

# Smoothing effect and Fredholm property for first-order hyperbolic PDEs

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## Abstract

We give an exposition of recent results on regularity and Fredholm properties for first-order one-dimensional hyperbolic PDEs. We show that large classes of boundary operators cause an effect that smoothness increases with time. This property is the key in finding regularizers (parametrics) for hyperbolic problems. We construct regularizers for periodic problems for dissipative first-order linear hyperbolic PDEs and show that these problems are modeled by Fredholm operators of index zero.

## 1 Introduction

In contrast to ODEs and parabolic PDEs, the Fredholm property and regularity behavior of hyperbolic problems are much less understood. In a recent series of papers [18, 20, 21], the latter two written jointly with Lutz Recke, we undertook a detailed analysis of this subject for first-order one-dimensional hyperbolic operators. The purpose of the present survey paper is to present some of our results and their extensions with emphasize on the smoothing phenomenon, construction of parametrics, and the Fredholmness of index zero.

An important step in local investigations of nonlinear differential equations (many ODEs and parabolic PDEs) is to establish the Fredholm solvability of their linearized versions. In the hyperbolic case this step is much more involved. Since the singularities of (semi-)linear hyperbolic equations propagate along characteristic curves, a solution cannot be more regular in the entire time-space domain than it is on the boundary. It can even be less regular which is known as the *loss-of-smoothness* effect. Therefore the Fredholm analysis of hyperbolic problems requires establishing an optimal regularity relation between the spaces of solutions and right-hand sides of the differential equations.

Proving a Fredholm solvability is typically based on the basic fact that any Fredholm operator is exactly a compact perturbation of a bijective operator. In the hyperbolic case,

using the compactness argument gets complicated because of the lack of regularity over the *whole* time-space domain.

Our approach is based on the fact that for a range of boundary operators, solutions improve smoothness *dynamically*, more precisely, they eventually become  $k$ -times continuously differentiable for each particular  $k$ . We prove such kind of results in Section 2. Note that in some interesting cases the smoothing phenomenon was shown earlier in [10, 13, 23, 25].

This phenomenon allows us in Section 3 to work out a regularization procedure via construction of a parametrix. We here present a quite general approach to proving the Fredholmness for first-order dissipative hyperbolic PDEs and apply it to the periodic problems. Our Fredholm results cover non-strictly hyperbolic systems with discontinuous coefficients, but they are new even in the case of strict hyperbolicity and smooth coefficients.

From a more general perspective, the smoothing effect and Fredholmness properties play an important role in the study of the Hopf bifurcation and periodic synchronizations in nonlinear hyperbolic PDEs [2, 22] via the Implicit Function Theorem and Lyapunov-Schmidt procedure [7, 15] and averaging procedure [6, 32].

From the practical point of view, our techniques cover the so-called traveling-wave models from laser dynamics [24, 30] (describing the appearance of self-pulsations of lasers and modulation of stationary laser states by time periodic electric pumping), population dynamics [9, 14, 36], and chemical kinetics [3, 4, 5, 38] (describing mass transition in terms of convective diffusion and chemical reaction and analysis of chemical processes in counterflow chemical reactors).

## 2 Smoothing effect

Here we describe classes of (initial-)boundary problems for first-order one-dimensional hyperbolic PDEs whose solutions improve their regularity in time.

Set

$$\Pi_T = \{(x, t) : 0 < x < 1, T < t < \infty\}.$$

We address the problem

$$(\partial_t + a(x, t)\partial_x + b(x, t))u = f(x, t), \quad (2.1)$$

$$u(x, 0) = \varphi(x), \quad (2.2)$$

$$\begin{aligned} u_j(0, t) &= (Ru)_j(t), & 1 \leq j \leq m \\ u_j(1, t) &= (Ru)_j(t), & m < j \leq n \end{aligned} \quad (2.3)$$

in the semi-strip  $\Pi_0$  and the problem (2.1), (2.3) in the strip  $\Pi_{-\infty}$ . Here  $u = (u_1, \dots, u_n)$ ,  $f = (f_1, \dots, f_n)$ , and  $\varphi = (\varphi_1, \dots, \varphi_n)$  are vectors of real-valued functions,  $b = \{b_{jk}\}_{j,k=1}^n$  and  $a = \text{diag}(a_1, \dots, a_n)$  are matrices of real-valued functions, and  $0 \leq m \leq n$  are fixed integers. Furthermore,  $R$  is an operator mapping  $C(\overline{\Pi_0})^n$  into  $C([0, \infty))^n$ , and similarly

for  $R$  in  $\Pi_{-\infty}$ . In Sections 2.1–2.3 we give examples of  $R$  as representatives of some classes of boundary operators ensuring smoothing solutions.

In the domain under consideration we assume that

$$a_j > 0 \text{ for all } j \leq m \quad \text{and} \quad a_j < 0 \text{ for all } j > m, \quad (2.4)$$

$$\inf_{x,t} |a_j| > 0 \text{ for all } j \leq n, \quad (2.5)$$

and

$$\text{for all } 1 \leq j \neq k \leq n \text{ there exists } p_{jk} \in C^1([0, 1] \times \mathbb{R}) \text{ such that } b_{jk} = p_{jk}(a_k - a_j). \quad (2.6)$$

Note that all these conditions are not restrictive neither from the practical nor from the theoretical points of view. In particular, condition (2.4) is true in traveling-wave models of laser and population dynamics as well as chemical kinetics, where the functions  $u_j$  for  $j \leq m$  (respectively,  $m + 1 \leq j \leq n$ ) describe “species” traveling to the right (respectively, to the left). Condition (2.5) means that all characteristics of the system (2.1) are bounded and the system (2.1) is, hence, non-degenerate. Finally, the condition (2.6) is a kind of Levy condition usually appearing to compensate non-strict hyperbolicity where the coefficients  $a_j$  and  $a_k$  for some  $j \neq k$  coincide at least at one point, say,  $(x_0, t_0)$ . In this case the lower-order terms with the coefficients  $b_{jk}$  and  $b_{kj}$  contribute to the system at  $(x_0, t_0)$  longitudinally to characteristic directions (keeping responsibility for the propagation of singularities), while in the strictly hyperbolic case we have a qualitatively different transverse contribution at that point. The purpose of (2.6) is to suppress propagation of singularities through the non-diagonal lower-order terms of (2.1).

We will impose the following smoothness assumptions on the initial data: The entries of  $a$ ,  $b$ , and  $f$  are  $C^\infty$ -smooth in all their arguments in the respective domains, while the entries of  $\varphi$  are assumed to be continuous functions only.

Let us introduce the system resulting from (2.1)–(2.3) (resp., from (2.1), (2.3)) via integration along characteristic curves. For given  $j \leq n$ ,  $x \in [0, 1]$ , and  $t \in \mathbb{R}$ , the  $j$ -th characteristic of (2.1) passing through the point  $(x, t)$  is defined as the solution  $\xi \in [0, 1] \mapsto \omega_j(\xi; x, t) \in \mathbb{R}$  of the initial value problem

$$\partial_\xi \omega_j(\xi; x, t) = \frac{1}{a_j(\xi, \omega_j(\xi; x, t))}, \quad \omega_j(x; x, t) = t. \quad (2.7)$$

Define

$$c_j(\xi, x, t) = \exp \int_x^\xi \left( \frac{b_{jj}}{a_j} \right) (\eta, \omega_j(\eta; x, t)) d\eta, \quad d_j(\xi, x, t) = \frac{c_j(\xi, x, t)}{a_j(\xi, \omega_j(\xi; x, t))}.$$

Due to (2.5), the characteristic curve  $\tau = \omega_j(\xi; x, t)$  reaches the boundary of  $\Pi_T$  in two points with distinct ordinates. Let  $x_j(x, t)$  denote the abscissa of that point whose ordinate is smaller. Straightforward calculations show that a  $C^1$ -map  $u : [0, 1] \times [0, \infty) \rightarrow \mathbb{R}^n$  is a

solution to (2.1)–(2.3) if and only if it satisfies the following system of integral equations

$$\begin{aligned}
u_j(x, t) &= (BSu)_j(x, t) \\
&\quad - \int_{x_j(x, t)}^x d_j(\xi, x, t) \sum_{\substack{k=1 \\ k \neq j}}^n b_{jk}(\xi, \omega_j(\xi; x, t)) u_k(\xi, \omega_j(\xi; x, t)) d\xi \\
&\quad + \int_{x_j(x, t)}^x d_j(\xi, x, t) f_j(\xi, \omega_j(\xi; x, t)) d\xi, \quad j \leq n,
\end{aligned} \tag{2.8}$$

where

$$(Bu)_j(x, t) = c_j(x_j(x, t), x, t) u_j(x_j(x, t), \omega_j(x_j(x, t); x, t)), \tag{2.9}$$

$$(Su)_j(x, t) = \begin{cases} (Ru)_j(t) & \text{if } t > 0, \\ \varphi_j(x) & \text{if } t = 0. \end{cases} \tag{2.10}$$

Here  $B$  is a shifting operator from  $\partial\Pi_0$  along characteristic curves of (2.1), while the operator  $S$  is used to denote the boundary operator on the whole  $\partial\Pi_0$ . Similarly, a  $C^1$ -map  $u : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}^n$  is a solution to (2.1), (2.3) if and only if it satisfies the system (2.8), where the definition of  $S$  is changed to  $S = R$ .

This motivates the following definition:

**Definition 2.1 (1)** A continuous function  $u$  is called a continuous solution to (2.1)–(2.3) in  $\overline{\Pi}_0$  if it satisfies (2.8) with  $S$  defined by (2.10).

**(2)** A continuous function  $u$  is called a continuous solution to (2.1), (2.3) in  $\overline{\Pi}_{-\infty}$  if it satisfies (2.8) with  $S = R$ .

Existence results for (continuous) solutions to the problems under consideration are obtained in [1, 16, 17, 19].

**Definition 2.2** A solution  $u$  to the problem (2.1)–(2.3) or (2.1), (2.3) is called *smoothing* if, for every  $k \in \mathbb{N}$ , there exists  $T > 0$  such that  $u_j \in C^k(\overline{\Pi}_T)$  for all  $j \leq n$ .

For the initial-boundary value problem (2.1)–(2.3) Definition 2.2 reflects a dynamic nature of the smoothing property stating that the regularity of solutions increases in time. The fact that the regularity cannot be uniform in the entire domain is a straightforward consequence of the propagation of singularities along characteristic curves. Moreover, switching from  $C^k$  to  $C^{k+1}$ -regularity is jump-like; this phenomenon is usually observed in the situations when solutions of hyperbolic PDEs change their regularity (see e.g., [25, 27, 29, 31]).

Note that, if the problem (2.1), (2.3) is subjected to periodic conditions in  $t$ , then Definition 2.2 implies that the smoothing solutions immediately meet the  $C^\infty$ -regularity in the entire domain.

Definition 2.2 captures the general nature of the smoothing phenomenon for hyperbolic PDEs. A more precise information can be extracted from the proof of Theorems 2.3, 2.4,

and 2.5 below: Reaching the  $C^k$ -regularity for solutions needs only a  $C^k$ -regularity for  $a$ ,  $b$ , and  $f$ . More exact regularity conditions for the boundary data, which also depend on  $k$ , can be derived from these proofs as well. These refinements are useful in some applications.

Definition 2.2 can be strengthened by admitting worse regularities for the initial data. One extension of this kind, when the initial data are strongly singular distributions concentrated at a finite number of points, can be found in [18]. In [18] we used a delta-wave solution concept. Another result in this direction [20, 21] concerns periodic problems and uses a variational setting of the problem (see also Theorem 3.2 (ii)). In [20, 21] we get an improvement of the solution regularity from being functionals to being functions.

In what follows we demonstrate the smoothing effect on generic examples of large classes of boundary operators and show which kinds of problems can be covered by our techniques. Our approach to establishing smoothing results is based on the consideration of the integral representation of the problems and the observation that the boundary and the integral parts of this representation have different influence on the regularity of solutions. Our main idea is to show that the integral part has a “self-improvement” property, while in many interesting cases the boundary part is not responsible for propagation of singularities. The latter contrasts to the case of the Cauchy problem where the solutions cannot be smoothing as the boundary term all the time “remembers” the regularity of the initial data. It is worthy to note that in the case of the problem (2.1)–(2.3) in  $\Pi_0$  the domain of influence of the initial conditions is determined by both parts of the integral system and is in general infinite. This makes the smoothing effect non-obvious.

## 2.1 Classical boundary conditions

Here we specify conditions (2.3) to

$$\begin{aligned} u_j(0, t) &= h_j(t), & 1 \leq j \leq m, \\ u_j(1, t) &= h_j(t), & m < j \leq n. \end{aligned} \tag{2.11}$$

and consider the problem (2.1), (2.2), (2.11).

**Theorem 2.3** *Assume that the data  $a_j$ ,  $b_{jk}$ ,  $f_j$ , and  $h_j$  are smooth in all their arguments and  $\varphi_j$  are continuous functions. Assume also (2.4), (2.5), and (2.6). Then any continuous solution to the problem (2.1), (2.2), (2.11) is smoothing.*

Note that in the case of smooth classical boundary conditions (2.11), the domain of influence of the initial data  $\varphi(x)$  on  $u_i$  for every  $i \leq n$  in general is unbounded (due to the lower-order terms in (2.1)). In spite of this, the influence of the initial data on the regularity of  $u$  becomes weaker and weaker in time causing the smoothing effect.

**Proof.** Suppose that  $u$  is a continuous solution to the problem (2.1)–(2.3) and show that the operator of the problem improves the regularity of  $u$  in time. The idea of the proof is similar to [18].

We start with an operator representation of  $u$ . To this end, introduce linear bounded operators  $D, F : C(\overline{\Pi_0})^n \rightarrow C(\overline{\Pi_0})^n$  by

$$\begin{aligned} (Du)_j(x, t) &= - \int_{x_j(x, t)}^x d_j(\xi, x, t) \sum_{\substack{k=1 \\ k \neq j}}^n (b_{jk} u_k)(\xi, \omega_j(\xi; x, t)) d\xi, \\ (Ff)_j(x, t) &= \int_{x_j(x, t)}^x d_j(\xi, x, t) f_j(\xi, \omega_j(\xi; x, t)) d\xi. \end{aligned}$$

Note that  $Ff$  is a smooth function in  $x, t$ . In this notation the integral system (2.8) can be written as

$$u = BSu + Du + Ff. \quad (2.12)$$

It follows that

$$u = BSu + (DBS + D^2)u + (I + D)Ff. \quad (2.13)$$

In the first step we prove that the right hand side of (2.13) restricted to  $\overline{\Pi_{T_1}}$  for some  $T_1 > 0$  is continuously differentiable in  $t$ . The  $C^1(\overline{\Pi_{T_1}})^n$ -regularity of  $u$  will then follow from the fact that  $u$  given by (2.8) satisfies (2.1) in the distributional sense. By the assumption (2.5), we can fix a large enough  $T_1 > 0$  such that the operator  $S$  in the right-hand side of (2.13) restricted to  $\overline{\Pi_{T_1}}$  does not depend on  $\varphi$  and, hence,  $Su = Ru = h$ , where  $h = (h_1, \dots, h_n)$ . We therefore arrive at the equality

$$u|_{\overline{\Pi_{T_1}}} = Bh + DBh + D^2u + (I + D)Ff, \quad (2.14)$$

where  $u|_{\overline{\Pi_{T_1}}}$  denotes the restriction of  $u$  to  $\overline{\Pi_{T_1}}$ . By the regularity assumption on  $a, b, f$ , and  $h$ , the function  $Bh + DBh + (I + D)Ff$  is smooth. We have reduced the problem to show that the operator  $D^2$  is smoothing, more specifically, that  $D^2u$  is  $C^1$ -smooth in  $t$  on  $\overline{\Pi_{T_1}}$ .

Notice that for  $t \geq T_1$  the function  $x_j(x, t)$  is a constant depending only on  $j$ . Below we therefore will drop the dependence of  $x_j$  on  $x$  and  $t$ . Fix a sequence  $u^l \in C^1(\overline{\Pi_0})^n$  such that

$$u^l \rightarrow u \text{ in } C(\overline{\Pi_0})^n \text{ as } l \rightarrow \infty. \quad (2.15)$$

By convergence in  $C(\Omega)^n$  here and below we mean the uniform convergence on any compact subset of  $\Omega$ . Then  $D^2u^l \rightarrow D^2u$  in  $C(\overline{\Pi_0})^n$  as well. It suffices to prove that  $\partial_t [D^2u^l]$  converges in  $C(\overline{\Pi_{T_1}})^n$  as  $l \rightarrow \infty$ . Given  $j \leq n$ , consider the following expression for  $(D^2u^l)_j(x, t)$ , obtained by change of the order of integration:

$$\begin{aligned} (D^2u^l)_j(x, t) & \\ &= \sum_{\substack{k=1 \\ k \neq j}}^n \sum_{\substack{i=1 \\ i \neq k}}^n \int_{x_j}^x \int_{\eta}^x d_{jki}(\xi, \eta, x, t) b_{jk}(\xi, \omega_j(\xi; x, t)) u_i^l(\eta, \omega_k(\eta; \xi, \omega_j(\xi; x, t))) d\xi d\eta \end{aligned} \quad (2.16)$$

with

$$d_{jki}(\xi, \eta, x, t) = d_j(\xi, x, t) d_k(\eta, \xi, \omega_j(\xi; x, t)) b_{ki}(\eta, \omega_k(\eta; \xi, \omega_j(\xi; x, t))).$$

It follows that

$$\begin{aligned}
& \partial_t \left[ (D^2 u^l)_j(x, t) \right] \\
&= \sum_{\substack{k=1 \\ k \neq j}}^n \sum_{\substack{i=1 \\ i \neq k}}^n \int_{x_j}^x \int_{\eta}^x \partial_t \left[ d_{jki}(\xi, \eta, x, t) b_{jk}(\xi, \omega_j(\xi; x, t)) \right] u_i^l(\eta, \omega_k(\eta; \xi, \omega_j(\xi; x, t))) d\xi d\eta \\
&+ \sum_{\substack{k=1 \\ k \neq j}}^n \sum_{\substack{i=1 \\ i \neq k}}^n \int_{x_j}^x \int_{\eta}^x d_{jki}(\xi, \eta, x, t) b_{jk}(\xi, \omega_j(\xi; x, t)) \\
&\times \partial_3 \omega_k(\eta; \xi, \omega_j(\xi; x, t)) \partial_t \omega_j(\xi; x, t) \partial_2 u_i^l(\eta, \omega_k(\eta; \xi, \omega_j(\xi; x, t))) d\xi d\eta, \tag{2.17}
\end{aligned}$$

where  $\partial_r g$  here and below denotes the derivative of  $g$  with respect to the  $r$ -th argument. The first summand in the right-hand side converges in  $C(\overline{\Pi}_{T_1})$ . Our task is therefore reduced to show the uniform convergence of all integrals in the second summand, whenever  $(x, t)$  varies on a compact subset of  $\overline{\Pi}_{T_1}$ . For this purpose we will transform the integrals as follows. Using (2.6) and the formulas

$$\partial_x \omega_j(\xi; x, t) = -\frac{1}{a_j(x, t)} \exp \int_{\xi}^x \left( \frac{\partial_t a_j}{a_j^2} \right) (\eta, \omega_j(\eta; x, t)) d\eta, \tag{2.18}$$

$$\partial_t \omega_j(\xi; x, t) = \exp \int_{\xi}^x \left( \frac{\partial_t a_j}{a_j^2} \right) (\eta, \omega_j(\eta; x, t)) d\eta, \tag{2.19}$$

we get

$$\begin{aligned}
& \int_{x_j}^x \int_{\eta}^x d_{jki}(\xi, \eta, x, t) b_{jk}(\xi, \omega_j(\xi; x, t)) \\
&\times \partial_3 \omega_k(\eta; \xi, \omega_j(\xi; x, t)) \partial_t \omega_j(\xi; x, t) \partial_2 u_i^l(\eta, \omega_k(\eta; \xi, \omega_j(\xi; x, t))) d\xi d\eta \\
&= \int_{x_j}^x \int_{\eta}^x d_{jki}(\xi, \eta, x, t) \partial_3 \omega_k(\eta; \xi, \omega_j(\xi; x, t)) \partial_t \omega_j(\xi; x, t) \\
&\times b_{jk}(\xi, \omega_j(\xi; x, t)) \left[ (\partial_{\xi} \omega_k)(\eta; \xi, \omega_j(\xi; x, t)) \right]^{-1} (\partial_{\xi} u_i^l)(\eta, \omega_k(\eta; \xi, \omega_j(\xi; x, t))) d\xi d\eta \\
&= \int_{x_j}^x \int_{\eta}^x d_{jki}(\xi, \eta, x, t) \partial_t \omega_j(\xi; x, t) (a_k a_j p_{jk})(\xi, \omega_j(\xi; x, t)) \\
&\times (\partial_{\xi} u_i^l)(\eta, \omega_k(\eta; \xi, \omega_j(\xi; x, t))) d\xi d\eta \\
&= \int_{x_j}^x \int_{\eta}^x \tilde{d}_{jki}(\xi, \eta, x, t) (\partial_{\xi} u_i^l)(\eta, \omega_k(\eta; \xi, \omega_j(\xi; x, t))) d\xi d\eta \\
&= - \int_{x_j}^x \int_{\eta}^x \partial_{\xi} \tilde{d}_{jki}(\xi, \eta, x, t) u_i^l(\eta, \omega_k(\eta; \xi, \omega_j(\xi; x, t))) d\xi d\eta \\
&+ \int_{x_j}^x \left[ \tilde{d}_{jki}(\xi, \eta, x, t) u_i^l(\eta, \omega_k(\eta; \xi, \omega_j(\xi; x, t))) \right]_{\xi=\eta}^{\xi=x} d\eta.
\end{aligned}$$

Here

$$\tilde{d}_{jki}(\xi, \eta, x, t) = d_{jki}(\xi, \eta, x, t) \partial_t \omega_j(\xi; x, t) (a_k a_j p_{jk})(\xi, \omega_j(\xi; x, t)).$$

Now, the desired convergence follows from (2.15).

In the second step we prove that there exists  $T_2 > T_1$  such that  $\partial_t u$  restricted to  $\bar{\Pi}_{T_2}$  is  $C^1$ -smooth in  $t$  on  $\bar{\Pi}_{T_2}$ . Once this is done, we differentiate (2.1) with respect to  $t$  and get  $\partial_{xt}^2 u \in C(\bar{\Pi}_{T_2})^n$ ; differentiating (2.1) with respect to  $x$ , we get  $\partial_x^2 u \in C(\bar{\Pi}_{T_2})^n$ . We will be able to conclude that  $u \in C^2(\bar{\Pi}_{T_2})^n$ , as desired. To prove the existence of  $T_2$ , let  $v = \partial_t u$ . Differentiation of (2.1) formally in  $t$  leads to

$$(\partial_t + a_j \partial_x)v_j + \sum_{k=1}^n b_{jk} v_k + \sum_{k=1}^n \partial_t b_{jk} u_k + \partial_t a_j \partial_x u_j = \partial_t f_j.$$

Combining this with (2.1), we obtain

$$\begin{aligned} & (\partial_t + a_j \partial_x)v_j + \sum_{k=1}^n b_{jk} v_k - \frac{\partial_t a_j}{a_j} v_j \\ &= \partial_t f_j - \sum_{k=1}^n \partial_t b_{jk} u_k + \frac{\partial_t a_j}{a_j} \left( \sum_{k=1}^n b_{jk} u_k - f_j \right) = G_j(f_j, \partial_t f_j, u). \end{aligned} \quad (2.20)$$

Here, for each  $j \leq n$ ,  $G_j$  is a certain linear function with smooth coefficients. Set

$$\tilde{c}_j(\xi, x, t) = \exp \int_x^\xi \left( \frac{b_{jj}}{a_j} - \frac{\partial_t a_j}{a_j^2} \right) (\eta, \omega_j(\eta; x, t)) d\eta, \quad \tilde{d}_j(\xi, x, t) = \frac{\tilde{c}_j(\xi, x, t)}{a_j(\xi, \omega_j(\xi; x, t))}$$

and introduce three linear operators  $\tilde{B}, \tilde{D}, \tilde{F} : C(\bar{\Pi}_0)^n \rightarrow C(\bar{\Pi}_0)^n$  by

$$\left( \tilde{B}u \right)_j(x, t) = \tilde{c}_j(x_j, x, t) u_j(x_j, \omega_j(x_j; x, t)), \quad (2.21)$$

$$\left( \tilde{D}u \right)_j(x, t) = - \int_{x_j}^x \tilde{d}_j(\xi, x, t) \sum_{\substack{k=1 \\ k \neq j}}^n (b_{jk} u_k)(\xi, \omega_j(\xi; x, t)) d\xi, \quad (2.22)$$

$$\left( \tilde{F}f \right)_j(x, t) = \int_{x_j}^x \tilde{d}_j(\xi, x, t) f_j(\xi, \omega_j(\xi; x, t)) d\xi. \quad (2.23)$$

Similarly to the above, our starting point is that for any  $T_2 \geq T_1$  the function  $v$  satisfies the following operator equation resulting from (2.20):

$$v|_{\bar{\Pi}_{T_2}} = \tilde{B}h' + \tilde{D}v + \tilde{F}G(f, \partial_t f, u),$$

and, hence, the equation

$$v|_{\bar{\Pi}_{T_2}} = \tilde{B}h' + \tilde{D}\tilde{B}h' + \tilde{D}^2 v + (I + \tilde{D})\tilde{F}G(f, \partial_t f, u), \quad (2.24)$$

where  $G = (G_1, \dots, G_n)$  and  $h' = (h'_1, \dots, h'_n)$ . Again, due to the assumption (2.5), we can fix  $T_2 > T_1$  such that the right-hand side of (2.24) does not depend on  $u$  and  $v$  in  $\bar{\Pi} \setminus \Pi_{T_1}$ . Due to Step 1, the function  $(I + \tilde{D})\tilde{F}G(f, \partial_t f, u)$  then meets the  $C_t^1$ -regularity. Moreover,  $\tilde{B}h' + \tilde{D}\tilde{B}h' \in C^\infty$ . We are thus left to show that the operator  $\tilde{D}^2$  is smoothing

in the above sense. As  $\tilde{D}$  is exactly the operator  $D$  with  $c_j$  and  $d_j$  replaced by the smooth functions  $\tilde{c}_j$  and  $\tilde{d}_j$ , the desired smoothing property of  $\tilde{D}^2$  follows from the proof of the smoothness of  $D^2$  and the fact that  $\tilde{D}^2 v$  in (2.24) does not depend on  $v$  in  $\bar{\Pi} \setminus \Pi_{T_1}$ .

Proceeding further by induction, assume that, given  $r \geq 2$ , there is  $T_r > 0$  such that  $u \in C^r(\bar{\Pi}_{T_r})^n$  and prove that  $u$  meets the  $C^{r+1}$ -regularity in  $t$  on  $\bar{\Pi}_{T_{r+1}}$  for some  $T_{r+1} > T_r$ . Set  $w = \partial_t^r u$ . Differentiating (2.1) and (2.3)  $r$ -times in  $t$ , we come to our starting operator equation for  $w$ , namely

$$w|_{\bar{\Pi}_{T_{r+1}}} = \tilde{B}h^{(r)} + \tilde{D}\tilde{B}h^{(r)} + \tilde{D}^2 w + (I + \tilde{D})\tilde{F}\tilde{G}(f, \partial_t f, \dots, \partial_t^r f, u, \partial_t u, \dots, \partial_t^{r-1} u), \quad (2.25)$$

where  $\tilde{G}$  is a vector of certain linear functions with smooth coefficients and the operators  $\tilde{B}$ ,  $\tilde{D}$ , and  $\tilde{F}$  are modified by  $\tilde{c}_j(\xi, x, t)$  in (2.21)–(2.23) changing to  $\tilde{c}_j(\xi, x, t) = \exp \int_x^\xi \left( \frac{b_{jj}}{a_j} - r \frac{\partial_t a_j}{a_j^2} \right) (\eta, \omega_j(\eta; x, t)) d\eta$ . Similarly to the above, fix  $T_{r+1} > T_r$  such that the right-hand side of (2.25) does not depend on  $u, \partial_t u, \dots, \partial_t^{r-1} u$ , and  $w$  in  $\bar{\Pi} \setminus \Pi_{T_r}$ . This ensures that the last two summands in (2.25) are  $C_t^1$ -functions. The first two summands are  $C_t^1$ -smooth by the regularity assumptions on the data. Finally, the  $C^{r+1}(\bar{\Pi}_{T_{r+1}})$ -regularity of  $u$  follows from the previous steps of the proof and suitable differentiations of the system (2.1).  $\square$

Theorem 2.3 can be extended over the boundary operators of the following kind (both linear and nonlinear). Given  $T > 0$ , in the domain  $\Pi_T$  let us consider the problem (2.1)–(2.3) with  $b_{jk} \equiv 0$  for all  $j \neq k$  (i.e., the system (2.1) is decoupled) and with (2.2) replaced by  $u(x, T) = \varphi(x)$  (the initial values are given at  $t = T$ ). This entails that the domain of influence of  $\varphi$  now depends only on the boundary conditions. For the latter it is supposed that, for every  $T > 0$  and  $\varphi(x)$ , the function  $\varphi(x)$  has a bounded domain of influence on  $u$ . In other words, for any decoupled system (2.1), if  $\varphi(x)$  has a singularity at some point  $x \in [0, 1]$ , then this singularity expands along a finite number of characteristic curves (we have a finite number of "reflections" from the boundary), and this number is bounded from above uniformly in  $x \in [0, 1]$ . This class of boundary operators is in detail described in [18], where the necessary and sufficient conditions for smoothing solutions are given. The results of [18] generalize the smoothing results obtained in [10, 13, 23, 25] for the system (2.1) with time-independent coefficients and (a kind of) Dirichlet boundary conditions.

## 2.2 Integral boundary conditions in age structured population models

Here we address another class of boundary operators admitting smoothing solutions. Though it covers a range of (partial) integral operators, we illustrate our smoothing result with an example from population dynamics.

Integral boundary conditions are usually used in continuous age structured population models to describe a fertility of populations. Let  $u(x, t)$  denote the density of a population of age  $x$  at time  $t$ . Then the dynamics of  $u$  can be described by the following model (see,

e.g. [11, 26, 36] and references therein):

$$(\partial_t + \partial_x + \mu)u = 0, \quad (x, t) \in \bar{\Pi}_0, \quad (2.26)$$

$$u(x, 0) = \varphi(x), \quad x \in [0, 1], \quad (2.27)$$

$$u(0, t) = h \left( \int_0^1 \gamma(x)u(x, t) dx \right), \quad t \in \mathbb{R}, \quad (2.28)$$

where  $\mu > 0$  is the mortality rate of the population and the functions  $h$  and  $\gamma$  describe the fertility of the population. Without losing potential applicability to the topic of population dynamics,  $h$  and  $\gamma$  are supposed to be  $C^\infty$ -smooth functions. The integral in (2.28) is a kind of the so-called “partial” integral, since  $u$  depends not only on the variable of integration  $x$ , but also on the free variable  $t$ . Therefore the right-hand side of (2.28) is not smoothing. Nevertheless, it turns out that it is regular enough to contribute into smoothing solutions.

**Theorem 2.4** *Assume that  $h$  and  $\gamma$  are  $C^\infty$ -smooth functions and  $\varphi$  is a continuous function. Then any continuous solution to the problem (2.26)–(2.28) is smoothing.*

**Proof.** It suffices to show the smoothing property starting from large enough  $t$ . Therefore, we can use the notation:

$$\begin{aligned} (Ru)(t) &= h \left( \int_0^1 \gamma(x)u(x, t) dx \right) \\ \omega(\xi; x, t) &= t + \xi - x \\ c(\xi, x, t) = \tilde{c}(\xi, x, t) &= e^{\mu(\xi-x)} \\ (Bu)(x, t) = (\tilde{B}u)(x, t) &= e^{-\mu x}u(0, t-x), \end{aligned}$$

the latter two being introduced for all large enough  $t$ . Integration along the characteristic curves implies that any continuous solution to (2.26)–(2.28) satisfies the operator equations  $u = BRu$  and  $u = Bu$  and, hence,

$$u = BRBu \quad (2.29)$$

whenever  $t > T_1$ , where  $T_1$  is chosen to be so large that the operator  $BRB$  moves away from the initial boundary (the right-hand side of (2.29) does not depend on  $\varphi$ ). Since

$$\begin{aligned} (BRBu)(t) &= e^{-\mu x}h \left( \int_0^1 \gamma(\xi)e^{-\mu\xi}u(0, t-x-\xi) d\xi \right) \\ &= e^{-\mu x}h \left( \int_{t-x-1}^{t-x} \gamma(t-x-\tau)e^{\mu(x-t+\tau)}u(0, \tau) d\tau \right), \end{aligned}$$

we obtain the  $C_t^1$ -smoothness of  $BRBu$  and, hence, of  $u$  on  $\bar{\Pi}_{T_1}$ . The  $C^1$ -smoothness of  $u$  on  $\bar{\Pi}_{T_1}$  now follows from (2.26).

Proceeding similarly to the proof of Theorem 2.3, in the second step we consider the following operator equation with respect to  $v = \partial_t u$ , obtained after differentiation of (2.26) and (2.28) with respect to  $t$  and integration along characteristic curves:

$$v|_{\overline{\Pi}_{T_2}} = B\partial_t RBv, \quad (2.30)$$

where

$$(\partial_t Rv)(t) = h' \left( \int_0^1 \gamma(x)u(x, t) dx \right) \int_0^1 \gamma(x)v(x, t) dx$$

and  $T_2 > T_1$  is fixed to satisfy the property that the right-hand side of (2.30) does not depend on  $u$  and  $v$  in  $\overline{\Pi}_0 \setminus \Pi_{T_1}$ . It follows that

$$\begin{aligned} v|_{\overline{\Pi}_{T_2}} &= e^{-\mu x} h' \left( \int_0^1 \gamma(\xi)u(\xi, t-x) d\xi \right) \int_0^1 \gamma(\xi)e^{-\mu\xi}v(0, t-x-\xi) d\xi \\ &= e^{-\mu x} h' \left( \int_0^1 \gamma(\xi)u(\xi, t-x) d\xi \right) \int_{t-x-1}^{t-x} \gamma(t-x-\tau)e^{\mu(x-t+\tau)}v(0, \tau) d\tau. \end{aligned}$$

To conclude that  $v \in C_t^1(\overline{\Pi}_{T_2})^n$ , it remains to note that  $u$  under the first integral in the right-hand side meets the  $C_t^1$ -regularity, while the second integral gives us the desired smoothing property.

In general, given  $T_r$  for  $r \geq 2$ , we choose  $T_{r+1} > T_r$  by the argument as above and for  $w = \partial_t^r u$  have the equation

$$w|_{\overline{\Pi}_{T_{r+1}}} = B\partial_t^r RBw, \quad (2.31)$$

where

$$\begin{aligned} (\partial_t^r Rw)(t) &= h' \left( \int_0^1 \gamma(x)u(x, t) dx \right) \int_0^1 \gamma(x)w(x, t) dx \\ &+ \frac{d^{r-1}}{dt^{r-1}} \left[ h' \left( \int_0^1 \gamma(x)u(x, t) dx \right) \right] \int_0^1 \gamma(x)\partial_t u(x, t) dx \\ &+ \frac{d^{r-2}}{dt^{r-2}} \left[ h' \left( \int_0^1 \gamma(x)u(x, t) dx \right) \int_0^1 \gamma(x)\partial_t u(x, t) dx \right]. \end{aligned}$$

Substituting the latter into (2.31) and changing variables under the integral of  $w$  similarly to the first two steps, we get the desired smoothing property for  $w$ . This completes the proof.  $\square$

## 2.3 Dissipative boundary conditions and periodic problems

Now we switch to boundary conditions having dissipative nature and fitting the smoothing property. A large class of dissipative boundary conditions for hyperbolic PDEs is described in [8].

To give an idea of the smoothing effect in this case, consider the following specification of (2.1):

$$\begin{aligned} u_j(0, t) &= h_j(z(t)), \quad 1 \leq j \leq m, \\ u_j(1, t) &= h_j(z(t)), \quad m < j \leq n, \end{aligned} \quad (2.32)$$

with

$$z(t) = (u_1(1, t), \dots, u_m(1, t), u_{m+1}(0, t), \dots, u_n(0, t)).$$

In the domain  $\Pi_{-\infty}$  we address the problem (2.1), (2.32) subjected to periodic boundary conditions

$$u(x, t + 2\pi) = u(x, t). \quad (2.33)$$

The problems of this kind appear in laser dynamics and chemical kinetics (in Section 3 we investigate a traveling-wave model of kind (2.1), (2.32), (2.33) from laser dynamics). Within this section, using the standard notation for the (sub)spaces of continuous functions, we assume that the functions have additional property of  $2\pi$ -periodicity in  $t$ . Write

$$h'_j(z) = \nabla_z h_j(z), \quad h'(z) = \{\partial_k h_j(z)\}_{j,k=1}^n.$$

**Theorem 2.5** *Assume that  $a_j, b_{jk}, f_j$ , and  $h_j$  are  $C^N$ -smooth functions in all their arguments and the conditions (2.4)–(2.6) are fulfilled. Moreover, the functions  $a_j, b_{jk}, f_j$  are supposed to be  $2\pi$ -periodic in  $t$ . If*

$$\exp \left\{ \int_x^{x_j} \left( \frac{b_{jj}}{a_j} - r \frac{\partial_t a_j}{a_j^2} \right) (\eta, \omega_j(\eta; x, t)) d\eta \right\} \sum_{k=1}^n |\partial_k h_j(z)| < 1 \quad (2.34)$$

for all  $j \leq n$ ,  $x \in [0, 1]$ ,  $t \in \mathbb{R}$ ,  $z \in \mathbb{R}^n$ , and  $r = 0, 1, \dots, N$ , then any continuous solution to the problem (2.1), (2.32), (2.33) belongs to  $C^N(\Pi_{-\infty})$ .

**Proof.** Any continuous solution to the problem (2.1), (2.32), (2.33) in  $\Pi_{-\infty}$  fulfills (2.12) with  $S = R$  and also satisfies the equation

$$u = Bu + Du + Ff \quad (2.35)$$

where the boundary conditions are not specified. Substituting (2.35) into (2.12), we obtain

$$u = BRu + (DB + D^2)u + (I + D)Ff. \quad (2.36)$$

We first show the bijectivity of  $I - BR \in \mathcal{L}(C_t^1(\overline{\Pi_{-\infty}})^n)$ . On the account of (2.19) and the definition of  $B$  given by (2.9), we have

$$\begin{aligned} (BRu)_j(x, t) &= c_j(x_j, x, t) h_j(z(\omega_j(x_j; x, t))) = c_j(x_j, x, t) h_j(0) \\ &+ \exp \left\{ \int_x^{x_j} \left( \frac{b_{jj}}{a_j} \right) (\eta, \omega_j(\eta; x, t)) d\eta \right\} \int_0^1 h'_j(\alpha z(\omega_j(x_j; x, t))) d\alpha \cdot z(\omega_j(x_j; x, t)) \end{aligned}$$

and

$$\begin{aligned} \partial_t [(BRu)_j(x, t)] &= \partial_t c_j(x_j, x, t) h_j(z(\omega_j(x_j; x, t))) \\ &+ h'_j(z(\omega_j(x_j; x, t))) \cdot z'(\omega_j(x_j; x, t)) \exp \left\{ \int_x^{x_j} \left( \frac{b_{jj}}{a_j} - \frac{\partial_t a_j}{a_j^2} \right) (\eta, \omega_j(\eta; x, t)) d\eta \right\}, \end{aligned}$$

where  $\cdot$  denotes the scalar product in  $\mathbb{R}^n$ . Taking into account (2.33), the bijectivity of  $I - BR \in \mathcal{L}(C_t^1(\overline{\Pi}_{-\infty})^n)$  now follows from the contractibility condition (2.34) with  $r = 0, 1$  and from the proof of the  $C^k$ -regularity result for solutions of first-order hyperbolic PDEs given in [31].

Now we claim that the operators  $DB$  and  $D^2$  in (2.36) are smoothing. The latter is smoothing by the proof in Theorem 2.3. Similar argument works also for  $DB$ . Indeed, by the definition of the operators  $D$  and  $B$  we have

$$\begin{aligned} (DBu^l)_j(x, t) & \\ &= \sum_{\substack{k=1 \\ k \neq j}}^n \int_x^{x_j} d_j(\xi, x, t) b_{jk}(\xi, \omega_j(\xi; x, t)) c_k(x_k, \xi, \omega_j(\xi; x, t)) u_k^l(x_k, \omega_k(x_k; \xi, \omega_j(\xi; x, t))) d\xi, \end{aligned} \quad (2.37)$$

where the sequence  $u^l$  is fixed to satisfy (2.15) with  $\Pi_0$  replaced by  $\Pi_{-\infty}$ . To show that  $\partial_t [DBu^l]$  converges uniformly on  $\overline{\Pi}_{-\infty}$ , we transform the integrals in (2.37) like to the case of  $D^2$ , that is, we differentiate (2.37) in  $t$ , use (2.6), and integrate by parts. In this way we get the smoothing property for  $DB$ . Turning back to the formula (2.36) and using in addition the fact that  $(I + D)Ff$  is  $C^\infty$ -smooth, we can rewrite (2.36) in the equivalent form

$$u = (I - BR)^{-1} [(DB + D^2)u + (I + D)Ff],$$

thereby reaching the  $C_t^1$ -regularity for  $u$ . Afterwards, the  $C^1$ -regularity of  $u$  is a straightforward consequence of the system (2.1).

Proceeding similarly to the proof of Theorem 2.3, we come to the formula for  $v = \partial_t u$ :

$$v = (I - \tilde{B}R'_z)^{-1} [(\tilde{D}\tilde{B} + \tilde{D}^2)v + (I + \tilde{D})\tilde{F}G(f, \partial_t f, u)],$$

where  $R'_z y = h'(z)y$ . The property that  $v \in C_t^1(\overline{\Pi}_{-\infty})^n$  follows from the bijectivity of  $I - \tilde{B}R'_z \in \mathcal{L}(C_t^1(\overline{\Pi}_{-\infty})^n)$ , which we have by condition (2.34) with  $r = 1, 2$  and the  $C_t^1$ -regularity of  $\tilde{D}\tilde{B} + \tilde{D}^2$  and  $(I + \tilde{D})\tilde{F}G(f, \partial_t f, u)$ . This entails  $u \in C_t^2(\overline{\Pi}_{-\infty})^n$ . It follows by (2.1) that  $u \in C^2(\overline{\Pi}_{-\infty})^n$ .

To complete the proof, we proceed by induction on the order of regularity of  $u$ . Assume that  $u \in C^r(\overline{\Pi}_{-\infty})^n$  for some  $r \geq 2$  and prove that  $u \in C^{r+1}(\overline{\Pi}_{-\infty})^n$ . Our starting formula for  $w = \partial_t^r u$  is as follows:

$$\begin{aligned} w &= (I - \tilde{B}R'_z)^{-1} [(\tilde{D}\tilde{B} + \tilde{D}^2)w \\ &\quad + (I + \tilde{D})\tilde{F}\tilde{G}(f, \partial_t f, \dots, \partial_t^r f, u, \partial_t u, \dots, \partial_t^{r-1} u) \\ &\quad + \tilde{B}\partial_t^{r-1} R'_z z' + \tilde{B}\partial_t^{r-2} (R'_z z')], \end{aligned}$$

where  $\partial_t^{r-1} R'_z = \{\partial_t^{r-1}(\partial_k h_j(z))\}_{j,k=1}^n$  and  $\tilde{B}, \tilde{D}$ , and  $\tilde{F}$  are modified by  $\tilde{c}_j(\xi, x, t)$  in (2.21)–(2.23) changing to  $\tilde{c}_j(\xi, x, t) = \exp \int_x^\xi \left( \frac{b_{jj}}{a_j} - r \frac{\partial_t a_j}{a_j^2} \right) (\eta, \omega_j(\eta; x, t)) d\eta$ . By the regularity assumptions on the data and the induction assumption, the last three summands in the square brackets are  $C_t^1$ -functions. Using in addition our smoothing argument for  $\tilde{D}\tilde{B} + \tilde{D}^2$  and the regularity properties of  $(I - \tilde{B}R'_z)^{-1}$ , we arrive at the desired conclusion.  $\square$

### 3 Fredholm solvability of periodic problems

In [20, 21] we suggest an approach to establish the Fredholm property for first-order hyperbolic operators. This is done by construction an equivalent regularization in the form of a parametrix. The construction is, implicitly but essentially, based on the smoothing effect investigated in Section 2. Consider the first-order one-dimensional hyperbolic system

$$(\partial_t + a(x)\partial_x + b(x))u = f(x, t), \quad x \in (0, 1), \quad (3.38)$$

subjected to periodic conditions (2.33) and reflection boundary conditions

$$\begin{aligned} u_j(0, t) &= \sum_{k=m+1}^n r_{jk}^0 u_k(0, t), \quad 1 \leq j \leq m, \\ u_j(1, t) &= \sum_{k=1}^m r_{jk}^1 u_k(1, t), \quad m < j \leq n. \end{aligned} \quad (3.39)$$

Here  $r_{jk}^0$  and  $r_{jk}^1$  are real numbers and the right-hand sides  $f_j : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  are supposed to be  $2\pi$ -periodic with respect to  $t$ .

The main result of this section states that the system (3.38), (2.33), (3.39) is solvable if and only if the right hand side is orthogonal to all solutions to the corresponding homogeneous adjoint system

$$-\partial_t u - \partial_x (a(x)u) + b^T(x)u = 0, \quad x \in (0, 1),$$

subjected to periodic conditions (2.33) and adjoint boundary conditions

$$\begin{aligned} a_j(0)u_j(0, t) &= - \sum_{k=1}^m r_{kj}^0 a_k(0)u_k(0, t), \quad m < j \leq n, \\ a_j(1)u_j(1, t) &= - \sum_{k=m+1}^n r_{kj}^1 a_k(1)u_k(1, t), \quad 1 \leq j \leq m. \end{aligned} \quad (3.40)$$

We will present our result in three steps. First we introduce appropriate function spaces for solutions. Then we decompose the operator of the problem into two parts, only one being responsible for propagation of singularities. Finally, based on this decomposition and the smoothing property, we construct a parametrix thereby establishing the Fredholm solvability.

When choosing the function spaces, note that the problem (3.38), (2.33), (3.39) describes the so-called traveling-wave models from laser dynamics [24, 30]. From the physical point of view, it is desirable to allow discontinuities in the coefficients and the right hand side of (3.38). This entails that the spaces of solutions should not be too small. On the other hand, they should not be too large, in order to admit embeddings into an algebra of functions with pointwise multiplication of its elements. The last property is important for potential applicability of our results to nonlinear problems, like describing such dynamic phenomena as Hopf bifurcation and periodic synchronizations. Finally, the solution

spaces capable to capture the Fredholm solvability need to have optimal regularity with respect to the function spaces of the right-hand side.

We now describe the scale of spaces  $V^\gamma$  (for the solutions) and  $W^\gamma$  (for the right-hand side) meeting all these properties. For  $\gamma \geq 0$ , let  $W^\gamma$  denote the vector space of all locally integrable functions  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}^n$  such that  $f(x, t) = f(x, t + 2\pi)$  for almost all  $x \in (0, 1)$  and  $t \in \mathbb{R}$  and that

$$\|f\|_{W^\gamma}^2 = \sum_{s \in \mathbb{Z}} (1 + s^2)^\gamma \int_0^1 \left\| \int_0^{2\pi} f(x, t) e^{-ist} dt \right\|^2 dx < \infty. \quad (3.41)$$

Here and in what follows  $\|\cdot\|$  is the Hermitian norm in  $\mathbb{C}^n$ . It is well known that  $W^\gamma$  is a Banach space with the norm (3.41); see, e.g. [12], [33, Chapter 5.10], and [35, Chapter 2.4].

Furthermore, for  $\gamma \geq 1$  and  $a \in L^\infty((0, 1); \mathbb{M}_n)$ , where  $\mathbb{M}_n$  denotes the space of real  $n \times n$  matrices, with  $\text{ess inf } |a_j| > 0$  for all  $j \leq n$  we will work with the function spaces

$$U^\gamma = \left\{ u \in W^\gamma : \partial_x u \in W^0, \partial_t u + a \partial_x u \in W^\gamma \right\}$$

endowed with the norms

$$\|u\|_{U^\gamma}^2 = \|u\|_{W^\gamma}^2 + \|\partial_t u + a \partial_x u\|_{W^\gamma}^2.$$

Remark that the space  $U^\gamma$  depends on  $a$  and is larger than the space of all  $u \in W^\gamma$  such that  $\partial_t u \in W^\gamma$  and  $\partial_x u \in W^\gamma$  (which does not depend on  $a$ ). For  $u \in U^\gamma$  there exist traces  $u(0, \cdot), u(1, \cdot) \in L_{loc}^2(\mathbb{R}; \mathbb{R}^n)$  (see [21]), and, hence, it makes sense to consider the closed subspaces in  $U^\gamma$

$$\begin{aligned} V^\gamma &= \{u \in U^\gamma : (3.39) \text{ is fulfilled}\}, \\ \tilde{V}^\gamma &= \{u \in U^\gamma : (3.40) \text{ is fulfilled}\}. \end{aligned}$$

Our next task is to decompose the operator of our problem into two parts in order to single out the part, denoted below by  $\mathcal{A}$ , which is bijective and at the same time is responsible for the propagation of singularities. If this decomposition is optimal, then after a regularization procedure the other part becomes smoothing and therefore meets the compactness property. Let

$$b^0 = \text{diag}(b_{11}, b_{22}, \dots, b_{nn}) \quad \text{and} \quad b^1 = b - b^0$$

denote the diagonal and the off-diagonal parts of the coefficient matrix  $b$ , respectively. Let us introduce operators  $\mathcal{A} \in \mathcal{L}(V^\gamma; W^\gamma)$ ,  $\tilde{\mathcal{A}} \in \mathcal{L}(\tilde{V}^\gamma; W^\gamma)$ , and  $\mathcal{B}, \tilde{\mathcal{B}} \in \mathcal{L}(W^\gamma)$  by

$$\begin{aligned} \mathcal{A}u &= \partial_t u + a \partial_x u + b^0 u, \\ \tilde{\mathcal{A}}u &= -\partial_t u - \partial_x (au) + b^0 u, \\ \mathcal{B}u &= b^1 u, \\ \tilde{\mathcal{B}}u &= (b^1)^T u. \end{aligned}$$

Remark that the operators  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\tilde{\mathcal{B}}$  are well-defined for  $a_j, b_{jk} \in L^\infty(0, 1)$ , while  $\tilde{\mathcal{A}}$  is well-defined under additional regularity assumptions with respect to the coefficients  $a_j$ , for example, for  $a_j \in C^{0,1}([0, 1])$ . Note that the operator equation

$$\mathcal{A}u + \mathcal{B}u = f$$

is an abstract representation of the problem (3.38), (2.33), (3.39).

Finally, for  $s \in \mathbb{Z}$  we introduce the complex  $(n - m) \times (n - m)$  matrices

$$R_s = \left[ \sum_{l=1}^m e^{is(\alpha_j(1) - \alpha_l(1) + \beta_j(1) - \beta_l(1))} r_{jl}^1 r_{lk}^0 \right]_{j,k=m+1}^n,$$

where

$$\alpha_j(x) = \int_0^x \frac{1}{a_j(y)} dy, \quad \beta_j(x) = \int_0^x \frac{b_{jj}(y)}{a_j(y)} dy.$$

The following theorem states, first, that the pair of spaces  $(V^\gamma, W^\gamma)$  gives an optimal regularity trade-off between the spaces of solutions and right-hand sides and, second, that  $\mathcal{A}$  meets the bijectivity property. The second desirable property for  $\mathcal{A}$  of being an optimal operator responsible for propagation of singularities will be a consequence of our Fredholmness result.

**Theorem 3.1** [21] *For every  $c > 0$  there exists  $C > 0$  such that the following is true: If*

$$a_j, b_{jj} \in L^\infty(0, 1) \quad \text{and} \quad \text{ess inf } |a_j| \geq c \quad \text{for all } j = 1, \dots, n, \quad (3.42)$$

$$\sum_{j=1}^n \|b_{jj}\|_\infty + \sum_{j=1}^m \sum_{k=m+1}^n |r_{jk}^0| + \sum_{j=m+1}^n \sum_{k=1}^m |r_{jk}^1| \leq \frac{1}{c},$$

and

$$|\det(I - R_s)| \geq c \quad \text{for all } s \in \mathbb{Z}, \quad (3.43)$$

then for all  $\gamma \geq 1$  the operator  $\mathcal{A}$  is an isomorphism from  $V^\gamma$  onto  $W^\gamma$  and

$$\|\mathcal{A}^{-1}\|_{\mathcal{L}(W^\gamma; V^\gamma)} \leq C.$$

Let

$$\langle f, u \rangle_{L^2} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \langle f(x, t), u(x, t) \rangle dx dt$$

denote the scalar product in the Hilbert space  $L^2((0, 1) \times (0, 2\pi); \mathbb{R}^n)$  and  $\langle \cdot, \cdot \rangle$  denote the Euclidean scalar product in  $\mathbb{R}^n$ . As usual, by  $BV(0, 1)$  we denote the Banach space of all functions  $h : (0, 1) \rightarrow \mathbb{R}$  with bounded variation, i.e. of all  $h \in L^\infty(0, 1)$  such that there exists  $C > 0$  with

$$\left| \int_0^1 h(x) \varphi'(x) dx \right| \leq C \|\varphi\|_{L^\infty(0,1)} \quad \text{for all } \varphi \in C_0^\infty(0, 1). \quad (3.44)$$

The norm of  $h$  in  $BV(0, 1)$  is the sum of the norm of  $h$  in  $L^\infty(0, 1)$  and of the smallest possible constant  $C$  in (3.44). We are prepared to formulate the main result of this section.

**Theorem 3.2** [21] *Suppose that conditions (3.42) and (3.43) are fulfilled for some  $c > 0$ . Suppose also that*

$$\begin{aligned} & \text{for all } j \neq k \text{ there is } p_{jk} \in BV(0,1) \text{ such that} \\ & a_k(x)b_{jk}(x) = p_{jk}(x)(a_j(x) - a_k(x)) \text{ for a.a. } x \in [0,1]. \end{aligned} \quad (3.45)$$

*Then the following is true:*

(i) *The operator  $\mathcal{A} + \mathcal{B}$  is a Fredholm operator with index zero from  $V^\gamma$  into  $W^\gamma$  for all  $\gamma \geq 1$ , and  $\ker(\mathcal{A} + \mathcal{B}) = \{u \in V^\gamma : (\mathcal{A} + \mathcal{B})u = 0\}$  does not depend on  $\gamma$ .*

(ii) (smoothing effect) *If  $a \in C^{0,1}([0,1]; \mathbb{M}_n)$ , then  $\ker(\mathcal{A} + \mathcal{B})^* = \ker(\tilde{\mathcal{A}} + \tilde{\mathcal{B}})$  and*

$$\{(\mathcal{A} + \mathcal{B})u : u \in V^\gamma\} = \left\{ f \in W^\gamma : \langle f, u \rangle_{L^2} = 0 \text{ for all } u \in \ker(\tilde{\mathcal{A}} + \tilde{\mathcal{B}}) \right\},$$

*where  $\ker(\tilde{\mathcal{A}} + \tilde{\mathcal{B}}) = \{u \in \tilde{V}^\gamma : (\tilde{\mathcal{A}} + \tilde{\mathcal{B}})u = 0\}$  does not depend on  $\gamma$ .*

Theorem 3.2 (ii) states that the kernel of the adjoint operator is actually defined on the classical function spaces. In other words, the kernel has much better regularity than ensured just by the formal definition of the adjoint operator. Here we encounter a smoothing effect for the solutions (of the adjoint hyperbolic problem), that are originally functionals. The proof of this effect in [20, 21] uses completely different techniques, based on a functional-analytic approach.

Finally, we outline the proof of Theorem 3.2 (i). As mentioned above, we construct a parametrix to the operator of the problem. By Theorem 3.1, the zero-order Fredholmness of the operator  $\mathcal{A} + \mathcal{B} \in \mathcal{L}(V^\gamma; W^\gamma)$  is equivalent to the zero-order Fredholmness of the operator  $I + \mathcal{B}\mathcal{A}^{-1} \in \mathcal{L}(W^\gamma)$ . Furthermore, we use the following Fredholmness criterion (see [34, Theorem 5.5] or [37, Proposition 5.7.1]).

**Lemma 3.3** *Let  $I$  denote the identity in a Banach space  $W$ . Suppose that  $\mathcal{D} \in \mathcal{L}(W)$  and  $\mathcal{D}^2$  is compact. Then  $I + \mathcal{D}$  is Fredholm.*

Setting  $\mathcal{D} = \mathcal{B}\mathcal{A}^{-1} \in \mathcal{L}(W^\gamma)$ , we prove that  $\mathcal{D}^2 \in \mathcal{L}(W^\gamma)$  is compact (while  $\mathcal{D}$  alone can hardly be compact, being a type of a partial integral operator). This actually means that  $\mathcal{D}^2$  has smoothing property. In fact,  $\mathcal{D}^2$  is basically the same as the operator  $D^2$ , that we used in the proof of Theorem 2.3.

Since  $I - \mathcal{D}^2 = (I - \mathcal{D})(I + \mathcal{D}) = (I + \mathcal{D})(I - \mathcal{D})$ , the operator  $I - \mathcal{D}$  is a parametrix of  $I + \mathcal{D}$ . It follows that the operator  $\mathcal{A} + \mathcal{B}$  admits an equivalent regularization in the form of the right parametrix  $\mathcal{A}^{-1}(I - \mathcal{B}\mathcal{A}^{-1})$ .

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## References

- [1] V. E. Abolinya, A. D. Myshkis, A mixed problem for an almost linear hyperbolic system on the plane, *Mat. Sb.* 50(92) (1960), 423-442.
- [2] T. A. Akramov, On the behavior of solutions to a certain hyperbolic problem, *Sib. Math. J.* 39 (1998), N 1, pp. 1-17.
- [3] T. A. Akramov, V. S. Belonosov, T. I. Zelenyak, M. M. Lavrentev, Jr., M. G. Slinko, and V. S. Sheplev, Mathematical Foundations of Modeling of Catalytic Processes: A Review, *Theoretical Foundations of Chemical Engineering* 34 (2000), N 3, pp. 295–306.
- [4] R. Aris, *The mathematical theory of diffusion and reaction in permeable catalysts. Vol. I: The theory of the steady state.* Oxford: Clarendon Press 444 p. (1975).
- [5] R. Aris, *The mathematical theory of diffusion and reaction in permeable catalysts. Vol. II: Questions of uniqueness, stability, and transient behaviour.* Oxford: Clarendon Press 217 p. (1975).
- [6] N. N. Bogoliubov, Yu. A. Mitropolskii, *Asymptotic method in the theory of nonlinear oscillations*, Gordon and Breach, New York, 1961.
- [7] S.-N. Chow, J. K. Hale, *Methods of Bifurcation Theory*, Grundlehren der Math. Wissenschaften 251, Springer-Verlag, New York-Berlin, 1982.
- [8] J.-M. Coron, G. Bastin, B. d'Andra-Novel, Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems, *SIAM J. Control Optim.* 47 (2008), No. 3, 1460-1498.
- [9] J. M. Cushing, *An introduction to structured population dynamics.* SIAM, Philadelphia, 193 p. (1998).
- [10] N. A. Ėltysheva, *On qualitative properties of solutions of some hyperbolic system on the plane*, *Matematicheskij Sbornik* 134 (1988), N 2, pp. 186–209.
- [11] K. P. Hadeler, K. Dietz, Nonlinear hyperbolic partial differential equations for the dynamics of parasite populations, *Comput. Math. Appl.* 9 (1983), 415-430.
- [12] L. Herrmann L., Periodic solutions of abstract differential equations: the Fourier method, *Czechoslovak Math. J.* 30(105) (1980), 177–206.
- [13] T. Hillen, Existence theory for correlated random walks on bounded domains, *Can. Appl. Math. Q.* 18 (2010), No. 1, pp. 1-40.
- [14] T. Hillen, K. P. Hadeler, Hyperbolic systems and transport equations in mathematical biology, in *Analysis and Numerics for Conservation Laws*, G. Warnecke, Springer, Berlin, 2005, 257–279.

- [15] H. Kielhöfer, *Bifurcation Theory. An Introduction with Applications to PDEs*, Appl. Math. Sciences 156, Springer-Verlag, New York-Berlin, 2004.
- [16] I. Kmit, Generalized solutions to singular initial-boundary hyperbolic problems with non-Lipshitz nonlinearities, *Bull. Cl. Sci. Math. Nat. Sci. Math.* 31 (2006), 87–99.
- [17] I. Kmit, Classical solvability of nonlinear initial-boundary problems for first-order hyperbolic systems, *International Journal of Dynamic Systems and Differential Equations* 1 (2008), N 3, pp. 191–195.
- [18] I. Kmit, Smoothing solutions to initial-boundary problems for first-order hyperbolic systems, *Applicable Analysis* 90 (2011), N 11, p. 1609 – 1634.
- [19] I. Kmit, G. Hörmann, Semilinear hyperbolic systems with nonlocal boundary conditions: reflection of singularities and delta waves, *J. of Analysis and its Applications* 20 (2001), No.3, 637–659.
- [20] I. Kmit, L. Recke, Fredholm Alternative for periodic-Dirichlet problems for linear hyperbolic systems, *J. Math. Anal. and Appl.* 335 (2007), No. 1, 355–370.
- [21] I. Kmit, L. Recke, Fredholmness and smooth dependence for linear time-periodic hyperbolic problems, *Journal of Differential Equations* 252 (2012), No. 2, 1962–1986.
- [22] Kmit, I., Recke, L. (2012). Periodic Solutions to Dissipative Hyperbolic Systems. II: Hopf Bifurcation for Semilinear Problems. In preparation.
- [23] M. M. Lavrent’ev Jr, N. A. Lyul’ko, *Increasing smoothness of solutions to some hyperbolic problems*, *Siberian Math. J.* 38 (1997), N 1, pp. 92–105.
- [24] M. Lichtner, M. Radziunas, and L. Recke, Well-posedness, smooth dependence and center manifold reduction for a semilinear hyperbolic system from laser dynamics, *Math. Methods Appl. Sci.* 30 (2007), 931–960.
- [25] N.A. Lyulko, The increasing smoothness properties of solutions to some hyperbolic problems in two independent variables, *Siberian Electronic Mathematical Reports* 7 (2010), 413–424.
- [26] P. Magal, S. Ruan, Center manifolds for semilinear equations with non-dense domain and applications to Hopf bifurcation in age structured models, *Mem. Am. Math. Soc.* 951 (2009), 1-71.
- [27] M. Oberguggenberger, *Propagation of singularities for semilinear hyperbolic initial-boundary value problems in one space dimension*, *J. Diff. Eqns.* 61 (1986), 1–39.
- [28] P. Popivanov, *Geometrical methods for solving of fully nonlinear partial differential equations* Mathematics and Its Applications, Sofia, 2006.

- [29] P. Popivanov, Nonlinear PDE. *Singularities, propagation, applications*, in *Nonlinear hyperbolic equations, spectral theory, and wavelet transformations*, S. Albeverio et al., Advances in Partial Differential Equations. Basel: Birkhuser. Oper. Theory, Adv. Appl. 145, 1–94 (2003).
- [30] M. Radziunas, H.-J. Wünsche, Dynamics of multisection DFB semiconductor lasers: traveling wave and mode approximation models, in *Optoelectronic Devices – Advanced Simulation and Analysis*, J. Piprek, eds., Springer, USA (2005), pp. 121–150.
- [31] J. Rauch, M. Reed, *Jump discontinuities of semilinear, strictly hyperbolic systems in two variables: Creation and propagation*, Comm. math. Phys. 81 (1981), 203–207.
- [32] A. M. Samoilenko, Elements of the mathematical theory of multi-frequency oscillations, Kluwer Acad. Publ. 1991.
- [33] J.C. Robinson, Infinite-Dimensional Dynamical Systems, Cambridge Texts in Appl. Math., Cambridge University Press (2001).
- [34] M. Schechter, Principles of Functional Analysis, second ed., Graduate Studies in Math. 36, American Mathematical Society, Providence, Rhode Island, 2002.
- [35] O. Vejvoda et al. Partial Differential Equations: Time-Periodic Solutions, Sijthoff Noordhoff (1981).
- [36] G. F. Webb, Theory of nonlinear age-dependent population dynamics, New York-Basel: Marcel Dekker, (1985).
- [37] E. Zeidler. Applied Functional Analysis. Main Principles and their Applications. Applied Math. Sciences 109, Springer, 1995.
- [38] T. I. Zelenyak, *On stationary solutions of mixed problems relating to the study of certain chemical processes*, Differ. Equations 2 (1966), pp. 98-102.