Counting Lattice Triangulations

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Abstract

We discuss the problem to count, or, more modestly, to estimate the number $f(m,n)$ of unimodular triangulations of the planar grid of size $m \times n$.

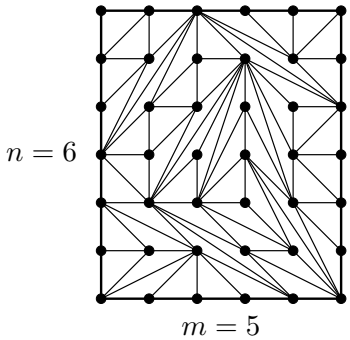
Among other tools, we employ recursions that allow one to compute the (huge) number of triangulations for small m and rather large n by dynamic programming; we show that this computation can be done in polynomial time if m is fixed, and present computational results from our implementation of this approach.

We also present new upper and lower bounds for large m and n, and we report about results obtained from a computer simulation of the random walk that is generated by flips.

1 Introduction

An innocent little combinatorial counting problem asks for the number of triangulations of a finite grid of size $m \times n$. That is, for $m, n \geq 1$ we define $P_{m,n} := \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}$, "the grid". Equivalently, the point configuration $P_{m,n}$ consists of all points of the integer lattice $\mathbb{Z}^2$ in the lattice rectangle $\text{conv}(P_{m,n}) = [0, m] \times [0, n]$ of area mn. Every triangulation of this rectangle point set that uses all the points in $P_{m,n}$ has $(m+1)(n+1) = |P_{m,n}|$ vertices, $2mn$ facets/triangles, and $3mn + m + n$ edges, $2(m + n)$ of them on the boundary, the other $3mn - m - n$ ones in the interior. All the triangles are minimal lattice triangles of area $\frac{1}{2}$ (that is, of determinant 1), which are referred to as *unimodular* triangles. The grid triangulations that use all the points are thus called *unimodular triangulations*. The number of unimodular triangulations of the grid $P_{m,n}$ will be denoted by $f(m, n)$.

As an example, our first figure shows one unimodular triangulation of the 5×6 grid:



To get started, one notes that of course $f(m,n) = f(n,m)$, one discovers with pleasure that $f(1,n) = \binom{2n}{n}$, and one works out by hand with a bit of pain that $f(2,2) = 64$ and $f(2,3) = 852$. Furthermore, one observes that the special triangulations that decompose into m vertical strips of width 1 yield the lower bound

$$f(m,n) \;\geq\; f(1,n)^m \;=\; \binom{2n}{n}^m \tag{1.1}$$

so for larger m and n the numbers $f(m,n)$ get huge *very soon*; for example, the bound yields $f(5,6) > 6 \cdot 10^{14}$.

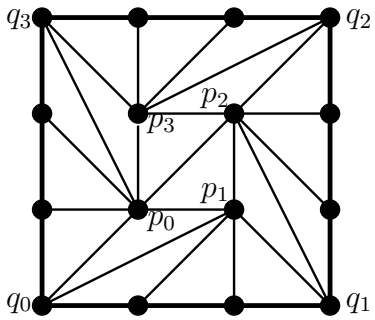
It is equally interesting to study/enumerate more general types of triangulations, such as triangulations of finite point sets that do not necessarily use all the points, triangulations of general convex or non-convex lattice polygons, triangulations of general position point sets, triangulations of higher-dimensional point sets, etc. None of these will appear here, but we refer interested readers to Lee [22] and De Loera, Rambau & Santos [12].

Lattice triangulations are basic combinatorial objects, and they are fundamental discrete geometric structures; so it is no surprise that they appear in various computational geometry contexts. However, lattice triangulations have also been studied intensively from different algebraic geometry angles. So, triangulations of a convex lattice polygon

- provide the data for Viro's [31] famous construction method of plane algebraic curves with prescribed combinatorics and topology, related to Hilbert's sixteenth problem;
- appear in Gel'fand–Kapranov–Zelevinsky's [16] theory of discriminants, where "regular" triangulations are in bijection with the vertices of the secondary polytope of the point configuration, and
- model torus-equivariant crepant resolution of singularities for toric threefolds, where "regular" triangulations correspond to projective desingularizations; see e. g., Kempf et al. [21, Chap. 3] and Dais [10].

The last two points pose the problem of counting or estimating the number $f^{\text{reg}}(m,n)$ of *regular* triangulations of $P_{m,n}$, that is, of triangulations of $\text{conv}(P_{m,n})$ whose triangles are the domains of linearity for a piecewise linear convex function (*lifting function*); see also, for example, Sturmfels [30, Chap. 8], Ziegler [32, Lect. 5]. Motivated by the toric variety considerations, one would like to know whether/that for large m and n most triangulations are non-regular. There is no proof, but a lot of evidence that this should be true (see, e.g., Table 2).

In this context, we must admit that despite the effort that has been put into studying this question (see e. g. Hastings [19, Chap. 2]), it has not become clear what non-regular triangulations really "look like." The "mother of all examples" is the whirlpool triangulation of the 3×3 grid:



Indeed, this unimodular triangulation is irregular, since a lifting function h would have to satisfy $h(p_i) + h(q_{i-1}) < h(p_{i-1}) + h(q_i)$ for $i = 0, \ldots, 3$ (all indices taken modulo 4), implying the contradiction $h(q_0) + \cdots + h(q_3) < h(q_0) + \cdots + h(q_3)$.

For $m = n = 3$, except for three symmetric copies this is the only example of a non-regular triangulation: We have $f(3,3) - f^{\text{reg}}(3,3) = 4$. This may lead one to the conjecture that some kind of "generalized whirlpools" are responsible for non-regularity. However, the pictures of "non-regular triangulations" that we present below do not support this intuition.

The plan for this paper is as follows: In Section 2 we face the challenge to ***count*** explicitly, trying to cope with the "combinatorial explosion." For this, we present a simple dynamic programming technique by which we get surprisingly far, and which on the specific problem of grid triangulations outruns the much more sophisticated general techniques such as Avis & Fukuda's [6] reverse search, Aichholzer's [1] path of a triangulation, or the oriented matroid technique of Rambau [27].

In Section 3 we ***estimate*** $f(m,n)$ for large m and n, trying to narrow the bounds for the asymptotics. It is interesting to compare with the situation for $N = (m+1)(n+1)$ points in the plane in general position, where the currently best available upper bound seems to be Santos & Seidel's [28] estimate that there are $o(59^N)$ triangulations. However, in our problem the N points are not at all in general position — and the upper bounds that we have to offer are much better: We report on a neat $O(2^{3N})$ upper bound by Anclin [5], which substantially improves on a previous $O(2^{4N})$ upper bound by Orevkov [25]. Based on explicit enumeration results from Section 2, we get a lower bound of $2^{2.055\,mn}$ when both m and n get large; note that (1.1) yields already a lower bound $2^{(1-o(1))2mn}$.

Finally, in Section 4 we ***sample*** lattice triangulations for large parameters m and n, and thus try to understand what typical lattice triangulations, as well as typical regular lattice triangulations, "look like." While we have some pictures to offer, proofs seem harder to come by. Indeed, the pictures display some long-range order; while this may make lattice triangulations interesting as a statistical physics model, it generates serious obstacles for any proof that the obvious Markov chain is rapidly mixing, and thus to application of the (by now) standard theory [8].

2 Explicit Values

There are several methods available to generate or count all triangulations of a finite set of points in $\mathbb{R}^2$. An approach that works for point sets in arbitrary dimensions is implemented in the software package TOPCOM by Rambau [27] (see also Pfeifle and Rambau [26]). It enumerates all triangulations in a purely combinatorial manner after the chirotope of the point set (oriented matroid data) has been computed.

The *reverse search algorithm* proposed by Avis and Fukuda [6] is a rather general enumeration scheme that can be specialized to triangulations of point configurations in $\mathbb{R}^2$ (see also Bespamyatnikh [7]). Since it was used to obtain some of the results reported in Section 4, and because it is based upon some structural properties that are relevant for our treatment later, we briefly describe the method here.

Let $\mathcal{T}$ be a *fine triangulation* of a point set $S \subset \mathbb{R}^2$; i.e., $\mathcal{T}$ is a set of triangles, for which the set of vertices equals S, such that the union of all triangles is the convex hull of S, and any two triangles intersect in a common (possibly empty) face. The unimodular triangulations of $P_{m,n}$ are precisely its fine triangulations. An edge of some triangle in $\mathcal{T}$ is *flippable* if it is contained in two triangles of $\mathcal{T}$ whose union is a strictly convex quadrangle. Replacing these two triangles by the two triangles into which the other diagonal cuts that quadrangle (*flipping* the edge) yields another fine triangulation of S. The graph on the fine triangulations of S defined via flipping is the *flip graph* of S.

Let us fix an arbitrary ordering of the points in S, inducing via lexicographical ordering a total order on the set of pairs of points, and thus, again via lexicographical ordering, a total order on the fine triangulations of S, which are identified with their sets of edges here. There is a distinguished fine triangulation $\mathcal{T}_0$ of S with respect to that ordering, namely the smallest *Delaunay triangulation* (i.e., a triangulation characterized by the condition that for every triangle the circumcircle contains no point from S in its interior).

Furthermore, for each fine triangulation $\mathcal{T} \neq \mathcal{T}_0$, there is a distinguished flippable edge (computable in $O(|S|)$ steps) such that, starting from any fine triangulation of S, iterated flipping of the respective distinguished edges eventually yields $\mathcal{T}_0$. This algorithmically defines a spanning tree in the flip graph of S, rooted at $\mathcal{T}_0$; in particular, the flip graph of a two dimensional finite point set is connected.

The basic idea of the reverse search method is to traverse that spanning tree from its root $\mathcal{T}_0$. At each iteration one chooses a leaf $\mathcal{T}$ of the current partial tree, and determines those among the triangulations adjacent to $\mathcal{T}$ on whose path to $\mathcal{T}_0$ in the spanning tree $\mathcal{T}$ lies. Properly implemented, the reverse search algorithm generates all fine triangulations in $O(|S| \cdot f(S))$ steps, where $f(S)$ is the number of fine triangulations of S; see [6].

Via the "secondary polytope" of a finite point set $S \subset \mathbb{R}^2$ [16] one can design a variant of the reverse search algorithm that generates *all regular*

triangulations of S in $\mathrm{O}(|S| \cdot F^{\mathrm{reg}}(S))$ steps, where $F^{\mathrm{reg}}(S)$ is the number of such triangulations. It is, unclear, however, if one can also generate all *regular fine* triangulations of S in a number of steps that is bounded by a polynomial in the number of such triangulations.

If one is interested in the number of triangulations of a two-dimensional set of points rather than in the explicit generation of all of them, then the *path of a triangulation method* due to Aichholzer [1] is more efficient than the reverse search algorithm.

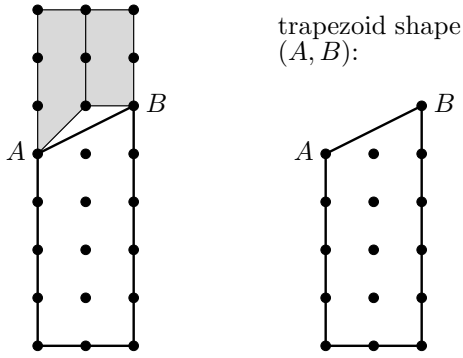
For counting the unimodular (fine) triangulations of the very special point sets $P_{m,n}$, however, different methods are much more efficient. These are described in the following sections.

2.1 Narrow Strips

Strips of width $m = 1$. For any lattice trapezoid of width 1, whose parallel vertical sides have lengths a and b, the number of unimodular triangulations is $g_1(a, b) = \binom{a+b}{a} = \binom{a+b}{b}$; indeed, a bijection between these triangulations and the a-subsets of $\{1, \ldots, a + b\}$ is established by top-down numbering the triangles of a triangulation $\mathcal{T}$ by $1, \ldots, a + b$, and mapping $\mathcal{T}$ to the a-set of all numbers of triangles whose vertical edges are on the left. In particular, we have

$$f(1, n) \;=\; \binom{2n}{n}. \tag{2.1}$$

Strips of width $m = 2$. For $f(2, n)$ we have no explicit formula, and we cannot evaluate the asymptotics precisely, but we have a "quadratic" recursion that can be evaluated efficiently: For this we enumerate the triangulations according to the highest "width 2 diagonal," which (if it exists) decomposes the rectangle into two width 1 strips, a single triangle, and a trapezoid of width 2 (see the left-hand figure below). Let $g_2(A, B)$ denote the number of triangulations of a trapezoid of width 2 with vertical edges of lengths A and B and horizontal base line, as in the right-hand figure below — where $A + B \equiv 1 \bmod 2$ implies that the midpoint of the diagonal is not a lattice point, and where we may assume $A < B$ by symmetry:



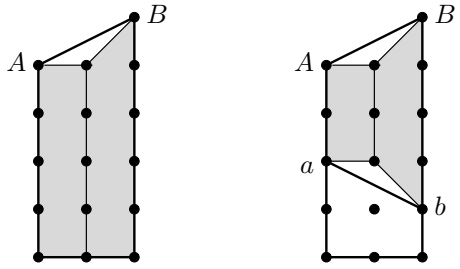
Thus we get

$$f(2,n) \;=\; \binom{2n}{n}^2 \;+\; 2\sum_{\substack{0\le A<B\le n\\ A+B\equiv 1 \bmod 2}} g_2(A,B)\binom{2n-\frac{3A+B+1}{2}}{n-A}\binom{2n-\frac{A+3B+1}{2}}{n-B}.$$

The binomial coefficients in this recursion correspond to triangulations of width 1 strips. So they could be rewritten in terms of g_1, as $g_1(n-A, n-\frac{A+B+1}{2})$ resp. $g_1(n-B, n-\frac{A+B+1}{2})$. A similar remark applies to the binomial coefficients that appear in the following.

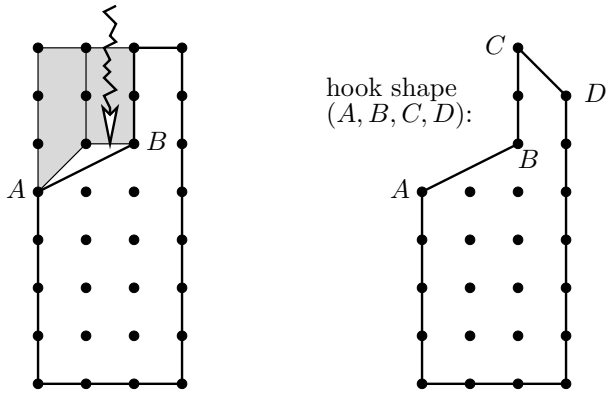
For $g_2(A,B) = g_2(B,A)$ we also get a recursion by considering the highest diagonal of width 2:

$$\begin{aligned} g_2(A,B) \;=\;& \binom{\frac{3A+B-1}{2}}{A}\binom{\frac{A+3B-1}{2}}{B} \\ &+ \sum_{\substack{0\le a\le A,\ 0\le b\le B\\ a+b\equiv 1 \bmod 2,\ a+b<A+B}} g_2(a,b)\binom{\frac{3A+B-3a-b}{2}-1}{A-a}\binom{\frac{A+3B-a-3b}{2}-1}{B-b}. \end{aligned}$$



Here the parameters A, B, a, b may be interpreted as the y-coordinates of certain lattice points. The shaded parts of the figure consist of strips of width 1, whose triangulations are counted by the binomial coefficients $g_1(\cdot,\cdot)$.

Strips of width $m = 3$. For $f(m,3)$ we have a recursion of order 4; it relies on the observation that if we screen the middle strip from the top for diagonals of width at least 2, then the first diagonal to find will be of width exactly 2, since any width 3 diagonal is flippable, and contained in a parallelogram that is bounded by two width 2 diagonals. The corresponding decomposition of our rectangle is indicated in the left figure below:



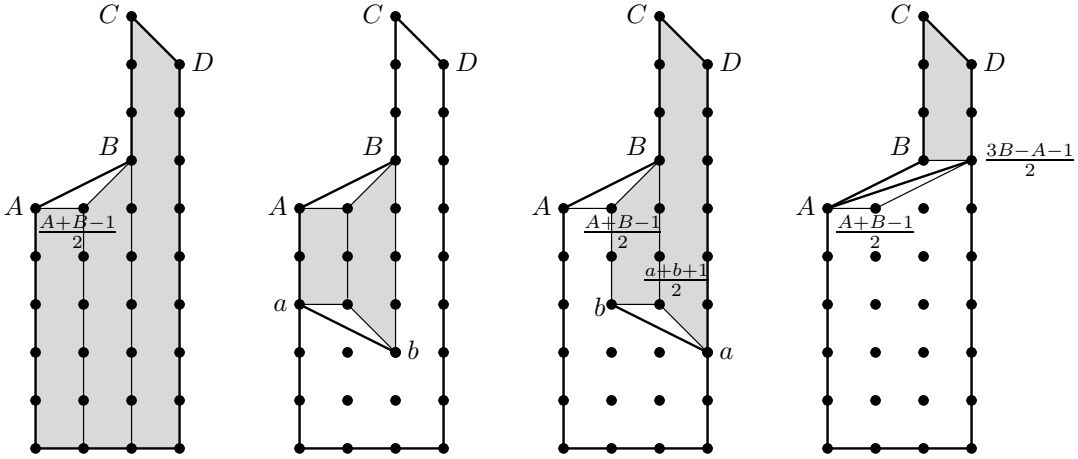
Therefore, we obtain

$$f(3,n) \;=\; \binom{2n}{n}^3 + 2 \sum_{\substack{0\le A,B\le n\\ A+B\equiv 1 \bmod 2}} h(A,B,n,n)\binom{2n-\frac{3A+B+1}{2}}{n-A}\binom{2n-\frac{A+3B+1}{2}}{n-B},$$

where $h(A,B,C,D)$ counts the number of triangulations in a "hook" shape as given in the right drawing in the figure above, which depends on four parameters $0 \le A,B,C,D \le n$, with $A+B \equiv 1 \bmod 2$ and $B \le C$. For the number of triangulations of such a hook shape we get a recursion

$$\begin{aligned} h(A,B,C,D) \;&=\; \binom{\frac{3A+B-1}{2}}{A}\binom{\frac{A+3B-1}{2}}{B}\binom{C+D}{C} \\ &+ \sum_{\substack{0\le a\le A,\ 0\le b\le B\\ a+b\equiv 1 \bmod 2,\ a+b<A+B}} h(a,b,C,D)\binom{\frac{3A+B-3a-b}{2}-1}{A-a}\binom{\frac{A+3B-a-3b}{2}-1}{B-b} \\ &+ \sum_{\substack{0\le a\le D,\ 0\le b\le \frac{A+B-1}{2}\\ a+b\equiv 1 \bmod 2,\ \frac{a+b+1}{2}\le B}} h(a,b,\tfrac{A+B-1}{2},A)\binom{D+C-\frac{3a+b+1}{2}}{D-a}\binom{\frac{A+3B-a-3b}{2}-1}{B-\frac{a+b+1}{2}} \\ &+ h(\tfrac{3B-A-1}{2},\tfrac{A+B-1}{2},\tfrac{A+B-1}{2},A)\binom{C+D-\frac{5B-A-1}{2}}{C-B} \quad \text{if } D \ge \tfrac{3B-A-1}{2} \ge 0. \end{aligned}$$

The four terms in this recursion correspond to the four cases depicted in the figure below, where the fourth case — of a long diagonal of width 3 — occurs only in the case where the second endpoint of the diagonal, which may be computed to have y-coordinate $\frac{3B-A-1}{2}$, comes to lie within the hook.

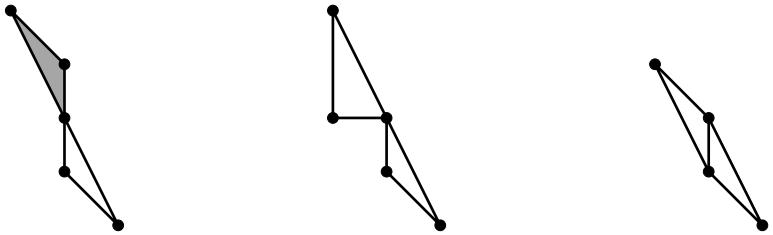


2.2 Strips of (fixed) width m.

We now describe a recursive strategy for the enumeration of unimodular triangulations of grids of arbitrary size. The method is applicable for triangulations of general finite point sets — but it is effective only in the special case

where the points lie on a small family of parallel (vertical, say) lines; in our case this is the situation of small (fixed) m and variable n. The key observation is that any triangulation may be dismantled by removing triangles from the upper boundary, while maintaining a lattice triangulation of a y-convex lattice polygon. Since for fixed m the number of such polygons in $P_{m,n}$ is bounded by a polynomial in n, this yields an efficient dynamical programming algorithm in this case.

Let $\Delta_1, \Delta_2 \subset \mathbb{R}^2$ be two triangles whose intersection $\Delta_1 \cap \Delta_2$ is a common (empty, zero-, or one-dimensional) face of both Δ_1 and Δ_2. We say that Δ_2 *lies above* Δ_1 ($\Delta_1 \prec \Delta_2$) if there are two points $(x, y_1) \in \Delta_1 \setminus \Delta_2$ and $(x, y_2) \in \Delta_2 \setminus \Delta_1$ on a vertical line with $y_1 < y_2$. For example, in our figure the shaded triangle lies above the other one; the other two pairs of triangles are incomparable.



Due to the convexity of the triangles and the intersection condition imposed on them, this is a well-defined asymmetric and irreflexive relation.

Lemma 2.1 *There is no sequence $\Delta_0, \ldots, \Delta_{t-1} \subset \mathbb{R}^2$ of triangles (such that the intersection of any two among them is a face of both) satisfying*

$$\Delta_0 \prec \Delta_1 \prec \ldots \prec \Delta_{t-1} \prec \Delta_0 \ . \tag{2.2}$$

Proof Suppose that $\Delta_0, \ldots, \Delta_{t-1} \subset \mathbb{R}^2$ is a minimal cyclic sequence of triangles (such that the intersection of any two among them is a face of both), i.e., it satisfies (2.2) (it is *cyclic*), but no subsequence of $\Delta_0, \ldots, \Delta_{t-1} \subset \mathbb{R}^2$ is cyclic. The orthogonal projections $x(\Delta_i)$ to the x-axis have the following three properties (where all indices are taken modulo t):
(a) $x(\Delta_i) \cap x(\Delta_{i+1}) \neq \emptyset$,
(b) $x(\Delta_i) \not\subseteq x(\Delta_{i-1}), x(\Delta_{i+1})$, and
(c) $x(\Delta_{i-1}) \cap x(\Delta_{i+1}) = \emptyset$,
where (a) follows immediately from the definition of $\prec$ and (b) as well as (c) are due to the minimality of the cycle.

But (a), (b), and (c) together imply that the intervals $x(\Delta_0), \ldots, x(\Delta_{t-1})$ "either run left-to-right or right-to-left"; in particular, we have $x(\Delta_0) \cap x(\Delta_i) = \emptyset$ for $i \in \{2, \ldots, t-1\}$, contradicting $\Delta_{t-1} \prec \Delta_0$. □

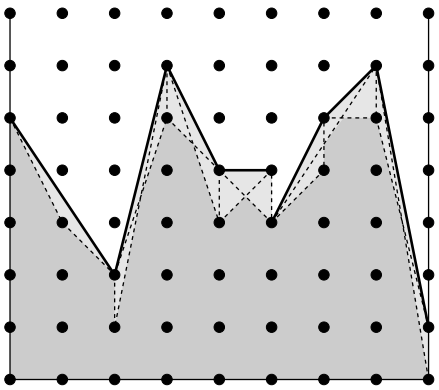
Of course, both in the definition of $\prec$ as well as in Lemma 2.1 one can replace "triangle" by "compact convex set."

The relation $\prec$ was defined with respect to parallel projection here. If one defines an ordering with respect to central rather than to parallel projection, then the analog to Lemma 2.1 (for arbitrary centers of projection) does not hold. For Delaunay triangulations, however, there is an analog to Lemma 2.1. (See De Floriani et al. [11] for dimension 2, and Edelsbrunner [13] for arbitrary dimensions.)

Due to Lemma 2.1, the relation $\prec$ induces a partial order on the set of triangles in $\mathbb{R}^2$, which we will also denote by $\prec$.

A sequence $\mathcal{T}_1, \ldots, \mathcal{T}_{2mn}$ of sets of triangles will be called an *admissible sequence* for $P_{m,n}$ if $\mathcal{T}_1$ is a unimodular triangulation of $P_{m,n}$, and if, for each $i = 2, 3, \ldots, 2nm$, we have $\mathcal{T}_i = \mathcal{T}_{i-1} \setminus \{\Delta\}$ for some $\prec$-maximal triangle Δ in $\mathcal{T}_{i-1}$. A subset $S \subset \mathbb{R}^2$ is called an *admissible shape* (of $P_{m,n}$) if it can be obtained as a union $S = \bigcup_{\Delta \in \mathcal{T}_i} \Delta$ for an admissible sequence $\mathcal{T}_1, \ldots, \mathcal{T}_{2mn}$ and some $i \in \{1, \ldots, 2mn\}$. Every admissible shape is y-convex (i.e., its intersection with any vertical line is connected). It is determined by its *upper boundary segments*, i.e., the sequence of line segments $[l^{(1)}, r^{(1)}], \ldots, [l^{(t)}, r^{(t)}]$ with $l_x^{(1)} = 0$, $l_x^{(t)} = n$, and $r_x^{(j-1)} = l_x^{(j)}$ for $j \in \{2, \ldots, t\}$, such that, for each point p in the relative interior of any of the segments, $p + (0, \varepsilon) \notin S$ holds for all $\varepsilon > 0$.

Let S be an admissible shape. We denote by $\mathcal{T}_{\max}(S)$ the set of all $\prec$-maximal unimodular triangles in S, that is, the finite set of all unimodular triangles that could be $\prec$-maximal in *some* unimodular triangulation of S. For example, the figure below indicates the 12 $\prec$-maximal triangles of the shaded admissible shape.



Any admissible shape S' that arises from S by removing some triangles contained in $\mathcal{T}_{\max}(S)$ is called an *admissible subshape* of S. These triangles have disjoint interiors, and they are uniquely determined for each admissible subshape S' of S (compare the proof of Lemma 2.5). Their number is denoted by $\#(S', S)$.

Since every unimodular triangulation of S contains at least one triangle from $\mathcal{T}_{\max}(S)$, we obtain the following inclusion-exclusion formula for the numbers $f(S)$ of unimodular triangulations of admissible shapes S.

Lemma 2.2 *Every admissible shape S has*

$$f(S) = \sum_{S'} (-1)^{\#(S',S)-1} f(S') \tag{2.3}$$

unimodular triangulations, where the sum is taken over all admissible proper subshapes S' of S.

Lemma 2.2 allows us to compute $f(m,n) = f([0,m] \times [0,n])$ via a dynamic programming approach: Determine the numbers $f(S)$ via (2.3) in some order such that every admissible shape appears after all its admissible subshapes. In order to analyze the running time of such an algorithm, we first need to estimate the number of admissible shapes.

Lemma 2.3 *Let $[l^{(1)}, r^{(1)}], \ldots, [l^{(t)}, r^{(t)}]$ be the sequence of upper boundary segments of an admissible shape S. Then*

$$l_y^{(j)} \in \{r_y^{(j-1)} - 1, r_y^{(j-1)}, r_y^{(j-1)} + 1\}$$

holds for each $2 \leq j \leq t$.

Proof This follows by induction on the number of triangles removed in order to obtain S: Every vertical edge of a triangle in a unimodular triangulation has length one, and thus removing a $\prec$-maximal triangle from an admissible shape never creates a vertical boundary part of height more than 1. □

Lemma 2.3 implies the following bound on the number of admissible shapes.

Lemma 2.4 *There are at most $(3n+2)^{m-1}(n+1)^2$ admissible shapes of $P_{m,n}$.*

Proof The upper boundary of an admissible shape has $n+1$ possible start and $n+1$ possible end points. At every interior x-coordinate ($x \in \{1, \ldots, m-1\}$) either a segment of the upper boundary ends and a new one starts ($3(n+1)-2$ possibilities, by Lemma 2.3) or a segment "passes through" (one possibility). □

The second important quantity for the analysis of the running time of the dynamic programming algorithm proposed above is the maximal number of summands that may occur in (2.3).

Lemma 2.5 *Every admissible shape S of $P_{m,n}$ has at most*

$$\Big(\frac{3+\sqrt{13}}{2}\Big)^m < 3.31^m$$

admissible subshapes.

Proof Let $(B_1, \ldots, B_t)$ be the sequence (from left to right) of upper boundary segments of S. Each triangle in $\mathcal{T}_{\max}(S)$ contains at least one of the edges $\{B_1, \ldots, B_t\}$.

Let $B \in \{B_1, \ldots, B_t\}$ be a segment of the upper boundary of S. There are at most two triangles in $\mathcal{T}_{\max}(S)$ containing B and one of its adjacent segments. Each other triangle in $\mathcal{T}_{\max}(S)$ that contains B must have its third vertex v in $B + \mathbb{R} \cdot (0, -1)$ on the line that is parallel to B at distance $\ell_2(B)^{-1}$, where $\ell_2(B)$ denotes the Euclidean length of B (because the triangle has area $\frac{1}{2}$). Since v must be integral and there is no integral point in the relative interior of B, there are at most two possibilities for v. Let us call one of them *the first triangle below* B, and the other one, *the second triangle below* B (if they exist).

For every admissible subshape T of S, define a word $w(T) \in \{0, \alpha, \beta, \gamma\}^\star$ by replacing B_i by

- 'α' if $S \setminus T$ contains the first triangle below B_i,
- 'β' if $S \setminus T$ contains the second triangle below B_i,
- 'γ' if $S \setminus T$ contains the triangle formed by B_i and B_{i-1}, and
- '0' otherwise.

Clearly, every 'γ' in $w(T)$ has a '0' as its left neighbor; furthermore, $w(\cdot)$ is an injective mapping. Therefore, the number of admissible subshapes of S is bounded from above by the function $\varphi(m)$ defined recursively via

$$\varphi(0) = 1\ , \qquad \varphi(1) = 3\ , \qquad \varphi(m) = 3\varphi(m-1) + \varphi(m-2) \quad (m \geq 2).$$

Using standard techniques (see, e. g., [29, Thm. 4.1.1]), one derives from this recursion that $\varphi(m) < \left(\frac{3+\sqrt{13}}{2}\right)^m$ for $m \geq 1$. □

Lemmas 2.4 and 2.5 imply that (for every m) the dynamic programming algorithm needs at most

$$3.31^m (3n+2)^{m-1} (n+1)^2 \ < \ 10^m (n+1)^{m+1}$$

arithmetic operations. The actual running time of an implementation heavily depends on the data structures for storing the admissible shapes and on the way by which one determines the admissible subshapes in that data structure. Therefore, here we include only the following rough statement.

Theorem 2.6 *For every fixed m, the function $f(m, n)$ can be computed in time bounded by a polynomial in n.*

In our implementation, we organize the admissible shapes as the leaves of a tree whose nodes are the prefixes of the sequences of upper boundaries segments of admissible shapes. The data structure allows quite efficient access to the admissible subshapes while not wasting too much memory. Nevertheless, the bottleneck in the computations is always memory. It is crucial to use an

ordering of the admissible shapes in which for each shape S the shapes of which it is an admissible subshape come as soon as possible after it; this allows one to keep in memory only a subset of the admissible shapes at each point of time.

Furthermore, in (2.3) there is no need to sum over *all* admissible subshapes S'. It suffices to consider those S' that arise from removing triangles in $\mathcal{T}_{\max}(S)$ that are contained in $\{(x,y) \in \mathbb{R}^2 : x \geq c\}$, where c is the maximal x-coordinate where $l_y^{(j)} = r_y^{(j-1)} + 1$ for some j and the upper boundary segments $[l^{(1)}, r^{(1)}], \ldots, [l^{(t)}, r^{(t)}]$ of S (if that maximum exists).

Of course, when computing the number of unimodular triangulations of $P_{m,n}$ by our method, we obtain as a byproduct the number of unimodular triangulations of several interesting polygons inside $P_{m,n}$, including $P_{m,1}$, ..., $P_{m,n-1}$.

The algorithm described above cannot only be used to calculate the number of unimodular triangulations of any admissible shape S in $P_{m,n}$; it can also be extended to produce a uniformly distributed random triangulation within the same asymptotic running time, and thus, in polynomial time (depending on n) for fixed m. For this, one just determines the numbers of triangulations of those admissible subshapes of S that arise from removing one single triangle; with respect to the corresponding random distribution one then chooses one of the $\prec$-maximal triangles in S at random, and proceeds with the subshape obtained by removing it.

2.3 Explicit values

We have implemented the algorithms described in Subsection 2.1 in C++, using the `gmp` library [17] for exact arithmetic, with the interface to it provided by the `polymake` system [15].

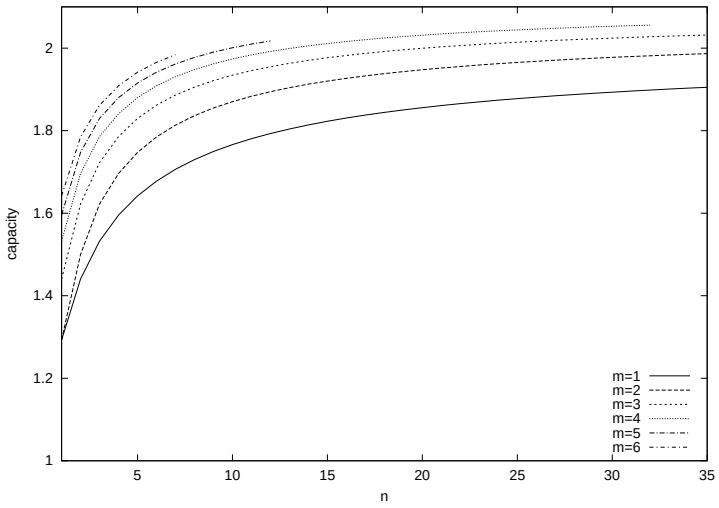
Results obtained by our code are compiled in the Appendix (see Tables 3, 4, 5, 6, and 7). The number of unimodular triangulations of $P_{m,n}$ asymptotically grows exponentially with mn (see also Section 3). Therefore, it is more convenient to view the function $f(m,n)$ on a logarithmic scale, normalized by mn.

Definition The *capacity* of the $m \times n$ grid is

$$c(m,n) \quad := \quad \frac{\log_2 f(m,n)}{mn}.$$

The following Figure shows the capacity functions $c(m,n)$ for $m \in \{1,\ldots,6\}$. The largest capacity we found is 2.055792 (for $m = 4$ and $n = 32$, see Table 5).

To give an impression of the amount of (machine) resources required for the calculations: The run of the admissible shape algorithm for $m = 6$ and $n = 7$ needed about three gigabytes of memory. The grid $P_{6,7}$ has 370252552 admissible shapes. We generated them in lexicographical order with respect



to the pairs of starting heights and volumes. This way, never more than 15% of the admissible shapes had to be kept in memory simultaneously. Notice that more than 400 megabytes (of the 3 gigabytes in total) where needed just to store the *numbers* of triangulations of the admissible shapes in the memory. The CPU time used for the computation was about 25 hours. Our computations were performed on a SUN UltraAX MP machine equipped with four 448 MHz UltraSPARC-II processors (of which we used only one) and 4 gigabytes main memory.

For very small parameters, Meyer [24] has enumerated all unimodular triangulations by Avis and Fukuda's reverse search method (sketched at the beginning of this section) and checked them for regularity. Table 1 shows the results. While for these small parameters irregular triangulations are quite rare, the picture changes drastically when m and n get larger, see Section 4.

$m \times n$	# triangulations	# irregular	fraction
3×3	46456	4	.000086
3×4	2822648	502	.000178
3×5	182881520	63528	.000347
4×4	736983568	1553020	.002107

Table 1: Number of regular triangulations of small grids.

3 Bounds

3.1 Patching

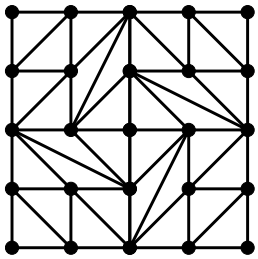
Any two unimodular triangulations of P_{m,n_1} and P_{m,n_2} can be patched to a unimodular triangulation of P_{m,n_1+n_2}. Thus we have the follwing super-multiplicativity relation, where $f^{\text{irreg}}(m,n)$ denotes the number of irregular unimodular triangulations of $P_{m,n}$.

Lemma 3.1 *For $m, n_1, n_2 \geq 1$ the following relations hold.*

(i) $$f(m, n_1 + n_2) \geq f(m, n_1) f(m, n_2)$$

(ii) $$f^{\text{irreg}}(m, n_1 + n_2) \geq f(m, n_1) f^{\text{irreg}}(m, n_2)$$

With respect to regular triangulations, patching is dangerous, as demonstrated by the following example of a non-regular triangulation of $P_{4,4}$ composed of four regular triangulations of $P_{2,2}$ (suggested by Francisco Santos).

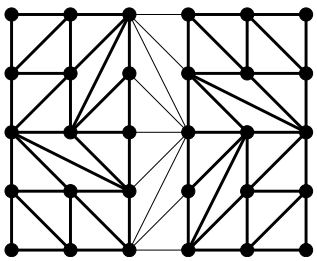


However, a much more general theorem by Goodman and Pach [18] says that any two regular triangulations of two disjoint convex polytopes $P_1, P_2 \subset \mathbb{R}^d$ can be extended to a regular triangulation of $\text{conv}(P_1 \cup P_2)$ without additional vertices. Thus also for the regular case we get a (slightly weaker) supermultiplicativity relation.

Lemma 3.2 *For $m, n_1, n_2 \geq 1$ we have*

$$f^{\text{reg}}(m, n_1 + n_2 + 1) \geq f^{\text{reg}}(m, n_1) f^{\text{reg}}(m, n_2) \ .$$

The following figure illustrates the patching of Lemma 3.2:



For $n_2 = 1$ we will (Lemma 3.5) strengthen the inequality in Lemma 3.2.

Let us fix some notations first. For any function $h : P_{m,n} \longrightarrow \mathbb{R}$ we denote by $H : P_{m,n} \longrightarrow \mathbb{R}^3$ the function with $H(x, y) = (x, y, h(x, y))$. The function h is called a *lifting function* of a triangulation $\mathcal{T}$ of $P_{m,n}$, if $\mathcal{T}$ is the image of the set of "lower facets" of the 3-polytope $\text{conv}\{H(x, y) : (x, y) \in P_{m,n}\}$ under orthogonal projection to the x, y-plane (deletion of the third coordinate).

A function h is a lifting function of $\mathcal{T}$ if and only if h is convex and piecewise linear, and its (maximal) domains of linearity are the triangles in $\mathcal{T}$. In

particular, one may add to h any convex piecewise linear function whose domains of linearity are unions of triangles of $\mathcal{T}$ in order to obtain another lifting function for $\mathcal{T}$.

A triangulation is regular if and only if it has a lifting function.

The following result shows that all unimodular triangulations of a strip of width 1 are regular; moreover one has a lot of freedom in choosing the respective lifting functions, which we will exploit below.

Lemma 3.3 *Let $\mathcal{T}$ be a unimodular triangulation of a lattice trapezoid with two parallel vertical or horizontal sides S_0 and S_1 at distance one. Every piecewise linear function $h_0 : S_0 \longrightarrow \mathbb{R}$ that is strictly convex on $S_0 \cap \mathbb{Z}^2$ can be extended to a lifting function for $\mathcal{T}$.*

Proof Let $p_0, \ldots, p_r$ be the integral points on S_0, and let $\{e_{i,1}, \ldots, e_{i,k(i)}\}$ be the edges of $\mathcal{T}$ connecting p_i to S_1. Let $S_0^+(i)$ and $S_0^-(i)$ be the closures of the two components of $S_0 \setminus \{p_i\}$. Then we may decompose the function h_0 as

$$h_0(x) = \sum_{i=0}^{r} \sum_{j=1}^{k(i)} h_{i,j}(x) \ ,$$

where each $h_{i,j}$ is a convex function on S_0, linear both on $S_0^+(i)$ and on $S_0^-(i)$, and having its unique break-point at p_i.

Now we extend each $h_{i,j}$ to a convex, piecewise-linear function defined on the entire trapezoid such that it has a break-line at the edge $e_{i,j}$ and is linear above and below this line. Then the sum of all $h_{i,j}$ is a lifting function for $\mathcal{T}$. □

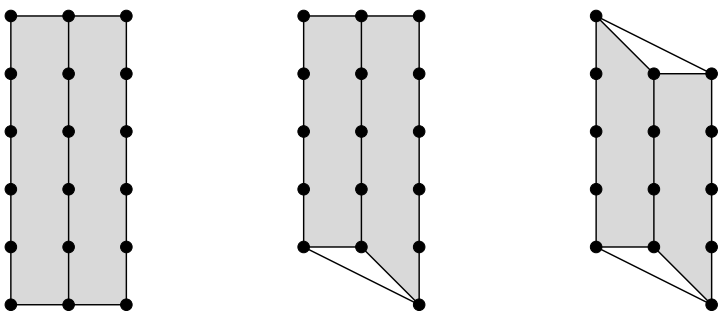
Proposition 3.4 *For $n \geq 1$ the following relations hold.*

(i) $$f^{\mathrm{reg}}(1,n) = f(1,n) = \tbinom{2n}{n}$$

(ii) $$f^{\mathrm{reg}}(2,n) = f(2,n)$$

Proof Part (i) follows immediately from Lemma 3.3(i) (and equation (2.1)).

Since patching two regular triangulations along a *single* edge preserves regularity, it suffices for the proof of part (ii) to show that every unimodular triangulation of shapes of one of the forms



is regular. But this can be derived from Lemma 3.3. One starts from an arbitrary prescribed strictly convex function on the middle column, and after the extensions to the two shaded vertical trapezoids of width one obtained from Lemma 3.3, one adds a piecewise linear function that is constant on the left strip, linear on the right strip, and sufficiently large on the right column of vertices. □

Similarly, one proves the following strengthening of Lemma 3.2 for $n_2 = 1$, as announced earlier.

Lemma 3.5 *For $m, n \geq 1$, we have*

$$f^{\text{reg}}(m, n+1) \geq f^{\text{reg}}(m, n) \cdot \binom{2n}{n} .$$

3.2 Limit capacities

In the following, we will show that the capacities $c(m, n)$ and $c^{\text{reg}}(m, n)$ (with $f(m, n) = 2^{c(m,n)mn}$ and $f^{\text{reg}}(m, n) = 2^{c^{\text{reg}}(m,n)mn}$) asymptotically behave well, which allows us to focus on their limits subsequently. Note that all capacities are bounded (see Theorem 3.9).

Proposition 3.6 *Let $m \geq 1$.*

(i) *The limit* $c_m := \lim_{n\to\infty} c(m, n)$ *exists.*

(ii) *The limit* $c_m^{\text{reg}} := \lim_{n\to\infty} c^{\text{reg}}(m, n)$ *exists.*

Proof Lemmas 3.1(i) and 3.2 imply by Fekete's lemma [23, Lemma 11.6] that

$$\lim_{n\to\infty} f(m, n)^{\frac{1}{n}} \qquad \text{and} \qquad \lim_{n\to\infty} f^{\text{reg}}(m, n-1)^{\frac{1}{n}}$$

exist. Therefore,

$$\lim_{n\to\infty} \frac{1}{m} \log f(m, n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{\log f(m, n)}{mn} = \lim_{n\to\infty} c(m, n)$$

and

$$\lim_{n\to\infty} \frac{n}{m(n-1)} \log f^{\text{reg}}(m, n-1)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{\log f^{\text{reg}}(m, n-1)}{m(n-1)} = \lim_{n\to\infty} c^{\text{reg}}(m, n)$$

exist as well. □

While the last proposition concerned the asymptotics of growing n for fixed m, the next result shows that also growing m and n simultaneously yields nice asymptotics.

Proposition 3.7 *Let $m \geq 1$.*

(i) *The limit $c := \lim_{m\to\infty} c(m,m)$ exists. It satisfies*

$$c = \lim_{m\to\infty} c_m \qquad \textit{and} \qquad c_{m_0} \leq c \quad (m_0 \in \mathbb{N}) \ .$$

(ii) *The limit $c^{\mathrm{reg}} := \lim_{m\to\infty} c^{\mathrm{reg}}(m,m)$ exists. It satisfies*

$$c^{\mathrm{reg}} = \lim_{m\to\infty} c^{\mathrm{reg}}_m \qquad \textit{and} \qquad c^{\mathrm{reg}}_{m_0} \leq c^{\mathrm{reg}} \quad (m_0 \in \mathbb{N}) \ .$$

Proof From Lemma 3.1(i) one derives (for $m_0, n_0 \geq 1$) the inequality

$$f(m,n) \geq f(m_0,n_0)^{\lfloor \frac{m}{m_0} \rfloor \lfloor \frac{n}{n_0} \rfloor}. \tag{3.1}$$

For integers $p, q \geq 1$ we define $\Phi(p,q) := 1 - \frac{p \bmod q}{p}$. We have $\Phi(q,q) = 1$ and $\lim_{p\to\infty} \Phi(p,q) = 1$ for all $q \in \mathbb{N}$.

Equation(3.1) then implies

$$c(m,n) \geq \Phi(m,m_0)\Phi(n,n_0)c(m_0,n_0) \ , \tag{3.2}$$

and, in particular,

$$c(m_0,n) \geq \Phi(n,n_0)c(m_0,n_0) \ . \tag{3.3}$$

Inequality (3.3) (together with $\lim_{n\to\infty} \Phi(n,n_0) = 1$) yields

$$c_m \geq c(m,n_0) \ . \tag{3.4}$$

Inequality (3.2) implies (together with $\lim_{m\to\infty} \Phi(m,m_0)\Phi(m,n_0) = 1$)

$$\liminf_{m\to\infty} c(m,m) \geq c(m_0,n_0) \ ,$$

and therefore,

$$\liminf_{m\to\infty} c(m,m) \geq c_{m_0} \ . \tag{3.5}$$

Finally, we obtain the following chain of inequalities, which, together with (3.5) proves part (i) of the proposition. The middle inequality is from (3.5), and the outer ones are due to (3.4).

$$\liminf_{m\to\infty} c_m \geq \liminf_{m\to\infty} c(m,m) \geq \limsup_{m\to\infty} c_m \geq \limsup_{m\to\infty} c(m,m)$$

Part (ii) is proved similarly, starting from Lemma 3.2. □

Note that a similar proof yields

$$c = \lim_{m\to\infty} c(\alpha m, \beta n) \qquad \text{and} \qquad c^{\mathrm{reg}} = \lim_{m\to\infty} c^{\mathrm{reg}}(\alpha m, \beta n)$$

for each pair $\alpha, \beta > 0$.

By Lemma 3.1(ii), the corresponding "irregular limit capacities" do exist as well, and they are equal to c_m ($m \geq 3$) and c, respectively. Therefore, we do not treat them explicitly.

3.3 Lower bounds

Proposition 3.8 *The following estimates hold.*

(i) $c_1 = c_1^{\text{reg}} = 2$

(ii) $c_2 = c_2^{\text{reg}} > 2.044$

(iii) $c_3 > 2.051$, $c_4 > 2.055$

(iv) $c_m > 2.048$ *for* $m \geq 5$

(v) $c_m \geq c_m^{\text{reg}} > 2$ *for* $m \geq 3$

Proof Part (i) follows from

$$f(1,n) = f^{\text{reg}}(1,n) = \binom{2n}{n} \approx \frac{2^{2n}}{\sqrt{2n}} \ .$$

Parts (ii) and (iii) are results of the computer calculations reported in Section 2, combined with Proposition 3.4(ii).

Lemma 3.1(i) applied to the "transposed grids" implies

$$c(m_1+m_2,n) \geq \frac{m_1}{m_1+m_2} c(m_1,n) + \frac{m_2}{m_1+m_2} c(m_2,n) \ , \tag{3.6}$$

and thus

$$c_{m_1+m_2} \geq \frac{m_1}{m_1+m_2} c_{m_1} + \frac{m_2}{m_1+m_2} c_{m_2} \ .$$

With (ii) and (iii), this yields (iv).

Similarly, Lemma 3.5 leads to

$$c_{m+1}^{\text{reg}} \geq \frac{m}{m+1} c_m^{\text{reg}} + \frac{1}{m+1} \cdot 2 \ .$$

With (ii) this proves part (v). □

Equation (3.6) implies that $c_{kn} \geq c_n$ for $k \geq 2$; for example, this implies that $c_4 \geq c_2$ — but it is not obvious that $c_3 \geq c_2$. Even stronger, one would assume that

$$2 = c_1 < c_2 < c_3 < \ \cdots \ \leq c \ ,$$

and

$$2 = c_1^{\text{reg}} < c_2^{\text{reg}} < c_3^{\text{reg}} < \ \cdots \ \leq c^{\text{reg}} \ ,$$

but neither monotonicity is proved.

3.4 Upper bounds

From general principles (see Ajtai et al. [4]) one gets that the capacity for any configuration of N points in the plane is finite. In the general position case (no three points on a line) the currently best upper bound is $o(2^{59N})$, due to Santos & Seidel [28]. However, in the very "degenerate" case of lattice triangulations, there are far fewer triangulations: Orevkov [25] obtained the bound $f(m,n) \leq 4^{3mn} = 2^{6mn}$. Very recently, this has been substantially improved by Anclin [5], as follows.

Theorem 3.9 (Anclin [5]) *For all $m, n \geq 1$,*

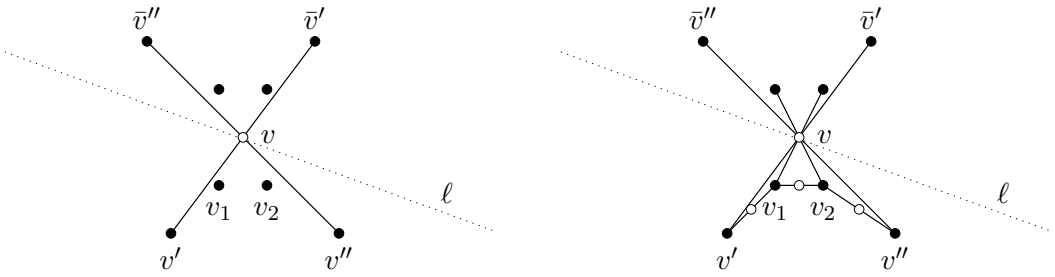
$$f(m,n) \;<\; 2^{3mn-m-n}.$$

Proof Our sketch relies on the essential ideas of Anclin's proof.

The first, crucial observation is that the midpoints of the edges in any unimodular triangulation are exactly the half-integral, not integral points in $\operatorname{conv}(P_{m,n})$. (Clearly the midpoint of every edge is half-integral; the converse may be derived from Pick's theorem, or from the fact that all unimodular triangles are equivalent to $\operatorname{conv}\{(0,0),(1,0),(0,1)\}$.) The number of these half-integral points in the interior of $P_{m,n}$ is $e = 3mn - m - n$.

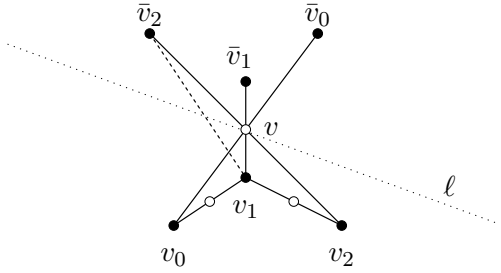
Now any triangulation is built as follows: The half-integral points are processed in an order that is given by a parallel *sweep*. (See de Berg et al. [9, Sect. 2.1] for a discussion and many further sweeping applications of this fundamental technique.)

Whenever a point is processed, we add to the partial triangulation a new edge with the given midpoint. The key *claim* is that at each such step, when a half-integral point v is processed, there are (at least one and) at most two possibilities for the new edge with midpoint v to be added. If this claim is true, then the number of triangulations is bounded by 2^e.



To prove the claim, one can verify the following: Let $[v', \bar{v}']$ and $[v'', \bar{v}'']$ be two potential edges with midpoint v that could be added, where v' and v'' are the endpoints below the sweep-line ℓ; let Q be the convex hull of the integral points in the triangle $[v, v', v'']$; then all the $k \geq 2$ vertices $v' = v_0, v_1, \ldots, v_{k-1}, v_k = v''$ of Q are "visible" from v, and any one of the edges $[v_i, \bar{v}_i]$ with midpoint v could potentially be added at this step. Furthermore, the midpoints $\frac{1}{2}(v_{i-1} + v_i)$ lie

below the sweep-line, and thus the edges $[v_{i-1}v_i]$ are present in the current partial triangulation.



Now assume that $[v_0, \bar{v}_0]$, $[v_1, \bar{v}_1]$, $[v_2, \bar{v}_2]$ are three *adjacent* edges with midpoint v that could be added when processing v. By central symmetry with respect to v we may assume that the midpoint w of $[v_1, \bar{v}_2]$ lies below the sweep-line. But the triangle $[v_1, v, \bar{v}_2]$ is then an empty triangle of area $\frac{1}{4}$, just like the triangle $[v_1, v, v_2]$. From this we conclude that in the current partial triangulation, the edge $[v_1, \bar{v}_2]$ must be present — which creates a crossing with the potential edge $[v_0, \bar{v}_0]$, and thus a contradiction. □

The following upper bounds on the limit capacities follow immediately from Theorem 3.9.

Corollary 3.10 *For all $m, n \geq 1$, the following inequalities hold:*

(i) $c^{\text{reg}}(m,n) \;\leq\; c(m,n) \;\leq\; 3 - \frac{1}{m} - \frac{1}{n}$

(ii) $c_m^{\text{reg}} \;\leq\; c_m \;\leq\; 3 - \frac{1}{m}$ (in particular, $c_2^{\text{reg}} \leq c_2 \leq 2.5$)

(iii) $c^{\text{reg}} \;\leq\; c \;\leq\; 3$

As Anclin noted, his proof works much more generally: For any partial triangulation of a not necessarily simple or convex lattice polygon, the number of completions is at most $2^{e'}$, where e' is the number of edges that are to be added.

4 Explicit Triangulations

For small grids, one can enumerate all unimodular triangulations by the reverse search algorithm sketched in Section 2. For larger grids, it is desirable to obtain from the huge set of unimodular triangulations "random" ones. There are ways to produce them, however, in most cases the probability distribution from which they are chosen is unknown.

4.1 Generating Random Triangulations

A standard way to compute complex random objects such as triangulations is to set up a random walk. In our case of unimodular triangulations of $P_{m,n}$ the method is described easily: First one determines any starting triangulation $\mathcal{T}$ of $P_{m,n}$. Then, the following operation is performed τ times: Choose one of the (inner) edges of $\mathcal{T}$ uniformly at random; if this edge is flippable (see Section 2), then with probability $\frac{1}{2}$ the current triangulation $\mathcal{T}$ is replaced by the one obtained from it by flipping that edge.

As the flip graph of the triangulations is connected (see Section 2), it follows from general principles that, with τ tending to infinity, the probability distribution defined by the output of this algorithm converges to the uniform distribution (with respect to the "total variation distance"); see, e.g., Jerrum and Sinclair [20] or Behrends [8]. Unfortunately, not much is known about the speed of convergence of the distribution. In particular, for general m and n it is not known whether there is a polynomial bound (in $n + m + \varepsilon^{-1}$) on the number τ of steps needed to guarantee that the total variation distance between the produced and the uniform distribution is at most ε (i.e., if the associated Markov chain is *rapidly mixing*). The only exception is the case $m = 1$: Here it follows from results of Felsner and Wernisch [14] that the Markov chain is indeed rapidly mixing.

Despite this lack of knowledge on the distribution of the output of the random walk algorithm, one still can use it in order to produce examples of "interesting" triangulations.

4.2 Empirical Results

For each $n \in \{10, \ldots, 20\}$, Meyer [24] generated 1000 random unimodular triangulations of $P_{n,n}$ by running the random walk described in Subsection 4.1 10^9 steps, recording every 10^6th triangulation; see Table 2.

The results support the conjecture that, for n tending to infinity, a random unimodular triangulation of $P_{n,n}$ is irregular with probablity one. A second observation is that the (expected value) of the average length of an edge in a random triangulation seems to grow very slowly with n.

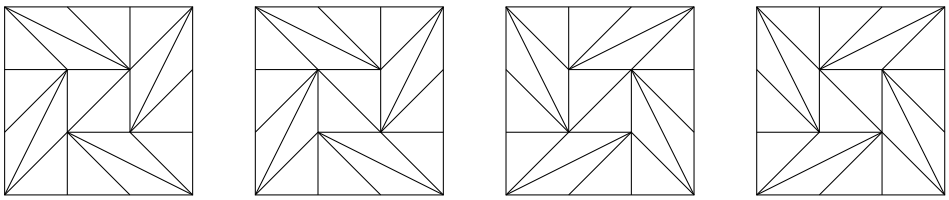
4.3 Obstructions to Regularity

All figures shown below have been produced by Meyer [24], who implemented procedures for checking regularity by solving linear programs using the CPLEX 6.6.1 library.

Proposition 3.4 shows that $P_{2,n}$ has only regular triangulations. The grid $P_{3,3}$ has precisely the following four (pairwise congruent) irregular unimodular triangulations.

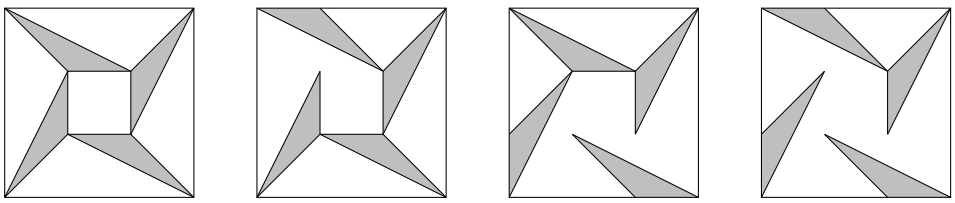
$m \times n$	irregularity	max. edge length	av. edge length
10×10	.355	5.538	1.614
11×11	.435	5.843	1.630
12×12	.559	6.118	1.645
13×13	.696	6.397	1.659
14×14	.782	6.650	1.670
15×15	.875	6.911	1.681
16×16	.927	7.151	1.690
17×17	.965	7.391	1.700
18×18	.971	7.618	1.708
19×19	.992	7.821	1.713
20×20	.997	8.060	1.723

Table 2: Results for random unimodular triangulations of large grids. The first column shows the (empirical) probability of irregularity, the second and third columns contain the (empirical) expected values of the maximal and the average edge length.

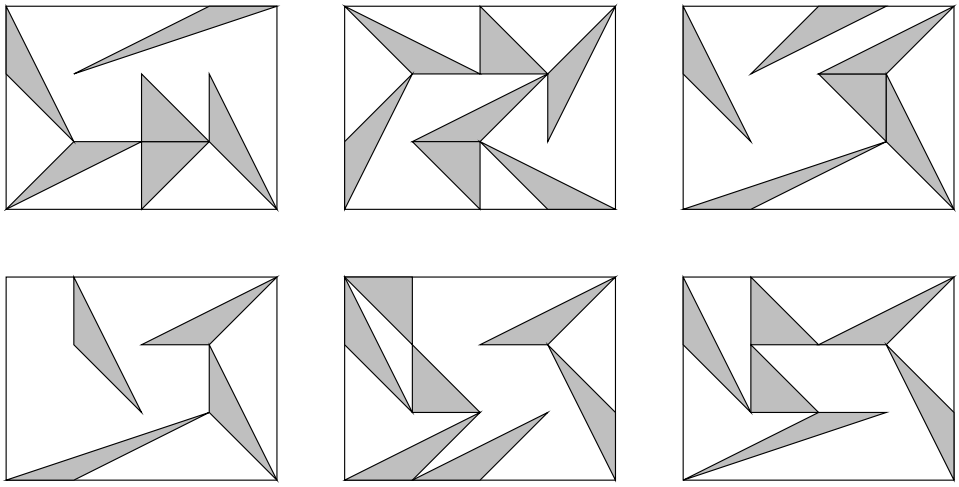


When trying to understand the reasons for irregularity, it seems useful to consider (smallest) forbidden patterns for regular triangulations. Let $\mathcal{T}$ be a set of unimodular triangles of $P_{m,n}$ (such that any two of them intersect in a common face). We denote by $S \subset P_{m,n}$ the set of all grid points covered by triangles in $\mathcal{T}$. The set $\mathcal{T}$ is called *regular* if there is a height function $h : S \longrightarrow \mathbb{R}$ such that for each triangle $\Delta \in \mathcal{T}$, all h-lifted points in $S \backslash \Delta$ lie strictly above the affine hull of the h-lifting of Δ. A subset of $\mathcal{T}$ is called a *minimal irregular configuration* if it is not regular, but all its proper subsets are regular. Clearly, a regular unimodular triangulation of $P_{m,n}$ cannot contain any minimal irregular configuration.

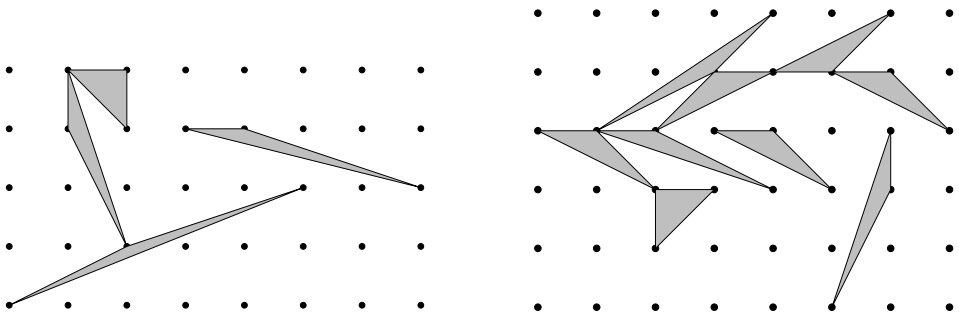
Examples for minimal irregular configurations of $P_{3,3}$ are the following:

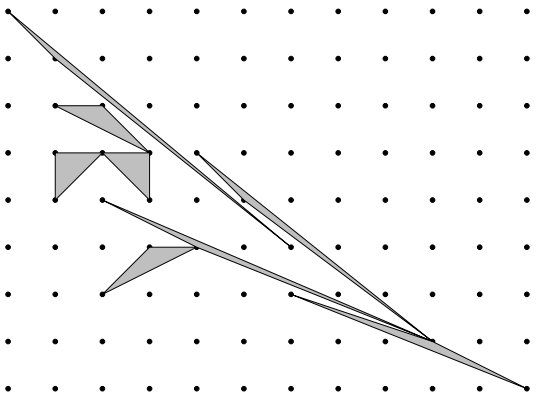


Already for $P_{3,4}$ many other minimal irregular configurations occur; some are depicted here:



While these figures still have some similarities with the nice "whirlpools" ones for $P_{3,3}$, the picture gets more and more complicated with growing grid sizes, as the following examples demonstrate:

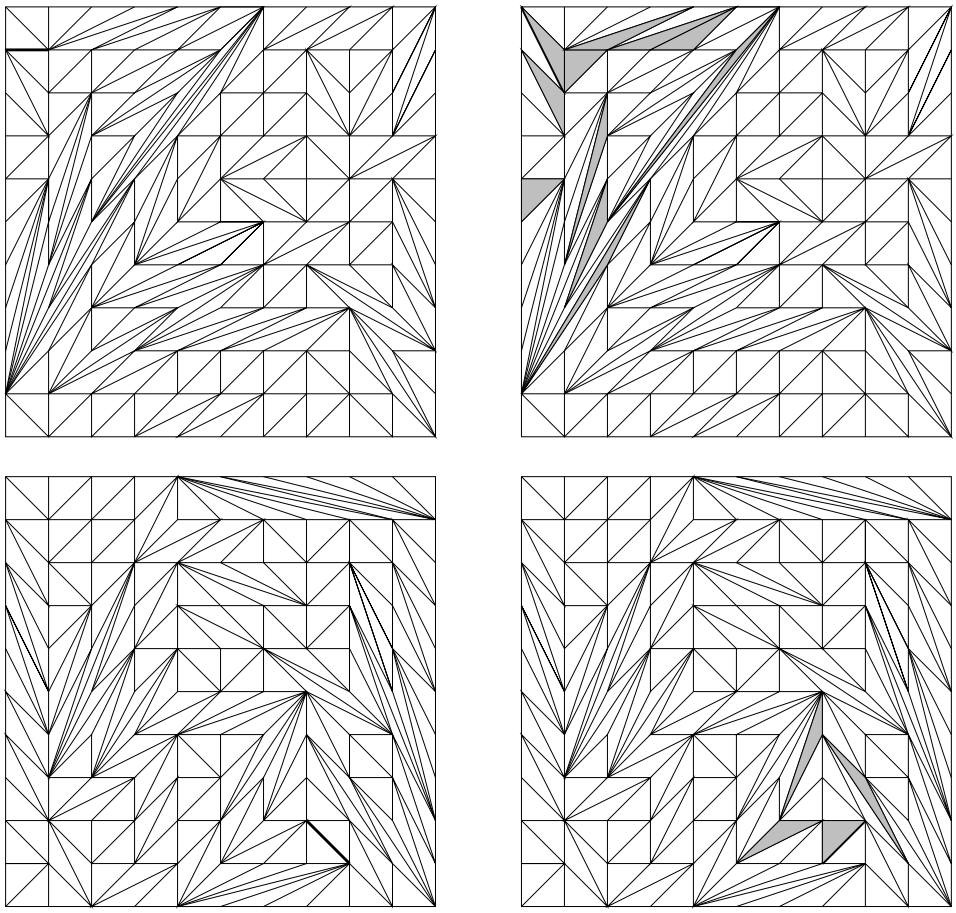




Viewing these figures, it seems unlikely that one can find any compact characterization of regularity for unimodular triangulations of $P_{m,n}$ in terms of forbidden substructures.

We close our zoo of "explicit triangulations" with some pairs of triangulations, found by Meyer's implementation of the random walk. In each of the

figures, the left triangulation is regular, but the right one is not, although it can be obtained from the left one by flipping just one edge (drawn bold in the upper left and in the lower right corner, respectively). For both irregular triangulations, a minimal irregular configuration contained in it is depicted as well.



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References

[1] O. Aichholzer, The path of a triangulation, in: Proc. 15th Annual ACM Workshop Computational Geometry (Miami Beach, Florida, June 13-15, 1999), ACM Press 1999, pp. 14-23.

[2] O. Aichholzer, Counting triangulations – Olympics, WWW page, created Oct. 5, 1999, last modified Nov. 20, 2001, `http://www.cis.TUGraz.at/igi/oaich/triangulations/counting/counting.html`

[3] M. Aigner and G. M. Ziegler, Proofs from THE BOOK, Springer-Verlag Heidelberg, second ed. 2001.

[4] M. Ajtai, V. Chvátal, M. Newborn and E. Szemerédi, Crossing-free subgraphs, *Annals Discrete Math.* **12** (1982), 9-12.

[5] E. Anclin, An upper bound for the number of planar lattice triangulations, Preprint, TU Berlin 2002, 4 pages, ArXiv: `math.CO/0212140`.

[6] D. Avis and K. Fukuda, Reverse search for enumeration, *Discrete Applied Math.* **65** (1996), 21-46.

[7] S. Bespamyatnikh, An efficient algorithm for enumeration of triangulations, *Comput. Geom.* **23** (2002), 271-279.

[8] E. Behrends, Introduction to Markov Chains, with Special Emphasis on Rapid Mixing, Vieweg, Braunschweig/Wiesbaden 2000.

[9] M. de Berg, M. van Kreveld, M. Overmars, and O. Schwarzkopf, Computational Geometry. Algorithms and Applications, Springer-Verlag, Heidelberg, second edition 2000.

[10] D. I. Dais, Resolving 3-dimensional toric singularities, *Séminaires & Congrés* **6**, Soc. Math. France, 2002, 155-186.

[11] L. De Floriani, B. Falcidieno, G. Nagy, and C. Pienovi, On sorting triangles in a Delaunay tessellation, *Algorithmica* , **6**, 1991, 522–532.

[12] J. De Loera, J. Rambau and F. Santos, Triangulations, Book in preparation.

[13] H. Edelsbrunner, An acyclicity theorem for cell complexes in d dimension, *Combinatorica* **10** (1990), 251–260.

[14] St. Felsner and L. Wernisch, Markov chains for linear extensions, the two-dimensional case, in: Proc. Eighth Annual ACM-SIAM Symposium on Discrete Algorithms (New Orleans, LA, 1997), ACM, New York, 1997, 239–247.

[15] E. Gawrilow and M. Joswig, polymake: an Approach to Modular Software Design in Computational Geometry, Proceedings of the 17th Annual Symposium on Computational Geometry (2001), 222–231;
Software package at `http://www.math.tu-berlin.de/polymake/`, version 1.5, September 2002.

[16] I. M. Gel'fand, M. M. Kapranov and A. V. Zelevinsky, Discriminants, Resultants, and Multidimensional Determinants, Birkhäuser, Boston 1994.

[17] GNU library for arbitrary precision arithmetic, `http://www.swox.com/gmp/`

[18] J. E. Goodman and J. Pach, Cell decomposition of polytopes by bending, *Israel J. Math.* **64**, 1988, 129–138.

[19] R. E. Hastings, Triangulations of point configurations and polytopes, Dissertation, Cornell University 1998, xii+103 pages, `ftp://cam.cornell.edu/pub/theses/hastings.ps.gz`.

[20] M. Jerrum and A Sinclair, The Markov Chain Monte Carlo Method: An Approach to Approximate Counting and Integration, in: Approximation Algorithms (D. Hochbaum, ed.), PWS Publishing, 482–520.

[21] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, Toroidal Embeddings I, *Lecture Notes in Math.* **339**, Springer-Verlag 1973.

[22] C. W. Lee, Subdivisions and triangulations of polytopes, in: Handbook on "Discrete and Computational Geometry" (J. E. Goodman and J. O'Rourke, eds.), CRC Press, Boca Raton FL 1997, 271–290.

[23] J. H. van Lint and R. M. Wilson, A Course in Combinatorics, Cambridge University Press, Cambridge 1992; second edition 2001.

[24] S. Meyer, Enumeration von Triangulierungen, Diplomarbeit, TU Berlin 2002, 87 Seiten.

[25] S. Yu. Orevkov, Asymptotic number of triangulations with vertices in $\mathbb{Z}^2$, *J. Combinatorial Theory*, Ser. A **86** (1999), 200-203.

[26] J. Pfeifle and J. Rambau, Counting triangulations using oriented matroids, to appear in: Algebra, Geometry, and Software Systems (M. Joswig and N. Takayama, eds.), Springer.

[27] J. Rambau, TOPCOM — Triangulations Of Point Configurations and Oriented Matroids, to appear in Proc. Int. Cong. Mathematical Software ICMS 2002;
Software package at `http://www.zib.de/rambau/TOPCOM/`, version 0.11.1, July 2002.

[28] F. Santos and R. Seidel, A better upper bound on the number of triangulations of a planar point set, to appear in: *J. Comb. Theory Ser. A.*

[29] R. P. Stanley, Enumerative Combinatorics I, *Cambridge Studies in Advanced Mathematics* **49**, Cambridge University Press 1997.

[30] B. Sturmfels, Gröbner Bases and Convex Polytopes, University Lecture Series, Vol. 8, Amer. Math. Soc., Providence, RI, 1995.

[31] O. Ya. Viro, Real plane algebraic curves: Constructions with controlled topology, *Leningrad Math. J.* **1** (1990), 1059–1134.

[32] G. M. Ziegler, Lectures on Polytopes, *Graduate Texts in Mathematics* **152**, Springer-Verlag New York 1995; revised edition, 1998.

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Appendix

Below we report on some values $f(m,n)$ and $c(m,n)$ that we have computed by the algorithms for narrow strips described in Subsection 2.1. Note that Aichholzer's results referred to in the captions of the tables have been obtained by a code that works for *general* point sets in the plane.

n	# unimodular triangulations of $P_{2,n}$	capacity
2	64	1.500000
3	852	1.622451
4	12170	1.696380
5	182132	1.747462
6	2801708	1.784822
7	43936824	1.813494
8	698607816	1.836244
9	11224598424	1.854774
10	181815529916	1.870184
11	2964167665340	1.883216
12	48580814410080	1.894393
13	799696199314500	1.904094
14	13212398835196240	1.912597
15	218976668040908248	1.920118
16	3639020246503687098	1.926820
17	60616163842958990268	1.932833
18	1011775545312594580868	1.938260
19	16918718677672553292440	1.943185
20	283368129709983000763876	1.947675
21	4752924784523774226889308	1.951787
22	79824154012907603962950312	1.955568
23	1342199498257069824064033644	1.959057
24	22592402326314503187343665228	1.962288
25	380653341141186360494812030908	1.965287
⋮	⋮	⋮
375	322379469795023850474295003353206721961082558051423084880244 622493239840267086773982523966267988496124181049802588221265 093372321162311900056540572387505992321869061059550930460251 208699393384897025341791259853666381631934919090333735984559 109034657337254608555310086159512051235323132000393390812675 635858586268866021116515894234547255346166243854345190625384 072567100646447104480055590853368584565574390095154916379988 529322533797949278877762995003200075959780	2.044130

Table 3: Results for $m = 2$ (up to $n = 15$ by Aichholzer [2]).)

n	# unimodular triangulations of $P_{3,n}$	capacity
3	46456	1.722619
4	2822648	1.785718
5	182881520	1.829755
6	12244184472	1.861743
7	839660660268	1.886238
8	58591381296256	1.905656
9	4140106747178292	1.921429
10	295372308876234428	1.934510
11	21234538315776214604	1.945546
12	1535939689343151109944	1.954989
13	111655493479477379881272	1.963164
14	8150727077307189203809876	1.970314
15	597087996550303632801161860	1.976623
16	438713502048958367585563692 12	1.982234
17	32317979789352667932687978092 60	1.987258
18	23860610519338038775657093219 4588	1.991783
19	17651135152017098450035730535703808	1.995882
20	13080292929840656303626948420423950 56	1.999613
21	97080539975603502667567153853690549804	2.003024
22	72151580478816506090755757731536095531 48	2.006154
23	53690568577690137148543684950579241584714 0	2.009039
24	39997858254082097021224132017959794867440460	2.011705
25	29827523066855578629893933286489275581388006 12	2.014178
26	22263854695021181497769393247709162080162655110 0	2.016478
27	16632293481947394846909242053460530217349594447732	2.018623
28	12434907458510562605577828215625071564189234074329 20	2.020627
29	93034737749193459157244717739574844241159902101217660	2.022506
30	69652448825424549370206198187020590537413772902550682 84	2.024269
31	52178955636741675340524432893461225988455279021120742 1244	2.025929
32	39111402471791798530057405675011481922023421912454706904712	2.027493
⋮	⋮	⋮
60	14029922650660567459529701025605687042597277225598781806575 4824810457824216053120262566533014115176725785594722 4	2.051236

Table 4: Results for $m = 3$ (up to $n = 10$ by Aichholzer [2].)

n	# unimodular triangulations of $P_{4,n}$	capacity
4	736983568	1.841066
5	208902766788	1.880202
6	61756221742966	1.908818
7	18792896208387012	1.930751
8	5831528022482629710	1.948080
9	1835933384812941453312	1.962138
10	584455230176565718869688	1.973785
11	187686028049755013528577884	1.983601
12	60685901262618326775192700244	1.991986
13	19731268926382148037209063600412	1.999235
14	6444884828545542240332780129017164	2.005567
15	2113222804656668311309302902100087020	2.011147
16	695163898467233943317499868644974218294	2.016103
17	229316915701559537858641762255000442720116	2.020535
18	75827610389461537709077484409103543784585710	2.024522
19	25126215170054967918515172517611569017605311400	2.028130
20	8341120564526486621411516194118239406742548614820	2.031409
21	2773492141234111587868660829757660194821668552146720	2.034405
22	923542836393484391634763396019705303948974557455679944	2.037151
23	307927840785137540445620416718491375007439099052396877524	2.039680
24	102788976776576952654837601573736966649525369087792041774754	2.042015
25	34348007173983906887977536976143285540616055151406837464386952	2.044178
26	11488735157368185493550332714289865763761071970295733672739011182	2.046188
27	3846114798702837748429637799631405899720860204464245538618538080224	2.048061
28	1288605098545745425026895981169173452565572617561190100930952999826626	2.049811
29	432053057804721606144370507673814019049397228524243323674624721189513152	2.051449
30	144960362040664009850310097114550102266012390064271163400409174321129109970	2.052986
31	48666988587816100417429461522083834520238266167053837281579839713435133670880	2.054431
32	16348321592766160525928861562545873132577437301251385744803248045014265044217096	2.055792

Table 5: Results for $m = 4$ (up to $n = 8$ by Aichholzer [2].)

n	# unimodular triangulations of $P_{5,n}$	capacity
1	252	1.595455
2	182132	1.747462
3	182881520	1.829755
4	208902766788	1.880202
5	260420548144996	1.915513
6	341816489625522032	1.941533
7	464476385680935656240	1.961547
8	645855159466371391947660	1.977388
9	913036902513499041820702784	1.990240
10	1306520849733616781789190513820	2.000871
11	1887591165891651253904039432371172	2.009821
12	2747848427721241461905176361078147168	2.017461

Table 6: Results for $m = 5$ (up to $n = 6$ by Aichholzer [2].)

n	# unimodular triangulations of $P_{6,n}$	capacity
1	924	1.641958
2	2801708	1.784822
3	12244184472	1.861743
4	61756221742966	1.908818
5	341816489625522032	1.941533
6	1999206934751133055518	1.965553
7	12169409954141988707186052	1.984082

Table 7: Results for $m = 6$.