

A TRACTABLE LIBOR MODEL WITH DEFAULT RISK

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ABSTRACT. We develop a model for the dynamic evolution of default-free and defaultable interest rates in a LIBOR framework. Utilizing the class of affine processes, this model produces positive LIBOR rates and spreads, while the dynamics are analytically tractable under defaultable forward measures. This leads to explicit formulas for CDS spreads, while semi-analytical formulas are derived for other credit derivatives. Finally, we give an application to counterparty risk.

1. INTRODUCTION

The current financial crisis has brought default risk to the forefront of attention, with large corporations and even countries being on the verge of bankruptcy. This has led to a renewed demand for credit derivatives, which can be used for hedging (or even for speculative) purposes; see the latest issue of the BIS Quarterly Review.

In this work, we present a tractable model for default-free and defaultable interest rates and study the pricing of credit derivatives in this framework. More precisely, we work in a discrete tenor framework and use the *LIBOR rate* as the risk-free rate. Of course, LIBOR is not considered risk-free any longer, see e.g. Crépey, Grbac, and Nguyen (2012) or Filipović and Trolle (2011), but one can simply replace the LIBOR with the “true” risk-free rate in today’s markets. Next, we consider a corporation that issues bonds which are subject to default risk. The riskiness of these bonds is reflected in their pre-default values, and we use them to derive the *defaultable LIBOR rate*, following Schönbucher (2000) and Eberlein, Kluge, and Schönbucher (2006). The defaultable LIBOR rate can be interpreted as the effective rate a corporation pays for borrowing money, which typically equals the LIBOR rate plus a (stochastic) spread (see also (3.3)).

Therefore, the aim of this paper is to develop an analytically tractable model for the joint evolution of default-free and defaultable LIBOR rates. The classical models for LIBOR rates, based on the seminal articles by Sandmann, Sondermann, and Miltersen (1995), Brace, Gątarek, and Musiela

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(1997) and Jamshidian (1997), are known to suffer from severe intractability problems. This has led to a multitude of approximation methods; see Gatarek, Bachert, and Maksymiuk (2006, Ch. 10) for an overview. These numerical problems are propagated in the defaultable framework; let us just mention that closed-form expressions do not exist even in the simple Brownian framework. Eberlein et al. (2006) use the so-called frozen drift approximation to derive prices for derivatives, but this is well-known to perform poorly for long maturities and high volatilities; we refer to Papapantoleon, Schoenmakers, and Skovmand (2011) for numerical experiments and alternative approximation schemes.

In order to overcome these problems, we work in the framework of the *affine LIBOR models* recently introduced by Keller-Ressel, Papapantoleon, and Teichmann (2011). These models are based on the wide and flexible class of affine processes, see Duffie, Filipović, and Schachermayer (2003), and have very appealing properties: LIBOR rates are positive, the model is arbitrage-free and the dynamics remain tractable under forward measures. The last property allows for semi-analytical pricing of many interest rate derivatives using Fourier transforms. We extend these models to the defaultable setting in a fashion that: (i) default-free and defaultable rates are positive, (ii) the market is free of arbitrage and (iii) the dynamics of rates are analytically tractable under restricted defaultable forward measures. In this framework, we can derive fully explicit formulas for CDS rates, while other credit derivatives admit semi-analytical pricing formulas.

The defaultable affine LIBOR model we develop belongs to the reduced form approach for modeling credit risk, where the default time is modeled as the first jump of a Cox process, with a given cumulative hazard process. This approach was studied by Elliott, Jeanblanc, and Yor (2000) and Jeanblanc and Le Cam (2008) from a theoretical perspective. For concrete examples of cumulative hazard process models we refer to Eberlein et al. (2006) and Kokholm and Nicolato (2010), and further references therein.

The paper is organized as follows. In Section 2 we recall the definition and some important properties of affine processes and provide an overview of the affine LIBOR models. In Section 3 we present the extension to the defaultable setting. In particular, we introduce defaultable LIBOR rates and discuss the no-arbitrage conditions in this set-up. This discussion follows along the lines of Eberlein et al. (2006). Then we construct a model for the dynamics of default intensities using a suitable family of exponentially-affine processes. The conditions for fitting the initial term structure of defaultable bonds are provided and analyzed. As an interesting consequence, we obtain that the hazard process in this model is an affine transformation of the driving affine process at tenor dates, which provides a direct link to hazard process models for credit risk. In Section 4 we derive valuation formulas for various credit derivatives. An explicit pricing formula is presented for credit default swaps and semi-analytical formulas are derived for options on defaultable bonds. Finally, we propose an application of the defaultable affine LIBOR model to counterparty risk and study the pricing of vulnerable options. The pricing formulas are again semi-analytical, based on Fourier transforms.

2. THE DEFAULT-FREE AFFINE LIBOR MODEL

We provide below a brief overview of the construction and the main results of the affine LIBOR model; for more details and proofs we refer to Keller-Ressel, Papapantoleon, and Teichmann (2011).

2.1. Affine processes. Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ denote a complete stochastic basis, where $\mathbf{F} = (\mathcal{F}_t)_{t \in [0, T]}$, and let $0 < T \leq \infty$ denote some, possibly infinite, time horizon. We consider a process X of the following type:

Assumption (A). Let $X = (X_t)_{0 \leq t \leq T}$ be a conservative, time-homogeneous, stochastically continuous Markov process taking values in $D = \mathbb{R}_{\geq 0}^d$, and $(\mathbb{P}_x)_{x \in D}$ a family of probability measures on (Ω, \mathcal{F}) , such that $X_0 = x$ \mathbb{P}_x -almost surely, for every $x \in D$. Setting

$$\mathcal{I}_T := \left\{ u \in \mathbb{R}^d : \mathbb{E}_x [e^{\langle u, X_T \rangle}] < \infty, \text{ for all } x \in D \right\}, \quad (2.1)$$

we assume that

- (i) $0 \in \mathcal{I}_T^\circ$, where \mathcal{I}_T° denotes the interior of \mathcal{I}_T ;
- (ii) the conditional moment generating function of X_t under \mathbb{P}_x has exponentially-affine dependence on x ; that is, there exist functions $\phi_t(u) : [0, T] \times \mathcal{I}_T \rightarrow \mathbb{R}$ and $\psi_t(u) : [0, T] \times \mathcal{I}_T \rightarrow \mathbb{R}^d$ such that

$$\mathbb{E}_x [\exp \langle u, X_t \rangle] = \exp (\phi_t(u) + \langle \psi_t(u), x \rangle), \quad (2.2)$$

for all $(t, u, x) \in [0, T] \times \mathcal{I}_T \times D$.

Here $\langle \cdot, \cdot \rangle$ denotes the inner product on \mathbb{R}^d , and \mathbb{E}_x the expectation with respect to \mathbb{P}_x . The filtration \mathbf{F} is the completed natural filtration of X .

The functions ϕ and ψ satisfy the so-called *generalized Riccati equations*

$$\frac{\partial}{\partial t} \phi_t(u) = F(\psi_t(u)), \quad \phi_0(u) = 0, \quad (2.3a)$$

$$\frac{\partial}{\partial t} \psi_t(u) = R(\psi_t(u)), \quad \psi_0(u) = u, \quad (2.3b)$$

for $(t, u) \in [0, T] \times \mathcal{I}_T$. The functions F and R are of Lévy–Khinchine form, that is

$$F(u) = \langle b, u \rangle + \int_D (e^{\langle \xi, u \rangle} - 1) m(d\xi), \quad (2.4a)$$

$$R_i(u) = \langle \beta_i, u \rangle + \left\langle \frac{\alpha_i}{2} u, u \right\rangle + \int_D (e^{\langle \xi, u \rangle} - 1 - \langle u, h^i(\xi) \rangle) \mu_i(d\xi), \quad (2.4b)$$

where $(b, m, \alpha_i, \beta_i, \mu_i)_{1 \leq i \leq d}$ are *admissible parameters* and $h^i : \mathbb{R}_{\geq 0}^d \rightarrow \mathbb{R}^d$ are suitable truncation functions. We refer to Duffie et al. (2003) for all the details.

We will later make use of the following results and definitions; here inequalities involving vectors are interpreted component-wise.

Lemma 2.1. *The functions ϕ and ψ satisfy the following:*

- (1) $\phi_t(0) = \psi_t(0) = 0$ for all $t \in [0, T]$.
- (2) \mathcal{I}_T is a convex set. Moreover, for each $t \in [0, T]$, the functions $\mathcal{I}_T \ni u \mapsto \phi_t(u)$ and $\mathcal{I}_T \ni u \mapsto \psi_t(u)$ are (componentwise) convex.

(3) $\phi_t(\cdot)$ and $\psi_t(\cdot)$ are order-preserving: let $(t, u), (t, v) \in [0, T] \times \mathcal{I}_T$, with $u \leq v$. Then

$$\phi_t(u) \leq \phi_t(v) \quad \text{and} \quad \psi_t(u) \leq \psi_t(v). \quad (2.5)$$

(4) $\psi_t(\cdot)$ is strictly order-preserving: let $(t, u), (t, v) \in [0, T] \times \mathcal{I}_T^\circ$, with $u < v$. Then $\psi_t(u) < \psi_t(v)$.

Definition 2.2. For any process $X = (X_t)_{0 \leq t \leq T}$ satisfying Assumption **(A)**, define

$$\gamma_X := \sup_{u \in \mathcal{I}_T \cap \mathbb{R}_{\geq 0}^d} \mathbf{E}_1[e^{\langle u, X_T \rangle}]. \quad (2.6)$$

Many results on affine processes can be extended to the time-inhomogeneous case, see Filipović (2005). The conditional moment generating function then takes the form

$$\mathbf{E}_x[\exp\langle u, X_r \rangle | \mathcal{F}_s] = \exp(\phi_{s,r}(u) + \langle \psi_{s,r}(u), X_s \rangle), \quad (2.7)$$

for all (s, r, u) such that $0 \leq s \leq r \leq T$ and $u \in \mathcal{I}_T$, with $\phi_{s,r}(u)$ and $\psi_{s,r}(u)$ now depending on both s and r . Assuming that X satisfies the ‘strong regularity condition’ (cf. Filipović (2005), Definition 2.9), $\phi_{s,r}(u)$ and $\psi_{s,r}(u)$ satisfy generalized Riccati equations with time-dependent right-hand sides:

$$-\frac{\partial}{\partial s} \phi_{s,r}(u) = F(s, \psi_{s,r}(u)), \quad \phi_{r,r}(u) = 0, \quad (2.8)$$

$$-\frac{\partial}{\partial s} \psi_{s,r}(u) = R(s, \psi_{s,r}(u)), \quad \psi_{r,r}(u) = u, \quad (2.9)$$

for all $0 \leq s \leq r \leq T$ and $u \in \mathcal{I}_T$.

We close this section with an example of an affine process on $\mathbb{R}_{\geq 0}$ that has already been used in the credit risk (Duffie and Gârleanu 2001) and term structure modeling literature (Filipović 2001).

Example 2.3. Let X be a Cox–Ingersoll–Ross process with jumps, defined by the SDE

$$dX_t = -\lambda(X_t - \theta)dt + 2\eta\sqrt{X_t}dW_t + dZ_t, \quad X_0 = x \in \mathbb{R}_{\geq 0},$$

where $\lambda, \theta, \eta \in \mathbb{R}_{\geq 0}$, W is a Brownian motion and Z is a compound Poisson process with constant intensity ℓ and exponentially distributed jumps with mean μ . This is an affine process on $\mathbb{R}_{\geq 0}$ with

$$F(u) = \lambda\theta u + \ell \frac{u}{\frac{1}{\mu} - u},$$

$$R(u) = 2\eta^2 u^2 - \lambda u.$$

The Riccati equations (2.3) can be solved explicitly, and we get that

$$\begin{aligned} \phi_t(u) &= -\frac{\lambda\theta}{2\eta^2} \log(1 - 2\eta^2 b(t)u) \\ &\quad - \frac{\ell\mu}{\lambda\mu - 2\eta^2} \log\left(\frac{\lambda(\mu u - 1)}{a(t)(\lambda\mu - 2\eta^2)u - \lambda + 2\eta^2 u}\right), \\ \psi_t(u) &= \frac{a(t)u}{1 - 2\eta^2 b(t)u}, \end{aligned}$$

with

$$a(t) = e^{-\lambda t} \quad \text{and} \quad b(t) = \frac{1 - e^{-\lambda t}}{\lambda}.$$

2.2. Ordered Martingales ≥ 1 . The construction of the affine LIBOR model is based on families of (parametrized) martingales greater than one, which are increasing in some parameter. We follow here the presentation of Papapantoleon (2010).

Let $X = (X_t)_{0 \leq t \leq T}$ be an affine process on $\mathbb{R}_{\geq 0}^d$ as described in the previous section, where from now on we restrict ourselves to a finite time horizon, i.e. $T < \infty$. Let $u \in \mathbb{R}_{\geq 0}^d$ and consider the random variable $Y_T^u := e^{\langle u, X_T \rangle}$. Then, from the tower property of conditional expectations we know immediately that $M^u = (M_t^u)_{0 \leq t \leq T}$, where

$$\begin{aligned} M_t^u &= \mathbb{E}[e^{\langle u, X_T \rangle} | \mathcal{F}_t] \\ &= \exp(\phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle), \end{aligned} \quad (2.10)$$

is a martingale. Moreover, it is obvious that $M_t^u \geq 1$ for all $t \in [0, T]$, while the ordering

$$u \leq v \implies M_t^u \leq M_t^v, \quad \forall t \in [0, T], \quad (2.11)$$

also follows directly.

2.3. Affine LIBOR models. Consider a discrete tenor structure $\mathcal{T} = \{0 = T_0 < T_1 < \dots < T_N \leq T\}$ and let $\delta_k := T_{k+1} - T_k$ for all $k \in \mathcal{K} \setminus \{N\}$, where $\mathcal{K} := \{1, \dots, N\}$. Let $B(\cdot, T_k)$ denote the price of a zero coupon bond with maturity T_k and $L(\cdot, T_k)$ denote the forward LIBOR rate settled at T_k and exchanged at T_{k+1} . They are related via

$$L(t, T_k) = \frac{1}{\delta_k} \left(\frac{B(t, T_k)}{B(t, T_{k+1})} - 1 \right). \quad (2.12)$$

Denote by \mathbb{P}_k the forward measure associated with the maturity T_k , i.e. the bond $B(\cdot, T_k)$ is the numeraire, for all $k \in \mathcal{K}$. Assume that X is an affine process under \mathbb{P}_N satisfying assumption **(A)**.

We know that discounted traded assets (e.g. bonds) should be martingales with respect to the terminal martingale measure, i.e.

$$\frac{B(\cdot, T_k)}{B(\cdot, T_N)} \in \mathcal{M}(\mathbb{P}_N), \quad \text{for all } k \in \mathcal{K}. \quad (2.13)$$

The affine LIBOR ansatz is thus to model quotients of bond prices using the \mathbb{P}_N -martingales M^u as follows:

$$\frac{B(t, T_1)}{B(t, T_N)} = M_t^{u_1} \quad (2.14)$$

\vdots

$$\frac{B(t, T_{N-1})}{B(t, T_N)} = M_t^{u_{N-1}}, \quad (2.15)$$

for all $t \in [0, T_1], \dots, t \in [0, T_{N-1}]$ respectively, while the initial values of the martingales M^{u_k} must satisfy:

$$M_0^{u_k} = \exp(\phi_T(u_k) + \langle \psi_T(u_k), x \rangle) = \frac{B(0, T_k)}{B(0, T_N)}, \quad (2.16)$$

for all $k \in \mathcal{K}$. Obviously we set $u_N = 0 \Leftrightarrow M_0^{u_N} = \frac{B(0, T_N)}{B(0, T_N)} = 1$.

We can show that under mild conditions on the underlying process X , an affine LIBOR model can fit any given term structure of initial LIBOR rates through the parameters u_1, \dots, u_N .

Proposition 2.4. *Suppose that $L(0, T_1), \dots, L(0, T_{N-1})$ is a tenor structure of non-negative initial LIBOR rates, and let X be a process satisfying assumption **(A)**, starting at the canonical value $\mathbf{1}$. The following hold:*

- (1) *If $\gamma_X > B(0, T_1)/B(0, T_N)$, then there exists a decreasing sequence $u_1 \geq u_2 \geq \dots \geq u_N = 0$ in $\mathcal{I}_T \cap \mathbb{R}_{\geq 0}^d$, such that*

$$M_0^{u_k} = \frac{B(0, T_k)}{B(0, T_N)}, \quad \text{for all } k \in \mathcal{K}. \quad (2.17)$$

In particular, if $\gamma_X = \infty$, then the affine LIBOR model can fit any term structure of non-negative initial LIBOR rates.

- (2) *If X is one-dimensional, the sequence $(u_k)_{k \in \mathcal{K}}$ is unique.*
(3) *If all initial LIBOR rates are positive, the sequence $(u_k)_{k \in \mathcal{K}}$ is strictly decreasing.*

In this model, forward prices have the following form:

$$\begin{aligned} 1 + \delta_k L(t, T_k) &= \frac{B(t, T_k)}{B(t, T_{k+1})} = \frac{M_t^{u_k}}{M_t^{u_{k+1}}} \\ &= \exp\left(\phi_{T_N-t}(u_k) - \phi_{T_N-t}(u_{k+1})\right. \\ &\quad \left. + \langle \psi_{T_N-t}(u_k) - \psi_{T_N-t}(u_{k+1}), X_t \rangle\right) \\ &= \exp\left(A_{T_N-t}(u_k, u_{k+1}) + \langle B_{T_N-t}(u_k, u_{k+1}), X_t \rangle\right), \end{aligned} \quad (2.18)$$

where we have defined

$$A_{T_N-t}(u_k, u_{k+1}) := \phi_{T_N-t}(u_k) - \phi_{T_N-t}(u_{k+1}), \quad (2.19)$$

$$B_{T_N-t}(u_k, u_{k+1}) := \psi_{T_N-t}(u_k) - \psi_{T_N-t}(u_{k+1}). \quad (2.20)$$

Using Proposition 2.4(1) and Lemma 2.1(3), we immediately deduce that *LIBOR rates are always non-negative* in the affine LIBOR models.

Proposition 2.5. *Suppose that $L(0, T_1), \dots, L(0, T_{N-1})$ is a tenor structure of non-negative initial LIBOR rates, and let X be a process satisfying assumption **(A)**. Let the bond prices be modelled by (2.14)–(2.15) and satisfying the initial conditions (2.16). Then the LIBOR rates $L(t, T_k)$ are non-negative a.s., for all $t \in [0, T_k]$ and $k \in \mathcal{K} \setminus \{N\}$.*

Forward measures in the affine LIBOR model are related to each other via quotients of the martingales M^u ; we have that

$$\frac{d\mathbb{P}_k}{d\mathbb{P}_{k+1}} \Big|_{\mathcal{F}_t} = \frac{B(0, T_{k+1})}{B(0, T_k)} \cdot \frac{M_t^{u_k}}{M_t^{u_{k+1}}} \quad (2.21)$$

for any $k \in \mathcal{K} \setminus \{N\}$, $t \in [0, T_k]$, where each \mathbb{P}_k is defined on $(\Omega, \mathcal{F}_{T_k})$. In addition, the density between the \mathbb{P}_k -forward measure and the terminal forward measure \mathbb{P}_N is given by the martingale M^{u_k} , as the defining equations (2.14)–(2.15) clearly dictate; we have

$$\frac{d\mathbb{P}_k}{d\mathbb{P}_N} \Big|_{\mathcal{F}_t} = \frac{B(0, T_N)}{B(0, T_k)} \cdot \frac{B(t, T_k)}{B(t, T_N)} = \frac{M_t^{u_k}}{M_0^{u_k}}. \quad (2.22)$$

The connection between the terminal and forward measures yields also the *martingale property of forward LIBOR rates*. We have that

$$1 + \delta_k L(\cdot, T_k) = \frac{M^{u_k}}{M^{u_{k+1}}} \in \mathcal{M}(\mathbb{P}_{k+1}) \quad (2.23)$$

because

$$\frac{M^{u_k}}{M^{u_{k+1}}} \cdot \frac{d\mathbb{P}_{k+1}}{d\mathbb{P}_N} \Big|_{\mathcal{F}} = M^{u_k} \in \mathcal{M}(\mathbb{P}_N). \quad (2.24)$$

Finally, we want to show that the model structure is preserved under any forward measure. Indeed, we calculate the conditional moment generating function of X_r under the forward measure \mathbb{P}_k , and get

$$\begin{aligned} & \mathbb{E}_k[e^{\langle v, X_r \rangle} | \mathcal{F}_s] \\ &= \exp \left(\phi_{r-s}(\psi_{T_N-r}(u_k) + v) - \phi_{r-s}(\psi_{T_N-r}(u_k)) \right. \\ & \quad \left. + \langle \psi_{r-s}(\psi_{T_N-r}(u_k) + v) - \psi_{r-s}(\psi_{T_N-r}(u_k)), X_s \rangle \right), \end{aligned} \quad (2.25)$$

which yields that X is a time-inhomogeneous *affine process under any forward measure* \mathbb{P}_k , for any $k \in \mathcal{K}$. In particular, setting $s = 0$, $r = t$, we get that

$$\mathbb{E}_k[e^{\langle v, X_t \rangle}] = \exp \left(\phi_t^k(v) + \langle \psi_t^k(v), x \rangle \right), \quad (2.26)$$

where

$$\phi_t^k(v) := \phi_t(\psi_{T_N-t}(u_k) + v) - \phi_t(\psi_{T_N-t}(u_k)), \quad (2.27)$$

$$\psi_t^k(v) := \psi_t(\psi_{T_N-t}(u_k) + v) - \psi_t(\psi_{T_N-t}(u_k)). \quad (2.28)$$

This also shows that the measure change from \mathbb{P}_k to \mathbb{P}_N is an exponential tilting (or Esscher transformation).

The main advantage of this modeling framework is that the affine structure of the driving process is preserved under any forward measure, which leads to semi-analytical pricing formulas for caps and swaptions using Fourier transforms. Moreover, in certain examples such as the CIR model, closed-form solutions similar to the Black–Scholes formula can be derived for both caps and swaptions; we refer to Keller-Ressel et al. (2011) for all the details.

3. THE DEFAULTABLE AFFINE LIBOR MODEL

In this section, we enlarge the market by adding *defaultable* bonds with zero recovery and maturities $T_k \in \mathcal{T}$. These are corporate bonds, and default risk means the risk of default of the corporation that issued the bond. The promised payoff at maturity T_k of such a bond is one currency unit, which is received by the bondholder (thereafter: she) if default does not occur before or at maturity. In case of default, she receives only a partial amount of

the promised payment depending on the recovery scheme that applies. Zero recovery means that in case of default she receives zero at maturity.

We denote the default time by τ . The time- t price of a defaultable bond with zero recovery, denoted by $B^0(t, T_k)$, can be written as

$$B^0(t, T_k) = \mathbf{1}_{\{\tau > t\}} \bar{B}(t, T_k), \quad (3.1)$$

where $\bar{B}(t, T_k)$, $t \in [0, T_k]$, denotes the pre-default value of the bond, which satisfies $\bar{B}(t, T_k) > 0$ and $\bar{B}(T_k, T_k) = 1$.

Let us introduce now a concept of defaultable forward LIBOR rates, by a straightforward generalization of the definition of the forward LIBOR rate and using pre-default bond prices instead of default-free ones. The following definitions are taken from Eberlein et al. (2006); see also Schönbucher (2000). For a detailed discussion of similar concepts and related defaultable FRAs we refer to Bielecki and Rutkowski (2002, §14.1.4, p. 431).

Definition 3.1. We define the *defaultable forward LIBOR rate* for the period $[T_k, T_{k+1}]$ prevailing at time $t \leq T_k$ by setting

$$\bar{L}(t, T_k) := \frac{1}{\delta_k} \left(\frac{\bar{B}(t, T_k)}{\bar{B}(t, T_{k+1})} - 1 \right). \quad (3.2)$$

Definition 3.2. The *forward credit spreads* between default-free and defaultable LIBOR rates are denoted by

$$S(t, T_k) := \bar{L}(t, T_k) - L(t, T_k), \quad (3.3)$$

while the associated *forward default intensities* are defined as

$$H(t, T_k) := \frac{\bar{L}(t, T_k) - L(t, T_k)}{1 + \delta_k L(t, T_k)}, \quad t \leq T_k, \quad (3.4)$$

with the convention $H(t, T_k) = H(T_k, T_k)$, for $t > T_k$.

Consequently, in terms of bond prices the following holds:

$$\begin{aligned} H(t, T_k) &= \frac{1}{\delta_k} \left(\frac{\bar{B}(t, T_k) B(t, T_{k+1})}{B(t, T_k) \bar{B}(t, T_{k+1})} - 1 \right) \\ \Leftrightarrow 1 + \delta_k H(t, T_k) &= \frac{\bar{B}(t, T_k)}{\bar{B}(t, T_{k+1})} \cdot \frac{B(t, T_{k+1})}{B(t, T_k)}. \end{aligned} \quad (3.5)$$

Each defaultable LIBOR rate can be expressed via the default-free LIBOR rate with the same tenor date and the corresponding default intensity as

$$\begin{aligned} 1 + \delta_k \bar{L}(t, T_k) &= (1 + \delta_k L(t, T_k))(1 + \delta_k H(t, T_k)) \\ \Leftrightarrow 1 + \delta_k H(t, T_k) &= \frac{1 + \delta_k \bar{L}(t, T_k)}{1 + \delta_k L(t, T_k)}. \end{aligned} \quad (3.6)$$

The aim of this work is to construct an analytically tractable framework for the joint evolution of default-free and defaultable LIBOR rates, where the requirement that riskier rates are higher than risk-free ones is respected, that is

$$L(t, T_k) \leq \bar{L}(t, T_k) \quad \forall t \in [0, T_k], \forall k \in \mathcal{K} \setminus \{N\}. \quad (3.7)$$

In order to fulfill the last requirement, we will follow the approach in Eberlein et al. (2006) and model default-free LIBOR rates and forward credit spreads,

or equivalently forward default intensities, as non-negative processes. In order to have an analytically tractable framework, we will extend the affine LIBOR model to the defaultable setting, i.e. we will model

$$1 + \delta_k H(\cdot, T_k) \quad (3.8)$$

such that: (i) it remains greater than one for all times, (ii) the model is free of arbitrage and (iii) the dynamics are of exponential-affine form.

3.1. The Cox construction of the default time. Here we describe the classical Cox process construction of the default time, which is modeled as the random time when an \mathbf{F} -adapted process crosses an independent trigger. This construction is also known as the canonical construction and provides a very simple and intuitive method to define the default event. It is widely used in credit risk modeling and the details can be found in many sources; we refer to Jeanblanc and Rutkowski (2000), Bielecki and Rutkowski (2002) and Eberlein et al. (2006).

Let $(\Omega, \mathcal{F}, \mathbb{P}_N)$ be a complete probability space such that a process X satisfying Assumption **(A)** is defined on it. Let \mathbf{F} denote the completed natural filtration of X , and assume that η is a random variable defined on $(\Omega, \mathcal{F}, \mathbb{P}_N)$, independent of \mathbf{F} and exponentially distributed with mean 1. Finally, let Γ be an \mathbf{F} -adapted, right-continuous, non-decreasing process such that $\Gamma_0 = 0$ and $\lim_{t \rightarrow \infty} \Gamma_t = \infty$.

Remark 3.3. Note that in order to define Γ and η with these properties one typically begins with a probability space where X is defined and then considers an enlarged space, obtained as the product space of the underlying probability space and another space supporting η . Here $(\Omega, \mathcal{F}, \mathbb{P}_N)$ is assumed to be already large enough to support the random variable η which is independent of \mathbf{F} .

Define a random time $\tau : \Omega \rightarrow \mathbb{R}_{\geq 0}$ by

$$\tau := \inf \{t \in \mathbb{R}_{\geq 0} : \Gamma_t \geq \eta\}.$$

This random time is not an \mathbf{F} -stopping time. Let us denote by $\mathcal{D}_t := \sigma(\mathbf{1}_{\{\tau \leq t\}} : t \geq 0)$ and set $\mathbf{D} = (\mathcal{D}_t)_{t \geq 0}$. Define the filtration $\mathbf{G} = (\mathcal{G}_t)_{t \geq 0}$ by setting $\mathcal{G}_t := \bigcap_{s > t} (\mathcal{F}_s \vee \mathcal{D}_s)$. Obviously, the random time τ is a \mathbf{G} -stopping time.

The following property can be easily proved: for all $0 \leq s \leq T_N$

$$\mathbb{P}_N(\tau > s | \mathcal{F}_{T_N}) = \mathbb{P}_N(\tau > s | \mathcal{F}_s) = e^{-\Gamma_s}. \quad (3.9)$$

Hence, the process Γ is by definition the \mathbf{F} -hazard process of the random time τ . Moreover, (3.9) entails the so-called \mathcal{H} -hypothesis, also known as the immersion property, namely:

(\mathcal{H}) Every \mathbf{F} -local martingale is a \mathbf{G} -local martingale,

which is equivalent to the following statements (cf. Brémaud and Yor 1978):

($\mathcal{H}1$) For any t , the σ -fields \mathcal{F}_{T_N} and \mathcal{G}_t are conditionally independent given \mathcal{F}_t , i.e.

$$\mathbb{E}_N[XY_t | \mathcal{F}_t] = \mathbb{E}_N[X | \mathcal{F}_t] \mathbb{E}_N[Y_t | \mathcal{F}_t],$$

for any bounded \mathcal{F}_{T_N} -measurable X and bounded \mathcal{G}_t -measurable Y_t .

($\mathcal{H}2$) For every bounded \mathcal{F}_{T_N} -measurable X

$$\mathbb{E}_N[X|\mathcal{G}_t] = \mathbb{E}_N[X|\mathcal{F}_t].$$

In the sequel, we shall use the following lemma which provides an expression for the conditional expectation with respect to the enlarged σ -algebras \mathcal{G}_s in terms of \mathcal{F}_s . The result is classical and can be found e.g. in Jeanblanc and Rutkowski (2000) or Bielecki and Rutkowski (2002).

Lemma 3.4. *Let Y be an integrable, \mathcal{F} -measurable random variable. Then for any $s \leq t$*

$$\mathbb{E}_N[\mathbf{1}_{\{\tau>t\}}Y|\mathcal{G}_s] = \mathbf{1}_{\{\tau>s\}} \frac{\mathbb{E}_N[\mathbf{1}_{\{\tau>t\}}Y|\mathcal{F}_s]}{\mathbb{E}_N[\mathbf{1}_{\{\tau>t\}}|\mathcal{F}_s]}.$$

We conclude this subsection with the following important remark.

Remark 3.5. In LIBOR modeling we consider the whole set of equivalent forward measures. Each \mathbb{P}_k , $k \in \mathcal{K} \setminus \{N\}$, was defined on $(\Omega, \mathcal{F}_{T_k})$ via (2.22). We now extend this definition to the σ -algebra \mathcal{G}_{T_k} using the same Radon-Nikodym derivative

$$\frac{d\mathbb{P}_k}{d\mathbb{P}_N} = \frac{M_{T_k}^{u_k}}{M_0^{u_k}}.$$

Using the \mathcal{H} -hypothesis, more precisely ($\mathcal{H}2$), we have

$$\left. \frac{d\mathbb{P}_k}{d\mathbb{P}_N} \right|_{\mathcal{G}_t} = \left. \frac{d\mathbb{P}_k}{d\mathbb{P}_N} \right|_{\mathcal{F}_t} = \frac{M_t^{u_k}}{M_0^{u_k}}.$$

Moreover, it easily follows that Γ is the \mathbf{F} -hazard process of τ under *all* measures \mathbb{P}_k , $k \in \mathcal{K}$. Applying the abstract Bayes' rule and ($\mathcal{H}1$) we obtain

$$\begin{aligned} \mathbb{P}_k(\tau > s|\mathcal{F}_s) &= \frac{\mathbb{E}_N[M_{T_k}^{u_k} \mathbf{1}_{\{\tau>s\}}|\mathcal{F}_s]}{M_s^{u_k}} \\ &= \frac{\mathbb{E}_N[M_{T_k}^{u_k}|\mathcal{F}_s] \mathbb{P}_N(\tau > s|\mathcal{F}_s)}{M_s^{u_k}} \\ &= \mathbb{P}_N(\tau > s|\mathcal{F}_s) = e^{-\Gamma_s}. \end{aligned} \quad (3.10)$$

3.2. No-arbitrage conditions: interplay between H and τ . Before proceeding with the construction of the defaultable affine LIBOR model, it is crucial to realize that we cannot choose H and τ arbitrarily. Here we follow the argumentation of Eberlein et al. (2006) closely; compare also Section 4.2 in Grbac (2010) and in particular Proposition 4.4, Lemma 4.5 and Remarks 4.3 and 4.6 for a discussion on the absence of arbitrage in this framework.

Let us begin by inspecting the relationship between τ and H that is necessarily satisfied in an arbitrage-free defaultable model.

Lemma 3.6. *Let $H(\cdot, T_k)$, $k \in \mathcal{K} \setminus \{N\}$, be the forward default intensities and τ the time of default. Then, in an arbitrage-free model we have*

$$1 + \delta_k H(t, T_k) = \frac{\mathbb{P}_k(\tau > T_k|\mathcal{F}_t)}{\mathbb{P}_{k+1}(\tau > T_{k+1}|\mathcal{F}_t)} = \frac{\mathbb{E}_k[e^{-\Gamma_{T_k}}|\mathcal{F}_t]}{\mathbb{E}_{k+1}[e^{-\Gamma_{T_{k+1}}}| \mathcal{F}_t]}. \quad (3.11)$$

Proof. On the one hand, the value of a defaultable bond at maturity is

$$B^0(T_k, T_k) = \mathbf{1}_{\{\tau > T_k\}} \bar{B}(T_k, T_k) = \mathbf{1}_{\{\tau > T_k\}}.$$

The time- t price of a contingent claim with payoff $\mathbf{1}_{\{\tau > T_k\}}$ at T_k , which we denote by $\pi_t(\mathbf{1}_{\{\tau > T_k\}})$, is given by the risk-neutral valuation formula under the forward measure \mathbb{P}_k , i.e.

$$\pi_t(\mathbf{1}_{\{\tau > T_k\}}) = B(t, T_k) \mathbb{E}_k[\mathbf{1}_{\{\tau > T_k\}} | \mathcal{G}_t]. \quad (3.12)$$

On the other hand, $B^0(t, T_k)$ denotes the time- t price of a defaultable bond. Hence, in order to have a consistent and arbitrage-free model, it should hold

$$B^0(t, T_k) = \pi_t(\mathbf{1}_{\{\tau > T_k\}}). \quad (3.13)$$

Now, (3.1), (3.12), (3.13) and Lemma 3.4 yield the following equality

$$\mathbf{1}_{\{\tau > t\}} \bar{B}(t, T_k) = \mathbf{1}_{\{\tau > t\}} B(t, T_k) \frac{\mathbb{P}_k(\tau > T_k | \mathcal{F}_t)}{\mathbb{P}_k(\tau > t | \mathcal{F}_t)},$$

and from (3.9) and (3.10), we obtain

$$\frac{\bar{B}(t, T_k)}{B(t, T_k)} = e^{\Gamma_t} \mathbb{P}_k(\tau > T_k | \mathcal{F}_t)$$

on the set $\{\tau > t\}$, for every $k \in \mathcal{K}$. Recalling (3.5) yields the first equality in (3.11). Moreover, using the tower property of conditional expectations and the properties of the hazard process, we get

$$\mathbb{E}_k[\mathbf{1}_{\{\tau > T_k\}} | \mathcal{F}_t] = \mathbb{E}_k[\mathbb{E}_k[\mathbf{1}_{\{\tau > T_k\}} | \mathcal{F}_{T_k}] | \mathcal{F}_t] = \mathbb{E}_k[e^{-\Gamma_{T_k}} | \mathcal{F}_t],$$

which combined with the first equality in (3.11) yields the second one. \square

Therefore, as soon as the default time τ is specified, equality (3.11) produces a formula for H and vice versa. In the spirit of Eberlein et al. (2006), we are going to “reverse engineer” the problem; that is, we shall specify the processes $H(\cdot, T_k)$, $k \in \mathcal{K} \setminus \{N\}$, satisfying certain conditions and then define an \mathbf{F} -adapted process Γ such that the relation between H and Γ given in (3.11) is satisfied. Finally, using the Cox construction, we know that a default time τ with \mathbf{F} -hazard process Γ exists.

Proposition 3.7. *Assume that the default intensities $H(\cdot, T_k)$, $k \in \mathcal{K} \setminus \{N\}$ satisfy the following assumption:*

$$\left(\prod_{i=0}^k \frac{1}{1 + \delta_i H(t, T_i)} \right)_{0 \leq t \leq T_k} \in \mathcal{M}(\mathbb{P}_{k+1}) \quad (3.14)$$

for every $k \in \mathcal{K} \setminus \{N\}$, with $H(t, T_i) = H(T_i, T_i)$, for $t > T_i$. Moreover, let Γ be any \mathbf{F} -adapted, right-continuous and non-decreasing process such that

$$\Gamma_{T_{k+1}} = \sum_{i=0}^k \ln(1 + \delta_i H(T_i, T_i)), \quad (3.15)$$

for every $k = 0, 1, \dots, N-2$. Then equation (3.11) is satisfied.

Remark 3.8. Note that $\Gamma_{T_{k+1}} \in \mathcal{F}_{T_k}$, thus using linear interpolation between tenor dates T_k and T_{k+1} provides a suitable example for Γ .

Proof. We begin by noting that, in order to satisfy (3.11), it suffices to specify the hazard process Γ only at the tenor points T_k . Inserting $t = T_k$ into (3.11), we get

$$\mathbb{E}_{k+1}[e^{-\Gamma_{T_{k+1}}}| \mathcal{F}_{T_k}] = e^{-\Gamma_{T_k}} \frac{1}{1 + \delta_k H(T_k, T_k)}.$$

This motivates us to define Γ as in (3.15) at tenor points T_k . Now it is easily checked that this combined with the martingale property (3.14) yields

$$1 + \delta_k H(t, T_k) = \frac{\mathbb{E}_k \left[\prod_{i=0}^{k-1} \frac{1}{1 + \delta_i H(T_i, T_i)} \middle| \mathcal{F}_t \right]}{\mathbb{E}_{k+1} \left[\prod_{i=0}^k \frac{1}{1 + \delta_i H(T_i, T_i)} \middle| \mathcal{F}_t \right]}, \quad (3.16)$$

which is exactly (3.11). \square

Finally, we will later make use of the following.

Definition 3.9. We denote by

$$\mathbb{H}(\cdot, T_k) := \prod_{i=0}^k \frac{1}{1 + \delta_i H(\cdot, T_i)}, \quad k \in \mathcal{K} \setminus \{N\}. \quad (3.17)$$

3.3. Modeling default intensities. Let us now turn our attention to the joint modeling of default-free and defaultable LIBOR rates. Any model for this evolution should satisfy some very basic requirements, dictated by economics, (mathematical) finance and practical applications. In particular:

- credit spreads should be positive;
- the model should be arbitrage-free;
- dynamics should be analytically tractable.

Combining these requirements with the considerations from the previous subsections, in order to have an arbitrage-free defaultable model that produces positive credit spreads, the processes $\mathbb{H}(\cdot, T_k)$, $k \in \mathcal{K} \setminus \{N\}$, should satisfy the following requirements :

- (A1) $1 + \delta_k H(\cdot, T_k) = \frac{\mathbb{H}(\cdot, T_{k-1})}{\mathbb{H}(\cdot, T_k)} \geq 1$,
- (A2) $\mathbb{H}(\cdot, T_k) \in \mathcal{M}(\mathbb{P}_{k+1})$.

Moreover, in order to have an analytically tractable model, we will employ the affine LIBOR model and extend it to the defaultable setting.

Remark 3.10. Note that (A1) immediately yields that $\mathbb{H}(\cdot, T_k)$ must be a $[0, 1]$ -valued process. In addition, (A2) is equivalent to

- (A2') $\mathbb{H}(\cdot, T_k) M^{u_{k+1}} \in \mathcal{M}(\mathbb{P}_N) \quad \forall k \in \mathcal{K} \setminus \{N\}$,

as a consequence of the relation between forward measures (2.22) and Jacod and Shiryaev (2003, Prop. III.3.8).

Proposition 3.11. *Assume that default-free LIBOR rates are modeled according to the affine LIBOR model. Let $(v_k)_{k \in \mathcal{K}}$ be a family of vectors in \mathbb{R}^d such that $v_1 \leq u_1$ and*

$$\phi_t(v_k) - \phi_t(u_k) \geq \phi_t(v_{k+1}) - \phi_t(u_{k+1}), \quad (3.18a)$$

$$\psi_t(v_k) - \psi_t(u_k) \geq \psi_t(v_{k+1}) - \psi_t(u_{k+1}), \quad (3.18b)$$

for all $t \in [0, T_k]$ and all $k \in \mathcal{K}$. Define a family of \mathbb{P}_N -martingales M^{v_k} , $k \in \mathcal{K}$, by

$$M_t^{v_k} = \exp\left(\phi_{T_N-t}(v_k) + \langle \psi_{T_N-t}(v_k), X_t \rangle\right), \quad t \leq T_k, \quad (3.19)$$

and model the \mathbb{H} -processes by setting

$$\mathbb{H}(t, T_k) := \frac{M_t^{v_{k+1}}}{M_t^{u_{k+1}}}, \quad k \in \mathcal{K} \setminus \{N\}, \quad t \leq T_k, \quad (3.20)$$

and $\mathbb{H}(t, T_k) = \mathbb{H}(T_k, T_k)$, for $t > T_k$. Then the family $\mathbb{H}(\cdot, T_k)$, $k \in \mathcal{K} \setminus \{N\}$, satisfies requirements **(A1)** and **(A2)**.

Proof. The specification obviously satisfies condition **(A2')**, or equivalently **(A2)**, i.e. $\mathbb{H}(\cdot, T_k)$ is a \mathbb{P}_{k+1} -martingale. Let us show that it fulfills **(A1)**. Firstly, by inserting $t = 0$ into (3.18b) and recalling (2.3b), we get

$$v_k - u_k \geq v_{k+1} - u_{k+1},$$

for all $k \in \mathcal{K}$. Secondly, since $v_1 \leq u_1$ by assumption, it follows $v_k \leq u_k$, for all $k \in \mathcal{K}$. Thus, we obtain

$$0 \leq \mathbb{H}(t, T_k) \leq 1, \quad \forall k, \forall t. \quad (3.21)$$

Moreover, (3.18a) and (3.18b) yield

$$\frac{M_t^{v_k}}{M_t^{u_k}} \geq \frac{M_t^{v_{k+1}}}{M_t^{u_{k+1}}} \quad (3.22)$$

for all $k \in \mathcal{K}$ and $t \in [0, T_k]$, which is equivalent to

$$\mathbb{H}(t, T_k) \geq \mathbb{H}(t, T_{k+1}), \quad \forall k, \forall t. \quad (3.23)$$

□

Remark 3.12. Let us briefly comment on the financial interpretation of the conditions on the families (u_k) and (v_k) , before we proceed with discussing some properties of the defaultable affine LIBOR model. These families satisfy the following conditions:

- (C1) $u_k \geq u_{k+1}$ for all $k \in \mathcal{K} \setminus \{N\}$, where $u_k \in \mathcal{I}_T \cap \mathbb{R}_{\geq 0}^d$ and $u_N = 0$
- (C2) $v_k \geq v_{k+1}$ for all $k \in \mathcal{K} \setminus \{N\}$, where $v_k \in \mathcal{I}_T$
- (C3) $u_k \geq v_k$ for all $k \in \mathcal{K}$
- (C4) the functions ϕ and ψ satisfy

$$\phi_t(v_k) - \phi_t(u_k) \geq \phi_t(v_{k+1}) - \phi_t(u_{k+1}), \quad (C4.a)$$

$$\psi_t(v_k) - \psi_t(u_k) \geq \psi_t(v_{k+1}) - \psi_t(u_{k+1}). \quad (C4.b)$$

Note that condition (C2), which was not stated explicitly above, follows by combining (C4.b) for $t = 0$ and (C1). The first condition ensures that default-free LIBOR rates are *non-negative*, while the second one ensures that defaultable LIBOR rates are *non-negative*, cf. (2.18) and (3.28), respectively. The third condition ensures that the processes $\mathbb{H}(\cdot, T_k)$ are $[0, 1]$ -valued, while the last condition ensures that forward default intensities H are *non-negative* (cf. (3.20)), thus the spreads between default-free and defaultable LIBOR rates are also non-negative. Note that (C4) ensures also that the hazard process Γ defined by (3.15) is non-decreasing. The first three conditions are automatically satisfied for any defaultable affine LIBOR model by

fitting the initial term structure of default-free and defaultable rates. The last condition has to be imposed in addition; it is automatically satisfied, for example, for independent affine processes, see Section 3.5.

3.4. Properties of the model. Next, we show that under mild conditions on the driving affine process the defaultable affine LIBOR model can fit any initial term structure of defaultable rates. This result also shows that conditions (C2) and (C3) are automatically satisfied.

Proposition 3.13. *Assume that the setting of Proposition 2.4 is in force. Suppose that $\bar{B}(0, T_1) \geq \bar{B}(0, T_2) \geq \dots \geq \bar{B}(0, T_N)$ is a tenor structure of initial defaultable bond prices such that $\bar{B}(0, T_k) \leq B(0, T_k)$ for every $k \in \mathcal{K}$, as well as $\bar{L}(0, T_k) \geq L(0, T_k)$, i.e.*

$$\frac{\bar{B}(0, T_k)}{\bar{B}(0, T_{k+1})} \geq \frac{B(0, T_k)}{B(0, T_{k+1})}.$$

Let X be a process satisfying assumption **(A)**, starting at the canonical value **1**. The following hold:

- (1) If $\gamma_X > B(0, T_1)/B(0, T_N)$, then there exists a decreasing sequence $v_1 \geq v_2 \geq \dots \geq v_N$ in \mathcal{I}_T , such that

$$M_0^{v_k} = \frac{\bar{B}(0, T_k)}{B(0, T_N)}, \quad \text{for all } k \in \mathcal{K}. \quad (3.24)$$

In particular, if $\gamma_X = \infty$, then the defaultable affine LIBOR model can fit any term structure of non-negative initial defaultable LIBOR rates. Moreover, for each $k \in \mathcal{K}$ it holds: $v_k \leq u_k$.

- (2) If X is one-dimensional, the sequence $(v_k)_{k \in \mathcal{K}}$ is unique.
(3) If all initial defaultable LIBOR rates are positive, then the sequence $(v_k)_{k \in \mathcal{K}}$ is strictly decreasing.

Remark 3.14. Note that since $\bar{B}(0, T_1) \leq B(0, T_1)$ by assumption, it follows that as soon as γ_X satisfies condition (1) above, it will automatically follow that $\gamma_X > \bar{B}(0, T_1)/B(0, T_N)$.

Proof. We have that the tenor structure of initial defaultable bond prices satisfies

$$\mathbb{H}(0, T_k) = \prod_{i=0}^k \frac{1}{1 + \delta_k H(0, T_i)} = \frac{\bar{B}(0, T_{k+1})}{B(0, T_{k+1})} \leq 1.$$

Recalling that $M_0^{u_{k+1}} = \frac{B(0, T_{k+1})}{B(0, T_N)}$ we obtain

$$\mathbb{H}(0, T_k) M_0^{u_{k+1}} = \frac{\bar{B}(0, T_{k+1})}{B(0, T_{k+1})} \frac{B(0, T_{k+1})}{B(0, T_N)} = \frac{\bar{B}(0, T_{k+1})}{B(0, T_N)}.$$

Note that by assumption

$$\frac{\bar{B}(0, T_1)}{B(0, T_N)} \geq \frac{\bar{B}(0, T_2)}{B(0, T_N)} \geq \dots \geq \frac{\bar{B}(0, T_N)}{B(0, T_N)} > 0, \quad (3.25)$$

where the last term $\frac{\bar{B}(0, T_N)}{B(0, T_N)} \leq 1$. Therefore, similarly to the proof of Proposition 2.4 (cf. Proposition 6.1 in Keller-Ressel et al. 2011), we can find a

decreasing sequence $(v_k)_{k \in \mathcal{K}}$ such that

$$M_0^{v_k} = \frac{\bar{B}(0, T_k)}{B(0, T_N)}.$$

More precisely, take u_+ as defined therein: let $u_+ \in \mathcal{I}_T \cap \mathbb{R}_{\geq 0}^d$ be such that

$$\mathbb{E}_1[e^{\langle u_+, X_T \rangle}] > \gamma_X - \varepsilon > \frac{B(0, T_1)}{B(0, T_N)},$$

where $\varepsilon > 0$ is small enough such that $\gamma_X - \varepsilon > \frac{B(0, T_1)}{B(0, T_N)}$. Note that u_+ must exist by definition of γ_X . Similarly, since $\inf_{v \in \mathbb{R}_{\leq 0}^d} \mathbb{E}_1[e^{\langle v, X_T \rangle}] = 0$, we can find some $\lambda < 0$ such that

$$\mathbb{E}_1[e^{\langle \lambda u_+, X_T \rangle}] < \frac{\bar{B}(0, T_N)}{B(0, T_N)} \leq 1.$$

We consider the function f defined in the aforementioned proposition and extend its domain to the interval $[\lambda, 1]$, i.e. we define

$$f : [\lambda, 1] \rightarrow \mathbb{R}_{\geq 0}, \quad \xi \mapsto f(\xi) = \mathbb{E}_1[e^{\langle \xi u_+, X_T \rangle}] = M_0^{\xi u_+}.$$

This function was already shown to be continuous and increasing. Moreover, $f(\lambda) < \bar{B}(0, T_N)/B(0, T_N)$ and $f(1) > \bar{B}(0, T_1)/B(0, T_N)$, since $f(1) > B(0, T_1)/B(0, T_N)$ by definition of u_+ . Thus, there exists a sequence $\lambda < \eta_N \leq \dots \leq \eta_1 < 1$ such that

$$f(\eta_k) = M_0^{\eta_k u_+} = \frac{\bar{B}(0, T_k)}{B(0, T_N)}, \quad k \in \mathcal{K}.$$

Setting $v_k := \eta_k u_+$ we obtain the desired decreasing sequence. Note that as soon as there exists k_0 such that $\frac{\bar{B}(0, T_{k_0})}{B(0, T_N)} < 1$, it follows that $v_k \in \mathbb{R}_{\leq 0}^d$, for all $k \geq k_0$.

Moreover, we have that $v_k \leq u_k$, since $\frac{\bar{B}(0, T_k)}{B(0, T_N)} \leq \frac{B(0, T_k)}{B(0, T_N)}$.

If X is one-dimensional, then any choice of u_+ and λ leads to the same parameters v_k , which shows (2).

Finally, if the initial defaultable LIBOR rates are positive, inequalities in (3.25) become strict and thus the sequence (v_k) becomes strictly decreasing (see again Proposition 2.4). \square

Remark 3.15. Note that from the assumption $\frac{\bar{B}(0, T_k)}{B(0, T_{k+1})} \geq \frac{B(0, T_k)}{B(0, T_{k+1})}$, it follows directly that

$$\frac{M_0^{v_k}}{M_0^{u_k}} \geq \frac{M_0^{v_{k+1}}}{M_0^{u_{k+1}}}. \quad (3.26)$$

Using (2.16) we get that

$$\begin{aligned} & \phi_{T_N}(v_k) - \phi_{T_N}(u_k) + \langle \psi_{T_N}(v_k) - \psi_{T_N}(u_k), \mathbf{1} \rangle \geq \\ & \phi_{T_N}(v_{k+1}) - \phi_{T_N}(u_{k+1}) + \langle \psi_{T_N}(v_{k+1}) - \psi_{T_N}(u_{k+1}), \mathbf{1} \rangle. \end{aligned} \quad (3.27)$$

which agrees with (3.18).

Obviously the defaultable affine LIBOR model inherits many properties from its default-free counterpart: the defaultable rates are non-negative and the dynamics have an exponential-affine structure.

Lemma 3.16. *The defaultable LIBOR rate $\bar{L}(\cdot, T_k)$ has the following form*

$$\begin{aligned} 1 + \delta_k \bar{L}(t, T_k) &= \frac{M_t^{v_k}}{M_t^{v_{k+1}}} \\ &= \exp \left(A_{T_N-t}(v_k, v_{k+1}) + \langle B_{T_N-t}(v_k, v_{k+1}), X_t \rangle \right) \geq 1, \end{aligned} \quad (3.28)$$

for all $T_k \in \mathcal{T}$ and $t \leq T_k$, where A and B are defined in (2.19)–(2.20).

Proof. We have that

$$1 + \delta_k H(t, T_k) = \frac{\mathbb{H}(t, T_{k-1})}{\mathbb{H}(t, T_k)} = \frac{M_t^{v_k} M_t^{u_{k+1}}}{M_t^{u_k} M_t^{v_{k+1}}}. \quad (3.29)$$

Using (2.18) and (3.6) we deduce

$$\begin{aligned} 1 + \delta_k \bar{L}(t, T_k) &= (1 + \delta_k L(t, T_k))(1 + \delta_k H(t, T_k)) \\ &= \frac{M_t^{u_k}}{M_t^{u_{k+1}}} \cdot \frac{M_t^{v_k} M_t^{u_{k+1}}}{M_t^{u_k} M_t^{v_{k+1}}} = \frac{M_t^{v_k}}{M_t^{v_{k+1}}}, \end{aligned}$$

which yields that the dynamics of defaultable rates are of exponential-affine form. Positivity follows from Lemma 2.1(3) and condition (C2). \square

Finally, we summarize below the main properties of the defaultable affine LIBOR model.

Proposition 3.17. *Suppose the conditions of Propositions 2.4 and 3.13 are satisfied and assume (3.18). Then the defaultable affine LIBOR model given by (2.14)–(2.15) and (3.19) with initial conditions (2.16) and (3.24) is free of arbitrage. The LIBOR rates $L(\cdot, T_k)$ and the defaultable LIBOR rates $\bar{L}(\cdot, T_k)$ are non-negative a.s., for all $k \in \mathcal{K} \setminus \{N\}$, and have exponentially-affine dynamics.*

Proof. The defaultable affine LIBOR model is free of arbitrage by Propositions 3.7 and 3.11. The other claims were already proved above. \square

Remark 3.18. Note that equation (3.15) implies that, in defaultable affine LIBOR models, the hazard process Γ is an affine transformation of the driving affine process X at tenor dates T_k , $k \in \mathcal{K}$. More precisely, we have

$$\begin{aligned} \Gamma_{T_{k+1}} &= \ln \left(\mathbb{H}(T_k, T_k)^{-1} \right) = \ln \left(\frac{M_{T_k}^{u_{k+1}}}{M_{T_k}^{v_{k+1}}} \right) \\ &= A_{T_N-T_k}(u_{k+1}, v_{k+1}) + \langle B_{T_N-T_k}(u_{k+1}, v_{k+1}), X_{T_k} \rangle, \end{aligned} \quad (3.30)$$

where A and B are defined in (2.19)–(2.20). In addition, we can embed this model in a Heath–Jarrow–Morton framework for defaultable bonds, by extending the tenor structure to a continuous term structure. This extension preserves the properties of the model, in particular (3.30) remains true. This provides a direct link between defaultable affine LIBOR models and intensity models for credit risk which are driven by affine processes; see Brigo and Mercurio (2006, Chapter 22) for a detailed overview of intensity models.

3.5. Example: independent affine processes. In this subsection, we provide an example of two families of processes $\{M^{u_k}; k \in \mathcal{K}\}$ and $\{M^{v_k}; k \in \mathcal{K}\}$ which satisfy our modeling requirements, in particular inequality (3.22). The construction relies on independent affine processes.

Let $d_1, d_2 \in \mathbb{N}$ with $d = d_1 + d_2$, where d is the dimension of the affine process X . The first d_1 and the last d_2 components of X are d_1 -, respectively d_2 -dimensional affine processes, assuming that the filtration \mathbf{F} is generated by X^1, X^2 ; see Proposition 4.8 in Keller-Ressel (2008). Let us denote them by X^1 and X^2 , respectively. In addition, we assume that X^1 and X^2 are mutually *independent*. Then, we have the following result

$$\begin{aligned} \mathbb{E}_N[e^{\langle u, X_{T_N} \rangle} | \mathcal{F}_t] &= \exp\left(\phi_{T_N-t}(u) + \langle \psi_{T_N-t}(u), X_t \rangle\right) \\ &= \exp\left(\phi_{T_N-t}^1(u^1) + \phi_{T_N-t}^2(u^2) \right. \\ &\quad \left. + \langle \psi_{T_N-t}^1(u^1), X_t^1 \rangle + \langle \psi_{T_N-t}^2(u^2), X_t^2 \rangle\right), \end{aligned} \quad (3.31)$$

where ϕ^i and ψ^i correspond to X^i , $i = 1, 2$, in the sense of Assumption **(A)**, while $u = (u^1, u^2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. See Keller-Ressel (2008, Proposition 4.7).

The families of \mathbb{P}_N -martingales $\{M^{u_k}\}$ and $\{M^{v_k}\}$ are constructed in the following way:

Step 1: We begin by constructing martingales $\{M^{u_k}; k \in \mathcal{K}\}$. First we apply Proposition 2.4 to the initial values of the LIBOR rates and the driving process X^1 . We obtain a decreasing sequence $\bar{u}_1 \geq \bar{u}_2 \geq \dots \geq \bar{u}_N = 0$, where $\bar{u}_k \in \mathbb{R}^{d_1}$, for every $k \in \mathcal{K}$. For each \bar{u}_k , let us denote $u_k := (\bar{u}_k, 0, \dots, 0) \in \mathbb{R}^d$. Then we have

$$\begin{aligned} M_t^{u_k} &= \exp\left(\phi_{T_N-t}^1(\bar{u}_k) + \langle \psi_{T_N-t}^1(\bar{u}_k), X_t^1 \rangle\right) \\ &= \exp\left(\phi_{T_N-t}(u_k) + \langle \psi_{T_N-t}(u_k), X_t \rangle\right), \end{aligned}$$

where the second equality follows from (3.31) and Lemma 2.1(1) applied to X^2 . The martingales $(M^{u_k})_{k \in \mathcal{K}}$, are used to model LIBOR rates.

Step 2: Next, we construct the processes $\frac{M^{v_k}}{M^{u_k}}$, $k \in \mathcal{K}$, by setting

$$\frac{M_t^{v_k}}{M_t^{u_k}} = \exp\left(\phi_{T_N-t}^2(\bar{w}_k) + \langle \psi_{T_N-t}^2(\bar{w}_k), X_t^2 \rangle\right) =: M_t^{\bar{w}_k},$$

where $\bar{w}_k \in \mathbb{R}^{d_2}$ are obtained by applying the same procedure as in the proof of Proposition 3.13 to the affine process X^2 and the initial values

$$M_0^{\bar{w}_k} = \mathbb{H}(0, T_{k-1}) = \frac{\bar{B}(0, T_k)}{B(0, T_k)} \leq 1, \quad k \in \mathcal{K}.$$

Note that $\bar{w}_k \leq 0$ by construction. Moreover, the sequence $0 \geq \bar{w}_1 \geq \bar{w}_2 \geq \dots \geq \bar{w}_N$ is decreasing, which follows from the initial conditions

$$\frac{\bar{B}(0, T_k)}{B(0, T_k)} \geq \frac{\bar{B}(0, T_{k+1})}{B(0, T_{k+1})}.$$

Consequently, applying the ordering (2.11) we directly conclude that

$$\frac{M_t^{v_k}}{M_t^{u_k}} \geq \frac{M_t^{v_{k+1}}}{M_t^{u_{k+1}}} \quad \text{since} \quad M_t^{\bar{w}_k} \geq M_t^{\bar{w}_{k+1}},$$

for every $k \in \mathcal{K} \setminus \{N\}$ and every $t \in [0, T_k]$. Hence, condition **(A1)** is satisfied.

Step 3: Finally, it remains to verify condition **(A2)**, which reads as follows: $\mathbb{H}(\cdot, T_{k-1})M^{u_k} = M^{v_k} \in \mathcal{M}(\mathbb{P}_N)$. We have

$$\begin{aligned} M_t^{v_k} &= M_t^{u_k} \cdot M_t^{\bar{w}_k} \\ &= \exp\left(\phi_{T_N-t}^1(\bar{u}_k) + \phi_{T_N-t}^2(\bar{w}_k) \right. \\ &\quad \left. + \langle \psi_{T_N-t}^1(\bar{u}_k), X_t^1 \rangle + \langle \psi_{T_N-t}^2(\bar{w}_k), X_t^2 \rangle\right) \\ &= \exp\left(\phi_{T_N-t}(v_k) + \langle \psi_{T_N-t}(v_k), X_t \rangle\right) \\ &= \mathbb{E}_N[e^{\langle v_k, X_{T_N} \rangle} | \mathcal{F}_t], \end{aligned}$$

where we defined $v_k := (\bar{u}_k, \bar{w}_k) \in \mathbb{R}^d$. Hence, $M^{v_k} \in \mathcal{M}(\mathbb{P}_N)$, for all $k \in \mathcal{K}$.

Remark 3.19. The existence of *dependent* affine processes that satisfy these requirements remains an open question. More generally, the construction of ordered martingales that satisfy inequality (3.22) seems to be non-trivial.

4. PRICING CREDIT DERIVATIVES

The pricing of credit derivatives in the defaultable affine LIBOR model is a (relatively) simple task due to the analytical tractability of the model. In particular, we can derive explicit expressions for derivatives with linear payoffs, such as credit default swaps, and semi-analytical formulas for products with non-linear payoffs, utilising the affine property and Fourier methods. Our formulas do not involve any approximation; compare with Eberlein et al. (2006) where approximations are necessary.

An essential tool for the pricing of credit derivatives are restricted defaultable forward measures, introduced by Schönbucher (2000) and further exploited by Eberlein et al. (2006). They are the restrictions of defaultable forward martingale measures, also called survival measures and defined on $(\Omega, \mathcal{G}_{T_k})$, to the sub- σ -fields \mathcal{F}_{T_k} for each $k \in \mathcal{K}$; see Bielecki and Rutkowski (2002, Defs. 15.2.1, 15.2.2).

Definition 4.1. The restricted defaultable forward martingale measure $\bar{\mathbb{P}}_k$ associated to the maturity T_k , $k \in \mathcal{K}$, is given on $(\Omega, \mathcal{F}_{T_k})$ by

$$\frac{d\bar{\mathbb{P}}_k}{d\mathbb{P}_k} \Big|_{\mathcal{F}_t} = \frac{B(0, T_k)}{\bar{B}(0, T_k)} \mathbb{P}_k(\tau > T_k | \mathcal{F}_t). \quad (4.1)$$

The explicit relation between the default time τ and the forward default intensity yields

$$\begin{aligned} \mathbb{P}_k(\tau > T_k | \mathcal{F}_t) &= \mathbb{E}_k[e^{-\Gamma_{T_k}} | \mathcal{F}_t] \\ &= \mathbb{E}_k[\mathbb{H}(T_{k-1}, T_{k-1}) | \mathcal{F}_t] = \mathbb{H}(t, T_{k-1}), \end{aligned}$$

hence we can deduce that

$$\frac{d\bar{\mathbb{P}}_k}{d\mathbb{P}_k} \Big|_{\mathcal{F}_t} = \frac{B(0, T_k)}{\bar{B}(0, T_k)} \cdot \frac{M_t^{v_k}}{M_t^{u_k}}. \quad (4.2)$$

Moreover, the density process between the restricted defaultable forward measures is described by

$$\frac{d\bar{\mathbb{P}}_k}{d\bar{\mathbb{P}}_{k+1}} \Big|_{\mathcal{F}_t} = \frac{\bar{B}(0, T_{k+1})}{\bar{B}(0, T_k)} \cdot \frac{M_t^{v_k}}{M_t^{v_{k+1}}}; \quad (4.3)$$

compare with expression (2.21) for the default-free forward measures.

These results clarify some important properties of the defaultable affine LIBOR model. On the one hand, it easily follows from (3.28) that the defaultable LIBOR rate $\bar{L}(\cdot, T_k)$ is a $\bar{\mathbb{P}}_{k+1}$ -martingale. On the other hand, we can deduce that the defaultable affine LIBOR model remains *analytically tractable*, in the sense that the driving process preserves the affine property under any restricted defaultable forward measure. Of course, as in the default-free case, it becomes time-inhomogeneous. Indeed, reasoning as in (2.25)–(2.28), we get that

$$\begin{aligned} \bar{\mathbb{E}}_k [e^{\langle w, X_t \rangle}] &= \mathbb{E}_N \left[e^{\langle w, X_t \rangle} \cdot \frac{d\bar{\mathbb{P}}_k}{d\mathbb{P}_k} \frac{d\mathbb{P}_k}{d\mathbb{P}_N} \Big|_{\mathcal{F}_t} \right] = \mathbb{E}_N \left[e^{\langle w, X_t \rangle} \frac{M_t^{v_k}}{M_0^{v_k}} \right] \\ &= \exp \left(\bar{\phi}_t^k(w) + \langle \bar{\psi}_t^k(w), x \rangle \right), \end{aligned} \quad (4.4)$$

where

$$\bar{\phi}_t^k(w) := \phi_t(\psi_{T_N-t}(v_k) + w) - \phi_t(\psi_{T_N-t}(v_k)), \quad (4.5)$$

$$\bar{\psi}_t^k(w) := \psi_t(\psi_{T_N-t}(v_k) + w) - \psi_t(\psi_{T_N-t}(v_k)). \quad (4.6)$$

4.1. Credit default swaps. *Credit default swaps* are credit derivatives used to provide protection against default of an underlying asset. Consider a maturity date T_m and a defaultable coupon bond with fractional recovery of treasury value as the underlying asset. The coupons with value c are promised to be paid at the dates T_1, \dots, T_m and, in case of default before maturity, a fixed fraction $\pi \in [0, 1)$ of the notional is received by the owner of the bond. The protection buyer in such a credit default swap pays a fixed amount \mathcal{S} periodically at dates T_0, T_1, \dots, T_{m-1} until default and the protection seller promises to make a payment that covers the loss if default happens, i.e.

$$1 - \pi(1 + c)$$

is paid to the protection buyer at T_{k+1} if default occurs in $(T_k, T_{k+1}]$, $k \in \{0, 1, 2, \dots, m-1\}$.

The value at time 0 of the fee payments is given by

$$\mathcal{S} \sum_{l=1}^m \bar{B}(0, T_{l-1}),$$

and the value of the default payment is given by

$$\begin{aligned} &\sum_{k=1}^m \left(B(0, T_k) \mathbb{E}_k \left[(1 - \pi(1 + c)) (\mathbf{1}_{\{\tau > T_{k-1}\}} - \mathbf{1}_{\{\tau > T_k\}}) \right] \right) \\ &= (1 - \pi(1 + c)) \sum_{k=1}^m \left(\bar{B}(0, T_k) \delta_{k-1} \bar{\mathbb{E}}_k [H(T_{k-1}, T_{k-1})] \right); \end{aligned}$$

see Eberlein et al. (2006, Lemma 4 and Section 6). The CDS rate, also known as the CDS spread, is defined as the level \mathcal{S} that makes the value of the credit default swap at inception equal to zero. We have that

$$\mathcal{S} = \frac{1 - \pi(1 + c)}{\sum_{l=1}^m \bar{B}(0, T_{l-1})} \sum_{k=1}^m \left(\bar{B}(0, T_k) \delta_{k-1} \bar{\mathbb{E}}_k [H(T_{k-1}, T_{k-1})] \right). \quad (4.7)$$

In the defaultable affine LIBOR model the forward default intensity has an exponential affine form, in particular we have from (3.29)

$$1 + \delta_{k-1} H(T_{k-1}, T_{k-1}) = \frac{M_{T_{k-1}}^{v_{k-1}} M_{T_{k-1}}^{u_k}}{M_{T_{k-1}}^{u_{k-1}} M_{T_{k-1}}^{v_k}} = e^{A_k + B_k \cdot X_{T_{k-1}}}, \quad (4.8)$$

where

$$\begin{aligned} A_k &:= \phi_{T_N - T_{k-1}}(v_{k-1}) - \phi_{T_N - T_{k-1}}(u_{k-1}) - \phi_{T_N - T_{k-1}}(v_k) + \phi_{T_N - T_{k-1}}(u_k), \\ B_k &:= \psi_{T_N - T_{k-1}}(v_{k-1}) - \psi_{T_N - T_{k-1}}(u_{k-1}) - \psi_{T_N - T_{k-1}}(v_k) + \psi_{T_N - T_{k-1}}(u_k). \end{aligned}$$

Using the affine property of X under restricted defaultable forward measures, we can deduce a closed-form expression for the CDS spread. We have, from (4.7), (4.8) and (4.4), that

$$\begin{aligned} \mathcal{S} &= \frac{1 - \pi(1 + c)}{\sum_{k=1}^m \bar{B}(0, T_{k-1})} \\ &\quad \times \sum_{k=1}^m \bar{B}(0, T_k) \left(\exp \left\{ A_k + \bar{\phi}_{T_{k-1}}^k(B_k) + \langle \bar{\psi}_{T_{k-1}}^k(B_k), X_0 \rangle \right\} - 1 \right). \end{aligned} \quad (4.9)$$

Remark 4.2. Analogous closed-form expressions for other credit derivatives with linear payoffs, such as total rate of returns swaps and asset swap packages, can be easily derived. See Kluge (2005, §4.6) for more details on credit derivatives in defaultable LIBOR models.

Remark 4.3. Note that when the processes driving the risk-free interest rates and the default intensities are independent (in other words, when the risk-free rates and the default time are independent), the CDS spread can be expressed as a function of the initial default-free and defaultable bond prices and is model-independent. This is a well-known property, discussed for example in Brigo and Mercurio (2006, § 21.3.5) for the continuous tenor case. Let us show it in our framework. We recall Example 3.5 and first note that

$$\bar{\mathbb{E}}_k [H(T_{k-1}, T_{k-1})] = H(0, T_{k-1}). \quad (4.10)$$

Then, it follows directly from (4.7) that

$$\begin{aligned} \mathcal{S} &= \frac{1 - \pi(1 + c)}{\sum_{l=1}^m \bar{B}(0, T_{l-1})} \sum_{k=1}^m \left(\bar{B}(0, T_k) \delta_{k-1} H(0, T_{k-1}) \right) \\ &= \frac{1 - \pi(1 + c)}{\sum_{l=1}^m \bar{B}(0, T_{l-1})} \sum_{k=1}^m \frac{\bar{B}(0, T_{k-1}) B(0, T_k) - \bar{B}(0, T_k) B(0, T_{k-1})}{B(0, T_{k-1})}. \end{aligned}$$

This formula can be used to bootstrap the initial defaultable bond prices from the CDS spreads quoted in the market. In order to show (4.10), we use

the independence and the martingale property of M^u and $M^{\bar{w}}$; we have

$$\begin{aligned} \bar{\mathbb{E}}_k [1 + \delta_{k-1} H(T_{k-1}, T_{k-1})] &= \mathbb{E}_N \left[\frac{M_{T_{k-1}}^{v_{k-1}} M_{T_{k-1}}^{u_k} M_{T_{k-1}}^{v_k}}{M_{T_{k-1}}^{u_{k-1}} M_{T_{k-1}}^{v_k} M_0^{v_k}} \right] \\ &= \frac{1}{M_0^{v_k}} \mathbb{E}_N \left[M_{T_{k-1}}^{\bar{w}_{k-1}} M_{T_{k-1}}^{u_k} \right] \\ &= \frac{1}{M_0^{v_k}} \mathbb{E}_N [M_{T_{k-1}}^{\bar{w}_{k-1}}] \mathbb{E}_N [M_{T_{k-1}}^{u_k}] \\ &= \frac{M_0^{\bar{w}_{k-1}}}{M_0^{v_k}} = 1 + \delta_{k-1} H(0, T_{k-1}). \end{aligned}$$

4.2. Options on defaultable bonds. We consider now options on defaultable bonds, and focus on a European call on a defaultable zero coupon bond, for simplicity. Options on defaultable fixed or floating coupon bonds can be treated similarly. Let T_i be the maturity and $K \in (0, 1)$ the strike of a call option on a defaultable zero coupon bond with maturity $T_m \geq T_i$. We follow Kluge (2005), and adopt the fractional recovery of treasury value scheme, which means that in case of default prior to maturity of the bond the owner receives the amount $\pi \in (0, 1)$ at maturity T_m ; see Bielecki and Rutkowski (2002) for details and alternative recovery schemes. We denote the price of this bond by $B^\pi(\cdot, T_m)$ and its time- t value equals

$$B^\pi(t, T_m) = \pi B(t, T_m) + (1 - \pi) \mathbf{1}_{\{\tau > t\}} \bar{B}(t, T_m).$$

The payoff of the option at maturity T_i is given by $\mathbf{1}_{\{\tau > T_i\}} (B^\pi(T_i, T_m) - K)^+$, which means that it is knocked out at default.

The price of this option, using (4.1), is provided by

$$\begin{aligned} \pi_0^{\text{CO}} &= B(0, T_i) \mathbb{E}_i [\mathbf{1}_{\{\tau > T_i\}} (B^\pi(T_i, T_m) - K)^+] \\ &= \bar{B}(0, T_i) \bar{\mathbb{E}}_i [(B^\pi(T_i, T_m) - K)^+] \\ &= \bar{B}(0, T_i) \bar{\mathbb{E}}_i [(\pi B(T_i, T_m) + (1 - \pi) \bar{B}(T_i, T_m) - K)^+] \\ &= \bar{B}(0, T_i) \bar{\mathbb{E}}_i \left[\left(\pi \prod_{l=i}^{m-1} (1 + \delta_l L(T_i, T_l))^{-1} \right. \right. \\ &\quad \left. \left. + (1 - \pi) \prod_{l=i}^{m-1} (1 + \delta_l \bar{L}(T_i, T_l))^{-1} - K \right)^+ \right]. \quad (4.11) \end{aligned}$$

Now, in the default-free and defaultable affine LIBOR models we have that

$$\begin{aligned} 1 + \delta_l L(T_i, T_l) &= \frac{M_{T_i}^{u_l}}{M_{T_i}^{u_{l+1}}} = \exp(A_{i,l} + \langle B_{i,l}, X_{T_i} \rangle), \\ 1 + \delta_l \bar{L}(T_i, T_l) &= \frac{M_{T_i}^{v_l}}{M_{T_i}^{v_{l+1}}} = \exp(\bar{A}_{i,l} + \langle \bar{B}_{i,l}, X_{T_i} \rangle), \end{aligned}$$

where

$$\begin{aligned} A_{i,l} &= A_{T_N - T_i}(u_l, u_{l+1}), & B_{i,l} &= B_{T_N - T_i}(u_l, u_{l+1}), \\ \bar{A}_{i,l} &= A_{T_N - T_i}(v_l, v_{l+1}), & \bar{B}_{i,l} &= B_{T_N - T_i}(v_l, v_{l+1}). \end{aligned}$$

Therefore, for the product terms in (4.11) we get that

$$\prod_{l=i}^{m-1} (1 + \delta_l L(T_i, T_l))^{-1} = \exp(A_i^m + \langle B_i^m, X_{T_i} \rangle),$$

and

$$\prod_{l=i}^{m-1} (1 + \delta_l \bar{L}(T_i, T_l))^{-1} = \exp(\bar{A}_i^m + \langle \bar{B}_i^m, X_{T_i} \rangle),$$

with the obvious definitions

$$A_i^m = - \sum_{l=i}^{m-1} A_{i,l}, \quad B_i^m = - \sum_{l=i}^{m-1} B_{i,l} \quad (4.12)$$

$$\bar{A}_i^m = - \sum_{l=i}^{m-1} \bar{A}_{i,l}, \quad \bar{B}_i^m = - \sum_{l=i}^{m-1} \bar{B}_{i,l}. \quad (4.13)$$

Therefore, returning to the option pricing problem, we have

$$\begin{aligned} \pi_0^{\text{CO}} &= \bar{B}(0, T_i) \bar{\mathbb{E}}_i \left[\left(\pi e^{A_i^m + \langle B_i^m, X_{T_i} \rangle} + (1 - \pi) e^{\bar{A}_i^m + \langle \bar{B}_i^m, X_{T_i} \rangle} - K \right)^+ \right] \\ &= \bar{B}(0, T_i) \bar{\mathbb{E}}_i \left[(e^{Y_1} + e^{Y_2} - K)^+ \right], \end{aligned} \quad (4.14)$$

where

$$Y_1 := \log \pi + A_i^m + \langle B_i^m, X_{T_i} \rangle, \quad (4.15)$$

$$Y_2 := \log(1 - \pi) + \bar{A}_i^m + \langle \bar{B}_i^m, X_{T_i} \rangle. \quad (4.16)$$

Now, the expression in (4.14) corresponds to the payoff of a spread option

$$g(x_1, x_2) = (e^{x_1} + e^{x_2} - K)^+, \quad (4.17)$$

whose Fourier transform is

$$\hat{g}(z) = K^{1+iz_1+iz_2} \frac{\Gamma(iz_2)\Gamma(1-iz_1-iz_2)}{\Gamma(1-iz_1)}, \quad (4.18)$$

for $z \in \mathcal{Y} := \{z \in \mathbb{C}^2 : \Im z_2 < 0, \Im(z_1 + z_2) > 1\}$; see Hubalek and Kallsen (2005) and Hurd and Zhou (2010). Here, Γ denotes the Gamma function. Therefore, using also Eberlein, Glau, and Papapantoleon (2010, Thm. 3.2), we have that the price of an option on a defaultable zero coupon bond admits the following semi-analytical expression:

$$\pi_0^{\text{CO}} = \frac{\bar{B}(0, T_i)}{4\pi^2} \int_{\mathbb{R}^2} \hat{g}(iR - w) M_Y(R + iw) dw, \quad (4.19)$$

for $iR \in \mathcal{Y}$ such that $M_Y(R) < \infty$, where M_Y denotes the moment generating function of the random vector $Y = (Y_1, Y_2)$. This can be computed

explicitly using the affine property of the driving process X under the measure $\bar{\mathbb{P}}_i$. We have

$$\begin{aligned} M_Y(w) &= \bar{\mathbb{E}}_i[e^{\langle w, Y \rangle}] = \bar{\mathbb{E}}_i[e^{w_1 Y_1 + w_2 Y_2}] \\ &= \mathcal{X} \bar{\mathbb{E}}_i[e^{\langle w_1 B_i^m + w_2 \bar{B}_i^m, X_{T_i} \rangle}] \\ &= \mathcal{X} \exp \left\{ \bar{\phi}_{T_i}^i(w_1 B_i^m + w_2 \bar{B}_i^m) + \langle \bar{\psi}_{T_i}^i(w_1 B_i^m + w_2 \bar{B}_i^m), X_0 \rangle \right\}, \end{aligned} \quad (4.20)$$

where

$$\mathcal{X} = \exp \left\{ w_1 (\log \pi + A_i^m) + w_2 (\log(1 - \pi) + \bar{A}_i^m) \right\}. \quad (4.21)$$

Remark 4.4. Similar semi-analytical expressions can be derived for other derivatives with non-linear payoffs, such as credit spread options. Credit default swaptions, also known as CDS options, are more difficult to handle, but expressions involving a high-dimensional integration can still be derived; see also Keller-Ressel, Papapanoleon, and Teichmann (2011, §7.3).

4.3. Vulnerable options. The term *credit risk* applies to two different types of risk: *reference risk* and *counterparty risk*. Reference risk is the risk associated with an underlying asset (reference) in a contract, whereas counterparty risk refers to any kind of risk associated with either of the counterparties involved in a contract; see Figure 4.1 for a graphical representation. Contingent claims with reference risk traded over-the-counter between default-free parties are labeled *credit derivatives* and the collective name *vulnerable claims* refers to contingent claims traded over-the-counter between default-prone parties with an underlying asset that is assumed to be default-free. The name *vulnerable* goes back to Johnson and Stulz (1987), who studied the impact of default risk of an option writer on option prices. We mention also some other papers studying counterparty risk such as Jarrow and Turnbull (1995), Hull and White (1995), Huge and Lando (1999) and Lotz and Schlögl (2000).

In this section, we study an application of the defaultable affine LIBOR model to the pricing of vulnerable options. Again, we obtain an analytical expression for the price of a vulnerable option which does not involve any approximations; compare with Grbac (2010, Section 4.3) where vulnerable options are studied in the defaultable Lévy LIBOR framework which requires “frozen drift”-type approximations.

A vulnerable European call option with maturity T_k and strike K on a default-free bond $B(\cdot, T_m)$, where $T_m \geq T_k$, has a payoff given by

$$\begin{aligned} \bar{C}_{T_k} &= C_{T_k} \mathbf{1}_{\{\tau > T_k\}} + q C_{T_k} \mathbf{1}_{\{\tau \leq T_k\}} \\ &= (C_{T_k} - q C_{T_k}) \mathbf{1}_{\{\tau > T_k\}} + q C_{T_k}, \end{aligned}$$

where $C_{T_k} = (B(T_k, T_m) - K)^+$ is the payoff at maturity T_k of a European call option written on a default-free bond with maturity T_m , τ is the default time of the writer of the option and q the recovery rate (in case of default the payoff of the option at maturity is reduced by a factor $q \in [0, 1]$).

Therefore, we give here a new interpretation to defaultable bonds $B^0(\cdot, T_k)$, which are now assumed to be issued by the *writer of the vulnerable option*.

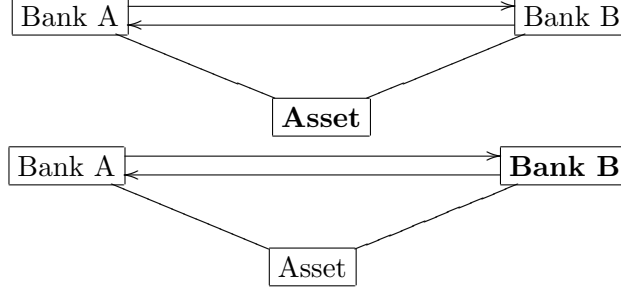


FIGURE 4.1. A graphical illustration of reference risk (top) vs. counterparty risk (bottom). The bold letters indicate where default risk lies in a contract between counterparties A and B.

Using the definition of the forward LIBOR rate, we can rewrite the payoff of the vulnerable option as follows

$$\begin{aligned} \bar{C}_{T_k} &= \mathbf{1}_{\{\tau > T_k\}}(1 - q)(B(T_k, T_m) - K)^+ + q(B(T_k, T_m) - K)^+ \\ &= \mathbf{1}_{\{\tau > T_k\}}(1 - q) \left(\prod_{l=k}^{m-1} (1 + \delta_l L(T_k, T_l))^{-1} - K \right)^+ \\ &\quad + q \left(\prod_{l=k}^{m-1} (1 + \delta_l L(T_k, T_l))^{-1} - K \right)^+ . \end{aligned}$$

Its value at time $t = 0$ is given by

$$\begin{aligned} \bar{C}_0 &= B(0, T_k) \mathbb{E}_k \left[\mathbf{1}_{\{\tau > T_k\}}(1 - q) \left(\prod_{l=k}^{m-1} (1 + \delta_l L(T_k, T_l))^{-1} - K \right)^+ \right. \\ &\quad \left. + q \left(\prod_{l=k}^{m-1} (1 + \delta_l L(T_k, T_l))^{-1} - K \right)^+ \right] \\ &= \bar{B}(0, T_k) \bar{\mathbb{E}}_k \left[(1 - q) \left(\prod_{l=k}^{m-1} (1 + \delta_l L(T_k, T_l))^{-1} - K \right)^+ \right] \\ &\quad + B(0, T_k) \mathbb{E}_k \left[q \left(\prod_{l=k}^{m-1} (1 + \delta_l L(T_k, T_l))^{-1} - K \right)^+ \right] \\ &= \bar{B}(0, T_k) (1 - q) \bar{\mathbb{E}}_k \left[(e^Z - K)^+ \right] + B(0, T_k) q \mathbb{E}_k \left[(e^Z - K)^+ \right] , \end{aligned}$$

where

$$Z := A_k^m + \langle B_k^m, X_{T_k} \rangle,$$

see equation (4.12). The payoff function of a call option

$$g(y) = (e^y - K)^+$$

has the Fourier transform

$$\widehat{g}(z) = \frac{K^{1+iz}}{iz(1+iz)},$$

for $z \in \mathcal{Z} := \{z \in \mathbb{C} : \Im z > 1\}$. Therefore, using Eberlein et al. (2010, Thm. 2.2), we obtain

$$\begin{aligned} \overline{C}_0 &= \frac{\overline{B}(0, T_k)(1-q)}{2\pi} \int_{\mathbb{R}} \widehat{g}(iR_1 - v) M_Z^{\overline{\mathbb{P}}_k}(R_1 + iv) dv \\ &\quad + \frac{B(0, T_k)q}{2\pi} \int_{\mathbb{R}} \widehat{g}(iR_2 - w) M_Z^{\mathbb{P}_k}(R_2 + iw) dw, \end{aligned}$$

for $iR_1, iR_2 \in \mathcal{Z}$ such that $M_Z^{\overline{\mathbb{P}}_k}(R_1) < \infty$ and $M_Z^{\mathbb{P}_k}(R_2) < \infty$. The moment generating function of Z under $\overline{\mathbb{P}}_k$ and \mathbb{P}_k respectively, is provided by

$$M_Z^{\overline{\mathbb{P}}_k}(v) = \exp \left\{ vA_{k,l}^m + \overline{\phi}_{T_k}^k(vB_{k,l}^m) + \langle \overline{\psi}_{T_k}^k(vB_{k,l}^m), X_0 \rangle \right\}$$

and

$$M_Z^{\mathbb{P}_k}(w) = \exp \left\{ wA_{k,l}^m + \phi_{T_k}^k(wB_{k,l}^m) + \langle \psi_{T_k}^k(wB_{k,l}^m), X_0 \rangle \right\}.$$

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