Periodic Solutions to Dissipative Hyperbolic Systems. I: Fredholm Solvability of Linear Problems

I. Kmit L. Recke

Institute of Mathematics, Humboldt University of Berlin,

Rudower Chaussee 25, D-12489 Berlin, Germany

and Institute for Applied Problems of Mechanics and Mathematics,

Ukrainian Academy of Sciences, Naukova St. 3b, 79060 Lviv, Ukraine

E-mail: kmit@informatik.hu-berlin.de

Institute of Mathematics, Humboldt University of Berlin,

Rudower Chaussee 25, D-12489 Berlin, Germany

E-mail: recke@mathematik.hu-berlin.de

Abstract

This paper concerns linear first-order hyperbolic systems in one space dimension of the type

$$\partial_t u_j + a_j(x,t)\partial_x u_j + \sum_{k=1}^{n} b_{jk}(x,t)u_k = f_j(x,t),\ x \in (0,1),\ j = 1,\dots,n,$$

with periodicity conditions in time and reflection boundary conditions in space. We state a kind of dissipativity condition (depending on the coefficients a_j and b_{jj} and the boundary reflection coefficients), which implies Fredholm solvability of the problem, i.e., either there is a nontrivial solution to the homogeneous problem (in this case the space of such solutions has finite dimension) or the nonhomogeneous problem is uniquely solvable for any right-hand side (in this case the solution depends continuously on the right-hand side). In particular, under those conditions no small denominator effects occur. Our results work for many non-strictly hyperbolic systems, but they are new even in the case of strict hyperbolicity.

Finally, in the case that all coefficients a_j are t-independent, we show that the solutions are C^∞-smooth if the data are C^∞-smooth.

Key words: first-order hyperbolic systems, time-periodic solutions, reflection boundary conditions, no small denominators, Fredholm solvability.

Mathematics Subject Classification: 35B10, 35L50, 47A53

1 Introduction

1.1 Problem and main results

This paper concerns general linear first-order hyperbolic systems in one space dimension of the type

$$\partial_t u_j + a_j(x,t)\partial_x u_j + \sum_{k=1}^{n} b_{jk}(x,t)u_k = f_j(x,t), \quad x \in (0,1),\ j = 1,\dots,n \tag{1.1}$$

with time-periodicity conditions

$$u_j(x,t+2\pi) = u_j(x,t), \quad x \in [0,1],\ j = 1,\dots,n \tag{1.2}$$

and reflection boundary conditions

$$\begin{aligned} u_j(0,t) &= \sum_{k=m+1}^{n} r_{jk}(t)u_k(0,t),\ j = 1,\dots,m, \\ u_j(1,t) &= \sum_{k=1}^{m} r_{jk}(t)u_k(1,t),\ j = m+1,\dots,n. \end{aligned} \tag{1.3}$$

Here $m < n$ are positive integers. Throughout the paper it is supposed that the functions $r_{jk} : \mathbb{R} \to \mathbb{R}$ and $a_j, b_{jk}, f_j : [0,1] \times \mathbb{R} \to \mathbb{R}$ are 2π-periodic with respect to t, and that the coefficients r_{jk}, a_j and b_{jk} are C^1-smooth. Additionally, we suppose that

$$a_j(x,t) \neq 0 \ \text{ for all } x \in [0,1], t \in \mathbb{R} \text{ and } j = 1,\dots,n \tag{1.4}$$

and that

$$\begin{aligned} &\text{for all } 1 \le j \neq k \le n \text{ there exists } \tilde{b}_{jk} \in C^1([0,1]\times\mathbb{R}) \text{ such that} \\ &b_{jk}(x,t) = \tilde{b}_{jk}(x,t)(a_k(x,t) - a_j(x,t)) \text{ for all } x \in [0,1] \text{ and } t \in \mathbb{R}. \end{aligned} \tag{1.5}$$

Roughly speaking, we will prove the following: If a certain dissipativity condition on the data a_j, b_{jj} and r_{jk} is satisfied (which is the case, for example, if the functions $|r_{jk}|$ with $1 \le j \le m$ and $m+1 \le k \le n$ or with $1 \le k \le m$ and $m+1 \le j \le n$ are sufficiently small (see (1.13)), then a Fredholm alternative is true for the system (1.1)–(1.3), i.e.,

- either the system (1.1)–(1.3) with $f = (f_1,\dots,f_n) = 0$ has a nontrivial solution (then the vector space of those solutions has a finite dimension),
- or for any continuous right-hand side f the system (1.1)–(1.3) has a unique solution $u = (u_1,\dots,u_n)$ (then the map $f \mapsto u$ is continuous with respect to the supremum norm).

In order to formulate our results more precisely, let us introduce the characteristics of the hyperbolic system (1.1). Given $j = 1,\dots,n$, $x \in [0,1]$, and $t \in \mathbb{R}$, the j-th

characteristic is defined as the solution $\xi \in [0,1] \mapsto \tau_j(\xi,x,t) \in \mathbb{R}$ of the initial value problem

$$\partial_\xi \tau_j(\xi,x,t) = \frac{1}{a_j(\xi,\tau_j(\xi,x,t))}, \quad \tau_j(x,x,t) = t. \tag{1.6}$$

Moreover, we denote

$$c_j(\xi,x,t) := \exp \int_x^\xi \frac{b_{jj}(\eta,\tau_j(\eta,x,t))}{a_j(\eta,\tau_j(\eta,x,t))}\, d\eta, \tag{1.7}$$

$$d_j(\xi,x,t) := \frac{c_j(\xi,x,t)}{a_j(\xi,\tau_j(\xi,x,t))}. \tag{1.8}$$

Straightforward calculations (see Section 2) show that a C^1-map $u : [0,1] \times \mathbb{R} \to \mathbb{R}^n$ is a solution to the PDE problem (1.1)–(1.3) if and only if it satisfies the following system of integral equations

$$\begin{aligned} u_j(x,t) = {} & c_j(0,x,t) \sum_{k=m+1}^{n} r_{jk}(\tau_j(0,x,t)) u_k(0,\tau_j(0,x,t)) \\ & - \int_0^x d_j(\xi,x,t) \sum_{\substack{k=1 \\ k\neq j}}^{n} b_{jk}(\xi,\tau_j(\xi,x,t)) u_k(\xi,\tau_j(\xi,x,t))\, d\xi \\ & + \int_0^x d_j(\xi,x,t) f_j(\xi,\tau_j(\xi,x,t))\, d\xi, \quad j = 1,\dots,m, \end{aligned} \tag{1.9}$$

$$\begin{aligned} u_j(x,t) = {} & c_j(1,x,t) \sum_{k=1}^{m} r_{jk}(\tau_j(1,x,t)) u_k(1,\tau_j(1,x,t)) \\ & + \int_x^1 d_j(\xi,x,t) \sum_{\substack{k=1 \\ k\neq j}}^{n} b_{jk}(\xi,\tau_j(\xi,x,t)) u_k(\xi,\tau_j(\xi,x,t))\, d\xi \\ & - \int_x^1 d_j(\xi,x,t) f_j(\xi,\tau_j(\xi,x,t))\, d\xi, \quad j = m+1,\dots,n. \end{aligned} \tag{1.10}$$

This motivates the following definition:

Definition 1.1 *(i) By $\mathcal{C}_n$ we denote the vector space of all continuous maps $u : [0,1] \times \mathbb{R} \to \mathbb{R}^n$ which satisfy (1.2), with the norm*

$$\|u\|_\infty := \max_{1\le j\le n} \max_{0\le x\le 1} \max_{t\in\mathbb{R}} |u_j(x,t)|.$$

(ii) A function $u \in \mathcal{C}_n$ is called a continuous solution to (1.1)–(1.3) if it satisfies (1.9) and (1.10).

(iii) A function $u \in C^1([0,1] \times \mathbb{R}; \mathbb{R}^n)$ is called a classical solution to (1.1)–(1.3) if it satisfies (1.1)–(1.3) pointwise.

Finally, we introduce $R^0, R^1 \in \mathcal{C}_n$ by

$$R_j^0(x,t) := \begin{cases} \exp\left(-\int_0^x \frac{b_{jj}(\xi,\tau_j(\xi,x,t))}{a_j(\xi,\tau_j(\xi,x,t))}d\xi\right) \sum_{k=m+1}^{n} |r_{jk}(\tau_j(0,x,t)| & \text{for } j=1,\dots,m, \\ \exp\int_x^1 \frac{b_{jj}(\xi,\tau_j(\xi,x,t))}{a_j(\xi,\tau_j(\xi,x,t))}d\xi \ \sum_{k=1}^{m} |r_{jk}(\tau_j(1,x,t)| & \text{for } j=m+1,\dots,n, \end{cases}$$

and

$$R_j^1(x,t) := \begin{cases} R_j^0(x,t)\ \exp\int_0^x \frac{\partial_t a_j(\xi,\tau_j(\xi,x,t))}{a_j(\xi,\tau_j(\xi,x,t))^2}d\xi & \text{for } j=1,\dots,m, \\ R_j^0(x,t)\ \exp\left(-\int_x^1 \frac{\partial_t a_j(\xi,\tau_j(\xi,x,t))}{a_j(\xi,\tau_j(\xi,x,t))^2}d\xi\right) & \text{for } j=m+1,\dots,n, \end{cases}$$

and write

$$S^0 := \max_{1\le j\le m} \max_{0\le x\le 1} \max_{t\in\mathbb{R}} R_j^0(x,t), \quad T^0 := \max_{m+1\le j\le n} \max_{0\le x\le 1} \max_{t\in\mathbb{R}} R_j^0(x,t), \tag{1.11}$$

$$S^1 := \max_{1\le j\le m} \max_{0\le x\le 1} \max_{t\in\mathbb{R}} R_j^1(x,t), \quad T^1 := \max_{m+1\le j\le n} \max_{0\le x\le 1} \max_{t\in\mathbb{R}} R_j^1(x,t). \tag{1.12}$$

Denote by $\mathcal{K}$ the vector space of all continuous solutions to (1.1)–(1.3) with $f = 0$.

Now we formulate our result:

Theorem 1.2 *Suppose (1.4), (1.5) and*

$$S^0 T^0 < 1. \tag{1.13}$$

Then the following is true:

(i) $\dim \mathcal{K} < \infty$.

(ii) The vector space of all $f \in \mathcal{C}_n$ *such that there exists a continuous solution to (1.1)–(1.3) is a closed subspace of codimension* $\dim \mathcal{K}$ *in* $\mathcal{C}_n$.

(iii) Either $\dim \mathcal{K} > 0$ *or for any* $f \in \mathcal{C}_n$ *there exists exactly one continuous solution* u *to (1.1)–(1.3). In the latter case the map* $f \in \mathcal{C}_n \mapsto u \in \mathcal{C}_n$ *is continuous.*

(iv) Suppose that the functions f_j *are continuously differentiable with respect to* t *and that*

$$S^1 T^1 < 1. \tag{1.14}$$

Then any continuous solution to (1.1)–(1.3) is a classical solution to (1.1)–(1.3).

(v) If all coefficients a_j *are* t*-independent and if all functions* a_j, b_{jk}, f_j *and* r_{jk} *are* C^∞*-smooth, then any continuous solution to (1.1)–(1.3) is* C^∞*-smooth.*

It is well-known that the Fredholm property of the linearization is a key for many local investigations of time-periodic solutions to nonlinear ODEs and parabolic PDEs. This is the case for Hopf bifurcation, for saddle node bifurcation or period doubling bifurcation of periodic solutions as well as for small periodic forcing of stationary or periodic solutions

(see, e.g. [4] for ODEs and [6] for parabolic PDEs). But almost nothing is known whether similar results are true for nonlinear dissipative hyperbolic PDEs.

The first aim of the present paper is to open the door for those local investigations of time-periodic solutions to nonlinear dissipative hyperbolic PDEs. In particular, in [9] we apply our results to prove a Hopf bifurcation theorem for semilinear dissipative hyperbolic PDEs.

The second aim is applications to semiconductor laser dynamics [11, 14, 15]. Phenomena like Hopf bifurcation (describing the appearance of selfpulsations of lasers) and periodic forcing of stationary solutions (describing the modulation of stationary laser states by time periodic electric pumping) and periodic solutions (describing the synchronization of selfpulsating laser states with small time periodic external optical signals, cf. [2, 16, 17, 18]) are essential for many applications of semiconductor laser devices in communication systems.

Remark that in [7] and [8] we proved similar results for the autonomous case, i.e., the case, when the coefficients a_j, b_{jk} and r_{jk} are t-independent. There the weak formulation of the problem (1.1)–(1.3) was a system of variational equations, and we used the method of Fourier series in anisotropic Sobolev spaces as in [19]. In the present paper the weak formulation of the problem (1.1)–(1.3) is the system (1.9)–(1.10) of integral equations, and we use the method of integration along characteristics in C-spaces. The corresponding dissipativity condition to (1.13) in the present paper is (1.7) in [7] and (1.11) in [8]. They imply that there is a uniform positive lower bound for the absolute values of the denominators in the Fourier coefficients of the solutions, i.e., that there are no small denominators.

Our paper is organized as follows:

In Section 1.2 we comment about the assumptions (1.5) and (1.13). In Section 2 we show that any classical solution to (1.1)–(1.3) is a continuous solution in the sense of Definition 1.1, and that any C^1-smooth continuous solution is a classical one. In Section 3 we introduce an abstract representation of the system (1.9)–(1.10). Moreover, we show that in the "diagonal" case, i.e., if $b_{jk} = 0$ for all $j \neq k$, there exists exactly one continuous solution to (1.1)–(1.3) for every $f \in \mathcal{C}_n$. The Fredholm solvability stated in the assertions (i)-(iii) of Theorem 1.2 is proved in Section 4, while the solution regularity given by the assertions (iv) and (v) is proved in Section 5.

1.2 Some Remarks

Remark 1.3 about assumption (1.13) Here we show that, if (1.13) is not fulfilled, the assertions of Theorem 1.2 are not true, in general. With this aim we consider the following example of a problem of the type (1.1)–(1.3) satisfying all but (1.13) assumptions of Theorem 1.2: Set $m = 1, n = 2, a_1(x,t) = -a_2(x,t) = \alpha = \text{const}, b_{jk}(x,t) = 0, f_j(x,t) = 0$, and $r_{12} = r_{21} = 1$. In this case the system (1.9)–(1.10) reads as

$$u_1(x,t) = u_2(0, t - \alpha x), \tag{1.15}$$

$$u_2(x,t) = u_1(1, t + \alpha(x-1)), \tag{1.16}$$

and $S^0 = T^0 = 1$, i.e., (1.13) is not satisfied. Inserting (1.16) into (1.15) and putting $x = 1$, we get

$$u_1(1,t) = u_1(1,t-2\alpha). \tag{1.17}$$

If $\alpha/2\pi$ is irrational, then the functional equation (1.17) does not have nontrivial continuous solutions. If

$$\frac{\alpha}{2\pi} = \frac{p}{q} \text{ with } p \in \mathbb{Z} \text{ and } q \in \mathbb{N},$$

then any $2\pi/q$-periodic function is a solution to (1.17). In other words,

$$\dim \mathcal{K} = \begin{cases} 0 & \text{if } \alpha/2\pi \notin \mathbb{Q}, \\ \infty & \text{if } \alpha/2\pi \in \mathbb{Q}. \end{cases}$$

Hence, the Fredholm solvability conclusion of Theorem 1.2 is failed if $\alpha/2\pi \in \mathbb{Q}$. Moreover, in the case $\alpha/2\pi \in \mathbb{Q}$ there exist continuous solutions to (1.1)–(1.3) which are not classical one's.

Remark 1.4 about assumption (1.5) Roughly speaking, assumption (1.5) means that a certain loss of strict hyperbolicity, caused by leading order coefficients a_j and a_k with $j \neq k$, must be compensated by a certain vanishing behavior of the corresponding lower order coefficients b_{jk}.

Let us show that, if (1.5) is not fulfilled, the assertions of Theorem 1.2 are not true, in general. With this aim we consider the following example of the problem (1.1)–(1.3) satisfying all but (1.5) assumptions of Theorem 1.2: Set $m = 1, n = 2, a_1(x,t) = a_2(x,t) = 1, b_{11}(x,t) = b_{12}(x,t) = b_{22}(x,t) = 0, b_{21}(x,t) = 3/2, f_1(x,t) = f_2(x,t) = 0$, and $r_{12}^0 = r_{21}^1 = 1/2$. In this case the system (1.9), (1.10) reads as

$$\begin{aligned} u_1(x,t) &= \frac{1}{2}u_2(0,t-x), \\ u_2(x,t) &= \frac{1}{2}u_1(1,t-x+1) + \frac{3}{2}\int_x^1 u_1(\xi,\xi-x+t)d\xi. \end{aligned}$$

It is easy to verify that any continuous 2π-periodic map $U : \mathbb{R} \to \mathbb{R}$ creates a solution

$$u_1(x,t) = U(t-x), \quad u_2(x,t) = \left(2 - \frac{3}{2}x\right) U(t-x)$$

to this system. In particular, we have $\dim \mathcal{K} = \infty$, and there exist continuous solutions to (1.1)–(1.3) which are not classical one's.

Let us remark that, surprisingly, an assumption of the type (1.5) is used also in quite another circumstances, for proving the spectrum-determined growth condition in L^p-spaces [3, 12, 13] and in C-spaces [10] for semiflows generated by initial value problems for hyperbolic systems of the type (1.1), (1.3).

Remark 1.5 about sufficient conditions for (1.13) The following sufficient conditions for (1.13) are obvious: First, if the coefficients a_j and b_{jj} are given and fixed, then (1.13) is fulfilled if

$$\max_{1\le j\le m} \max_{m+1\le k\le n} \max_{t\in\mathbb{R}} |r_{jk}(t)| \text{ is sufficiently small}$$

and/or if

$$\max_{m+1\le j\le n}\max_{1\le k\le m}\max_{t\in\mathbb{R}}|r_{jk}(t)| \text{ is sufficiently small.}$$

Second, if the coefficients r_{jk} are given and fixed, then (1.13) is fulfilled if

$$\max_{1\le j\le m}\min_{0\le x\le 1}\min_{t\in\mathbb{R}}\frac{b_{jj}(x,t)}{a_j(x,t)} \text{ is sufficiently large}$$

and/or if

$$\max_{m+1\le j\le n}\max_{0\le x\le 1}\max_{t\in\mathbb{R}}\frac{b_{jj}(x,t)}{a_j(x,t)} \text{ is sufficiently small.}$$

In particular, this is the case if for all $x\in[0,1]$ and $t\in\mathbb{R}$ we have

$$a_j(x,t)<0 \text{ for } j=1,\dots,m \text{ and } a_j(x,t)>0 \text{ for } j=m+1,\dots,n \tag{1.18}$$

and

$$\max_{1\le j\le n}\max_{0\le x\le 1}\max_{t\in\mathbb{R}} b_{jj}(x,t) \text{ is negative and sufficiently small.}$$

It is easy to verify (see (2.3)) that $a_j(x,t)\partial_x\tau_j(\xi,x,t)<0$ for all $j=1,\dots,n$, $\xi,x\in[0,1]$, and $t\in\mathbb{R}$. Therefore, the functions $u_j(0,\tau_j(0;x,t))$ with indices j such that $a_j(x,t)<0$ describe waves traveling to the left, and the functions $u_j(1,\tau_j(1;x,t))$ with indices j such that $a_j(x,t)>0$ describe waves traveling to the right. Hence, it is natural to prescribe reflection boundary conditions at the left interval end $x=0$ for those indices j such that $a_j(x,t)>0$ and at the right interval end $x=1$ for those indices j such that $a_j(x,t)<0$. Therefore, in most of the applications (1.18) is true.

2 Integration along characteristics

Let us show that a C^1-function $u:[0,1]\times\mathbb{R}\to\mathbb{R}^n$ satisfies the system (1.1) of first-order partial differential equations, the time-periodicity conditions (1.2) and the boundary conditions (1.3) if and only if it satisfies the system (1.9)-(1.10) of integral equations. The type of calculations is well-known, so we do this for the convenience of the reader.

Standard results about initial value problems for ordinary differential equations yield that the functions $\tau_j:[0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$ are well-defined by (1.6), and they are C^1-smooth. Moreover, it holds

$$\tau_j(\xi,x,t+2\pi)=\tau_j(\xi,x,t)+2\pi, \tag{2.1}$$

$$\tau_j(x,\xi,\tau_j(\xi,x,t))=t \tag{2.2}$$

and

$$\partial_x\tau_j(\xi,x,t) = -\frac{1}{a_j(x,t)}\exp\int_\xi^x\frac{\partial_t a_j(\eta,\tau_j(\eta,x,t))}{a_j(\eta,\tau_j(\eta,x,t))^2}d\eta, \tag{2.3}$$

$$\partial_t\tau_j(\xi,x,t) = \exp\int_\xi^x\frac{\partial_t a_j(\eta,\tau_j(\eta,x,t))}{a_j(\eta,\tau_j(\eta,x,t))^2}d\eta \tag{2.4}$$

for all $j = 1, \dots, n$, $\xi, x \in [0,1]$, and $t \in \mathbb{R}$. From (2.3) and (2.4) follows

$$(\partial_t + a_j(x,t)\partial_x)\, \varphi(\tau_j(\xi,x,t)) = 0 \tag{2.5}$$

for all $j = 1, \dots, n$, $x \in [0,1]$, $t \in \mathbb{R}$ and any C^1-function $\varphi : \mathbb{R} \to \mathbb{R}$.

Now, let us show that any C^1-solution to (1.9)-(1.10) is a solution to (1.1)-(1.3). Let u be a C^1-solution to (1.9)-(1.10). Then (2.5) yields

$$\begin{aligned}
&(\partial_t + a_j(x,t)\partial_x)\, (r_{jk}(\tau_j(0,x,t))u_k(0,\tau_j(0,x,t))) \\
&\quad = (\partial_t + a_j(x,t)\partial_x)\, (r_{jk}(\tau_j(1,x,t))u_k(1,\tau_j(1,x,t))) \\
&\quad = (\partial_t + a_j(x,t)\partial_x)\, (b_{jk}(\xi,\tau_j(\xi,x,t))u_k(\xi,\tau_j(\xi,x,t))) \\
&\quad = (\partial_t + a_j(x,t)\partial_x)\, f_j(\xi,\tau_j(\xi,x,t)) = 0,
\end{aligned}$$

and (1.7), (1.8), and (2.5) imply

$$\begin{aligned}
(\partial_t + a_j(x,t)\partial_x)\, c_j(\xi,x,t) &= -b_{jj}(x,t)c_j(\xi,x,t), \\
(\partial_t + a_j(x,t)\partial_x)\, d_j(\xi,x,t) &= -b_{jj}(x,t)d_j(\xi,x,t).
\end{aligned}$$

Hence the partial differential equations (1.1) are satisfied. The time-periodicity conditions (1.2) follow directly from (1.9), (1.10), and (2.1), while the boundary conditions (1.3) follow from (1.9), (1.10), and (2.2).

Finally, let us show that any C^1-solution to (1.1)-(1.3) is a solution to (1.9)-(1.10). Let u be a C^1-solution to (1.1)-(1.3). Then

$$\begin{aligned}
\frac{d}{d\xi} u_j(\xi,\tau_j(\xi,x,t)) &= \partial_x u_j(\xi,\tau_j(\xi,x,t)) + \frac{\partial_t u_j(\xi,\tau_j(\xi,x,t))}{a_j(\xi,\tau_j(\xi,x,t))} \\
&= \frac{1}{a_j(\xi,\tau_j(\xi,x,t))} \left(-\sum_{k=1}^{n} b_{jk}(\xi,\tau_j(\xi,x,t))u_k(\xi,\tau_j(\xi,x,t)) + f_j(\xi,\tau_j(\xi,x,t)) \right).
\end{aligned}$$

This is a linear inhomogeneous ordinary differential equation for the function $u_j(\cdot,\tau_j(\cdot,x,t))$, and the variation of constants formula (with initial condition at $\xi = 0$) gives

$$\begin{aligned}
u_j(x,t) &= u_j(x_j,\tau_j(x_j,x,t)) \exp \int_0^x \left(-\frac{b_{jj}(\xi,\tau_j(\xi,x,t))}{a_j(\xi,\tau_j(\xi,x,t))} \right) d\xi \\
&\quad - \int_0^x \exp \int_\xi^x \left(-\frac{b_{jj}(\eta,\tau_j(\eta,x,t))}{a_j(\eta,\tau_j(\eta,x,t))} \right) d\eta \\
&\quad \times \sum_{k \neq j} \frac{b_{jk}(\xi,\tau_j(\xi,x,t))u_k(\xi,\tau_j(\xi,x,t)) + f_j(\xi,\tau_j(\xi,x,t))}{a_j(\xi,\tau_j(\xi,x,t))} d\xi.
\end{aligned}$$

Here and in what follows we use the notation

$$x_j := \begin{cases} 0 & \text{for} \quad j = 1, \dots, m, \\ 1 & \text{for} \quad j = m+1, \dots, n. \end{cases} \tag{2.6}$$

Inserting the boundary conditions (1.3) for $j = 1, \dots, m$, we get (1.9) and inserting (1.3) for $j = 1, \dots, m$, we get (1.10).

3 Abstract representation of (1.9)–(1.10)

The system (1.9)–(1.10) can be written as the operator equation

$$u = Cu + Du + Ff, \tag{3.1}$$

where the linear bounded operators $C, D, F : \mathcal{C}_n \to \mathcal{C}_n$ are defined as follows:

Denote by $\mathcal{C}_m$ the space of all continuous maps $v : [0,1] \times \mathbb{R} \to \mathbb{R}^m$ with $v(x, t+2\pi) = v(x,t)$ for all $x \in [0,1]$ and $t \in \mathbb{R}$, with the norm

$$\|v\|_\infty := \max_{1\le j\le m} \max_{0\le x\le 1} \max_{t\in\mathbb{R}} |v_j(x,t)|.$$

Similarly we define the space $\mathcal{C}_{n-m}$. The spaces $\mathcal{C}_n$ and $\mathcal{C}_m \times \mathcal{C}_{n-m}$ will be identified, i.e., elements $u \in \mathcal{C}_n$ will be written as $u = (v, w)$ with $v \in \mathcal{C}_m$ and $w \in \mathcal{C}_{n-m}$. Define linear bounded operators $K : \mathcal{C}_{n-m} \to \mathcal{C}_m$ and $L : \mathcal{C}_m \to \mathcal{C}_{n-m}$ by

$$\begin{aligned}
(Lv)_j(x,t) &:= c_j(1,x,t) \sum_{k=1}^{m} r_{jk}(\tau_j(1,x,t)) v_k(1, \tau_j(1,x,t)), \quad j = m+1, \dots, n, \\
(Kw)_j(x,t) &:= c_j(0,x,t) \sum_{k=m+1}^{n} r_{jk}(\tau_j(0,x,t)) w_k(0, \tau_j(0,x,t)), \quad j = 1, \dots, m.
\end{aligned} \tag{3.2}$$

Then the operator C is defined as

$$Cu := (Kw, Lv) \text{ for } u = (v, w). \tag{3.3}$$

The operators D and F are given by

$$(Du)_j(x,t) := -\int_{x_j}^{x} d_j(\xi, x, t) \sum_{\substack{k=1 \\ k\ne j}}^{n} b_{jk}(\xi, \tau_j(\xi,x,t)) u_k(\xi, \tau_j(\xi,x,t)) d\xi, \tag{3.4}$$

$$(Ff)_j(x,t) := \int_{x_j}^{x} d_j(\xi, x, t) f_j(\xi, \tau_j(\xi,x,t)) d\xi. \tag{3.5}$$

Lemma 3.1 *Suppose (1.13). Then $I - C$ is an isomorphism on $\mathcal{C}_n$.*

Proof. Let $f = (g, h) \in \mathcal{C}_n$ with $g \in \mathcal{C}_m$ and $h \in \mathcal{C}_{n-m}$ be arbitrary given. We have $u = Cu + f$ if and only if $v = Kw + g, \quad w = Lv + h$, i.e., if and only if

$$v = K(Lv + h) + g, \quad w = Lv + h.$$

Hence, if

$$\|K\|_{\mathcal{L}(\mathcal{C}_{n-m};\mathcal{C}_m)} \|L\|_{\mathcal{L}(\mathcal{C}_m;\mathcal{C}_{n-m})} < 1, \tag{3.6}$$

then $I - C$ is an isomorphism from $\mathcal{C}_n$ onto $\mathcal{C}_n$. In order to prove (3.6) we use (1.11) and get the estimate

$$\begin{aligned}
\|Kw\|_\infty &= \max_{1\le j\le m} \max_{0\le x\le 1} \max_{t\in\mathbb{R}} \left| c_j(0,x,t) \sum_{k=m+1}^{n} r_{jk}(\tau_j(0,x,t)) w_k(0, \tau_j(0,x,t)) \right| \\
&\le \|w\|_\infty \max_{1\le j\le m} \max_{0\le x\le 1} \max_{t\in\mathbb{R}} R_j^0(x,t) = S^0 \|w\|_\infty \quad \text{for all } w \in \mathcal{C}_{n-m}.
\end{aligned} \tag{3.7}$$

Similarly,

$$\|Lv\|_\infty \le \|v\|_\infty \max_{m+1\le j\le n} \max_{0\le x\le 1} \max_{t\in\mathbb{R}} R_j^0(x,t) = T^0\|v\|_\infty \quad \text{for all } v\in\mathcal{C}_m. \tag{3.8}$$

Hence, assumption (1.13) yields (3.6). □

4 Fredholm property

In this section we prove the assertions (i)–(iii) of Theorem 1.2. Hence, we suppose that the assumptions (1.4), (1.5) and (1.13) are satisfied.

We have to show that the operator $I-C-D$ is Fredholm of index zero from $\mathcal{C}_n$ to $\mathcal{C}_n$. Unfortunately, the operator D is not compact from $\mathcal{C}_n$ to $\mathcal{C}_n$, in general, because it is a partial integral operator (cf. [1]). But by Lemma 3.1, the operator $I-C-D$ is Fredholm of index zero from $\mathcal{C}_n$ to $\mathcal{C}_n$ if and only if

$$I-(I-C)^{-1}D \text{ is Fredholm of index zero from } \mathcal{C}_n \text{ to } \mathcal{C}_n, \tag{4.1}$$

and for proving (4.1) we use the following Fredholmness criterion (cf., e.g. [5, Theorem XII.5.2]):

Lemma 4.1 *Let W be a Banach space, I the identity in W, and $A: W\to W$ a linear bounded operator with A^2 being compact. Then $I+A$ is a Fredholm operator of index zero.*

Now, for (4.1) it is sufficient to show that the operator $(I-C)^{-1}D(I-C)^{-1}D$ is compact from $\mathcal{C}_n$ to $\mathcal{C}_n$, i.e., that

$$D(I-C)^{-1}D \text{ is compact from } \mathcal{C}_n \text{ to } \mathcal{C}_n. \tag{4.2}$$

Because of $D(I-C)^{-1}D = D^2 + DC(I-C)^{-1}D$, the statement (4.2) will be proved if we show that

$$D^2 \text{ and } DC \text{ are compact from } \mathcal{C}_n \text{ to } \mathcal{C}_n. \tag{4.3}$$

Let us denote by $\mathcal{C}_n^1$ the Banach space of all $u\in\mathcal{C}_n$, which are C^1-smooth, with the norm $\|u\|_\infty+\|\partial_x u\|_\infty+\|\partial_t u\|_\infty$. By the Arcela-Ascoli theorem, $\mathcal{C}_n^1$ is compactly embedded into $\mathcal{C}_n$. Hence, for (4.3) it suffices to show that

$$D^2 \text{ and } DC \text{ map } \mathcal{C}_n \text{ continuously into } \mathcal{C}_n^1. \tag{4.4}$$

The definitions (3.2), (3.3) and (3.4) imply that for all $u\in\mathcal{C}_n^1$ we have $\partial_x u, \partial_t u\in\mathcal{C}_n^1$ and

$$\begin{aligned} \partial_x Cu &= C_{11}u + C_{12}\partial_t u, \quad \partial_t Cu = C_{21}u + C_{22}\partial_t u,\\ \partial_x Du &= D_{11}u + D_{12}\partial_t u, \quad \partial_t Du = D_{21}u + D_{22}\partial_t u \end{aligned} \tag{4.5}$$

with linear bounded operators $C_{jk}, D_{jk} : \mathcal{C}_n \to \mathcal{C}_n$, which are defined by

$$(C_{11}u)_j(x,t) := \begin{cases} \displaystyle\sum_{k=m+1}^{n} \partial_x c_{jk}(x,t) u_k(1, \tau_j(1,x,t)) & \text{for } j = 1, \dots, m, \\ \displaystyle\sum_{k=1}^{m} \partial_x c_{jk}(x,t) u_k(0, \tau_j(1,x,t)) & \text{for } j = m+1, \dots, n, \end{cases}$$

$$(C_{12}u)_j(x,t) := \begin{cases} \displaystyle\sum_{k=m+1}^{n} \partial_x \tau_j(1,x,t) c_{jk}(x,t) u_k(1, \tau_j(1,x,t)) & \text{for } j = 1, \dots, m, \\ \displaystyle\sum_{k=1}^{m} \partial_x \tau_j(0,x,t) c_{jk}(x,t) u_k(0, \tau_j(1,x,t)) & \text{for } j = m+1, \dots, n, \end{cases}$$

$$(C_{21}u)_j(x,t) := \begin{cases} \displaystyle\sum_{k=m+1}^{n} \partial_t c_{jk}(x,t) u_k(1, \tau_j(1,x,t)) & \text{for } j = 1, \dots, m, \\ \displaystyle\sum_{k=1}^{m} \partial_t c_{jk}(x,t) u_k(0, \tau_j(1,x,t)) & \text{for } j = m+1, \dots, n, \end{cases}$$

$$(C_{22}u)_j(x,t) := \begin{cases} \displaystyle\sum_{k=m+1}^{n} \partial_t \tau_j(1,x,t) c_{jk}(x,t) u_k(1, \tau_j(1,x,t)) & \text{for } j = 1, \dots, m, \\ \displaystyle\sum_{k=1}^{m} \partial_t \tau_j(0,x,t) c_{jk}(x,t) u_k(0, \tau_j(1,x,t)) & \text{for } j = m+1, \dots, n \end{cases}$$

and

$$(D_{11}u)_j(x,t) := \sum_{\substack{k=1 \\ k \neq j}}^{n} \int_{x_j}^{x} \partial_x d_{jk}(\xi,x,t) u_k(\xi, \tau_j(\xi,x,t)) d\xi,$$

$$(D_{12}u)_j(x,t) := \sum_{\substack{k=1 \\ k \neq j}}^{n} \int_{x_j}^{x} \partial_x \tau_j(\xi,x,t) d_{jk}(\xi,x,t) u_k(\xi, \tau_j(\xi,x,t)) d\xi,$$

$$(D_{21}u)_j(x,t) := \sum_{\substack{k=1 \\ k \neq j}}^{n} \int_{x_j}^{x} \partial_t d_{jk}(\xi,x,t) u_k(\xi, \tau_j(\xi,x,t)) d\xi,$$

$$(D_{22}u)_j(x,t) := \sum_{\substack{k=1 \\ k \neq j}}^{n} \int_{x_j}^{x} \partial_t \tau_j(\xi,x,t) d_{jk}(\xi,x,t) u_k(\xi, \tau_j(\xi,x,t)) d\xi.$$

Here

$$c_{jk}(x,t) := \begin{cases} r_{jk}(\tau_j(1,x,t) c_j(1,x,t) & \text{for } j = 1, \dots, m,\ k = m+1, \dots, n, \\ r_{jk}(\tau_j(0,x,t) c_j(0,x,t) & \text{for } j = m+1, \dots, n,\ k = 1, \dots, m \end{cases}$$

and

$$d_{jk}(\xi,x,t) := -d_j(\xi,x,t) b_{jk}(\xi, \tau_j(\xi,x,t)). \tag{4.6}$$

By (4.5) we get for all $u \in \mathcal{C}_n^1$

$$\partial_x D^2 u = D_{11} Du + D_{12}(D_{11}u + D_{12}\partial_t u), \quad \partial_t D^2 u = D_{21} Du + D_{22}(D_{21}u + D_{22}\partial_t u),$$
$$\partial_x DCu = D_{11} Cu + D_{12}(C_{11}u + C_{12}\partial_t u), \quad \partial_t DCu = D_{21} Cu + D_{22}(C_{21}u + C_{22}\partial_t u).$$

Now, taking into account the density of $\mathcal{C}_n^1$ in $\mathcal{C}_n$, in order to show (4.3) it suffices to prove the following statement:

Lemma 4.2 *There exists a positive constant such that for all $u \in \mathcal{C}_n^1$ we have*

$$\|D_{12}^2 \partial_t u\|_\infty + \|D_{22}^2 \partial_t u\|_\infty + \|D_{12} C_{12} \partial_t u\|_\infty + \|D_{22} C_{22} \partial_t u\|_\infty \le \text{const } \|u\|_\infty.$$

Proof. For any $j = 1, \dots, n$ and $u \in \mathcal{C}_n^1$ we have

$$\begin{aligned}
&(D_{12}^2 \partial_t u)_j(x,t) \\
&= \sum_{\substack{k=1 \\ k \neq j}}^{n} \sum_{\substack{l=1 \\ l \neq k}}^{n} \int_{x_j}^{x} \int_{x_k}^{\xi} d_{jkl}(\xi, \eta, x, t) b_{jk}(\xi, \tau_j(\xi, x, t)) \partial_t u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi, x, t))) d\eta d\xi
\end{aligned} \tag{4.7}$$

with

$$d_{jkl}(\xi, \eta, x, t) := \partial_t \tau_j(\xi, x, t) \partial_t \tau_k(\eta, \xi, \tau_j(\xi, x, t))) d_j(\xi, x, t) d_{kl}(\eta, \xi, \tau_j(\xi, x, t)).$$

On the other hand, (1.6), (2.3) and (2.4) imply (for all $\xi, \eta, x \in [0,1]$ and $t \in \mathbb{R}$ with $\partial_t u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi, x, t))) \neq 0$)

$$\begin{aligned}
&\frac{\frac{d}{d\xi} u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi, x, t)))}{\partial_t u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi, x, t)))} \\
&\quad = \partial_x \tau_k(\eta, \xi, \tau_j(\xi, x, t)) + \partial_t \tau_k(\eta, \xi, \tau_j(\xi, x, t)) \partial_\xi \tau_j(\xi, x, t) \\
&\quad = \partial_t \tau_k(\eta, \xi, \tau_j(\xi, x, t)) \left(\frac{1}{a_j(\xi, \tau_j(\xi, x, t))} - \frac{1}{a_k(\xi, \tau_j(\xi, x, t))} \right).
\end{aligned} \tag{4.8}$$

Hence, (1.5), (2.3) and (2.4) yield that for all $\xi, \eta, x \in [0,1]$ and $t \in \mathbb{R}$ it holds

$$\begin{aligned}
&d_{jkl}(\xi, x, t) b_{jk}(\xi, \tau_j(\xi, x, t)) \partial_t u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi, x, t))) \\
&= \tilde{d}_{jkl}(\xi, \eta, x, t) \tilde{b}_{jk}(\xi, \tau_j(\xi, x, t)) \frac{d}{d\xi} u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi, x, t)))
\end{aligned} \tag{4.9}$$

with

$$\tilde{d}_{jkl}(\xi, \eta, x, t) := a_j(\xi, \tau_j(\xi, x, t)) a_k(\xi, \tau_j(\xi, x, t)) \partial_t \tau_j(\xi, x, t) d_j(\xi, x, t) d_{kl}(\eta, \xi, \tau_j(\xi, x, t)).$$

Remark that the values $\tilde{b}_{jk}(x,t)$ are not uniquely defined for (x,t) with $a_j(x,t) = a_k(x,t)$ by the condition (1.5), but, anyway, the right-hand side (and, hence, the left-hand side) of (4.9) does not depend on the choice of $\tilde{b}_{jk}$ because $\frac{d}{d\xi} u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi, x, t))) = 0$ if $a_j(x,t) = a_k(x,t)$ (cf. (4.8)).

Let us check if for all $j \neq k$ and $k \neq l$ the partial derivatives $\partial_\xi \tilde{d}_{jkl}$ exist and are continuous: For the factor $a_j(\xi, \tau_j(\xi, x, t))$ this is the case because a_j and τ_j are C^1-smooth, the same for the factor $a_k(\xi, \tau_j(\xi, x, t))$. For the factor $\partial_t \tau_j(\xi, x, t)$ this is the case because $\partial_x \tau_j$ is C^1-smooth (cf. (1.4) and (2.3)). Finally, for the factors $d_j(\xi, x, t)$ and $d_{jk}(\eta, \xi, \tau_j(\xi, x, t))$ this follows from (1.8) and (4.6).

Applying Fubini's theorem and partial integration, we get, for example for the terms in (4.7) with $1 \leq j, k \leq m$,

$$\begin{aligned}
&\left| \int_0^x \int_0^\xi d_{jkl}(\xi, \eta, x, t) b_{jk}(\xi, \tau_j(\xi, x, t)) \partial_t u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi, x, t))) d\eta d\xi \right| \\
&\quad = \left| \int_0^x \int_\eta^x \tilde{d}_{jkl}(\xi, \eta, x, t) \tilde{b}_{jk}(\xi, \tau_j(\xi, x, t)) \frac{d}{d\xi} u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi, x, t))) d\xi d\eta \right| \\
&\quad \leq \left| \int_0^x \int_\eta^x \frac{d}{d\xi} \left(\tilde{d}_{jkl}(\xi, \eta, x, t) \tilde{b}_{jk}(\xi, \tau_j(\xi, x, t)) \right) u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi, x, t))) d\xi d\eta \right| \\
&\quad + \left| \int_0^x \left[\tilde{d}_{jkl}(\xi, \eta, x, t) \tilde{b}_{jk}(\xi, \tau_j(\xi, x, t)) u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi, x, t)) \right]_{\xi=\eta}^{\xi=x} d\eta \right| \\
&\quad \leq \text{ const } \|u\|_\infty.
\end{aligned}$$

Similarly one can handle the other terms in (4.7) in order to get $\|D_{12}^2 \partial_t u\|_\infty \leq \text{const } \|u\|_\infty$. The estimate $\|D_{22}^2 \partial_t u\|_\infty \leq \text{const } \|u\|_\infty$ can be proven in an analogous way.

It remains to show that

$$\|D_{12} C_{12} \partial_t u\|_\infty + \|D_{22} C_{22} \partial_t u\|_\infty \leq \text{const } \|u\|_\infty \text{ for all } u \in \mathcal{C}_n^1.$$

Let us do this for the term $D_{22} C_{22} \partial_t u$ (the calculations for $D_{12} C_{12} \partial_t u$ are similar). For any $u \in \mathcal{C}_n^1$ we have (using notation (2.6))

$$(D_{22} C_{22} \partial_t u)_j(x, t) = \sum_{\substack{k=1 \\ k \neq j}}^n \sum_{l=1}^n \int_{x_j}^x e_{jkl}(\xi, x, t) b_{jk}(\xi, \tau_j(\xi, x, t)) \partial_t u_l(x_j, \tau_k(x_j, \xi, \tau_j(\xi, x, t))) d\xi$$

with

$$\begin{aligned}
&e_{jkl}(\xi, x, t) \\
&\quad := \begin{cases} -\partial_t \tau_j(\xi, x, t) \partial_t \tau_k(1, \xi, \tau_j(\xi, x, t)) d_j(\xi, x, t) c_{kl}(\xi, \tau_j(\xi, x, t)) \text{ for } l = 1, \dots m, \\ -\partial_t \tau_j(\xi, x, t) \partial_t \tau_k(0, \xi, \tau_j(\xi, x, t)) d_j(\xi, x, t) c_{kl}(\xi, \tau_j(\xi, x, t)) \text{ for } l = m+1, \dots n. \end{cases}
\end{aligned}$$

Using (4.8), we get for $l = 1, \dots, m$

$$\begin{aligned}
&e_{jkl}(\xi, x, t) b_{jk}(\xi, \tau_j(\xi, x, t)) \partial_t u_l(1, \tau_k(1, \xi, \tau_j(\xi, x, t))) \\
&= \tilde{e}_{jkl}(\xi, x, t) \tilde{b}_{jk}(\xi, \tau_j(\xi, x, t)) \frac{d}{d\xi} u_l(1, \tau_k(1, \xi, \tau_j(\xi, x, t)))
\end{aligned}$$

with

$$\tilde{e}_{jkl}(\xi, \eta, x, t) := -a_j(\xi, \tau_j(\xi, x, t)) a_k(\xi, \tau_j(\xi, x, t)) \partial_t \tau_j(\xi, x, t) d_j(\xi, x, t) c_{kl}(\eta, \xi, \tau_j(\xi, x, t)).$$

Hence, we can integrate by parts in order to get

$$\begin{aligned}
&\left|\int_{x_j}^{x} e_{jkl}(\xi,x,t)b_{jk}(\xi,\tau_j(\xi,x,t))\partial_t u_l(1,\tau_k(1,\xi,\tau_j(\xi,x,t)))d\xi\right| \\
&\quad= \left|\int_{x_j}^{x} \tilde{e}_{jkl}(\xi,x,t)\tilde{b}_{jk}(\xi,\tau_j(\xi,x,t))\frac{d}{d\xi}u_l(1,\tau_k(1,\xi,\tau_j(\xi,x,t)))d\xi\right| \\
&\quad\leq \text{ const } \|u\|_\infty.
\end{aligned}$$

Similarly one can proceed in the case $l = m+1,\dots,n$. □

Remark 4.3 about smoothness assumptions on the coefficients b_{jk} In fact, for our results we do not need to assume that the partial derivatives $\partial_x b_{jj}$ exist. We need that b_{jj} is continuous and that the partial derivatives $\partial_t b_{jj}$ exist and are continuous.

What concerns the coefficients b_{jk} with $j \neq k$, in the proof of Lemma 4.2 we only used that

$$\left|\int_{x_j}^{x} \tilde{b}_{jk}(\xi,\tau_j(\xi,x,t))\frac{d}{d\xi}u_l(1,\tau_k(1,\xi,\tau_j(\xi,x,t)))d\xi\right| \leq \text{ const } \|u\|_\infty \text{ for all } u \in \mathcal{C}_n^1. \quad (4.10)$$

For that the assumption $\tilde{b}_{jk} \in \mathcal{C}_n^1$ (cf. (1.5)) is sufficient, but not necessary. For example, if a_j, a_k and b_{jk} and, hence, $\tilde{b}_{jk}$ are t-independent, then for (4.10) it is sufficient that $\tilde{b}_{jk} \in BV(0,1)$.

5 Solution regularity

In this section we prove the assertions (iv) and (v) of Theorem 1.2. Hence, we suppose that the assumptions (1.4), (1.5), (1.13) and (1.14) are satisfied. Remark that (1.13) implies (1.14) if all coefficients a_j are t-independent.

To prove assertion (iv), assume that the functions f_j are continuously differentiable with respect to t. Let u be a continuous solution to to (1.1)–(1.3). We have to show that the partial derivatives $\partial_x u$ and $\partial_t u$ exist and are continuous. For that it is sufficient to show that $\partial_t u$ exists and is continuous, since then (1.9) and (1.10) imply that also $\partial_x u$ exists and is continuous.

Because of (3.1) we have

$$(I-C)u = D(C+D)u + (I+D)Ff. \quad (5.1)$$

Denote by $\tilde{\mathcal{C}}_n^1$ the subspace of all $v \in \mathcal{C}_n$ such that the partial derivative $\partial_t v$ exists and is continuous. By assumption we have $f \in \tilde{\mathcal{C}}_n^1$. Moreover, from (3.4) and (3.5) it follows that the operators D and F map $\tilde{\mathcal{C}}_n^1$ into $\tilde{\mathcal{C}}_n^1$. Therefore, (4.4) implies that the right-hand side of (5.1) belongs to $\tilde{\mathcal{C}}_n^1$. Hence, it remains to prove the following fact:

Lemma 5.1 *If for $\tilde{u} \in \mathcal{C}_n$ and $\tilde{f} \in \tilde{\mathcal{C}}_n^1$ it holds $\tilde{u} = C\tilde{u} + \tilde{f}$, then $\tilde{u} \in \tilde{\mathcal{C}}_n^1$.*

Proof. We proceed as in the proof of Lemma 3.1. In particular, we use the Banach spaces $\mathcal{C}_m$ and $\mathcal{C}_{n-m}$ and the linear bounded operators $K : \mathcal{C}_{n-m} \to \mathcal{C}_m$ and $L : \mathcal{C}_m \to \mathcal{C}_{n-m}$, which are introduced there. Further, by $\tilde{\mathcal{C}}^1_m$ we denote the space of all $v \in \mathcal{C}_m$ such that the partial derivatives $\partial_t v$ exist and are continuous. Similarly the space $\tilde{\mathcal{C}}^1_{n-m}$ is introduced. Then we have $\mathcal{C}_n = \mathcal{C}_m \times \mathcal{C}_{n-m}$ and $\tilde{\mathcal{C}}^1_n = \tilde{\mathcal{C}}^1_m \times \tilde{\mathcal{C}}^1_{n-m}$.

Now, suppose that there are given $\tilde{u} \in \mathcal{C}_n$ and $\tilde{f} \in \tilde{\mathcal{C}}^1_n$ such that

$$\tilde{u} = C\tilde{u} + \tilde{f}. \tag{5.2}$$

Then $\tilde{u} = (\tilde{v}, \tilde{w})$ with certain $\tilde{v} \in \tilde{\mathcal{C}}_m$, $\tilde{w} \in \tilde{\mathcal{C}}_{n-m}$ and $\tilde{f} = (\tilde{g}, \tilde{h})$ with certain $\tilde{g} \in \tilde{\mathcal{C}}^1_m$, $\tilde{h} \in \tilde{\mathcal{C}}^1_{n-m}$. We have to show that $\tilde{u} \in \tilde{\mathcal{C}}^1_n$, i.e., that

$$\tilde{v} \in \tilde{\mathcal{C}}^1_m,\ \tilde{w} \in \tilde{\mathcal{C}}^1_{n-m}. \tag{5.3}$$

Because of (3.3) and (5.2) we have $\tilde{v} = K(L\tilde{v} + \tilde{h}) + \tilde{g},\ \tilde{w} = L\tilde{v} + \tilde{h}$. Moreover, (3.2) yields that K maps $\tilde{\mathcal{C}}^1_{n-m}$ into $\tilde{\mathcal{C}}^1_m$ and that L maps $\tilde{\mathcal{C}}^1_m$ into $\tilde{\mathcal{C}}^1_{n-m}$. Hence, for proving (5.3) it suffices to show that

$$(I - KL)^{-1} g \in \tilde{\mathcal{C}}^1_m \text{ for any } g \in \tilde{\mathcal{C}}^1_m. \tag{5.4}$$

For any $\gamma > 0$ the spaces $\tilde{\mathcal{C}}^1_m$ are Banach spaces with the norms

$$\|v\|_{\tilde{\mathcal{C}}^1_m} := \|v\|_\infty + \gamma \|\partial_t v\|_\infty.$$

Hence, for proving (5.4) it suffices to show that there exists $\gamma > 0$ such that

$$\|KL\|_{\mathcal{L}(\tilde{\mathcal{C}}^1_m)} < 1$$

with the norm corresponding to γ. For that we have to show that there exists a constant $c < 1$ such that

$$\|KLv\|_\infty + \gamma \|\partial_t KLv\|_\infty \le c\,(\|v\|_\infty + \gamma \|\partial_t v\|_\infty) \text{ for all } v \in \tilde{\mathcal{C}}^1_m.$$

Because of (3.7) and (3.8), this estimate will be proved if we show that

$$\|\partial_t KLv\|_\infty \le \frac{c - S^0 T^0}{\gamma} \|v\|_\infty + c \|\partial_t v\|_\infty \quad \text{for all } v \in \tilde{\mathcal{C}}^1_m. \tag{5.5}$$

For proving (5.5), let us calculate $\partial_t KLv$. Similarly to (4.5) we have for all $v \in \tilde{\mathcal{C}}^1_m$ and $w \in \tilde{\mathcal{C}}^1_{n-m}$

$$\partial_t Kw = K_1 w + K_2 \partial_t w, \quad \partial_t Lv = L_1 v + L_2 \partial_t v$$

with linear bounded operators $K_1, K_2 : \mathcal{C}_{n-m} \to \mathcal{C}_m$ and $L_1, L_2 : \mathcal{C}_m \to \mathcal{C}_{n-m}$ defined as follows (cf. (3.3) and (4.5)):

$$(K_1 w)_j(x,t) \ := \ \sum_{k=m+1}^{n} \partial_t c_{jk}(x,t) w_k(1, \tau_j(1,x,t)),$$

$$
\begin{aligned}
(K_2 w)_j(x,t) &:= \sum_{k=m+1}^{n} \partial_t \tau_j(1,x,t) c_{jk}(x,t) w_k(1,\tau_j(1,x,t)),\\
(L_1 v)_j(x,t) &:= \sum_{k=1}^{m} \partial_t c_{jk}(x,t) v_k(0,\tau_j(0,x,t)),\\
(L_2 v)_j(x,t) &:= \sum_{k=m+1}^{n} \partial_t \tau_j(0,x,t) c_{jk}(x,t) v_k(0,\tau_j(0,x,t)).
\end{aligned}
$$

Therefore, for $v \in \tilde{\mathcal{C}}_m^1$ it holds

$$
\begin{aligned}
\|\partial_t K L v\|_\infty &= \|K_1 L v + K_2(L_1 v + L_2 \partial_t v)\|_\infty\\
&\le \|K_1 L + K_2 L_1\|_{\mathcal{L}(\mathcal{C}_m)} \|v\|_\infty + \|K_2 L_2\|_{\mathcal{L}(\mathcal{C}_m)} \|\partial_t v\|_\infty.
\end{aligned}
$$

Now, (1.12) and (2.4) yield $\|K_{22}\|_{\mathcal{L}(\mathcal{C}_{n-m};\mathcal{C}_m)} \le S^1$ and $\|L_{22}\|_{\mathcal{L}(\mathcal{C}_m;\mathcal{C}_{n-m})} \le T^1$. Hence, we get

$$
\|\partial_t K L v\|_\infty \le \|K_1 L + K_2 L_1\|_{\mathcal{L}(\mathcal{C}_m)} \|v\|_\infty + S^1 T^1 \|\partial_t v\|_\infty. \tag{5.6}
$$

By assumptions (1.13) and (1.14) we have $S^0 T^0 < 1$ and $S^1 T^1 < 1$. Fix c such that $\max\{S^0 T^0, S^1 T^1\} < c < 1$. Then choose γ so small that

$$
\|K_1 L + K_2 L_1\|_{\mathcal{L}(\mathcal{C}_m)} \le \frac{c - S^0 T^0}{\gamma}.
$$

Finally, (5.6) implies (5.5). □

The proof of assertion (iv) is therewith complete.

To prove assertion (v) of Theorem 1.2, suppose that all coefficients a_j are t-independent. Then (2.4) yields that $\partial_t \tau_j(\xi,x,t) = 1$. Therefore in (4.5) we have

$$
C_{22} = C \text{ and } D_{22} = D. \tag{5.7}
$$

Let u be a continuous solution to (1.1)–(1.3), i.e., a solution to (1.9)–(1.10), and suppose that all functions a_j, b_{jk}, f_j and r_{jk} are C^∞-smooth.

First we show by induction that all partial derivatives $\partial_t^k u, k = 1,2,\dots$ exist and are continuous.

For $k = 1$ this follows from assertion (iv) of Theorem 1.2.

Now suppose that all partial derivatives $\partial_t u, \dots, \partial_t^k u$ exist and are continuous. Then as in (4.5) one gets (cf. (5.7))

$$
\partial_t^k C u = \sum_{j=0}^{k-1} C_j \partial_t^j u + C \partial_t^k u,\ \partial_t^k D u = \sum_{j=0}^{k-1} D_j \partial_t^j u + D \partial_t^k u
$$

with linear bounded operators $C_j, D_j : \mathcal{C}_n \to \mathcal{C}_n$ such that $C_j v, D_j v \in \tilde{\mathcal{C}}_n^1$ for all $v \in \tilde{\mathcal{C}}_n^1$. Now we proceed as in (5.1): From $(I - C - D)u = Ff$ it follows

$$
(I - C - D)\partial_t^k u = \partial_t^k F f - \sum_{j=0}^{k-1} \big(C_j \partial_t^j u + D_j \partial_t^j u\big) =: R_k \in \tilde{\mathcal{C}}_n^1
$$

and, hence,

$$(I-C)\partial_t^k u = D\left((C+D)\partial_t^k u + R_k\right) + R_k \in \tilde{\mathcal{C}}_n^1.$$

By Lemma 5.1, $\partial_t^k u \in \tilde{\mathcal{C}}_n^1$, i.e., $\partial_t^{k+1} u$ exists and is continuous.

Finally we show that the partial derivative $\partial_x^k \partial_t^l u$ exists and is continuous for all $k, l \in \mathbb{N}$. From (1.1) it follows

$$\partial_x u_j(x,t) = \frac{1}{a_j(x)} \left(f_j(x,t) - \partial_t u_j(x,t) - \sum_{k=1}^{n} b_{jk}(x,t) u_k(x,t) \right). \tag{5.8}$$

All partial derivatives with respect to t of the right-hand side (and, hence, of the left-hand side) of (5.8) exist and are continuous, i.e., $\partial_x \partial_t^l u_j$ exists and is continuous for all $l \in \mathbb{N}$. Therefore the partial derivative with respect to x of the right-hand side (and, hence, of the left-hand side) of (5.8) exists and is continuous, i.e., $\partial_x^2 u_j$ exists and is continuous and

$$\partial_x^2 u_j = \frac{1}{a_j^2} \left(a_j \partial_x \left(f_j - \partial_t u_j - \sum_{k=1}^{n} b_{jk} u_k \right) - \partial_x a_j \left(f_j - \partial_t u_j - \sum_{k=1}^{n} b_{jk} u_k \right) \right). \tag{5.9}$$

Again, all partial derivatives with respect to t of the right-hand side of (5.9) exist and are continuous. Hence, $\partial_x^2 \partial_t^l u_j$ exists and is continuous for all $l \in \mathbb{N}$. Therefore the partial derivative with respect to x of the right-hand side of (5.9) exists and is continuous, i.e., $\partial_x^3 u_j$ exists and is continuous. By continuation of this procedure we get the claim.

Acknowledgments

The first author was supported by the Alexander von Humboldt Foundation. Both authors acknowledge support of the DFG Research Center MATHEON mathematics for key technologies (project D8).

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