THE EXISTENCE OF DOMINATING LOCAL MARTINGALE MEASURES

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Abstract. We prove that for locally bounded processes, the absence of arbitrage of the first kind is equivalent to the existence of a dominating local martingale measure. This is related to results from the theory of filtration enlargements.

Contents

1. Introduction

Let S be a d-dimensional stochastic process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. We think of S as the price process of a given number of assets. The Fundamental Theorem of Asset Pricing (Delbaen and Schachermayer, [DS94]) is certainly among the most important achievements in financial mathematics. It states that for locally bounded S, there exists an equivalent probability measure Q such that S is a Q-local martingale if and only if the two conditions (NA1) and (NA) are satisfied. Here (NA) and (NA1) are two notions of arbitrage that we will define later. In a related work, Delbaen and Schachermayer [DS95b] proved that for a continuous process S, the condition (NA) implies the existence of an *absolutely continuous* local martingale measure Q.

Here we "complete" this program, by proving that for predictable S, (NA1) is equivalent to the existence of a *dominating* local martingale measure Q. Of course we have to be very careful in defining the notion of a dominating martingale measure in the first place. It turns out that Föllmer's measure ([Föl72]) associated to a nonnegative supermartingale appears naturally in this context.

For non-predictable locally bounded S, (NA1) is equivalent to the existence of a dominating local martingale measure that needs to satisfy an additional condition.

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Denote by

$$\mathcal{K}_1 = \{1 + (H \cdot S)_\infty : H \text{ is 1-admissible and } (H \cdot S)_t \text{ converges as } t \to \infty\}$$

all portfolios that are attainable with starting wealth 1 and using only 1-admissible strategies. A strategy H is called 1-admissible if its stochastic integral $H \cdot S$ with respect to S exists and if a.s. for all t

$$(H \cdot S)_t \geq -1.$$

If S is not a semimartingale, then we can only use simple strategies.

A family of random variables $\mathcal{X}$ is called bounded in probability, or bounded in L^0, if

$$\lim_{M \to \infty} \sup_{X \in \mathcal{X}} P(|X| \geq M) = 0.$$

Definition. We say that there is **no arbitrage of the first kind (NA1)** if $\mathcal{K}_1$ is bounded in probability. If there is no $X \in \mathcal{K}_1$ with $X \geq 1$ and $P(X > 1) > 0$, we say that there is **no arbitrage (NA)**. If both (NA1) and (NA) hold, we say that there is **no free lunch with vanishing risk (NFLVR)**.

Heuristically, (NA) says that it is not possible to make a profit without taking a risk. (NA1) states that it is not possible to make infinite profit with a bounded credit line. This is why (NA1) is also referred to as "no unbounded profit with bounded risk" (NUPBR), cf. Karatzas and Kardaras [KK07].

Definition. If Z is a right-continuous and a.s. càdlàg positive supermartingale with

$$\lim_{t \to \infty} Z_t > 0$$

such that

$$((1 + (H \cdot S)_t)Z_t : t \geq 0)$$

is a supermartingale for every 1-admissible strategy H, then we call Z a **supermartingale density**.

Our main results are then:

Theorem 1.1. *Let S be a d-dimensional, right-continuous, and adapted process. Then (NA1) holds if and only if there exists a supermartingale density for S.*

Theorem 1.2. *Let S be predictable. If Z is a supermartingale density for S, then Z determines a probability measure $P^Z \gg P$ such that S is a local martingale under P^Z. Conversely, if $Q \gg P$ is a dominating local martingale measure for S, then S admits a supermartingale density.*

Actually the current formulation is slightly too simple, but it describes the essential result. We will have to reformulate Theorem 1.2 once we defined Föllmer's measure.

For optional processes that are not predictable, we can give an easy counterexample to Theorem 1.2. Recall that a stopping time T is **accessible** under a probability measure P if there exists an increasing sequence (T_n) of stopping times, such that P-a.s. $T_n < T$ for every n, and such that $\lim_{n \to \infty} T_n = T$ P-a.s. In this case, (T_n) is called an **announcing sequence** for T. The correct formulation for non-predictable S is then

Theorem 1.3. *Let S be locally bounded. Then (NA1) holds if and only if there exists a dominating measure $Q \gg P$ such that S is a Q-local martingale, and such that the hitting time*

$$T = \inf \left\{ t \geq 0 : \left. \frac{dP}{dQ} \right|_{\mathcal{F}_t} = 0 \right\}$$

is accessible under Q.

Recently there has been an increased interest in Föllmer's measure, which is motivated from problems in mathematical finance: It appears naturally in the construction and study of "strict local martingales", i.e. local martingales that are not martingales. These are used to model bubbles in financial markets, cf. Jarrow, Protter and Shimbo [JPS10]. A pioneering work on the relation between Föllmer's measure and strict local martingales is Delbaen and Schachermayer [DS95a]. Other references are Pal and Protter [PP10] and Kardaras, Kreher and Nikeghbali [KKN11]. The work most related to this one is Ruf [Ruf10], where it is shown that in a diffusion setting, (NA1) implies the existence of a dominating local martingale measure. All these works have in common that they study the Föllmer measure of strictly positive local martingales. In an upcoming work of Carr, Fisher and Ruf [CFR11], they study the Föllmer measure of a local martingale which is not strictly positive. To the best of our knowledge, here the Föllmer measures of supermartingales which are not local martingales are used as local martingale measures for the first time.

Another related work is Kardaras [Kar10], where it is shown that (NA1) is equivalent to the existence of a finitely additive equivalent local martingale measure. Here we construct actual probability measures rather than only finitely additive measures.

This work is motivated by insights from the theory of enlargements of filtrations, cf. Amendinger, Imkeller and Schweizer [AIS98], Ankirchner's thesis [Ank05], and Ankirchner, Dereich and Imkeller [ADI06]. In these works it was shown that if M is a continuous local martingale in a given filtration $(\mathcal{F}_t)$, then under an enlarged filtration $(\mathcal{G}_t)$, assuming suitable conditions, M is of the form

$$M = \tilde{M} + \int_0^{\cdot} \alpha_s d\langle \tilde{M} \rangle_s.$$

Here $\tilde{M}$ is a $(\mathcal{G}_t)$-local martingale. Therefore it is a natural question to ask whether there exists an equivalent measure Q that "eliminates" the drift, i.e. under which M is a $(\mathcal{G}_t)$-local martingale. The answer to this question is in general negative. However Ankirchner observed that if it is possible to do utility maximization in the large filtration, then the **information drift** α must be locally square integrable with respect to $\tilde{M}$ (cf. [Ank05], Theorem 9.2.7). Here we show that this condition is also sufficient, we relate it to (NA1), and we show that this allows to construct dominating local martingale measures. We do all this for general locally bounded, d-dimensional adapted processes, not even assuming the semimartingale property.

Section 2 describes our motivation coming from the theory of enlargement of filtrations in more detail. In Section 2 we also explore of which form the supermartingales associated to the dominating local martingale measure should be. In Section 3 we prove that the existence of these supermartingale densities is equivalent to (NA1). In Section 4 we prove that for predictable S, a supermartingale Z is a supermartingale density if and only if S is a local martingale under the Föllmer measure P^Z. For non-predictable S we prove the more complicated Theorem 1.3. In Section 5 we return to the theory of filtration enlargements and examine how Jacod's criterion relates to our results.

Remark. That the existence of "supermartingale deflators" is equivalent to (NA1) (i.e. Theorem 1.1) was already shown in [KK07]. However we completed our proof without being aware of their result. The proof here is very different from their proof. Also, we do not need to assume that S is a semimartingale. Because we think of Z as the "density" of a dominating local martingale measure, we prefer the term supermartingale density.

2. Motivation

In this section we show that under enlarged filtrations, generally there exists no equivalent local martingale measure. Then we recall that nonetheless there often is a dominating local martingale measure. Finally we argue that often the condition (NA1) is satisfied. This convinces us that (NA1) should be in some relation to the existence of dominating local martingale measures.

Then we assume that such a dominating measure exists, and we examine its Kunita-Yoeurp decomposition under P.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space with $P(A) \in \{0,1\}$ for every $A \in \mathcal{F}_0$. Define $\mathcal{F}_\infty := \vee_{t\geq 0}\mathcal{F}_t$.

Let S be a one-dimensional semimartingale that describes a complete market (i.e. for every $F \in L^\infty(\mathcal{F}_\infty)$ there exists a predictable process H, integrable with respect to S, such that $F = F_0 + \int_0^\infty H_s dS_s$ for some constant $F_0 \in \mathbb{R}$). Let L be a random variable that is $\mathcal{F}_\infty$-measurable. Assume that L is not P-a.s. constant. Define the initially enlarged filtration

$$(\mathcal{G}_t = \mathcal{F}_t \vee \sigma(L) : t \geq 0).$$

This is a toy model for insider trading: At time 0, the insider has the additional knowledge of L. It turns out that there exists no equivalent local martingale measure for S under $(\mathcal{G}_t)$. Since L is not constant, there exists $A \in \sigma(L)$ such that $P(A) \in (0,1)$. Assume Q is an equivalent $(\mathcal{G}_t)$-local martingale measure for S. Consider the $(Q, (\mathcal{F}_t))$-martingale

$$N_t = E_Q(1_A|\mathcal{F}_t), \qquad t \geq 0.$$

Since the market is complete, 1_A can be replicated. That is, there exists a $(\mathcal{F}_t)$-predictable strategy H such that $N_\cdot = Q(A) + \int_0^\cdot H_s dS_s$. But then $\int_0^\cdot H_s dS_s$ is a bounded $(Q, (\mathcal{G}_t))$-local martingale. Hence it is a martingale, and we obtain

$$0 = E_Q(1_{A^c}1_A) = E_Q\left(1_{A^c}\left(Q(A) + \int_0^\infty H_s dS_s\right)\right) = Q(A^c)Q(A) > 0$$

which is absurd. The last step follows because Q is assumed to be equivalent to P.

So we see that already in the easiest insider trading models, there generally does not exist an equivalent local martingale measure any more. By the Fundamental Theorem of Asset Pricing, one of the two conditions (NA) or (NA1) has to be violated if S is locally bounded.

One of problems treated in the theory of filtration enlargements is the following: Let $(\mathcal{G}_t)$ be a filtration enlargement of $(\mathcal{F}_t)$, i.e. $\mathcal{F}_t \subseteq \mathcal{G}_t$ for every $t \geq 0$. Let $\mathcal{S}$ be a family of $(\mathcal{F}_t)$-semimartingales. Under which conditions are all $S \in \mathcal{S}$ also $(\mathcal{G}_t)$-semimartingales? **Hypothèse** (H') is said to be satisfied if *all* $(\mathcal{F}_t)$-semimartingales are $(\mathcal{G}_t)$-semimartingales.

One of the most celebrated criterions that guarantee Hypothèse (H') to be satisfied is Jacod's criterion ([Jac85]). Here we give an equivalent formulation, first found in Föllmer and Imkeller [FI93] and later generalized and carefully studied in Ankirchner, Dereich and Imkeller [ADI07]: Let L be a random variable and define the initial enlargement

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(L).$$

Define the product space

$$\overline{\Omega} = \Omega \times \Omega, \qquad \overline{\mathcal{G}} = \mathcal{F}_\infty \otimes \sigma(L)$$

and the filtration

$$\overline{\mathcal{G}}_t = \mathcal{F}_t \otimes \sigma(L).$$

On $\overline{\Omega}$ we define two measures: $\overline{Q} = P|_{\mathcal{F}_\infty} \otimes P|_{\sigma(L)}$ and

$$\overline{P} = P \circ \psi^{-1} \qquad \text{where}$$
$$\psi : \Omega \to \overline{\Omega}, \qquad \psi(\omega) = (\omega, \omega).$$

We have the following result, which in this setting is just a reformulation of Jacod's criterion:

Theorem (Theorem 1 in [ADI07]). *If $\overline{P} \ll \overline{Q}$, then Hypothèse (H') holds, i.e. any $(\mathcal{F}_t)$-semimartingale is a $(\mathcal{G}_t)$-semimartingale.*

In this formulation it is quite obvious why Jacod's criterion works: Under the measure $\overline{Q}$, the additional information L is independent of $\mathcal{F}_\infty$. Therefore any $(\mathcal{F}_t)$-martingale will stay a $(\overline{\mathcal{G}}_t)$-martingale under $\overline{Q}$ (if we embed random variables from Ω to $\overline{\Omega}$ by setting $\overline{X}(\omega, \omega') = X(\omega)$). The assumption is that the martingale measure $\overline{Q}$ *dominates* the actual measure $\overline{P}$. Then an application of Girsanov's theorem implies that the martingale stays a $\overline{P}$-semimartingale. This can be transferred back to the original space and to the original measure. So the message is:
Observation: One of the most famous criteria in filtration enlargements implies the existence of a dominating martingale measure under which any $(\mathcal{F}_t)$-martingale is a $(\mathcal{G}_t)$-martingale.

How does this relate to our first example? In fact it is not hard to see that Jacod's criterion is always satisfied as long as L takes its values in a discrete space. So Jacod's criterion may be satisfied even though there is no equivalent local martingale measure in the large filtration.

Let us also remark that there are many article devoted to calculating the additional utility of an insider. It is shown e.g. in Ankirchner's thesis ([Ank05], Theorem 12.6.1), that assuming Jacod's criterion, the maximal expected logarithmic utility under $(\mathcal{G}_t)$ is given by

$$\sup_{X \in \mathcal{K}_1(\mathcal{G}_t)} E(\log(X)) = \sup_{X \in \mathcal{K}_1(\mathcal{F}_t)} E(\log(X)) + I(L, \mathcal{F}_\infty)$$

where $\mathcal{K}_1(\mathcal{G}_t)$ and $\mathcal{K}_1(\mathcal{F}_t)$ are defined as above, using $(\mathcal{G}_t)$- respectively $(\mathcal{F}_t)$-predictable strategies, and $I(L, \mathcal{F}_\infty)$ is the mutual information between L and $\mathcal{F}_\infty$. And this quantity may be finite. But in the next section we will prove the following result:

Proposition. *S satisfies (NA1) if and only if there exists an unbounded increasing function U such that the maximal expected utility is finite, i.e. such that*

$$\sup_{X \in \mathcal{K}_1} E(U(X)) < \infty.$$

Now if we gather all our observations, we see that under enlarged filtrations there are generally no equivalent local martingale measures. However there are often *dominating* local martingale measures. Also, the maximal expected utility is often finite if we assume the existence of a dominating local martingale measure, and this implies that under the large filtration we still have (NA1). So (NA1) seems to be related to the existence of a dominating martingale measure. In this work we prove that in a certain sense, the two conditions are equivalent.

Now we only work under one given filtration $(\mathcal{F}_t)$, and we assume that S is a local martingale under Q with $P \ll Q$. Define $\gamma_t = dP/dQ|_{\mathcal{F}_t}$. We assume that γ is right-continuous. Then $T = \inf\{t \geq 0 : \gamma_t = 0\}$ is a stopping time, and we can define the adapted process

$$Z_t = \frac{1}{\gamma_t} 1_{\{t<T\}}.$$

Let H be 1-admissible for S under Q, that is Q-a.s. for any $t \geq 0$

$$\int_0^t H_s dS_s \geq -1.$$

Let $t, s \geq 0$ and let $A_t \in \mathcal{F}_t$. We have

$$\begin{aligned} E_P(1_{A_t} Z_{t+s}(1 + (H \cdot S)_{t+s})) &= E_Q\left(1_{A_t} \frac{1_{\{t+s<T\}}}{\gamma_{t+s}}(1 + (H \cdot S)_{t+s})\gamma_{t+s}\right) \\ &\leq E_Q\left(1_{A_t} 1_{\{t<T\}}(1 + (H \cdot S)_{t+s})\right) \\ &\leq E_Q\left(1_{A_t} 1_{\{t<T\}}(1 + (H \cdot S)_t)\right) \\ &= E_P(1_{A_t} Z_t(1 + (H \cdot S)_t)) \end{aligned}$$

using in the second line that $1_{A_t}(1+(H\cdot S)_{t+s})$ is nonnegative, and in the third line that $1+(H\cdot S)$ is a nonnegative Q-local martingale and thus a Q-supermartingale. This indicates that Z should be a supermartingale density. Of course here we only considered strategies that are 1-admissible under Q, and there might be strategies that are 1-admissible under P but not under Q. The way out is to only consider those strategies until time $T-$. We will make this rigorous later.

Note that the couple (Z, T) is the **Kunita-Yoeurp decomposition** of Q with respect to P. The Kunita-Yoeurp decomposition is a progressive Lebesgue decomposition on filtered probability spaces. It was introduced in Kunita [Kun76] in a Markovian context, and generalized to arbitrary filtered probability spaces in Yoeurp [Yoe85]. Namely we have for every $t \geq 0$

(1) $P(T = \infty) = 1$
(2) $Q(\cdot \cap \{T \leq t\})$ and P are singular on $\mathcal{F}_t$.
(3) $Q(\cdot \cap \{T > t\})$ is absolutely continuous with respect to P on $\mathcal{F}_t$, and for $A \in \mathcal{F}_t$

$$Q(A \cap \{T > t\}) = E_P(1_A Z_t).$$

So our program will be as follows: First find a supermartingale density Z, and then find a measure Q and a stopping time T, such that (Z, T) is the Kunita-Yoeurp decomposition of Q with respect to P. It turns out that the second part was already solved by [Yoe85], and that Q will be the Föllmer measure of Z. After studying the interplay of S and Z, we can then apply the generalized Girsanov theorem of Yoeurp to obtain that under Q, S will be a local martingale until time T.

3. Existence of Supermartingale Densities

Now that we motivated why we are interested in finding supermartingale densities, let us prove Theorem 1.1. We are in the following setting: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is a filtered probability space with a right-continuous filtration. We do not require $(\mathcal{F}_t)$ to be complete. Note that stochastic integration is possible for non-complete filtrations, cf. Jacod and Shiryaev [JS03], Definition I.1.2. S is a d-dimensional stochastic process that is adapted to $(\mathcal{F}_t)$ and right-continuous. If S is a semimartingale, then we define

$$\mathcal{K}_1 = \left\{1 + (H \cdot S)_\infty : H \text{ is 1-admissible and } \lim_{t\to\infty} (H \cdot S)_t \text{ exists a.s.}\right\}.$$

The stochastic integral is to be understood in the sense of vector integration, cf. [JS03]. If S is not a semimartingale, $\mathcal{K}_1$ is defined similarly, but only using simple strategies of the form

$$H_t = \sum_{j=1}^n F_j 1_{(T_j, T_{j+1}]}(t)$$

for stopping times $0 \leq T_0 < T_1 < \cdots < T_n < \infty$ and bounded random variables $F_j \in \mathcal{F}_{T_j}$. In both cases it is understood that H is a row vector with components H^i and S is a column vector with components S^i, and

$$HdS = \sum_{i=1}^d H^i dS^i.$$

The structure of the proof is as follows:

- Show that under (NA1) we can find $Q \sim P$ such that $\mathcal{K}_1$ is bounded in $L^1(Q)$.
- Show that if $\mathcal{X}$ is a L^1-bounded family of adapted two-step processes of the form (X_0, X_1), and if $\mathcal{X}$ satisfies certain stability assumptions, then there exists a random variable Z, such that (ZX_0, X_1) is a supermartingale.
- Use induction to prove the result in finite discrete time.
- Use a compactness argument to construct the "skeleton" $(Z_q : q \in \mathbb{Q}_+ \cup \{\infty\})$: If no supermartingale density Z indexed by $\mathbb{Q}_+$ exists, then there must already exist finitely many times $t_1, \ldots, t_n$ such that for the processes $(1 + (H \cdot S)_{t_i})_{i=1,\ldots,n}$ there is no supermartingale density. This contradicts the previous step.
- Use standard results for supermartingales to define Z on all of $\mathbb{R}_+$.

3.1. **Discrete time.** We start by proving a de la Vallée-Poussin type theorem for families of random variables that are bounded in L^0.

Proposition 3.1. *A family of random variables $\mathcal{X}$ is bounded in probability if and only if there exists an increasing, nonnegative and unbounded function U on $[0,\infty)$, such that*

$$\sup_{X\in\mathcal{X}} E(U(|X|)) = C < \infty.$$

In this case, U can be chosen concave and such that $U(0)=0$.

Proof. First, assume that such a U exists. Then

$$\sup_{X\in\mathcal{X}} P(|X| \geq M) \leq \sup_{X\in\mathcal{X}} P(U(|X|) \geq U(M)) \leq \sup_{X\in\mathcal{X}} \frac{E(U(|X|))}{U(M)} = \frac{C}{U(M)}.$$

Since U is unbounded, the right hand side converges to zero as M tends to ∞.

Conversely, assume that $\mathcal{X}$ is bounded in probability. We need to construct an unbounded, nonnegative, concave, increasing function U with $U(0)=0$, and such that $E(U(|X|))$ is bounded on $\mathcal{X}$. The idea is of course to work with the function $F_{\mathcal{X}}(M) = \sup_{X\in\mathcal{X}} P(|X| \geq M)$. Our construction is inspired by the proof of the de la Vallé-Poussin Theorem. That is, we will construct a function U of the form

$$U(x) = \int_0^x g(y)dy \qquad \text{where} \qquad g(y) = g_n, y \in [n-1,n)$$

for a decreasing sequence of positive numbers g_n. This U will be increasing, concave, $U(0)=0$. It will be unbounded if and only if $\sum_{n=1}^{\infty} g_n = \infty$.

If U is of this form, we have by monotone convergence and Fubini (because all the terms are nonnegative):

$$\begin{aligned}
E(U(|X|)) &= \sum_{n=1}^{\infty} E(U(|X|)1_{\{|X|\in[n-1,n)\}}) \leq \sum_{n=1}^{\infty} U(n)P(|X| \in [n-1,n)) \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{n} g_k P(|X| \in [n-1,n)) = \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} g_k P(|X| \in [n-1,n)) \\
&= \sum_{k=1}^{\infty} g_k P(|X| \geq k-1) \leq \sum_{k=1}^{\infty} g_k F_{\mathcal{X}}(k-1)
\end{aligned}$$

The proof is therefore complete if we can find a decreasing sequence (g_k) of nonnegative numbers, such that

$$\sum_{k=1}^{\infty} g_k = \infty \text{ but } \sum_{k=1}^{\infty} g_k F_{\mathcal{X}}(k-1) < \infty.$$

To do so, let $n \in \mathbb{N}$. Since $(F_{\mathcal{X}}(k))$ converges to zero, it also converges to zero in the Cesàro sense. Therefore there exists $K_n \in \mathbb{N}$, such that

$$\frac{1}{K_n}\sum_{k=1}^{K_n} F_{\mathcal{X}}(k-1) \leq \frac{1}{n}.$$

We choose such a $K_n \geq n$. Define the sequence

$$g_k^n = \begin{cases} \frac{1}{nK_n} & k \leq K_n \\ 0 & k > K_n \end{cases}.$$

This is a decreasing sequence. For every n, we define such a $(g_k^n)_k$. Then also the sequence (g_k) consisting of the terms

$$g_k = \sum_{n=1}^{\infty} g_k^n = \sum_{n=n_k}^{\infty} \frac{1}{nK_n} \leq \sum_{n=n_k}^{\infty} \frac{1}{n^2} < \infty$$

is decreasing. n_k is of course the smallest n for which $g_k^n \neq 0$. We have by Fubini

$$\sum_{k=1}^{\infty} g_k = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} g_k^n = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} g_k^n = \sum_{n=1}^{\infty}\sum_{k=1}^{K_n} \frac{1}{nK_n} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

and at the same time

$$\sum_{k=1}^{\infty} g_k F_{\mathcal{X}}(k-1) = \sum_{n=1}^{\infty}\sum_{k=1}^{K_n} \frac{F_{\mathcal{X}}(k-1)}{nK_n} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

which completes the proof. □

In what follows, we will occasionally need this continuity result:

Proposition 3.2. *Let g be a proper (i.e. $g > -\infty$ and there exists x s.t. $g(x) < \infty$), convex, and lower semi-continuous function on $\mathbb{R}$. Then the map*

$$X \mapsto E(g(X))$$

is lower semi-continuous on $(L^1, \sigma(L^1, L^\infty))$. $\sigma(L^1, L^\infty)$ denotes the weak topology of L^1, i.e. the coarsest topology on L^1 for which all the maps $X \mapsto E(XY)$ with $Y \in L^\infty$ are continuous.

Proof. Since g is proper, convex, and lower semi-continuous, it is equal to its convex biconjugate (cf. Rockafellar [Roc70], Theorem 12.2). That is,

$$g(x) = \sup_{y\in\mathbb{R}}\{xy - g^*(y)\} \qquad \text{where} \qquad g^*(y) = \sup_{x\in\mathbb{R}}\{xy - g(x)\}.$$

So let (X^n) be a sequence in L^1 that converges to some X in the weak topology. Then

$$\liminf_{n\to\infty} E(g(X^n)) = \liminf_{n\to\infty} E\left(\sup_y\{X^n y - g^*(y)\}\right).$$

g^* is lower semi-continuous and $y \mapsto X^n y$ is continuous. X^n is measurable. Therefore the y are defined in a measurable way, i.e.

$$E\left(\sup_y\{X^n y - g^*(y)\}\right) = \sup_{Y \text{r.v.}} E\left(\{X^n Y - g^*(Y)\}\right)$$

and so

$$\begin{aligned}
\liminf_{n\to\infty} E(g(X^n)) &\geq \sup_Y \liminf_{n\to\infty} E(X^n Y - g^*(Y)) \\
&\geq \lim_{m\to\infty} \sup_{|Y|\leq m} \liminf_{n\to\infty} E(X^n Y - g^*(Y)) \\
&= \lim_{m\to\infty} \sup_{|Y|\leq m} E(XY - g^*(Y)).
\end{aligned}$$

g is proper, therefore there exists M such that $g^*(M) < \infty$. So for $m \geq |M|$, the expression in the expectation is bounded from below by $MX - g^*(M)$ which is in L^1. By monotone convergence we conclude

$$\liminf_{n\to\infty} E(g(X^n)) \geq E(g(X)).$$

□

Now we restrict ourselves to families of nonnegative random variables: Ultimately we are only interested in those, and our method of proof is based on techniques from Kramkov and Schachermayer [KS99], which only work for nonnegative random variables. A similar result should hold in general.

Proposition 3.3. *Let $\mathcal{X}$ be a family of random variables. If there exists a strictly positive random variable Z such that*

$$\mathcal{X} \ni X \mapsto E(|X|Z)$$

is bounded by some constant $C < \infty$, then $\mathcal{X}$ is bounded in probability.

If $\mathcal{X}$ is a convex *family of* nonnegative *random variables, then such a Z exists* if and only if *$\mathcal{X}$ is bounded in probability.*

Proof. We will relate the existence of Z to the existence of U as described in Proposition 3.1.

(1) First assume that Z exists. We show that then there exists a lower semi-continuous, convex function V on $\mathbb{R}$, such that $V|_{(-\infty,0]} = \infty$ and $\lim_{y\to 0} V(y) = \infty$, for which $E(V(Z)) < \infty$. We could again use a construction inspired by the proof of the de la Vallée-Poussin Theorem. But here we will argue differently. Define

$$\hat{V}(y) = \frac{1}{\sqrt{F_Z(y)}}$$

where $F_Z(y) = P(Z \le y)$ is the cumulative distribution function of Z. By convention, $1/0 = \infty$. Let V be the convex hull of $\hat{V}$ (i.e. V is the largest lower semi-continuous, convex function that satisfies $V(y) \le \hat{V}(y)$ for all y). Then V is positive, because $1 \le 1/\sqrt{F_Z(y)}$. for all y. Since the constant function 1 is lower semi-continuous and convex, V must be bounded from below by 1. Also $V(0) = \infty$: Let $n \in \mathbb{N}$ and choose $y_n > 0$ such that for $y \le y_n$

$$\frac{1}{\sqrt{F_Z(y)}} > n.$$

This is possible, because we assumed Z to be strictly positive. Then the affine function f with $f(0) = n$ and $f(y_n) = 0$ is everywhere smaller than $\hat{V}$, and therefore $V \ge f$. But this implies $V(0) \ge n$.

Let us calculate $E(V(Z))$.

$$\begin{aligned} E(V(Z)) &= \int_0^\infty P(V(Z) > x)dx \le \int_0^\infty P(\hat{V}(Z) > x)dx \\ &\le 1 + \int_1^\infty P\left(\frac{1}{\sqrt{F_Z(Z)}} > x\right) dx = 1 + \int_1^\infty P\left(F_Z(Z) < \frac{1}{x^2}\right) dx \\ &= 1 + \int_1^\infty P\left(Z < q_Z\left(\frac{1}{x^2}\right)\right) dx \end{aligned}$$

where the quantile function is defined as $q_Z(y) = \inf\{z : F_Z(z) \ge y\}$.

By right-continuity of the cumulative distribution function we have $F_Z(q_Z(y)) \ge y$. And because the distribution of Z can have at most countably many atoms, $F_Z(q_Z(y)) = y$ except for at most countably many y. So finally we obtain

$$E(V(Z)) \le 1 + \int_1^\infty F_Z\left(q_Z\left(\frac{1}{x^2}\right)\right) dx = 1 + \int_1^\infty \frac{1}{x^2} dx < \infty.$$

Now define

$$U : \mathbb{R}_+ \to \mathbb{R}, \qquad U(x) = \inf_{y>0}\{V(y) + xy\}.$$

This is the conjugate of V as it is used in utility maximization. It is an increasing, concave, and upper semi-continuous function. It is also nonnegative, since V is nonnegative. To see that it is unbounded, let $n \in \mathbb{N}$ and choose $x > 0$ such that

$$V\left(\frac{1}{\sqrt{x}}\right) > n \text{ and } \sqrt{x} > n.$$

Note that V is decreasing. So for $0 < y \leq 1/\sqrt{x}$ we have

$$V(y) + xy \geq V\left(\frac{1}{\sqrt{x}}\right) + 0 > n$$

on the other side for $y > 1/\sqrt{x}$ and because V is bounded from below by 1:

$$V(y) + xy \geq 1 + \sqrt{x} > n$$

Hence $U(x) = \inf_{y>0}\{V(y) + xy\} > n$.

It remains to show that $E(U(|X|))$ is bounded on $\mathcal{X}$. But this is easy:

$$\sup_{X\in\mathcal{X}} E(U|X|)) = \sup_{X\in\mathcal{X}} \inf_{Y>0} E(V(Y) + |X|Y) \leq \sup_{X\in\mathcal{X}} E(V(Z) + Z|X|) < \infty.$$

(2) Now let $\mathcal{X}$ be convex and consist only of nonnegative random variables. We assume that Z does not exist, and we show that then no nonnegative, concave, upper semi-continuous, increasing and unbounded U can exist such that

$$\sup_{X\in\mathcal{X}} E(U(X)) < \infty.$$

Let U be such a function, and again define its conjugate in the sense of utility maximization as

$$V(y) = \sup_{x\geq 0}\{U(x) - xy\}.$$

We assume U to be proper, i.e. $U < \infty$ and $U(x) > -\infty$ for at least one x. This is justified because the function U that we constructed in Proposition 3.1 was finite on $\mathbb{R}_+$. Then

$$U(x) = \inf_{y\geq 0}\{V(y) + xy\}.$$

Note that for all $y \geq 0$ we have $V(y) \geq U(0) \geq 0$, and that $V(0) = \infty$, since U is unbounded. By definition

$$\sup_{X\in\mathcal{X}} E(U(X)) = \sup_{X\in\mathcal{X}} \inf_{Y\geq 0} E(V(Y) + XY) = \sup_{X\in\mathcal{X}} \inf_{Y>0} E(V(Y) + XY)$$

where the last step is true because $V(0) = \infty$. And since we assumed that Z does not exist, we have

$$\inf_{Y>0} \sup_{X\in\mathcal{X}} E(V(Y) + XY) \geq \inf_{Y>0} \sup_{X\in\mathcal{X}} E(XY) = \infty.$$

So if we can apply a minimax theorem to exchange the sup and the inf, then the proof is complete. To do so, we will first have to restrict the Y to a compact set: For given n define

$$A_n = \{Y \in L^1 : 0 \leq Y \leq n\}.$$

A_n is uniformly integrable and therefore relatively compact in $(L^1, \sigma(L^1, L^\infty))$. It is also closed, and therefore compact. A_n is also convex. For given m define

$$B_m = \{X \in L^\infty : \text{ there exists } X' \in \mathcal{X} : 0 \leq X \leq X' \wedge m\}.$$

This is a convex subset of L^∞: Let X_1 and X_2 be such that

$$0 \leq X_i \leq X_i' \wedge m.$$

Then for all $\lambda \in [0,1]$:

$$0 \leq \lambda X_1 + (1-\lambda)X_2 \leq \lambda[X_1' \wedge m] + (1-\lambda)[X_2' \wedge m] \leq [\lambda X_1' + (1-\lambda)X_2'] \wedge m.$$

Because $\mathcal{X}$ is convex, B_m is also convex.

To apply Sion's Minimax Theorem, it suffices now to show that

$Y \mapsto E(V(Y) + XY)$ is lower semi-continuous and convex for every X and

$X \mapsto E(V(Y) + XY)$ is continuous and concave for every Y.

Convexity / concavity are of course no problem. The continuity of the map $X \mapsto E(V(Y)+XY)$ is fine as well because the Y are in L^1 (they are even bounded). For the map with fixed X, the term $Y \mapsto E(XY)$ is even continuous (X is in L^∞). The term $Y \mapsto E(V(Y))$ is lower semi-continuous by Proposition 3.2. Therefore by the Minimax Theorem (cf. Appendix, Theorem B.1):

$$\sup_{X\in B_m} \inf_{Y\in A_n} E\left(\{V(Y)+XY\}\right) = \inf_{Y\in A_n} \sup_{X\in B_m} E\left(\{V(Y)+XY\}\right). \tag{1}$$

For every n, the right hand side of (1) diverges to infinity as m tends to ∞: Let $Y \in A_n$. If $P(Y=0) > 0$, then $E(V(Y)) = \infty$, so

$$\sup_{X\in B_m} E(V(Y)+XY) = \infty.$$

Otherwise let $C > 0$. By our assumption there exists $X \in \mathcal{X}$ such that $E(XY) > C$. By monotone convergence,

$$\lim_{m\to\infty} E((X \wedge m)Y) = E(XY) > C.$$

Since this is true for every C

$$\lim_{m\to\infty} \sup_{X\in B_m} E(XY) = \infty$$

for every strictly positive $Y \in A_n$. Let now $0 < C < \infty$ and define

$$C_m = \left\{ Y \in A_n : \sup_{X\in B_m} E(V(Y)+XY) \leq C \right\}.$$

We just showed that

$$\cap_{m=1}^{\infty} C_m = \emptyset.$$

Let us verify that the C_m are closed, because then they are closed subsets of the compact space A_n with empty intersection. Therefore already a finite intersection would have to be empty. And this would then imply

$$\lim_{m\to\infty} \inf_{Y\in A_n} \sup_{X\in B_m} E(V(Y)+XY) > C.$$

Since this would be true for all $C < \infty$, we would have shown that the right hand side in (1) diverges to infinity. To see that C_m is closed, let (Y_k) be a sequence in C_m that converges to Y. We want to show that $Y \in C_m$, i.e. that $\sup_{X\in B_m} E(V(Y)+XY) \leq C$. But

$$\begin{aligned} C &\geq \liminf_{n\to\infty} \sup_{X\in B_m} E(V(Y_n)+XY_n) \\ &\geq \sup_{X\in B_m} \liminf_{n\to\infty} E(V(Y_n)+XY_n) \\ &\geq \sup_{X\in B_m} E(V(Y)+XY) \end{aligned}$$

where the last step follows by lower semi-continuity of $Y \mapsto E(V(Y)+XY)$.

Since this is true for every n, we have on the right hand side of (1)

$$\lim_{n\to\infty}\lim_{m\to\infty}\inf_{Y\in A_n}\sup_{X\in B_m} E(V(Y)+XY)=\infty$$

and therefore also for the left hand side of (1)

$$\lim_{n\to\infty}\lim_{m\to\infty}\sup_{X\in B_m}\inf_{Y\in A_n} E\left(V(Y)+XY\right)=\infty.$$

Let us show that this implies

$$\sup_{X\in\mathcal{X}}\inf_{Y\geq 0} E\left(V(Y)+XY\right)=\sup_{X\in\mathcal{X}} E(U(X))=\infty.$$

Getting rid of the m is easy: For every $Z\in B_m$ there exists $X\in\mathcal{X}$ such that $Z\leq X\wedge m$. Therefore for any $Y\in A_n$

$$E(V(Y)+ZY)\leq E(V(Y)+(X\wedge m)Y)\leq E(V(Y)+XY)$$

and therefore

$$\sup_{X\in\mathcal{X}} E(V(Y)+XY)\geq\sup_{X\in B_m} E(V(Y)+XY)$$

for any $Y\in A_n$ and any m. So we have

$$\lim_{n\to\infty}\sup_{X\in\mathcal{X}}\inf_{Y\in A_n} E(V(Y)+XY)=\infty.$$

Now let us deal with n. Note that

$$x\mapsto \operatorname{arginf}_{y\geq 0}\{V(y)+xy\}$$

is a decreasing function of x which converges to zero as x tends to infinity. So there exists some $f_n\geq 0$ such that for $x\geq f_n$

$$\operatorname{arginf}_{y\geq 0}\{V(y)+xy\}\leq n.$$

Now let $C>0$ and choose $X\in\mathcal{X}$ such that

$$\begin{aligned}
C &\leq \inf_{Y\in A_n} E(V(Y)+XY)\\
&= \inf_{0\leq Y\leq n}\{E((V(Y)+XY)1_{[f_n,\infty)}(X))+E((V(Y)+XY)1_{[0,f_n)}(X))\}\\
&= \inf_{0\leq Y\leq n} E((V(Y)+XY)1_{[f_n,\infty)}(X))+\inf_{0\leq Y\leq n} E((V(Y)+XY)1_{[0,f_n)}(X))\\
&\leq \inf_{0\leq Y} E((V(Y)+XY)1_{[f_n,\infty)}(X))+V(1)+E(X1_{[0,f_n)}(X))\\
&\leq \inf_{0\leq Y} E(V(Y)+XY)+V(1)+f_n.
\end{aligned}$$

In the last step we used that $V\geq 0$. The constant $V(1)+f_n$ does not depend on C or X. This shows that also

$$X\mapsto\inf_{0\leq Y} E(V(Y)+XY)$$

is unbounded on $\mathcal{X}$, so the proof is complete:

$$\infty=\sup_{X\in\mathcal{X}}\inf_{0\leq Y} E(V(Y)+XY)=\sup_{X\in\mathcal{X}} E(U(X)).$$

Since this is true for every such U, $\mathcal{X}$ cannot be bounded in probability.

□

Remark. While it should be possible to prove the result without the nonnegativity-condition, convexity is necessary: Let $\{A_k^n : 1\leq k\leq 2^n, n\in\mathbb{N}\}$ be an increasing sequence of partitions of Ω, such that for every n and k

$$P(A_k^n)=2^{-n}.$$

This can for example be realized on $\Omega = [0,1]$. Define the random variables
$$X_k^n = 1_{A_k^n} 2^{2n}.$$
Then $(X_k^n : n,k)$ is bounded in probability. Let $Z \geq 0$ be such that $(ZX_k^n : n,k)$ is bounded in L^1. Then there exists $C > 0$ such that
$$E(ZX_k^n) = E(1_{A_k^n} Z) 2^{2n} \leq C.$$
Summing over all k, we obtain
$$E(Z) = \sum_{k=1}^{2^n} E(1_{A_k^n} Z) \leq 2^n \frac{C}{2^{2n}} = C 2^{-n}.$$
Since this has to hold for all n, $Z = 0$.

Remark. One way to interpret the previous result is as follows: Let $\mathcal{X}$ be a convex family of nonnegative random variables. Then $\mathcal{X}$ is bounded in probability if and only if there exists a measure $Q \sim P$, such that $\mathcal{X}$ is $L^1(Q)$-bounded. We might ask if we can choose Q such that something stronger than $L^1(Q)$-boundedness holds. In general this is not possible: Even if we impose all the structure conditions that our portfolio variables satisfy, and if we assume the stronger condition (NFLVR), and if S is a continuous semimartingale, there might still not even be an absolutely continuous $Q \ll P$, such that $\mathcal{K}_1$ is uniformly integrable under Q. To see this, choose an increasing sequence of partitions $(A_k^n : 1 \leq k \leq 2^n, n \in \mathbb{N})$ of $\mathbb{R}$, such that if $N(dx)$ denotes the standard normal distribution, for every n,k we have
$$\int_{A_k^n} N(dx) = 2^{-n}.$$
Now let S be a standard Brownian motion. Define the random variables
$$X_k^n = 1_{A_k^n}(S_1) 2^n.$$
Then every X_k^n is a bounded random variable, and $E(X_k^n) = 1$ for all n,k. By the predictable representation property of Brownian motion, $X_k^n \in \mathcal{K}_1$ for every n,k. Now let $Q \ll P$, and let $g \geq 0$ be such that $\lim_{x\to\infty} g(x)/x = \infty$. If we show that for any such g, $(g(X_k^n))_{n,k}$ is unbounded in $L^1(Q)$, then $\mathcal{K}_1$ is not uniformly integrable under Q by de la Vallée-Poussin's theorem. Let $C > 0$. Choose $n \in \mathbb{N}$ such that $g(2^n) \geq C 2^n$, and choose k such that $Q(S_1 \in A_k^n) \geq 2^{-n}$ (there must exist such a k since Q has total mass 1). Then
$$E_Q(g(X_k^n)) \geq E_Q(1_{A_k^n}(S_1) 2^n C) \geq 2^{-n} 2^n C = C$$
which shows that $E_Q(g(\cdot))$ is unbounded on $\mathcal{K}_1$.

The next Lemma will be the main step towards the supermartingale property. The conditions are slightly technical, but in the (simple) proof it will become immediately clear why we chose them in this way.

Lemma 3.4. *Let $\mathcal{X}$ be a L^1-bounded family of nonnegative processes indexed by $\{0,1\}$, adapted to a filtration $(\mathcal{F}_0, \mathcal{F}_1)$. Assume that for every $X \in \mathcal{X}$*
$$\{X_0 = 0\} \subseteq \{X_1 = 0\}.$$
Also suppose that for every disjoint $A, B \in \mathcal{F}_0$ and $X^A, X^B \in \mathcal{X}$ there exists $X^{AB} \in \mathcal{X}$ such that
$$1_A \frac{X_1^A}{X_0^A} + 1_B \frac{X_1^B}{X_0^B} \leq 1_{A\cup B} X_1^{AB} \quad \text{and} \quad 1_A \frac{X_1^A}{X_0^A} + 1_B \frac{X_1^B}{X_0^B} \leq 1_{A\cup B} \frac{X_1^{AB}}{X_0^{AB}} \tag{2}$$
(we agree that $0/0 = 0$). Further assume that for every $X^1, X^2 \in \mathcal{X}$ there exists $X^0 \in \mathcal{X}$ such that
$$X_0^1 \frac{X_1^2}{X_0^2} \leq X_1^0. \tag{3}$$

Then there exists a $\mathcal{F}_0$-measurable random variable $Z \in L^1_+(\mathcal{F}_0)$, such that $(X_0 Z, X_1)$ is a supermartingale for every $X \in \mathcal{X}$. Z can be chosen such that for every $X \in \mathcal{X}$

$$E(Z) \le \sup_{X\in\mathcal{X}} \max_{i=0,1} E(X_i) \qquad \text{and} \qquad E(X_0 Z) \le \sup_{X\in\mathcal{X}} \max_{i=0,1} E(X_i). \tag{4}$$

Proof. Define the set function μ on $\mathcal{F}_0$ as follows:

$$\mu(A) := \sup_{X\in\mathcal{X}} E\left(1_A \frac{X_1}{X_0}\right).$$

Since $\mathcal{X}$ is bounded in L^1, μ is finite by condition (2). The conditions of the Lemma are chosen exactly so that μ becomes a finite measure. First note that it is finitely additive: Let $A \cap B = \emptyset$. Then by (2)

$$\begin{aligned}\mu(A)+\mu(B) &= \sup_{X^A\in\mathcal{X}} E\left(1_A \frac{X_1^A}{X_0^A}\right) + \sup_{X^B\in\mathcal{X}} E\left(1_B \frac{X_1^B}{X_0^B}\right)\\ &= \sup_{(X^A,X^B)\in\mathcal{X}^2} E\left(1_A \frac{X_1^A}{X_0^A} + 1_B\frac{X_1^B}{X_0^B}\right) \le \sup_{(X^A,X^B)\in\mathcal{X}^2} E\left(1_{A\cup B}\frac{X_1^{AB}}{X_0^{AB}}\right)\\ &\le \sup_{X\in\mathcal{X}} E\left(1_{A\cup B}\frac{X_1}{X_0}\right) = \mu(A\cup B).\end{aligned}$$

$\mu(A\cup B) \le \mu(A)+\mu(B)$ is obvious. Now let (A_n) be a sequence of disjoint sets in $\mathcal{F}_0$. Then

$$\mu(\cup_{n=1}^\infty A_n) = \sup_{X\in\mathcal{X}} \sum_{n=1}^\infty E\left(1_{A_n}\frac{X_1}{X_0}\right) \le \sum_{n=1}^\infty \sup_{X^n\in\mathcal{X}} E\left(1_{A_n}\frac{X_1^n}{X_0^n}\right) = \sum_{n=1}^\infty \mu(A_n).$$

But for every finitely additive positive set function the inverse inequality is true: For every $N \in \mathbb{N}$

$$\mu(\cup_{n=1}^\infty A_n) \ge \mu(\cup_{n=1}^N A_n) = \sum_{n=1}^N \mu(A_n).$$

Letting $N\to\infty$, we obtain that μ is in fact a finite positive measure on $\mathcal{F}_0$. It is absolutely continuous with respect to P. Therefore there exists $Z \in L^1_+(\mathcal{F}_0, P)$, such that

$$\sup_{X\in\mathcal{X}} E\left(1_A\frac{X_1}{X_0}\right) = \mu(A) = E(1_A Z).$$

This holds for every $A \in \mathcal{F}_0$, and therefore for all $\mathcal{F}_0$-measurable step functions. Let $G \ge 0$ be $\mathcal{F}_0$-measurable and let G_n be a sequence of step functions monotonically increasing to X. Then

$$E(GZ) = \lim_{n\to\infty}\sup_{X\in\mathcal{X}} E\left(G^n\frac{X_1}{X_0}\right) \le \lim_{n\to\infty}\sup_{X\in\mathcal{X}} E\left(G\frac{X_1}{X_0}\right) = \sup_{X\in\mathcal{X}} E\left(G\frac{X_1}{X_0}\right)$$

and

$$E(GZ) = \lim_{n\to\infty}\sup_{X\in\mathcal{X}} E\left(G^n\frac{X_1}{X_0}\right) \ge \sup_{X\in\mathcal{X}}\lim_{n\to\infty} E\left(G^n\frac{X_1}{X_0}\right) = \sup_{X\in\mathcal{X}} E\left(G\frac{X_1}{X_0}\right)$$

which proves the equality for all nonnegative $G \in \mathcal{F}_0$. In particular for any $X \in \mathcal{X}$ and $A \in \mathcal{F}_0$

$$E(1_A X_0 Z) = \sup_{\tilde{X}\in\mathcal{X}} E\left(1_A X_0 \frac{\tilde{X}_1}{\tilde{X}_0}\right) \ge E\left(1_A X_0\frac{X_1}{X_0}\right) = E(1_A X_1).$$

So $(X_0 Z, X_1)$ is a supermartingale if $E(X_0 Z) < \infty$. But the bound stated in (4) follows immediately from assumption (3):

$$E(X_0 Z) = \sup_{\tilde{X}\in\mathcal{X}} E\left(X_0\frac{\tilde{X}_1}{\tilde{X}_0}\right) \le \sup_{X^0\in\mathcal{X}} E\left(X_1^0\right).$$

□

Remark. It is also possible to prove this result by a compactness argument. The proof based on the Radon-Nikodym theorem seems more elegant to us, which is why we only gave this one here.

Remark. Note that some type of stability assumption is necessary for the previous Lemma to hold. Even for a uniformly integrable and convex family of processes $\mathcal{X}$, the proposition may fail without any stability assumptions: Let again $\{A_k^n : 1 \leq k \leq 2^n, n \in \mathbb{N}\}$ be an increasing sequence of partitions of Ω, such that for every n and k we have $P(A_k^n) = 2^{-n}$. Define the random variables

$$X_k^n = 1_{A_k^n} \frac{2^n}{n}.$$

Then $(X_k^n : k, n)$ is uniformly integrable: Let $M > 2^{n_0}$. Then

$$\sup_{n,k} E(|X_k^n| 1_{\{|X_k^n| \geq M\}}) \leq E(|X_1^{n_0}|) = \frac{1}{n_0}.$$

From the de la Vallée-Poussin Theorem and Jensen's inequality we obtain that also the convex hull $\mathcal{X}$ of the X_k^n is uniformly integrable. Define $\mathcal{F}_0 = \mathcal{F}_1 = \sigma(A_k^n : 1 \leq k \leq 2^n, n \in \mathbb{N})$. The processes we consider are all of the form $(1, X)$ for some $X \in \mathcal{X}$. Now assume that there exists $Z > 0$ such that $E(1_A X Z) \leq P(A)$ for all $A \in \mathcal{F}_0$ and $X \in \mathcal{X}$. This is even a weaker statement than what we showed in the Proposition. But it still is not possible, because then

$$E(1_{A_k^n} Z X_k^n) = E(1_{A_k^n} Z) \frac{2^n}{n} \leq 2^{-n}$$

and thus $E(1_{A_k^n} Z) \leq n 2^{-2n}$. Summing over all k, we obtain $E(Z) \leq n 2^{-n}$. This has to hold for all n, and therefore $E(Z) = 0$ - a contradiction to $Z > 0$.

Corollary 3.5. *Let $\mathcal{X}$ be a L^1-bounded family of nonnegative processes indexed by $\{0, \ldots, n\}$, adapted to a filtration $(\mathcal{F}_k : 0 \leq k \leq n)$. Assume that for every $X \in \mathcal{X}$ and $k \leq n-1$*

$$\{X_k = 0\} \subseteq \{X_{k+1} = 0\}$$

also suppose that for every disjoint $A, B \in \mathcal{F}_k$ and $X^A, X^B \in \mathcal{X}$ there exists $X^{AB} \in \mathcal{X}$ such that

$$1_A \frac{X_{k+1}^A}{X_k^A} + 1_B \frac{X_{k+1}^B}{X_k^B} \leq 1_{A \cup B} \frac{X_{k+1}^{AB}}{X_k^{AB}} \quad \text{and} \quad 1_A \frac{X_{k+1}^A}{X_k^A} + 1_B \frac{X_{k+1}^B}{X_k^B} \leq 1_{A \cup B} X_{k+1}^{AB}$$

and that for every $X^1, X^2 \in \mathcal{X}$ there exists $X^0 \in \mathcal{X}$ such that

$$\frac{X_{k+1}^2}{X_k^2} X_k^1 \leq X_{k+1}^0.$$

Then there exists a strictly positive and adapted process $(Z_k : 0 \leq k \leq n)$, such that ZX is a supermartingale for every $X \in \mathcal{X}$. Z can be chosen such that that every $X \in \mathcal{X}$

$$\max_{k=0\ldots,n} E(Z_k) \leq \sup_{X \in \mathcal{X}} \max_{k=0,\ldots,n} E(X_k) \vee 1 \qquad \text{and} \qquad \max_{k=0\ldots,n} E(Z_k X_k) \leq \sup_{X \in \mathcal{X}} \max_{k=0,\ldots,n} E(X_k) \vee 1.$$

Proof. We will prove the result by induction. For $n = 1$, this is just Proposition 3.4: Take $Z_0 = Z, Z_1 = 1$.

Now assume the result is true for n. Let $\mathcal{X}$ be a family of processes indexed by $\{0, \ldots, n+1\}$. Assume that $\mathcal{X}$ satisfies all the requirements. Then $(X_1, \ldots, X_{n+1})$ satisfies all the requirements as well, and therefore there exists a strictly positive supermartingale $(Z_1, \ldots, Z_{n+1})$ as stated. It therefore suffices to construct a suitable Z_0. This Z_0 will have to satisfy

$$E(Z_0) \leq \sup_{X \in \mathcal{X}} \max_{k=0,\ldots,n} E(X_k) \vee 1$$

and

$$E(Z_0 X_0) \leq \sup_{X \in \mathcal{X}} \max_{k=0,\ldots,n} E(X_k) \vee 1$$

and

$$E\left(1_{A_0} Z_1 X_1\right) \leq E(1_{A_0} Z_0 X_0)$$

for all $A_0 \in \mathcal{F}_0$, $X \in \mathcal{X}$. We obtain such a Z_0 by applying Proposition 3.4 to the family of processes

$$\{(X_0, Z_1 X_1) : X \in \mathcal{X}\}.$$

It is straightforward to show that this family satisfies all the requirements stated in Proposition 3.4. Therefore we obtain Z_0, and we are done. □

The previous results can be combined to show that in the case of finite discrete time, the set of attainable portfolios is bounded in probability if and only if there exists a strictly positive supermartingale denisity Z:

Corollary 3.6. *Let $0 \leq t_0 < \cdots < t_n \leq \infty$. Define*

$$\mathcal{K}_1^{t_n} = \left\{ 1 + (H \cdot S)_{t_n} : H \textit{ is 1-admissible and } \lim_{t \to t_n} (H \cdot S)_t \textit{ exists a.s.} \right\}.$$

Then $\mathcal{K}_1^{t_n}$ is bounded in probability if and only if there exists a strictly positive supermartingale $(Z_{t_i} : i = 0, \dots, n)$, such that for every 1-admissible H for which $\lim_{t\to t_n}(H \cdot S)_t$ exists,

$$((1 + (H \cdot S)_{t_i}) Z_{t_i} : i = 0, \dots, n)$$

is a supermartingale.

Proof. First assume that $\mathcal{K}_1$ is bounded in probability, and apply Proposition 3.3 to obtain $Q \sim P$, such that $\mathcal{K}_1$ is bounded in $L^1(Q)$.

It remains to show that the portfolio processes satisfy the stability properties required in Corollary 3.5. For this purpose let $k < n$ and $A, B \in \mathcal{F}_{t_k}$ be two disjoint sets. Also, let H^A and H^B be two 1-admissible strategies (under Q, but since Q is equivalent to P, the set of 1-admissible strategies is the same under the two measures). Denote $X^A_{t_k} = 1 + (H^A \cdot S)_{t_k}$ and similarly for B. Define the strategy

$$H_t^{AB} = \begin{cases} 0, & t \leq t_k \\ 1_A \frac{H_t^A}{X^A_{t_k}} 1_{\{X^A_{t_k} > 0\}} + 1_B \frac{H_t^B}{X^B_{t_k}} 1_{\{X^B_{t_k} > 0\}}, & t > t_k \end{cases}.$$

Then H^{AB} is 1-admissible and satisfies for $l > k$ (since $1 + (H^{AB} \cdot S)_{t_k} = 1$):

$$\begin{aligned} \frac{1 + (H^{AB} \cdot S)_{t_l}}{1 + (H^{AB} \cdot S)_{t_k}} = 1 + (H^{AB} \cdot S)_{t_l} &\geq 1_A 1_{\{X^A_{t_k} > 0\}} + 1_B 1_{\{X^B_{t_k} > 0\}} + (H^{AB} \cdot S)_{t_l} \\ &= 1_A \frac{1}{X^A_{t_k}} (X^A_{t_k} + (H^A \cdot S)_{t_l} - (H^A \cdot S)_{t_k}) \\ &\quad + 1_B \frac{1}{X^B_{t_k}} (X^B_{t_k} + (H^B \cdot S)_{t_l} - (H^B \cdot S)_{t_k}) \\ &= 1_A \frac{X^A_{t_l}}{X^A_{t_k}} + 1_B \frac{X^B_{t_l}}{X^B_{t_k}}. \end{aligned}$$

Since outside of $A \cup B$, the right hand side vanishes, we can multiply the left hand side with $1_{A\cup B}$ to get the first required stability assumption (2).

If H^1 and H^2 are 1-admissible, then define the strategy

$$H_t^0 = \begin{cases} H_t^1, & t \leq t_k \\ H_t^2 \frac{X^1_{t_k}}{X^2_{t_k}} 1_{\{X^2_{t_k} > 0\}}, & t > t_k \end{cases}.$$

Then for $l > k$

$$\begin{aligned}
1 + (H^0 \cdot S)_{t_l} &= 1 + (H^0 \cdot S)_{t_k} + ((H^0 \cdot S)_{t_l} - (H^0 \cdot S)_{t_k}) \\
&= X^1_{t_k} + \frac{X^1_{t_k}}{X^2_{t_k}} 1_{\{X^2_{t_k} > 0\}} (X^2_{t_l} - X^2_{t_k}) \\
&\geq X^1_{t_k} \left(1 + \frac{1_{\{X^2_{t_k} > 0\}}}{X^2_{t_k}} X^2_{t_l} - 1 \right) = \frac{X^2_{t_l}}{X^2_{t_k}} X^1_{t_k}
\end{aligned}$$

which proves the second stability assumption (3).

Finally we need to show that for any 1-admissible H, $\{1+(H \cdot S)_{t_k} = 0\} \subseteq \{1+(H \cdot S)_{t_{k+1}} = 0\}$. But this is clear, because in this case $\tilde{H}_t = H_t 1_{\{1+(H\cdot S)_{t_k}=0\}} 1_{t \in (t_k, t_{k+1}]}$ is a strategy which satisfies

$$\int_{t_k}^{t_{k+1}} \tilde{H}_s dS_s \geq 0.$$

Therefore also $n\tilde{H}$ is 1-admissible for every n. If the integral was strictly positive with positive probability, then $\mathcal{K}_1$ could not be bounded in probability.

So we can apply Corollary 3.5 to obtain a strictly positive Q-supermartingale Z such that for every 1-admissible H

$$Z(1 + (H \cdot S))$$

is a Q-supermartingale. If we define $\tilde{Z}_{t_k} = E_P(dQ/dP|\mathcal{F}_{t_k})$, then $Z\tilde{Z}$ is a strictly positive P-supermartingale, such that for every 1-admissible H,

$$Z\tilde{Z}(1 + (H \cdot S))$$

is a P-supermartingale.

Conversely assume that such a supermartingale Z exists. Then in particular for every $X \in \mathcal{K}_1^{t_n}$, $E(XZ_n) \leq 1$. Thus $\mathcal{K}_1^{t_n}$ is bounded in probability by Proposition 3.3. □

3.2. **Continuous time.** In this section, we want to transfer the results from the previous section to the setting of continuous time processes. By Proposition 3.3, we can find an equivalent probability measure Q, such that $\mathcal{K}_1$ is bounded in $L^1(Q)$.

To translate Corollary 3.6 to the continuous time setting is more complicated. Luckily we are interested in finding a *supermartingale* Z, such that for every portfolio process X, XZ is a supermartingale. For a supermartingale, it is essentially sufficient to define its skeleton $(Z_q : q \in \mathbb{Q}_+)$. This allows us to use a compactness argument to reduce to the finite discrete time case.

We will need the notion of convex compactness. It was defined by Zitkovic [Ž10]:

Definition. Let X be a topological vector space. A closed convex subset $C \subseteq X$ is called **convexly compact** if for any family of closed convex subsets $\{F_\alpha : \alpha \in A\}$ of C,

$$\cap_{\alpha \in A} F_\alpha = \emptyset$$

implies that there exists $\alpha_1, \ldots, \alpha_n \in A$, such that

$$\cap_{i=1}^n F_{\alpha_i} = \emptyset.$$

Zitkovic [Ž10] then proves the following result:

Proposition 3.7. *Let $\mathcal{X}$ be a convex set of nonnegative random variables, closed with respect to convergence in probability. Then $\mathcal{X}$ is convexly compact in L^0 (the space of real-valued random variables, equipped with the topology of convergence in probability) if and only if it is bounded in probability.*

In Proposition A.2 we prove a Tychonoff theorem for countable families of convexly compact subsets of metric spaces. We will use this in the proof of the following Lemma.

Lemma 3.8. *Let Q be a probability measure that is equivalent to P, such that $\mathcal{K}_1$ is bounded in $L^1(Q)$. Then there exists a nonnegative Q-supermartingale $(\bar{Z}_q : q \in \mathbb{Q}_+ \cup \{\infty\})$ with $Z_\infty > 0$, such that for every 1-admissible H for which $\lim_{t\to\infty}(H \cdot S)_t$ exists, the process*

$$(\bar{Z}_q(1 + (H \cdot S)_q) : q \in \mathbb{Q}_+ \cup \{\infty\})$$

is a Q-supermartingale.

Proof. Define $M = \sup_{X\in\mathcal{K}_1} E_Q(X) \vee 1$, and let $\mathcal{X}$ be the set of all processes of the form $1 + H \cdot S$ for 1-admissible H such that $\lim_{t\to\infty}(H \cdot S)_t$ exists. Introduce the following class of processes:

$$C = \{(Z_q : q \in \mathbb{Q}_+ \cup \{\infty\}) : Z_\infty = 1, Z_q \geq 0, Z_q \in \mathcal{F}_q \text{ and } E_Q(Z_q) \leq M \text{ for all } q\}.$$

By Proposition 3.7 and Proposition A.2, this is a convexly compact set in $\prod_{q\in\mathbb{Q}_+\cup\{\infty\}} L^0(\mathcal{F}_q, Q)$ if this space is equipped with the product topology (and all the single L^0-spaces are equipped with the topology of convergence in probability). Define for given $q, r \in \mathbb{Q}_+ \cup \{\infty\}$:

$$C(q, r) = \{Z \in C : E_Q(Z_{q+r}X_{q+r}/X_q|\mathcal{F}_q) \leq Z_q \text{ for all } X \in \mathcal{X}\}.$$

These are convex subsets of a convexly compact set. By Fatou's lemma, they are also closed: Let $E(Z^n_{q+r}X_{q+r}/X_q|\mathcal{F}_q) \leq Z^n_q$ for all n, and let Z^n converge to Z in the product topology. Then

$$Z_q = \lim_{n\to\infty} Z^n_q \geq E_Q\left(\liminf_{n\to\infty} Z^n_{q+r}\frac{X_q + r}{X_q}\middle| \mathcal{F}_q\right) = E_Q\left(Z_{q+r}\frac{X_{q+r}}{X_q}\middle| \mathcal{F}_q\right).$$

So if

$$\cap_{q\in\mathbb{Q}_+, r\in\mathbb{Q}_+\cup\{\infty\}} C(q, r)$$

was empty, then already a finite intersection would have to be empty. But this is impossible due to Corollary 3.6: If a finite intersection was empty, then there would be some $0 \leq t_1 < \cdots < t_n \leq \infty$ for which it is impossible to find a strictly positive $(Z_{t_1}, \ldots, Z_{t_n})$ such that

$$(X_{t_i}Z_{t_i} : i = 1, \ldots, n)$$

is a supermartingale for every $X \in \mathcal{X}$. But Corollary 3.6 gives us exactly such a Z!

So let Z be in the intersection of all $C(q, r)$. Then for any $q \in \mathbb{Q}_+$ and $r \in \mathbb{Q}_+ \cup \{\infty\}$:

$$E_Q(Z_{q+r}1/1|\mathcal{F}_q) \leq Z_q$$

which shows that Z is a Q-supermartingale. Since $Z_\infty = 1$, Z_q must be strictly positive for every $q \geq 0$. The same argument with X_{q+r}/X_q replacing $1/1$ shows that XZ is a Q-supermartingale for every $X \in \mathcal{X}$. □

Corollary 3.9. *There exists a nonnegative Q-supermartingale Z with $Z_\infty > 0$, such that for every 1-admissible H for which $\lim_{t\to\infty}(H \cdot S)_t$ exists,*

$$(Z_t(1 + (H \cdot S)_t) : t \geq 0)$$

is a Q-supermartingale.

Z can be chosen right-continuous for every $\omega \in \Omega$, *and such that it a.s. possesses left limits at every t.*

Proof. Let $\bar{Z}$ be the supermartingale given by Lemma 3.8. This is a supermartingale indexed by $\mathbb{Q}_+ \cup \{\infty\}$. Since $(\mathcal{F}_t)$ is right-continuous, for every $t \geq 0$ there exists a $(Q-)$null set $N_t \in \mathcal{F}_t$, such that for $\omega \in \Omega\backslash N_t$

$$\lim_{\substack{r\to s-\\ r\in\mathbb{Q}}} \bar{Z}(\omega)_r \text{ and } \lim_{\substack{r\to s+\\ r\in\mathbb{Q}}} \bar{Z}(\omega)_r$$

exist for every $s \leq t$ (cf. e.g. Ethier and Kurtz [EK86], right before Proposition 2.2.9). So we define

$$Z_t(\omega) = \begin{cases} \lim_{\substack{s\to t+\\ s\in\mathbb{Q}}} \bar{Z}_s(\omega), & \omega \in \Omega\backslash N_t \\ 0, & \text{otherwise} \end{cases}.$$

Then Z is adapted because $(\mathcal{F}_t)$ is right-continuous. It is right-continuous by definition. However it may not have left limits everywhere. Nonetheless outside of the null set $N = \cup_{n\in\mathbb{N}} N_n$ it has left limits at every $t > 0$. Z is also a supermartingale: Using Fatou's Lemma in the first step and Corollary 2.2.10 of [EK86] in the second step, we obtain (since N_{t+s} is a null set)

$$E_Q(Z_{t+s}|\mathcal{F}_t) \le \liminf_{\substack{r\to(t+s)+\\ r\in\mathbb{Q}}} E_Q(\bar{Z}_r|\mathcal{F}_t) = \liminf_{\substack{r\to(t+s)+\\ r\in\mathbb{Q}}} \liminf_{\substack{u\to t+\\ u\in\mathbb{Q}}} E_Q(\bar{Z}_r|\mathcal{F}_u) \le \liminf_{\substack{r\to(t+s)+\\ r\in\mathbb{Q}}} \liminf_{\substack{u\to t+\\ u\in\mathbb{Q}}} \bar{Z}_u = Z_t.$$

If we recall that for every 1-admissible H, $1 + (H \cdot S)$ is almost surely right-continuous, then the same argument shows that also

$$Z(1 + (H \cdot S))$$

is a Q-supermartingale. □

Proof of Theorem 1.1. Assume $\mathcal{K}_1$ is bounded in probability and let $(\tilde{Z}_t)$ be a right-continuous and a.s. càdlàg version of the martingale

$$\tilde{Z}_t = E_P\left(\frac{dQ}{dP}\middle|\mathcal{F}_t\right).$$

This exists because $(\mathcal{F}_t)$ is right-continuous, cf. Corollary 2.2.11 of [EK86]. Then $Z\tilde{Z}$ is as required: It is a.s. càdlàg, it is a P-supermartingale, strictly positive at ∞. From Theorem VI-6 of Dellacherie and Meyer [DM80], we obtain that $\lim_{t\to\infty} Z_t\tilde{Z}_t$ exists and satisfies

$$\lim_{t\to\infty} Z_t\tilde{Z}_t \ge Z_\infty\tilde{Z}_\infty > 0.$$

For every 1-admissible strategy H for which $\lim_{t\to\infty}(H\cdot S)_t$ exists, $Z\tilde{Z}(1 + (H \cdot S))$ is a supermartingale. But in fact the limit exists for every 1-admissible strategy: Let H be 1-admissible. Let $t, s \in \mathbb{R}_+$. Define the restricted strategy $H^r_u = H_u, r \le t + s$, $H^r_u = 0, u > t + s$. Then of course for H^r the limit exists, and therefore

$$\begin{aligned} E(Z_{t+s}\tilde{Z}_{t+s}(1 + (H \cdot S)_{t+s})|\mathcal{F}_t) &= E(Z_{t+s}\tilde{Z}_{t+s}(1 + (H^r \cdot S)_{t+s})|\mathcal{F}_t) \\ &\le Z_t\tilde{Z}_t(1 + (H^r \cdot S)_t) \\ &= Z_t\tilde{Z}_t(1 + (H \cdot S)_t). \end{aligned}$$

So for every 1-admissible strategy, $Z\tilde{Z}(1 + (H \cdot S))$ is a supermartingale. It is also nonnegative. Therefore it must a.s. converge as $t \to \infty$. Since $\tilde{Z}Z$ converges to a strictly positive limit, $(H \cdot S)$ must converge as well.

Conversely assume that such a supermartingale Z exists. Then for every 1-admissible H, $(H \cdot S)_\infty = \lim_{t\to\infty}(H \cdot S)_t$ must exist and satisfy

$$E(Z_\infty(1 + (H \cdot S)_\infty)) \le 1.$$

In particular, $Z_\infty\mathcal{K}_1$ is norm-bounded in L^1 and Z_∞ is strictly positive. By Proposition 3.3, $\mathcal{K}_1$ must be bounded in probability. □

Corollary 3.10. *In the previous proof we showed that if $\mathcal{K}_1$ is bounded in probability, then $\lim_{t\to\infty}(H \cdot S)_t$ exists for every 1-admissible strategy H.*

A strategy H is called admissible if it is C-admissible for some $C > 0$. If H is C-admissible, then $(1/C)H$ is 1-admissible. Therefore the limit exists for every admissible strategy.

Remark. Of course we can introduce trading constraints without any problem, as long as the set that we constrain to is a convex cone at any given time. All we needed were the stability assumptions that $\mathcal{K}_1$ needs to satisfy, and in that case they will still be satisfied.

Corollary 3.11. *Let S be locally bounded. If (NA1) holds, then S is a semimartingale.*

Proof. We assume that S is one-dimensional. Otherwise we can repeat the arguments for every component of S. Since local semimartingales are semimartingales (cf. Protter [Pro04], Theorem II.6), it suffices to show the statement in the case when S is globally bounded. So assume $|S| \leq C$, and let Z be a supermartingale density. Then the strategy $H_t \equiv 1$ is C-admissible, and therefore

$$(C + (H \cdot S))Z = CZ + SZ$$

is a semimartingale. In particular SZ is a semimartingale. Since $Z_t > 0$ for every t, $1/Z$ is a semimartingale by Itô's formula, which completes the proof after another application of Itô's formula to the product $(SZ)(1/Z)$. □

It it not possible to extend this result to the unbounded case. Just take a one-dimensional Lévy-process with jumps that are unbounded both from above and from below, and add to it any independent process which is not a semimartingale. Then their sum is not a semimartingale. But since in this case there are no admissible simple strategies, $\mathcal{K}_1 = \{1\}$ is of course bounded in probability.

4. Construction of a Dominating Martingale Measure

4.1. **Föllmer's measure.** Now let Z be a supermartingale density. As is to be expected, it is much more delicate to construct a dominating measure, than it is to construct an absolutely continuous measure. The reason for this is that by the Radon-Nikodym theorem, the space of absolutely continuous finite measures is in one-to-one correspondence with $L^1(P)$, a relatively simple space.

We want to construct a dominating measure and a stopping time T, such that (Z, T) is the Kunita-Yoeurp decomposition of Q with respect to P. Recall that (Z, T) is the Kunita-Yoeurp decomposition of Q with respect to P if

(1) $P(T = \infty) = 1$
(2) $Q(\cdot \cap \{T \leq t\})$ and P are singular on $\mathcal{F}_t$.
(3) $Q(\cdot \cap \{T > t\})$ is absolutely continuous with respect to P on $\mathcal{F}_t$, and for $A \in \mathcal{F}_t$

$$Q(A \cap \{T > t\}) = E_P(1_A Z_t). \tag{5}$$

It is clear that it is not possible to find (Q, T) on every filtered probability space. For one, the space could be too small. Consider e.g. an Ω that consists of one single point, and define $\mathcal{F} = \mathcal{F}_t = \{\emptyset, \Omega\}, t \geq 0$. Then (e^{-t}) is a continuous positive supermartingale. Nonetheless there exists only one probability measure on Ω, and therefore we cannot possibly expect to find Q and T such that $(e^{-\cdot}, T)$ is the Kunita-Yoeurp decomposition of Q with respect to P. This is reminiscent of the Dambis Dubins-Schwarz Theorem without the assumption $\langle M \rangle_\infty = \infty$ (cf. Revuz and Yor [RY99], Theorem V.1.7). The way to deal with this is to enlarge the space.

But even if we assume the space to be large enough, there might still be problems. Namely assume that the filtration $(\mathcal{F}_t)$ is complete with respect to P and that $E_P(Z_0) = 1$. Then Q would have to be absolutely continuous with respect to P on $\mathcal{F}_0$. But $\mathcal{F}_0$ contains all P-null sets. Thus Q has to be absolutely continuous with respect to P. But then $Z_t = E_P(Z|\mathcal{F}_t)$ with $Z = dQ/dP$, i.e. Z has to be a uniformly integrable martingale under P. So for supermartingales Z, the filtration $(\mathcal{F}_t)$ should not be completed. The solution to this problem is to assume that $(\mathcal{F}_t)$ is the right-continuous modification of a standard system.

If Ω is large enough and if $(\mathcal{F}_t)$ is the right-continuous modification of a standard system, then the problem of constructing Q and T has been solved by Yoeurp [Yoe85] with the help of Föllmer's measure. Let us describe that solution in detail.

First we enlarge Ω. Define $\overline{\Omega} := \Omega \times (0, \infty]$ and $\overline{\mathcal{F}} = \mathcal{F} \otimes \mathcal{B}(0, \infty]$. $\mathcal{B}(0, \infty]$ denotes the Borel σ-algebra of $(0, \infty]$. Also define $\overline{P} = P \otimes \delta_\infty$ where δ_∞ is the Dirac measure at ∞. The filtration

$(\overline{\mathcal{F}}_t)$ is defined as

$$\overline{\mathcal{F}}_t = \cap_{s>t} \mathcal{F}_s \otimes \sigma((0,r] : r \leq s).$$

Random variables X on Ω are embedded into $\overline{\Omega}$ by setting

$$\overline{X}(\omega, \zeta) = X(\omega).$$

Definition (cf. [RY99], p. 182)**.** A filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{P})$ is an **enlargement** of $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ if there exists a measurable map $\pi : \tilde{\Omega} \to \Omega$, such that $\pi^{-1}(\mathcal{F}_t) \subseteq \tilde{\mathcal{F}}_t$ and $\tilde{P} \circ \pi^{-1} = P$. In this case, random variables are embedded from $(\Omega, \mathcal{F})$ into $(\tilde{\Omega}, \tilde{\mathcal{F}})$ by setting $\tilde{X}(\tilde{\omega}) = X(\pi(\tilde{\omega}))$.

Note that $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t), \overline{P})$ is an enlargement of $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$: Define $\pi(\omega, \zeta) = \omega$. Then for $A_t \in \mathcal{F}_t$

$$\pi^{-1}(A_t) = A_t \times (0, \infty] \in \overline{\mathcal{F}_t}$$

and therefore $\pi^{-1}(\mathcal{F}_t) \subseteq \overline{\mathcal{F}_t}$. For any set $A \in \mathcal{F}$

$$\overline{P} \circ \pi^{-1}(A) = P \otimes \delta_\infty(A \times (0, \infty]) = P(A).$$

And for a random variable X on $(\Omega, \mathcal{F})$

$$\overline{X}(\omega, \zeta) = X(\omega) = X(\pi(\omega, \zeta)).$$

Now that we enlarged the space, let us remove the second problem in constructing (Q, T). We assume that the filtration $(\mathcal{F}_t)$ is the right-continuous modification of a standard system $(\mathcal{F}_t^0)$, i.e.

1. For every $t \geq 0$, $\mathcal{F}_t^0$ is σ-isomorphic to the Borel σ-algebra of a Polish space. That is, there exists a Polish space $(\mathcal{X}_t, \mathcal{B}_t)$, and a bijective map $\pi : \mathcal{F}_t^0 \to \mathcal{B}_t$, such that π^{-1} preserves countable set operations.
2. If $(t_i)_{i \geq 0}$ is an increasing sequence of positive times, and if $(A_i)_{i \geq 0}$ is a decreasing sequence of atoms of $\mathcal{F}_{t_i}^0$, then $\cap_{i \geq 0} A_i \neq \emptyset$.
3. $\mathcal{F}_t = \cap_{s>t} \mathcal{F}_s^0$.

Path spaces equipped with the canonical filtration are standard systems only if we allow for explosion in finite time, cf. Meyer [Mey72]. Note that if $(\mathcal{F}_t)$ is the right-continuous modification of a standard system $(\mathcal{F}_t^0)$, then $(\overline{\mathcal{F}}_t)$ is the right-continuous modification of the standard system

$$\overline{\mathcal{F}}_t^0 = \mathcal{F}_t^0 \otimes \sigma((0,s] : s \leq t).$$

Now we can proceed to construct $(\overline{Q}, \overline{T})$ on $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t))$. In fact it suffices to construct $\overline{Q}$, because we define

$$\overline{T}(\omega, \zeta) = \zeta.$$

Then $\overline{P}(\overline{T} = \infty) = 1$. Note however that in general we cannot hope to get a unique $\overline{Q}$ on $\overline{\mathcal{F}}$. $\overline{Q}$ has to satisfy for every $t \geq 0$

$$1 = \overline{Q}(\overline{\Omega}) = \overline{Q}(\overline{\Omega} \cap \{t < \overline{T}\}) + \overline{Q}(\overline{\Omega} \cap \{t \geq \overline{T}\}) = E_{\overline{P}}(\overline{Z}_t) + \overline{Q}(t \geq \overline{T}).$$

So if $\overline{Z}$ is not a martingale (i.e. its expectation is not constant), then $\overline{Q}$ and $\overline{P}$ become singular at time $\overline{T}$. Therefore knowing $\overline{P}, \overline{Z}, \overline{T}$, in general we can only uniquely determine $\overline{Q}$ on the σ-field

$$\begin{aligned} \overline{\mathcal{F}}_{\overline{T}-} &= \sigma(\overline{\mathcal{F}}_0, \overline{A}_t \cap \{\overline{T} > t\} : \overline{A}_t \in \overline{\mathcal{F}}_t, t \geq 0) \\ &= \sigma(A_t \times (t, \infty] : A_t \in \mathcal{F}_t, t \geq 0). \end{aligned}$$

For the second equality we refer to [Föl72]. Note that $\overline{\mathcal{F}}_{T-}$ is exactly the predictable sigma-algebra on $\Omega \times (0, \infty]$.

Under the assumption that $(\mathcal{F}_t)$ is the right-continuous modification of a standard system, Föllmer constructs a measure P^Z on $(\overline{\Omega}, \overline{\mathcal{F}}_{T-})$, which satisfies for every $t \geq 0$ and every $\overline{A}_t \in \overline{\mathcal{F}}_t$:

$$P^Z(\overline{A}_t \cap \{\overline{T} > t\}) = \frac{1}{E_{\overline{P}}(\overline{Z}_0)} E_{\overline{P}}(\overline{Z}_t 1_{\overline{A}_t})$$

But this is exactly the relation (5).

Now that we described how to enlarge the space and how to obtain the measure and stopping time corresponding to the given supermartingale Z, we omit the notation $\overline{(\cdot)}$: We assume as given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ with a right-continuous filtration, a positive right-continuous supermartingale Z with $Z_\infty > 0$, a stopping time T, and a measure Q on

$$\mathcal{F}_{T-} = \sigma(\mathcal{F}_0, A_t \cap \{T > t\}, A_t \in \mathcal{F}_t, t > 0)$$

such that (T, Z) is the Kunita-Yoeurp decomposition of Q with respect to P. Note that the assumption $Z_\infty > 0$ P-a.s. guarantees $Q \gg P|_{\mathcal{F}_{T-}}$. Namely let $A_t \in \mathcal{F}_t$. Then

$$P(A_t \cap \{T > t\}) = E_P\left(\frac{Z_\infty}{Z_\infty} 1_{A_t \cap \{T = \infty\}}\right) = E_Q\left(\frac{1}{Z_\infty} 1_{A_t \cap \{T = \infty\}}\right) \leq E_Q\left(\frac{1_{\{Z_\infty > 0\}}}{Z_\infty} 1_{A_t \cap \{T > t\}}\right).$$

By the monotone class theorem, this inequality extends to arbitrary $A \in \mathcal{F}_{T-}$.

We also assume that Z is the supermartingale density for a given d-dimensional right-continuous adapted process S. There is one major difficulty in determining whether or not S is a local martingale under Q: Q is only defined on the filtration $\mathcal{F}_{T-}$. That is, we only know the behavior of S under Q up to time $T-$.

Definition. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space, and let M be a $(\mathcal{F}_t)$-adapted process. Let T be a $(\mathcal{F}_t)$-stopping time. Assume that there exists an increasing sequence of stopping times $T_n \uparrow T$ P-a.s., such that for every n, $M^{T_n} = M_{T_n \wedge \cdot}$ is a uniformly integrable P-martingale. Then we say that M is a **P-local martingale until time T**.

Define for a given process X the process X^{T-} stopped at $T-$:

$$X_t^{T-} = X_t 1_{\{t < T\}} + \lim_{s \uparrow t} X_s 1_{\{t \geq T\}}.$$

Yoeurp obtains the following generalized Girsanov theorem for processes stopped at time $T-$.

Theorem 4.1 (Theorem 8 of [Yoe85])**.** *Let P and Q be two probability measures that are not singular on $\mathcal{F}_0$. Let (Z, T) be the Kunita-Yoeurp decomposition of Q with respect to P. Let M be a P-local martingale. Assume that the oblique bracket $\langle M, Z\rangle$ exists under P. Then*

$$M^{T-} - \frac{1}{Z_-} \cdot \langle M, Z\rangle^{T-}$$

is a Q-local martingale until time T.

If S is predictable, knowing Q on $\mathcal{F}_{T-}$ determines the behavior of S under Q until time T (cf. [JS03], Proposition I.2.4). Therefore we can determine whether or not a predictable S is a local martingale under Q up to time T . Otherwise, things get more complicated, because in general $T-$ is not a stopping time. Therefore we first treat the predictable case.

4.2. **The predictable case.** We still assume $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, Z, S, T, and Q to be as described above. In addition, we assume that S is *predictable*. We examine the structure of S and Z closer. This allows us to apply Yoeurp's generalized Girsanov theorem to deduce that under the dominating measure associated to Z, S is a local martingale until time T.

First we observe that S is P-a.s. locally bounded. Namely, let for $C \geq 0$: $R = \inf\{t \geq 0 : |S_t| \geq C\}$. Since S is right-continuous and adapted, this is a stopping time. Then this is a predictable stopping time, so it is announced by a sequence of stopping times (R_n), i.e. $R_n < R$ P-a.s. on $\{R > 0\}$ and $\lim_{n\to\infty} R_n = R$ P-a.s. (cf. [JS03], I.2.16). This allows us to construct a localizing sequence (U_n) for S, such that the stopped process S^{U_n} is P-a.s.

bounded. By Corollary 3.11, this implies that S is a semimartingale. But then S is even a special semimartingale (cf. [JS03], I.4.23 (iv)), i.e. there exists a unique decomposition

$$S = S_0 + M + A \tag{6}$$

with a local martingale M with $M_0 = 0$, and with a predictable process of finite variation A with $A_0 = 0$. This implies that

$$M = S - S_0 - A$$

is predictable. But any predictable right-continuous local martingale is continuous (cf. [JS03], I.2.31). Therefore S is of the form (6) with M continuous and A predictable.

Let us examine the structure that S and Z must have in this situation more closely. It turns out that S must satisfy the *structure condition* as described in Schweizer [Sch95]. Note however that this result is not a special case of [Sch95], because we do not assume that A is continuous, we do not suppose that Z is locally square integrable, and we do not suppose that ZS is a local martingale. Nonetheless our result is of course strongly related to [Sch95].

Definition. Let

$$S = S_0 + M + A$$

be a d-dimensional special semimartingale with locally square-integrable M. Define

$$B_t = \sum_{i=1}^{d} \langle M^i \rangle_t \qquad \text{and for } 1 \le i, j \le d: \qquad \sigma_t^{ij} = \frac{d\langle M^i, M^j \rangle_t}{dB_t}.$$

(σ exists because of the Kunita-Watanabe inequality). S satisfies the **structure condition** if for every $1 \le i \le d$:

$$dA^i \ll d\langle M^i \rangle \qquad \text{with } \alpha_t^i = \frac{dA_t^i}{d\langle M^i \rangle_t} \text{ predictable}$$

and if there exists a predictable process $\lambda_t = (\lambda_t^1, \ldots, \lambda_t^d)^* \in L^2_{\text{loc}}(M)$ (x^* denotes the transpose of x), such that

$$(\sigma_t \lambda_t)_i = \alpha_t^i \sigma_t^{ii}, \qquad i = 1, \ldots, d.$$

In this case λ might not be uniquely determined, but the stochastic integral $\int \lambda dM$ does not depend on the choice of λ (cf. [Sch95]).

Proposition 4.2. *Suppose that Z is a supermartingale density for the d-dimensional predictable right-continuous process S. Then S is P-a.s. continuous, it satisfies the structure condition, and*

$$dZ_t = Z_{t-}(-\lambda_t^* dM_t + dL_t - dC_t) \tag{7}$$

where λ is as in the definition of the structure condition, L is a local martingale strongly orthogonal to M (i.e. LM is a local martingale), C is increasing, and $\Delta(L - C) > -1$.

Conversely, if a predictable S satisfies the structure condition, and if Z is defined by (7), then Z is a supermartingale density for S.

In particular, for predictable S, the structure condition is equivalent to (NA1).

Proof. Let

$$Z = Z_0 + N - C$$

be the Doob-Meyer decomposition of the supermartingale Z, i.e. N is a local martingale, and C is increasing and predictable. The semimartingale decomposition of S is given by

$$S = S_0 + M + A$$

with a continuous local martingale M and a predictable A of finite variation.

Because M is continuous, there exists a predictable process $\beta \in L^2_{\mathrm{loc}}(M)$, such that

$$N = \beta \cdot M + L$$

where L is a local martingale which is strongly orthogonal to all components of M (cf. [JS03], Theorem III.4.11). Let H be a 1-admissible strategy. Let us apply the integration by parts formula to $Z(1 + (H \cdot S))$:

$$\begin{aligned} d[Z(1 + (H \cdot S))] &= (1 + (H \cdot S)_-)dZ + Z_- d(H \cdot S) + d[(H \cdot S), Z] \\ &= (1 + (H \cdot S)_-)(\beta dM + dL - dC) + Z_- H(dM + dA) \\ &\quad + d[(H \cdot M) + (H \cdot A), (\beta \cdot M) + L - C] \\ &= (1 + (H \cdot S)_-)(\beta dM + dL) + Z_- H dM \\ &\quad + d[(H \cdot M), L] - d[(H \cdot M), C] + d[(H \cdot A), (\beta \cdot M) + L] \\ &\quad - (1 + (H \cdot S)_-)dC + Z_- H dA + d[(H \cdot M), (\beta \cdot M)] - d[(H \cdot A), C]. \end{aligned}$$

All the terms in the first two lines are local martingales. For the quadratic covariation terms, this is true because L is strongly orthogonal to M, and because A and C are predictable (cf. [JS03], Proposition I.4.49, c)). Let us write $X \sim Y$, if $X - Y$ is a local martingale. Then we obtain from Proposition I.4.49 a) of [JS03]:

$$d[Z(1 + (H \cdot S))] \sim -(1 + (H \cdot S)_-)dC + Z_- H dA + d[(H \cdot M), (\beta \cdot M)] - \Delta(H \cdot A)dC.$$

Because M is continuous, $\Delta(H \cdot S) = H\Delta S = H\Delta A = \Delta(H \cdot A)$, so that

$$\begin{aligned} d[Z(1 + (H \cdot S))] &\sim -(1 + (H \cdot S))dC + Z_- H dA + d[(H \cdot M), (\beta \cdot M)] \\ &\sim -(1 + (H \cdot S))dC + Z_- H dA + d\langle (H \cdot M), (\beta \cdot M)\rangle \\ &= -(1 + (H \cdot S))dC + \sum_{i=1}^d H^i \left(Z_- dA^i + \sum_{j=1}^d \sigma^{ij} \beta^j dB \right) \end{aligned}$$

where the last step follows from Theorem III.4.5 of [JS03] (B and σ are as in the definition of the structure condition). Assume that for some i, the term in the big brackets is not identically zero. We show that then we can construct a 1-admissible H, such that the part of finite variation in the Doob-Meyer decomposition of the supermartingale $(1 + H \cdot S)Z$ is locally increasing with positive probability. This is a contradiction.

First observe that all the A^i must be continuous, because under (NA1), necessarily $dA^i \ll d\langle M^i \rangle$. This is a well known fact, cf. e.g. Ankirchner's thesis [Ank05], Lemma 9.1.2 (otherwise one could choose a strategy H^i which satisfies $H^i \cdot M^i \equiv 0$, but for which $H^i \cdot A^i$ is increasing; this would contradict $\mathcal{K}_1$ being bounded in probability).

Now assume that there exists i for which the P-a.s. continuous process

$$D^i_t = \int_0^t \left(Z_{s-} dA^i_s + \sum_{j=1}^d \sigma^{ij}_s \beta^j_s dB_s \right)$$

is not P-evanescent. By the predictable Radon-Nikodym theorem of Delbaen and Schachermayer (cf. [DS95b], Theorem 2.1 b)), there exists a predictable γ^i with values in $\{-1, 1\}$, such that

$$\int_0^\cdot \gamma^i_s dD^i_s = V^i$$

where V^i denotes the variation of D^i. In particular, $\gamma^i \in L^2_{\mathrm{loc}}(M^i)$. Define the admissible (S is continuous!) strategy

$$H^{i,n}_t = n(1 + (H^{i,n} \cdot S^i)_{t-})\gamma^i_t = n(1 + (H^{i,n} \cdot S^i)_t)\gamma^i_t, \qquad H^{j,n} \equiv 0, j \neq i.$$

Then we have

$$\begin{aligned} d[Z(1+(H^n\cdot S))] &\sim -(1+(H^{i,n}\cdot S^i))dC^i + n(1+(H^{i,n}\cdot S^i))\gamma^i dD^i \\ &= (1+(H^i\cdot S^i))(ndV^i - dC^i). \end{aligned}$$

The first bracket is positive by choice of H^i (since S^i is continuous). V^i and C^i are increasing. By assumption, V^i is not identically zero. Hence we can choose n large enough, such that $nV^i - C^i$ becomes locally increasing with positive probability - a contradiction to $Z(1+H^n\cdot S)$ being a supermartingale.

So now we know that for some predictable α^i,

$$0 = \left(Z_- dA^i + \sum_{j=1}^d \sigma^{ij}\beta^j dB\right) = \left(Z_-\alpha^i d\langle M^i\rangle + \sum_{j=1}^d \sigma^{ij}\beta^j dB\right) = \left(Z_-\alpha^i\sigma^{ii} + (\sigma\beta)^i\right) dB$$

so that

$$Z_-\alpha^i\sigma^{ii} = (\sigma\beta)^i \quad dB(\omega)P(d\omega) - \text{a.e..}$$

If we change α and β on $dB(\omega)P(d\omega)$-null sets, then this does not change S and Z. Therefore we can assume that the equality holds identically. Because $Z_- > 0$, we can define $\lambda^i = \beta^i/Z_-$, and obtain

$$\alpha^i\sigma^{ii} = (\sigma\lambda)^i.$$

It remains to verify $\lambda \in L^2_{\text{loc}}(M)$. But this is obvious, because $\beta \in L^2_{\text{loc}}(M)$:

$$\infty > \sum_{i,j=1}^d \int_0^t \beta^i_s\sigma^{ij}_s\beta^j_s dB_s = \sum_{i,j=1}^d \int_0^t Z^2_{s-}\lambda^i_s\sigma^{ij}_s\lambda^j_s dB_s \geq \left\{\inf_{0\leq s\leq t} Z^2_{s-}\right\} \sum_{i,j=1}^d \int_0^t \lambda^i_s\sigma^{ij}_s\lambda^j_s dB_s$$

and the infimum is strictly positive, because Z is a positive supermartingale.

For now we obtained Z of the form

$$dZ = -\lambda Z_- dM + dL - dC.$$

The statement of the proposition follows by defining $d\tilde{L} = 1/Z_- dL$ and similarly for $\tilde{C}$. □

Corollary 4.3. *ZS^i is a local supermartingale if and only if $S^i \geq 0$ on the support of the measure dC. If $S^i \geq 0$ identically, ZS^i is an actual supermartingale, since positive local supermartingales are supermartingales by Fatou's lemma.*

ZS^i is a local martingale if and only if $S^i = 0$ on the support of the measure dC.

Proof. We just apply integration by parts to ZS^i:

$$\begin{aligned} d(ZS^i) &= Z_- dS^i + S^i_- dZ + d[S^i, Z] \\ &\sim Z_-\alpha^i\sigma^{ii}dB - Z_- S^i dC + d\langle S^i, Z\rangle \\ &= Z_-\alpha^i\sigma^{ii}dB - Z_- S^i dC + \sum_{j=1}^d Z_-\sigma^{ij}\lambda^j dB \\ &= -Z_- S^i dC. \end{aligned}$$

This is the sum of a local martingale and a process of finite variation. It is therefore a local supermartingale if and only if the process of finite variation is decreasing. This is the case if and only if S^i is nonnegative whenever C increases.

Since $S^i dC$ is predictable and of finite variation, it is only a local martingale if it vanishes. □

We call a process X a **maximal element** of a family of processes $\mathcal{X}$, if there exists no $\tilde{X} \in \mathcal{X}$, such that

$$P(\tilde{X}_t \geq X_t \text{ for all } t \geq 0) = 1 \quad \text{and} \quad P(\text{there exists } t \geq 0 \text{ such that } \tilde{X}_t > X_t) > 0.$$

Corollary 4.4. *If Z is maximal among the supermartingale densities with a given initial value, then it is a local martingale.*

Proof. Since Z is a supermartingale density, it is of the form

$$dZ = Z_-(-\lambda dM + dL - dC)$$

for some local martingale L that is orthogonal to M and for some decreasing predictable C. If C was not 0, then we could define

$$d\tilde{Z} = \tilde{Z}_-(\lambda dM + dL), \qquad \tilde{Z}_0 = Z_0$$

to obtain a supermartingale density with the same initial value, such that $\tilde{Z}$ is strictly larger than Z with positive probability. □

Corollary 4.5. *Let Z be a supermartingale density for S, and let Q be the dominating measure associated to Z. Let T be the associated stopping time, i.e. $(Z/E_P(Z_0), T)$ is the Kunita-Yoeurp decomposition of Q with respect to P. Then S is a Q-local martingale until time T.*

Proof. It suffices to show that for every $i = 1, \ldots, d$, $(S^i)^T$ is a local martingale until time T. Since S is predictable and admits a supermartingale density, it must be continuous. Therefore $(S^i)^T = (S^i)^{T-}$. Let

$$S^i = M^i + \alpha^i \cdot \langle M^i \rangle$$

be the semimartingale decomposition of S^i under P (recall that by Proposition 4.2, S satisfies the structure condition). We can apply Theorem 4.1 and Proposition 4.2 to obtain that under Q

$$\begin{aligned}
(M^i)^{T-} - \frac{1}{Z_-} \cdot \langle M^i, Z \rangle^{T-} &= (M^i)^T - \frac{1}{Z_-} \cdot \left(-Z_- \sum_{j=1}^d \sigma^{ij} \lambda^j \right) dB^T \\
&= (M^i)^T + \alpha^i \sigma^{ii} dB^T \\
&= (M^i)^T + \alpha^i d\langle M^i \rangle^T \\
&= (S^i)^T
\end{aligned}$$

is a local martingale until time T. □

Corollary 4.6 ("Predictable Weak Fundamental Theorem of Asset Pricing")**.** *Let $(\mathcal{F}_t)$ be the right-continuous modification of a standard system. Let S be a predictable right-continuous stochastic process. Then S satisfies (NA1) if and and only there exists an enlarged probability space $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t), \overline{P})$ and a dominating measure $\overline{Q} \gg \overline{P}$ with Kunita-Yoeurp decomposition $(\overline{Z}, \overline{T})$ with respect to $\overline{P}$, such that $\overline{S}$ is a $\overline{Q}$-local martingale until time $\overline{T}$.*

Proof. We just showed the existence of $\overline{Q}$ assuming (NA1). Conversely, let us assume that $\overline{Q}$ exists. Then $\overline{S}$ is a predictable right-continuous $\overline{Q}$-local martingale until time $\overline{T}$. Therefore it is $\overline{Q}$-a.s. continuous until time $\overline{T}$. Let H be a P 1-admissible strategy. Since $\overline{Q}$ and $\overline{P}$ are equivalent until time $\overline{T}-$, this implies $(\overline{H} \cdot \overline{S})^{\overline{T}} = (\overline{H} \cdot \overline{S})^{\overline{T}-} \geq -1$ $\overline{Q}$-a.s. Now we can repeat the arguments from the introduction, to obtain that

$$\overline{Z}_t = \frac{1}{\overline{\gamma}_t} 1_{\{t < \overline{T}\}} \qquad \text{where} \qquad \overline{\gamma}_t = \frac{d\overline{P}}{d\overline{Q}}\bigg|_{\overline{\mathcal{F}}_t}$$

is a supermartingale density for $\overline{S}$. This implies that $\overline{S}$ satisfies (NA1) under $\overline{P}$. Since $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t), \overline{P})$ is an enlargement of $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, this implies that S satisfies (NA1) under P. □

4.3. **The optional case.** Let us start the treatment of the nonpredictable case with two negative results.

Example. (1) If S is optional and if Q is a dominating local martingale measure for S, then S does not need to satisfy (NA1). Namely, define

$$S_t = e^t 1_{\{t<T\}}, t \in [0,1]$$

where under Q, T is exponentially distributed with parameter 1. It is not hard to see that this is a martingale in its natural filtration. One way to verify this is to regard S as the stochastic exponential of the martingale $-N_t + t$, where N is a standard Poisson process. Since time is finite, this means that S is a uniformly integrable martingale. So the measure $dP = S_1 dQ$ is absolutely continuous with respect to Q. Let us examine the distribution of S_t under P:

$$E_P(f(S_t)) = E_Q(f(e^t 1_{\{t<T\}})e^t 1_{\{t<T\}}) = f(e^t)e^t Q(t < T) = f(e^t).$$

The only process with margins $\text{law}(S_t) = e^t$ is the deterministic process e^t. So clearly S does not satisfy (NA1) under P, although Q is a dominating martingale measure for S. Note that S is not predictable, because T is totally inaccessible under Q and $(\mathcal{F}_t)$.

(2) If S is optional and satisfies (NA1), and if $Q \gg P$ is a dominating measure with Kunita-Yoeurp decomposition (Z, T) such that T is not accessible under Q, then there exists an optional process $\tilde{S}$, P-indistinguishable of S, such that $\tilde{S}$ is not a Q-local martingale until time T:

Assume that (T^n) is a localizing sequence for S under Q, such that $T_n \le n$. Since T is not accessible, this means that there exists $n \in \mathbb{N}$ for which $Q(T_n = T) > 0$. In particular, on the set $A = \{T_n = T\}$ we have $T < \infty$. Let $x \in \mathbb{R}^d$ and define

$$\tilde{S}_t = S_t + x 1_{t \ge T}.$$

Since $T = \infty$ P-a.s., $\tilde{S}$ is P-indistinguishable of S. We have

$$E_Q(S_0) = E_Q(\tilde{S}_t^{T_n}) = E_Q(S_t^{T_n}) + xQ(t \wedge T_n \ge T).$$

Choosing t large enough such that $Q(t \wedge T_n \ge T)$ and letting x vary through $\mathbb{R}^d$, we obtain a contradiction.

So in the non-predictable case, in general it is not a well-posed question whether Q determines a dominating local martingale measure. However for a *given* version S, it should be possible to extend Q from $\mathcal{F}_{T-}$ to $\mathcal{F}_T$ in a way that makes S a Q-local martingale until time T. But since this extension of Q is quite arbitrary (it is not uniquely determined by Z), and since it is not stable with respect to taking a P-indistinguishable version $\tilde{S}$ of S, we do not take this approach here.

We rather note that the counterexamples only worked because T was not accessible under Q. If T is accessible, then we can stop S before T, thereby not worrying about what happens to S at time T.

The following is a slight extension of [Föl72], Proposition (2.1).

Proposition 4.7. *Let (Z, T) be the Kunita-Yoeurp decomposition of Q with respect to P. Then T is accessible for Q if and only if Z is a P-local martingale.*

Proof. If Z is a P-local martingale, and if T_n is a localizing sequence for T, then

$$1 = E_P(Z_{T_n \wedge n}) = Q(T_n \wedge n < T)$$

(cf. [Yoe85], Proposition 4). At the same time $T_n \wedge n \uparrow \infty$ P-a.s., so by Corollary 5 of [Yoe85], $T_n \wedge n \uparrow T$ Q-a.s., which proves that T is announced by the sequence $(T_n \wedge n)$ under Q.

Conversely, assume T is announced by T_n under Q. Then

$$1 = Q(T_n \wedge n \wedge t < T) = E_P(Z_{T_n \wedge n \wedge t}) = E_P(Z_t^{T_n \wedge n}).$$

Because the stopped process $Z^{T_n\wedge n}$ remains a P-supermartingale by the optional sampling theorem, it is a supermartingale with constant expectation. Thus it is a martingale, i.e. Z is a local martingale (we apply once again Corollary 5 of [Yoe85] to obtain $T_n \wedge n \uparrow \infty$ P-a.s.). □

So we should look for supermartingale densities that are local martingales. This problem was solved by Kardaras [Kar09]. Kardaras defines the notion of an equivalent local martingale deflator. To go together with our notion of supermartingale densities, we call them local martingale densities:

Definition. Let S be a right-continuous stochastic process. A **local martingale density** is a local martingale Z, such that for every 1-admissible strategy H,

$$(1 + (H \cdot S))Z$$

is a local martingale.

Theorem 1.1 of [Kar09] states that the existence of local martingale densities is equivalent to (NA1). The proof is in the spirit of the article [KK07]. We leave it to future research to give an independent proof in the spirit of Theorem 1.1.

Proposition 4.8. *Let S be right-continuous, locally bounded and assume S satisfies (NA1). Let Z be a local martingale density for S. If $(T, Z/E_P(Z_0))$ is the Kunita-Yoeurp decomposition of the associated dominating measure Q, then S is a Q-local martingale until time T.*

Proof. It suffices to argue for each component of S separately, so we assume S to be one-dimensional. Let (T_n) be a localizing sequence for Z under P that announces T under Q. Let (R^n) be a localizing sequence such that $|S^{R_n}|$ is bounded by C_n. By replacing R_n with $R_n \wedge n$, we can assume that $T_n \wedge R_n$ is finite. Let $t, s \geq 0$ and $A_t \in \mathcal{F}_t$. Then by the monotone class theorem combined with Proposition 4 of [Yoe85],

$$\begin{aligned}
&E_Q\left[\left(2C_n + 1_{A_t}(S^{T_n\wedge R_n}_{t+s} - S^{T_n\wedge R_n}_t)\right)\right]\\
&= E_Q\left[\left(2C_n + 1_{A_t}(S^{T_n\wedge R_n}_{t+s} - S^{T_n\wedge R_n}_t)\right) 1_{\{T_n\wedge R_n<T\}}\right]\\
&= \frac{1}{E_P(Z_0)} E_P\left[\left(2C_n + 1_{A_t}(S^{T_n\wedge R_n}_{t+s} - S^{T_n\wedge R_n}_t)\right) Z_{T_n\wedge R_n\wedge(t+s)} 1_{\{T_n\wedge R_n<\infty\}}\right]\\
&= \frac{1}{E_P(Z_0)} E_P\left[\left(2C_n + 1_{A_t}(S^{T_n\wedge R_n}_{t+s} - S^{T_n\wedge R_n}_t)\right) Z^{T_n\wedge R_n}_{t+s}\right] \leq \frac{2C_n}{E_P(Z_0)}.
\end{aligned}$$

At the same time $E_P(2C_n Z^{T_n\wedge R_n}_{t+s}) = 2C_n$, so that

$$E_Q\left[1_{A_t}(S^{T_n\wedge R_n}_{t+s} - S^{T_n\wedge R_n}_t)\right] \leq 0.$$

The same holds if we replace 1_{A_t} with -1_{A_t}, which proves that S is a Q-local martingale until time T: $(R_n \wedge T_n)$ converges to T Q-a.s., because it diverges to ∞ P-a.s.. □

Corollary 4.9 ("Optional Fundamental Theorem of Asset Pricing")**.** *Let $(\mathcal{F}_t)$ be the right-continuous modification of a standard system. Let S be a locally bounded, right-continuous and adapted stochastic process. Then S satisfies (NA1) if and and only there exists an enlarged probability space $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t), \overline{P})$ and a dominating measure $\overline{Q} \gg \overline{P}$ with Kunita-Yoeurp decomposition $(\overline{Z}, \overline{T})$ with respect to $\overline{P}$, such that $\overline{S}$ is a $\overline{Q}$-local martingale until time $\overline{T}$, and such that $\overline{T}$ is accessible under $\overline{Q}$.*

Proof. We just proved the existence of $\overline{Q}$ assuming (NA1). So now we assume that $\overline{Q}$ exists. Let H be a P 1-admissible strategy. Let $(\overline{T}_n)$ be a sequence of stopping times announcing $\overline{T}$ under $\overline{Q}$. Since $\overline{Q}$ is equivalent to $\overline{P}$ on $\overline{\mathcal{F}}_{\overline{T}_-}$, and since $\overline{T}_n < \overline{T}$ $\overline{Q}$-a.s., $(\overline{H} \cdot \overline{S})^{\overline{T}_n} \geq -1$ $\overline{Q}$-a.s. So we can repeat the arguments from the introduction, to obtain that for every n,

$$(1 + (\overline{H} \cdot \overline{S})^{\overline{T}_n})\overline{Z}^{\overline{T}_n}$$

is a $\overline{P}$-supermartingale. This means that

$$(1 + (\overline{H} \cdot \overline{S}))\overline{Z}$$

is a local $\overline{P}$-supermartingale. Since it is $\overline{P}$-a.s. nonnegative, it is an actual supermartingale by Fatou's lemma. This implies that $\overline{S}$ satisfies (NA1) under $\overline{P}$. Since $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t), \overline{P})$ is an enlargement of $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, this implies that S satisfies (NA1) under P. □

5. Relation to Filtration Enlargements

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space, and let $(\mathcal{G}_t)$ be a filtration enlargement of $(\mathcal{F}_t)$, i.e. for every $t \geq 0$: $\mathcal{F}_t \subseteq \mathcal{G}_t$. Recall that Hypothèse (H') is said to be satisfied if all $(\mathcal{F}_t)$-semimartingales are $(\mathcal{G}_t)$-semimartingales. It turns out that this question is closely related to the existence of supermartingale densities.

First observe that if we want to verify whether Hypothèse (H') is satisfied, it suffices to show that all bounded $(\mathcal{F}_t)$-martingales stay $(\mathcal{G}_t)$-semimartingales: Clearly we only need to verify that $(\mathcal{F}_t)$-local martingales stay $(\mathcal{G}_t)$-local martingales. But every local martingale M can be decomposed as

$$M = M' + M''$$

where $M'_0 = 0$ and M' has bounded jumps, and M'' is a process of finite variation (cf. [JS03], Theorem I.4.17). Then M'' trivially is a $(\mathcal{G}_t)$-semimartingale. But M' is locally bounded. Since local $(\mathcal{G}_t)$-semimartingales are $(\mathcal{G}_t)$-semimartingales, it therefore suffices to verify that bounded $(\mathcal{F}_t)$-martingales stay $(\mathcal{G}_t)$-semimartingales.

Let us now give the classical formulation of Jacod's criterion: Let X be a random variable taking its values in a Lusin space $\mathcal{X}$, and define the initial enlargement

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(X).$$

Since X takes its values in a Lusin space, the regular conditional distributions

$$P_t(\omega, dx) = P(X \in dx | \mathcal{F}_t)(\omega)$$

exist. Denote by P_X the distribution of X. Jacod's criterion states that Hypothèse (H') is satisfied as long as for every $t \geq 0$

$$P_t(\omega, dx) \ll P_X(dx) \text{ a.s..}$$

We have the following result:

Proposition 5.1. *Assume Jacod's condition is satisfied. Define for a given bounded $(\mathcal{F}_t)$-martingale $\mathcal{K}_1$ by using 1-admissible simple $(\mathcal{G}_t)$-predictable strategies of the form*

$$H_t(\omega) = \sum_{k=1}^{n} F^k(\omega, X(\omega)) 1_{(t_k, t_{k+1}]}(t)$$

for bounded $F^k \in \mathcal{F}_{t_k} \otimes \sigma(X)$.

Then there exists one single *supermartingale density Z* for every *bounded $(\mathcal{F}_t)$-martingale S.*

In particular, Hypothése (H') is satisfied.

Proof. (1) Define for every $t \geq 0$

$$Y_t(\omega, x) = \frac{dP_t(\omega, \cdot)}{dP_X}(x).$$

Let $t, s \geq 0$ and let us show that $\{(\omega, x) : Y_t(\omega, x) = 0\} \subseteq \{(\omega, x) : Y_{t+s}(\omega, x) = 0\}$ $P \otimes P_X$-a.s. Note that $Y_{t+s} \geq 0$ $P \otimes P_X$-a.s. Further we have by Fubini's Theorem and the tower property

$$\begin{aligned}\int_{\Omega\times\mathcal{X}} 1_{\{Y_t(\omega,x)=0\}} Y_{t+s}(\omega,x) P\otimes P_X(d\omega,dx) &= \int_\Omega\int_{\mathcal{X}} 1_{\{Y_t(\omega,x)=0\}} P_{t+s}(\omega,dx)P(d\omega)\\ &= \int_\Omega\int_{\mathcal{X}} 1_{\{Y_t(\omega,X(\omega))=0\}} P(d\omega)\\ &= \int_\Omega\int_{\mathcal{X}} 1_{\{Y_t(\omega,x)=0\}} P_t(\omega,dx)P(d\omega)\\ &= \int_\Omega 0 P(d\omega)\\ &= 0\end{aligned}$$

since of course $Y_t(\omega,x) > 0$ $P_t(\omega,\cdot)$-a.s.

(2) Define

$$\tilde{Z}_t(\omega,x) = \frac{1}{Y_t(\omega,x)} 1_{\{Y_t(\omega,x)>0\}}.$$

Then $Z_t(\omega) = \tilde{Z}_t(\omega,X(\omega))$ will be the requested supermartingale density: Let S be a bounded $(\mathcal{F}_t)$-martingale, and let H be a simple 1-admissible strategy for S (under $(\mathcal{G}_t)$). Then H is of the form

$$H_t(\omega) = \sum_{k=1}^n F^k(\omega,X(\omega)) 1_{(t_k,t_{k+1}]}(t) =: \sum_{k=1}^n H_t^k(\omega).$$

We assume that the F^k are bounded for *every* $x\in\mathcal{X}$, such that for every fixed $t\geq 0$ a.s. for *every* x

$$\sum_{k=1}^n F^k(\omega,x)(S_{t_{k+1}\wedge t}(\omega) - S_{t_k\wedge t}(\omega)) \geq -1.$$

An inspection of the proof of Corollary 3.11 shows that this will be sufficient to obtain the semimartingale property of S under $(\mathcal{G}_t)$. Let $t,s\geq 0$, let $A_t\in\mathcal{F}_t$ and let B be a borel subset of $\mathcal{X}$. Then by the tower property

$$\begin{aligned}&E\left(1_{A_t}1_B(X)(1+(H\cdot S)_{t+s})Z_{t+s}\right)\\ &= \int_\Omega 1_{A_t}(\omega)\int_{\mathcal{X}} 1_B(x)\left(1+(H\cdot S)_{t+s}(\omega,x)\right)\right) \tilde{Z}_{t+s}(\omega,x)P_{t+s}(\omega,dx)P(d\omega)\\ &= \int_\Omega 1_{A_t}(\omega)\int_{\mathcal{X}} 1_B(x)\left(1+(H\cdot S)_{t+s}(\omega,x)\right)\right) \frac{Y_{t+s}(\omega,x)}{Y_{t+s}(\omega,x)} 1_{\{Y_{t+s}(\omega,x)>0\}} P_X(dx)P(d\omega)\\ &= \int_\Omega 1_{A_t}(\omega)\int_{\mathcal{X}} 1_B(x)\left(1+(H\cdot S)_{t+s}(\omega,x)\right)\right) 1_{\{Y_{t+s}(\omega,x)>0\}} P_X(dx)P(d\omega)\\ &\leq \int_\Omega 1_{A_t}(\omega)\int_{\mathcal{X}} 1_B(x)\left(1+(H\cdot S)_{t+s}(\omega,x)\right)\right) 1_{\{Y_t(\omega,x)>0\}} P_X(dx)P(d\omega).\end{aligned}$$

In the last step we used that $1_B(x)\left(1+(H\cdot S)_{t+s}(\omega,x)\right)\right)$ is $P_X\otimes P$-a.s. nonnegative, and that $\{Y_t(\omega,x)>0\}\supseteq\{Y_{t+s}(\omega,x)>0\}$ under $P_X\otimes P$. Using the martingale property of $(H \cdot S)(\cdot, x)$ with respect to $(\mathcal{F}_t)$, we obtain

$$\begin{aligned}
&\int_\Omega 1_{A_t}(\omega) \int_{\mathcal{X}} 1_B(x) \left(1 + (H \cdot S)_{t+s}(\omega, x)\right)\right) 1_{\{Y_t(\omega,x)>0\}} P_X(dx) P(d\omega) \\
&\quad = \int_{\mathcal{X}} 1_B(x) \int_\Omega 1_{A_t}(\omega) \left(1 + (H \cdot S)_{t+s}(\omega, x)\right)\right) 1_{\{Y_t(\omega,x)>0\}} P(d\omega) P_X(dx) \\
&\quad = \int_{\mathcal{X}} 1_B(x) \int_\Omega 1_{A_t}(\omega) \left(1 + (H \cdot S)_t(\omega, x)\right)\right) 1_{\{Y_t(\omega,x)>0\}} P(d\omega) P_X(dx) \\
&\quad = \int_\Omega 1_{A_t}(\omega) \int_{\mathcal{X}} 1_B(x) \left(1 + (H \cdot S)_t(\omega, x)\right)\right) \frac{Y_t(\omega, x)}{Y_t(\omega, x)} 1_{\{Y_t(\omega,x)>0\}} P_X(dx) P(d\omega) \\
&\quad = \int_\Omega 1_{A_t}(\omega) \int_{\mathcal{X}} 1_B(x) \left(1 + (H \cdot S)_t(\omega, x)\right)\right) \tilde{Z}_t(\omega, x) P_t(\omega, dx) P(d\omega) \\
&\quad = E\left[1_{A_t} 1_B(X)(1 + (H \cdot S)_t) Z_t\right]
\end{aligned}$$

which after an application of the monotone class theorem proves that $(1 + (H \cdot S))Z$ is a $(\mathcal{G}_t)$-supermartingale.

(3) It remains to show that Z is P-a.s. strictly positive. But this is nearly trivial: Clearly it suffices to show that $Y_t(\omega, X(\omega))$ is P-a.s. positive, and we have by the tower property

$$\begin{aligned}
E(1_{\{Y_t(\cdot,X(\cdot))=0\}}) &= \int_\Omega \int_X 1_{\{Y_t(\omega,x)=0\}} P_t(\omega, dx) P(d\omega) \\
&= \int_\Omega \int_X 1_{\{Y_t(\omega,x)=0\}} Y_t(\omega, x) P_X(dx) P(d\omega) \\
&= 0.
\end{aligned}$$

□

It is also possible to prove that Z is a supermartingale density for the stochastic integrals under general 1-admissible $(\mathcal{G}_t)$-predictable strategies.

Conversely, let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space and let $\mathcal{G}_t \supseteq \mathcal{F}_t$ be a filtration enlargement. Assume that $\mathcal{G}_t$ is countably generated for every $t \geq 0$. Then in particular, the regular conditional probabilities

$$P_t(\omega, \cdot)|_{\mathcal{G}_t} = P(\cdot|\mathcal{F}_t)|_{\mathcal{G}_t}(\omega)$$

exist. We say that the generalized Jacod condition is satisfied if

$$P_{t+s}|_{\mathcal{G}_t}(\omega, \cdot) \ll P_t|_{\mathcal{G}_t}(\omega, \cdot) \text{ a.s.}$$

for every $t, s \geq 0$. It is known that neither Jacod's condition, nor the generalized Jacod condition are necessary for Hypothèse (H') to hold. But if we assume something stronger than Hypothèse (H'), then the generalized Jacod condition does become necessary:

Proposition 5.2. *Assume that there exists a strictly positive $(\mathcal{G}_t)$-supermartingale Z such that for every bounded nonnegative $(\mathcal{F}_t)$-martingale M, ZM is a $(\mathcal{G}_t)$-supermartingale. Then the generalized Jacod condition is satisfied.*

Proof. (1) Let $A \in \mathcal{F}$. Then

$$(M_t^A := E_P(1_A|\mathcal{F}_t) : t \geq 0)$$

is a bounded and nonnegative $(\mathcal{F}_t)$ martingale. By our assumption, $M^A Z$ is a $(\mathcal{G}_t)$-supermartingale. Now fix $t, s \geq 0$. Let $A \in \mathcal{F}_{t+s}$ and $B \in \mathcal{G}_t$. Then we have for every

n

$$\begin{aligned} E\left(1_A 1_B \frac{Z_{t+s}}{Z_t} 1_{\{Z_t \geq 1/n\}}\right) &= E\left(\frac{1_B 1_{\{Z_t\geq 1/n\}}}{Z_t} M^A_{t+s} Z_{t+s}\right) \\ &\leq E\left(\frac{1_B 1_{\{Z_t\geq 1/n\}}}{Z_t} M^A_t Z_t\right) \\ &= E(1_A E(1_B 1_{\{Z_t\geq 1/n\}}|\mathcal{F}_t)). \end{aligned}$$

Applying monotone convergence on both sides, we obtain

$$E\left(1_A 1_B \frac{Z_{t+s}}{Z_t}\right) \leq E(1_A E_P(1_B|\mathcal{F}_t)).$$

The same inequality holds if we replace Z_{t+s}/Z_t by a version $\tilde{Z}_{t+s}/\tilde{Z}_t$ that is strictly positive for **every** ω. Since the inequality holds for every $A \in \mathcal{F}_{t+s}$, this implies

$$\int 1_B(\omega') \frac{\tilde{Z}_{t+s}}{\tilde{Z}_t}(\omega') P_{t+s}(\omega, d\omega') \leq P_t(\omega, B) \text{ for a.e. } \omega.$$

This looks promising. The only problem is that the null set outside of which this inequality holds may depend on B.

(2) Now we use the assumption that $\mathcal{G}_t$ is countably generated. This means that we can find an increasing sequence of finite partitions

$$\mathcal{P}^n = \cup_{k=1}^{K_n} B^n_k$$

of Ω such that

$$\mathcal{G}_t = \sigma(\mathcal{P}^n : n \geq 0).$$

Increasing means of course that $\mathcal{P}^n \subseteq \mathcal{P}^{n+1}$.

Since $\cup_{n\geq 0}\sigma(\mathcal{P}^n)$ is countable, we can choose a null set N such that for every $\omega \in \Omega\backslash N$ and $B \in \cup_n \sigma(\mathcal{P}^n)$:

$$\int 1_B(\omega') \frac{\tilde{Z}_{t+s}}{\tilde{Z}_t}(\omega') P_{t+s}(\omega, d\omega') \leq P_t(\omega, B).$$

Note that $B \in \cup_n \sigma(\mathcal{P}^n)$ is stable by finite intersection (it even is an algebra). By the monotone class theorem, this inequality holds for every $B \in \sigma(\mathcal{P}^n : n \geq 0) = \mathcal{G}_t$. Since $\tilde{Z}_{t+s}/\tilde{Z}_t(\omega') > 0$ for every (ω'), the proof is complete.

□

Corollary 5.3. *Let $(\mathcal{F}_t)$ be a filtration for which a continuous local martingale M has the predictable representation theory. Assume that under $(\mathcal{G}_t)$, M is of the form*

$$M_t = \tilde{M}_t + \int_0^t \alpha_s d\langle \tilde{M}\rangle_s$$

for a $(\mathcal{G}_t)$ local martingale $\tilde{M}$ and some α with

$$\int_0^T \alpha_s^2 d\langle \tilde{M}\rangle_s < \infty \text{ a.s. for every } T \geq 0.$$

Then the generalized Jacod condition has to hold. This was previously shown by Imkeller, Pontier and Weisz ([IPW01]*) for the case of initial enlargements and under the stronger assumption*

$$E\left(\int_0^\infty \alpha_s^2 d\langle \tilde{M}\rangle_s\right) < \infty.$$

Appendix A. Convex Compactness

We needed a version of Tychonoff's Theorem for convex compactness. Since we are only interested in a countable product of metric spaces, we only prove the result in this case. Let us recall the following definitions:

Definition. (1) A is a **directed set** if it is partially ordered and such that for every $a, b \in A$ there exists $c \in A$ with $a \leq c$, $b \leq c$.
(2) Let X be a topological space. A **net** in X is a map from a directed set A to X.
(3) A net $\{x_\alpha\}_{\alpha \in A}$ in X **converges** to a point $x \in X$ if for every open neighborhood U of X there exists $\alpha \in A$, such that for every $\alpha' \geq \alpha$: $x_{\alpha'} \in U$.

Example. In the case where $A = \mathbb{N}$, a net in X is just a sequence in X.

Zitkovic [Ž10] introduces the notation $\mathrm{Fin}(A)$, which denotes all non-empty finite subsets of a given set A. If B is a subset of a vector space, $\mathrm{conv}(B)$ denotes convex hull of B. Zitkovic then gives the following definition:

Definition. Let $\{x_\alpha\}_{\alpha \in A}$ be a net in a vector space X. A net $\{y_\beta\}_{\beta \in B}$ is called a **subnet of convex combinations of** $\{x_\alpha\}_{\alpha \in A}$ if there exists a map $D : B \to \mathrm{Fin}(A)$ such that

(1) $y_\beta \in \mathrm{conv}\{x_\alpha : \alpha \in D(\beta)\}$ for every $\beta \in B$, and
(2) for every $\alpha \in A$ there exists $\beta \in B$ such that for every $\alpha' \in \cup_{\beta' \geq \beta} D(\beta')$: $\alpha' \geq \alpha$.

Remark. Let $\{y_\beta\}_{\beta \in B}$ be a subnet of convex combinations of $\{x_\alpha\}_{\alpha \in A}$, and let $\{z_\gamma\}_{\gamma \in C}$ be a subnet of convex combinations of $\{y_\beta\}_{\beta \in B}$. Then $\{z_\gamma\}_{\gamma \in C}$ is a subnet of convex combinations of $\{x_\alpha\}_{\alpha \in A}$.

Proof. Let $D_B : B \to \mathrm{Fin}(A)$ and $D_C : C \to \mathrm{Fin}(B)$ be two maps as described in the definition of a subnet of convex combinations. Define

$$D : C \to \mathrm{Fin}(A), \qquad D(\gamma) = \cup_{\beta \in D_C(\gamma)} D_B(\beta).$$

Then for every $\gamma \in C$:

$$z_\gamma \in \mathrm{conv}\{y_\beta : \beta \in D_C(\gamma)\} \subseteq \mathrm{conv}\{x_\alpha : \alpha \in \cup_{\beta \in D_C(\gamma)} D_B(\beta)\} = \mathrm{conv}\{x_\alpha : \alpha \in D(\gamma)\}.$$

So the first condition of the definition is satisfied. As for the second one, let $\alpha \in A$. Then there exists $\beta \in B$, such that for $\alpha' \in \cup_{\beta' \geq \beta} D_B(\beta')$: $\alpha' \geq \alpha$. For this given β, there exists $\gamma \in C$, such that for $\beta' \in \cup_{\gamma' \geq \gamma} D_C(\gamma')$: $\beta' \geq \beta$. Thus for

$$\alpha' \in \cup_{\gamma' \geq \gamma} D(\gamma') = \cup_{\gamma' \geq \gamma} \cup_{\beta' \in D_C(\gamma')} D_B(\beta') \subseteq \cup_{\beta' \geq \beta} D_B(\beta')$$

we have $\alpha' \geq \alpha$. □

Zitkovic [Ž10] proves

Proposition A.1. *A closed and convex subset C of a topological vector space X is convexly compact if and only if for any net $\{x_\alpha : \alpha \in A\}$ in C there exists a a subnet $\{y_\beta : \beta \in B\}$ of convex combinations, such that $\{y_\beta\}$ converges to some $y \in X$.*

We will use this insight to prove the following weak version of Tychonoff's Theorem for convexly compact sets:

Proposition A.2. *Let $\{X_n : n \in \mathbb{N}\}$ be a* countable *family of convexly compact* metric *spaces. Then*

$$\prod_{n \in \mathbb{N}} X_n$$

is convexly compact in the product topology.

Proof. Let $\{(x_\alpha(n))_{n\in\mathbb{N}} : \alpha \in A\}$ be a net in $\prod_{n\in\mathbb{N}} X_n$. Then in particular,

$$\{x_\alpha(1) : \alpha \in A\}$$

is a net in X_1. Therefore there exists a subnet of convex combinations

$$\{(y^1_\beta(n))_{n\in\mathbb{N}} : \beta \in B_1\}$$

such that $\{y^1_\beta(1) : \beta \in B_1\}$ converges to some $y(1)$. Using the remark above, we can inductively construct for every k a subnet of convex combinations

$$\{(y^k_\beta(n))_{n\in\mathbb{N}} : \beta \in B_k\}$$

of $\{x_\alpha : \alpha \in A\}$, such that for every $l = 1, \dots, k$, $\{y^k_\beta(l) : \beta \in B_k\}$ converges to $y(l) \in X_l$. We denote the corresponding maps from B_k to $\mathrm{Fin}(A)$ by D_k. Now take the directed set

$$\mathbb{N} \times A \text{ with the partial order } (k,\alpha) \le (k',\alpha') \text{ iff } k \le k' \text{ and } \alpha \le \alpha'.$$

Let

$$A(k,\alpha) = \left\{ y^k_\beta : \text{ for every } \alpha' \in D_k(\beta) : \alpha' \ge \alpha \text{ and for } l = 1,\dots,k : d_l(y^k_\beta(l), y(l)) \le \frac{1}{k} \right\}.$$

By our construction of the $(y^k_\beta(n))$, all these sets are non-empty. For every $(k,\alpha) \in \mathbb{N}\times A$ define $\beta(k,\alpha)$ to be such that $y^k_{\beta(k,\alpha)}$ is in $A(k,\alpha)$ - note that in general this is only possible if we assume the Axiom of Choice! Set

$$z_{(k,\alpha)} = y^k_{\beta(k,\alpha)} \text{ and } D((k,\alpha)) = D_k(\beta(k,\alpha)).$$

Then by construction, $\{z_{(k,\alpha)} : (k,\alpha) \in \mathbb{N}\times A\}$ is a subnet of convex combinations of $\{x_\alpha : \alpha \in A\}$, which converges to some

$$(y(n))_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$$

in the product topology. Therefore $\prod_n X_n$ is convexly compact in the product topology. □

Remark. It should be possible to prove a version of this result for general products of convexly compact spaces.

If it is possible to reduce Zitkovic's criterion to sequences rather than nets in the case of a metric space X, then we can also prove this countable metric Tychonoff result without relying on the Axiom of Choice. However it is not immediate that this is possible: The proof that a metric space is sequentially compact if and only if it is compact uses the concept of total boundedness. It is not clear how to translate this concept to the setting of convex compactness.

Appendix B. Minimax Theorem

We used Sion's minimax theorem in the proof of Proposition 3.3. Using the notion of convex compactness, we can in fact prove a version of the minimax theorem that is stronger than all versions we could find in the literature. It is also stronger than the minimax theorem in [Ž10], even though here we do not obtain the existence of a saddle point. However we should point out that the proof here is just a reproduction of Komiya's [Kom88] work. This proof is extremely elegant, and robust enough to be applied without change to the more general version stated here. First, a definition:

Definition. A function $f : D \to \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ is called **quasi-convex** if for every $x, y \in D$:

$$f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\}.$$

Similarly it is called **quasi-concave** if

$$f(\lambda x + (1 - \lambda y)) \ge \min\{f(x), f(y)\}.$$

Theorem B.1. *Let $\mathcal{Y}$ be a convexly compact subset of a topological vector space, and let $\mathcal{X}$ be a convex subset of a topological vector space. Let f be a function on $\mathcal{X} \times \mathcal{Y}$, taking its values in the extended real line $\mathbb{R} \cup \{-\infty\} \cup \{\infty\}$. Assume that*

(1) *$f(y, \cdot)$ is upper semi-continuous and quasi-concave on X for each $y \in \mathcal{Y}$*
(2) *$f(\cdot, x)$ is lower semi-continuous and quasi-convex on Y for each $x \in \mathcal{X}$.*

Then

$$\inf_{y \in \mathcal{Y}} \sup_{x \in \mathcal{X}} f(y, x) = \sup_{x \in \mathcal{X}} \inf_{y \in \mathcal{Y}} f(y, x).$$

Remark. Komiya states this Theorem for real-valued f and convex and compact Y, so our generalization consists in using functions with values in the extended real line, and in replacing convexity and compactness by convex compactness.

Before we give the proof of Theorem B.1, let us make the following simple observation:

Proposition B.2. *Let Y be a convexly compact subset of a topological vector space, and let $F : Y \to \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ be lower semi-continuous and quasi-convex. Then there exists $y_0 \in Y$, such that $F(y_0) \leq F(y)$ for all $y \in Y$.*

Proof. In the case where $F(y) = \infty$ for all y, there is nothing to show. Otherwise define $a := \inf_{y \in Y} F(y)$. There are two possibilities: either $a = -\infty$, or $a \in \mathbb{R}$.

First assume $a = -\infty$. Define for every given $n \in \mathbb{N}$

$$C_n = \{y \in Y : F(y) \leq -n\}.$$

By assumption, $C_n \neq \emptyset$. Also, C_n is closed (since F is lower semi-continuous), and convex (since F is quasi-convex). Since $C_n \subseteq C_m$ for $n \geq m$, every finite intersection of the C_n is nonempty. By convex compactness of Y, also

$$\cap_{n \in \mathbb{N}} C_n \neq \emptyset.$$

But for $y \in \cap_n C_n$, $F(y) \leq -n$ for all n, and therefore $F(y) = -\infty$.

In case $a \in \mathbb{R}$, define

$$C_n = \left\{ y \in Y : F(y) \leq a + \frac{1}{n} \right\}.$$

Then we can proceed as before. □

Proof of Theorem B.1. ($\geq$) is clear. As for ($\leq$), the proof is based on two observations:

(1) Let $-\infty < a < \inf_{y \in \mathcal{Y}} \sup_{x \in \mathcal{X}} f(y, x)$. If the infimum equals $-\infty$, then we cannot find such an a. But in this case there is nothing to show in the first place. Otherwise, define for every $x \in \mathcal{X}$

$$C(x, a) = \{y \in \mathcal{Y} : f(y, x) \leq a\}.$$

Note that this is a closed set: Let $(y_n) \in C(x, a)$ converge to y. Then by lower semi-continuity of $f(\cdot, x)$

$$a \geq \liminf_{n \to \infty} f(y_n, x) \geq f(y, x)$$

so that $y \in C(x, a)$. Also, $C(x, a)$ is convex: Let $y_1, y_2 \in C(x, a)$. Then by quasi-convexity

$$f(\lambda y_1 + (1 - \lambda) y_2, x) \leq \max\{f(y_1, x), f(y_2, x)\} \leq a.$$

By our choice of a, we have

$$\cap_{x \in \mathcal{X}} C(x, a) = \emptyset.$$

This is an empty intersection over a family of closed convex subsets of the convexly compact space $\mathcal{Y}$. Thus there must exist $x_1, \dots, x_n \in X$ such that

$$\cap_{i=1}^n C(x_i, a) = \emptyset.$$

But this means that for any $y \in \mathcal{Y}$ there is $i \in \{1, \dots, n\}$, such that $y \notin C(x_i, a)$, i.e. such that

$$f(y, x_i) > a.$$

In particular,

$$\inf_{y \in \mathcal{Y}} \max_{i=1,\dots,n} f(y, x_i) > a.$$

(2) If $x_1, x_2 \in \mathcal{X}$ are such that

$$-\infty < a < \inf_{y \in \mathcal{Y}} \max\{f(y, x_1), f(y, x_2)\}$$

then there also exists a single $x_0 \in \mathcal{X}$ such that

$$a < \inf_{y \in \mathcal{Y}} f(y, x_0).$$

By induction, this generalizes to $x_1, \dots, x_n$ for every fixed n, which completes the proof in combination with the first observation: Because then there exists x_0 such that

$$\inf_{y \in \mathcal{Y}} f(y, x_0) > a.$$

In particular

$$\sup_{x \in \mathcal{X}} \inf_{y \in \mathcal{Y}} f(y, x) > a.$$

Let us show the statement for $n = 2$ (we do not include do the induction here, it can be found in [Kom88], Lemma 2). Assume that it is not true, i.e. that

$$\inf_{y \in \mathcal{Y}} f(y, x) \leq a$$

for every $x \in \mathcal{X}$. We will work with sets of the form $C(x, b)$, where again

$$C(x, b) = \{y \in Y : f(y, x) \leq b\}.$$

Namely, let

$$a < b < \inf_{y \in \mathcal{Y}} \max\{f(y, x_1), f(y, x_2)\}.$$

By assumption, $C(x, b) \neq \emptyset$ for any $x \in \mathcal{X}$. In fact, even $C(x, a) \neq \emptyset$ for any $x \in \mathcal{X}$: By lower semi-continuity and quasi-convexity of $f(\cdot, x)$ and convex compactness of $\mathcal{Y}$, we can actually replace every inf by a min (cf. Proposition B.2). The assumption then reads as $\min_{y \in \mathcal{Y}} f(y, x) \leq a$, i.e. for every $x \in \mathcal{X}$ there exists y_x, such that $f(y_x, x) \leq a$. But then $y_x \in C(x, a)$. Recall that all $C(x, c)$ are closed for any $x \in \mathcal{X}$ and any $c \in \mathbb{R}$. Since for every $y \in \mathcal{Y}$ either $f(y, x_1) > a$ or $f(y, x_2) > a$, we have $C(x_1, b) \cap C(x_2, b) = \emptyset$.

Let $z \in [x_1, x_2]$ be a convex combination of x_1 and x_2, write $z = \lambda x_1 + (1 - \lambda) x_2$. Then for all $y \in \mathcal{Y}$,

$$f(y, z) = f(y, \lambda x_1 + (1 - \lambda) x_2) \geq \min\{f(y, x_1), f(y, x_2)\}$$

by quasi-convexity of $f(y, \cdot)$. This shows that

$$C(z, b) \subseteq C(x_1, b) \cup C(x_2, b).$$

Recall that $C(z, b)$ is not only closed, but also convex by quasi-convexity of $f(\cdot, z)$. But every convex set is path-connected, and therefore connected. So $C(z, b)$ is a connected set that is included in the union of two disjoint closed sets. Then it is already included in one of these closed sets.

So for every $z \in [x_1, x_2]$:

$$C(z,a) \subseteq C(z,b) \subseteq C(x_1,b) \quad \text{or} \quad C(z,a) \subseteq C(z,b) \subseteq C(x_2,b).$$

We define

$$I = \{z \in [x_1,x_2] : C(z,a) \subseteq C(x_1,b)\} \quad \text{and}$$
$$J = \{z \in [x_1,x_2] : C(z,a) \subseteq C(x_2,b)\}.$$

I and J are both nonempty, $I \cap J = \emptyset$, and $I \cup J = [x_1, x_2]$. If we can show that I is closed in $[x_1, x_2]$, then we obtain a contradiction. But this is in fact easy to show: Let (z_n) be a sequence in I, converging to some $z \in [x_1, x_2]$. We need to show that $z \in I$, i.e. that

$$C(z,a) \subseteq C(x_1,b).$$

Let $y \in C(z,a)$. Then $f(y,z) \leq a < b$. By upper semi-continuity of $f(y,\cdot)$

$$\limsup_{n\to\infty} f(y,z_n) \leq f(y,z) < b.$$

So there exists m such that $f(y,z_m) < b$. But this means that

$$y \in C(z_m,b) \subseteq C(x_1,b)$$

and the proof is complete.

□

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