{aligned}
$$

and this tends to zero for $r \to \infty$. So we get $\langle (C(\lambda^r,\omega^r) - C(0,1))\, w^r, \varphi\rangle \to 0$. Similarly we have

$$
\begin{aligned}
&\int_0^1 \int_{x_j}^x \int_0^{2\pi} \left(\frac{c_j(\xi,x,\lambda^r)}{a_j(\xi,\lambda^r)} \partial_{u_k} f_j(\xi,\lambda^r,u^r(\xi,\tau_j(\xi,x,t,\lambda^r,\omega^r))) w_k^r((\xi,\tau_j(\xi,x,t,\lambda^r,\omega^r) \right. \\
&\quad \left. - \frac{c_j(\xi,x,0)}{a_j(\xi,0)} \partial_{u_k} f_j(\xi,0,u^r(\xi,\tau_j(\xi,x,t,0,1)))(\xi,\tau_j(\xi,x,t,0,1) \right) \varphi_j(x,t)dtd\xi dx \\
&\quad = \int_0^1 \int_{x_j}^x \int_0^{2\pi} \left(\frac{c_j(\xi,x,\lambda^r)}{a_j(\xi,\lambda^r)} \partial_{u_k} f_j(\xi,\lambda^r,u^r(\xi,\tau))\varphi_j(x,t_j(\xi,x,\tau,\lambda^r,\omega^r)) \right. \\
&\quad \left. - \frac{c_j(\xi,x,0)}{a_j(\xi,0)} \partial_{u_k} f_j(\xi,0,u^r(\xi,\tau))\varphi_j(x,t_j(\xi,x,\tau,0,1)) \right) w_k^r(x,\tau)d\tau d\xi dx,
\end{aligned}
$$

and this tends to zero for $r \to \infty$. Therefore, $\langle (\partial_u F(\lambda^r,\omega^r,u^r) - \partial_u F(0,1,0))\, w^r, \varphi\rangle$ tends to zero.

In a third step we show that without loss of generality we can assume that

$$
w^r \rightharpoonup 0 \text{ in } \mathcal{C}_n. \tag{3.20}
$$

Indeed, because of $\dim \operatorname{im} P < \infty$ there exist $g^* \in \operatorname{im} P$ and a subsequence of $w^1, w^2, \ldots$ (which will be denoted by $w^1, w^2, \ldots$ again) such that

$$
P\,(I - C(0,1) - \partial_u F(0,1,0))\, w^r \to g^*. \tag{3.21}
$$

On the other hand, (3.18) and (3.19) imply that

$$
(I-P)\,(I - C(0,1) - \partial_u F(0,1,0))\, w^r \rightharpoonup 0 \text{ in } \mathcal{C}_n. \tag{3.22}
$$

But (3.21) and (3.22) imply

$$
(I - C(0,1) - \partial_u F(0,1,0))\, w^r \rightharpoonup g^* \text{ in } \mathcal{C}_n. \tag{3.23}
$$

Therefore, it follows (cf. Lemma 2.2 (i))

$$w^r - (I - C(0,1))^{-1}\partial_u F(0,1,0))w^r \rightharpoonup (I - C(0,1))^{-1}g^* \text{ in } \mathcal{C}_n \tag{3.24}$$

and

$$\begin{aligned}\partial_u F(0,1,0))w^r - \partial_u F(0,1,0))(I - C(0,1))^{-1}\partial_u F(0,1,0))w^r \\ \rightharpoonup \partial_u F(0,1,0))(I - C(0,1))^{-1}g^* \text{ in } \mathcal{C}_n.\end{aligned} \tag{3.25}$$

In [26, formula (4.2)] we showed that $\partial_u F(0,1,0))(I-C(0,1))^{-1}\partial_u F(0,1,0))$ is a compact operator from $\mathcal{C}_n$ into $\mathcal{C}_n$. Therefore, because of assumption (3.16) without loss of generality we can assume that $\partial_u F(0,1,0))(I - C(0,1))^{-1}\partial_u F(0,1,0))w^r$ converges strongly in $\mathcal{C}_n$. Hence, (3.25) yields that $\partial_u F(0,1,0))w^r$ converges weakly in $\mathcal{C}_n$, and, therefore, (3.24) yields that w^r converges weakly in $\mathcal{C}_n$, i.e., $w^r \rightharpoonup w^* \in \ker Q$. Inserting this into (3.23) we get

$$(I - C(0,1) - \partial_u F(0,1,0))\, w^* = g^*,$$

i.e., $g^* \in \ker P \cap \operatorname{im} P$, i.e., $g^* = 0$. Therefore, $w^* \in \ker Q \cap \operatorname{im} Q$, i.e., $w^* = 0$.

In a fourth step we mention that assumption (3.16) contradicts to (3.20) if we would know that

$$\text{the set } \{w^r :\ r \in \mathbb{N}\} \text{ is precompact in } \mathcal{C}_n. \tag{3.26}$$

Hence, it remains to show (3.26).

In a fifth step we show that for (3.26) it is sufficient to prove that

$$\text{the set } \left\{\left((I - C(\lambda^r,\omega^r))^{-1}\partial_u F(\lambda^r,\omega^r,0)\right)^2 w^r :\ r \in \mathbb{N}\right\} \text{ is precompact in } \mathcal{C}_n. \tag{3.27}$$

Indeed, denote $g^r := (I - C(\lambda^r,\omega^r) - \partial_u F(\lambda^r,\omega^r,0))w^r$. Because of (3.18) and (3.21) we have $\|g^r\|_\infty \to 0$. Hence, Lemma 2.2 (i) yields

$$\|(I - C(\lambda^r,\omega^r))^{-1}(\partial_u F(\lambda^r,\omega^r,0)(I - C(\lambda^r,\omega^r))^{-1}g^r + g^r)\|_\infty \to 0.$$

Here we used that the operators $\partial_u F(\lambda^r,\omega^r,0)$ are uniformly bounded with respect to r in the uniform operator norm (cf. (2.14)), i.e.,

$$\|\partial_u F(\lambda^r,\omega^r,0)\|_{\mathcal{L}(\mathcal{C}_n)} \le \text{const.} \tag{3.28}$$

On the other hand, we have

$$\begin{aligned} w^r &= (I - C(\lambda^r,\omega^r))^{-1}\left(\partial_u F(\lambda^r,\omega^r,0)w^r + g^r\right) \\ &= (I - C(\lambda^r,\omega^r))^{-1}\left(\partial_u F(\lambda^r,\omega^r,0)(I - C(\lambda^r,\omega^r))^{-1}(\partial_u F(\lambda^r,\omega^r,0)w^r + g^r) + g^r\right). \end{aligned}$$

Hence $\|w^r - ((I - C(\lambda^r,\omega^r))^{-1}\partial_u F(\lambda^r,\omega^r,0))^2\, w^r\|_\infty \to 0.$

In a sixth step we show that for (3.27) it is sufficient to prove that the operators $\partial_u F(\lambda^r,\omega^r,0)^2$ and $\partial_u F(\lambda^r,\omega^r,0)C(\lambda^r,\omega^r)$ map $\mathcal{C}_n$ continuously into $\mathcal{C}_n^1$ and that the corresponding operator norms are uniformly bounded with respect to r:

$$\|\partial_u F(\lambda^r,\omega^r,0)^2\|_{\mathcal{L}(\mathcal{C}_n;\mathcal{C}_n^1)} + \|\partial_u F(\lambda^r,\omega^r,0)C(\lambda^r,\omega^r)\|_{\mathcal{L}(\mathcal{C}_n;\mathcal{C}_n^1)} \le \text{const.} \tag{3.29}$$

Indeed, if (3.29) is true, then the sequence $\partial_u F(\lambda^r,\omega^r,0)^2 w^r$ is bounded in $\mathcal{C}_n^1$. Hence, Lemma 2.2 (i) yields that the sequence $(I-C(\lambda^r,\omega^r))^{-1}\partial_u F(\lambda^r,\omega^r,0)^2 w^r$ is bounded in $\mathcal{C}_n^1$. Similarly, because of Lemma 2.2 (i) and (3.28) the sequence $(I-C(\lambda^r,\omega^r))^{-1}\partial_u F(\lambda^r,\omega^r,0)w^r$ is bounded in $\mathcal{C}_n$. Hence, if (3.29) is true, then the sequence

$$(I-C(\lambda^r,\omega^r))^{-1}\partial_u F(\lambda^r,\omega^r,0)C(\lambda^r,\omega^r)(I-C(\lambda^r,\omega^r))^{-1}\partial_u F(\lambda^r,\omega^r,0)w^r$$

is bounded in $\mathcal{C}_n^1$. Therefore, the sequence

$$\begin{aligned}
&(I-C(\lambda^r,\omega^r))^{-1}\partial_u F(\lambda^r,\omega^r,0)(I-C(\lambda^r,\omega^r))^{-1}\partial_u F(\lambda^r,\omega^r,0)w^r\\
&\quad = (I-C(\lambda^r,\omega^r))^{-1}\partial_u F(\lambda^r,\omega^r,0)^2 w^r\\
&\quad +(I-C(\lambda^r,\omega^r))^{-1}\partial_u F(\lambda^r,\omega^r,0)C(\lambda^r,\omega^r)(I-C(\lambda^r,\omega^r))^{-1}\partial_u F(\lambda^r,\omega^r,0)w^r
\end{aligned}$$

is bounded in $\mathcal{C}_n^1$. Hence, the Arcela-Ascoli theorem yields (3.27).

In a seventh step we show that the operators $\partial_u F(\lambda^r,\omega^r,0)^2$ really map $\mathcal{C}_n$ continuously into $\mathcal{C}_n^1$ and that the corresponding operator norms are uniformly bounded with respect to r, i.e., that

$$\|\partial_x\partial_u F(\lambda^r,\omega^r,0)^2 u\|_\infty + \|\partial_t\partial_u F(\lambda^r,\omega^r,0)^2 u\|_\infty \le \text{const}\|u\|_\infty \text{ for all } u\in\mathcal{C}_n^1. \tag{3.30}$$

Indeed, because of (2.15) we have for all $u\in\mathcal{C}_n^1$

$$\begin{aligned}
\partial_x\partial_u F(\lambda^r,\omega^r,0)u &= G^r u + H^r\partial_t u,\\
\partial_t\partial_u F(\lambda^r,\omega^r,0)u &= \partial_u F(\lambda^r,\omega^r,0)\partial_t u,
\end{aligned} \tag{3.31}$$

where the linear bounded operators $G^r, H^r : \mathcal{C}_n \to \mathcal{C}_n$ are defined by $G^r := \partial_u F_0^x(\lambda^r,\omega^r,0)$ and $H^r := F_1^x(\lambda^r,\omega^r,0)$. Therefore,

$$\begin{aligned}
&\partial_t\partial_u F(\lambda^r,\omega^r,0)^2 u = \partial_u F(\lambda^r,\omega^r,0)^2\partial_t u,\\
&\partial_x\partial_u F(\lambda^r,\omega^r,0)^2 u = G^r\partial_u F(\lambda^r,\omega^r,0)u + H^r\partial_u F(\lambda^r,\omega^r,0)\partial_t u.
\end{aligned}$$

hence,

$$\begin{aligned}
&\|\partial_t\partial_u F(\lambda^r,\omega^r,0)u\|_\infty + \|\partial_x\partial_u F(\lambda^r,\omega^r,0)u\|_\infty\\
&\qquad \le \text{const}(\|u\|_\infty + \|\partial_u F(\lambda^r,\omega^r,0)^2\partial_t u\|_\infty + \|H^r\partial_u F(\lambda^r,\omega^r,0)\partial_t u\|_\infty).
\end{aligned}$$

Here we used (3.28) and the fact that the operators G^r and H^r are uniformly bounded with respect to r in the uniform operator norm. Hence, in order to prove (3.30) we have to show that

$$\|\partial_u F(\lambda^r, \omega^r, 0)^2 \partial_t u\|_\infty + \|H^r \partial_u F(\lambda^r, \omega^r, 0)\partial_t u\|_\infty \le \text{const}\|u\|_\infty \text{ for all } u \in \mathcal{C}_n^1. \tag{3.32}$$

Let us start with $\partial_u F(\lambda^r, \omega^r, 0)^2 \partial_t u$. Because of (2.21) we have

$$\begin{aligned}&(\partial_u F(\lambda^r, \omega^r, 0)^2 \partial_t u)_j(x,t)\\ &\quad= \int_{x_j}^x \frac{c_j(\xi, x, \lambda^r)}{a_j(\xi, \lambda^r)} \sum_{\substack{k=1\\k\neq j}}^n \partial_{u_k} b_j(\xi, \lambda^r, 0) v_k^r(\xi, \tau_j(\xi, x, t, \lambda^r, \omega^r)))d\xi\end{aligned} \tag{3.33}$$

with

$$v_k^r(\xi, \tau) := \int_{x_j}^\xi \frac{c_k(\eta, \xi, \lambda^r)}{a_k(\eta, \lambda^r)} \sum_{\substack{l=1\\l\neq k}}^n \partial_{u_l} b_k(\eta, \lambda^r, 0)\partial_t u_l(\eta, \tau_k(\eta, \xi, \tau, \lambda^r, \omega^r))d\eta. \tag{3.34}$$

Inserting (3.34) into (3.33) we get the integrals

$$\begin{aligned}&\int_{x_j}^x \int_{x_j}^\xi d_{jkl}^r(\xi, \eta, x)\partial_t u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi, x, t, \lambda^r, \omega^r), \lambda^r, \omega^r))d\eta d\xi\\ &\quad= \int_{x_j}^x \int_\eta^{x_j} d_{jkl}^r(\xi, \eta, x)\partial_t u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi, x, t, \lambda^r, \omega^r), \lambda^r, \omega^r))d\xi d\eta\end{aligned} \tag{3.35}$$

with $j \neq k$ and $l \neq k$ and

$$d_{jkl}^r(\xi, \eta, x) := \frac{c_j(\xi, x, \lambda^r)c_k(\eta, \xi, \lambda^r)\partial_{u_k} b_j(\xi, \lambda^r, 0)\partial_{u_l} b_k(\eta, \lambda^r, 0)}{a_j(\xi, \lambda^r)a_k(\eta, \lambda^r)}.$$

Moreover, from (2.3) it follows

$$\begin{aligned}&\frac{d}{d\xi} u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi, x, t, \lambda^r, \omega^r), \lambda^r, \omega^r))\\ &\quad= \omega^r \left(\frac{1}{a_j(\xi, \lambda^r)} - \frac{1}{a_k(\xi, \lambda^r)}\right) \partial_t u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi, x, t, \lambda^r, \omega^r), \lambda^r, \omega^r)).\end{aligned} \tag{3.36}$$

By (3.36) and assumption (1.6), the right-hand side of (3.35) equals

$$\frac{1}{\omega^r} \int_0^x \int_\eta^x \frac{a_j(\xi, \lambda^r)a_k(\xi, \lambda^r)}{a_k(\xi, \lambda^r) - a_j(\xi, \lambda^r)} d_{jkl}^r(\xi, \eta, x) \frac{d}{d\xi} u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi, x, t, \lambda^r, \omega^r), \lambda^r, \omega^r))d\xi d\eta.$$

Integrating by parts in the inner integral (with respect to ξ) we see that the absolute values of these integrals can be estimated by a constant times $\|u\|_\infty$, where the constant does not depend on x, t, r, and u.

Now, let us consider $H^r\partial_u F(\lambda^r,\omega^r,0)\partial_t u$. We have

$$
\begin{aligned}
&(H^r\partial_u F(\lambda^r,\omega^r,0)\partial_t u)_j(x,t)\\
&\quad = -\frac{\omega^r}{a_j(x,\lambda^r)}\int_{x_j}^{x}\frac{c_j(\xi,x,\lambda^r)}{a_j(\xi,\lambda^r)}\sum_{\substack{k=1\\k\neq j}}^{n}\partial_{u_k}b_j(\xi,\lambda^r,0)v_k^r(\xi,\tau_j(\xi,x,t,\lambda^r,\omega^r)))d\xi
\end{aligned}
$$

with (3.34). Proceeding as above one shows that these integrals can be estimated by a constant times $\|u\|_\infty$, where the constant does not depend on x, t, r, and u.

Finally, in the last step we show that the operators $\partial_u F(\lambda^r,\omega^r,0)C(\lambda^r,\omega^r)$ really map $\mathcal{C}_n$ continuously into $\mathcal{C}_n^1$ and that the corresponding operator norms are uniformly bounded with respect to r. Indeed, because of $\partial_t C(\lambda^r,\omega^r)u = C(\lambda^r,\omega^r)\partial_t u$ (cf. (2.15)) and of (3.31) we have for all $u\in\mathcal{C}_n^1$ that

$$
\begin{aligned}
\partial_x\partial_u F(\lambda^r,\omega^r,0)C(\lambda^r,\omega^r)u &= G^rC(\lambda^r,\omega^r)u + H^rC(\lambda^r,\omega^r)\partial_t u,\\
\partial_t\partial_u F(\lambda^r,\omega^r,0)C(\lambda^r,\omega^r)u &= \partial_u F(\lambda^r,\omega^r,0)C(\lambda^r,\omega^r)\partial_t u.
\end{aligned}
$$

It remains to show that

$$
\|H^rC(\lambda^r,\omega^r)\partial_t u\|_\infty + \|\partial_u F(\lambda^r,\omega^r,0)C(\lambda^r,\omega^r)\partial_t u\|_\infty \le \text{const } \|u\|_\infty \text{ for all } u\in\mathcal{C}_n^1.
$$

Because of (2.9) and (2.21) we have

$$
\left(\partial_u F(\lambda^r,\omega^r,0)C(\lambda^r,\omega^r)\partial_t u\right)_j(x,t) = \sum_{\substack{k=1\\k\neq j}}^{m}\sum_{l=m+1}^{n} f_{jkl}^r(x,t) + \sum_{\substack{k=m+1\\k\neq j}}^{n}\sum_{l=1}^{m} f_{jkl}^r(x,t)
$$

with

$$
\begin{aligned}
&f_{jkl}^r(x,t)\\
&\quad := \int_x^{x_j}\frac{r_{kl}c_j(\xi,x,\lambda^r)c_k(x_k,\xi,\lambda^r)\partial_{u_k}b_k(\xi,\lambda^r,0)}{a_j(\xi,\lambda^r)}\partial_t u_l(x_k,\tau_k(x_k,\xi,\tau_j(\xi,x,t,\lambda^r,\omega^r),\lambda^r,\omega^r))d\xi.
\end{aligned}
$$

Using (3.36) and assumption (1.6), one can integrate by parts in this integral in order to see that the absolute values of these integrals can be estimated by a constant times $\|u\|_\infty$, where the constant does not depend on x, t, r, and u. □

Now we are well-prepared for solving (3.15) locally with respect to w by an implicit-function-theorem-type argument.

For $\delta > 0$ we denote

$$
\mathcal{B}_\delta := \{v\in\mathcal{C}_n:\ \|v\|_\infty \le \delta\}.
$$

Lemma 3.4 *(i) There exists $\delta_3 \in (0, \delta_2)$ such that for all $\lambda \in [-\delta_3, \delta_3]$, $\omega \in [1-\delta_3, 1+\delta_3]$, and $v \in \mathcal{B}_{\delta_3} \cap \operatorname{im} Q$ there exists exactly one solution $w = \hat{w}(\lambda, \omega, v)$ to (3.15) with $w \in \mathcal{B}_{\delta_3} \cap \ker Q$.*

(ii) There exists $c_3 > 0$ such that for all $\lambda \in [-\delta_3, \delta_3]$, $\omega \in [1-\delta_3, 1+\delta_3]$, and $v \in \mathcal{B}_{\delta_3} \cap \operatorname{im} Q$ it holds

$$\|\hat{w}(\lambda, \omega, v)\|_\infty \le c_3 \|v\|_\infty^2.$$

(iii) The map $(\lambda, \omega, v) \in [-\delta_3, \delta_3] \times [1 - \delta_3, 1 + \delta_3] \times (\mathcal{B}_{\delta_3} \cap \operatorname{im} Q) \mapsto \hat{w}(\lambda, \omega, v) \in \mathcal{C}_n$ is continuous.

(iv) For all $\lambda \in [-\delta_3, \delta_3]$ and $\omega \in [1-\delta_3, 1+\delta_3]$ the maps $v \in \mathcal{B}_{\delta_3} \cap \operatorname{im} Q \mapsto \hat{w}(\lambda, \omega, v) \in \mathcal{C}_n$ are C^∞-smooth.

Proof. Consider the map $\mathcal{F} : [-\delta_0, \delta_0] \times \mathbb{R} \times \operatorname{im} Q \times \ker Q \to \ker P$ defined by

$$\mathcal{F}(\lambda, \omega, v, w) := (I - P)((I - C(\lambda, \omega))(v + w) - F(\lambda, \omega, v + w)). \tag{3.37}$$

Obviously, the map $\mathcal{F}$ has properties analogous to (2.12)–(2.14), in particular, it is continuous,

$$\mathcal{F}(\lambda, \omega, \cdot, \cdot) \text{ is } C^\infty\text{-smooth} \tag{3.38}$$

and

$$(\lambda, \omega) \in [-\delta_3, \delta_3] \times \mathbb{R} \mapsto \partial_w \mathcal{F}(\lambda, \omega, v, w) w_1 \in \mathcal{C}_n \text{ is continuous} \tag{3.39}$$

for all $v \in \operatorname{im} Q$ and $w, w_1 \in \ker Q$. Because of Lemma 2.2 (ii) and Lemma 3.3, for all $\lambda \in [-\delta_2, \delta_2]$ and $\mu \in [1 - \delta_2, 1 + \delta_2]$ the operator

$$\partial_w \mathcal{F}(\lambda, \omega, 0, 0) = (I - P)(I - C(\lambda, \omega) - \partial_u F(\lambda, \omega, 0))$$

is an injective Fredholm operator from $\ker Q$ into $\ker P$. Its index is zero because the index of $I - P$ (as an operator from $\mathcal{C}_n$ into $\ker P$) is -2, the index of $I - C(\lambda, \omega) - \partial_u F(\lambda, \omega, 0)$ (as an operator from $\mathcal{C}_n$ into $\mathcal{C}_n$) is zero, and the index of the embedding $w \in \ker Q \mapsto w \in \mathcal{C}_n$ is 2. Hence, $\partial_w \mathcal{F}(\lambda, \omega, 0, 0)$ is an isomorphism from $\ker Q$ onto $\ker P$, and Lemma 3.3 yields that for all $\lambda \in [-\delta_2, \delta_2]$ and $\omega \in [1 - \delta_2, 1 + \delta_2]$ it holds

$$\|\partial_w \mathcal{F}(\lambda, \omega, 0, 0)^{-1}\|_{\mathcal{L}(\ker P; \ker Q)} \le c_2. \tag{3.40}$$

Furthermore,

$$(\lambda, \omega) \in [-\delta_3, \delta_3] \times \mathbb{R} \mapsto \partial_w \mathcal{F}(\lambda, \omega, 0, 0)^{-1} w \in \mathcal{C}_n \text{ is continuous for all } w \in \ker Q. \tag{3.41}$$

Indeed, on the account of (3.39) and (3.40), we get that

$$\begin{aligned}
&\left(\partial_w \mathcal{F}(\lambda, \omega, 0, 0)^{-1} - \partial_w \mathcal{F}(\lambda^0, \omega^0, 0, 0)^{-1}\right) w \\
&= \partial_w \mathcal{F}(\lambda, \omega, 0, 0)^{-1} \left(\partial_w \mathcal{F}(\lambda^0, \omega^0, 0, 0) - \partial_w \mathcal{F}(\lambda, \omega, 0, 0)\right) \partial_w \mathcal{F}(\lambda^0, \omega^0, 0, 0)^{-1} w
\end{aligned}$$

tends to zero for $(\lambda, \omega) \to (\lambda^0, \omega^0)$.

We have to solve (3.15), i.e., $\mathcal{F}(\lambda, \omega, v, w) = 0$. For $\lambda \in [-\delta_2, \delta_2]$ and $\omega \in [1-\delta_2, 1+\delta_2]$ this is equivalent to the fixed point problem

$$\mathcal{G}(\lambda, \omega, v, w) := w - \partial_w \mathcal{F}(\lambda, \omega, 0, 0)^{-1} \mathcal{F}(\lambda, \omega, v, w) = w. \tag{3.42}$$

Moreover, it holds

$$\begin{aligned}
\mathcal{G}(\lambda, \omega, v, w^1) - \mathcal{G}(\lambda, \omega, v, w^2) &= \int_0^1 \partial_w \mathcal{G}(\lambda, \omega, v, sw^1 + (1-s)w^2)(w_1 - w_2) ds \\
= \partial_w \mathcal{F}(\lambda, \omega, 0, 0)^{-1} &\int_0^1 \left(\partial_w \mathcal{F}(\lambda, \omega, 0, 0) - \partial_w \mathcal{F}(\lambda, \omega, v, sw^1 + (1-s)w^2)\right)(w^1 - w^2) ds,
\end{aligned}$$

where

$$\begin{aligned}
&\left(\partial_w \mathcal{F}(\lambda, \omega, v, sw^1 + (1-s)w^2) - \partial_w \mathcal{F}(\lambda, \omega, 0, 0)\right)(w^1 - w^2) \\
&\quad = (I-P)\left(\partial_u F(\lambda, \omega, v + sw^1 + (1-s)w^2) - \partial_u F(\lambda, \omega, 0)\right)(w^1 - w^2) \\
&\quad = (I-P) \int_0^1 \partial_u^2 F(\lambda, \omega, r(v + sw^1 + (1-s)w^2))(v + sw^1 + (1-s)w^2, w^1 - w^2) dr.
\end{aligned}$$

Hence, (2.14) and (3.40) yield that there exists $\delta_3 \in (0, \delta_2)$ such that for all $\lambda \in [-\delta_3, \delta_3]$, $\omega \in [1-\delta_3, 1+\delta_3]$, and $v \in \mathcal{B}_{\delta_3} \cap \operatorname{im} Q$ we have

$$\|\mathcal{G}(\lambda, \omega, v, w^1) - \mathcal{G}(\lambda, \omega, v, w^2)\|_\infty \le \frac{1}{2}\|w^1 - w^2\|_\infty \text{ for all } w^1, w^2 \in \mathcal{B}_{\delta_3} \cap \ker Q.$$

In other words: For those λ, ω, and v the map $\mathcal{G}(\lambda, \omega, v, \cdot)$ is strictly contractive on $\mathcal{B}_{\delta_3} \cap \ker Q$. In order to apply Banach's fixed point theorem we have to show that $\mathcal{G}(\lambda, \omega, v, \cdot)$ maps $\mathcal{B}_{\delta_3} \cap \ker Q$ into itself if δ_3 is chosen sufficiently small.

Using (3.40) again, for all $\lambda \in [-\delta_3, \delta_3]$, $\omega \in [1-\delta_3, 1+\delta_3]$, and $v \in \mathcal{B}_{\delta_3} \cap \operatorname{im} Q$ we get

$$\begin{aligned}
\|\mathcal{G}(\lambda, \omega, v, w)\|_\infty &\le \|\mathcal{G}(\lambda, \omega, v, w) - \mathcal{G}(\lambda, \omega, v, 0)\|_\infty + \|\mathcal{G}(\lambda, \omega, v, 0)\|_\infty \\
&\le \frac{1}{2}\|w\|_\infty + c_2 \|\mathcal{F}(\lambda, \omega, v, 0)\|_\infty \text{ for all } w \in \mathcal{B}_{\delta_3} \cap \ker Q. \qquad (3.43)
\end{aligned}$$

Moreover, because of $v \in \ker L_0$ it holds

$$\begin{aligned}
&\mathcal{F}(\lambda, \omega, v, 0) = (I-P)((I - C(\lambda, \omega))v - F(\lambda, \omega, v)) \\
&\quad = (I-P)(\partial_u F(\lambda, \omega, 0)v - F(\lambda, \omega, v)) = (I-P) \int_0^1 s \partial_u^2 F(\lambda, \omega, sv)(v, v) ds.
\end{aligned}$$

Hence, (2.14) yields that there exists $c_3 > 0$ such that for all $\lambda \in [-\delta_3, \delta_3]$, $\omega \in [1-\delta_3, 1+\delta_3]$, and $v \in \mathcal{B}_{\delta_3} \cap \operatorname{im} Q$ we have

$$\|\mathcal{F}(\lambda, \omega, v, 0)\|_\infty \le \frac{c_3}{2c_2}\|v\|_\infty^2, \tag{3.44}$$

and, because of (3.43), it follows that $\mathcal{G}(\lambda,\omega,v,w) \in \mathcal{B}_{\delta_3} \cap \operatorname{im} Q$ for all $w \in \mathcal{B}_{\delta_3} \cap \operatorname{im} Q$ if $\delta_3 \le 1/c_3$.

Now, Banach's fixed point theorem gives a unique in $\mathcal{B}_{\delta_3} \cap \ker Q$ solution $w = \hat{w}(\lambda,\omega,v)$ to (3.42) for all $\lambda \in [-\delta_3,\delta_3]$, $\omega \in [1-\delta_3, 1+\delta_3]$, and $v \in \mathcal{B}_{\delta_3} \cap \operatorname{im} Q$. Hence, assertion (i) of the lemma is proved.

Moreover, (3.43) and (3.44) yield assertion (ii).

Assertion (iii) follows from the uniform contraction principle (cf. [7, Theorem 1.244]): First note that the contraction constant of $\mathcal{G}(\lambda,\omega,v,\cdot)$ does not depend on (λ,ω,v). Moreover, for all $\lambda,\lambda^0 \in [\delta_2,\delta_2]$, $\omega,\omega^0 \in [1-\delta_2,1+\delta_2]$, $v,v^0 \in \mathcal{B}_\delta \cap \operatorname{im} Q$, and $w,w^0 \in \mathcal{B}_\delta \cap \ker Q$ the difference

$$
\begin{aligned}
&\mathcal{G}(\lambda,\omega,v,w) - \mathcal{G}(\lambda^0,\omega^0,v^0,w^0)\\
&= w - w^0 - \partial_w \mathcal{F}(\lambda,\omega,0,0)^{-1}\left(\mathcal{F}(\lambda,\omega,v,w) - \mathcal{F}(\lambda^0,\omega^0,v^0,w^0)\right)\\
&+ \left(\partial_w \mathcal{F}(\lambda^0,\omega^0,0,0)^{-1} - \partial_w \mathcal{F}(\lambda,\omega,0,0)^{-1}\right)\mathcal{F}(\lambda^0,\omega^0,v^0,w^0),
\end{aligned}
$$

tends to zero in $\mathcal{C}_n$ for $(\lambda,\omega,v,w) \to (\lambda^0,\omega^0,v^0,w^0)$ because of the continuity of $\mathcal{F}$, (3.40), and (3.41). Hence, $\mathcal{G}$ is continuous.

Finally, assertion (iv) follows from the classical implicit function theorem. □

3.4 Smoothness with respect to x and t

The aim of this subsection is to prove the following result:

Lemma 3.5 *For all $\lambda \in [-\delta_3,\delta_3]$, $\omega \in [1-\delta_3,1+\delta_3]$, and $v \in \mathcal{B}_{\delta_3} \cap \operatorname{im} Q$ the map*

$$(x,t) \in [0,1] \times \mathbb{R} \mapsto [\hat{w}(\lambda,\omega,v)](x,t) \in \mathbb{R}^n$$

is C^∞-smooth.

Proof. For $\varphi \in \mathbb{R}$ and $u \in \mathcal{C}_n$ we define $S_\varphi u \in \mathcal{C}_n$ by $[S_\varphi u](x,t) := u(x,t+\varphi)$. Take $\lambda \in [-\delta_3,\delta_3]$, $\omega \in [1-\delta_3,1+\delta_3]$, and $v \in \mathcal{B}_{\delta_3} \cap \operatorname{im} Q$. It is easy to see that for all $u \in \mathcal{C}_n$ it holds

$$S_\varphi C(\lambda,\omega)u = C(\lambda,\omega)S_\varphi u,\ S_\varphi F(\lambda,\omega,u) = F(\lambda,\omega,S_\varphi u).$$

Moreover, the definitions (3.6), (3.10), and (3.12) imply that for all $u \in \mathcal{C}_n$ it holds

$$
\begin{aligned}
PS_\varphi u &= \sum_{k=1}^{2}\langle S_\varphi u, \tilde{A}\mathbf{w_k}\rangle \tilde{\mathbf{v}}_\mathbf{k} = \sum_{k=1}^{2}\langle u, \tilde{A}S_{-\varphi}\mathbf{w_k}\rangle \tilde{\mathbf{v}}_\mathbf{k}\\
&= \langle u, \tilde{A}(\cos\varphi \mathbf{w_1} - \sin\varphi \mathbf{w_2})\rangle \tilde{\mathbf{v}}_\mathbf{1} + \langle u, \tilde{A}(\cos\varphi \mathbf{w_2} + \sin\varphi \mathbf{w_1})\rangle \tilde{\mathbf{v}}_\mathbf{2}\\
&= \langle u, \tilde{A}\mathbf{w_1}\rangle(\cos\varphi \tilde{\mathbf{v}}_\mathbf{1} + \sin\varphi \tilde{\mathbf{v}}_\mathbf{2}) + \langle u, \tilde{A}\mathbf{w_2}\rangle(\cos\varphi \tilde{\mathbf{v}}_\mathbf{2} - \sin\varphi \tilde{\mathbf{v}}_\mathbf{1})\\
&= S_\varphi P u.
\end{aligned}
\tag{3.45}
$$

Similarly one shows $QS_\varphi u = S_\varphi Qu$. Hence, applying S_φ to the identity

$$(I - P)((I - C(\lambda, \omega))(v + \hat{w}(\lambda, \omega, v)) - F(\lambda, \omega, v + \hat{w}(\lambda, \omega, v))) = 0, \tag{3.46}$$

we get $(I-P)((I-C(\lambda, \omega))(S_\varphi v+S_\varphi \hat{w}(\lambda, \omega, v))-F(\lambda, \omega, S_\varphi v+S_\varphi \hat{w}(\lambda, \omega, v))) = 0$. Therefore, the uniqueness assertion of Lemma 3.4 yields

$$S_\varphi \hat{w}(\lambda, \omega, v) = \hat{w}(\lambda, \omega, S_\varphi v). \tag{3.47}$$

But all functions $v \in \ker L_0 = \operatorname{im} Q$ are C^∞-smooth, hence the maps $\varphi \in \mathbb{R} \mapsto S_\varphi v \in \mathcal{C}_n$ are C^∞-smooth for all $v \in \operatorname{im} Q$. By Lemma 3.4 (iv), $\varphi \in \mathbb{R} \mapsto S_\varphi \hat{w}(\lambda, \omega, v) \in \mathcal{C}_n$ is C^∞-smooth, i.e.,

$$t \in \mathbb{R} \mapsto \hat{w}(\lambda, \omega, v)(x, t) \in \mathbb{R}^n \text{ is } C^\infty\text{-smooth.} \tag{3.48}$$

On the other hand, (3.46) can be rewritten as follows:

$$\begin{aligned}
&(I - C(\lambda, \omega))\hat{w}(\lambda, \omega, v) - F(\lambda, \omega, v + \hat{w}(\lambda, \omega, v)) \\
&\quad = (C(\lambda, \omega) - I)v + P\left((I - C(\lambda, \omega))(v + \hat{w}(\lambda, \omega, v)) - F(\lambda, \omega, v + \hat{w}(\lambda, \omega, v))\right).
\end{aligned} \tag{3.49}$$

The right-hand side of (3.49) is C^∞-smooth with respect to (x, t). Moreover, from (2.9), (2.10), and (3.48) it follows that $C(\lambda, \omega)\hat{w}(\lambda, \omega, v)$ and $F(\lambda, \omega, v + \hat{w}(\lambda, \omega, v))$ are C^∞-smooth with respect to (x, t). Hence, (3.49) yields that $\hat{w}(\lambda, \omega, v)$ is C^∞-smooth with respect to (x, t). □

3.5 Differentiability with respect to λ and ω

In this subsection we show that the map $\hat{w}$ is C^2-smooth. For that we use the well-known fact (see, e.g. [7, Section 1.11.3]) that the fiber contraction principle can be used to show that functions, which are fixed points of certain operators in C-spaces, are smooth.

Remark that in the particular case, when all coefficient functions $a_j(x, \lambda)$ are λ-independent (and, hence, the problem (1.1)–(1.3) depends on the bifurcation parameter λ via the terms $b_{jk}(x, \lambda, u)$ only), the maps $\lambda \mapsto \hat{w}(\lambda, \omega, v)$ are C^∞-smooth because in this case the characteristics $\tau_j(\xi, x, t, \lambda, \omega)$ do not depend on λ and, hence, the maps $\lambda \mapsto C(\lambda, \omega)$ and $\lambda \mapsto F(\lambda, \omega, u)$ are C^∞-smooth (cf. (2.9) and (2.10)). But also in this case the question if the maps $\omega \mapsto \hat{w}(\lambda, \omega, v)$ are differentiable, remains to be difficult.

Lemma 3.6 *The map*

$$(\lambda, \omega, v) \in [-\delta_3, \delta_3] \times [1 - \delta_3, 1 + \delta_3] \times (\mathcal{B}_{\delta_3} \cap \operatorname{im} Q) \mapsto \hat{w}(\lambda, \omega, v) \in \mathcal{C}_n$$

is C^2-smooth.

Proof. For $k = 0, 1, 2, \dots$ we define maps $w_k : [-\delta_3, \delta_3] \times [1-\delta_3, 1+\delta_3] \times (\mathcal{B}_{\delta_3} \cap \operatorname{im} Q) \to \ker Q$ by means of

$$w_0 := 0,\ w_{k+1}(\lambda, \omega, v) := w_k(\lambda, \omega, v) - \partial_w \mathcal{F}(\lambda, \omega, 0, 0)^{-1} \mathcal{F}(\lambda, \omega, v, w_k(\lambda, \omega, v)). \tag{3.50}$$

As it follows from the proof of Lemma 3.4,

$$w_k(\lambda, \omega, v) \to \hat{w}(\lambda, \omega, v) \text{ in } \mathcal{C}_n \text{ for } k \to \infty \text{ uniformly with respect to } \lambda, \omega, v. \tag{3.51}$$

Now we are going to show that all functions $w_1, w_2, \dots$ are C^2-smooth and that the sequences of all their partial derivatives up to the second order converge in $\mathcal{C}_n$ uniformly with respect to λ, ω, and v. Then the classical theorem of calculus [11, Theorem 8.6.3] yields that $\hat{w}$ is C^2-smooth (and, hence, the lemma is proved). Here and in what follows we identify the two-dimensional vector space $\operatorname{im} Q$ with $\mathbb{R}^2$ (by fixing a certain basis in $\operatorname{im} Q$), and the partial derivatives are taken with respect to the corresponding coordinates.

For $l = 1, 2, \dots$ we denote

$$\tilde{\mathcal{C}}_n^l := \{u \in \mathcal{C}_n : \partial_t^j u \in \mathcal{C}_n \text{ for } j = 1, 2, \dots, l\}.$$

This is a Banach space with the norm

$$\|u\|_l := \sum_{j=0}^{l} \|\partial_t^j u\|_\infty.$$

In a first step we show (by induction with respect to k) that for all $\lambda \in [-\delta_3, \delta_3]$, $\omega \in [1-\delta_3, 1+\delta_3]$, $v \in \mathcal{B}_{\delta_3} \cap \operatorname{im} Q$, and $k, l = 1, 2, \dots$ the function $w_k(\lambda, \omega, v)$ belongs to $\tilde{\mathcal{C}}_n^l$, and $w_k(\lambda, \omega, v)$ depends continuously (with respect to the norm in $\tilde{\mathcal{C}}_n^l$) on λ, ω, and v.

For $k = 0$ this claim is obvious.

To do the induction step we proceed as in the proof of Lemma 3.5. For all $\varphi \in \mathbb{R}$ we have

$$S_\varphi w_{k+1}(\lambda, \omega, v) = S_\varphi w_k(\lambda, \omega, v) - \partial_w \mathcal{F}(\lambda, \omega, 0, 0)^{-1} \mathcal{F}(\lambda, \omega, v, S_\varphi w_k(\lambda, \omega, v)). \tag{3.52}$$

By induction assumption the map $\varphi \in \mathbb{R} \mapsto S_\varphi w_k(\lambda, \omega, v) \in \mathcal{C}_n$ is C^∞-smooth, and all its derivatives depend continuously (with respect to $\|\cdot\|_\infty$) on λ, ω, and v. Hence, by (3.38) and (3.41), the right-hand side of (3.52) is C^∞-smooth with respect to φ, and all its derivatives with respect to φ depend continuously (with respect to $\|\cdot\|_\infty$) on λ, ω, and v. Therefore, also the left-hand side of (3.52) has this property, i.e., all partial derivatives $\partial_t^j w_{k+1}(\lambda, \omega, v)$ exist in $\mathcal{C}_n$ and depend continuously (with respect to $\|\cdot\|_\infty$) on λ, ω, and v.

In a second step we show that the sequence $\partial_t w_k(\lambda, \omega, v)$, $k = 1, 2, \dots$, converges in $\mathcal{C}_n$ uniformly with respect to λ, ω, and v. Indeed, from (2.15) and (3.45) (which implies that $\partial_t P u = P \partial_t u$ for all $u \in \tilde{\mathcal{C}}_n^1$) it follows that for all $w \in \tilde{\mathcal{C}}_n^1 \cap \ker Q$ we have

$$\partial_t \mathcal{F}(\lambda, \omega, v, w) = \partial_w \mathcal{F}(\lambda, \omega, v, w) \partial_t w + \partial_v \mathcal{F}(\lambda, \omega, v, w) \partial_t v. \tag{3.53}$$

Hence,

$$\begin{aligned}
&\partial_t w_{k+1}(\lambda,\omega,v)\\
&\quad = \left(I-\partial_w\mathcal{F}(\lambda,\omega,0,0)^{-1}\partial_w\mathcal{F}(\lambda,\omega,v,w_k(\lambda,\omega,v))\right)\partial_t w_k(\lambda,\omega,v)\\
&\quad -\partial_w\mathcal{F}(\lambda,\omega,0,0)^{-1}\partial_v\mathcal{F}(\lambda,\omega,v,w_k(\lambda,\omega,v))\partial_t v.
\end{aligned}$$

But (3.51) implies

$$\partial_w\mathcal{F}(\lambda,\omega,0,0)^{-1}\partial_v\mathcal{F}(\lambda,\omega,v,w_k(\lambda,\omega,v))\partial_t v \to \partial_w\mathcal{F}(\lambda,\omega,0,0)^{-1}\partial_v\mathcal{F}(\lambda,\omega,v,\hat{w}(\lambda,\omega,v))\partial_t v$$

in $\mathcal{C}_n$ for $k\to\infty$ uniformly with respect to λ,ω, and v. Moreover, from the proof of Lemma 3.4 it follows that

$$\|I-\partial_w\mathcal{F}(\lambda,\omega,0,0)^{-1}\partial_w\mathcal{F}(\lambda,\omega,v,w_k(\lambda,\omega,v))\|_{\mathcal{L}(\ker Q)}\le\frac{2}{3}$$

for large k. Hence, Lemma A.1 yields the claim of the second step. Here we apply Lemma A.1 with U chosen as the Banach space of all continuous maps $u:[-\delta_3,\delta_3]\times[1-\delta_3,1+\delta_3]\times(\mathcal{B}_{\delta_3}\cap\operatorname{im}Q)\to\ker Q$ with the usual maximum norm.

In a third step we show (again by induction with respect to k) that the partial derivatives of $w_k(\lambda,\omega,v)$ with respect to the two components of v exist in $\mathcal{C}_n$ and depend continuously (with respect to the norm $\|\cdot\|_\infty$ in $\mathcal{C}_n$) on λ, ω, and v. Indeed, by induction assumption $\partial_{v_j}w_k(\lambda,\omega,v)$ exists in $\mathcal{C}_n$. Hence, the partial derivative with respect to v_j of $w_{k+1}(\lambda,\omega,\cdot)=w_k(\lambda,\omega,\cdot)-\partial_w\mathcal{F}(\lambda,\omega,0,0)^{-1}\mathcal{F}(\lambda,\omega,\cdot,w_k(\lambda,\omega,\cdot))$ exists in $\mathcal{C}_n$. Here we used (3.38) again.

Similarly one shows that all partial derivatives of the type $\partial_t^l\partial_{v_j}^m w_k(\lambda,\omega,v)$ exist in $\mathcal{C}_n$ and depend continuously (with respect to the norm in $\mathcal{C}_n$) on λ, ω, and v.

In a fourth step we show that the partial derivatives $\partial_{v_j}w_k(\lambda,\omega,v)$ converge in $\mathcal{C}_n$ for $k\to\infty$ uniformly with respect to λ,ω, and v. This follows, as above, from

$$\begin{aligned}
&\partial_{v_j} w_{k+1}(\lambda,\omega,v)\\
&\quad = \left(I-\partial_w\mathcal{F}(\lambda,\omega,0,0)^{-1}\partial_w\mathcal{F}(\lambda,\omega,v,w_k(\lambda,\omega,v))\right)\partial_{v_j} w_k(\lambda,\omega,v)\\
&\quad -\partial_w\mathcal{F}(\lambda,\omega,0,0)^{-1}\partial_{v_j}\mathcal{F}(\lambda,\omega,v,w_k(\lambda,\omega,v))
\end{aligned}$$

and Lemma A.1.

In a fifth step we use step one and show by induction with respect to k that the partial derivatives $\partial_\lambda w_k(\lambda,\omega,v)$ exist, belong to $\mathcal{C}_n$ and depend continuously (with respect to $\|\cdot\|_\infty$) on λ, ω, and v:

For $k=0$ the claim is obvious.

In order to do the induction step, first let us do some preliminary calculations. It follows directly from the definitions (2.9) and (2.10) that for all $\omega\in\mathbb{R}$ and $u\in\tilde{\mathcal{C}}_n^1$ the maps $\lambda\in[-\delta_0,\delta_0]\mapsto C(\lambda,\omega)u\in\mathcal{C}_n$ and $\lambda\in[-\delta_0,\delta_0]\mapsto F(\lambda,\omega,u)\in\mathcal{C}_n$ are differentiable and

$$\begin{aligned}
\partial_\lambda C(\lambda,\omega)u &= C_0^\lambda(\lambda,\omega)u+C_1^\lambda(\lambda,\omega)\partial_t u,\\
\partial_\lambda F(\lambda,\omega,u) &= F_0^\lambda(\lambda,\omega,u)+F_1^\lambda(\lambda,\omega,u)\partial_t u,
\end{aligned}\tag{3.54}$$

where the operators $C_0^\lambda(\lambda,\omega), C_1^\lambda(\lambda,\omega) \in \mathcal{L}(\mathcal{C}_n)$ are defined as follows: For $j = 1,\dots,m$ we set

$$\begin{aligned}\left(C_0^\lambda(\lambda,\omega)u\right)_j(x,t) &:= \partial_\lambda c_j(0,x,\lambda)\sum_{k=m+1}^{n} r_{jk}u_k(0,\tau_j(0,x,t,\lambda,\omega)),\\ \left(C_1^\lambda(\lambda,\omega)v\right)_j(x,t) &:= \partial_\lambda \tau_j(0,x,t,\lambda,\omega)c_j(0,x,\lambda)\sum_{k=m+1}^{n} r_{jk}v_k(0,\tau_j(0,x,t,\lambda,\omega)),\end{aligned}$$

and for $j = m+1,\dots,n$ we set

$$\begin{aligned}\left(C_0^\lambda(\lambda,\omega)u\right)_j(x,t) &:= \partial_\lambda c_j(1,x,\lambda)\sum_{k=1}^{m} r_{jk}u_k(1,\tau_j(1,x,t,\lambda,\omega)),\\ \left(C_1^\lambda(\lambda,\omega)v\right)_j(x,t) &:= \partial_\lambda \tau_j(1,x,t,\lambda,\omega)c_j(1,x,\lambda)\sum_{k=1}^{m} r_{jk}v_k(1,\tau_j(1,x,t,\lambda,\omega)).\end{aligned}$$

Further, the map $F_0^\lambda : [-\delta_0,\delta_0]\times\mathbb{R}\times\mathcal{C}_n \to \mathcal{C}_n$ is defined by

$$\begin{aligned}(F_0^\lambda(\lambda,\omega,u))_j(x,t) &:= \int_{x_j}^{x} \partial_\lambda\left(\frac{c_j(\xi,x,\lambda)}{a_j(\xi,\lambda)}\right) f_j(\xi,\lambda,u(\xi,\tau_j(\xi,x,t,\lambda,\omega)))d\xi\\ &\quad + \int_{x_j}^{x} \frac{c_j(\xi,x,\lambda)}{a_j(\xi,\lambda)}\partial_\lambda f_j(\xi,\lambda,u(\xi,\tau_j(\xi,x,t,\lambda,\omega)))d\xi\end{aligned}$$

and $F_1^x(\lambda,\omega,u) \in \mathcal{L}(\mathcal{C}_n)$ is defined by

$$\begin{aligned}&(F_1^x(\lambda,\omega,u)v)_j(x,t) :=\\ &\int_{x_j}^{x} \frac{\partial_\lambda\tau_j(\xi,x,t,\lambda,\omega)c_j(\xi,x,\lambda)}{a_j(\xi,\lambda)}\sum_{k=1}^{n}\partial_{u_k} f_j(\xi,\lambda,u(\xi,\tau_j(\xi,x,t,\lambda,\omega)))v_k(\xi,\tau_j(\xi,x,t,\lambda,\omega))d\xi.\end{aligned}$$

Hence, for all $\omega \in \mathbb{R}$, $v \in \operatorname{im} Q$, and $w \in \tilde{\mathcal{C}}_n^1 \cap \ker Q$ the map $\lambda \in [-\delta_0,\delta_0] \mapsto \mathcal{F}(\lambda,\omega,v,w) \in \mathcal{C}_n$ is differentiable and its derivative is

$$\begin{aligned}\partial_\lambda\mathcal{F}(\lambda,\omega,v,w) &= -(I-P)\big(C_0^\lambda(\lambda,\omega)(v+w) + F_0^\lambda(\lambda,\omega,v+w)\\ &\quad +(C_1^\lambda(\lambda,\omega) + F_1^\lambda(\lambda,\omega,v+w))(\partial_t v + \partial_t w)\big). \qquad (3.55)\end{aligned}$$

Remark that the map $\mathcal{F} : [-\delta_0,\delta_0]\times\mathbb{R}\times\operatorname{im}Q\times\ker Q \to \ker P$ defined in (3.37) is not differentiable (with respect to the norm $\|\cdot\|_\infty$ in $\ker P$ and $\ker Q$), in general. But its restriction to $[-\delta_0,\delta_0]\times\mathbb{R}\times\operatorname{im}Q\times(\tilde{\mathcal{C}}_n^1\cap\ker Q)$ is differentiable (with respect to the norms $\|\cdot\|_\infty$ in $\ker P$ and $\|\cdot\|_1$ in $\ker Q$). Therefore, the notation $\partial_\lambda\mathcal{F}$ should be used with care: It is not the partial derivative of $\mathcal{F}$, but of its restriction to $[-\delta_0,\delta_0]\times\mathbb{R}\times\operatorname{im}Q\times(\tilde{\mathcal{C}}_n^1\cap\ker Q)$.

In particular, it is a map from $[-\delta_0, \delta_0] \times \mathbb{R} \times \operatorname{im} Q \times (\tilde{\mathcal{C}}^1_n \cap \ker Q)$ into $\mathcal{C}_n$. The same care is needed if using the notations $\partial_\lambda C(\lambda,\omega)u$ and $\partial_\lambda F(\lambda,\omega,u)$.

Now let us do the induction step. By the induction assumption, for a fixed k the partial derivative $\partial_\lambda w_k(\lambda,\omega,v)$ exists in $\mathcal{C}_n$ and depends continuously on λ, ω, and v. Define

$$R_k(\lambda,\omega,v) := \partial_w \mathcal{F}(\lambda,\omega,0,0)w_k(\lambda,\omega,v) - \mathcal{F}(\lambda,\omega,v,w_k(\lambda,\omega,v)).$$

Because of $w_k(\lambda,\omega,v) \in \tilde{\mathcal{C}}^1_n$ the partial derivative $\partial_\lambda R_k(\lambda,\omega,v)$ exists in $\mathcal{C}_n$ and depends continuously on λ, ω, and v. In fact, it holds

$$\begin{aligned}\partial_\lambda R_k(\lambda,\omega,v) = \partial_\lambda\partial_w \mathcal{F}(\lambda,\omega,0,0)w_k(\lambda,\omega,v) + \partial_w \mathcal{F}(\lambda,\omega,0,0)\partial_\lambda w_k(\lambda,\omega,v)\\ -\partial_\lambda \mathcal{F}(\lambda,\omega,v,w_k(\lambda,\omega,v)) - \partial_w \mathcal{F}(\lambda,\omega,v,w_k(\lambda,\omega,v))\partial_\lambda w_k(\lambda,\omega,v).\end{aligned}$$

Here $\partial_\lambda\partial_w \mathcal{F}(\lambda,\omega,0,0) \in \mathcal{L}(\tilde{\mathcal{C}}^1_n \cap \ker Q; \mathcal{C}_n)$ is the derivative of the map $\lambda \in [-\delta, \delta_0] \mapsto \partial_w \mathcal{F}(\lambda,\omega,0,0) \in \mathcal{L}(\tilde{\mathcal{C}}^1_n \cap \ker Q; \mathcal{C}_n)$ or, the same, the derivative of the map $w \in \tilde{\mathcal{C}}^1_n \cap \ker Q \mapsto \partial_\lambda \mathcal{F}(\lambda,\omega,0,w) \in \mathcal{C}_n$ at $w = 0$, i.e. (cf. (3.55)),

$$\partial_\lambda\partial_w \mathcal{F}(\lambda,\omega,0,0)w = (P - I)\left((C_0^\lambda(\lambda,\omega) + \partial_u F_0^\lambda(\lambda,\omega,0))w + (C_1^\lambda(\lambda,\omega) + F_1^\lambda(\lambda,\omega,0))\partial_t w\right).$$

It is easy to check (using the definitions of $C_0^\lambda, C_1^\lambda, F_0^\lambda$, and F_1^λ) that for all $w \in \tilde{\mathcal{C}}^1_n \cap \ker Q$ it holds

$$(\lambda,\omega) \in [-\delta_3, \delta_3] \times \mathbb{R} \mapsto \partial_\lambda\partial_w \mathcal{F}(\lambda,\omega,0,0)w \in \mathcal{C}_n \text{ is continuous,} \tag{3.56}$$

and that for all $\lambda \in [-\delta_0, \delta_0]$, $\omega \in \mathbb{R}$, and $w \in \tilde{\mathcal{C}}^1_n \cap \ker Q$ it holds

$$\|\partial_\lambda\partial_w \mathcal{F}(\lambda,\omega,0,0)w\|_\infty \le \text{const } \|w\|_1, \tag{3.57}$$

where the constant does not depend on w, λ, and ω (as long as ω varies in a bounded interval).

We have

$$\partial_w \mathcal{F}(\lambda,\omega,0,0)w_{k+1}(\lambda,\omega,v) = R_k(\lambda,\omega,v). \tag{3.58}$$

The induction claim, we have to prove, is that the partial derivative $\partial_\lambda w_{k+1}(\lambda,\omega,v)$ exists in $\mathcal{C}_n$ and depends continuously on λ, ω, and v. The candidate for $\partial_\lambda w_{k+1}(\lambda,\omega,v)$ is

$$\partial_w \mathcal{F}(\lambda,\omega,0,0)^{-1}\left(\partial_\lambda R_k(\lambda,\omega,v) - \partial_w\partial_\lambda \mathcal{F}(\lambda,\omega,0,0)w_{k+1}(\lambda,\omega,v)\right).$$

This candidate depends continuously on λ, ω, and v because of (3.41) and (3.56). In order to show that this candidate is really $\partial_\lambda w_{k+1}(\lambda,\omega,v)$ we denote, for the sake of shortness, $w_k(\lambda) := w_k(\lambda,\omega,v)$, $R_k(\lambda) := R_k(\lambda,\omega,v)$, $\partial_w \mathcal{F}(\lambda) := \partial_w \mathcal{F}(\lambda,\omega,0,0)$, $\partial_\lambda\partial_w \mathcal{F}(\lambda) := \partial_\lambda\partial_w \mathcal{F}(\lambda,\omega,0,0)$ and calculate

$$\begin{aligned}&\partial_w \mathcal{F}(\lambda+\mu)\left(w_{k+1}(\lambda+\mu) - w_{k+1}(\lambda) - \mu\partial_w \mathcal{F}(\lambda)^{-1}\left(\partial_\lambda R_k(\lambda) - \partial_\lambda\partial_w \mathcal{F}(\lambda)w_{k+1}(\lambda)\right)\right)\\ &\quad = R_k(\lambda+\mu) - R_k(\lambda) - \mu\partial_w \mathcal{F}(\lambda+\mu)\partial_w \mathcal{F}(\lambda)^{-1}\partial_\lambda R_k(\lambda)\\ &\quad - \left(\partial_w \mathcal{F}(\lambda+\mu) - \partial_w \mathcal{F}(\lambda) - \mu\partial_w \mathcal{F}(\lambda+\mu)\partial_w \mathcal{F}(\lambda)^{-1}\partial_\lambda\partial_w \mathcal{F}(\lambda)\right)w_{k+1}.\end{aligned}$$

The right-hand side is $o(\mu)$ in $\mathcal{C}_n$ for $\mu \to 0$ because of (3.39). Hence, the same is true for the left-hand side. Now, Lemma 3.3 yields the claim of the fifth step.

In a sixth step we show that the partial derivatives $\partial_\lambda w_k(\lambda, \omega, v)$ converge in $\mathcal{C}_n$ for $k \to \infty$ uniformly with respect to λ, ω, and v. Indeed, from (3.58) it follows

$$\begin{aligned}
&\partial_w \mathcal{F}(\lambda, \omega, 0, 0)\left(\partial_\lambda w_{k+1}(\lambda, \omega, v) - \partial_\lambda w_k(\lambda, \omega, v)\right)) \\
&\quad + \partial_w \partial_\lambda \mathcal{F}(\lambda, \omega, 0, 0)\left(w_{k+1}(\lambda, \omega, v) - w_k(\lambda, \omega, v)\right) \\
&\quad = -\partial_w \mathcal{F}(\lambda, \omega, v, w_k(\lambda, \omega, v))\partial_\lambda w_k(\lambda, \omega, v) - \partial_\lambda \mathcal{F}(\lambda, \omega, v, w_k(\lambda, \omega, v)).
\end{aligned}$$

Therefore, $\partial_\lambda w_{k+1}(\lambda, \omega, v) - (I - \partial_w \mathcal{F}(\lambda, \omega, 0, 0)^{-1} \partial_w \mathcal{F}(\lambda, \omega, v, w_k(\lambda, \omega, v)))\, \partial_\lambda w_k(\lambda, \omega, v)$ converges in $\mathcal{C}_n$ for $k \to \infty$ uniformly with respect to λ, ω, and v (here we use (3.55) and the claim of step two). Hence, Lemma A.1 yields the claim of step six.

Similarly one shows that all partial derivatives $\partial_\omega w_k(\lambda, \omega, v)$ exist in $\mathcal{C}_n$ and converge in $\mathcal{C}_n$ for $k \to \infty$ uniformly with respect to λ, ω, and v.

Now we consider partial derivatives of the second order.

In a seventh step we show that the partial derivatives $\partial_t^2 w_k(\lambda, \omega, v)$ converge in $\mathcal{C}_n$ for $k \to \infty$ uniformly with respect to λ, ω, and v. From (3.53) we get

$$\begin{aligned}
&\partial_t^2 \mathcal{F}(\lambda, \omega, v, w) = \partial_w \mathcal{F}(\lambda, \omega, v, w)\partial_t^2 w + \partial_w^2 \mathcal{F}(\lambda, \omega, v, w)(\partial_t w, \partial_t w) \\
&\quad + 2\partial_v \partial_w \mathcal{F}(\lambda, \omega, v, w)(\partial_t v, \partial_t w) + \partial_v^2 \mathcal{F}(\lambda, \omega, v, w)(\partial_t v, \partial_t v) + \partial_v \mathcal{F}(\lambda, \omega, v, w)\partial_t^2 v
\end{aligned} \quad (3.59)$$

for all $w \in \tilde{\mathcal{C}}_n^2 \cap \ker Q$. Hence, it follows from (3.58) that the sequence $\partial_t^2 w_{k+1}(\lambda, \omega, v) - (I - \partial_w \mathcal{F}(\lambda, \omega, 0, 0)^{-1} \partial_w \mathcal{F}(\lambda, \omega, v, w_k(\lambda, \omega, v)))\, \partial_t^2 w_k(\lambda, \omega, v)$ converges in $\mathcal{C}_n$ for $k \to \infty$ uniformly with respect to λ, ω, and v, and Lemma A.1 yields the claim of step seven.

Similarly one shows that partial derivatives $\partial_t \partial_{v_j} w_k(\lambda, \omega, v)$ and $\partial_{v_i} \partial_{v_j} w_k(\lambda, \omega, v)$ converge in $\mathcal{C}_n$ for $k \to \infty$ uniformly with respect to λ, ω, and v.

In step eight we show that the partial derivatives $\partial_t \partial_\lambda w_k(\lambda, \omega, v)$ exist in $\mathcal{C}_n$ and converge in $\mathcal{C}_n$ for $k \to \infty$ uniformly with respect to λ, ω, and v. We have

$$\begin{aligned}
&\partial_w \mathcal{F}(\lambda, \omega, 0, 0)\left(\partial_t w_{k+1}(\lambda, \omega, v) - \partial_t w_k(\lambda, \omega, v)\right) \\
&\quad = -\partial_v \mathcal{F}(\lambda, \omega, v, w_k(\lambda, \omega, v))\partial_t v - \partial_w \mathcal{F}(\lambda, \omega, v, w_k(\lambda, \omega, v))\partial_t w_k(\lambda, \omega, v).
\end{aligned} \quad (3.60)$$

Hence, we can proceed as in steps five and six, replacing (3.50) by (3.60) to get the claim of step eight.

Similarly one shows that all partial derivatives $\partial_t \partial_\omega w_k(\lambda, \omega, v)$, $\partial_{v_j} \partial_\lambda w_k(\lambda, \omega, v)$, and $\partial_{v_j} \partial_\omega w_k(\lambda, \omega, v)$ exist in $\mathcal{C}_n$ and converge in $\mathcal{C}_n$ for $k \to \infty$ uniformly with respect to λ, ω, and v.

In step nine we show (by induction with respect to k and using steps seven and eight) that the partial derivatives $\partial_\lambda^2 w_k(\lambda, \omega, v)$ exist, belong to $\mathcal{C}_n$ and depend continuously (with respect to $\|\cdot\|_\infty$) on λ, ω, and v:

Similarly to step five one can show that the restriction of the map $\mathcal{F}$ to $[-\delta_0,\delta_0]\times\mathbb{R}\times \operatorname{im} Q\times(\tilde{\mathcal{C}}_n^2\cap\ker Q)$ is twice differentiable (with respect to the norm $\|\cdot\|_2$ in $\tilde{\mathcal{C}}_n^2\cap\ker Q$). We denote the second partial derivative with respect to λ of this restriction by $\partial_\lambda^2\mathcal{F}$, i.e., $\partial_\lambda^2\mathcal{F}$ is a continuous map from $[-\delta_0,\delta_0]\times\mathbb{R}\times\operatorname{im} Q\times(\tilde{\mathcal{C}}_n^2\cap\ker Q)$ into $\mathcal{C}_n$. By induction assumption the second partial derivative with respect to λ of R_k exists in $\mathcal{C}_n$ (because of $w_k(\lambda,\omega,v)\in\tilde{\mathcal{C}}_n^2$). It follows from (3.58) that

$$\partial_w\mathcal{F}(\lambda,\omega,0,0)\partial_\lambda w_{k+1}(\lambda,\omega,v)=\partial_\lambda R_k(\lambda,\omega,v)-\partial_w\partial_\lambda\mathcal{F}(\lambda,\omega,0,0)w_k(\lambda,\omega,v). \tag{3.61}$$

Hence, if $\partial_\lambda^2 w_{k+1}(\lambda,\omega,v)$ exists in $\mathcal{C}_n$, then it holds

$$\begin{aligned}\partial_w\mathcal{F}(\lambda,\omega,0,0)\partial_\lambda^2 w_{k+1}(\lambda,\omega,v)&=\partial_\lambda^2 R_k(\lambda,\omega,v)-\partial_w\partial_\lambda^2\mathcal{F}(\lambda,\omega,0,0)w_k(\lambda,\omega,v)\\ &-\partial_w\partial_\lambda\mathcal{F}(\lambda,\omega,0,0)\partial_\lambda w_k(\lambda,\omega,v)-\partial_w\partial_\lambda\mathcal{F}(\lambda,\omega,0,0)\partial_\lambda w_{k+1}(\lambda,\omega,v).\end{aligned} \tag{3.62}$$

Here $\partial_w\partial_\lambda^2\mathcal{F}(\lambda,\omega,0,0)\in\mathcal{L}(\tilde{\mathcal{C}}_n^2\cap\ker Q;\mathcal{C}_n)$ is the derivative of the map $\lambda\in[-\delta,\delta_0]\mapsto\partial_\lambda\partial_w\mathcal{F}(\lambda,\omega,0,0)\in\mathcal{L}(\tilde{\mathcal{C}}_n^2\cap\ker Q;\mathcal{C}_n)$ or, the same, the derivative of the map $w\in\tilde{\mathcal{C}}_n^2\cap\ker Q\mapsto\partial_\lambda^2\mathcal{F}(\lambda,\omega,0,w)\in\mathcal{C}_n$ at $w=0$ (cf. (3.55)).

We use the notation $w_k(\lambda)$, $R_k(\lambda)$, $\partial_w\mathcal{F}(\lambda)$, $\partial_\lambda\partial_w\mathcal{F}(\lambda)$, introduced above, and $\partial_\lambda^2\partial_w\mathcal{F}(\lambda):=\partial_w\partial_\lambda^2\mathcal{F}(\lambda,\omega,0,0)$. Because of (3.62) the candidate for $\partial_\lambda^2 w_{k+1}(\lambda)$ is

$$\partial_w\mathcal{F}(\lambda)^{-1}(\partial_\lambda^2 R_k(\lambda)-\partial_w\partial_\lambda\mathcal{F}(\lambda)\partial_\lambda w_k(\lambda)-\partial_w\partial_\lambda\mathcal{F}(\lambda)\partial_\lambda w_{k+1}(\lambda)-\partial_w\partial_\lambda^2\mathcal{F}(\lambda)w_k(\lambda)).$$

In order to show that this candidate is really $\partial_\lambda^2 w_{k+1}(\lambda)$ we use (3.61) and calculate

$$\begin{aligned}&\partial_w\mathcal{F}(\lambda+\mu)\Big[\partial_\lambda w_{k+1}(\lambda+\mu)-\partial_\lambda w_{k+1}(\lambda)\\ &\quad-\mu\partial_w\mathcal{F}(\lambda)^{-1}\left(\partial_\lambda^2 R_k(\lambda)-\partial_w\partial_\lambda\mathcal{F}(\lambda)\partial_\lambda w_k(\lambda)-\partial_w\partial_\lambda\mathcal{F}(\lambda)\partial_\lambda w_{k+1}(\lambda)-\partial_w\partial_\lambda^2\mathcal{F}(\lambda)w_k(\lambda)\right)\Big]\\ &\quad=\Big[\partial_\lambda R_k(\lambda+\mu)-\partial_\lambda R_k(\lambda)-\mu\partial_w\mathcal{F}(\lambda+\mu)\partial_w\mathcal{F}(\lambda)^{-1}\partial_\lambda^2 R_k(\lambda)\Big]\\ &\quad-\Big[\partial_w\partial_\lambda\mathcal{F}(\lambda+\mu)-\partial_w\partial_\lambda\mathcal{F}(\lambda)-\mu\partial_w\mathcal{F}(\lambda+\mu)\partial_w\mathcal{F}(\lambda)^{-1}\partial_w\partial_\lambda^2\mathcal{F}(\lambda)\Big]w_k(\lambda)\\ &\quad-\Big[\partial_w\partial_\lambda\mathcal{F}(\lambda+\mu)\left(w_k(\lambda+\mu)-w_k(\lambda)\right)-\mu\partial_w\mathcal{F}(\lambda+\mu)\partial_w\mathcal{F}(\lambda)^{-1}\partial_w\partial_\lambda\mathcal{F}(\lambda)\partial_\lambda w_k(\lambda)\Big]\\ &\quad-\Big[\partial_w\mathcal{F}(\lambda+\mu)-\partial_w\mathcal{F}(\lambda)-\mu\partial_w\mathcal{F}(\lambda+\mu)\partial_w\mathcal{F}(\lambda)^{-1}\partial_\lambda\partial_w\mathcal{F}(\lambda+\mu)\Big]w_{k+1}(\lambda).\end{aligned}$$

The right-hand side is $o(\mu)$ in $\mathcal{C}_n$ for $\mu\to 0$ (here we use (3.57)). Hence, the same is true for the left-hand side, and Lemma 3.3 yields the claim of step nine.

In step ten we show that the partial derivatives $\partial_\lambda^2 w_k(\lambda,\omega,v)$ converge in $\mathcal{C}_n$ for $k\to\infty$ uniformly with respect to λ,ω, and v. It follows from (3.58) that

$$\partial_w\mathcal{F}(\lambda,\omega,0,0)\left(\partial_\lambda^2 w_{k+1}(\lambda,\omega,v)-\partial_\lambda^2 w_k(\lambda,\omega,v)\right)$$

$$
\begin{aligned}
&+2\partial_w\partial_\lambda\mathcal{F}(\lambda,\omega,0,0)\left(\partial_\lambda w_{k+1}(\lambda,\omega,v)-\partial_\lambda w_k(\lambda,\omega,v)\right)\\
&+\partial_w\partial_\lambda^2\mathcal{F}(\lambda,\omega,0,0)\left(w_{k+1}(\lambda,\omega,v)-w_k(\lambda,\omega,v)\right)\\
&=-\partial_w\mathcal{F}(\lambda,\omega,v,w_k(\lambda,\omega,v))\partial_\lambda^2 w_k(\lambda,\omega,v)-2\partial_w\partial_\lambda\mathcal{F}(\lambda,\omega,v,w_k(\lambda,\omega,v))\partial_\lambda w_k(\lambda,\omega,v)\\
.\quad&-\partial_\lambda^2\mathcal{F}(\lambda,\omega,v,w_k(\lambda,\omega,v)).
\end{aligned}
$$

Therefore, $\partial_\lambda^2 w_{k+1}(\lambda,\omega,v)-\left(I-\partial_w\mathcal{F}(\lambda,\omega,0,0)^{-1}\partial_w\mathcal{F}(\lambda,\omega,v,w_k(\lambda,\omega,v))\right)\partial_\lambda^2 w_k(\lambda,\omega,v)$ converges in $\mathcal{C}_n$ for $k\to\infty$ uniformly with respect to λ,ω, and v. Hence, Lemma A.1 yields the claim of step ten.

Similarly one shows that the remaining second order partial derivatives $\partial_\omega^2 w_k(\lambda,\omega,v)$ and $\partial_\omega\partial_\lambda w_k(\lambda,\omega,v)$ exist in $\mathcal{C}_n$ and converge in $\mathcal{C}_n$ for $k\to\infty$ uniformly with respect to λ,ω, and v. $\square$

Remark 3.7 *In fact the map $\hat{w}$ is not only C^2-smooth, as it is claimed in Lemma 3.6, but C^∞-smooth. In order to prove this rigorously, one has to handle "higher order analogues" of formulas like (3.59) and (3.62), which are getting more and more complicated.*

4 The bifurcation equation

In this section we finish the proof of Theorem 1.2 by inserting the solution $w=\hat{w}(\lambda,\omega,v)$ of the infinite dimensional part (3.15) of the Liapunov-Schmidt system into the finite dimensional part (3.14)

$$
P((I-C(\lambda,\omega))(v+\hat{w}(\lambda,\omega,v))-F(\lambda,\omega,v+\hat{w}(\lambda,\omega,v)))=0 \tag{4.1}
$$

and by solving this so-called bifurcation equation with respect to $\lambda\approx 0$ and $\omega\approx 1$ for given small $v\neq 0$.

4.1 Local solution of (4.1)

Because of Lemma 3.2 the functions $\mathbf{v_1}=\operatorname{Re}\mathbf{v}$ and $\mathbf{v_2}=\operatorname{Im}\mathbf{v}$ constitute a basis in $\ker L_0$. Hence, the variable $v\in\ker L_0$ can be represented as

$$
v=\operatorname{Re}(\zeta\mathbf{v}) \text{ with } \zeta\in\mathbb{C}.
$$

Moreover, because of $\mathbf{w_1}=\operatorname{Re}\mathbf{w}$ and $\mathbf{w_2}=\operatorname{Im}\mathbf{w}$ and of the definition (3.12) of the projection P we have $Pu=0$ for $u\in\mathcal{C}_n$ if and only if $\langle u,\tilde{\mathbf{A}}\mathbf{w}\rangle=0$. Hence, (4.1) is equivalent to the one-dimensional complex equation

$$
\Phi(\lambda,\omega,\zeta)=0 \tag{4.2}
$$

with

$$\Phi(\lambda,\omega,\zeta) :=$$
$$\left\langle (I - C(\lambda,\omega))\left(\mathrm{Re}(\zeta\mathbf{v}) + \hat{w}(\lambda,\omega,\mathrm{Re}(\zeta\mathbf{v}))\right) - F\left(\lambda,\omega,\mathrm{Re}(\zeta\mathbf{v}) + \hat{w}(\lambda,\omega,\mathrm{Re}(\zeta\mathbf{v}))\right), \tilde{A}\mathbf{w}\right\rangle.$$

Here and in what follows $\langle\cdot,\cdot\rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ is the Hermitian scalar product in $\mathbb{C}^n$. Remark that $\Phi(\lambda,\omega,\cdot) : \mathbb{C} \to \mathbb{C}$ is C^∞-smooth in the sense of real differentiability, i.e., as a map from a two-dimensional real vector space into itself. It follows from (3.47) that

$$e^{i\varphi}\Phi(\lambda,\omega,\zeta) = \Phi\left(\lambda,\omega,e^{i\varphi}\zeta\right). \tag{4.3}$$

Therefore, it suffices to solve (4.2) for real ζ. To shorten notation, we will write

$$\bar{w}(\lambda,\omega,r) := \hat{w}(\omega,\lambda,r\,\mathrm{Re}\,\mathbf{v}) \text{ for } r \in \mathbb{R}.$$

Moreover, because of $\hat{w}(\lambda,\omega,0) = 0$ we have $\Phi(\lambda,\omega,0) = 0$ for all λ and ω. Hence, it suffices to solve the so-called reduced or scaled bifurcation equation

$$\Psi(\lambda,\omega,r) = 0 \tag{4.4}$$

for small $r \in \mathbb{R}$. Here the map Ψ is defined by

$$\Psi(\lambda,\omega,r) := \begin{cases} \dfrac{1}{r}\Phi(\lambda,\omega,r) & \text{for} \quad r \neq 0, \\ \lim\limits_{s\to 0}\dfrac{1}{s}\Phi(\lambda,\omega,s) & \text{for} \quad r = 0, \end{cases}$$

which leads to

$$\begin{aligned} \Psi(\lambda,\omega,r) &= \int_0^1 \partial_\zeta\Phi(\lambda,\omega,rs)\,ds \\ &= \int_0^1 \left\langle \left(I - C(\lambda,\omega) - \partial_u F\left(\lambda,\omega,rs\,\mathrm{Re}\,\mathbf{v} + \bar{w}(\lambda,\omega,rs)\right)\right)\left(\mathrm{Re}\,\mathbf{v} + \partial_r\bar{w}(\lambda,\omega,rs)\right), \tilde{A}w\right\rangle ds. \end{aligned}$$

Because of Lemma 3.4(ii) we have

$$\partial_r\bar{w}(\lambda,\omega,0) = 0. \tag{4.5}$$

This together with Lemma 3.2 yields $\Psi(0,1,0) = 0$. In order to solve (4.4) with respect to λ and ω close to the solution $\lambda = 0$, $\omega = 1$, $r = 0$ by means of the implicit function theorem we have to calculate the partial derivatives $\partial_\lambda\Psi(0,1,0)$ and $\partial_\omega\Psi(0,1,0)$.

For $\lambda \approx 0$ denote by $\hat{\nu}(\lambda)$ the eigenvalue close to $\nu = -i$ of the eigenvalue problem (1.10)–(1.11) (which is the complex conjugate to the eigenvalue close to $\mu = i$ of the eigenvalue problem (1.9)), and let $\hat{w}(\lambda) : [0,1] \to \mathbb{C}^n$ be the corresponding eigenfunction, normalized by (cf. (1.14))

$$\sum_{j=1}^n \int_0^1 v_j^0(x)\overline{\hat{w}_j(\lambda)(x)}\, dx = 2.$$

Further, define $\hat{\mathbf{w}}(\lambda) : [0,1] \times \mathbb{R} \to \mathbb{C}^n$ by $\hat{\mathbf{w}}(\lambda)(x,t) := \hat{w}(\lambda)(x)e^{-it}$. Then we have $\hat{w}(0) = w^0$ and, hence, $\hat{\mathbf{w}}(\mathbf{0}) = \mathbf{w}$. Because $\hat{w}(\lambda)$ satisfies the adjoint boundary conditions (1.11), the function $\hat{\mathbf{w}}(\lambda)$ satisfies the adjoint boundary conditions (3.4). Moreover, $(I - C(\lambda,1) - \partial_u F(\lambda,1,0)) \operatorname{Re} \mathbf{v}$ satisfies the boundary conditions (1.2). Hence, we get from Lemma 3.1 that

$$\begin{aligned}
&\left\langle (I - C(\lambda,1) - \partial_u F(\lambda,1,0)) \operatorname{Re} \mathbf{v}, \tilde{A}(\lambda,1)\hat{\mathbf{w}}(\lambda) \right\rangle \\
&= \langle A(\lambda,1)\,(I - C(\lambda,1) - \partial_u F(\lambda,1,0)) \operatorname{Re} \mathbf{v}, \hat{\mathbf{w}}(\lambda)\rangle \\
&= \langle (A(\lambda,1) + B(\lambda)) \operatorname{Re} \mathbf{v}, \hat{\mathbf{w}}(\lambda)\rangle = \left\langle \operatorname{Re} \mathbf{v}, \left(\tilde{A}(\lambda,1) + \tilde{B}(\lambda)\right) \hat{\mathbf{w}}(\lambda) \right\rangle \\
&= \langle \operatorname{Re} \mathbf{v}, (i + \nu(\lambda))\, \hat{\mathbf{w}}(\lambda)\rangle = \overline{\nu(\lambda)} - i.
\end{aligned}$$

But $(A(0,1) + B(0)) \operatorname{Re} \mathbf{v} = 0$ (cf. Lemma (3.2)), hence

$$\begin{aligned}
\partial_\lambda \Psi(0,1,0) &= \frac{d}{d\lambda} \left\langle (I - C(\lambda,1) - \partial_u F(\lambda,1,0)) \operatorname{Re} \mathbf{v}, \tilde{A}\mathbf{w} \right\rangle \Big|_{\lambda=0} \\
&= \frac{d}{d\lambda} \left\langle (I - C(\lambda,1) - \partial_u F(\lambda,1,0)) \operatorname{Re} \mathbf{v}, \tilde{A}(\lambda,1)\hat{\mathbf{w}}(\lambda) \right\rangle \Big|_{\lambda=0} \\
&= \langle (\partial_\lambda A(0,1) + \partial_\lambda B(0)) \operatorname{Re} \mathbf{v}, \mathbf{w}\rangle \\
&= \frac{1}{2} \sum_{j=1}^n \int_0^1 \left(\partial_\lambda a_j(x,0) v_j^0(x) + \sum_{k=1}^n \partial_{u_k} \partial_\lambda b_j(x,0,0) v_k^0(x) \right) \overline{w_j^0(x)} dx.
\end{aligned}$$

In particular, it holds

$$\operatorname{Re} \partial_\lambda \Psi(0,1,0) = \operatorname{Re} \nu'(0) = \alpha \tag{4.6}$$

(cf. (1.15)). Similarly we have

$$\begin{aligned}
\partial_\omega \Psi(0,1,0) &= \frac{d}{d\omega} \left\langle [I - C(0,\omega) - \partial_u F(0,\omega,0)] \operatorname{Re} \mathbf{v}, \tilde{A}\mathbf{w} \right\rangle \Big|_{\omega=1} \\
&= \langle \partial_\omega A(0,1) \operatorname{Re} \mathbf{v}, \mathbf{w}\rangle = -i \langle \operatorname{Re} \mathbf{v}, \mathbf{w}\rangle = -i.
\end{aligned} \tag{4.7}$$

Hence, the transversality condition (1.15) yields

$$\det \frac{\partial(\operatorname{Re}\Psi, \operatorname{Im}\Psi)}{\partial(\lambda,\omega)}(0,1,0) = -\alpha \neq 0,$$

therefore the implicit function theorem works. Using, moreover, that (4.3) yields $\Phi(\lambda,\omega,-\zeta) = -\Phi(\lambda,\omega,\zeta)$ and, hence, $\Psi(\lambda,\omega,-r) = \Psi(\lambda,\omega,r)$, we get

Lemma 4.1 *There exist* $\varepsilon_0 > 0$ *and* $\delta > 0$ *such that for all* $r \in [0, \varepsilon_0]$ *there exists exactly one solution* $\lambda = \hat{\lambda}(r) \in [-\delta, \delta]$*,* $\omega = \hat{\omega}(r) \in [1-\delta, 1+\delta]$ *to (4.4). Moreover, the map* $r \in [0, \varepsilon_0] \mapsto (\hat{\lambda}(r), \hat{\omega}(r)) \in \mathbb{R}^2$ *is* C^2*-smooth, and it holds* $\hat{\lambda}(0) = 0$*,* $\hat{\omega}(0) = 1$*, and*

$$\hat{\lambda}'(0) = \hat{\omega}'(0) = 0. \tag{4.8}$$

Now the local existence and uniqueness assertions of Theorem 1.2 are proved with

$$[\hat{u}(\varepsilon)](x,t) = \left[\varepsilon \operatorname{Re} \mathbf{v} + \bar{w}\left(\hat{\lambda}(\varepsilon), \hat{\omega}(\varepsilon), \varepsilon\right)\right](x,t) = \varepsilon \operatorname{Re}\left(e^{-it} v^0(x)\right) + O(\varepsilon^2).$$

4.2 Bifurcation formula

In this subsection we will prove the so-called bifurcation formula (cf. notation (1.15) and (1.20))

$$\hat{\lambda}''(0) = \frac{\beta}{\alpha}. \tag{4.9}$$

This formula determines the so-called bifurcation direction, i.e., if $\alpha\beta > 0$, then the bifurcating periodic solutions to (1.1)–(1.3) exist for small $\lambda > 0$, and, if $\alpha\beta < 0$, then they exist for small $\lambda < 0$. Moreover, it turns out that, as in the case of Hopf bifurcation for ODEs and parabolic PDEs, this formula determines the stability of the bifurcating periodic solutions to (1.1)–(1.3): If $\beta > 0$ (the so-called supercritical case) and if the real parts of all eigenvalues $\mu \neq \pm i$ of (1.9) with $\lambda = 0$ have negative real parts (this is a reinforcement of the nonresonance assumption (1.16)), then the bifurcating periodic solutions to (1.1)–(1.3) are asymptotically orbitally stable, if $\beta < 0$ (the so-called subcritical case) then they are unstable. Remark that in the present paper we do not deal with the stability question because we do not have suitable criteria for nonlinear stability of time-periodic solutions to semilinear dissipative hyperbolic PDEs. This will be the topic for future work.

In the following calculations we will use a standard approach (cf., e.g., [22, Chapter I.9]) as well as its concrete realization for hyperbolic systems (see [14]).

In order to prove (4.9), we differentiate the identity $\Psi(\hat{\lambda}(r), \hat{\omega}(r), r) = 0$ twice with respect to r at $r = 0$. Using (4.8), we get $\partial_\lambda \Psi(0,1,0)\hat{\lambda}''(0) + \partial_\omega \Psi(0,1,0)\hat{\omega}''(0) + \partial_r^2 \Psi(0,1,0) = 0$. Hence (4.6) and (4.7) yield $\alpha \hat{\lambda}''(0) + \operatorname{Re} \partial_r^2 \Psi(0,1,0) = 0$. On the other hand, differentiating the identity $r\Psi(\lambda, \omega, r) = \Phi(\lambda, \omega, r)$ three times with respect to r at $\lambda = r = 0$ and $\omega = 1$, we get $3\partial_r^2 \Psi(0,1,0) = \partial_r^3 \Phi(0,1,0)$. Hence, in order to show (4.9) we have to show

$$\beta = -\frac{1}{3} \operatorname{Re} \partial_r^3 \Phi(0,1,0). \tag{4.10}$$

To get a formula for $\partial_r^3 \Phi(0,1,0)$ we denote $\tilde{w}(r) := \bar{w}(0,1,r)$ and calculate

$$\Phi(0,1,r) = \langle (I - C(0,1))\left(r \operatorname{Re} \mathbf{v} + \tilde{w}(r)\right) - F\left(0,1,r \operatorname{Re} \mathbf{v} + \tilde{w}(r)\right), \tilde{A}\mathbf{w} \rangle$$

$$
\begin{aligned}
&= \langle A\left((I - C(0,1))\left(r\,\mathrm{Re}\,\mathbf{v} + \tilde{w}(r)\right) - F\left(0,1,r\,\mathrm{Re}\,\mathbf{v} + \tilde{w}(r)\right)\right), \mathbf{w}\rangle \\
&= \langle A\left(r\,\mathrm{Re}\,\mathbf{v} + \tilde{w}(r)\right) - AF(0,1,r\,\mathrm{Re}\,\mathbf{v} + \tilde{w}(r)), \mathbf{w}\rangle \\
&= -\langle B\left(r\,\mathrm{Re}\,\mathbf{v} + \tilde{w}(r)\right) + AF(0,1,r\,\mathrm{Re}\,\mathbf{v} + \tilde{w}(r)), \mathbf{w}\rangle\,. \qquad (4.11)
\end{aligned}
$$

Here we used that for all $u \in \mathcal{C}_n$ the function $C(0,1)u + F(0,1,u)$ satisfies the boundary conditions (1.2) and that also $\mathbf{v}$ and $\tilde{w}(r)$ satisfy those boundary conditions. For $\tilde{w}(r)$ this follows from

$$
\begin{aligned}
&P\left[(I - C(0,1))(r\,\mathrm{Re}\,\mathbf{v} + \tilde{w}(r)) - F(0,1,r\,\mathrm{Re}\,\mathbf{v} + \tilde{w}(r))\right] \\
&= \sum_{k=1}^{2} \langle ((I - C(0,1))(r\,\mathrm{Re}\,\mathbf{v} + \tilde{w}(r)) - F(0,1,r\,\mathrm{Re}\,\mathbf{v} + \tilde{w}(r))), \tilde{A}\mathbf{w_k}\rangle \tilde{\mathbf{v}}_\mathbf{k} \qquad (4.12)
\end{aligned}
$$

(cf. (3.12) and (3.15)) and from the fact, that the functions $\tilde{\mathbf{v}}_\mathbf{k}$ satisfy the boundary conditions (1.2) (cf. (3.10). Now we differentiate three times (4.11) with respect to r at $r = 0$, use $B + A\partial_u F(0) = 0$ and get

$$
\begin{aligned}
&\partial_r^3 \Phi(0,1,0) \\
&= -\left\langle A\partial_u^3 F(0,1,0)(\mathrm{Re}\,\mathbf{v}, \mathrm{Re}\,\mathbf{v}, \mathrm{Re}\,\mathbf{v}) + 3A\partial_u^2 F(0,1,0)(\mathrm{Re}\,\mathbf{v}, \tilde{w}''(0)), \mathbf{w}\right\rangle. \qquad (4.13)
\end{aligned}
$$

Let us calculate $\tilde{w}''(0)$. For that we differentiate the identity (4.12) twice with respect to r at $r = 0$ and get

$$
\begin{aligned}
&P\left[(I - C(0,1) - \partial_u F(0,1,0))\tilde{w}''(0) - \partial_u^2 F(0,1,0)(\mathrm{Re}\,\mathbf{v}, \mathrm{Re}\,\mathbf{v})\right] \\
&= -\sum_{k=1}^{2} \langle A\partial_u^2 F(0,1,0)(\mathrm{Re}\,\mathbf{v}, \mathrm{Re}\,\mathbf{v}), \mathbf{w_k}\rangle \tilde{\mathbf{v}}_\mathbf{k}. \qquad (4.14)
\end{aligned}
$$

But $[A\partial_u^2 F(0,1,0)(\mathrm{Re}\,\mathbf{v}, \mathrm{Re}\,\mathbf{v})](x,t)$ is of the type $c_1(x)e^{2it} + c_0(x) + c_{-1}(x)e^{-2it}$ (cf. (3.5)), and $\mathbf{w_k}(x,t)$ is of the type $d_1(x)e^{it} + d_{-1}(x)e^{-it}$, hence $\langle A\partial_u^2 F(0,1,0)(\mathrm{Re}\,\mathbf{v}, \mathrm{Re}\,\mathbf{v}), \mathbf{w_k}\rangle = 0$. Applying A to both sides of (4.14) and using Lemma 3.1, we get

$$
(A + B)\tilde{w}''(0) = A\partial_u^2 F(0,1,0)(\mathrm{Re}\,\mathbf{v}, \mathrm{Re}\,\mathbf{v}). \qquad (4.15)
$$

Because of (3.5) the j-th component of the right-hand side of (4.15) calculated at the point (x,t) is

$$
\begin{aligned}
&-\sum_{k,l=1}^{n} b_{jkl}(x)\,\mathrm{Re}(v_k^0(x)e^{-it})\,\mathrm{Re}(v_l^0(x)e^{-it}) \\
&= -\frac{1}{4}\sum_{k,l=1}^{n} b_{jkl}(x)\left(v_k^0(x)v_l^0(x)e^{-2it} + 2v_k^0(x)\overline{v_l^0(x)} + \overline{v_k^0(x)v_l^0(x)}e^{2it}\right).
\end{aligned}
$$

Here we used notation (1.17). The j-th component of the left-hand side of (4.15) calculated at the point (x,t) is

$$\partial_t \tilde{w}''_j(0)(x,t) + a_j(x,0)\partial_x \tilde{w}''_j(0)(x,t) + \sum_{k=1}^n b_{jk}(x)\tilde{w}''_k(0)(x,t).$$

We are going to solve (4.15) with respect to $\tilde{w}''(0)$ by means of the ansatz

$$\tilde{w}''_j(0)(x,t) = y_j(x)e^{-2it} + z_j(x) + \overline{y_j(x)}e^{2it}$$

with C^1-functions $y_j : [0,1] \to \mathbb{C}$ and $z_j : [0,1] \to \mathbb{R}$ which satisfy the boundary conditions corresponding to (1.2). Then (4.15) is equivalent to

$$\begin{aligned} a_j(x,0)\frac{d}{dx}y_j(x) - 2iy_j(x) + \sum_{k=1}^n b_{jk}(x)y_k(x) &= -\frac{1}{4}\sum_{k,l=1}^n b_{jkl}(x)v_k^0(x)v_l^0(x), \\ y_j(0) &= \sum_{k=m+1}^n r_{jk}y_k(0),\ j=1,\dots,m \\ y_j(1) &= \sum_{k=1}^m r_{jk}y_k(1), j=m+1,\dots,n, \end{aligned}$$

$$\begin{aligned} a_j(x,0)\frac{d}{dx}z_j(x) + \sum_{k=1}^n b_{jk}(x)z_k(x) &= -\frac{1}{2}\sum_{k,l=1}^n b_{jkl}(x)v_k^0(x)\overline{v_l^0(x)}, \\ z_j(0) &= \sum_{k=m+1}^n r_{jk}z_k(0), j=1,\dots,m \\ z_j(1) &= \sum_{k=1}^m r_{jk}z_k(1), j=m+1,\dots,n. \end{aligned}$$

In other words: The functions y_j and z_j have to be those as introduced in Section 1.1.

Now we calculate the two terms in (4.13): The first term is

$$-\left\langle A\partial_u^2 F(0,1,0)(\operatorname{Re}\mathbf{v},\operatorname{Re}\mathbf{v},\operatorname{Re}\mathbf{v}),\mathbf{w}\right\rangle = \frac{3}{8}\sum_{j,k,l,r=1}^n \int_0^1 b_{jklr}v_k^0 v_l^0 \overline{v_r^0}\,\overline{w_j^0}dx.$$

The second term is

$$-3\left\langle A\partial_u^2 F(0,1,0)(\operatorname{Re}\mathbf{v},\tilde{w}''(0)),\mathbf{w}\right\rangle = \frac{3}{2}\sum_{j,k,l=1}^n \int_0^1 b_{jkl}\left(v_k^0 z_l + \overline{v_k^0}y_l\right)\overline{w_j^0}dx.$$

Therefore, (4.10) and (4.13) yield the formula (1.20) of Section 1.1.

5 Example

In this section we present a simple and, hence, academic example of a problem of the type (1.1)–(1.3) such that all assumptions (1.4)–(1.6), (1.8), (1.12)–(1.16), and (1.21) of Theorem 1.2 are satisfied. The equations depend, besides of the frequency parameter ω and the bifurcation parameter λ, on an additional real parameter γ which can be chosen in such a way that supercritical as well as subcritical Hopf bifurcation occurs:

$$\begin{aligned} &\omega\partial_t u_1 - \partial_x u_1 + \lambda u_1 - u_2 + \gamma u_1^3 = \omega\partial_t u_2 + \partial_x u_2 = 0,\ x \in (0,1), \\ &u_1(0,t) = 0, \quad u_1(1,t) = u_2(1,t). \end{aligned} \tag{5.1}$$

Using the notation of Section 1, we have $m = 1$, $n = 2$, $a_1(x,\lambda) = -1$, $a_2(x,\lambda) = 1$, $b_1(x,\lambda,u_1,u_2) = \lambda u_1 - u_2 + \gamma u_1^3$, $b_2(x,\lambda,u_1,u_2) = 0$, $r_{12} = 0$, and $r_{21} = 1$.

Obviously, the conditions (1.4)–(1.6) and (1.8) are fulfilled. Condition (1.21) is fulfilled also because the choice $r_{12} = 0$ implies $R_0 = 0$ (cf. (1.18)).

In order to check the remaining conditions (1.12)–(1.16) we consider the eigenvalue problem (see (1.9))

$$\begin{aligned} &-v_1' + (\lambda - \mu)v_1 - v_2 = v_2' - \mu v_2 = 0,\ x \in (0,1), \\ &v_1(0) = v_1(1) - v_2(1) = 0 \end{aligned} \tag{5.2}$$

and the adjoint eigenvalue problem (see (1.10))

$$\begin{aligned} &w_1' + (\lambda - \nu)w_1 = -w_2' - w_1 - \nu w_2 = 0,\ x \in (0,1), \\ &w_2(0) = w_1(1) - w_2(1) = 0. \end{aligned} \tag{5.3}$$

It is easy to verify that there do not exist real eigenvalues μ to (5.2) and that (5.2) is equivalent to

$$\begin{aligned} v_1(x) &= \frac{c}{\lambda - 2\mu}\left(e^{\mu x} - e^{(\lambda-\mu)x}\right),\ v_2(x) = ce^{\mu x}, \\ e^{\lambda - 2\mu} &= 2\mu - \lambda + 1. \end{aligned} \tag{5.4}$$

Here $c = v_2(0)$ is a nonzero complex constant. Setting $\lambda - 2\mu = a + ib$ with $a \in \mathbb{R}$ and (without loss of generality) $b > 0$, we get

$$b = \sqrt{e^{2a} - (1-a)^2}$$

and

$$\sin\sqrt{e^{2a} - (1-a)^2} = -\sqrt{1 - e^{-2a}(1-a)^2}. \tag{5.5}$$

It is easy to see that equation (5.5) has (besides of the solution $a = 0$) a countable number of solutions $0 < a_0 < a_1 \dots$ tending to ∞. Hence, the spectrum of (5.2) consists of countably many geometrically simple eigenvalues

$$\mu_j^{\pm}(\lambda) = \frac{1}{2}\left(\lambda - a_j \pm i\sqrt{e^{2a_j} - (1-a_j)^2}\right).$$

If $\lambda = a_0$, then the eigenvalue pair $\mu_0^\pm(a_0)$ is on the imaginary axis:

$$\pm i\omega_0 := \mu_0^\pm(a_0) = \pm\frac{i}{2}\sqrt{e^{2a_0} - (1-a_0)^2}.$$

The real parts of all other eigenvalues are negative, hence the nonresonance condition (1.16) is fulfilled. Obviously, the transversality condition (1.15) is fulfilled, since

$$\alpha = \frac{d}{d\lambda}\mathrm{Re}\mu_0^\pm(\lambda)|_{\lambda=a_0} = \frac{1}{2}.$$

In order to check (1.13), we first calculate $w_1(x)$ and $w_2(x)$ using (5.3) and the fact that $\nu = \bar{\mu}$. We get

$$w_1(x) = de^{(-i\omega_0 - a_0)x},\ w_2(x) = \frac{d}{a_0 + 2i\omega_0}\left(e^{-(i\omega_0+a_0)x} - e^{i\omega_0 x}\right).$$

Here $d = w_1(0)$ is again a nonzero complex constant. Hence,

$$\begin{aligned} \int_0^1 (v_1\overline{w}_1 + v_2\overline{w}_2)\ dx &= \frac{2c\bar{d}}{a_0 - 2i\omega_0}\left(\int_0^1 e^{(2i\omega_0 - a_0)x}dx - 1\right) \\ &= \frac{2c\bar{d}}{(a_0 - 2i\omega_0)^2}\left(e^{a_0 - 2i\omega_0} - e^{2i\omega_0 - a_0}\right) \neq 0. \end{aligned} \tag{5.6}$$

Here we used that

$$e^{a_0 - 2i\omega_0} = 2i\omega_0 - a_0 + 1 \tag{5.7}$$

(cf. (5.4)).

In order to calculate β we have to normalize v_1, v_2, w_1 and w_2 according to (1.14). Using (5.6), we get

$$c\bar{d} = \frac{(a_0 - 2i\omega_0)^2}{e^{a_0 - 2i\omega_0} - e^{2i\omega_0 - a_0}}. \tag{5.8}$$

Further, we have $b_{jkl} = 0$ for all indices j, k, and l, $b_{1111} = 6\gamma$ and $b_{jklr} = 0$ if one of the indices j, k, l, r is not equal to one. Hence,

$$\beta = -\frac{3\gamma}{4}\,\mathrm{Re}\int_0^1 v_1^0 v_1^0 \overline{v_1^0 w_1^0}\,dx.$$

Moreover, using (5.7) and (5.8), we get

$$\left|v_1^0(x)\right|^2 v_1^0(x)\overline{w_1^0(x)} = \frac{|c|^2(e^{a_0 - 2i\omega_0} + e^{2a_0})}{|a_0 - 2i\omega_0|^2|1 + e^{a_0 - 2i\omega_0}|^2}\left|1 - e^{(a_0 - 2i\omega_0)x}\right|^2\left(1 - e^{(2i\omega_0 - a_0)x}\right).$$

Equation (5.7) yields also

$$\int_0^1 \left|1 - e^{(a_0 - 2i\omega_0)x}\right|^2 \left(1 - e^{(2i\omega_0 - a_0)x}\right) dx = 5 + \frac{e^{2a_0} - 1}{2a_0} + \frac{e^{4i\omega_0} - 1}{4i\omega_0} + e^{2i\omega_0 - a_0}$$

and

$$\begin{aligned} &\mathrm{Re}\left(\left(e^{a_0 - 2i\omega_0} + e^{2a_0}\right)\left(5 + \frac{e^{2a_0} - 1}{2a_0} + \frac{e^{4i\omega_0} - 1}{4i\omega_0} + e^{2i\omega_0 - a_0}\right)\right) \\ &= \left(e^{2a_0} - a_0 + 1\right)\left(5 + \frac{e^{2a_0} - 1}{2a_0}\right). \end{aligned}$$

Therefore,

$$\beta = -\frac{3\gamma|c|^2(e^{a_0} - a_0 + 1)}{4|a_0 - 2i\omega_0|^2|1 + e^{a_0 - 2i\omega_0}|^2}\left(5 + \frac{e^{2a_0} - 1}{2a_0}\right).$$

Now, because of $a_0 > 0$ it follows $\beta > 0$ (supercritical Hopf bifurcation) for $\gamma < 0$ and $\beta < 0$ (subcritical Hopf bifurcation) for $\gamma > 0$.

A Appendix

In this appendix we present a simple linear version of the so-called fiber contraction principle (see, e.g., [7, Section 1.11.3]):

Lemma A.1 *Let U be a Banach space and $u_1, u_2, \ldots \in U$ a converging sequence. Further, let $A_1, A_2, \ldots \in \mathcal{L}(U)$ be a sequence of linear bounded operators on U such that there exists $c < 1$ such that for all $u \in U$ it holds*

$$\|A_n u\| \le c\|u\| \text{ for all } n = 1, 2, \ldots \tag{A.1}$$

and

$$A_1 u, A_2 u, \ldots \text{ converges in } U. \tag{A.2}$$

Finally, let $v_1, v_2, \ldots \in U$ be a sequence such that for all $n = 1, 2, \ldots$ we have

$$v_{n+1} = A_n v_n + u_n. \tag{A.3}$$

Then the sequence $v_1, v_2, \ldots$ converges in U.

Proof. Because of (A.2) there exists $A \in \mathcal{L}(U)$ such that for all $u \in U$ we have $A_n u \to Au$ in U for $n \to \infty$. Moreover, (A.1) yields that $\|Au\| \le c\|u\|$ for all $u \in U$. Because of $c < 1$ there exists exactly one $v \in U$ such that $v = Av + u$, where $u \in U$ is the limit of the sequence $u_1, u_2, \ldots \in U$.

Let us show that $v_k \to v$ in U for $k \to \infty$. Because of (A.3) we have

$$v_{n+1} - v = A_n(v_n - v) + (A_n - A)\, v + u_n - u. \tag{A.4}$$

Using the notation $r_n := \|v_n - v\|$ and $s_n := \|\,(A_n - A)\, v\| + \|u_n - u\|$, we get from (A.1) and (A.4)

$$r_{n+1} \leq cr_n + s_n \leq c^2 r_{n-1} + cs_{n-1} + s_n \leq \ldots \leq c^n r_1 + \sum_{m=0}^{n} c^{n-m} s_m. \tag{A.5}$$

Take $s_0 > 0$ such that $s_n \leq s_0$ for all n. Then (A.5) implies for $0 \leq l \leq n$

$$r_{n+1} \leq c^n \left(r_1 + s_0 \sum_{m=0}^{l} c^{-m} \right) + \frac{1}{1-c} \max_{m>l} s_m. \tag{A.6}$$

But we have

$$\max_{m>l} s_m \to 0 \text{ for } l \to \infty. \tag{A.7}$$

Given $\varepsilon > 0$, fix $l \in \mathbb{N}$ sufficiently large, in order to meet the estimate

$$\frac{1}{1-c} \max_{m>l} s_m < \frac{\varepsilon}{2}. \tag{A.8}$$

Moreover, choose $n_0 \in \mathbb{N}$ so large that

$$c^n \left(r_1 + s_0 \sum_{m=0}^{l} c^{-m} \right) < \frac{\varepsilon}{2} \text{ for all } n \geq n_0. \tag{A.9}$$

Then (A.6)–(A.9) yield that $|r_{n+1}| < \varepsilon$ for all $n \geq n_0$, as desired. $\square$

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References

[1] R. Abraham, J. E. Marsden, T. Ratiu, *Manifolds, Tensor Analysis, and Applications.* Appl. Math. Sciences **75**, Springer, 1988.

[2] T. A. Akramov, On the behavior of solutions to a certain hyperbolic problem. *Sib. Math. J.* **39** (1998), 1-17.

[3] H. Amann, J. Escher, *Analysis II.* Birkhäuser, 2008.

[4] A. A. Andronov, S. E. Khaikin, *Teorija Kolebanij* (Russian). ONTI, Moscow, 1937.

[5] J. Appell, A. Vignoli, P. P. Zabrejko, Implicit function theorems and nonlinear integral equations. *Expo. Math.* **14** (1996), 384-424.

[6] E. Barbera, C. Curro, G. Valenti, Wave features of a hyperbolic prey-predator model. *Math. Methods Appl. Sciences* **33** (2010), 1504-1515.

[7] C. Chicone, *Ordinary Differential Equations with Applications. Second Edition.* Texts in Appl. Math. **34**, Springer, 2006.

[8] C. Chicone, Y. Latuschkin, *Evolution Semigroups in Dynamical Systems and Differential Equations.* Math. Surveys Monographs **70**, American Mathematical Society, Providence, 1999.

[9] S.-N. Chow, J. K. Hale, *Methods of Bifurcation Theory.* Grundlehren der Math. Wissenschaften **251**, Springer, 1982.

[10] M. G. Crandall, P. H. Rabinowitz, The Hopf bifurcation theorem in infinite dimensions. *Arch. Rat. Mech. Anal.* **67** (1978), 53–72.

[11] J. Dieudonne, *Foundations of Modern Analysis.* Academic Press, 1960.

[12] N.A. Ëltysheva, On qualitative properties of solutions of some hyperbolic systems on the plane. *Mat. Sb., N. Ser.* **135(177)** (1988), 186–209.

[13] K.-J. Engel, R. Nagel, *A Short Course on Operator Semigroups.* Universitext, Springer, 2006.

[14] P. Gajewski, *Hopf-Bifurkations-Formeln und Anwendungen auf hyperbolische Systeme.* Diploma Thesis, Humboldt University of Berlin, 2011.

[15] M. Golubitsky, P.H. Rabinowitz, A sketch of the Hopf bifurcation theorem. In: *Selected Works of Eberhard Hopf with Commentaries,* C.S. Morawetz, J.B. Serrin and Y.G. Sinai, eds., AMS, Providence, 2002, 111-118.

[16] J. K. Hale, G. Raugel, A modified Poincare method for the persistence of periodic orbits and applications. *J. Dyn. Diff. Equat.* **22** (2010), 3-68.

[17] M. Haragus, G. Iooss, *Local Bifurcations, Center Manifolds, and Normal Forms in Infinite-Dimensional Dynamical Systems*. Springer, 2011.

[18] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*. Lecture Notes in Math. **840**, Springer, 1981.

[19] T. Hillen, A Turing model with correlated random walk. *J. Math. Biol.* **35** (1996), 49-72.

[20] E. Hopf, Abzweigung einer periodischen Lösung von einer stationären Lösung eines Differentialsystems. *Berichte Math.-Phys. Kl. Sächs. Akad. Wiss.* **94** (1942), 3–22.

[21] W. Horsthemke, Spatial instabilities in reaction random walks with direction-independent kinetics. *Phys. Rev. E* **60** (1999), 2651-2663.

[22] H. Kielhöfer, *Bifurcation Theory. An Introduction with Applications to PDEs*. Appl. Math. Sciences **156**, Springer, 2004.

[23] I. Kmit, Smoothing effect and Fredholm property for first-order hyperbolic PDEs. *Operator Theory: Advances and Applications* **231**, Basel: Birkhäuser, 2013, 219-238.

[24] I. Kmit, L. Recke, Fredholm alternative for periodic-Dirichlet problems for linear hyperbolic systems. *J. Math. Anal. Appl.* **335** (2007), 355–370.

[25] I. Kmit, L. Recke, Fredholmness and smooth dependence for linear hyperbolic periodic-Dirichlet problems. *J. Differ. Equations* **252** (2012), 1962-1986.

[26] I. Kmit, L. Recke, Periodic solutions to dissipative hyperbolic systems. I: Fredholm solvability of linear problems. E-print: *http://arxiv.org/abs/1108.2882*, 18 pages (2013).

[27] M. Lichtner, A spectral mapping theorem for linear hyperbolic systems. *Proc. Amer. Math. Soc.* **136 (6)** (2008), 2091–2101.

[28] M. Lichtner, Principle of linearized stability and smooth center manifold theorem for semilinear hyperbolic systems. Preprint **1155** (2006), Weierstraß Institute for Applied Analysis and Stochastics Berlin.

[29] M. Lichtner, M. Radziunas, L. Recke, Well-posedness, smooth dependence and center manifold reduction for a semilinear hyperbolic system from laser dynamics. *Math. Methods Appl. Sci.* **30** (2007), 931-960.

[30] Z. Liu, P. Magal, S. Ruan, Hopf bifurcation for non-densely defined Cauchy problems. *Z. Angew. Math. Phys. (ZAMP)* **62** (2011), 191–222.

[31] Z.-H. Luo, B.-Z. Guo, O. Mogul, *Stability and Stabilization of Infinite Dimensional Systems with Applications*. Springer, 1999.

[32] T. Ma, S. Wang, *Bifurcation Theory and Applications.* World Scientific Series Nonl. Science **A 53**, World Scientific, 2005.

[33] P. Magal, S. Ruan, *Center manifolds for semilinear equations with non-dense domain and applications to Hopf bifurcation in age structured models.* Memoirs AMS, vol. **202**, No. **951**, Providence, 2009.

[34] R. Magnus, The implicit function theorem and multi-bumb solutions of periodic partial differential equations. *Proc. Royal Soc. Edinburgh* **136A** (2006), 559–583.

[35] C. Martel, J. M. Vega, Dynamics of a hyperbolic system that applies at the onset of the oscillatory instability. *Nonlinearity* **11** (1998), 105-142.

[36] A. F. Neves, X.-B. Lin, A multiplicity theorem for hyperbolic systems. *J. Differ. Equations* **76** (1988), 339-352.

[37] A. F. Neves, H. de Souza Ribeiro, O. Lopes, On the spectrum of evolution operators generated by hyperbolic systems. *J. Functional Anal.* **670** (1986), 320-344.

[38] D. Peterhof, B. Sandstede, All-optical clock recovery using multisection distributed-feedback lasers. *J. Nonlinear Sci.* **9** (1999), 575-613.

[39] M. Radziunas, Numerical bifurcation analysis of traveling wave model of multisection semiconductor lasers. *Physica D* **213** (2006), 575–613.

[40] M. Radziunas, H.-J. Wünsche, Dynamics of multisection DFB semiconductor lasers: traveling wave and mode approximation models. *In: Optoelectronic Devices–Advanced Simulation and Analysis*, ed. by J. Piprek, Springer, 2005, 121–150.

[41] L. Recke, O. Omel'chenko, Boundary layer solutions to problems with infinite dimensional singular and regular perturbations. *J. Differ. Equations* **245 (12)** (2008), 3806–3822.

[42] L. Recke, K. R. Schneider, V. V. Strygin, Spectral properties of coupled wave equations. *Z. Angew. Math. Phys. (ZAMP)* **50** (1999), 925-933.

[43] M. Renardy, Bifurcation from rotating waves. *Arch. Rat. Mech. Anal.* **79** (1982), 49–84.

[44] M. Renardy, On the linear stability of hyperbolic PDEs and viscoelastic flows. *Z. Angew. Math. Phys. (ZAMP)* **45** (1994), 854-865.

[45] J. Sieber, Numerical bifurcation analysis for multisection semiconductor lasers. *SIAM J. Appl. Dyn. Syst.* **1** (2002), 248-270.

[46] A. Vanderbauwhede, G. Iooss, Center manifold theory in infinite dimensions. *Dyn. Rep., Expo. Dyn. Syst., New Ser.* **1** (1992), 125-163.