Rational Shapes of the Local Volatility Surface

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Abstract

A robust implementation of a Dupire type local volatility model is an important issue for every option trading floor. In the present note we provide new analytic insights into the asymptotic behavior of local volatility in the wings. We present a general approximation formula and specialize it to the Heston model, showing that local variance is linear in the wings. This further justifies the choice of certain local volatility parametrizations.

1 Introduction

A robust implementation of a Dupire type local volatility model [13, 14] is an important issue for every equity option trading floor. Typically, this (inverse) problem is solved in a two step procedure : (i) a smooth parametrization of the implied volatility surface; (ii) computation of the local volatility based on the resulting call price surface. Point (i), and in particular how to extrapolate the implied volatility in extreme strike regimes, is widely recognized as important risk management issue. To our knowledge, this was first discussed in the Quant Congress 2000 presentation *Rational Shapes of the* [implied] *Volatility Surface* [19]. In the context of the Heston stochastic volatility model [23], it is seen that implied volatility squared (in short: *implied variance*) grows (asymptotically) linearly in log-strike. This and related matters were then studied by numerous authors, starting with Lee [24], see also [17] and the references therein. Subsequently, this has inspired parametrizations of the implied volatility surface, notably the so-called *SVI*, short for *stochastic-volatility inspired*, parametrization, cf. [18],[20].

In the present note we deal with point (ii), aiming at an understanding of typical shapes of the local volatility surface, by giving novel insights into the behavior of local volatility at extreme strikes. At the heart of our discussion lies a novel saddle point-based formula for local volatility, which can be used in a wide range of maturities and strikes. When applied to the Heston model, it reveals that *local variance* is – similar to implied variance – asymptotically linear in log-strike. As an immediate application, this provides a justification to use SVI-type parametrization also for the local variance surface.

Understanding the local volatility surface in extreme regimes, with robust numerical algorithms, can play an important role in model risk management. Indeed, assume one wants to quantify the model risk attached to a given path-dependent option, subject to consistency with today's option

[*]supported by MATHEON
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Maturity	Log spot, 10^3 MC paths	
	Minimum	Maximum
0.25	-0.71	0.36
1	-1.52	0.59
5	-4.13	1.14

Table 1: Minimum and maximum values of $\log(S_t/S_0)$ attained by a Monte Carlo (MC) simulation under Heston local variance (2.4), with 12 time steps per year and Heston parameters as in Figure 3. Even with only 10^3 trajectories, values are reached for which computing local variance by Dupire's formula becomes numerically challenging.

prices (i.e. the implied volatility surface). A simple, universally applicable, test of model risk is to compute the price under both stochastic and local volatility. Of course, it is assumed that both models are calibrated to the same market. (In fact, in judging model risk one may take a calibrated stochastic volatility model, say Heston, as reference here and construct the matching local volatilities by Dupire's formula, details of which are recalled below.) If we get the same price (typical example: variance swap) this points to a low model risk; a high difference however (typical example: volatility swap) indicates a high sensitivity of the option to the choice of model for the underlying price process. (These examples are discussed in detail in [18].) Relatedly, Reghai [28] suggests to define a "Toxicity Index" as

$$I = \frac{|SV - LV|}{|SV + LV|},$$

where SV and LV are the prices of the exotic under the stochastic volatility and the local volatility model; a value of I away from 0 is a warning flag for a particularly 'toxic' (i.e. highly model risk-sensitive) product, with all its consequences for hedging decisions.

The trouble with this simple test is that the local volatility in Heston (or other stochastic volatility models) is not explicitly known. Typically, one would then use either *Dupire's formula*[1] ("$\sigma^2_{\mathrm{loc}}(K,T) = 2\partial_T C / K^2 \partial_{KK} C$", with Fourier pricing of the respective derivatives of the call prices), or Monte Carlo-based computation of local variances as *conditional expectations* ("$\sigma^2_{\mathrm{loc}}(K,T) = \mathbb{E}[\sigma^2_{\mathrm{stoch}}\,(T)|S_T = K]$"). Clearly then, if the price process under local volatility makes a large excursion, local volatility from Dupire's formula cannot be trusted since Fourier pricing is known to become quickly unstable for extreme strikes (the remedy is a careful contour shift; see Figures 1 and 2); similarly, conditioning on very unlikely events of the form $\{S_T \in [K, K + \delta K]\}$ when $K \gg S_0$ is numerically difficult (unless one uses importance sampling or Malliavin techniques [22]); related numerical problems when computing local volatility by Monte Carlo simulation also arise in the Carmona and Nadtochyi formula in the Heston model [5, p. 20].)

These numerical instabilities can be overcome with our extrapolation analytics: the typical result for stochastic volatility reads $\sigma^2_{\mathrm{loc}}(K,T) \sim (const) \log K$ (cf. Theorem 1 and also formula (1.7) for Heston, comment 6 below for the Stein–Stein model). The resulting method is straight-forward patching: fix a reasonable region I in the (K,T) plane on which local volatility can be (reliably) computed with either of the above "classical" methods; for a strike-maturity point outside the above region, we use our approximate formula (the constant can be fully expressed in terms of the Heston

[1] Throughout, we work under the appropriate forward measure to avoid drift terms.

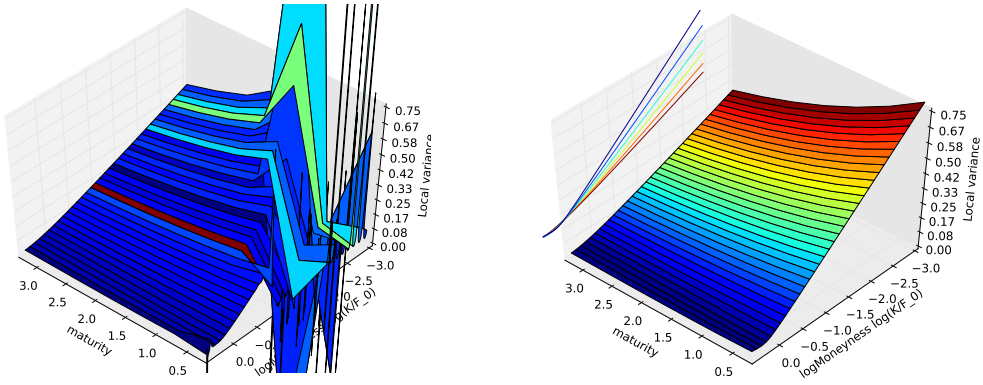


Figure 1: Local variance for Heston model computed with Dupire's formula (left wing). Left hand surface: call price derivatives computed via 1D integration of Heston characteristic function on a fixed integration contour ($0.5 + i\mathbb{R}$ here). Right hand surface: computed on adaptive contour with shift into saddle point $\hat{s}(k,T) + i\mathbb{R}$ (cf. (1.8) below).

model parameters, see Theorem 1; the semi-explicit formula (1.7) is to be preferred for accuracy). This yields a parametric, globally defined local volatility surface which then serves as the basis of a reliable Monte Carlo simulation for option prices under local volatility.

Let us introduce in some details the mathematical ingredients necessary for the discussion to come. Dupire's formula [13, 14]

$$\sigma^2_{\mathrm{loc}}(K,T) = \frac{2\partial_T C}{K^2 \partial_{KK} C} \tag{1.1}$$

implies that any arbitrage free call price surface

$$C = C(K,T) = C_{\mathrm{BS}}(K,T;\sigma_{\mathrm{BS}}(K,T))$$

which arises from a (not necessarily Markovian) Itô diffusion is obtained from the one-factor ("Dupire's local vol") model

$$dS_t/S_t = \sigma_{\mathrm{loc}}(S_t,t)dW_t. \tag{1.2}$$

(Note that spot remains fixed in the present discussion.) Local volatility can be thought of as a Markovian projection, term coined in [27], of a higher dimensional model (e.g. Heston); the first component then forms an Itô diffusion of the form

$$dS_t/S_t = \sigma_{\mathrm{stoch}}(t,\omega)dW_t.$$

Indeed, it is known (see e.g. [18] and the references therein) that

$$\sigma^2_{\mathrm{loc}}(K,T) = \mathbb{E}[\sigma_{\mathrm{stoch}}\ (T)^2 | S_T = K].$$

We emphasize (again!) that even for stochastic volatility models with fully explicit Markovian specification, sampling from the corresponding local volatility models requires substantial computational effort.

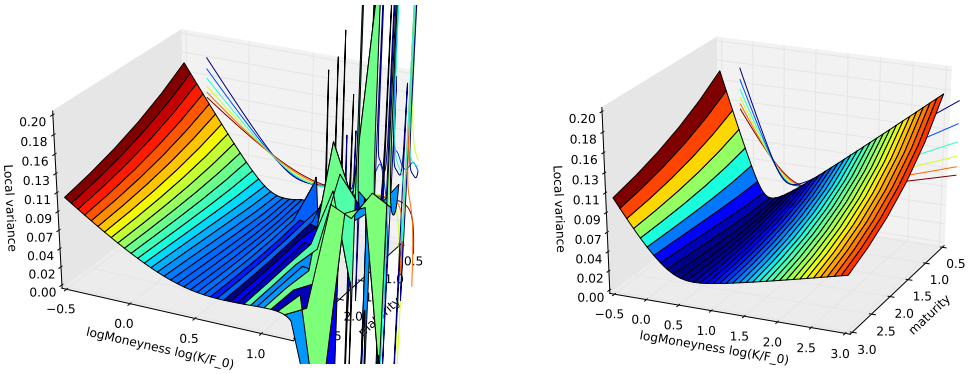


Figure 2: As in Figure 1, now right wing.

The analysis of implied, local, and stochastic volatility and their interplay has been subject of countless works; a very small selection relevant to the present discussion is [1, 2, 4, 16, 21, 24]. Our key contribution here is formula (1.7) below which allows for approximation of $\sigma^2_{\mathrm{loc}}(K,T)$ when K is large (and similarly, K is small). The main ingredient to this formula is a known moment generating function (mgf) of the log-price (X_t) (under the pricing measure),

$$M(s,T) := \exp(m(s,T)) := \mathbb{E}\exp(sX_T),$$

assumed to be finite in some (maximal) interval $(s_-(T), s_+(T))$ with critical exponents s_- and s_+ defined as

$$s_-(T) := \inf\{s : M(s,T) < \infty\}, \qquad s_+(T) := \sup\{s : M(s,T) < \infty\}.$$

We also assume that call prices have sufficient regularity to make Dupire's formula (1.1) well-defined, and that the mgf blows up at the upper critical moment:

$$\lim_{s\uparrow s_+(T)} M(s,T) = \infty. \tag{1.3}$$

This holds, e.g., in the Heston model [23], with log-price $X_t = \log(S_t/S_0)$, where

$$\begin{aligned} dS_t &= S_t\sqrt{V_t}dW_t, \qquad S_0 = s_0 > 0,\\ dV_t &= (a+bV_t)dt + c\sqrt{V_t}dB_t, \qquad V_0 = v_0 > 0, \end{aligned}$$

with $a \geq 0$, $b \leq 0$, $c > 0$, $d\langle W, B\rangle_t = \rho dt$, and we shall assume $\rho \leq 0$, which is typical in equity markets. Before moving on to a more general discussion, we formulate an asymptotic formula for local volatility in the Heston model.

Theorem 1. *For Heston, the following asymptotic behavior of local variance holds:*[2]

$$\lim_{k\to\infty} \frac{\sigma^2_{\mathrm{loc}}(k,T)}{k} = \frac{2}{s_+(s_+-1)R_1/R_2}, \tag{1.4}$$

[2]By a common abuse of notation, we write $\sigma^2_{\mathrm{loc}}(k,T)$ instead of $\sigma^2_{\mathrm{loc}}(e^k,T)$ when we wish to express the local vol as a function of log-strike k.

where $k = \log(K/S_0)$, $s_+ \equiv s_+(T)$ *and*

$$\begin{aligned} R_1 &= Tc^2 s_+(s_+-1)\left[c^2(2s_+-1) - 2\rho c(s_+\rho c + b)\right] \\ &\quad - 2(s_+\rho c + b)\left[c^2(2s_+-1) - 2\rho c(s_+\rho c + b)\right] \\ &\quad + 4\rho c\left[c^2 s_+(s_+-1) - (s_+\rho c + b)^2\right], \end{aligned} \tag{1.5}$$

$$R_2 = 2c^2 s_+(s_+-1)\left[c^2 s_+(s_+-1) - (s_+\rho c + b)^2\right]. \tag{1.6}$$

As will be discussed in Section 2, this result originates in the saddle point based approximation formula

$$\sigma^2_{\text{loc}}(k,T) \approx \left.\frac{2\frac{\partial}{\partial T}m(s,T)}{s(s-1)}\right|_{s=\hat{s}(k,T)} \tag{1.7}$$

where $\hat{s} = \hat{s}(k,T)$ is determined as solution of the saddle point equation

$$\frac{\partial}{\partial s}m(s,T) = k. \tag{1.8}$$

As such, formula (1.7) is clearly not restricted to the Heston model. Consider for instance

Example 2 (Time-dependent Black-Scholes)**.** *Assume risk-neutral dynamics of the form*

$$dS_t = S_t\sqrt{v(t)}dW_t.$$

We find $m(s,T) = \frac{1}{2}s(s-1)\int_0^T v(t)dt$*, and then, correctly,* $\sigma^2_{\text{loc}}(k,T) = v(T)$*. (In this example the evaluation of* $\hat{s}(k,T)$ *plays no role, since the fraction on the right hand side of* (1.7) *does not depend on it.)*

We expect our approximation formula (1.7) to work whenever (1.3) holds (also assuming that call prices are smooth enough to make (1.1) a well-defined quantity). The essence of the argument is given in Section 2. A rigorous proof in the Heston case requires some careful tail-estimates, similar to [16], and is omitted (details are found in [15]). The asymptotic equivalence of (1.4) and (1.7) is shown in Section 3.

Some additional comments are in order.

1. Equation (1.8) is solvable for large k, since (1.3) implies $\lim_{s\uparrow s_+}\partial/\partial s\, m(s,T) = \infty$.

2. The solution $\hat{s}(k,T)$ to the saddle point equation (1.8) can itself be used to stabilize the numerical evaluation Dupire's formula in models with know mgf (see (2.4) below): by shifting the integration contour into the saddle point $\hat{s}(k,T) + i\mathbb{R}$, the integrands in (2.4) will be highly concentrated, and the performances of any numerical 1D quadrature will be strongly enhanced (see Figures 1 and 2, where we are using adaptive Gauss-Lobatto after shifting the integration contour).

3. We have $\hat{s}(k,T) \uparrow s_+(T)$ as $k \to \infty$; hence, in models with moment explosion [1, 24], where $s_+(T) < \infty$, the denominator in (1.7) may be replaced by $s_+(T)(s_+(T)-1)$. While this is correct to first order, it is often preferable to use (1.7) as it is, and to calculate $\hat{s}(k,T)$ by (numerically) solving (1.8). The accuracy of the resulting approximation is illustrated in Section 3 for the Heston model.

4. There is a version of our approximation formula (1.7) for *small* values of K (i.e. $K \downarrow 0$, or $k \downarrow -\infty$), which requires that the mgf blows up at the lower critical moment $s_-(T)$. If $k < 0$ and $|k|$ is large, equation (1.8) has a unique solution $\hat{s}_-(k,T) < 0$. Then the approximation (1.7) holds, if $\hat{s}$ is replaced by $\hat{s}_-$.

5. (Jump models). There are extensions of Dupire's work to jump diffusions and also pure jump models; see e.g. [6] ("local Lévy models"), [3] and the references therein. In particular, Dupire's formula (which may be written as a PDE) becomes a PIDE which features an integral term involving the second derivative of C w.r.t. strike, times a kernel depending on K, integrated against all strikes in $(0,\infty)$. Another difficulty in the jump setting is the potential lack of immediate smoothing. For instance, the variance gamma model satisfies the above PIDE only in viscosity sense; in fact, call prices in the Variance gamma model may not be twice differentiable in K for small times, as was noted in [9]. It is a general remark, then, that in a general jump setting, Dupire's formula as stated in (1.1), may be ill-defined. If call prices are smooth enough to make (1.1), and the resulting local volatility process (1.2) well-defined, the latter may still fail to recreate the correct marginals of the original price process. May that be as it is, *it is industry standard to use Dupire's formula* (1.1)*, after some smoothing of the call prices seen in the market.* Since jumps cannot be ruled out from market data, this leads, unsurprisingly, to local volatility surfaces which blow up at the short end. (Ad-hoc fixes - such as freezing the vol surface some distance away from the short end - are being reported from practitioners.)

   In this spirit, it is interesting to see what formula (1.7) gives when applied to jump models. The situation is particularly simple in exponential Lévy models, which have the property that $m(s,T)$ is linear in T; thus, the numerator in (1.7) may be replaced by $2m(s,1)$. According to type of singularity of the mgf at the critical moment, the local volatility can display different behaviors. It is shown in [15] that the local variance $\sigma^2_{\text{loc}}(k,T)$ is asymptotic to $c_T \times k^{1/2}$ in the double exponential Lévy model [8] (and the constant c_T exhibits a $1/\sqrt{T}$-blowup as $T \downarrow 0$), while it has logarithmic $\log(k/T)$ wings in the Variance gamma model [26] (when T is large enough). The quality of the fit obtained using (1.7), as well as the techniques that can be used to handle the cases of mgfs with no or slow blow-up, are also discussed in [15].

6. (Stochastic volatility models other than Heston). The linear asymptotic behavior in (1.4) is likely to hold in large classes of stochastic volatility models. Relying on scaling properties and on large deviation principles, it is shown in [11] that also in the correlated Stein-Stein [30] (or Schöbel-Zhu [29]) model

$$\begin{aligned} dX_t &= -\frac{1}{2}\sigma_t^2 dt + \sigma_t dW_t, \qquad X_0 = 0, \\ d\sigma_t &= (a + b\sigma_t)dt + c dZ_t, \qquad \sigma_0 > 0, \end{aligned}$$

   the local variance is linear in the 'wings'. To see this, set $\varepsilon^2 = 1/k$, and rescale the Stein-Stein variables as $X_t^\varepsilon = \varepsilon^2 X_t, \sigma_t^\varepsilon = \varepsilon\,\sigma_t$. Then,

$$\begin{aligned} \sigma^2_{\text{loc}}(k,T) = \mathbb{E}\left[\sigma_T^2 | X_T = k\right] &= \varepsilon^{-2}\mathbb{E}\left[(\sigma_T^\varepsilon)^2 | X_T^\varepsilon = \varepsilon^2 k\right] \\ &= k\mathbb{E}\left[(\sigma_T^\varepsilon)^2 | X_T^\varepsilon = 1\right]. \end{aligned}$$

Hence, computing $\lim_{k\to\infty}\sigma^2_{\rm loc}(k,T)/k$ is equivalent to evaluating the asymptotics $\lim_{\varepsilon\to 0}\sigma^2_{{\rm loc},\varepsilon}(1,T)$, where $\sigma_{{\rm loc},\varepsilon}$ is the local volatility in the rescaled problem. On the other hand, $(X^\varepsilon,\sigma^\varepsilon)$ was seen in [12] to satisfy an LDP so that, under the above conditioning, $(X^\varepsilon,\sigma^\varepsilon)$ will center around the least-energy path arriving at the target subspace $(1,\cdot)$ at time T, say at the point $(1,\sigma^*_T)$.[3] Then

$$\lim_{k\to\infty}\frac{\sigma^2_{\rm loc}(k,T)}{k}=(\sigma^*_T)^2.$$

The *Pontryagin maximum principle* leads to (first order optimality) conditions (*Hamiltonian ODEs*, subject to suitable terminal and transversality conditions). Remarkably enough, these equations can be fully solved, see [12], and so give σ^*_T, and hence the asymptotic slope of local variance in the Stein–Stein model explicitly in terms of the model parameters.

2 Saddle point asymptotics

As is well known [7, 25], we can recover the call price C and the probability density $D(\cdot,T)$ of S_T by Laplace-Fourier inversion from the mgf:

$$C(K,T)=S_0-\frac{K}{2}+\frac{e^k}{2i\pi}\int_{-i\infty}^{i\infty}e^{-ks}\frac{M(s,T)}{s(s-1)}ds, \tag{2.1}$$

$$D(x,T)=\frac{1}{2i\pi}\int_{-i\infty}^{i\infty}e^{-(s+1)\log x}M(s,T)ds, \tag{2.2}$$

where the integration runs on the imaginary axis $i\mathbb{R}$. Now differentiate the call price under the integral sign:

$$\partial_T C(K,T)=\frac{e^k}{2i\pi}\int_{-i\infty}^{i\infty}\frac{\partial_T m(s,T)}{s(s-1)}e^{-ks}M(s,T)ds. \tag{2.3}$$

By Dupire's formula, we have

$$\sigma^2_{\rm loc}(k,T)=\frac{2\partial_T C(K)}{K^2D(K,T)}=\frac{2\int_{-i\infty}^{i\infty}\frac{\partial_T m(s,T)}{s(s-1)}e^{-ks}M(s,T)ds}{\int_{-i\infty}^{i\infty}e^{-ks}M(s,T)ds}. \tag{2.4}$$

Both integrands in (2.4) have a singularity at $s=s_+$, since $M(s,T)$ gets infinite there. The singular behavior of $M(s,T)$ dominates the asymptotics of both integrals. The resulting asymptotic factor cancels, and only the contribution of $2\frac{\partial_T m(s,T)}{s(s-1)}$ remains. This is the idea behind (1.7).

To implement it, we analyze both integrals in (2.4) by a saddle point approximation [10]. If M features an exponential blow-up at the critical moment s_+, its validity can be justified rather universally. Examples include the Heston model, double exponential Lévy, and Black-Scholes. (Note that the critical moment is $s_+=\infty$ for Black-Scholes.) If the saddle point method is not applicable (because of insufficient blow-up), different arguments are required; see Section 3 in [15].

So let us proceed with the saddle point analysis of (2.4). For both integrals, we only use the factor $e^{-ks}M(s)$ to find the location of the (approximate) saddle point $\hat{s}=\hat{s}(k,T)$. The saddle point equation is (1.8), obtained by equating the derivative of $e^{-ks}M(s)$ to zero. We move the

[3] This argument is made rigorous in [11].

integration contour through the saddle point. Then, for large k, only a small part $|\Im(s)| \leq h(k)$ of the contour, around the saddle point, matters asymptotically. (The choice of the function h depends on the singular expansion of M.) The integral can be approximated via a local expansion of the integrand. Let us carry this out for the denominator of (2.4). (In the following formulas we write m'' for $\partial^2 m/\partial s^2$.)

$$\begin{aligned}\int_{\hat{s}-i\infty}^{\hat{s}+i\infty} e^{-ks} M(s,T)ds &\sim \int_{\hat{s}-ih(k)}^{\hat{s}+ih(k)} e^{-ks} M(s,T)ds \\ &\sim \int_{\hat{s}-ih(k)}^{\hat{s}+ih(k)} \exp\left(-ks + m(\hat{s},T) + k(s-\hat{s}) + \tfrac{1}{2}m''(\hat{s},T)(s-\hat{s})^2\right) ds \\ &= e^{m(\hat{s},T)-k\hat{s}} \int_{\hat{s}-ih(k)}^{\hat{s}+ih(k)} \exp\left(\tfrac{1}{2}m''(\hat{s},T)(s-\hat{s})^2\right) ds. \end{aligned} \tag{2.5}$$

In the Taylor expansion of the exponent we have used the equation $m'(\hat{s},T) = k$. Now the crucial observation is that the numerator of (2.4) admits a similar approximation, where the only new ingredient is the factor $2\frac{\partial_T m(s,T)}{s(s-1)}$:

$$\begin{aligned} &2\int_{-i\infty}^{i\infty} \frac{\partial_T m(s,T)}{s(s-1)} e^{-ks} M(s,T)ds \\ &\sim 2\int_{\hat{s}-ih(k)}^{\hat{s}+ih(k)} \frac{\partial_T m(s,T)}{s(s-1)} e^{-ks} M(s,T)ds \\ &\sim 2e^{m(\hat{s},T)-k\hat{s}} \int_{\hat{s}-ih(k)}^{\hat{s}+ih(k)} \frac{\partial_T m(\hat{s},T)}{\hat{s}(\hat{s}-1)} (1+o(1)) \exp\left(\tfrac{1}{2}m''(\hat{s},T)(s-\hat{s})^2\right) ds \\ &\sim 2\frac{\partial_T m(\hat{s},T)}{\hat{s}(\hat{s}-1)} e^{m(\hat{s},T)-k\hat{s}} \int_{\hat{s}-ih(k)}^{\hat{s}+ih(k)} \exp\left(\tfrac{1}{2}m''(\hat{s},T)(s-\hat{s})^2\right) ds. \end{aligned} \tag{2.6}$$

Dividing (2.6) by (2.5) concludes the derivation. Summarizing, we note that the asymptotics of $\sigma^2_{\mathrm{loc}}(k)$ are governed by the local expansions at $s = \hat{s}$ of the integrands in (2.4). The respective first terms of both expansions agree, and thus cancel, except for the factor (1.7).

3 Local volatility at extreme strikes in the Heston model

In this section we focus on the Heston model, testing our approximation formula and asymptotic result and explaining how (1.4) is obtained by specializing (1.7).

Figure 3 compares the local variance in the Heston model and its saddle-point based approximation (1.7). It is seen that formula (1.7) provides a robust approximation of local volatility in a wide range of strikes, in particular for short maturities. For some visual comparison, Figure 4 shows the boundaries of the region in the (k,T) plane where the relative accuracy of the approximation becomes better than a fixed threshold, here 5%, that is to say

$$\frac{|\sigma^2_{\mathrm{loc}}(k,T) - \frac{\partial_T m(s,T)}{s(s-1)}|_{s=\hat{s}(k,T)}|}{\sigma^2_{\mathrm{loc}}(k,T)} < 5\%$$

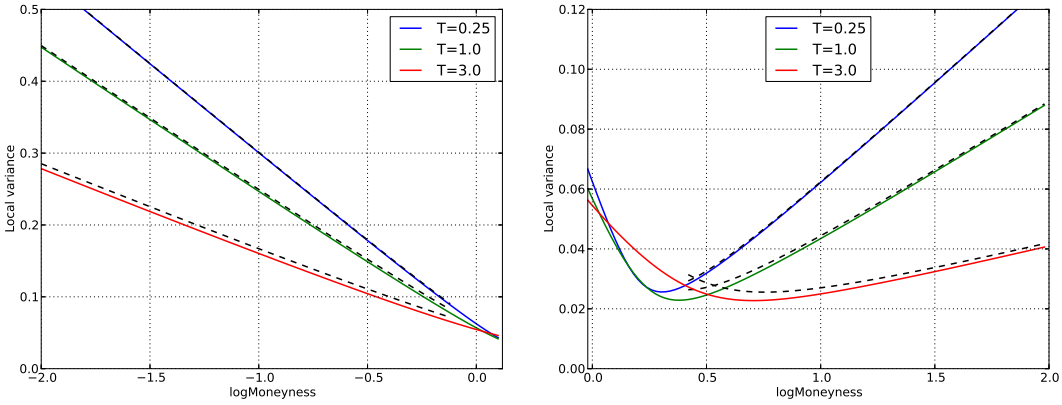


Figure 3: Local variance $\sigma^2_{\text{loc}}(k,T)$ (left and right wing) in the Heston model with parameters $a = 0.0428937$, $b = -0.6067$, $c = 0.2928$, $s_0 = 1$, $v_0 = 0.0654$, $\rho = -0.7571$. The solid lines show the local variance computed with Dupire's formula and adaptive integration contour as in Figures 1 and 2. The approximation (1.7) is dashed.

in the outer region. Maturities range down to $T = 0.25$. Note that the right hand side of (1.7) can be easily evaluated numerically, by using the explicit expression [23] of the Heston mgf in (1.8). The evaluation of the saddle point $\hat{s}(k,T)$ requires the inversion of the derivative $\partial_s m(s,T)$ (again explicit) in (1.8): this root-finding can be performed by simple bisection or by Newton-Raphson, using the approximate expression of the saddle point in (3.5) as an initial guess. The involved number of iterations (Table 2) stays bounded – and small – across maturities and log-moneyness. We can consider (1.7) as an 'almost' explicit approximation formula.

Maturity	Number of function evaluations	
	Minimum	Maximum
0.25	7	8
1	7	9
10	11	12

Table 2: Number of evaluations of $\partial_s m(s,T)$ required to find the saddle point $\hat{s}(k,T)$ with a relative precision of 10^{-8}, using Newton-Raphson method and starting from the explicit approximate saddle point $\hat{s}$ from (3.5). Values sampled over k ranging from 0.1 to 3.

When the local volatility is replaced with the almost-explicit approximation (1.7) in the wings, we are of course expecting a good fit of the implied volatility output. In Figure 5, we compare the implied volatilities obtained by Monte Carlo when using Heston local volatility (computed via Dupire's formula and adaptive integration contour for the integrals in (2.4), as in the right pane of Figures 1 and 2), and when replacing the numerical evaluation of local volatility with formula (1.7) on the outer region of Figure 4. For the 5% approximation of local volatility (the yellow curve), the

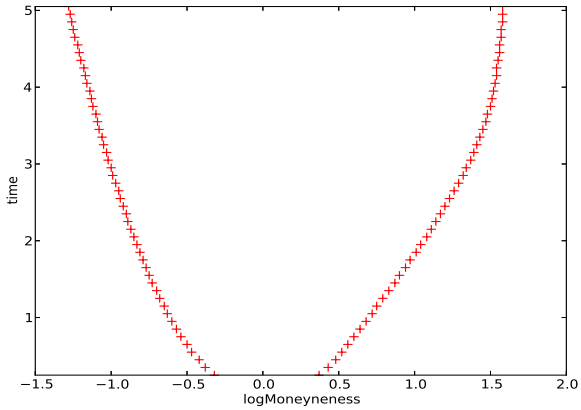


Figure 4: Boundaries of the region where the percentage difference $\left|\sigma^2_{\rm loc}(k,T) - \frac{\partial_T m(s,T)}{s(s-1)}|_{s=\hat{s}(k,T)}\right|/\sigma^2_{\rm loc}(k,T)$ between local variance and its saddle point approximation is less than 5%. Heston parameters as in Figure 3.

fit of implied volatility is within 10 basis points (the Monte Carlo error is around 20 basis points for all curves); the message of the red line in the right hand pane is that formula (1.7) should not be used *everywhere* (notably, not too close to the money for large maturities).

We will now show that the right hand side of (1.7) is indeed asymptotically equivalent to the right hand side of (1.4). This requires us to know that

$$2\,\frac{\partial}{\partial T} m(s,T)\Big|_{s=\hat{s}(k,T)} \sim 2k/\sigma, \tag{3.1}$$

where $\sigma = \sigma(T)$ is the so-called critical slope, defined as

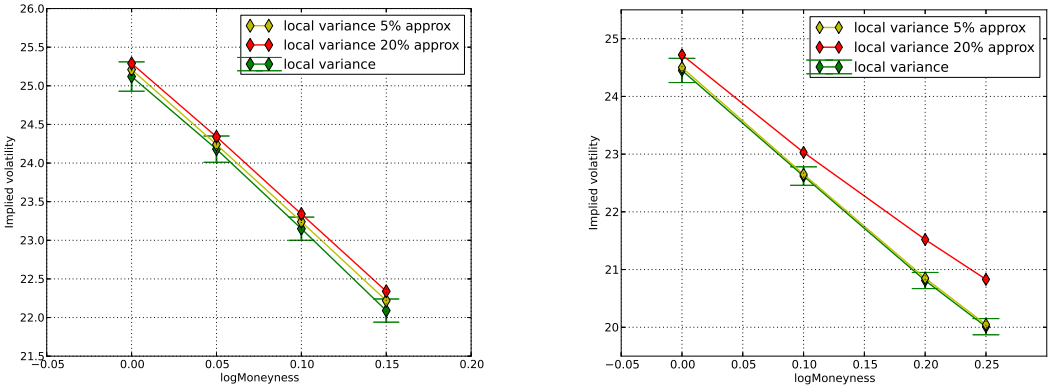
$$\begin{aligned} \sigma(T) &= -\frac{\partial T^*}{\partial s}(s_+(T)), \\ T^*(s) &= \sup\{t \geq 0 : \mathbb{E}[e^{sX_t}] < \infty\}. \end{aligned} \tag{3.2}$$

In fact, while the computation of the critical exponent s_+ in the Heston model requires simple numerics, the critical slope can be computed in closed form [16]; we have $\sigma(T) = R_1/R_2$, where $R_i = R_i(b,c,\rho,s_+(T))$, $i = 1,2$, are defined in (1.5)–(1.6).

Since $\hat{s}(k,T) \to s_+(T)$ as $k \to \infty$, if (3.1) is verified, the right hand side of (1.7) then satisfies

$$\frac{2\frac{\partial}{\partial T} m(s,T)}{s(s-1)}\Big|_{s=\hat{s}(k,T)} \sim \frac{2}{\sigma(T)s_+(T)(s_+(T)-1)} \times k, \qquad k \to \infty,$$

which is the formula from Theorem 1. Let us now discuss validity of (3.1). The argument which follows nicely illustrates how formula (1.7) can be used in stochastic volatility models of affine type. From an asymptotic analysis of the Riccati equations [16], $m(s,t)$ is know to satisfy (writing



(a) $T = 0.25$ (b) $T = 1$

Figure 5: Implied volatilities obtained when Heston local volatility is computed via Dupire's formula and adaptive integration contour in (2.4) (green), and when this numerical evaluation is replaced with formula (1.7) on the outer region of Figure 4 (yellow), respectively on the region where the accuracy of (1.7) becomes better than 20% (red). Monte Carlo parameters: 10^5 trajectories, 12 time steps per year.

$s_+ = s_+(T)$ when T is fixed)

$$m(s,t) \sim \frac{v_0}{\frac{c^2}{2}\sigma(s_+ - s)}, \qquad s \uparrow s_+,$$

$$\frac{\partial}{\partial s} m(s,t) \sim \frac{v_0}{\frac{c^2}{2}\sigma(s_+ - s)^2}, \qquad s \uparrow s_+, \tag{3.3}$$

$$\frac{\partial}{\partial T} m(s,T) \sim \frac{v_0}{\frac{c^2}{2}(\sigma(s_+ - s))^2}, \qquad s \uparrow s_+. \tag{3.4}$$

Equation (1.8) leads to

$$\hat{s} = s_+ - \beta k^{-1/2} + o(k^{-1/2}), \tag{3.5}$$

since

$$\frac{\partial}{\partial s} m(s,t) \sim \frac{v_0}{\frac{c^2}{2}\sigma(s_+ - \hat{s})^2} = k \implies s_+ - \hat{s} \sim \beta k^{-1/2}$$

with $\beta = \frac{\sqrt{2v_0}}{c\sqrt{\sigma}}$. Substitution then yields

$$\frac{\partial}{\partial T} m(s,T)|_{s=\hat{s}} \sim \frac{v_0}{\frac{c^2}{2}\sigma^2\beta^2/k} = k/\sigma,$$

which concludes our derivation of (3.1).

A numerical example of the convergence of the slope of local variance $\sigma^2_{\text{loc}}(k,T)/k$ to its asymptotic value is show in Figure 6 (for the left wing, i.e. the most interesting one in the presence of negatively correlated stochastic volatility).

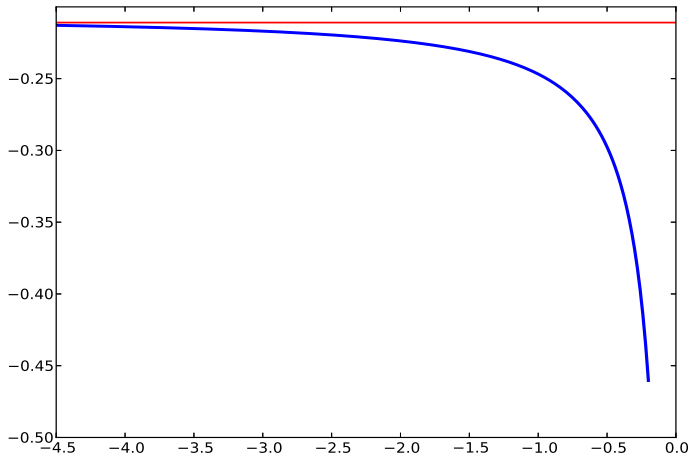


Figure 6: Convergence of the slope of local variance to its asymptotic value (left wing). Blue line: $k \mapsto \frac{\sigma^2_{\mathrm{loc}}(k,T)}{k}$, $T = 1$. Red line: asymptotic value from (1.4). Heston parameters as in Figure 3. This gives numerical confirmation of the analytic result (1.4). For actual numerics, the approximation formula (1.7) is to be preferred.

4 Conclusions

We proposed a new formula that expresses local volatility for extreme strikes as a computable function of commonly available model information. In the Heston model this leads to a proof that local volatility (squared) behaves asymptotically linear in log-strike (which is qualitatively similar to Lee's result [24] for implied volatility). This suggests concrete parametrizations of the local volatility surface. (In contrast to ad-hoc specifications of the implied volatility surface there is no danger of introducing arbitrage.) We also investigated the form of local volatility *defined via Dupire's formula* when the underlying exhibits jumps; qualitatively different behavior (compared to diffusion models such as Heston) is seen.

Our results also help the process of quantifying model risk for path-dependent options. Indeed, as we explained in the introduction, using our formula local volatility surfaces can be constructed in a (globally) robust fashion. Prices of exotic options under local volatility can then be compared with the prices obtained in the matching stochastic volatility model.

All asymptotics results are supported by numerical examples based on our novel and generic approximation formula (1.7).

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