Analysis and characterization of asynchronous state transition graphs using extremal states

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Abstract. Logical modeling of biological regulatory networks gives rise to a representation of the system's dynamics as a so-called state transition graph. Analysis of such a graph in its entirety allows for a comprehensive understanding of the functionalities and behavior of the modeled system. However, the size of the vertex set of the graph is exponential in the number of the network components making analysis costly, motivating development of reduction methods. In this paper, we present results allowing for a complete description of an asynchronous state transition graph of a Thomas network solely based on the analysis of the subgraph induced by certain extremal states. Utilizing this notion, we compare the behavior of a simple multi-valued network and a corresponding Boolean network and analyze the conservation of dynamical properties between them. Understanding the relation between such coarser and finer models is a necessary step towards meaningful network reduction as well as model refinement methods.

1 Introduction

Logical modeling, allowing for a mathematically rigorous description of a dynamical system based only on coarse qualitative information, has long since proved its worth in the context of systems biology research, see e.g. [1–3]. The components of a system are represented by variables adopting values from a finite value range signifying different activity levels of a component. Incorporation of knowledge about the topology of a given network, i.e., interactions between components, and a basic understanding about the effect different components can exhibit on the state of the network, allow the construction of a finite state transition graph representing the dynamical behavior of the modeled system. Different variants of this modeling paradigm are in use, however, in this paper, we will focus on the formalism introduced by R. Thomas in the early 1970s [4].

Systems biology is a rapidly growing field of research. As more and more data on biological systems becomes available, models of biological regulatory systems become feasible not only for small isolated networks but for much more comprehensive systems. In addition, a better understanding of the functionalities of a network component may call for a more refined

representation of its activity levels and may lead from a Boolean representation to a multi-valued model. Both, network and value range size, lead to a state space explosion, making comprehensive analysis of the state transition graph difficult and costly.

To deal with this problem, a wide range of approaches for complexity reduction have been developed. One possibility is to contract the model in some manner, e.g., by eliminating or merging certain components and interactions [5, 6] or by getting a more compact representation of the state transition graph [7]. In other approaches the focus is on modularization of the network and analysis and reintegration of modules [8–10]. Lastly, a very natural approach is to first consider only partial information on the system and derive characteristics or constraints of the dynamical behavior. For example, based only on certain characteristics of the network topology, in particular, cycles in the network structure, it is possible to extract crucial dynamical characteristics relating to oscillations and multistability [10–13]. In any case, a crucial point in all these methods is to understand in how far analysis results of the reduced model can be transfered to the original system.

In this paper, we focus on methods that allow reducing the cost of analysis for multi-valued models. We start by giving a description of the state transition graph of a model only based on the network topology, and a subgraph of the state transition graph derived from states with extremal properties, i.e., states whose component values are the upper and lower bound of the value range. A similar idea allows also the analysis of certain cycles in the state transition graph just by focussing on a subset of its states.

In a second reduction approach, we simplify a multi-valued model by considering a corresponding Boolean representation. It has been shown that a dynamics preserving transformation of a multi-valued into a Boolean model is possible [14]. However, complete preservation of all model properties will not lead to a reduction in complexity. Here, we transform the multi-valued model by keeping the network structure and changing all value ranges to the set $\{0, 1\}$. In most cases, there are different possibilities to translate the multi-valued into Boolean dynamics. Here, we will focus on a very restricted class of models which allows for a straight-forward translation. Once a Boolean model is obtained from the multi-valued model, again the crucial question is in how far dynamical properties of the original model are preserved in the reduced model. We will show that even for the restricted class of multi-valued models having a natural Boolean representation, there is only a weak correspondence in their dynamic characteristics.

Our results are at this point mostly of theoretical interest. On the one hand, they provide ways to exploit certain regularities or symmetries in the models for reduction purposes. On the other hand, they show problems that can arise when utilizing reduction methods. Although the focus is on theoretical questions, these results also illustrate nicely the potential and possible relevance of mathematical research regarding the complexity of model reduction. In particular, they stress the importance of a mathematical investigation of property conservation for reduction methods. Furthermore, they motivate research in the different capabilities of Boolean and multi-

valued modeling. Only a good understanding of the latter will allow for suitable modeling choices in applications.

The paper is organized as follows. After a short introduction to the formalism of R. Thomas, we introduce the concept and application of extremal states in Section 3. The section is divided in two parts, one describing the representation of the state transition graph based on extremal states and one focusing on cycles in the state transition graph. In Section 4, we then investigate conservation of properties when transforming a multi-valued into a Boolean model. General observations concerning reachability and steady states are presented in a first subsection, followed by detailed results on trap cycles in a second subsection. We conclude the paper with final remarks and ideas for future work.

2 Preliminaries

We briefly recall the basic concepts of the Thomas formalism and introduce our notation. For additional information and background, we refer to [15, 16].

An *interaction graph* $I = (V, E, \varepsilon, \vartheta)$ is a directed graph (V, E) with a finite set of vertices V corresponding to regulatory components, and a set $E \subseteq V \times V$ of edges corresponding to regulatory interactions. The set E may contain loops of the form (u, u), but no multiple edges between the same pair of vertices. The function $\varepsilon : E \longrightarrow \{+, -\}$ specifies whether an interaction (u, v) is activating, $\varepsilon(u, v) = '+'$, or inhibiting, $\varepsilon(u, v) = '-'$. The function $\vartheta : E \longrightarrow \mathbb{N}$ defines for each $(u, v) \in E$ the minimum activity level $\vartheta(u, v) > 0$ of component u needed by (u, v) to become effective. Thus, with each component $u \in V$ we associate the set of possible values $X_u := \{0, \ldots, \max_u\}$, where $\max_u := \max\{\vartheta(u, v) \mid (u, v) \in E\}$. The *state space* X is given as the Cartesian product $X := X_I := \prod_{u \in V} X_u$.

For $u \in V$ we denote by $\mathrm{Pre}(u) := \{v \in V \mid (v, u) \in E\}$ the set of predecessors of u in the graph (V, E). The set of *resources* for u in a state $x \in X$ is defined as the set of predecessors of u that, in state x, have a present activating or an absent inhibiting effect on u:

$$\mathrm{Res}_I(u, x) := \{v \in \mathrm{Pre}(u) \mid (\varepsilon(v, u) = '+' \wedge x_v \geq \vartheta(v, u)) \vee (\varepsilon(v, u) = '-' \wedge x_v < \vartheta(v, u))\}.$$

A *model* $M = (I, K)$ is composed of an interaction graph I and *logical parameters* $K(u, \omega) \in \{0, \ldots, \max_u\}$, for all components $u \in V$ and resources $\omega \subseteq Pre(u)$. These parameters are used to define a function $\delta_M : X \times V \longrightarrow \{-1, 0, 1\}$ that indicates how a component $u \in V$ can change in a given state $x \in X$:

$$\delta_M(x, u) := \begin{cases} 1 & \text{if } x_u < K(u, \mathrm{Res}_I(u, x)), \\ 0 & \text{if } x_u = K(u, \mathrm{Res}_I(u, x)), \\ -1 & \text{if } x_u > K(u, \mathrm{Res}_I(u, x)). \end{cases}$$

The resulting dynamics of the model $M = (I, K)$ is described by the *asynchronous state transition graph* $T_M := (X, S_M)$, where

$$S_M := \bigcup_{x \in X} \{(x, x + \delta_M(x, u)e^u) \mid u \in V : \delta_M(x, u) \neq 0\}.$$

Here, e^u denotes the u-th unit vector in X.

Finally, for $x \in X$, the number of outgoing edges $\mathrm{outdeg}(x) := \#\{(x, y) \mid (x, y) \in S_M\} = \#\{u \in V \mid \delta_M(x, u) \neq 0\}$ is called the *outdegree* of x in T_M.

Example 1. On the left of Fig. 2, we see an example for an interaction graph with three components. Consider state $(0, 0, 0)$ in component 0. Even though the state values of components 1 and 2 are both below the threshold of the edges $(1, 0)$ and $(2, 0)$, the resources are $\mathrm{Res}_I(0, (0, 0, 0)) = \{1\}$, because the edge $(1, 0)$ has negative sign. Furthermore, in the example, $K(0, \mathrm{Res}_I(0, \{1\})) = 2$ and therefore $\delta_M((0, 0, 0), 0) = 1$. This results in an edge from $(0, 0, 0)$ to $(1, 0, 0)$ in the state transition graph, which can be seen in the center of Fig. 2.

3 Characterizing state transition graphs by their extremal states

3.1 Extremal states and rows

The state space, and consequently the state transition graph, is exponentially large in the number $|V| = d$ of network components and usually is a complex object. However, it turns out that for the type of models considered in this paper all information inherent in the state transition graph is already encoded in an easy to define subgraph. The key notion needed to state this more precisely is that of extremal states and rows.

Definition 1. *Let T_M be the state transition graph of a model M with state space X. Then a state $x = (x_u)_{u \in V} \in X$ is called extremal state, if $x_u \in \{0, max_u\}$ for all $u \in V$.*

A $(k+1)$-tuple $(x^0, \ldots, x^k) \in X^{k+1}$ is called u-row of X if $u \in V$, $k = max_u$, $x^0_u = 0$ and $x^l = x^0 + le^u$ for all $l \in \{1, \ldots, k\}$. The u-row is called extremal, if x^0 is an extremal state. Lastly, an (extremal) u-row of the graph T is the subgraph induced by an (extremal) u-row of X.

The extremal states can be visualized as the corners of the d-dimensional cuboid given by $\prod_{u \in V} X_u$ and the extremal rows as its edges, i.e., its 1-dimensional facets. Clearly, a u-row is extremal if it contains an extremal state. In the following, we will often talk about *extremal rows*, i.e., we do not always specify the index u.

In the Boolean case where $X_u = \{0, 1\}$, for all $u \in V$, every state is an extremal state. If the components of a model can take more than two values the extremal states are distinguished and, in general, not all states lie on extremal rows. However, the following theorem states that the extremal rows carry all the dynamical information inherent in a given model.

Theorem 1. *For any model $M = (I, K)$, the state transition graph T_M is uniquely determined by I and the extremal rows of T_M.*

In agreement with the observations above, the statement is not of interest in the case of Boolean networks, since in that case the subgraph induced by the extremal rows coincides with the state transition graph. However, in the multi-valued case, the more values are used to model the activity range of a network component, the larger the corresponding reduction. In the general case, it can be easily shown that there are $2^d + 2^{d-1} \sum_{u \in V} (|X_u| - 2)$ states lying on extremal rows, while the total number of states is $|X| = \prod_{u \in V} |X_u|$.

The statement can be rephrased in the following way. Given an interaction graph, knowing the behavior of a corresponding model on all extremal rows is enough to reconstruct the behavior of the model in the entire state space. The reason for that is that the resources of the components in any given state can be discovered by considering a suitable extremal state. In the following, we develop this argument in a few elementary lemmata and provide a description that allows us to immediately see the implications for the topology of the state transition graph. We will show that we can construct the full state transition graph by basically copying the extremal rows. For this purpose, we introduce the following notions based on graph isomorphism.

Definition 2. *Let $M = (I, K)$ be a model and T_M the corresponding state transition graph. Two u-rows $(x^0, \ldots, x^k)$ and $(y^0, \ldots, y^k)$ in T_M are called* isomorphic*, if $(x^i, x^j) \in S_M$ if and only if $(y^i, y^j) \in S_M$, for all $i, j \in \{0, \ldots, k\}$.*

It follows immediately from the definition of the state transition graph that two u-rows $(x^0, \ldots, x^k)$ and $(y^0, \ldots, y^k)$ are isomorphic if and only if $\delta_M(x^i, u) = \delta_M(y^i, u)$ for all $i \in \{0, \ldots, k\}$. Based on this observation, it is easy to see that we can formulate conditions for two rows to be isomorphic using the notion of resources.

In the remainder of this paper, we consider a model $M = (I, K)$ with corresponding state transition graph T. Clearly, in a u-row, there are at most two different sets of resources for u. If u does not influence itself, i.e., there is no loop (u, u) in the interaction graph, then the resources are the same for all states of the u-row. If there is such a loop then it follows from the definition of resources that all states of the u-row with the u-th entry below the loop threshold yield the same set of resources for u. The same is true for all states of the u-row with the u-th entry larger or equal to the threshold value. We summarize this observation in a lemma.

Lemma 1. *Let $(x^0, \ldots, x^k)$ be a u-row of X.*
If $(u, u) \notin E$, then $\mathrm{Res}(u, x^i) = \mathrm{Res}(u, x^0) = \mathrm{Res}(u, x^k)$ for all $i \in \{0, \ldots, k\}$.
If $(u, u) \in E$, then $\mathrm{Res}(u, x^i) = \mathrm{Res}(u, x^0)$ for all $i < \vartheta(u, u)$ and $\mathrm{Res}(u, x^i) = \mathrm{Res}(u, x^k)$ for all $i \geq \vartheta(u, u)$.

Using this lemma, it is now easy to describe the topology of all the rows in the state transition graph.

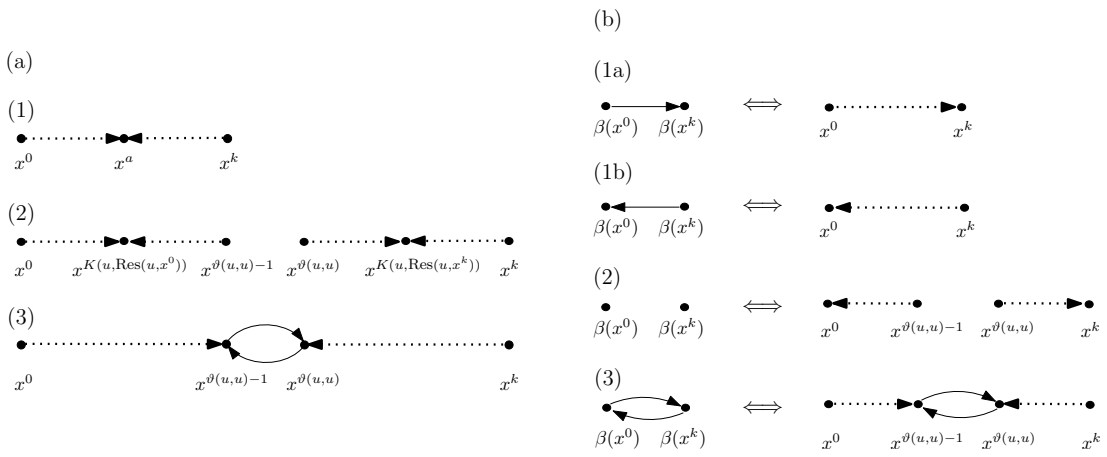

Fig. 1: On the left, (1)-(3) illustrate the possible topologies of rows in a state transition graph as given in Proposition 1. On the right, row topology and correspondence between multi-valued and Boolean models as stated in Proposition 2. Not all vertices of a row are displayed. Dotted arrows represent directed paths while solid arrows indicate edges.

Proposition 1. *For each u-row $(x^0, \ldots, x^k)$ of T exactly one of the following statements holds:*

(1) There exists $a \in \{0, \ldots, k\}$ such that (x^i, x^{i+1}) for $i \in \{0, \ldots, a-1\}$ and (x^i, x^{i-1}) for $i \in \{a+1, \ldots, k\}$ are the only edges in the u-row.
This is the case, if $(u,u) \notin E$ and $K(u, \mathrm{Res}(u, x^0)) = K(u, \mathrm{Res}(u, x^k)) = a$,
or if $(u,u) \in E$, $K(u, \mathrm{Res}(u, x^0)), K(u, \mathrm{Res}(u, x^k)) < \vartheta(u,u)$ and $K(u, \mathrm{Res}(u, x^0)) = a$,
or if $(u,u) \in E$, $K(u, \mathrm{Res}(u, x^0))$, $K(u, \mathrm{Res}(u, x^k)) \geq \vartheta(u,u)$ and $K(u, \mathrm{Res}(u, x^k)) = a$.

(2) There exists $a \in \{0, \ldots, k\}$ such that
(x^i, x^{i+1}) for $i \in \{0, \ldots, K(u, \mathrm{Res}\, u, x^0) - 1\} \cup \{a, \ldots, K(u, \mathrm{Res}\, u, x^k) - 1\}$ and
(x^i, x^{i-1}) for $i \in \{K(u, \mathrm{Res}\, u, x^0) + 1, \ldots, a-1\} \cup \{K(u, \mathrm{Res}\, u, x^k) + 1, \ldots, k\}$
are the only edges in the u-row.
This is the case, if $(u,u) \in E$, $a = \vartheta(u,u)$ and $K(u, \mathrm{Res}(u, x^0)) < \vartheta(u,u) \leq K(u, \mathrm{Res}(u, x^k))$.

(3) There exists $a \in \{0, \ldots, k\}$ such that (x^i, x^{i+1}) for $i \in \{0, \ldots, a-1\}$ and (x^i, x^{i-1}) for $i \in \{a, \ldots, k\}$ are the only edges in the u-row.
This is the case, if $(u,u) \in E$, $a = \vartheta(u,u)$ and $K(u, \mathrm{Res}(u, x^0)) \geq \vartheta(u,u) > K(u, \mathrm{Res}(u, x^k))$.

The proposition is illustrated in Figure 1(a), which shows the different types of row topology that correspond to the three cases listed. One can observe that the absence of a loop, and therefore the absence of autoregulation, at a component always leads to the first case, whereas the presence of a loop only leads to the first case, if the logical parameters of the row lie on the same side of the activation level of the loop. The second case is always associated with positive,

the third case with negative autoregulation, both of which result in a loop in the interaction graph.

The resources and corresponding parameter values determine the structure of the state transition graph. The next lemma translates a condition phrased in terms of resources into such structural knowledge.

Lemma 2. *Let $x, y \in X$ such that there exists a component $u \in V$ with*

$$\mathrm{Res}(u, x)\backslash\{u\} = \mathrm{Res}(u, y)\backslash\{u\}.$$

Then the u-row $(x^0, \ldots, x^k)$ containing x is isomorphic to the u-row $(y^0, \ldots, y^k)$ containing y.

Proof. It is sufficient to show that $K(u, \mathrm{Res}(u, x^i)) = K(u, \mathrm{Res}(u, y^i))$ for all $i \in \{0, \ldots, k\}$. Due to the fact that $x^i_u = i = y^i_u$, this implies $\delta(x^i, u) = \delta(y^i, u)$ for all $i \in \{0, \ldots, k\}$. We even show that $\mathrm{Res}(u, x^i) = \mathrm{Res}(u, y^i)$ for all $i \in \{0, \ldots, k\}$.

In case $u \notin \mathrm{Pre}(u)$, we have $\mathrm{Res}(u, x^i) = \mathrm{Res}(u, x^0)$ and $\mathrm{Res}(u, y^i) = \mathrm{Res}(u, y^0)$ for all $i \in \{0, \ldots, k\}$. In case $u \in \mathrm{Pre}(u)$, we have $\mathrm{Res}(u, x^i) = \mathrm{Res}(u, x^0), \mathrm{Res}(u, y^i) = \mathrm{Res}(u, y^0)$ for all $i < \vartheta(u, u)$, and $\mathrm{Res}(u, x^i) = \mathrm{Res}(u, x^0)\Delta\{u\}, \mathrm{Res}(u, y^i) = \mathrm{Res}(u, y^0)\Delta\{u\}$ for all $i \geq \vartheta(u, u)$, where Δ denotes the symmetric difference. Thus, we only need to show that $\mathrm{Res}(u, x^0) = \mathrm{Res}(u, y^0)$. The definition of the resources and the assumption of the lemma immediately yield that $\mathrm{Res}(u, x^0)\backslash\{u\} = \mathrm{Res}(u, x)\backslash\{u\} = \mathrm{Res}(u, y)\backslash\{u\} = \mathrm{Res}(u, y^0)\backslash\{u\}$. Furthermore, we have by definition $u \in \mathrm{Res}(u, x^0)$ if and only if $u \in \mathrm{Res}(u, y^0)$, which completes the proof.

Lastly, we observe in the following lemma that the extremal states already yield all possible sets of resources.

Lemma 3. *Let $u \in V$. Then for all $\omega \subseteq \mathrm{Pre}(u)$ there is an extremal state $x \in X$ such that $\mathrm{Res}(u, x) = \omega$.*

Proof. We define a state x by $x_v := 0$ for $v \notin \mathrm{Pre}(u)$, $x_v := 0$ for $v \notin \omega$ and $\varepsilon(v, u) = {}'+'$, $x_v := \max_v$ for $v \in \omega$ and $\varepsilon(v, u) = {}'+'$, $x_v := \max_v$ for $v \notin \omega$ and $\varepsilon(v, u) = {}'-'$, and $x_v := 0$ for $v \in \omega$ and $\varepsilon(v, u) = {}'-'$. Then x is an extremal state and $\mathrm{Res}(u, x) = \omega$.

Clearly, in general, the extremal state corresponding to ω is not unique. We can now prove the theorem.

Proof (of Theorem 1). Let $(x^0, \ldots, x^k)$ be an arbitrary u-row of T. Then there exists an extremal state y such that $\mathrm{Res}(u, y) = \mathrm{Res}(u, x^0)$ according to Lemma 3. It follows from Lemma 2 that the u-row containing y is isomorphic to $(x^0, \ldots, x^k)$. The u-row containing y is extremal since y is extremal. The assertion follows.

3.2 Extremal states in cycles

In the preceding section, we have seen that the extremal rows carry all information concerning the state transition graph. The underlying idea of exploiting the characteristics of extremal states can also be applied to substructures of the state transition graph, e.g., the attractors.

Definition 3. *A strongly connected component A of T is called attractor, if there is no path starting in A leaving A. If the cardinality of A is 1, A is called a stable state. If the cardinality of A is larger than one, then A is called cyclic attractor. If in addition* $\mathrm{outdeg}(a) = 1$ *for all $a \in A$, than A is called trap cycle.*

In the following we will focus on trap cycles in T. Again, we will see that important properties of a trap cycle are already encoded in states of the cycle that have certain extremal properties. We will make this statement more precise using the following terminology. Note that when talking about cycles, indices are always to be understood modulo the cycle length.

Definition 4. *A cycle $A = \langle a^1, \ldots, a^n \rangle$ is a path of length $n \geq 2$ in T such that $a^i \neq a^j$ for $i \neq j$ and $(a^n, a^1) \in S$. For all $i \in \{1, \ldots, n\}$, we call the unique $u^i \in V$ satisfying $a^{i+1} = a^i + \delta(a^i, u^i)e^{u^i}$ the direction of the cycle in a^i, where e^{u^i} is again the u^i-th unit vector. Furthermore, the corners $y^1, \ldots, y^m$, $m \leq n$, of the cycle are the states $y^j = a^{i_j}$, with $i_j < i_{j+1}$ for all $j < m$, which satisfy $u^{i_j - 1} \neq u^{i_j}$ or $\delta(a^{i_j - 1}, u^{i_j - 1}) \neq \delta(a^{i_j}, u^{i_j - 1})$.*

Basically, the corners of a cycle are the states where the path constituting the cycle changes direction on the grid of the state space. The condition $\delta(a^{i_j - 1}, u^{i_j - 1}) \neq \delta(a^{i_j}, u^{i_j - 1})$ can be removed when we are only interested in cycles of length greater than 2. We will see in the following proposition that in this case it is also a sufficient condition for a cycle state to be a corner.

A cycle is uniquely defined by its corners. Clearly, we can write $y^{j+1} = y^j + k \cdot \kappa e^{u_{i_j}}$ with $\kappa = \delta(y^j, u_{i_j}) \in \{-1, +1\}$ and suitable k. Then $a^{i_j + l} = y^j + l \cdot \kappa e^{u_{i_j}}$ for all $l \leq k$. In the next lemma, we summarize some easy observations.

Lemma 4. *For a cycle $A = \langle a^1, \ldots, a^n \rangle$ holds:*

(1) If $n = 2$, both elements of the cycle are corners and $u^1 = u^2$.
(2) If $n > 2$: $\forall\, i \in \{1, \ldots, n\} : \delta(a^{i-1}, u^{i-1}) \neq \delta(a^i, u^{i-1}) \Rightarrow u^{i-1} \neq u^i$
If A is a trap cycle, we even have

$$\forall\, i \in \{1, \ldots, n\} : u^{i-1} \neq u^i \Leftrightarrow \delta(a^{i-1}, u^{i-1}) \neq \delta(a^i, u^{i-1}).$$

Proof. If $n = 2$, then $a^2 = a^1 + \kappa e^u$ for some $u \in V$ and $\kappa \in \{-1, +1\}$ and the statement follows.

If $n > 2$ and $\delta(a^{i-1}, u^{i-1}) \neq \delta(a^i, u^{i-1})$, then the assumption $u^{i-1} = u^i$ immediately leads to the conclusion that $a^{i+1} = a^{i-1}$ contradicting A being a cycle. If A is a trap cycle and $u^i \neq u^{i+1}$, then $\delta(a^{i+1}, u^i) = 0 \neq \delta(a^i, u^i)$ since $\mathrm{outdeg}(a^i) = 1$.

A cycle consists of two states if and only if it has exactly two corners. If a cycle has more than two elements, it has at least four corners.

Again, the terminology and characterizations using corners is only interesting in the multi-valued case, since for a Boolean model all states of a cycle are corners.

The corners of a cycle have some extremal property since they constitute the endpoints of row segments belonging to the cycle. The cycle may at some point touch the corresponding row again, but only after leaving it by executing a change in direction. As long as the cycle proceeds along some u-row in the state transition graph, very similar conditions concerning resources and the values of the state transformation function hold for the cycle states. In the next lemma, we list some useful observations that follow easily from the basic definitions.

Lemma 5. *(1) Let $x, y \in X$ be in the same u-row. If $v \in V\backslash\{u\}$ and $\mathrm{Res}(v,x) = \mathrm{Res}(v,y)$, then $\delta(x,v) = \delta(y,v)$.*

(2) Let $x, y \in X$ with $y = x \pm e^u$. If $v \in V\backslash\{u\}$ and $\delta(x,v) \neq \delta(y,v)$, then $\vartheta(u,v) = max\{x_u, y_u\}$.

(3) Let $x, ..., x + i \cdot e^u \in X$ for some $i \in \mathbb{Z}$. If $v \in V\backslash\{u\}$ and $\delta(x,v) = \delta(x + i \cdot e^u, v) =: \kappa$, then $\delta(x + l \cdot e^u, v) = \kappa$ for all $l \in \{0, \ldots, i\}$.

With the understanding of the dynamical behavior in the rows of the state transition graph, our goal is now to determine whether a given cycle is a trap cycle simply by checking conditions pertaining the corners. For the remainder of this section, we consider a cycle $A = \langle a^1, \ldots, a^n \rangle$ with corners $y^1, \ldots, y^m$. Let $j \in \{1, \ldots, m\}$. For the corners y^j, y^{j+1} let $(x^0, \ldots, x^{max_u})$ be the u-row containing y^j and y^{j+1}, i.e., we find $i, k \in \{0, \ldots, max_u\}$ with $y^j = x^i$ and $y^{j+1} = x^k$.

Lemma 6. *If $i < k$, then $\delta(x^l, u) = 1$ for all $i \leq l < k$ and $\delta(x^k, v) \neq 0$ for some $v \in V$ with $x^k + \delta(x^k, v)e^v \neq x^{k+1}$ (in case $k < max_u$). If, in addition,* $\mathrm{outdeg}(x^i) = \mathrm{outdeg}(x^k) = 1$*, then holds:*

$$\mathrm{outdeg}(x^l) = 1 \ \forall i \leq l < k \quad \Longleftrightarrow \quad \vartheta(u,v) = k\,.$$

If $k < i$, then $\delta(x^l, u) = -1$ for all $i \geq l > k$ and $\delta(x^k, v) \neq 0$ for some $v \in V$ with $x^k + \delta(x^k, v)e^v \neq x^{k-1}$ (in case $k > 0$). If, in addition, $\mathrm{outdeg}(x^i) = \mathrm{outdeg}(x^k) = 1$*, then holds:*

$$\mathrm{outdeg}(x^l) = 1 \ \forall i \geq l > k \quad \Longleftrightarrow \quad \vartheta(u,v) = k + 1$$

Proof. Let $i < k$. Since A includes a path from x^i to x^k along the u-row, we have $\delta(x^l, u) = 1$ for all $i \leq l < k$. Since $x^k = y^{j+1}$ is a corner of the cycle, we know that there exists $v \in V$ with $x^k + \delta(x^k, v)e^v \neq x^{k+1}$. Let us now assume $\mathrm{outdeg}(x^i) = \mathrm{outdeg}(x^k) = 1$. Then $\delta(x^i, w) = \delta(x^k, w) = 0$ for $w \neq u, v$. Also, $\delta(x^l, w) = 0$ for all $w \neq u, v$, as follows from the third statement of Lemma 5. This implies that $\mathrm{outdeg}(x^l) \leq 2$.

From the properties of x^k, we can now infer that $\delta(x^k, u) \neq 1$. If $u = v$, then $\mathrm{outdeg}(x^l) = 1$ for all $i \leq l < k$ according to our preceding observation. Furthermore, $\vartheta(u,v) = k$ from the

second statement of Lemma 5, and the assertion follows. If $u \neq v$, then with the help of Lemma 5 we obtain for all $i \leq l < k$:

$$\delta(x^l, v) = \delta(x^i, v) = 0 \;\Leftrightarrow\; \mathrm{Res}(v, x^l) = \mathrm{Res}(v, x^i) \neq \mathrm{Res}(v, x^k) \;\Leftrightarrow\; \vartheta(u, v) \in \{l+1, \ldots, k\}$$

The statement of the lemma follows, since $\delta(x^l, w) = 0$ for all $w \neq u, v$.
The case $k < i$ can be proved similarly.

The lemma shows how the defining characteristic of a trap cycle, every state having outdegree one, can be verified using only the corners of the cycle and information about the thresholds of certain edges in the interaction graph. We use this result to give a characterization of trap cycles. Similar ideas have been used to find necessary conditions pertaining the interaction graph for the existence of trap cycles in [12].

Theorem 2. *Let $A = \langle a^1, \ldots, a^n \rangle$ be a cycle in T with corners $y^1 = a^{i_1}, \ldots, y^m = a^{i_m}$. A is a trap cycle if and only if for all $j \in \{1, \ldots, m\}$ both of the following conditions are true:*

(1) $(u^{i_{j-1}}, u^{i_j}) \in E$ *and*

$$y^j_{u^{i_{j-1}}} = \begin{cases} \vartheta(u^{i_{j-1}}, u^{i_j}), & \delta(y^{j-1}, u^{i_{j-1}}) = +1 \\ \vartheta(u^{i_{j-1}}, u^{i_j}) - 1, & \delta(y^{j-1}, u^{i_{j-1}}) = -1 \end{cases},$$

(2) $\mathrm{outdeg}(y^j) = 1$.

Proof. It is sufficient to show that, for every $j \in \{1, \ldots, m\}$, $\mathrm{outdeg}(a^{i_{j-1}+l}) = 1$ for all l satisfying $i_{j-1} + l \leq i_j$ if and only if the conditions (1) and (2) hold for j .

Let $j \in \{1, \ldots, m\}$ and set $u := u^{i_{j-1}}$ and $v := u^{i_j}$. From the definition of corners it follows that $u^{i_{j-1}+l} = u$ and $\delta(a^{i_{j-1}+l}, u) = \delta(a^{i_{j-1}}, u)$ for all l satisfying $i_{j-1} + l \leq i_j$.

If A is a trap cycle, then (2) holds by definition and (1) follows from Lemma 6. If (1) and (2) hold, in particular, for j and $j-1$, then Lemma 6 yields $\mathrm{outdeg}(a^{i_{j-1}+l}) = 1$ for all l satisfying $i_{j-1} + l \leq i_j$. This concludes the proof.

The statement is obvious for cycles of length 2. In general, the theorem illustrates that it is possible to analyze certain dynamical features of a model without exhaustively generating the corresponding subgraph of the state transition graph. Rather, we employ information gleaned from the interaction graph and the extremal states. Here, we infer a general statement about the outdegree of every cycle state by just looking at the corners of the cycle. For the moment, we still need to calculate the state transition graph to find the cycle. However, preliminary results indicate that we may extend these ideas to obtain criteria that allow the identification of trap cycles, or at least a set of possible trap cycles, directly from the underlying interaction graph.

4 Comparing multi-valued and Boolean models

In the preceding sections we introduced methods to reduce the complexity of the state transition graph analysis of a multi-valued model by identifying states and subgraphs that carry the pertinent information. Now, we study a reduction obtained by transforming a multi-valued into a Boolean model. We will focus on a class of models where such a transformation is straightforward. The analysis of the dynamical properties of two corresponding models will allow us to evaluate in how far the Boolean version can be used to analyze characteristics of the more complex multi-valued model. Again, our results rely on the analysis of extremal states.

With any multi-valued interaction graph $I = (V, E, \varepsilon, \vartheta)$ we can associate the Boolean interaction graph $I^b = (V, E, \varepsilon, \vartheta_b)$, where $\vartheta_b(e) = 1$, for all $e \in E$. Now we can compare a model $M = (I, K)$ with a Boolean model $M^b = (I^b, K^b)$, for suitable K^b. Since we are interested in finding close correspondences between the two models, we consider the case that the logical parameters of the multi-valued model M only take extremal values. This leads to a restricted dynamical behavior. Mostly, within every row, all transitions are included in paths that lead to the endpoints of the row, since the logical parameters dictate that components tend to extremal values. The only exception can be generated by a negative autoregulation, i.e., a loop in the interaction graph which induces cycles of length two within the corresponding rows. We will give a more precise description of the different possible structures of the extremal rows, and therefore according to Theorem 1 all rows, in Proposition 2.

The restriction of the parameter values also allows us to naturally associate a Boolean model to the multi-valued model. We choose K^b such that

$$K(u, \omega) = K^b(u, \omega) \cdot \max_u, \text{ for all } u \in V \text{ and } \omega \subseteq Pre(u)\,. \tag{A}$$

As we will see, even under this strong hypothesis, the relationship between the dynamics of the two models is not so easy to describe.

We start by associating the extremal states in X with corresponding states in X^b. Let $Y \subseteq X$ denote the set of extremal states in X. Then there is a bijective map $\beta : Y \longrightarrow X^b, x \longmapsto x^b$, with $x^b_v = 0$, if $x_v = 0$, and $x^b_v = 1$, if $x_v = max_v$. The states x and $\beta(x)$ are called *associated*. From the basic definitions in Section 2, we immediately obtain the following properties.

Lemma 7. *For all $u \in V$ and $x \in Y$, the following holds:*

(1) $\mathrm{Res}_I(u, x) = \mathrm{Res}_{I^b}(u, \beta(x))$
(2) $\delta_M(x, u) = \delta_{M^b}(\beta(x), u)$
(3) $\mathrm{outdeg}(x) = \mathrm{outdeg}(\beta(x))$

Our basic assumption (A) implies the following lower bound on $\mathrm{outdeg}(x)$, for an arbitrary state $x \in X$:

$$\mathrm{outdeg}(x) = \#\{v \in V \mid x_v \neq K(v, Res(v, x))\} \geq \#\{v \in V \mid x_v \notin \{0, max_v\}\}$$

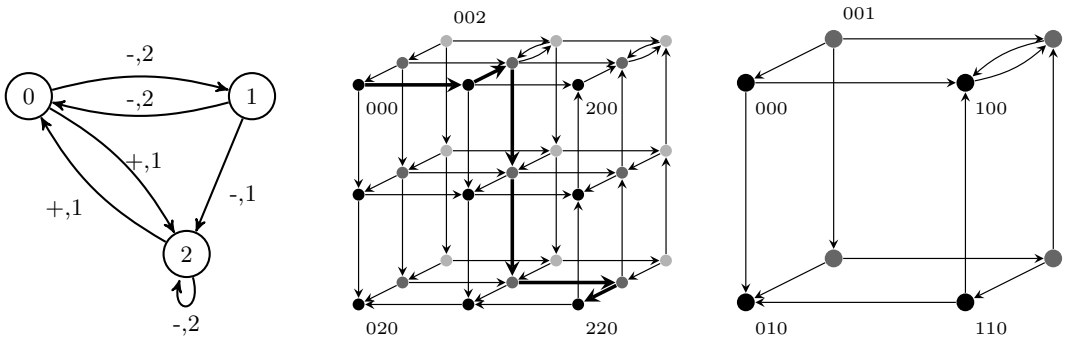

Fig. 2: On the left, an interaction graph, in the center the state transition graph for a corresponding model with parameters $K(0,\{1\}) = K(0,\{2\}) = K(0,\{1,2\}) = K(1,\{0\}) = K(2,\{0,1,2\}) = 2$, and $K(u,\omega) = 0$, otherwise, and on the right the state transition graph of the corresponding Boolean model. A path illustrating the different reachability properties as explained in the text is indicated by heavy edges.

4.1 Stable states and connectivity

Now it is easy to prove that the stable states in both models correspond to each other.

Theorem 3. *If x is a stable state in T, then x is extremal and the associated state $\beta(x)$ is stable in T^b. Conversely, if y is a stable state in T^b, the associated extremal state $\beta^{-1}(y)$ is stable in T.*

Proof. $\Rightarrow$: If $x \in X$ is stable in T, then $\mathrm{outdeg}(x) = 0$. Since $\mathrm{outdeg}(x) \geq \#\{v \in V \mid x_v \notin \{0, max_v\}\}$, we get that $x \in Y$ is extremal. Thus $\beta(x) \in X^b$ is well-defined. By Lemma 7, $\mathrm{outdeg}(\beta(x)) = \mathrm{outdeg}(x) = 0$, and so $\beta(x)$ is stable in T^b.

$\Leftarrow$: If $y \in X^b$ is stable in T^b, the associated state $\beta^{-1}(y) \in X$ is extremal in T and $\mathrm{outdeg}(\beta^{-1}(y)) = \mathrm{outdeg}(y) = 0$, which implies that $\beta^{-1}(y)$ is stable in T.

Next we ask whether reachability between two extremal states carries over to the associated states. First suppose that there is a path P in T connecting two extremal states x and $\tilde{x}$. As the following example shows, this does *not* imply that there is a path in T^b connecting the associated states $\beta(x)$ and $\beta(\tilde{x})$.

Example 2. Consider the model given in Figure 2 and the corresponding state transition graphs T and T^b also given in the figure. Although there is a path in T connecting 000 and 220, there is no path in T^b connecting the associated states 000 and 110.

Conversely, if $y, \tilde{y}$ are two states in X^b lying in the same strongly connected component of T^b, the associated states $\beta^{-1}(y),\ \beta^{-1}(\tilde{y})$ in X need not belong to the same strongly connected

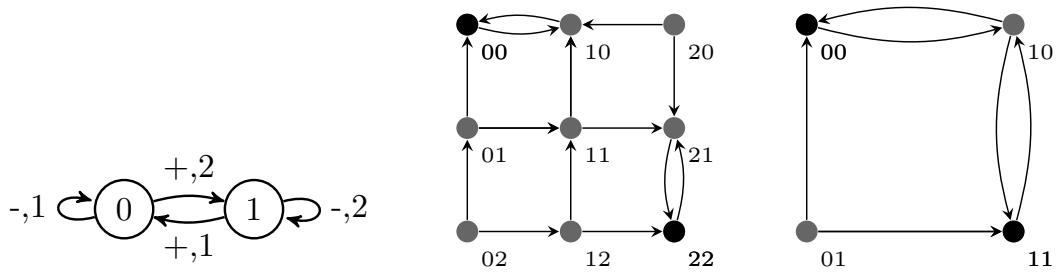

Fig. 3: On the left, an interaction graph, in the center the state transition graph for a corresponding model with parameters $K(0,\{0\}) = K(0,\{1\}) = K(0,\{0,1\}) = K(1,\{0,1\}) = 2$, and $K(u,\omega) = 0$, otherwise, and on the right the state transition graph of the corresponding Boolean model.

component of T. As the next example illustrates, we cannot even conclude that there is a path in T from $\beta^{-1}(y)$ to $\beta^{-1}(\tilde{y})$, or from $\beta^{-1}(\tilde{y})$ to $\beta^{-1}(y)$.

Example 3. Consider the model given in Figure 3 and the corresponding state transition graphs T and T^b also depicted in the figure. While the states 00 and 11 belong to the same strongly connected component of T^b, we have in T the trap cycles $(00, 10)$ and $(21, 22)$.

In Section 3, we have seen that the state transition graph T of a multi-valued model M is completely determined by the interaction graph I and the extremal rows of T. Hypothesis (A) restricts the structure of an extremal row in T which, in addition, can be derived from the structure of the associated row in T^b. More precisely, we get the following characterization based on Proposition 1 and Lemma 7:

Proposition 2. *Let M and M^b be a multi-valued resp. Boolean model satisfying hypothesis (A). For any extremal u-row $(x^0, \ldots, x^k)$ in T, exactly one of the following cases holds:*

(1a) *(x^i, x^{i+1}), for $i \in \{0, \ldots, k-1\}$, are the only edges in the u-row of T, and $(\beta(x^0), \beta(x^k))$ is the only edge in the associated u-row of T^b.*
This holds if $K(u, \mathrm{Res}(u, x^0)) = K(u, \mathrm{Res}(u, x^k)) = \max_u$.

(1b) *(x^i, x^{i-1}), for $i \in \{1, \ldots, k\}$, are the only edges in the u-row of T, and $(\beta(x^k), \beta(x^0))$ is the only edge in the associated u-row of T^b.*
This holds if $K(u, \mathrm{Res}(u, x^0)) = K(u, \mathrm{Res}(u, x^k)) = 0$.

(2) *There exists $a \in \{0, \ldots, k\}$ such that (x^i, x^{i+1}), for $i \in \{a, \ldots, k-1\}$, and (x^i, x^{i-1}), for $i \in \{0, \ldots, a-1\}$, are the only edges in the u-row of T, and there are no edges in the associated u-row of T^b.*
This holds if $(u,u) \in E$, $a = \vartheta(u,u)$ and $K(u, \mathrm{Res}(u, x^0)) = 0 < \vartheta(u,u) \leq K(u, \mathrm{Res}(u, x^k)) = \max_u$.

(3) There exists $a \in \{0,\dots,k\}$ such that (x^i, x^{i+1}), for $i \in \{0,\dots,a-1\}$, and (x^i, x^{i-1}), for $i \in \{a,\dots,k\}$, are the only edges in the u-row of T, and $(\beta(x^0), \beta(x^k))$, $(\beta(x^k), \beta(x^0))$ are both edges in the associated u-row of T^b. This holds if $(u,u) \in E$, $a = \vartheta(u,u)$ and $K(u, \mathrm{Res}(u, x^0)) = \max_u \geq \vartheta(u,u) > K(u, \mathrm{Res}(u, x^k)) = 0$.

Note that the conditions given in (1a) and (1b) do not depend on whether or not a loop (u,u) exists. This is due to the fact that this loop does not have any impact on the dynamical behavior within the row if the parameter equalities given in the conditions hold.

The possible row structures and correspondence between the Boolean and multi-valued case are illustrated in Figure 1(b).

4.2 Trap cycles

Among the cyclic attractors, trap cycles satisfy the most rigid conditions on their structure, since every state in a trap cycle has only one successor in the state transition graph. In this section we will investigate the question whether the strong conditions posed for trap cycles allow us to recover the trap cycles of the multi-valued model in the Boolean one.

The extremal properties of the parameters of the considered multi-valued model imply that if $\delta(x,v) = 0$ for some state $x \in X$ and $v \in V$, then $x_v = 0$ or $x_v = max_v$. Consequently, the states of a trap cycle have at most one component value which is not extremal, since they only have one successor. This is obviously also true in the Boolean case, since then every state is extremal. We rephrase this observation in the following lemma, for which we need to recall the notion of direction u^i of the cycle in a state a^i introduced in Definition 4.

Lemma 8. *Let $A := \langle a^1, \dots, a^n \rangle$ be a trap cycle in T. For all i, the states a^i and a^{i+1} are contained in an extremal row in direction u^i.*

In the following we will distinguish between trap cycles consisting of two and those including more states, since the two cases exhibit markedly different characteristics.

Trap cycles with two elements As we have seen, trap cycle states have to lie on extremal rows of the state transition graph in both the Boolean and the multi-valued model. Lemma 8 yields that a trap cycle with two elements has to lie on an extremal row. Due to Proposition 2, we know precisely the structure of this extremal row. This makes it easy to see that 2-cycles on extremal rows in the multi-valued model induce 2-cycles in the Boolean model.

Lemma 9. *Let $x \in X$ and $\langle x, x + e^u \rangle$ be a cycle with two elements on an extremal u-row in T. Then $\langle y, y + e^u \rangle$ is a cycle in T^b for y defined as follows:*

$$y_u = 0,$$
$$y_v = \begin{cases} 0 & \text{if } x_v = 0 \\ 1 & \text{if } x_v = \max_v \end{cases} \qquad \text{for all } v \in V \backslash \{u\}.$$

Furthermore, x and $x + e^u$ lie on the extremal row determined by $\beta^{-1}(y)$ and $\beta^{-1}(y + e^u)$.

Proof. It is sufficient to show that $\delta_{M^b}(y, u) = -\delta_{M^b}(y + e^u, u) = +1$ to prove that $\langle y, y + e^u\rangle$ is a cycle in T^b.

From Proposition 2, we know that $(u, u) \in E$ and $x_u = \vartheta(u, u) - 1$. It follows with the definition of y that $\mathrm{Res}_I(u, x+e^u) = \mathrm{Res}_I(u, x)\Delta\{u\} = \mathrm{Res}_{I^b}(u, y)\Delta\{u\} = \mathrm{Res}_{I^b}(u, y+e^u)$, where Δ again denotes the symmetric difference. Since $\delta_M(x, u) = +1$, we have $K(u, \mathrm{Res}(u, x)) > x_u$, and thus $K(u, \mathrm{Res}(u, x)) = \max_u$. It follows that $K^b(u, \mathrm{Res}(u, x)) = 1$ and therefore $\delta_{M^b}(y, u) = +1$. Similar reasoning proves $\delta_{M^b}(y + e^u, u) = -1$.

The additional statement follows from the definitions of β and y.

Since we are interested in trap cycles, we formulate a corollary based on the preceding two lemmata.

Corollary 1. *Let $\langle x, x + e^u\rangle$ be a trap cycle in T. Then $\langle y^1, y^2\rangle$ with $y^1 := \beta(x - x_u \cdot e^u)$ and $y^2 := \beta(x + (\max_u - x_u) \cdot e^u)$ defines a cycle in T^b.*

Thus, every trap cycle in the multi-valued model can be rediscovered as a cycle in the Boolean model. However, as the following example shows, the Boolean cycle need not be a trap cycle.

Example 4. Consider again the model given in Figure 3 and the corresponding state transition graphs T and T^b. In T, we see two trap cycles consisting of two elements, namely $\langle 00, 10\rangle$ and $\langle 21, 22\rangle$. The corresponding 2-cycles $\langle 00, 10\rangle$ and $\langle 10, 11\rangle$ in T^b are both no trap cycles.

Let us also consider the question whether all 2-cycles in T^b originate in cycles in T.

Lemma 10. *Let $\langle y, y + e^u\rangle$ be a cycle in T^b. Let $x \in X$ be defined as $x_u := \vartheta(u, u) - 1$ and $x_v := \max_v \cdot y_v$ for all $v \in V\backslash\{u\}$. Then the following two statements hold.*

(1) $\langle x, x + e^u\rangle$ is a cycle in T.
(2) If $\langle y, y + e^u\rangle$ is a trap cycle in T^b, then $\langle x, x + e^u\rangle$ is a trap cycle in T.

Proof. Since $\langle y, y + e^u\rangle$ is a cycle, $(u, u) \in E$ according to Proposition 2. Therefore, x is well defined, and x and $x + e^u$ lie by definition on the extremal row with endpoints $\beta^{-1}(y)$ and $\beta^{-1}(y + e^u)$.

To show (1), we need to prove $\delta(x, u) = -\delta(x + e^u, u) = +1$. Since $\delta(y, u) = -\delta(y + e^u, u) = 1$, we have $K^b(u, \mathrm{Res}(u, y)) = 1$ and $K^b(y + e^u, u) = 0$. From the definition of x it follows that $\mathrm{Res}(x, u) = \mathrm{Res}(y, u)$ and $x_u < \max_u = \max_u \cdot K^b(u, \mathrm{Res}(u, y)) = K(u, \mathrm{Res}(u, x))$. Consequently, $\delta(x, u) = +1$. Similar reasoning shows $\delta(x + e^u, u) = -1$.

To show (2), let us now additionally assume that $\langle y, y + e^u\rangle$ is a trap cycle, i.e., $\delta(y, v) = \delta(y+e^u, v) = 0$ for all $v \neq u$. Then by construction we have $\delta(\beta^{-1}(y), v) = \delta(\beta^{-1}(y+e^u), v) = 0$. Lemma 5 then yields $\delta(x, v) = 0 = \delta(x + e^u, v)$ and therefore $\langle x, x + e^u\rangle$ is a trap cycle in T.

We summarize the results of this section in the following theorem.

Theorem 4. *There is an injective map from the set of 2-state trap cycles in T to the set of 2-cycles in T^b. The image B of a trap cycle A in T lies in the row associated to the row containing A, but B need not be a trap cycle. Conversely, there is an injective map from the set of 2-state trap cycles in T^b to the set of 2-state trap cycles in T.*

The theorem shows that we generally cannot recover the trap cycles of the multi-valued system by focusing on the trap cycles of the Boolean model, at least if we consider the case of cycles consisting of two states.

Trap cycles comprising more than two states As in the preceding case, we can deduce certain extremal properties a trap cycle in T has to satisfy due to the extremal choice of parameters. Recall the definition of corners of a cycle given in Definition 4.

Lemma 11. *Let $A = \langle x^1, \dots x^n \rangle$ be a trap cycle in T with $n > 2$. Then all corners y^j of A are extremal states.*

Proof. Let $y^j = x^{i_j}$ be a corner and assume y^j is not an extremal state, i.e., there exists $u \in V$ with $y^j_u \notin \{0, \max_u\}$. Since the parameters are all extremal, we then have $\delta(y^j, u) \neq 0$. It follows that the direction $u^{i_j} = u$, since $\mathrm{outdeg}(y^j) = 1$. However, by definition, $y^j = x^{i_j - 1} + \delta(x^{i_j - 1}, u^{i_j - 1}) e^{u^{i_j - 1}}$ and $u^{i_j - 1} \neq u^{i_j}$. Thus, $x^{i_j - 1}_u = y^j_u \notin \{0, \max_u\}$ and consequently $\mathrm{outdeg}(x^{i_j - 1}) \geq 2$, which is a contradiction.

We can now easily show how to rediscover trap cycles of the multi-valued in the Boolean model. Recall that a cycle has more than two states if and only if it has more than two corners.

Lemma 12. *Let A be a trap cycle in T with the corners $y^1, \dots, y^m$ and $m > 2$. Then $A^b := \langle \beta(y^1), \dots, \beta(y^m) \rangle$ defines a trap cycle in T^b.*

Proof. First, we show that A^b is a cycle. For all j, we have $y^{j+1} = y^j + \delta(y^j, u^{i_j}) \cdot \max_{u^{i_j}} e^{u^{i_j}}$ according to Lemma 11. From the definition of β it follows easily that

$$\beta(y^{j+1}) = \beta(y^j + \delta(y^j, u^{i_j}) \cdot \max_{u^{i_j}} e^{u^{i_j}}) = \beta(y^j) + \delta(y^j, u^{i_j}) e^{u^{i_j}} = \beta(y^j) + \delta(\beta(y^j), u^{i_j}) e^{u^{i_j}}.$$

Thus, there is an edge between y^j and y^{j+1} for all $j \in \{1, \dots, m\}$, and A^b is a cycle. Since, according to the third statement in Lemma 7, $\mathrm{outdeg}(\beta(y^j)) = \mathrm{outdeg}(y^j) = 1$, we have that A^b is a trap cycle.

This result differs from the one for trap cycles with two states, where the trap set property of a trap cycle in the multi-valued model was not necessarily preserved when reduced to the Boolean representation. However, every 2-state trap cycle in the Boolean model corresponds to a trap cycle in the expanded model. Interestingly, this is not the case when considering larger trap cycles.

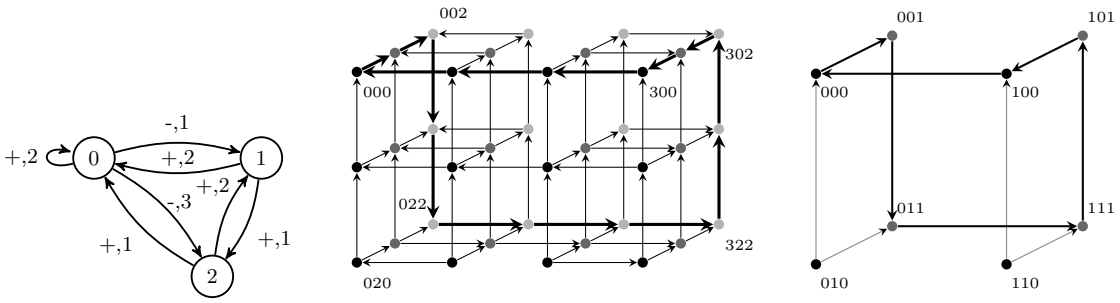

Fig. 4: On the left an interaction graph, in the center the state transition graph for a corresponding model with parameters $K(0,\{0,1\}) = K(0,\{0,2\}) = K(0,\{1,2\}) = K(0,\{0,1,2\}) = 3$, $K(1,\{0,2\}) = K(2,\{0\}) = K(2,\{1\}) = K(2,\{0,1\}) = 2$ and $K(u,\omega) = 0$, otherwise, and on the right the state transition graph of the corresponding Boolean model. Heavier edges indicate the cycles discussed in the text.

Lemma 13. *Let $A^b := \langle a^1, \dots a^n \rangle$ be a trap cycle in T^b and $n > 2$. Then, the states $y^j := \beta^{-1}(a^j)$ are the corners of a cycle A in T. Moreover, A is a trap cycle if and only if for all j and directions u^j*

$$(u^{j-1}, u^j) \in E \quad \textit{and} \quad \vartheta(u^{j-1}, u^j) = \begin{cases} 1, & \delta(y^{j-1}, u^{j-1}) = -1 \\ \max_{u^{j-1}}, & \delta(y^{j-1}, u^{j-1}) = +1 \end{cases} .$$

Proof. The first statement follows immediately from the correspondence between the row structures of the models as in Proposition 2.

Since every corner of A has outdegree 1 according to Lemma 7, the condition characterizing A as a trap cycle can be directly derived from Theorem 2.

Example 5. Consider the model given in Figure 4 and the corresponding state transition graphs T and T^b also depicted in the figure. In T, we see emphasized a cycle that corresponds to a trap cycle in T^b, indicated by heavier edges, which is not a trap cycle itself. Here, the cycle in T is still part of an attractor. In general, however, this need not be the case.

Let us again summarize our observations in a theorem.

Theorem 5. *There is an injective map from the set of trap cycles with more than 2 elements in T to the set of trap cycles in T^b. Conversely, there is an injective map from the set of trap cycles with more than two elements in T^b to the set of cycles in T, which however does not necessarily result in a trap cycle of T.*

The results of this section illustrate nicely that complexity reduction of a multi-valued model via Boolean representation is not necessarily a straight-forward method. Crucial characteristics

of the more complex model can be destroyed, and artefacts generated by the more restricted representation make analysis of the original model via the Boolean one precarious. A deeper understanding is needed to develop reduction techniques allowing for a controlled loss of information.

5 Conclusion

Without doubt, reduction methods play a major role in the study of large and complex biological systems. Not only do they allow for efficient analysis, but also they may be used to gain a deeper understanding of essential properties of a network by identifying objects, such as network modules or subsets of state space, carrying essential information on the network in condensed form.

In this paper, we showed that the dynamics of a model is to a good part already determined by the behavior in the extremal states. This is illustrated by a construction method for the state transition graph based only on the extremal rows and the information inherent in the interaction graph. The same concept holds for the analysis of cycles, where the distinguished states with extremal properties are the corners of the cycle. As mentioned before, the results are as yet more of theoretical interest. When describing the state transition graph by extremal rows, the reduction factor strongly depends on the size of the component value ranges. Most currently available models are either Boolean or only allow for a few values, which however may change in the future. Regardless of this, our results highlight ways to exploit certain regularities or symmetries inherent in a model for reduction purposes. In future work, we plan to build on these results by exploiting the information inherent in the interaction graph to identify interesting subgraphs in the state transition graph for further analysis. As a first step, we will try to construct a set of the only possible candidates for trap cycles in state space from the interaction graph. This approach will also be applicable to finding constraints on the possible parameter sets corresponding to a given interaction graph based on the existence of trap cycles.

The methods and results presented here only hold for the class of logical models derived from relatively simple interaction graphs. That is, we only consider interactions with a nonambiguous impact when crossing the unique threshold, which does not depend on the state of the system. This condition gives the models a certain kind of regularity that we exploit for our results. If we allow for a more general class of models, some of those properties will be lost. In the most extreme case, we would consider arbitrary update functions on the state space, which would destroy all symmetries in the corresponding state transition graph. However, it might be interesting to investigate whether we can find a useful way to describe the destruction of symmetries in the state transition graph with the relaxation of the model restrictions.

In the second part of the paper, we focused on reduction of multi-valued models by a Boolean representation. Our results mostly illustrate the difficulties of this approach. Although we consider a restricted class of multi-valued models highly suited for Boolean representation,

property conservation between the two models is very limited. A clearer understanding and classification of the network characteristics governing the inheritance of properties is needed to obtain meaningful results for such reduction methods.

A second aspect arising from these considerations concerns the modeling choices that need to be made in application. A Boolean representation is often chosen due to practical reasons such as model complexity or lack of information that would allow to distinguish between activity levels. A thorough understanding of the dynamical characteristics tied to the choice of value range would allow for sensible modeling decisions based both on efficiency and the best possible representation of the biological system.

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