

OPTIMAL CONTROL OF MAXWELL'S EQUATIONS WITH REGULARIZED STATE CONSTRAINTS

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Abstract. This paper is devoted to an optimal control problem of Maxwell's equations in the presence of pointwise state constraints. The control is given by a divergence-free three-dimensional vector function representing an applied current density. To cope with the divergence-free constraint on the control, we consider a vector potential ansatz. Due to the lack of regularity of the control-to-state mapping, existence of Lagrange multipliers cannot be guaranteed. We regularize the optimal control problem by penalizing the pointwise state constraints. Optimality conditions for the regularized problem can be derived straightforwardly. It also turns out that the solution of the regularized problem enjoys higher regularity which then allows us to establish its convergence towards the solution of the unregularized problem. The second part of the paper focuses on the numerical analysis of the regularized optimal control problem. Here the state and the control are discretized by Nédélec's curl-conforming edge elements. Employing the higher regularity property of the optimal control, we establish an a priori error estimate for the discretization error in the $\mathbf{H}(\mathbf{curl})$ -norm. The paper ends by numerical results including a numerical verification of our theoretical results.

Key words. Optimal control with PDEs, Maxwell's equations, vector potential, pointwise state constraints, Nédélec's curl-conforming edge elements, a priori error estimates.

AMS subject classifications. 49K20, 35Q61, 65K10, 49M05.

1. Introduction. Maxwell's equations describe the interaction between magnetic and electric fields given by a coupled system of first-order partial differential equations (see e.g. [17]). If implicit time-stepping is used (cf. [7, 11, 23]), then we are faced with a degenerate elliptic boundary value problem of the following type:

$$(1.1) \quad \begin{cases} \mathbf{curl} \alpha \mathbf{curl} \mathbf{y} + \beta \mathbf{y} = \mathbf{j} & \text{in } \Omega \\ \mathbf{y} \times \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded, simply connected Lipschitz polyhedral domain with a connected boundary $\partial\Omega$. Further the coefficients $\alpha, \beta \in L^\infty(\Omega)$ satisfy

$$(1.2) \quad \underline{\alpha} \leq \alpha(x) \leq \bar{\alpha} \quad \text{a.e. in } \Omega \quad \underline{\beta} \leq \beta(x) \leq \bar{\beta} \quad \text{a.e. in } \Omega,$$

with some constants $0 < \underline{\alpha} < \bar{\alpha} < \infty$ and $0 < \underline{\beta} < \bar{\beta} < \infty$. As far as the analysis and the numerical treatment of the solution to (1.1) are concerned, we refer to [1, 2, 4, 5, 7, 8, 9, 11, 12, 13, 14, 18, 19, 20, 21, 23]. Maxwell's equations play an essential role in many modern technologies and applications including fusion energy, magnetohydrodynamics (MHD), electromagnetic induction heating, signal processing, magnetic levitation, and many more. Optimal control of Maxwell's equations is truly delicate and calls for a careful study. The present paper is devoted to the mathematical and numerical analysis of a class of state-constrained optimal control problems governed by (1.1) with \mathbf{j} as the control representing an applied current density. In view of the charge conservation law, the applied current \mathbf{j} needs to satisfy the following consistency condition:

$$(1.3) \quad \operatorname{div} \mathbf{j} = 0 \text{ in } \Omega \quad \mathbf{j} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega.$$

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We consider a *vector potential ansatz* to cope with the divergence-free constraint (1.3). More precisely, since

$$\{\mathbf{j} \in \mathbf{L}^2(\Omega) \mid \operatorname{div} \mathbf{j} = 0 \text{ in } \Omega, \mathbf{j} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\} = \{\operatorname{curl} \mathbf{u} \mid \mathbf{u} \in \mathbf{H}(\operatorname{curl}), \mathbf{u} \times \mathbf{n} = 0 \text{ on } \partial\Omega\}$$

(cf. [10]), we are justified to employ the ansatz

$$(1.4) \quad \mathbf{j} = \operatorname{curl} \mathbf{u},$$

and use the vector potential $\mathbf{u} \in \mathbf{H}_0(\operatorname{curl}) := \{\mathbf{q} \in \mathbf{H}(\operatorname{curl}) \mid \mathbf{q} \times \mathbf{n} = 0 \text{ on } \partial\Omega\}$ as our control. Based on (1.4), we concentrate throughout this paper on the following optimal control problem:

$$(P) \quad \min_{(\mathbf{u}, \mathbf{y}) \in \mathbf{H}_0(\operatorname{curl}) \times \mathbf{H}_0(\operatorname{curl})} J(\mathbf{u}, \mathbf{y}) := \frac{1}{2} \int_{\Omega} |\mathbf{y} - \mathbf{y}_d|^2 dx + \frac{\tau}{2} \int_{\Omega} |\operatorname{curl} \mathbf{y} - \mathbf{y}_c|^2 dx + \frac{\kappa}{2} \|\mathbf{u}\|_{\mathbf{H}(\operatorname{curl})}^2$$

subject to

$$(1.5) \quad \begin{cases} \operatorname{curl} \alpha \operatorname{curl} \mathbf{y} + \beta \mathbf{y} = \operatorname{curl} \mathbf{u} & \text{in } \Omega \\ \mathbf{y} \times \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

and to the pointwise state constraints

$$(1.6) \quad \mathbf{y}_a(x) \leq \mathbf{y}(x) \leq \mathbf{y}_b(x) \quad \text{a.e. in } \Omega.$$

Here, the cost parameters $\kappa > 0$, $\tau \geq 0$ and $\mathbf{y}_d, \mathbf{y}_c, \mathbf{y}_a, \mathbf{y}_b \in \mathbf{L}^2(\Omega)$ are given data, where \mathbf{y}_d and \mathbf{y}_c represent the desired state and the desired curl of the state, respectively. Further $\mathbf{y}_a(x) \leq \mathbf{y}_b(x)$ holds a.e. in Ω . Provided that the feasible set of (P) is not empty, the optimal control problem (P) admits a unique solution. Let us remark that the pointwise state constraints in (1.6) are of importance for instance to avoid undesired singularities in the optimal state $\bar{\mathbf{y}}$ of (P) (see Costabel et al. [8, 9] concerning singularities in the solution of Maxwell's equations). In particular, if $\mathbf{y}_a, \mathbf{y}_b \in \mathbf{L}^\infty(\Omega)$, then (1.6) guarantees that $\bar{\mathbf{y}}$ lives also in $\mathbf{L}^\infty(\Omega)$.

One of the main complexities involved in (P) is raised by the lack of regularity of the control-to-state mapping $\mathbf{u} \mapsto \mathbf{y}$ associated with the Maxwell's equations (1.5). Even if \mathbf{y}_a and \mathbf{y}_b are assumed to be continuous, (1.6) cannot be considered in the space of continuous functions since the control-to-state mapping is not well-defined as an operator from $\mathbf{H}_0(\operatorname{curl})$ to $\mathcal{C}(\bar{\Omega})$. For this reason we do not know how to derive existence of Lagrange multipliers under a reasonable assumption as the cone of nonnegative functions of the space associated with the state constraints has an empty interior.

We regularize the optimal control problem (P) by penalizing the pointwise state constraints. The corresponding regularized optimal control problem reads as follows:

$$(P^\gamma) \quad \begin{cases} \min_{(\mathbf{u}, \mathbf{y}) \in \mathbf{H}_0(\operatorname{curl}) \times \mathbf{H}_0(\operatorname{curl})} J^\gamma(\mathbf{u}, \mathbf{y}) \\ \text{subject to (1.5),} \end{cases}$$

where $J^\gamma(\mathbf{u}, \mathbf{y}) := J(\mathbf{u}, \mathbf{y}) + \frac{\gamma}{2} (\|\mathbf{max}(0, \mathbf{y} - \mathbf{y}_b)\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{max}(0, \mathbf{y}_a - \mathbf{y})\|_{\mathbf{L}^2(\Omega)}^2)$. Here, for every three-dimensional vector function $\mathbf{q} = (q_1, q_2, q_3)$, $\mathbf{max}(0, \mathbf{q}) = (\max(0, q_1), \max(0, q_2), \max(0, q_3))$ with $\max(0, \cdot)$ in the pointwise sense. In literature, the penalization method above is also known as the Moreau-Yosida regularization. This type of regularization was introduced by Ito and Kunisch [15, 16] to deal with measure-valued Lagrange multipliers of $H^1(\Omega)$ -elliptic optimal control

problems with pointwise state constraints. See Casas [6] for the analysis of $H^1(\Omega)$ -elliptic state-constrained optimal control problems. As an approximation of (P), Moreau–Yosida regularization seems to be quite suitable due to the availability of its Karush–Kuhn–Tucker (KKT) type optimality conditions with regular Lagrange multipliers.

To the best knowledge of the author, this article is the first contribution towards optimal control of Maxwell's equations in the presence of pointwise state constraints. Our main theoretical results established in this paper are as follows: First, by using a Helmholtz–decomposition, we show that the optimal control $\bar{\mathbf{u}}$ of (P) satisfies $\bar{\mathbf{u}} \in \mathbf{H}_0(\mathbf{curl}) \cap \mathbf{H}^{\frac{1}{2}+\epsilon}(\Omega)$ with some $\epsilon > 0$. Further, exploiting the optimality conditions for the regularized problem (P^γ) , we prove that the optimal control \mathbf{u}^γ of (P^γ) possesses the following regularity property:

$$(1.7) \quad \mathbf{u}^\gamma, \mathbf{curl} \mathbf{u}^\gamma \in \mathbf{H}^{\frac{1}{2}+\epsilon}(\Omega).$$

Here we see that \mathbf{u}^γ enjoys better regularity than $\bar{\mathbf{u}}$. With (1.7) at hand and using certain regularity properties of the control–to–state mapping, we are able to show the strong convergence of $(\mathbf{u}^\gamma)_{\gamma>0}$ towards $\bar{\mathbf{u}}$ with respect to the $\mathbf{H}(\mathbf{curl})$ -topology as $\gamma \rightarrow \infty$. The last part of this paper deals with the numerical analysis of (P^γ) . We discretize both the control and the state by Nédélec's edge elements of the first family (see [20]). Employing (1.7), we are able to establish an a priori estimate for the $\mathbf{H}(\mathbf{curl})$ -error between \mathbf{u}^γ and its finite element approximation. The rate of convergence is $0.5 + \epsilon$ and seems to be optimal.

The rest of this paper is organized as follows: in the next section we introduce our notation and present a regularity result for (P). Section 3 is devoted to the mathematical analysis of the regularized optimal control problem (P^γ) . Here, we discuss the regularity property of \mathbf{u}^γ and prove the strong $\mathbf{H}(\mathbf{curl})$ -convergence of $(\mathbf{u}^\gamma)_{\gamma>0}$ as $\gamma \rightarrow \infty$. In Section 4, we consider a finite element approximation of (P^γ) based on an edge element discretization, and an a priori error estimate for the discretization error is derived. Finally, in Section 5, we present a report on numerical tests including a numerical verification of the error estimate.

2. Preliminaries. We begin by introducing the notation used in this article. Thereafter we prove a regularity result for the solution to the optimal control problem (P).

2.1. Notation. Throughout the paper, c denotes a generic positive constant which can take different values on different occasions. If V is a Hilbert space, then we use the notation $\|\cdot\|_V$ and $(\cdot, \cdot)_V$ for a standard norm and a standard inner product used in V . If V is continuously embedded in another normed function space Y , then we write $V \hookrightarrow Y$. We use a bold typeface to indicate a three-dimensional vector function or a Hilbert space of three-dimensional vector functions. The following spaces are used throughout the paper:

$$\begin{aligned} \mathbf{H}(\mathbf{curl}) &= \{\mathbf{q} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl} \mathbf{q} \in \mathbf{L}^2(\Omega)\} & \mathbf{H}_0(\mathbf{curl}) &= \{\mathbf{q} \in \mathbf{H}(\mathbf{curl}) \mid \mathbf{q} \times \mathbf{n} = 0 \text{ on } \partial\Omega\} \\ \mathbf{H}^s(\mathbf{curl}) &= \{\mathbf{q} \in \mathbf{H}^s(\Omega) \mid \mathbf{curl} \mathbf{q} \in \mathbf{H}^s(\Omega)\} & \mathbf{H}_0^s(\mathbf{curl}) &= \mathbf{H}^s(\mathbf{curl}) \cap \mathbf{H}_0(\mathbf{curl}) \\ \mathbf{H}(\mathbf{div}) &= \{\mathbf{q} \in \mathbf{L}^2(\Omega) \mid \mathbf{div} \mathbf{q} \in \mathbf{L}^2(\Omega)\} & \mathbf{H}_0(\mathbf{div}) &= \{\mathbf{q} \in \mathbf{H}(\mathbf{div}) \mid \mathbf{q} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\} \\ \mathbf{H}_{0,0}(\mathbf{div}) &= \{\mathbf{q} \in \mathbf{H}_0(\mathbf{div}) \mid \mathbf{div} \mathbf{q} = 0\}, \end{aligned}$$

where s is a nonnegative real number. The spaces $\mathbf{H}(\mathbf{curl})$, $\mathbf{H}(\mathbf{div})$, $\mathbf{H}^s(\mathbf{curl})$, with $s \geq 0$, are endowed with the following norms:

$$\begin{aligned} \|\mathbf{q}\|_{\mathbf{H}(\mathbf{curl})} &= (\|\mathbf{q}\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{curl} \mathbf{q}\|_{\mathbf{L}^2(\Omega)}^2)^{\frac{1}{2}} \\ \|\mathbf{q}\|_{\mathbf{H}(\mathbf{div})} &= (\|\mathbf{q}\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{div} \mathbf{q}\|_{\mathbf{L}^2(\Omega)}^2)^{\frac{1}{2}} \\ \|\mathbf{q}\|_{\mathbf{H}^s(\mathbf{curl})} &= (\|\mathbf{q}\|_{\mathbf{H}^s(\Omega)}^2 + \|\mathbf{curl} \mathbf{q}\|_{\mathbf{H}^s(\Omega)}^2)^{\frac{1}{2}}. \end{aligned}$$

2.2. Regularity result. In the upcoming lemma, we recall an embedding result important for our subsequent analysis.

LEMMA 2.1 (Amrouche et al. [3]). *There exists a constant $\epsilon > 0$ such that*

$$(2.1) \quad \mathbf{H}_0(\mathbf{curl}) \cap \mathbf{H}(\text{div}) \hookrightarrow \mathbf{H}^{\frac{1}{2}+\epsilon}(\Omega) \quad \text{and} \quad \mathbf{H}(\mathbf{curl}) \cap \mathbf{H}_0(\text{div}) \hookrightarrow \mathbf{H}^{\frac{1}{2}+\epsilon}(\Omega).$$

If Ω is convex, then (2.1) holds with $\epsilon = \frac{1}{2}$.

Let us further introduce the solution operator $\mathcal{S} : \mathbf{H}_0(\mathbf{curl}) \longrightarrow \mathbf{H}_0(\mathbf{curl})$ that assigns to every element $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl})$ the solution $\mathbf{y} \in \mathbf{H}_0(\mathbf{curl})$ of

$$(2.2) \quad (\alpha \mathbf{curl} \mathbf{y}, \mathbf{curl} \mathbf{q})_{L^2(\Omega)} + (\beta \mathbf{y}, \mathbf{q})_{L^2(\Omega)} = (\mathbf{curl} \mathbf{u}, \mathbf{q})_{L^2(\Omega)} \quad \forall \mathbf{q} \in \mathbf{H}_0(\mathbf{curl}).$$

Employing the solution operator \mathcal{S} , we can rephrase the optimal control problem (P) as follows:

$$(P) \quad \min_{\mathbf{u} \in \mathcal{U}^{feas}} f(\mathbf{u}) := J(\mathbf{u}, \mathcal{S}\mathbf{u}),$$

where the feasible set $\mathcal{U}^{feas} \subset \mathbf{H}_0(\mathbf{curl})$ is given by the following convex and closed set:

$$\mathcal{U}^{feas} := \{\mathbf{u} \in \mathbf{H}_0(\mathbf{curl}) \mid \mathbf{y}_a(x) \leq (\mathcal{S}\mathbf{u})(x) \leq \mathbf{y}_b(x) \text{ a.e. in } \Omega\}.$$

We assume that \mathcal{U}^{feas} is not empty. For instance, if $\mathbf{y}_a(x) \leq 0$ and $\mathbf{y}_b(x) \geq 0$ hold a.e. in Ω , then $\mathcal{U}^{feas} \neq \emptyset$ since $\mathbf{0} \in \mathcal{U}^{feas}$. In view of $\mathcal{U}^{feas} \neq \emptyset$, existence and uniqueness of an optimal solution $\bar{\mathbf{u}} \in \mathbf{H}_0(\mathbf{curl})$ of (P) follow from classical arguments. This solution turns out to satisfy a higher regularity property as demonstrated in the following theorem:

THEOREM 2.2. *The optimal solution of (P) satisfies $\bar{\mathbf{u}} \in \mathbf{H}_0(\mathbf{curl}) \cap \mathbf{H}^{\frac{1}{2}+\epsilon}(\Omega)$ with $\epsilon > 0$ as in Lemma 2.1.*

Proof. To demonstrate the higher regularity property of $\bar{\mathbf{u}}$, we make use of the following Helmholtz–decomposition:

$$(2.3) \quad \mathbf{H}_0(\mathbf{curl}) = \nabla H_0^1(\Omega) \oplus \mathbf{X}_{N,0},$$

with $\mathbf{X}_{N,0} := \{\mathbf{z} \in \mathbf{H}_0(\mathbf{curl}) \mid (\mathbf{z}, \nabla \phi)_{L^2(\Omega)} = 0 \quad \forall \phi \in H_0^1(\Omega)\} = \{\mathbf{z} \in \mathbf{H}_0(\mathbf{curl}) \cap \mathbf{H}(\text{div}) \mid \text{div} \mathbf{z} = 0\}$. According to (2.3), there exists a unique pair $(\bar{\phi}, \bar{\mathbf{z}}) \in H_0^1(\Omega) \times \mathbf{X}_{N,0}$ such that

$$(2.4) \quad \bar{\mathbf{u}} = \nabla \bar{\phi} + \bar{\mathbf{z}}.$$

Now, for any $\delta \in \mathbb{R}$, let us introduce

$$(2.5) \quad \bar{\mathbf{u}}_\delta := \delta \nabla \bar{\phi} + \bar{\mathbf{z}}.$$

Obviously, since $\mathbf{curl} \nabla \bar{\phi} = 0$, we have

$$(2.6) \quad \mathbf{curl} \bar{\mathbf{u}}_\delta = \mathbf{curl} \bar{\mathbf{z}} = \mathbf{curl} \bar{\mathbf{u}} \quad \implies \quad \mathcal{S}\bar{\mathbf{u}}_\delta = \mathcal{S}\bar{\mathbf{u}} \quad \forall \delta \in \mathbb{R}.$$

As a consequence, $\bar{\mathbf{u}}_\delta \in \mathcal{U}^{feas}$ holds for all $\delta \in \mathbb{R}$, and hence

$$(2.7) \quad f(\bar{\mathbf{u}}) \leq f(\bar{\mathbf{u}}_\delta) \quad \implies \quad \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 \leq \|\bar{\mathbf{u}}_\delta\|_{L^2(\Omega)}^2.$$

As $(\bar{\mathbf{z}}, \nabla \bar{\phi})_{\mathbf{L}^2(\Omega)} = 0$, the latter inequality along with (2.4)–(2.5) implies that

$$(2.8) \quad \begin{aligned} 0 &\geq \|\bar{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}^2 - \|\bar{\mathbf{u}}_\delta\|_{\mathbf{L}^2(\Omega)}^2 = \|\nabla \bar{\phi}\|_{\mathbf{L}^2(\Omega)}^2 + \|\bar{\mathbf{z}}\|_{\mathbf{L}^2(\Omega)}^2 - \delta^2 \|\nabla \bar{\phi}\|_{\mathbf{L}^2(\Omega)}^2 - \|\bar{\mathbf{z}}\|_{\mathbf{L}^2(\Omega)}^2 \\ &= (1 - \delta^2) \|\nabla \bar{\phi}\|_{\mathbf{L}^2(\Omega)}^2. \end{aligned}$$

Since (2.8) holds for any $\delta \in \mathbb{R}$, it follows that $\nabla \bar{\phi} = 0$ and thus, by Lemma 2.1, we come to the conclusion that $\bar{\mathbf{u}} = \bar{\mathbf{z}} \in \mathbf{X}_{N,0} \hookrightarrow \mathbf{H}^{\frac{1}{2}+\epsilon}(\Omega)$. \square

3. Analysis of (P^γ) . Analogously to (P), the regularized optimal control problem (P^γ) can be equivalently expressed as follows:

$$(P^\gamma) \quad \min_{\mathbf{u} \in \mathbf{H}_0(\mathbf{curl})} f^\gamma(\mathbf{u}) := J^\gamma(\mathbf{u}, \mathcal{S}\mathbf{u}).$$

As (P^γ) belongs to a class of unconstrained linear–quadratic optimization problems, its sufficient and necessary optimality condition at the optimal solution \mathbf{u}^γ is given by

$$(3.1) \quad f^{\gamma'}(\mathbf{u}^\gamma)\mathbf{q} = 0 \quad \forall \mathbf{q} \in \mathbf{H}_0(\mathbf{curl}).$$

This variational equality yields the following characterization for the solution \mathbf{u}^γ :

THEOREM 3.1. *Let $\gamma > 0$. Then $\mathbf{u}^\gamma \in \mathbf{H}_0(\mathbf{curl})$ is the optimal solution of (P^γ) if and only if there exist $\mathbf{p}^\gamma \in \mathbf{H}_0(\mathbf{curl})$ and $\lambda_b^\gamma, \lambda_a^\gamma \in \mathbf{L}^2(\Omega)$ such that*

$$(3.2) \quad \begin{aligned} (\alpha \mathbf{curl} \mathbf{p}^\gamma, \mathbf{curl} \mathbf{q})_{\mathbf{L}^2(\Omega)} + (\beta \mathbf{p}^\gamma, \mathbf{q})_{\mathbf{L}^2(\Omega)} &= (\mathcal{S}\mathbf{u}^\gamma - \mathbf{y}_d + \lambda_b^\gamma - \lambda_a^\gamma, \mathbf{q})_{\mathbf{L}^2(\Omega)} \\ &\quad + \tau(\mathbf{curl} \mathcal{S}\mathbf{u}^\gamma - \mathbf{y}_c, \mathbf{curl} \mathbf{q})_{\mathbf{L}^2(\Omega)} \quad \forall \mathbf{q} \in \mathbf{H}_0(\mathbf{curl}) \end{aligned}$$

$$(3.3) \quad (\mathbf{curl} \mathbf{u}^\gamma, \mathbf{curl} \mathbf{q})_{\mathbf{L}^2(\Omega)} + (\mathbf{u}^\gamma, \mathbf{q})_{\mathbf{L}^2(\Omega)} = -\kappa^{-1}(\mathbf{p}^\gamma, \mathbf{curl} \mathbf{q})_{\mathbf{L}^2(\Omega)} \quad \forall \mathbf{q} \in \mathbf{H}_0(\mathbf{curl})$$

$$(3.4) \quad \lambda_b^\gamma = \gamma \max(0, \mathcal{S}\mathbf{u}^\gamma - \mathbf{y}_b) \quad \lambda_a^\gamma = \gamma \max(0, \mathbf{y}_a - \mathcal{S}\mathbf{u}^\gamma).$$

Proof. By the Lax-Milgram lemma, (3.2) admits a unique solution $\mathbf{p}^\gamma \in \mathbf{H}_0(\mathbf{curl})$. Setting $\mathcal{S}\mathbf{q}$ with $\mathbf{q} \in \mathbf{H}_0(\mathbf{curl})$ as a test function in (3.2), we obtain

$$\begin{aligned} (\alpha \mathbf{curl} \mathbf{p}^\gamma, \mathbf{curl} \mathcal{S}\mathbf{q})_{\mathbf{L}^2(\Omega)} + (\beta \mathbf{p}^\gamma, \mathcal{S}\mathbf{q})_{\mathbf{L}^2(\Omega)} &= (\mathcal{S}\mathbf{u}^\gamma - \mathbf{y}_d + \lambda_b^\gamma - \lambda_a^\gamma, \mathcal{S}\mathbf{q})_{\mathbf{L}^2(\Omega)} \\ &\quad + \tau(\mathbf{curl} \mathcal{S}\mathbf{u}^\gamma - \mathbf{y}_c, \mathbf{curl} \mathcal{S}\mathbf{q})_{\mathbf{L}^2(\Omega)} \quad \forall \mathbf{q} \in \mathbf{H}_0(\mathbf{curl}). \end{aligned}$$

According to the definition of \mathcal{S} , we have

$$(\alpha \mathbf{curl} \mathcal{S}\mathbf{q}, \mathbf{curl} \mathbf{p}^\gamma)_{\mathbf{L}^2(\Omega)} + (\beta \mathcal{S}\mathbf{q}, \mathbf{p}^\gamma)_{\mathbf{L}^2(\Omega)} = (\mathbf{curl} \mathbf{q}, \mathbf{p}^\gamma)_{\mathbf{L}^2(\Omega)},$$

from which we deduce that

$$(\mathcal{S}\mathbf{u}^\gamma - \mathbf{y}_d + \lambda_b^\gamma - \lambda_a^\gamma, \mathcal{S}\mathbf{q})_{\mathbf{L}^2(\Omega)} + \tau(\mathbf{curl} \mathcal{S}\mathbf{u}^\gamma - \mathbf{y}_c, \mathbf{curl} \mathcal{S}\mathbf{q})_{\mathbf{L}^2(\Omega)} = (\mathbf{p}^\gamma, \mathbf{curl} \mathbf{q})_{\mathbf{L}^2(\Omega)} \quad \forall \mathbf{q} \in \mathbf{H}_0(\mathbf{curl}).$$

By the latter variational equality, we see that (3.1) is equivalent to (3.3). \square

Next, we investigate the regularity property of the solution to the regularized problem (P^γ) . Exploiting the structure of the optimality system in Theorem 3.1, we are able to extract a regularity result for the optimal solution \mathbf{u}^γ of (P^γ) which is better than Theorem 2.2.

THEOREM 3.2. *Let $\epsilon > 0$ be as in Lemma 2.1. Then*

$$\mathbf{u}^\gamma \in \mathbf{H}_0^{\frac{1}{2}+\epsilon}(\mathbf{curl})$$

holds for all $\gamma > 0$. Furthermore, there exists a positive constant c independent of γ such that

$$(3.5) \quad \|\mathbf{u}^\gamma\|_{\mathbf{H}^{\frac{1}{2}+\epsilon}(\Omega)} \leq c \|\mathbf{u}^\gamma\|_{\mathbf{H}(\mathbf{curl})}.$$

Proof. Setting $\mathbf{q} = \nabla\phi$ with $\phi \in \mathcal{C}_0^\infty(\Omega)$ in the variational equality (3.3) results in

$$(\mathbf{u}^\gamma, \nabla\phi)_{\mathbf{L}^2(\Omega)} = 0 \quad \forall \phi \in \mathcal{C}_0^\infty(\Omega),$$

from which we deduce that

$$(3.6) \quad \mathbf{u}^\gamma \in \mathbf{H}_0(\mathbf{curl}) \cap \mathbf{H}(\text{div})$$

and

$$(3.7) \quad \|\mathbf{u}^\gamma\|_{\mathbf{H}(\mathbf{curl}) \cap \mathbf{H}(\text{div})} = \|\mathbf{u}^\gamma\|_{\mathbf{H}(\mathbf{curl})}.$$

Further, the variational equality (3.3) also implies that

$$\begin{aligned} (\mathbf{curl} \mathbf{u}^\gamma, \mathbf{curl} \mathbf{q})_{\mathbf{L}^2(\Omega)} &= -(\mathbf{u}^\gamma, \mathbf{q})_{\mathbf{L}^2(\Omega)} - \kappa^{-1}(\mathbf{p}^\gamma, \mathbf{curl} \mathbf{q})_{\mathbf{L}^2(\Omega)} \\ &= -(\mathbf{u}^\gamma + \kappa^{-1} \mathbf{curl} \mathbf{p}^\gamma, \mathbf{q})_{\mathbf{L}^2(\Omega)} \quad \forall \mathbf{q} \in \mathcal{C}_0^\infty(\Omega)^3. \end{aligned}$$

Therefore, the distributional definition of the curl yields

$$(3.8) \quad \mathbf{curl} \mathbf{curl} \mathbf{u}^\gamma = -\mathbf{u}^\gamma - \kappa^{-1} \mathbf{curl} \mathbf{p}^\gamma \in \mathbf{L}^2(\Omega) \implies \mathbf{curl} \mathbf{u}^\gamma \in \mathbf{H}(\mathbf{curl}).$$

In addition, as $\mathbf{u}^\gamma \in \mathbf{H}_0(\mathbf{curl})$, we obtain $\mathbf{curl} \mathbf{u}^\gamma \in \mathbf{H}_{0,0}(\text{div})$ such that

$$(3.9) \quad \mathbf{curl} \mathbf{u}^\gamma \in \mathbf{H}_{0,0}(\text{div}) \cap \mathbf{H}(\mathbf{curl}).$$

In view of (3.6) and (3.9), Lemma 2.1 ensures that $\mathbf{u}^\gamma \in \mathbf{H}_0^{\frac{1}{2}+\epsilon}(\mathbf{curl})$. Also, the estimate (3.5) is an immediate consequence of Lemma 2.1 along with (3.7). In conclusion, the assertion is valid. \square

Invoking Theorem 3.2, we obtain the following convergence result:

LEMMA 3.3. *Let $\epsilon > 0$ be as in Lemma 2.1. Then*

$$\begin{array}{llll} \mathbf{u}^\gamma & \rightarrow & \bar{\mathbf{u}} & \text{strongly in } \mathbf{L}^2(\Omega) \quad \text{as } \gamma \rightarrow \infty \\ \mathbf{curl} \mathbf{u}^\gamma & \rightharpoonup & \mathbf{curl} \bar{\mathbf{u}} & \text{weakly in } \mathbf{L}^2(\Omega) \quad \text{as } \gamma \rightarrow \infty. \end{array}$$

Proof. Since $\bar{\mathbf{u}} \in \mathcal{U}^{feas}$ and \mathbf{u}^γ is the optimal solution of (P^γ) , we have

$$(3.10) \quad f(\mathbf{u}^\gamma) \leq f^\gamma(\mathbf{u}^\gamma) \leq f^\gamma(\bar{\mathbf{u}}) = f(\bar{\mathbf{u}}) \quad \forall \gamma > 0,$$

from which we deduce that

$$\frac{\kappa}{2} \|\mathbf{u}^\gamma\|_{\mathbf{H}(\mathbf{curl})}^2 \leq f(\bar{\mathbf{u}}) \quad \forall \gamma > 0.$$

By the latter inequality, Theorem 3.2 guarantees that $(\mathbf{u}^\gamma)_{\gamma>0}$ is uniformly bounded in $\mathbf{H}_0(\mathbf{curl}) \cap \mathbf{H}^{\frac{1}{2}+\epsilon}(\Omega)$. For this reason, there exists a subsequence of $(\mathbf{u}^\gamma)_{\gamma>0}$, denoted w.l.o.g. by $(\mathbf{u}^\gamma)_{\gamma>0}$, such that

$$(3.11) \quad \mathbf{u}^\gamma \rightharpoonup \tilde{\mathbf{u}} \quad \text{weakly in } \mathbf{H}^{\frac{1}{2}+\epsilon}(\Omega) \text{ as } \gamma \rightarrow \infty,$$

$$(3.12) \quad \mathbf{u}^\gamma \rightharpoonup \tilde{\mathbf{u}} \quad \text{weakly in } \mathbf{H}_0(\mathbf{curl}) \text{ as } \gamma \rightarrow \infty,$$

and hence

$$(3.13) \quad \mathcal{S}\mathbf{u}^\gamma \rightharpoonup \mathcal{S}\tilde{\mathbf{u}} \quad \text{weakly in } \mathbf{H}_0(\mathbf{curl}) \text{ as } \gamma \rightarrow \infty.$$

Thanks to the weak lower semicontinuity of $f : \mathbf{H}_0(\mathbf{curl}) \rightarrow \mathbb{R}$, (3.12) and (3.10) yield

$$(3.14) \quad f(\tilde{\mathbf{u}}) \leq \liminf_{\gamma \rightarrow \infty} f(\mathbf{u}^\gamma) \leq \limsup_{\gamma \rightarrow \infty} f(\mathbf{u}^\gamma) \leq f(\bar{\mathbf{u}}).$$

Let us now demonstrate that $\tilde{\mathbf{u}} \in \mathcal{U}^{feas}$. By the definition of f^γ , we infer from (3.10) that

$$\|\mathbf{max}(0, \mathcal{S}\mathbf{u}^\gamma - \mathbf{y}_b)\|_{L^2(\Omega)}^2 + \|\mathbf{max}(0, \mathbf{y}_a - \mathcal{S}\mathbf{u}^\gamma)\|_{L^2(\Omega)}^2 \leq 2\gamma^{-1} f(\bar{\mathbf{u}}).$$

In view of the weak lower semicontinuity of $\mathcal{M} : L^2(\Omega) \rightarrow \mathbb{R}$, $\mathcal{M}(\mathbf{q}) = \|\mathbf{max}(0, \mathbf{q})\|_{L^2(\Omega)}^2$, we obtain from the latter inequality and (3.13) that

$$\mathbf{max}(0, \mathcal{S}\tilde{\mathbf{u}} - \mathbf{y}_b) = \mathbf{max}(0, \mathbf{y}_a - \mathcal{S}\tilde{\mathbf{u}}) = 0,$$

and thus $\tilde{\mathbf{u}} \in \mathcal{U}^{feas}$. Consequently, since $\bar{\mathbf{u}}$ is the optimal solution of (P), (3.14) implies that $\bar{\mathbf{u}} = \tilde{\mathbf{u}}$. In conclusion, the assertion follows from the latter identity along with (3.11) and the compactness of the embedding $\mathbf{H}^{\frac{1}{2}+\epsilon}(\Omega) \hookrightarrow L^2(\Omega)$. \square

REMARK 3.4. Lemma 3.3 does not necessarily guarantee that $\mathbf{u}^\gamma \rightarrow \bar{\mathbf{u}}$ strongly in $\mathbf{H}_0(\mathbf{curl})$ as $\gamma \rightarrow \infty$. To establish the strong convergence of $(\mathbf{u}^\gamma)_{\gamma>0}$ in $\mathbf{H}_0(\mathbf{curl})$, we require some higher regularity properties of the solution operator \mathcal{S} which we shall address in the upcoming propositions.

PROPOSITION 3.5. Let $\beta \in C^{0,1}(\bar{\Omega})$. Then the solution operator \mathcal{S} is well-defined and bounded from $\mathbf{H}_0(\mathbf{curl})$ to $\mathbf{H}_0(\mathbf{curl}) \cap \mathbf{H}^{\frac{1}{2}+\epsilon}(\Omega)$ with $\epsilon > 0$ as in Lemma 2.1.

Proof. Let $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl})$ and $\mathbf{y} = \mathcal{S}\mathbf{u}$. Since β is away from zero and Lipschitz-continuous, we have $\frac{1}{\beta} \in C^{0,1}(\bar{\Omega})$ and consequently $\frac{1}{\beta}\phi \in H_0^1(\Omega)$ holds for all $\phi \in C_0^\infty(\Omega)$. Then setting $\mathbf{q} = \nabla(\frac{1}{\beta}\phi)$ with $\phi \in C_0^\infty(\Omega)$ in the variational equality (2.2) results in

$$(\beta\mathbf{y}, \nabla(\frac{1}{\beta}\phi))_{L^2(\Omega)} = (\mathbf{curl} \mathbf{u}, \nabla(\frac{1}{\beta}\phi))_{L^2(\Omega)} = 0 \quad \forall \phi \in C_0^\infty(\Omega).$$

Since $\nabla\phi = \nabla(\beta\frac{1}{\beta}\phi) = \beta\nabla(\frac{1}{\beta}\phi) + (\nabla\beta)\frac{1}{\beta}\phi$, we obtain

$$0 = (\beta\mathbf{y}, \nabla(\frac{1}{\beta}\phi))_{L^2(\Omega)} = (\mathbf{y}, \beta\nabla(\frac{1}{\beta}\phi))_{L^2(\Omega)} = (\mathbf{y}, \nabla\phi)_{L^2(\Omega)} - (\frac{1}{\beta}\nabla\beta \cdot \mathbf{y}, \phi)_{L^2(\Omega)} \quad \forall \phi \in C_0^\infty(\Omega),$$

or equivalently

$$(\mathbf{y}, \nabla \phi)_{L^2(\Omega)} = \left(\frac{1}{\beta} \nabla \beta \cdot \mathbf{y}, \phi \right)_{L^2(\Omega)} \quad \forall \phi \in C_0^\infty(\Omega).$$

Consequently, by the distributional definition of the divergence, we obtain

$$(3.15) \quad \operatorname{div} \mathbf{y} = - \underbrace{\frac{1}{\beta} \nabla \beta}_{\in L^\infty} \cdot \underbrace{\mathbf{y}}_{\in L^2} \in L^2(\Omega).$$

Hence, thanks to Lemma 2.1, $\mathbf{y} \in \mathbf{H}_0(\mathbf{curl}) \cap \mathbf{H}(\operatorname{div}) \hookrightarrow \mathbf{H}^{\frac{1}{2}+\epsilon}(\Omega)$. Also (3.15) implies that

$$\begin{aligned} \|\mathbf{y}\|_{\mathbf{H}(\mathbf{curl}) \cap \mathbf{H}(\operatorname{div})} &= (\|\mathbf{y}\|_{L^2(\Omega)}^2 + \|\mathbf{curl} \mathbf{y}\|_{L^2(\Omega)}^2 + \|\operatorname{div} \mathbf{y}\|_{L^2(\Omega)}^2)^{\frac{1}{2}} \leq c \|\mathbf{y}\|_{\mathbf{H}(\mathbf{curl})} = c \|\mathbf{S}\mathbf{u}\|_{\mathbf{H}(\mathbf{curl})} \\ &\leq c \|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl})} \end{aligned}$$

with $c > 0$ independent of \mathbf{y} and \mathbf{u} . This completes the proof. \square

PROPOSITION 3.6. *Let $\alpha, \beta \in C^{0,1}(\overline{\Omega})$. Then the solution operator \mathbf{S} is well-defined and bounded from $\mathbf{H}_0(\mathbf{curl})$ to $\mathbf{H}_0^{\frac{1}{2}+\epsilon}(\mathbf{curl})$ with $\epsilon > 0$ as in Lemma 2.1.*

Proof. Let $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl})$ and $\mathbf{S}\mathbf{u} = \mathbf{y}$. From variational equality (2.2), \mathbf{y} satisfies

$$(\alpha \mathbf{curl} \mathbf{y}, \mathbf{curl} \mathbf{q})_{L^2(\Omega)} = (-\beta \mathbf{y} + \mathbf{curl} \mathbf{u}, \mathbf{q})_{L^2(\Omega)} \quad \forall \mathbf{q} \in C_0^\infty(\Omega)^3,$$

from which we infer that

$$(3.16) \quad \mathbf{curl}(\alpha \mathbf{curl} \mathbf{y}) = -\beta \mathbf{y} + \mathbf{curl} \mathbf{u} \in L^2(\Omega) \implies \alpha \mathbf{curl} \mathbf{y} \in \mathbf{H}(\mathbf{curl}).$$

Since α is away from zero and Lipschitz-continuous, $\frac{1}{\alpha}$ is also Lipschitz-continuous such that

$$\mathbf{curl} \mathbf{y} = \underbrace{\frac{1}{\alpha}}_{\in C^{0,1}} \underbrace{\alpha \mathbf{curl} \mathbf{y}}_{\in \mathbf{H}(\mathbf{curl})} \in \mathbf{H}(\mathbf{curl}),$$

and consequently Lemma 2.1 yields

$$(3.17) \quad \mathbf{curl} \mathbf{y} \in \mathbf{H}_{0,0}(\operatorname{div}) \cap \mathbf{H}(\mathbf{curl}) \hookrightarrow \mathbf{H}^{\frac{1}{2}+\epsilon}(\Omega).$$

Furthermore

$$(3.18) \quad \begin{aligned} \|\mathbf{curl} \mathbf{y}\|_{\mathbf{H}(\operatorname{div}) \cap \mathbf{H}(\mathbf{curl})} &= (\|\mathbf{curl} \mathbf{y}\|_{L^2(\Omega)}^2 + \|\mathbf{curl} \mathbf{curl} \mathbf{y}\|_{L^2(\Omega)}^2)^{\frac{1}{2}} \\ &\leq c (\|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl})}^2 + \|\mathbf{curl} \mathbf{curl} \mathbf{y}\|_{L^2(\Omega)}^2)^{\frac{1}{2}} \end{aligned}$$

with a constant $c > 0$ independent of \mathbf{y} and \mathbf{u} . Further, as $\frac{1}{\alpha} \in C^{0,1}(\overline{\Omega})$ and $\alpha \mathbf{curl} \mathbf{y} \in \mathbf{H}(\mathbf{curl})$, we have

$$(3.19) \quad \begin{aligned} \mathbf{curl} \mathbf{curl} \mathbf{y} &= \mathbf{curl} \frac{1}{\alpha} \alpha \mathbf{curl} \mathbf{y} = \nabla \left(\frac{1}{\alpha} \right) \times \alpha \mathbf{curl} \mathbf{y} + \frac{1}{\alpha} \mathbf{curl} \alpha \mathbf{curl} \mathbf{y} \\ &\stackrel{(3.16)}{=} \underbrace{\nabla \left(\frac{1}{\alpha} \right)}_{\in L^2(\Omega)} \times \alpha \mathbf{curl} \mathbf{y} + \frac{1}{\alpha} (-\beta \mathbf{y} + \mathbf{curl} \mathbf{u}). \end{aligned}$$

Using (3.19) in (3.18), we infer that

$$(3.20) \quad \|\mathbf{curl} \mathbf{y}\|_{\mathbf{H}(\text{div}) \cap \mathbf{H}(\mathbf{curl})} \leq c \|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl})}$$

holds with a positive constant c independent of \mathbf{y} and \mathbf{u} . Combining (3.17), (3.20) and Proposition 3.5, we see that the assertion is valid. \square

We are now in the position to establish the convergence of $(\mathbf{u}_\gamma)_{\gamma>0}$ with respect to the $\mathbf{H}_0(\mathbf{curl})$ -topology:

THEOREM 3.7. *Assume that $\alpha, \beta \in \mathcal{C}^{0,1}(\overline{\Omega})$. Then*

$$(3.21) \quad \mathbf{u}^\gamma \rightarrow \overline{\mathbf{u}} \quad \text{strongly in } \mathbf{H}_0(\mathbf{curl}) \text{ as } \gamma \rightarrow \infty.$$

Proof. We recall from Lemma 3.3 that

$$(3.22) \quad \mathbf{u}^\gamma \rightarrow \overline{\mathbf{u}} \quad \text{strongly in } \mathbf{L}^2(\Omega) \quad \text{as } \gamma \rightarrow \infty$$

$$(3.23) \quad \mathbf{curl} \mathbf{u}^\gamma \rightharpoonup \mathbf{curl} \overline{\mathbf{u}} \quad \text{weakly in } \mathbf{L}^2(\Omega) \quad \text{as } \gamma \rightarrow \infty.$$

Then Proposition 3.6 together with the compactness of the embedding $\mathbf{H}_0^{\frac{1}{2}+\epsilon}(\mathbf{curl}) \hookrightarrow \mathbf{H}_0(\mathbf{curl})$ implies that

$$(3.24) \quad \mathcal{S} \mathbf{u}^\gamma \rightarrow \mathcal{S} \overline{\mathbf{u}} \quad \text{strongly in } \mathbf{H}_0(\mathbf{curl}) \quad \text{as } \gamma \rightarrow \infty.$$

Now, according to (3.14), $\lim_{\gamma \rightarrow \infty} f(\mathbf{u}^\gamma) = f(\overline{\mathbf{u}})$ such that (3.22) and (3.24) yield

$$\lim_{\gamma \rightarrow \infty} \|\mathbf{curl} \mathbf{u}^\gamma\|_{\mathbf{L}^2(\Omega)} = \|\mathbf{curl} \overline{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}.$$

From the latter norm convergence and the weak convergence (3.23), it follows that that $\mathbf{curl} \mathbf{u}^\gamma \rightarrow \mathbf{curl} \overline{\mathbf{u}}$ strongly in $\mathbf{L}^2(\Omega)$ as $\gamma \rightarrow \infty$. In conclusion, the assertion is valid. \square

REMARK 3.8. *In the case of $\tau = 0$ or equivalently*

$$f(\mathbf{u}) = \frac{1}{2} \|\mathcal{S} \mathbf{u} - \mathbf{y}_d\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\kappa}{2} \|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl})}^2,$$

the compactness of \mathcal{S} from $\mathbf{H}_0(\mathbf{curl})$ to $\mathbf{L}^2(\Omega)$ is sufficient for the convergence (3.21). Thus the Lipschitz-continuity assumption $\alpha \in \mathcal{C}^{0,1}(\overline{\Omega})$ in Theorem 3.7 can be dropped, if $\tau = 0$.

4. Finite element approximation of (P^γ) . This section is concerned with the numerical analysis of (P^γ) . Based on an edge element discretization for the state and the control, we will soon propose a finite element approximation of (P^γ) . Our ultimate goal is to set up an a priori estimate for the error between the optimal solution \mathbf{u}^γ and its finite element approximation. The key point to establish the error estimate lies in the regularity $\mathbf{u}^\gamma \in \mathbf{H}_0^{\frac{1}{2}+\epsilon}(\mathbf{curl})$ as proven in Theorem 3.2. Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations $\mathcal{T}_h = \{T\}$ consisting of tetrahedron T such that $\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} T$. Here h stands for the maximal diameter of all elements T . We define the space of lowest order edge elements of Nédélec's first family by

$$\mathbf{V}_h := \{\mathbf{q}_h \in \mathbf{H}_0(\mathbf{curl}) \mid \mathbf{q}_h|_T = \mathbf{a}_T + \mathbf{b}_T \times x \text{ with } \mathbf{a}_T, \mathbf{b}_T \in \mathbb{R}^3\}.$$

The finite element approximation of (P^γ) we focus on reads as follows:

$$(P_h^\gamma) \quad \min_{\mathbf{u}_h \in \mathbf{V}_h} f_h^\gamma(\mathbf{u}_h) := \frac{1}{2} \|\mathcal{S}_h \mathbf{u}_h - \mathbf{y}_d\|_{L^2(\Omega)}^2 + \frac{\tau}{2} \|\mathbf{curl}(\mathcal{S}_h \mathbf{u}_h) - \mathbf{y}_c\|_{L^2(\Omega)}^2 + \frac{\kappa}{2} \|\mathbf{u}_h\|_{\mathbf{H}(\mathbf{curl})}^2 \\ + \frac{\gamma}{2} (\|\mathbf{max}(0, \mathcal{S}_h \mathbf{u}_h - \mathbf{y}_b)\|_{L^2(\Omega)}^2 + \|\mathbf{max}(0, \mathbf{y}_a - \mathcal{S}_h \mathbf{u}_h)\|_{L^2(\Omega)}^2),$$

where $\mathcal{S}_h : \mathbf{H}_0(\mathbf{curl}) \rightarrow \mathbf{V}_h$ denotes the solution operator associated with the discrete state equation. More precisely, it assigns to every element $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl})$ the solution $\mathbf{y}_h \in \mathbf{V}_h$ of

$$(4.1) \quad (\alpha \mathbf{curl} \mathbf{y}_h, \mathbf{curl} \mathbf{q}_h)_{L^2(\Omega)} + (\beta \mathbf{y}_h, \mathbf{q}_h)_{L^2(\Omega)} = (\mathbf{curl} \mathbf{u}, \mathbf{q}_h)_{L^2(\Omega)} \quad \forall \mathbf{q}_h \in \mathbf{V}_h.$$

We denote the optimal solution of (P_h^γ) by \mathbf{u}_h^γ . As in the continuous case, the sufficient and necessary optimality condition at the optimal solution \mathbf{u}_h^γ is given by

$$(4.2) \quad f_h^{\gamma'}(\mathbf{u}_h^\gamma) \mathbf{u}_h = 0 \quad \forall \mathbf{u}_h \in \mathbf{V}_h.$$

4.1. Error analysis. We start by providing a well-known error estimate for the Nédélec interpolation operator $\mathbf{\Pi}_h$ (see [20] for its definition):

THEOREM 4.1 (Alonso and Valli [1, Proposition 5.6] or [19, Theorem 5.41]). *Let $\frac{1}{2} < s \leq 1$. Then there exists a positive constant c , independent of h and \mathbf{u} , such that*

$$(4.3) \quad \|\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}\|_{\mathbf{H}(\mathbf{curl})} \leq ch^s \|\mathbf{u}\|_{\mathbf{H}^s(\mathbf{curl})} \quad \forall \mathbf{u} \in \mathbf{H}_0^s(\mathbf{curl}).$$

Let us further introduce a projection operator $\mathbf{\Phi}_h : \mathbf{H}_0(\mathbf{curl}) \rightarrow \mathbf{V}_h$ defined by

$$(4.4) \quad (\mathbf{\Phi}_h \mathbf{u}, \mathbf{q}_h)_{\mathbf{H}(\mathbf{curl})} = (\mathbf{u}, \mathbf{q}_h)_{\mathbf{H}(\mathbf{curl})} \quad \forall \mathbf{q}_h \in \mathbf{V}_h.$$

It is evident that the projection $\mathbf{\Phi}_h$ satisfies the following optimal error estimate:

$$(4.5) \quad \|\mathbf{u} - \mathbf{\Phi}_h \mathbf{u}\|_{\mathbf{H}(\mathbf{curl})} \leq \inf_{\mathbf{q}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{q}_h\|_{\mathbf{H}(\mathbf{curl})} \quad \forall \mathbf{u} \in \mathbf{H}_0(\mathbf{curl}).$$

COROLLARY 4.2. *Let $\alpha, \beta \in \mathcal{C}^{0,1}(\overline{\Omega})$. Then there exists a constant $c > 0$, independent of h and \mathbf{u} , such that*

$$(4.6) \quad \|\mathcal{S} \mathbf{u} - \mathcal{S}_h \mathbf{u}\|_{\mathbf{H}(\mathbf{curl})} \leq ch^{\frac{1}{2} + \epsilon} \|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl})} \quad \forall \mathbf{u} \in \mathbf{H}_0(\mathbf{curl}),$$

with $\epsilon > 0$ as in Lemma 2.1.

Proof. Using the projection $\mathbf{\Phi}_h$ together with (4.5), we can verify that

$$(4.7) \quad \|\mathcal{S} \mathbf{u} - \mathcal{S}_h \mathbf{u}\|_{\mathbf{H}(\mathbf{curl})} \leq c \inf_{\mathbf{q}_h \in \mathbf{V}_h} \|\mathcal{S} \mathbf{u} - \mathbf{q}_h\|_{\mathbf{H}(\mathbf{curl})} \quad \forall \mathbf{u} \in \mathbf{H}_0(\mathbf{curl})$$

holds with a constant $c > 0$ depending only on $\underline{\alpha}, \overline{\alpha}$ and $\underline{\beta}, \overline{\beta}$.

Since $\alpha, \beta \in \mathcal{C}^{0,1}(\overline{\Omega})$, Proposition 3.6 ensures that \mathcal{S} is well-defined and bounded as an operator from $\mathbf{H}_0(\mathbf{curl})$ to $\mathbf{H}_0^{\frac{1}{2} + \epsilon}(\mathbf{curl})$ with $\epsilon > 0$ as in Lemma 2.1. Inserting $\mathbf{q}_h = \mathbf{\Pi}_h \mathcal{S} \mathbf{u}$ in (4.7) results in

$$\|\mathcal{S} \mathbf{u} - \mathcal{S}_h \mathbf{u}\|_{\mathbf{H}(\mathbf{curl})} \leq c \|\mathcal{S} \mathbf{u} - \mathbf{\Pi}_h \mathcal{S} \mathbf{u}\|_{\mathbf{H}(\mathbf{curl})} \underbrace{\leq}_{(4.3)} ch^{\frac{1}{2} + \epsilon} \|\mathcal{S} \mathbf{u}\|_{\mathbf{H}^{\frac{1}{2} + \epsilon}(\mathbf{curl})} \leq ch^{\frac{1}{2} + \epsilon} \|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl})},$$

with a positive constant c independent of h and \mathbf{u} .

□

Next, we split the objective functionals f^γ and f_h^γ into

$$(4.8) \quad f^\gamma(\mathbf{u}) = f(\mathbf{u}) + \gamma l(\mathbf{u}) \quad \text{and} \quad f_h^\gamma(\mathbf{u}) = f_h(\mathbf{u}) + \gamma l_h(\mathbf{u}),$$

where

$$\begin{aligned} l(\mathbf{u}) &:= \frac{1}{2}(\|\mathbf{max}(0, \mathcal{S}\mathbf{u} - \mathbf{y}_b)\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{max}(0, \mathbf{y}_a - \mathcal{S}\mathbf{u})\|_{\mathbf{L}^2(\Omega)}^2) \\ l_h(\mathbf{u}) &:= \frac{1}{2}(\|\mathbf{max}(0, \mathcal{S}_h\mathbf{u} - \mathbf{y}_b)\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{max}(0, \mathbf{y}_a - \mathcal{S}_h\mathbf{u})\|_{\mathbf{L}^2(\Omega)}^2) \\ f_h(\mathbf{u}) &:= \frac{1}{2}\|\mathcal{S}_h\mathbf{u} - \mathbf{y}_d\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\tau}{2}\|\mathbf{curl}(\mathcal{S}_h\mathbf{u}) - \mathbf{y}_c\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\kappa}{2}\|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl})}^2. \end{aligned}$$

LEMMA 4.3. *It holds that*

$$(\mathbf{max}(0, \mathbf{y}) - \mathbf{max}(0, \mathbf{w}), \mathbf{w} - \mathbf{y})_{\mathbf{L}^2(\Omega)} \leq 0 \quad \forall \mathbf{y}, \mathbf{w} \in \mathbf{L}^2(\Omega).$$

Proof. For any $a, b \in \mathbb{R}$, we have $(\max(0, a) - \max(0, b))(b - \max(0, b)) \leq 0$. For this reason,

$$(\mathbf{max}(0, \mathbf{y}) - \mathbf{max}(0, \mathbf{w}), \mathbf{w} - \mathbf{max}(0, \mathbf{w}))_{\mathbf{L}^2(\Omega)} \leq 0 \quad \forall \mathbf{y}, \mathbf{w} \in \mathbf{L}^2(\Omega).$$

Using this inequality, we obtain

$$\begin{aligned} &(\mathbf{max}(0, \mathbf{y}) - \mathbf{max}(0, \mathbf{w}), \mathbf{w} - \mathbf{y})_{\mathbf{L}^2(\Omega)} \\ &= (\mathbf{max}(0, \mathbf{y}) - \mathbf{max}(0, \mathbf{w}), \mathbf{w})_{\mathbf{L}^2(\Omega)} + (\mathbf{max}(0, \mathbf{w}) - \mathbf{max}(0, \mathbf{y}), \mathbf{y})_{\mathbf{L}^2(\Omega)} \\ &= (\mathbf{max}(0, \mathbf{y}) - \mathbf{max}(0, \mathbf{w}), \mathbf{w} - \mathbf{max}(0, \mathbf{w}))_{\mathbf{L}^2(\Omega)} + (\mathbf{max}(0, \mathbf{w}) - \mathbf{max}(0, \mathbf{y}), \mathbf{y} - \mathbf{max}(0, \mathbf{y}))_{\mathbf{L}^2(\Omega)} \\ &\quad - \|\mathbf{max}(0, \mathbf{y}) - \mathbf{max}(0, \mathbf{w})\|^2 \leq 0. \end{aligned}$$

□

By Lemma 4.3, we obtain

$$(4.9) \quad \begin{aligned} &(l'_h(\mathbf{u}) - l'_h(\mathbf{v}))(\mathbf{v} - \mathbf{u}) = (\mathbf{max}(0, \mathcal{S}_h\mathbf{u} - \mathbf{y}_b) - \mathbf{max}(0, \mathcal{S}_h\mathbf{v} - \mathbf{y}_b), \mathcal{S}_h\mathbf{v} - \mathcal{S}_h\mathbf{u})_{\mathbf{L}^2(\Omega)} \\ &+ (\mathbf{max}(0, \mathbf{y}_a - \mathcal{S}_h\mathbf{u}) - \mathbf{max}(0, \mathbf{y}_a - \mathcal{S}_h\mathbf{v}), \mathcal{S}_h\mathbf{u} - \mathcal{S}_h\mathbf{v})_{\mathbf{L}^2(\Omega)} \leq 0 \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}). \end{aligned}$$

Now we have all ingredients to establish the following error estimate result:

THEOREM 4.4. *Let $\gamma > 0$ and assume that $\alpha, \beta \in \mathcal{C}^{0,1}(\overline{\Omega})$. Then there exists a constant $c > 0$, independent of h , such that*

$$\|\mathbf{u}_h^\gamma - \mathbf{u}^\gamma\|_{\mathbf{H}(\mathbf{curl})} \leq ch^{\frac{1}{2}+\epsilon} \|\mathbf{u}^\gamma\|_{\mathbf{H}^{\frac{1}{2}+\epsilon}(\mathbf{curl})}$$

where $\epsilon > 0$ is defined as in Lemma 2.1. In particular, if Ω is convex, then we obtain

$$\|\mathbf{u}_h^\gamma - \mathbf{u}^\gamma\|_{\mathbf{H}(\mathbf{curl})} \leq ch \|\mathbf{u}^\gamma\|_{\mathbf{H}^{\frac{1}{2}+\epsilon}(\mathbf{curl})}.$$

Proof. We set $\mathbf{w}_h = \mathbf{u}_h^\gamma - \mathbf{\Pi}_h \mathbf{u}^\gamma$. The linear–quadratic structure of f_h yields $f_h''(\mathbf{u}^\gamma)[\mathbf{w}_h, \mathbf{w}_h] = f_h'(\mathbf{u}^\gamma)\mathbf{w}_h - f_h'(\mathbf{\Pi}_h \mathbf{u}^\gamma)\mathbf{w}_h$. By the latter identity, we have

$$(4.10) \quad \kappa \|\mathbf{w}_h\|_{\mathbf{H}(\mathbf{curl})}^2 \leq f_h''(\mathbf{u}_h^\gamma)[\mathbf{w}_h, \mathbf{w}_h] = f_h'(\mathbf{u}_h^\gamma)\mathbf{w}_h - f_h'(\mathbf{\Pi}_h \mathbf{u}^\gamma)\mathbf{w}_h =: I + II.$$

Using (4.2) and (4.9), the first term I can be estimated as follows:

$$(4.11) \quad \begin{aligned} I &= f_h'(\mathbf{u}_h^\gamma)\mathbf{w}_h - f_h^{\gamma'}(\mathbf{u}_h^\gamma)\mathbf{w}_h = -\gamma l_h'(\mathbf{u}_h^\gamma)\mathbf{w}_h \\ &= \gamma (l_h'(\mathbf{\Pi}_h \mathbf{u}^\gamma) - l_h'(\mathbf{u}_h^\gamma))\mathbf{w}_h - \gamma l_h'(\mathbf{\Pi}_h \mathbf{u}^\gamma)\mathbf{w}_h \leq -\gamma l_h'(\mathbf{\Pi}_h \mathbf{u}^\gamma)\mathbf{w}_h. \end{aligned}$$

From (4.11) and (3.1), it follows then:

$$(4.12) \quad \begin{aligned} I + II &\leq -f_h^{\gamma'}(\mathbf{\Pi}_h \mathbf{u}^\gamma)\mathbf{w}_h = f^{\gamma'}(\mathbf{u}^\gamma)\mathbf{w}_h - f_h^{\gamma'}(\mathbf{\Pi}_h \mathbf{u}^\gamma)\mathbf{w}_h \\ &= (f^{\gamma'}(\mathbf{u}^\gamma) - f^{\gamma'}(\mathbf{\Pi}_h \mathbf{u}^\gamma))\mathbf{w}_h + (f^{\gamma'}(\mathbf{\Pi}_h \mathbf{u}^\gamma) - f_h^{\gamma'}(\mathbf{\Pi}_h \mathbf{u}^\gamma))\mathbf{w}_h. \end{aligned}$$

In view of (4.3) along with the regularity $\mathbf{u}^\gamma \in \mathbf{H}_0^{\frac{1}{2}+\epsilon}(\mathbf{curl})$, we deduce that

$$(4.13) \quad \begin{aligned} (f^{\gamma'}(\mathbf{u}^\gamma) - f^{\gamma'}(\mathbf{\Pi}_h \mathbf{u}^\gamma))\mathbf{w}_h &\leq c \|\mathbf{u}^\gamma - \mathbf{\Pi}_h \mathbf{u}^\gamma\|_{\mathbf{H}(\mathbf{curl})} \|\mathbf{w}_h\|_{\mathbf{H}(\mathbf{curl})} \\ &\leq ch^{\frac{1}{2}+\epsilon} \|\mathbf{u}^\gamma\|_{\mathbf{H}^{\frac{1}{2}+\epsilon}(\mathbf{curl})} \|\mathbf{w}_h\|_{\mathbf{H}(\mathbf{curl})}, \end{aligned}$$

with a constant $c > 0$ independent of h . Furthermore, using Corollary 4.2, we can also infer that

$$(4.14) \quad \begin{aligned} (f^{\gamma'}(\mathbf{\Pi}_h \mathbf{u}^\gamma) - f_h^{\gamma'}(\mathbf{\Pi}_h \mathbf{u}^\gamma))\mathbf{w}_h &\leq ch^{\frac{1}{2}+\epsilon} \|\mathbf{\Pi}_h \mathbf{u}^\gamma\|_{\mathbf{H}(\mathbf{curl})} \|\mathbf{w}_h\|_{\mathbf{H}(\mathbf{curl})} \\ &\leq ch^{\frac{1}{2}+\epsilon} \|\mathbf{u}^\gamma\|_{\mathbf{H}^{\frac{1}{2}+\epsilon}(\mathbf{curl})} \|\mathbf{w}_h\|_{\mathbf{H}(\mathbf{curl})}. \end{aligned}$$

Combining (4.10), (4.12)–(4.14), we come to the conclusion that

$$\|\mathbf{u}_h^\gamma - \mathbf{\Pi}_h \mathbf{u}^\gamma\|_{\mathbf{H}(\mathbf{curl})} = \|\mathbf{w}_h\|_{\mathbf{H}(\mathbf{curl})} \leq ch^{\frac{1}{2}+\epsilon} \|\mathbf{u}^\gamma\|_{\mathbf{H}^{\frac{1}{2}+\epsilon}(\mathbf{curl})},$$

which along with Theorem 4.1 yields the assertion. \square

5. Numerical experiments. In this section, we report on some numerical results including a numerical verification of Theorem 4.4. For the numerical solution of (\mathbf{P}_h^γ) , we utilize standard gradient descent. The iterations are stopped as soon as the $\mathbf{H}(\mathbf{curl})$ –norm of the gradient drops below 10^{-4} . Note that the computation of $f_h^{\gamma'}(\mathbf{u}_n)$ in the n^{th} iteration of the gradient method is given by the following steps:

Step 1. Find the solution $\mathbf{y}_n \in \mathbf{V}_h$ of

$$(\alpha \mathbf{curl} \mathbf{y}_n, \mathbf{curl} \mathbf{q}_h)_{\mathbf{L}^2(\Omega)} + (\beta \mathbf{y}_n, \mathbf{q}_h)_{\mathbf{L}^2(\Omega)} = (\mathbf{curl} \mathbf{u}_n, \mathbf{q}_h) \quad \forall \mathbf{q}_h \in \mathbf{V}_h.$$

Set $\boldsymbol{\lambda}_{b,n} = \gamma \max(0, \mathbf{y}_n - \mathbf{y}_b)$ and $\boldsymbol{\lambda}_{a,n} = \gamma \max(0, \mathbf{y}_a - \mathbf{y}_n)$.

Step 2. Find the solution $\mathbf{p}_n \in \mathbf{V}_h$ of

$$\begin{aligned} (\alpha \mathbf{curl} \mathbf{p}_n, \mathbf{curl} \mathbf{q}_h)_{\mathbf{L}^2(\Omega)} + (\beta \mathbf{p}_n, \mathbf{q}_h)_{\mathbf{L}^2(\Omega)} &= (\mathbf{y}_n - \mathbf{y}_d, \mathbf{q}_h)_{\mathbf{L}^2(\Omega)} + \tau (\mathbf{curl} \mathbf{y}_n - \mathbf{y}_c, \mathbf{q}_h)_{\mathbf{L}^2(\Omega)} \\ &\quad + (\boldsymbol{\lambda}_{b,n} - \boldsymbol{\lambda}_{a,n}, \mathbf{q}_h)_{\mathbf{L}^2(\Omega)} \quad \forall \mathbf{q}_h \in \mathbf{V}_h. \end{aligned}$$

Step 3. Find the solution $\mathbf{v}_n \in \mathbf{V}_h$ of

$$(\mathbf{curl} \mathbf{v}_n, \mathbf{curl} \mathbf{q}_h)_{L^2(\Omega)} + (\mathbf{v}_n, \mathbf{q}_h)_{L^2(\Omega)} = (\mathbf{curl} \mathbf{p}_n, \mathbf{q}_h)_{L^2(\Omega)} + \kappa(\mathbf{u}_n, \mathbf{q}_h)_{\mathbf{H}(\mathbf{curl})} \quad \forall \mathbf{q}_h \in \mathbf{V}_h.$$

Then we have $f_h^{\gamma'}(\mathbf{u}_n) = \mathbf{v}_n$.

We numerically solve the linear systems in the above steps using the multigrid preconditioned CG Solver from the open source software NETGEN/NGSolve ([22]) with a precision of 10^{-9} . Further the computational domain Ω used throughout our experiment is given by the following nonconvex L-shaped domain:

$$(5.1) \quad \Omega = \{(-0.5, 0.5) \times (-0.5, 0.5) \times (0, 0.5)\} \setminus [0, 0.5]^3.$$

Using NETGEN, the L-shaped domain (5.1) was triangulated with a regular mesh of mesh size h . The regular meshes used in our experiment are depicted in Fig. 5.1. Lastly we point out that all computations reported on in the upcoming subsections were implemented in C++.

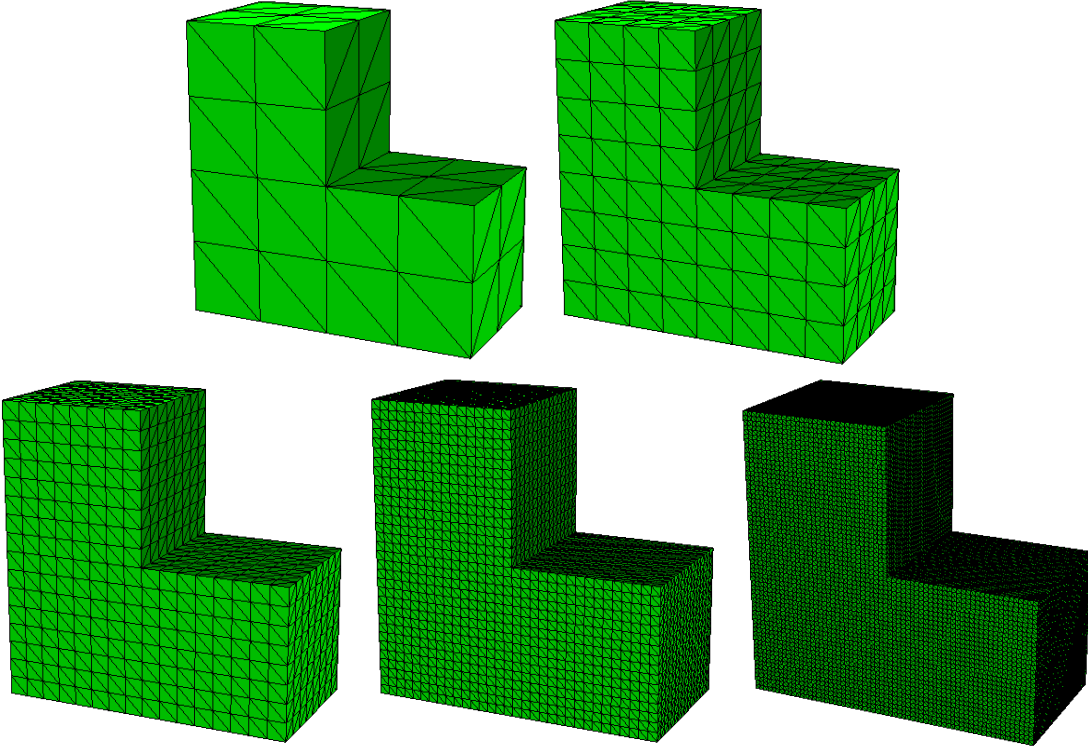


FIG. 5.1. Regular meshes for the L-shaped domain (5.1).

5.1. Numerical verification of Theorem 4.4. Let us consider the following test problem:

$$(5.2) \quad \min_{\mathbf{u} \in \mathbf{H}_0(\mathbf{curl})} \frac{1}{2} \|\mathcal{S}\mathbf{u} - \mathcal{S}\mathbf{u}_d\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\mathbf{curl} \mathcal{S}\mathbf{u} - \mathbf{curl} \mathcal{S}\mathbf{u}_d\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\mathbf{u} - \mathbf{u}_d\|_{\mathbf{H}(\mathbf{curl})}^2 \\ + \frac{\gamma}{2} (\|\max(0, \mathcal{S}\mathbf{u} - \mathcal{S}\mathbf{u}_d)\|_{L^2(\Omega)}^2 + \|\max(0, \mathcal{S}\mathbf{u}_d - \mathcal{S}\mathbf{u})\|_{L^2(\Omega)}^2),$$

where $\mathbf{u}_d \in \mathbf{H}_0(\mathbf{curl})$ is given data satisfying $\operatorname{div} \mathbf{u}_d = 0$. By the presence of the shift control \mathbf{u}_d in (5.2), we see that the optimal solution of (5.2) for any $\gamma > 0$ is given by $\mathbf{u}_d = \mathbf{u}^\gamma$.

EXAMPLE 1. In (5.2) we choose $\alpha(x) = |x_1| + |x_2| + \exp(x_3)$, $\beta = 1$, $\gamma = 10^3$ and

$$\mathbf{u}_d(x) = 10^2 \begin{pmatrix} (x_2^2 - 0.5|x_2|)(x_3^2 - 0.5x_3) \\ (x_1^2 - 0.5|x_1|)(x_3^2 - 0.5x_3) \\ (x_1^2 - 0.5|x_1|)(x_2^2 - 0.5|x_2|) \end{pmatrix}.$$

Table 5.1 displays the $\mathbf{H}(\mathbf{curl})$ -error between the exact solution \mathbf{u}^γ of (5.2) and the numerical solution \mathbf{u}_h^γ of the finite element approximation of (5.2). As the mesh size h decreases, we observe that the numerical solution becomes closer to the exact one. Further, to detect the order of convergence, we employ the following quantity:

$$\text{EOC} = \frac{\log \|\mathbf{u}_{h_1}^\gamma - \mathbf{u}^\gamma\|_{\mathbf{H}(\mathbf{curl})} - \log \|\mathbf{u}_{h_2}^\gamma - \mathbf{u}^\gamma\|_{\mathbf{H}(\mathbf{curl})}}{\log h_1 - \log h_2},$$

for two consecutive mesh sizes h_1 and h_2 . According to Theorem 4.4, since the computational domain (5.1) features a nonconvex structure, we can only expect convergence of the rate $0.5 + \epsilon$. This theoretical prediction is confirmed by our numerical results (see the last row of Table 5.1). The experimental order of convergence EOC approximates indeed $0.5 + \epsilon$.

TABLE 5.1
Example 1: Convergence behavior of \mathbf{u}_h^γ towards \mathbf{u}^γ for decreasing h .

| $h/\sqrt{2}$ | 2^{-2} | 2^{-3} | 2^{-4} | 2^{-5} | 2^{-6} |
|---|----------|----------|----------|----------|----------|
| $\ \mathbf{u}_h^\gamma - \mathbf{u}^\gamma\ _{\mathbf{H}(\mathbf{curl})}$ | 0.513619 | 0.356033 | 0.234443 | 0.156735 | 0.104703 |
| EOC | - | 0.528688 | 0.602776 | 0.580907 | 0.582024 |

EXAMPLE 2. In (5.2) we consider

$$\alpha(x) = \begin{cases} 5 & \text{if } (x_1, x_2) \in [-0.5, 0) \times (0, 0.5] \\ 8 & \text{if } (x_1, x_2) \in [-0.5, 0)^2 \\ 1 & \text{otherwise} \end{cases} \quad \beta(x) = \begin{cases} 8 & \text{if } (x_1, x_2) \in [-0.5, 0) \times (0, 0.5] \\ 5 & \text{if } (x_1, x_2) \in [-0.5, 0)^2 \\ 1 & \text{otherwise.} \end{cases}$$

Further, we set $\gamma = 10^3$ and

$$\mathbf{u}_d(x) = \begin{pmatrix} \sin(2\pi x_2) \sin(2\pi x_3) \\ \sin(2\pi x_1) \sin(2\pi x_3) \\ (x_1^2 - 0.5|x_1|)(x_2^2 - 0.5|x_2|) \end{pmatrix}.$$

Example 2 deals with a case where the coefficients α and β are given by piecewise constant functions which are no longer continuous. We are interested in the convergence behavior of this particular case which is not covered by our analysis. The $\mathbf{H}(\mathbf{curl})$ -error between the numerical and analytical solutions is presented in Table 5.2. For decreasing mesh size h , convergence of \mathbf{u}_h^γ towards \mathbf{u}^γ can be detected, and its convergence behavior turns out to be similar to the previous example. The experimental order of convergence approximates also $0.5 + \epsilon$.

TABLE 5.2
Example 2: Convergence behavior of \mathbf{u}_h^γ towards \mathbf{u}^γ for decreasing h

| $h/\sqrt{2}$ | 2^{-2} | 2^{-3} | 2^{-4} | 2^{-5} | 2^{-6} |
|---|----------|----------|----------|----------|----------|
| $\ \mathbf{u}_h^\gamma - \mathbf{u}^\gamma\ _{\mathbf{H}(\text{curl})}$ | 1.147901 | 0.758734 | 0.404816 | 0.301656 | 0.192041 |
| EOC | - | 0.597333 | 0.686842 | 0.643847 | 0.651489 |

5.2. Convergence with respect to γ . We close this paper by providing a numerical report which exhibits the action of the regularization parameter. Consider now the following test problem:

$$(5.3) \quad \min_{\mathbf{u} \in \mathbf{H}_0(\text{curl})} \frac{1}{2} \|\mathbf{S}\mathbf{u} - \mathbf{100}\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\text{curl } \mathbf{S}\mathbf{u}\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\mathbf{u}\|_{\mathbf{H}(\text{curl})}^2 + \frac{\gamma}{2} (\|\max(0, \mathbf{S}\mathbf{u} - \mathbf{S}\mathbf{u}_d)\|_{L^2(\Omega)}^2 + \|\max(0, \mathbf{S}\mathbf{u}_d - \mathbf{S}\mathbf{u})\|_{L^2(\Omega)}^2),$$

with $\mathbf{u}_d \in \mathbf{H}_0(\text{curl})$ satisfying $\text{div } \mathbf{u}_d = 0$. We should underline that, for every $\gamma > 0$, the solution \mathbf{u}^γ of (5.3) is unknown. But, the exact solution of the associated unregularized problem is known and is given by $\bar{\mathbf{u}} = \mathbf{u}_d$ (see Lemma A.1 in the appendix). In what follows, the mesh size is set to be fixed: $h/\sqrt{2} = 2^{-5}$.

EXAMPLE 3. In (5.3) we choose $\alpha = \beta = 1$ and

$$\mathbf{u}_d(x) = \begin{pmatrix} (x_2^2 - 0.5|x_2|)(x_3^2 - 0.5|x_3|) \\ (x_1^2 - 0.5|x_1|)(x_3^2 - 0.5|x_3|) \\ (x_1^2 - 0.5|x_1|)(x_2^2 - 0.5|x_2|) \end{pmatrix}.$$

The numerical solutions $\text{curl } \mathbf{u}_h^\gamma$ at $\gamma = 10^3$ and $\gamma = 10^6$ are depicted in Figure 5.2. Here we observe that $\text{curl } \mathbf{u}_h^\gamma$ at $\gamma = 10^6$ exhibits a similar structure to $\text{curl } \bar{\mathbf{u}}$ (see Figure 5.3). Furthermore, as presented in Table 5.3, the numerical solution of the regularized problem \mathbf{u}_h^γ becomes closer to $\bar{\mathbf{u}}$ as the regularization parameter γ increases. This confirms our convergence result in Theorem 3.7.

TABLE 5.3
Example 3: Convergence behavior of \mathbf{u}_h^γ towards $\bar{\mathbf{u}}$ for increasing γ .

| γ | 10^3 | $5 \cdot 10^3$ | 10^4 | $5 \cdot 10^4$ | 10^5 | $5 \cdot 10^5$ | 10^6 |
|--|----------|----------------|----------|----------------|----------|----------------|----------|
| $\ \mathbf{u}_h^\gamma - \bar{\mathbf{u}}\ _{\mathbf{H}(\text{curl})}$ | 0.680050 | 0.524371 | 0.419899 | 0.212261 | 0.165994 | 0.118289 | 0.107309 |

EXAMPLE 4. In (5.3) we choose $\mathbf{u}_d = (0, 0, \pi^{-1} \sin(2\pi x_1) \sin(2\pi x_2))^T$, $\beta = 1$, and α is given by the following piecewise constant function:

$$\alpha(x) = \begin{cases} 2 & \text{if } (x_1, x_2) \in [-0.5, 0) \times (0, 0.5] \\ 8 & \text{if } (x_1, x_2) \in [-0.5, 0]^2 \\ 1 & \text{otherwise.} \end{cases}$$

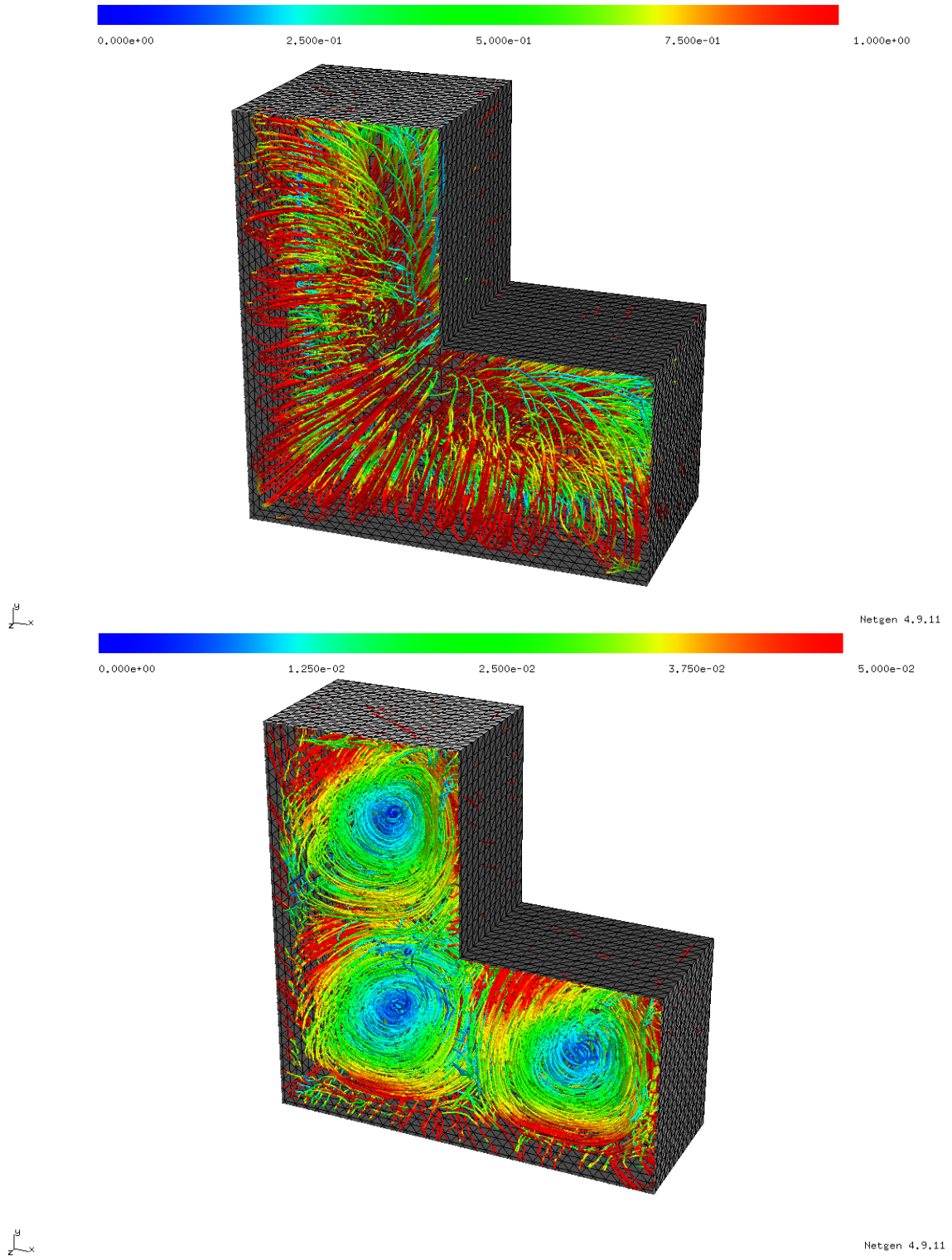
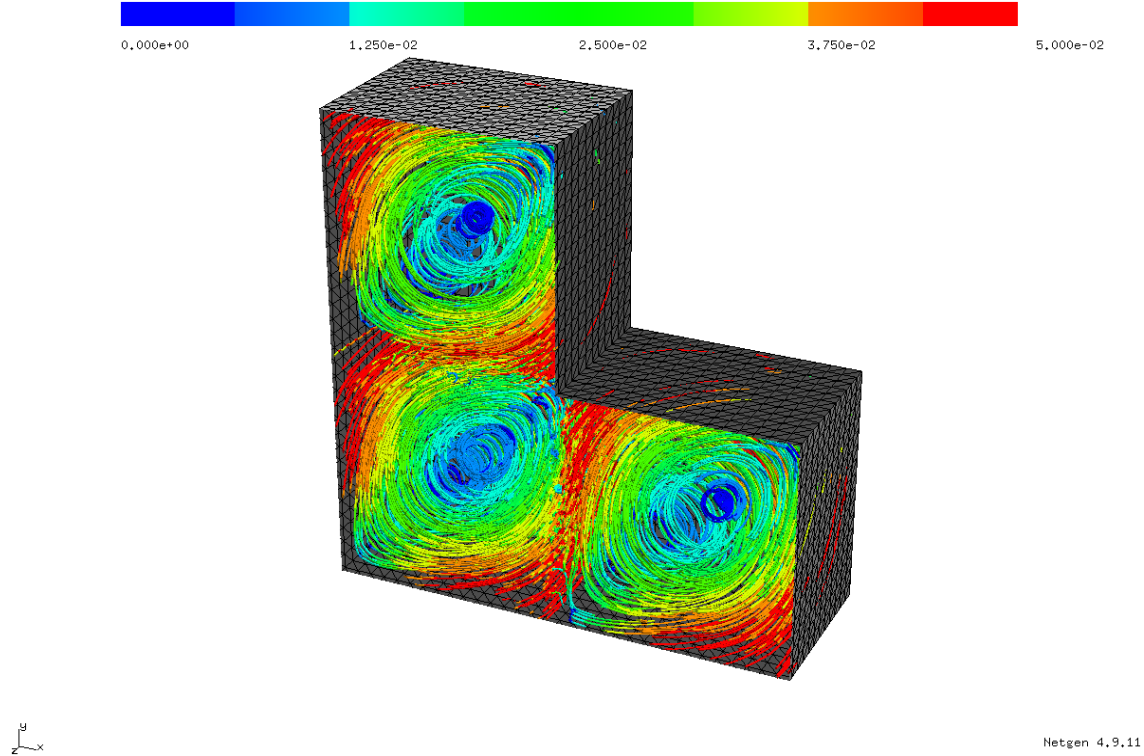


FIG. 5.2. Example 3: Field lines of $\text{curl } \mathbf{u}_h^\gamma$ at $\gamma = 10^3$ (upper plot) and 10^6 (lower plot).

FIG. 5.3. Example 3: *Field lines of $\mathbf{curl} \bar{\mathbf{u}}$.*

Our last example deals with a discontinuous coefficient α . For this particular case, we also detect a convergence behavior with respect to the regularization parameter γ . Analogously to the previous example, the $\mathbf{H}(\mathbf{curl})$ -error due to the regularization $\|\mathbf{u}_h^\gamma - \bar{\mathbf{u}}\|_{\mathbf{H}(\mathbf{curl})}$ becomes smaller and smaller as the regularization parameter γ gets larger. Also, we depict in Figure 5.4 the numerical solutions $\mathbf{curl} \mathbf{u}_h^\gamma$ at $\gamma = 10^3$ and $\gamma = 10^6$. Analogously to Example 3, we can observe here that $\mathbf{curl} \mathbf{u}_h^\gamma$ at $\gamma = 10^6$ approaches the exact solution of the unregularized optimal control problem.

TABLE 5.4
Example 4: *Convergence behavior of \mathbf{u}_h^γ towards $\bar{\mathbf{u}}$ for increasing γ .*

| γ | 10^3 | $5 \cdot 10^3$ | 10^4 | $5 \cdot 10^4$ | 10^5 | $5 \cdot 10^5$ | 10^6 |
|--|----------|----------------|----------|----------------|----------|----------------|----------|
| $\ \mathbf{u}_h^\gamma - \bar{\mathbf{u}}\ _{\mathbf{H}(\mathbf{curl})}$ | 0.994781 | 0.964027 | 0.934454 | 0.818984 | 0.762306 | 0.675294 | 0.667683 |

Appendix A.

LEMMA A.1. *Let $\mathbf{u}_d \in \mathbf{H}_0(\mathbf{curl})$ satisfying $\operatorname{div} \mathbf{u}_d = 0$. Then the optimal solution $\bar{\mathbf{u}}$ of the unregularized problem (P) associated with (5.3) is given by $\bar{\mathbf{u}} = \mathbf{u}_d$.*

Proof. Due to the construction of (5.3), the optimal state $\bar{\mathbf{y}}$ of the unregularized problem

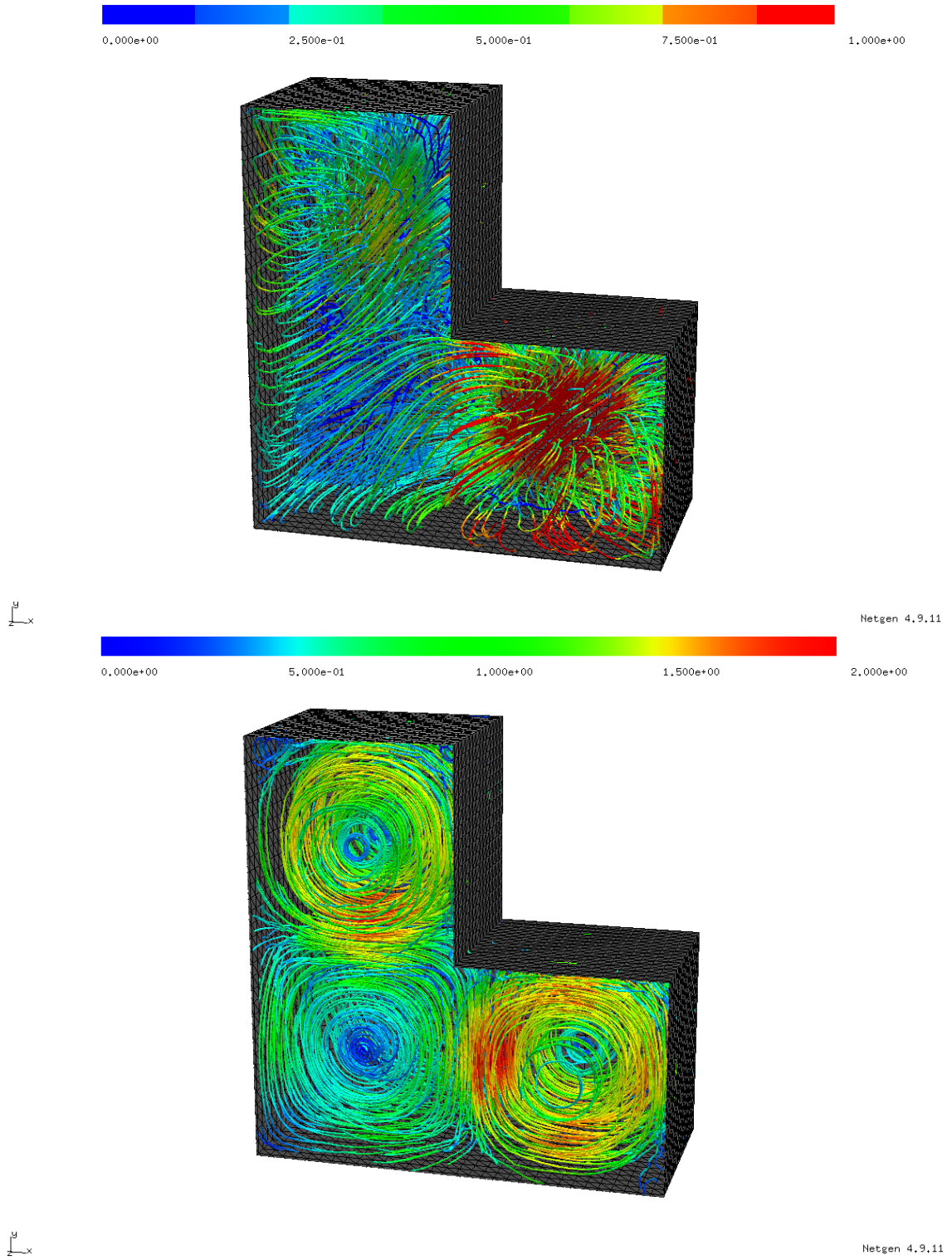
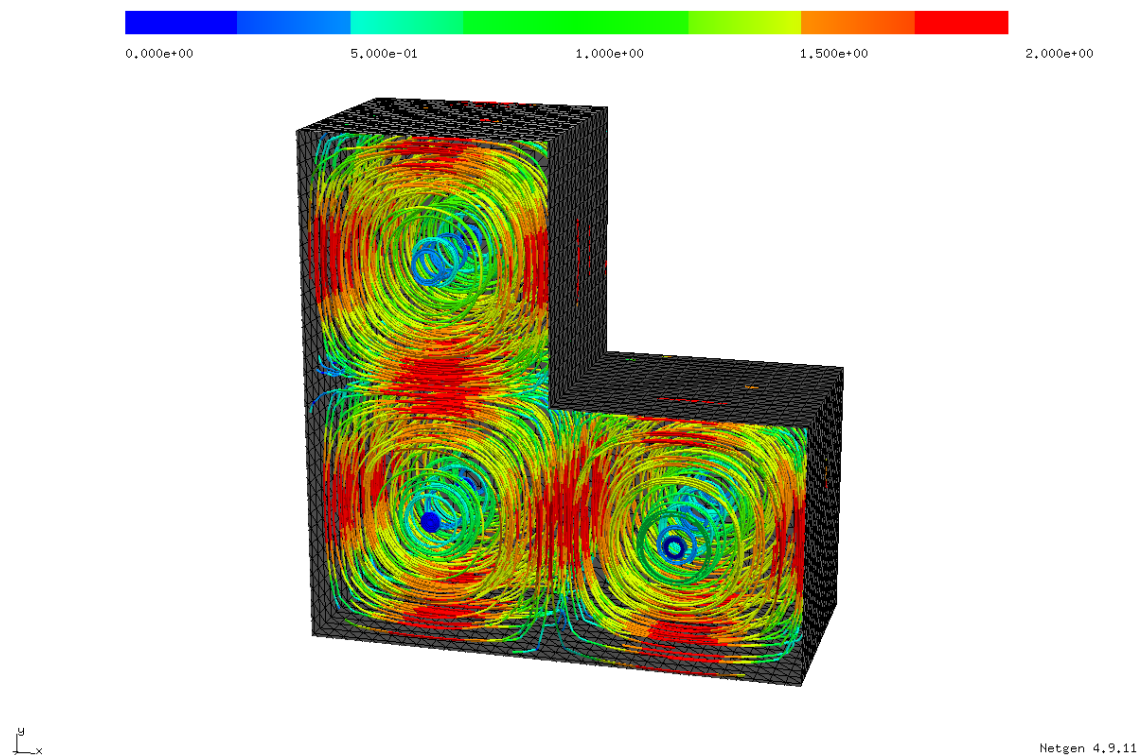


FIG. 5.4. Example 4: Field lines of $\text{curl } \mathbf{u}_h^\gamma$ at $\gamma = 10^3$ (upper plot) and 10^6 (lower plot).

FIG. 5.5. Example 4: *Field lines of $\mathbf{curl} \bar{\mathbf{u}}$.*

associated with (5.3) is given by $\bar{\mathbf{y}} = \mathcal{S}\bar{\mathbf{u}} = \mathcal{S}\mathbf{u}_d$, which yields by the definition of \mathcal{S} that

$$(\mathbf{curl}(\bar{\mathbf{u}} - \mathbf{u}_d), \mathbf{q})_{\mathbf{L}^2(\Omega)} = 0 \quad \forall \mathbf{q} \in \mathcal{C}_0^\infty(\Omega)^3.$$

Thus, $\mathbf{curl}(\bar{\mathbf{u}} - \mathbf{u}_d) = 0$ holds. Then, as Ω is simply connected with a connected boundary and $\operatorname{div} \bar{\mathbf{u}} = 0$ (see the proof of Theorem 2.2) as well as $\operatorname{div} \bar{\mathbf{u}}_d = 0$, we infer that

$$\|\bar{\mathbf{u}} - \mathbf{u}_d\|_{\mathbf{L}^2(\Omega)} \leq c \|\mathbf{curl}(\bar{\mathbf{u}} - \mathbf{u}_d)\|_{\mathbf{L}^2(\Omega)} = 0;$$

cf. [19, Corollary 3.51] for the latter Friedrich inequality. □

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