A BSDE approach to the Skorokhod embedding problem for the Brownian motion with drift

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December 11, 2007

Abstract

We solve Skorokhod's embedding problem for Brownian motion with linear drift $(W_t + \kappa t)_{t\geq 0}$ by means of techniques of stochastic control theory. The search for a stopping time T such that the law of $W_T + \kappa T$ coincides with a prescribed law μ possessing the first moment is based on solutions of backward stochastic differential equations of quadratic type. This new approach generalizes an approach by Bass [Bas] of the classical version of Skorokhod's embedding problem using martingale representation techniques.

2000 AMS subject classifications: Primary 60G40, 60G44; secondary 60J65, 60F05, 60F17.

Key words and phrases: Skorokhod embedding, Brownian motion, diffusion, stopping time, control theory, BSDE, quadratic growth, Malliavin calculus.

Introduction

Brownian motion owes its high significance in the entire scope of models of stochastic dynamics to its appearance in the Donsker-Varadhan invariance principle. The latter underpins the fact that the law of the dynamics of any linearly interpolated random walk $S_k = \sum_{i=1}^k X_i$ with i.i.d. centered increments X_i with variance σ^2, speeded up in time with a factor n and rescaled in space by the factor $\sqrt{n}\sigma$ converges to the law of a Brownian motion, irrespective of what the individual law of the X_i may be. In his famous 1961 *embedding formulation* Skorokhod ([Sko65]) approaches the invariance principle from the reverse angle. He embeds the random walk by law into a given Brownian motion W in the following way. He first chooses a stopping time T_1 of W such that W_{T_1} possesses the law μ of X_1, then a stopping time T_2

of the Brownian motion $W_{T_1+\cdot} - W_{T_1}$ such that $W_{T_2+T_1} - W_{T_1}$ possesses again the law μ of X_2. Consequently the law of W_{T_2} coincides with the one of S_2. This clearly provides a recipe to find a sequence of stopping times $(T_n)_{n\in\mathbf{N}}$ such that the law of S_n finds itself *embedded* into the Brownian motion as the law of W at time $T_1 + \cdots + T_n$ for any $n \in \mathbf{N}$. Its main ingredient, the construction of a stopping time T for a given Brownian motion W such that the law of W_T realizes a given centered probability measure μ possessing a second moment is today generally known as *Skorokhod's embedding problem*. Since its original appearance it has been rephrased many times, and a variety of approaches have been developed. For an overview see Oblói [Obl].

The embedding problem may obviously be generalized from Brownian motion to transient diffusion processes. In this setting (see [Obl]) Hall [Hal] developed a first solution in 1969. The subject was treated again by Falkner and Grandits [GF], and Peskir [Pes] in 2000. According to [Obl] the methodology of the embeddings considered in these approaches may be summarized as follows. Viewed with its scale function, the given diffusion turns into a continuous local martingale. Using the time change technique by Dambis, Dubins and Schwarz the latter may be transformed into a Brownian motion, for which the version of Skorokhod's embedding due to Azéma and Yor (see [AY], or [Obl], Chapter 5) applies. Then the previous steps have to be taken in the reverse direction.

In the present paper we start with the observation that the Skorokhod embedding problem may be viewed as the weak version of a *stochastic control problem*: a Brownian motion has to be steered in such a way that it takes the prescribed law of some random variable. This observation will be put into a concept of solving the Skorokhod embedding problem in the framework of a particular transient diffusion, the Brownian motion with linear drift, which is based on the powerful tool of backward stochastic differential equations (BSDE). The precise formulation of the problem we solve by means of BSDE techniques is this.

Problem (*). *Let μ be a probability measure on $\mathbb{R}$ with $\int |x| d\mu(x) < \infty$ and $(W_t + \kappa t + c)_{t\geq 0}$ a Brownian motion with linear drift of slope κ (w.l.o.g $\kappa > 0$) and starting point c defined on a probability space $(\Omega, \mathscr{F}, P)$. If $(\mathscr{F}_t)$ denotes the filtration generated by W, completed by the P-null sets of $\mathscr{F}$, find an $(\mathscr{F}_t)$ stopping time T such that $W_T + \kappa T + c$ has law μ and $\mathbb{E}T < \infty$.*

Our method to solve (*) for Brownian motion with linear drift is in spirit much related to an approach of the original Skorokhod embedding problem for Brownian motion by Bass [Bas], where the solution algorithm consists in the following steps.

1. Find a $\sigma(W_1)$-measurable random variable ξ with law μ.
2. Use the martingale representation property of Brownian motion to write $\xi - E(\xi)$ as a stochastic integral $\int_0^1 Z_s dW_s$.

3. Use random time change of the Dambis, Dubins and Schwarz type with the scale process $\int_0^\cdot Z_s^2 ds$ to transform the martingale $\int_0^\cdot Z_s dW_s$ into a Brownian motion B. Then there is a stopping time S of B such that $B_S = \xi$, whence B_S has law μ.

4. To finally transform this embedding with respect to B into one with respect to W, use explicit descriptions of solutions of an associated partial differential equation which provides a representation of the stopping time as a functional of the corresponding Brownian motion.

Working with Brownian motion with linear drift instead requires a modification of this algorithm where martingale representation comes into play. We replace it - in the terms of Peng [**?**] - with *nonlinear martingale representation* appearing in the disguise of BSDE. To understand this modification more precisely, note first that to find an explicit representation

$$\xi - E(\xi) = \int_0^1 Z_s \, dW_s$$

amounts to finding the solution (Y, Z) of a BSDE with generator $f = 0$, i.e.

$$Y_t = \xi - \int_t^1 Z_s \, dW_s.$$

Now observe that the random time change in the quadratic variation scale $S = \int_0^\cdot Z_s^2 ds$ which turns the martingale $\int_0^\cdot Z_s \, dW_s$ into a Brownian motion will of course turn this scale process into a linear drift. This is the reason why for generalizing the second step in Bass [Bas] to Brownian motion with linear drift of slope κ we have to solve the nonlinear martingale representation problem in terms of the BSDE

$$Y_t = \xi - \int_t^1 Z_s \, dW_s - \kappa \int_t^1 Z_s^2 \, ds \tag{1}$$

with the quadratic generator $f(s, y, z) = -\kappa z^2$. This particular BSDE that has been studied in the literature already, for example in [BLSM07].

Alternatively, we might think of eliminating first the linear drift with slope κ by means of Girsanov's theorem providing an equivalent measure Q, then following the algorithm proposed by Bass [Bas], and finally returning to the original measure P. Martingale representations with respect to Q will then correspond exactly to solving non-linear BSDEs (1) with respect to P.

Here is an outline of the following presentation of the Skorokhod embedding problem for Brownian motion with linear drift along these lines of reasoning. In section 1 we use BSDE related techniques to extend steps 1 - 3 of the algorithm sketched above to our setting. This way, we construct a Brownian motion B and a corresponding stopping time S, with $B_S + \kappa S + c = \xi$ (Lemma 1.1). An explicit solution of Equation (1) and therefore closed formulas for Y and Z in terms of an associated partial differential equation will be given in section 2 (Lemma 2.2). An adaptation of arguments from [Bas] yields an ODE providing a representation of the stopping time S in the filtration of B (Theorem 2.3). To finally obtain the desired embedding into W in section 3, B has to be replaced with W in the ODE. We close the presentation by describing the domain of possible starting points and discussing some integrability properties of the stopping time.

1 A weak solution of the Skorokhod embedding problem

Let μ be a probability measure on $\mathbb{R}$. In this section we will show that starting from a solution of a simple quadratic BSDE one can construct a Brownian motion and a stopping time such that the distribution of the Brownian motion stopped at this time is equal to μ. Let us first recall the definition of a BSDE.

Let ξ be an $\mathcal{F}_1$-measurable random variable, and let $f : \Omega \times [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that for all $y, z \in \mathbb{R}$ the mapping $f(\cdot,\cdot,y,z)$ is predictable. A solution of the BSDE with *terminal condition* ξ and *generator* f is defined to be a pair of predictable processes (Y,Z) such that almost surely we have $\int_0^1 Z_s^2 ds < \infty$, $\int_0^1 |f(s,Y_s,Z_s)|ds < \infty$, and for all $t \in [0,1]$

$$Y_t = \xi - \int_t^1 Z_s dW_s + \int_t^1 f(s,Y_s,Z_s)ds.$$

The solution processes (Y,Z) are often shown to satisfy some integrability properties. To this end one usually verifies whether they belong to the following function spaces. Let $p \geq 1$. We denote by $\mathcal{H}^p$ the set of all $\mathbb{R}$-valued predictable processes ζ such that $E\int_0^1 |\zeta_t|^p dt < \infty$, and by $\mathcal{S}^p$ the set of all $\mathbb{R}$-valued predictable processes δ satisfying $E\left(\sup_{s\in[0,1]} |\delta_s|^p\right) < \infty$.

Let us now come back to problem of finding a Brownian motion B and a stopping time T such that $B_T + \kappa T + c$ has the distribution μ.

Let $\hat{F}$ be the distribution function of μ and $\hat{F}^{-1}(y) = \inf\{x : \hat{F}(x) \geq y\}$. With Φ denoting the distribution function of the standard normal distribution we define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \hat{F}^{-1}(\Phi(x))$ and set ξ as $\xi = g(W_1) = \hat{F}^{-1}(\Phi(W_1))$. Note that g is nondecreasing, left-continuous, measurable and not identically constant. Hence ξ is a $\sigma(W_1)$-measurable random variable with law μ .

As mentioned in our outline ξ will now serve as the terminal value in the following quadratic BSDE

$$Y_t = \xi - \int_t^1 Z_s\, dW_s - \kappa \int_t^1 Z_s^2\, ds, \qquad 0 \leq t \leq 1, \tag{2}$$

with $\kappa > 0$. As is shown in the next lemma, Y_t can be interpreted as a *time changed* Brownian motion with drift κt that is stopped as it runs into the random variable ξ.

Lemma 1.1. *Suppose (Y,Z) is a solution of (2), $\mathbb{E}(\xi) < \infty$ and $Z \in \mathcal{H}^2$. Then there exists a Brownian motion B and a stopping time S, with $\mathbb{E}(S) < \infty$, such that $B_S + \kappa S + Y_0 = \xi$, i.e. the process $B_s + \kappa s + Y_0$ stopped at S has law μ.*

Proof: Since $Z \in \mathcal{H}^2$, the process $M_t = \int_0^t Z_s\, dW_s$, $0 \le t \le 1$, is a martingale. We extend our probablility space such that it accommodates another Brownian motion $\tilde{B}$, which is independent of M. Let $S = \langle M \rangle_1$. We time-change M using the stopping times

$$\tau_s = \begin{cases} \inf\{t \ge 0 : \langle M \rangle_t > s\}, & 0 \le s < S, \\ 1, & s \ge S, \end{cases}$$

and obtain the Brownian motion $B_s = \tilde{B}_s - \tilde{B}_{s \wedge S} + M_{\tau_s}$, $0 \le s < \infty$. B is adapted to the filtration $(\mathscr{G}_s)$ defined by $(\mathscr{G}_s) = (\mathscr{F}_{\tau_s})$. Since $S = \langle M \rangle_1$ is a $(\mathscr{G}_s)$ stopping time we find $B_S + \kappa S + Y_0 = M_1 + \kappa \int_0^1 Z_s^2\, ds + Y_0 = \xi$. Furthermore

$$\begin{aligned} \mathbb{E}(S) = \mathbb{E}(\int_0^1 Z_s^2\, ds) &= \mathbb{E}(\frac{1}{\kappa} Y_1 - \frac{1}{\kappa} \int_0^1 Z_s\, dW_s - \frac{1}{\kappa} Y_0) \\ &= \frac{1}{\kappa} \mathbb{E}(\xi) - \frac{1}{\kappa} Y_0 \qquad (3) \\ &< \infty, \end{aligned}$$

and hence the proof is complete. □

Remark 1.2. *We remark that it is a priori not clear that the stopping time S of Lemma 1.1 is also a stopping time with respect to $(\mathscr{F}_s^B)$, the right continuous completion of the filtration $(\sigma(B_u : u \le s))$. However, it will follow from the functional dependence of S on the paths of the Brownian motion B shown in the next section.*

In Lemma 1.1 it is *assumed* that there exists a solution (Y, Z) of (2). The next goal of this section therefore is to show that one can explicitly construct a solution of (2), if ξ satisfies the integrability condition $\mathbb{E}(e^{-2\kappa\xi}) < \infty$. We will also see that the constructed solution is such that $Z \in \mathcal{H}^2$, and hence it satisfies the assumptions of Lemma 1.1.

If $e^{-2\kappa\xi}$ is integrable, then we can define the martingale $N_t = \mathbb{E}\left[\exp(-2\kappa\, \xi) | \mathscr{F}_t\right]$, $0 \le t \le 1$. The martingale representation property implies that there exists a predictable and locally square integrable process H such that $N_t = N_0 + \int_0^t H_s dW_s$. It is straightforward to show that the two processes defined by

$$Y_t = -\frac{1}{2\kappa} \ln N_t, \text{ and } Z_t = -\frac{1}{2\kappa} \frac{H_t}{N_t}, \quad 0 \le t \le 1, \qquad (4)$$

are a solution of (2). Moreover, we have the following.

Lemma 1.3. *Let ξ and $e^{-2\kappa\xi}$ be integrable. Then $Y \in \mathcal{S}^1$, $Z \in \mathcal{H}^2$ and (Y, Z) solves (2).*

Proof: By Ito's formula, applied to $\ln N$, we have

$$-\frac{1}{2\kappa}[\ln N_t - \ln N_0] = -\frac{1}{2\kappa}\left[\int_0^t \frac{H_s}{N_s} dW_s - \frac{1}{2}\int_0^t \left(\frac{H_s}{N_s}\right)^2 ds\right], \qquad (5)$$

which means that $Y_t - Y_0 = \int_0^t Z_s dW_s + \kappa \int_0^t Z_s^2 ds$. Since $Y_1 = \xi$, this implies that (Y, Z) is a solution of (2). In order to show that Z belongs to $\mathcal{H}^2$, observe that

$$\kappa \int_0^t Z_s^2 ds = -\frac{1}{2\kappa}[\ln N_t - \ln N_0] - \int_0^t Z_s dW_s, \qquad (6)$$

and by Jensen's inequality

$$\ln N_t = \ln \mathbb{E}(\exp(-2\kappa\xi)|\mathcal{F}_t) \geq -2\kappa\mathbb{E}(\xi|\mathcal{F}_t).$$

Using this in (6), we obtain

$$\kappa \int_0^t Z_s^2 ds \leq \mathbb{E}(\xi|\mathcal{F}_t) + \frac{1}{2\kappa}\ln N_0 - \int_0^t Z_s dW_s. \tag{7}$$

There exists an increasing sequence of stopping times τ_n with $\tau_n \to 1$, a.s. and such that $E\int_0^{\tau_n} Z_s^2 ds < \infty$. This implies $E\int_0^{\tau_n} Z_s^2 ds \leq \frac{1}{\kappa}\mathbb{E}(\xi) + \frac{1}{2\kappa^2}\ln N_0$, and monotone convergence yields that Z belongs to $\mathcal{H}^2$.

Now notice that Equation (6) implies

$$\sup_{0\leq t\leq 1} |Y_t| \leq |Y_0| + \kappa\int_0^1 Z_s^2 ds + \sup_{0\leq t\leq 1} |\int_0^t Z_s dW_s|, \tag{8}$$

and with the Burkholder-Davis-Gundy Inequality we obtain that for a constant $C \in \mathbb{R}_+$,

$$\mathbb{E}\sup_{0\leq t\leq 1} |Y_t| \leq C\left(1 + \mathbb{E}\int_0^1 Z_s^2 ds\right) < \infty, \tag{9}$$

which shows that Y belongs to the space $\mathcal{S}^1$. □

Remark 1.4. *We want to comment on the uniqueness of solutions of quadratic BSDE, such as* (2)*. Kobylanski* [*Kob*] *proved existence and uniqueness under the restriction that the terminal variable* ξ *is bounded. Recently Briand and Hu* [*BH07*] *gave a uniqueness result under the assumptions that the generator* f *is convex in the variable* z *and that* $\exp(p\,|\xi|)$ *is integrable for all* $p \geq 1$*. The integrability condition imposed on* μ *in Lemma 2.2 translates to* $\mathbb{E}(\exp(-2\kappa\xi)) < \infty$ *for our choice of* ξ*, i.e. we cannot draw on the uniqueness results just mentioned. In fact, refer to [AIP07] Section 1.3.1 for an example of a BSDE with generator* $\frac{1}{2}z^2$*,* ξ *fulfilling* $\mathbb{E}(\exp(\gamma|\xi|)) < \infty$ *for an arbitrary* $\gamma > 0$ *and two different solutions.*

2 Path dependence of the Skorokhod stopping times

The aim of this section is to find the analytic relation of the paths of the Brownian motion B and the stopping time S constructed in Lemma 1.1 of the previous section. Describing S as a functional of the Brownian paths, will allow us, in the next section, to solve the Skorokhod stopping problem not only for B, but for any *arbitrary* Brownian motion.

We first show that the BSDE solution processes (Y, Z), defined in (4), are deterministic functions of W. To this end we introduce the auxiliary function

$$F(t,x) = \mathbb{E}[\exp(-2\kappa\, g(W_1 - W_t + x))], \qquad t \in [0,1) \text{ and } x \in \mathbb{R}. \tag{10}$$

Note that F is finite for all $0 < t < 1$ and $x \in \mathbb{R}$, and $F(0,0) = \mathbb{E}[\exp(-2\kappa g(W_1)] < \infty$. This can be deduced from the following fact: If I is a bounded interval in $\mathbb{R}$, J a compact subset of $(0,1)$, then there exists a constant $0 < K < \infty$ such that

$$\exp(-(z-x)^2/2t) \leq \exp(-z^2/2), \tag{11}$$

for all $x \in I$, $t \in J$, $|z| > K$.

The next lemma collects some further properties of the function F we will need in order to write the processes Y and Z as functions of W. Some of the properties are only needed later and will enable us to deduce an ODE that establishes the link between the Brownian paths and the Skorokhod stopping time. The lemma follows from straightforward adaptations of Lemma 1 and Lemma 2 in [Bas], and therefore the proof is omitted.

Lemma 2.1. (*Properties of F*)
The function F defined in (10) has the following properties.

1. *$F(t,x)$ is strictly decreasing in x, for $0 \le t < 1$.*

2. *$F \in \mathcal{C}^{1,2}((0,1) \times \mathbb{R})$.*

3. *For $t \in (0,1)$ the function F_x can be expressed as the Lebesgue-Stieltjes integral*

$$F_x(t,x) = -\int \varphi_{1-t}(z-x)\, d\bar{g}(z),$$

   where φ_s is the density of the $\mathcal{N}(0,s)$ distribution and $\bar{g}$ the nondecreasing function defined by $\bar{g}(z) = -\exp(-2\kappa g(z))$.

4. *On compact subsets of $(0,1) \times \mathbb{R}$, F (resp. F_x) is bounded above (resp. bounded below), bounded below away from 0 (resp. bounded above away from 0), and uniformly Lipschitz in t and x.*

5. *For each $t \in (0,1)$, let $F^{-1}(t,\cdot)$ be the inverse of $F(t,\cdot)$. Then on compact subsets of its domain, F^{-1} is uniformly Lipschitz in t and jointly continuous in t and y.*

6. *The function*

$$h(t,x) = -\frac{1}{2\kappa}\frac{F_x(t,x)}{F(t,x)}, \tag{12}$$

   is well defined for all $0 < t < 1$ and $x \in \mathbb{R}$. On compact subsets of $(0,1) \times \mathbb{R}$, h is bounded above, bounded below away from 0, and uniformly Lipschitz in t and x.

Note that our function F (resp. F_x) is the analogue of the function b (resp. a) in [Bas]. In the next lemma we give a representation of the solution (Y,Z) of (2), as functions of the Brownian motion W. Recall that $\xi = g(W_1)$ and that we assumed that $\kappa > 0$.

Lemma 2.2. *Let μ be such $\int |x| d\mu(x) < \infty$ and $\int \exp(-2\kappa\, x)\, \mu(dx) < \infty$. Let F be defined as in (10). Then the processes Y and Z, defined in (4), satisfy for almost all ω,*

$$Y_t = -\frac{1}{2\kappa} \ln F(t, W_t), \text{ for all } t \in [0,1], \tag{13}$$

and

$$Z_t = -\frac{1}{2\kappa}\frac{F_x(t,W_t)}{F(t,W_t)}, \text{ for all } t \in (0,1). \tag{14}$$

Proof: Note that the martingale $N_t = \mathbb{E}[e^{-2\kappa\xi}|\mathcal{F}_t]$ can be written as $N_t = F(t, W_t)$ for all $t \in [0, 1]$. Thus, the very definition of Y implies (13).

By Lemma 2.1, we have $F \in \mathcal{C}^{1,2}((0,1) \times \mathbb{R})$, and hence with Ito's formula we find for $\varepsilon > 0$ and $t < 1$

$$N_t - N_\varepsilon - \int_\varepsilon^t F_x(u, W_u)\, dW_u \;=\; \int_\varepsilon^t F_u(u, W_u)\, du + \frac{1}{2}\int_\varepsilon^t F_{xx}(u, W_u)\, du.$$

On the left hand side we have a local martingale, whereas on the right hand side we have a process of bounded variation. Hence we know that $\int_\varepsilon^t F_u(u, W_u)\, du + \frac{1}{2}\int_\varepsilon^t F_{xx}(u, W_u)\, du = 0$, $\mathbb{P}$-a.s. and therefore $N_t = N_\varepsilon + \int_\varepsilon^t F_x(u, W_u)\, dW_u$. Again, the very definition of Z, implies that almost surely we have $Z_t = -\frac{1}{2\kappa}\frac{F_x(t,W_t)}{F(t,W_t)}$ on $(0, 1)$. □

The next theorem will use the information gathered so far to show that the stopping time S, defined in Lemma 1.1, is actually an $(\mathscr{F}_s^B)$ stopping time. We remark that the proof is a generalisation of the proof of Proposition 3 in [Bas].

Theorem 2.3. *Let (Y, Z) be defined as in (4). Then the associated stopping time S defined in Lemma 1.1 is an $(\mathscr{F}_s^B)$ stopping time.*

Proof: Let M be as in Lemma 1.1 and $S_t = \langle M \rangle_t$. By the properties of our explicit solution we find that the stopping times τ_s reduce to $\tau_s = S^{-1}(s)$, for all $0 \le s < S$, and satisfy the equation

$$\frac{\partial}{\partial s}\tau_s = \frac{1}{S'(S^{-1}(s))} = \frac{1}{h^2(\tau_s, W_{\tau_s})}, \tag{15}$$

for all $0 < s < S$. From the forward version of (2) we get $W_{\tau_s} = F^{-1}(\tau_s, \exp(-2\kappa(M_{\tau_s} + \kappa\int_0^{\tau_s} Z_u^2\, du + Y_0))) = F^{-1}(\tau_s, \exp(-2\kappa(B_s + \kappa s + Y_0)))$ and thus, for all $0 < s < S$, we can write (15) as

$$\frac{\partial}{\partial s}\tau_s \;=\; \frac{1}{h^2(\tau_s, F^{-1}(\tau_s, \exp(-2\kappa(B_s + \kappa s + Y_0))))}. \tag{16}$$

Therefore, for each ω, (16) is a differential equation, which is solved at least by τ_s, with $0 < s < S$. Standard results of the theory of ordinary differential equations show that our solution is $\mathbb{P}$-almost surely unique. Moreover it can be constructed via Picard iteration and therefore τ_s is measurable with respect to $\mathscr{F}_s^B$.

Because by monotone convergence we have $\lim_{t\to 1} S_t = S_1$ it is sufficient to show $\{S_t \le s\} \in \mathscr{F}_s^B$, for $t \in (0, 1)$ in order to see that $S = S_1$ is even an $(\mathscr{F}_s^B)$ stopping time. But by measurability of τ_s we have $\{S_t \le s\} = \{\tau_s < t\}^c \in \mathscr{F}_s^B$. □

3 The solution of the Skorokhod embedding problem

Now we are finally set to present a solution to Problem (*).

Theorem 3.1. *Let $\kappa > 0$ and μ a probability measure on $\mathbb{R}$ such that $\int |x| d\mu(x) < \infty$ and $\int \exp(-2\kappa\, x)\, \mu(dx) < \infty$, and define $c = -\frac{1}{2\kappa} \ln \int \exp(-2\kappa\, x)\, \mu(dx)$. Let W be a Brownian motion on a probability space $(\Omega, \mathbb{P}, \mathscr{F})$ and $(\mathscr{F}_s^W)$ its right continuous and completed filtration.*
Then there exists an $(\mathscr{F}_s^W)$ stopping time T, with finite expectation, such that the process

$$W_s + \kappa s + c$$

stopped at T has law μ.

Proof: Let Y and Z be defined as in (4), and let B and S be the associated Brownian motion and stopping time constructed in Lemma 1.1. We instantly deduce $c = Y_0 = -\frac{1}{2\kappa} \ln \mathbb{E}(\exp(-2\kappa\xi))$, and $B_S + \kappa S + Y_0 = \xi$.

In order to obtain a stopping time for W we use Equation (16). By replacing B with W we readjust the Brownian paths and still have the $\mathbb{P}$-almost sure unique and global solution, which we now call σ_s.

$$\frac{\partial}{\partial s}\sigma_s = \frac{1}{h^2(\sigma_s, F^{-1}(\sigma_s, \exp(-2\kappa(W_s + \kappa s + Y_0))))}.$$

Again with Lemma 2.1 we know that $\frac{\partial}{\partial s}\sigma_s > 0$, therefore σ_s is strictly increasing and we can define σ_t^{-1} for $t < 1$. Set $T = \lim_{t\to 1} \sigma_t^{-1}$, allowing for $T = \infty$. The same arguments as above yield $\{\sigma_t^{-1} \le s\} \in \mathscr{F}_s^W$, which leads to T being a $(\mathscr{F}_s^W)$ stopping time.
The law of (B, S) is the same as the law of (W, T), hence $W_T + \kappa T + Y_0$ has distribution μ. Since particularly $S \overset{d}{=} T$, we know from Equation (3)

$$\mathbb{E}(T) = \frac{1}{\kappa}\mathbb{E}(\xi) + \frac{1}{2\kappa^2} \ln \mathbb{E}(\exp(-2\kappa\, \xi)).$$

□

The domain of possible starting points

The following Corollary extends Theorem 3.1.

Corollary 3.2. *Let μ satisfy $\int |x| d\mu(x) < \infty$ and $\int \exp(-2\kappa\, x)\, d\mu(x) \le \exp(-2\kappa c)$, for $\kappa > 0$, for some $c \in \mathbb{R}$. Let W be a Brownian motion on a probability space $(\Omega', \mathbb{P}', \mathscr{F}')$ and $(\mathscr{F}_s^W)$ its right continuous and completed filtration. Then there exists an $(\mathscr{F}_s^W)$ stopping time T', with finite expectation such that the process $W_s + \kappa s + c$ stopped at T' has law μ.*

Proof: We can choose $\bar{c} > c$ fulfilling

$$\int \exp(-2\kappa\, x)\, \mu(dx) = \exp(-2\kappa\bar{c}).$$

Then by Theorem 3.1 we obtain a stopping time T such that $W_T + \kappa T + \bar{c}$ has law μ. Set $a = \bar{c} - c > 0$ and $b = -\kappa$. By standard results about first hitting times of Brownian motion we know that the stopping time $\bar{T} = \inf\{t \geq 0 : W_t = a + bt\}$, $a > 0$ and $b < 0$, has finite expectation. Thus we define $T' = T + \bar{T}$ and use the strong Markov property of Brownian motion to show that $W_{T'} + \kappa T' + c$ has the same distribution as $W_T + \kappa T + \bar{c}$, i.e. has law μ. Obviously $\mathbb{E}T' < \infty$. □

It can be shown that $c = -\frac{1}{2\kappa} \ln \int \exp(-2\kappa\, x)\, d\mu(x)$ is the largest starting point that allows to stop the Brownian motion with drift of slope κ so that it has distribution μ. As stated in [GF] for $c = 0$, an exponential change of the space variable shows that the existence of a stopping time T, such that $W_T + \kappa T + c$ has law μ already implies

$$\int \exp(-2\kappa x)\, d\mu(x) \leq \exp(-2\kappa c).$$

Integrability properties of the stopping time

The stopping time T of Theorem 3.1, that solves the embedding problem, has been shown to be integrable. In the remainder we will show that if μ has a finite moment of order $p \geq 1$, then T is also L^p-integrable.

Theorem 3.3. *Let $p \geq 1$. If $\int |x|^p d\mu(x) < \infty$ and $\int \exp(-2\kappa\, x)\, \mu(dx) < \infty$, then the stopping time T of Theorem 3.1 solving the embedding problem, satisfies*

$$E(T^p) < \infty.$$

Proof: Let (Y, Z) be defined as in (4), $S = \int_0^1 Z_s^2 ds$, and B the Brownian motion constructed in Section 1 so that $B_S + \kappa S + Y_0$ has distribution μ. Since (B, S) has the same law as (W, T), all we need to show is that $\mathbb{E}\left(\int_0^1 Z_s^2 ds\right)^p$ is finite.

To this end notice that Inequality (7) and the inequality $|a+b+c|^p \leq 3^{p-1}(|a|^p + |b|^p + |c|^p)$, imply

$$\left(\int_0^1 Z_s^2 ds\right)^p \leq \frac{1}{\kappa^p} 3^{p-1} \left(|\xi|^p + |\frac{1}{2\kappa} \ln N_0|^p + \left|\int_0^1 Z_s dW_s\right|^p\right).$$

The Burkholder-Davis-Gundy Inequality further yields that for a constant $C_1 \in \mathbb{R}_+$,

$$\mathbb{E}\left(\int_0^1 Z_s^2 ds\right)^p \leq C_1 \left(\mathbb{E}|\xi|^p + |\frac{1}{2\kappa} \ln N_0|^p + \mathbb{E}\left(\int_0^1 Z_s^2 ds\right)^{p/2}\right). \tag{17}$$

Now let $C_2 \in \mathbb{R}_+$ be such that $a \leq \frac{a^2}{2C_1} + C_2$ for all $a \geq 0$. Then $\mathbb{E}\left(\int_0^1 Z_s^2 ds\right)^{p/2} \leq \frac{1}{2C_1}\mathbb{E}\left(\int_0^1 Z_s^2 ds\right)^p + C_2$, and combining this with (17) we obtain for some constant C_3

$$\begin{aligned} \frac{1}{2}\mathbb{E}\left(\int_0^1 Z_s^2 ds\right)^p &\leq C_3 \left(1 + \mathbb{E}|\xi|^p + |\frac{1}{2\kappa} \ln N_0|^p\right) \\ &= C_3 \left(1 + \int |x|^p d\mu(x) + \left|\frac{1}{2\kappa} \ln \int \exp(-2\kappa\, x)\, \mu(dx)\right|\right) \\ &< \infty, \end{aligned}$$

and hence $E(T^p) < \infty$. □

Remark 3.4. *If $\int |x|^p d\mu(x) < \infty$ and $\int \exp(-2\kappa\, x)\, \mu(dx) < \infty$, then the process Y, defined in (4), belongs to $\mathcal{S}^p$.*

Proof: Observe that from Equation (6) we may deduce

$$\sup_{0\le t\le 1} |Y_t|^p \le 3^{p-1}\left(|Y_0| + \kappa^p \left(\int_0^1 Z_s^2 ds\right)^p + \sup_{0\le t\le 1} |\int_0^t Z_s dW_s|^p\right),$$

and hence the Burkholder-Davis-Gundy Inequality implies that there exist constants C_4 and C_5 such that

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le 1} |Y_t|^p &\le C_4\left(|Y_0|^p + \mathbb{E}\left(\int_0^1 Z_s^2 ds\right)^p + \mathbb{E}\left(\int_0^t Z_s^2 ds\right)^{p/2}\right)\\ &\le C_5\left(1 + \mathbb{E}\left(\int_0^1 Z_s^2 ds\right)^p\right) < \infty.\end{aligned}$$

and hence the result. □

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