Energy Minimizers of the Coupling of a Cosserat Rod to an Elastic Continuum ⋆

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Abstract: We formulate the static mechanical coupling of a geometrically exact Cosserat rod to an elastic continuum. The coupling conditions accommodate for the difference in dimension between the two models. Also, the Cosserat rod model incorporates director variables, which are not present in the elastic continuum model. Two alternative coupling conditions are proposed, which correspond to two different configuration trace spaces. For both we show existence of solutions of the coupled problems. We also derive the corresponding conditions for the dual variables and interpret them in mechanical terms.

Keywords: variational analysis, Cosserat rods, directors, nonlinear elasticity, coupling conditions, energy minimization

1. INTRODUCTION

In this article we analyze the coupling of a three-dimensional continuum model to a geometrically exact one-dimensional rod with an orthonormal director frame (Cosserat rod). We assume that both objects are governed by hyperelastic material laws, and consider the static case only. We then propose two different coupling conditions and show that the coupled problem has solutions for both of them. Existence of more than one set of plausible coupling conditions is a general feature of heterogeneous models (cf. Blanco et al. (2011)). The proof uses the direct method of the calculus of variations. For each coupling condition we also obtain the corresponding dual conditions, which we interpret as coupling conditions for the forces and moments.

The coupling of mechanical models of differing dimensions has been treated both in the engineering and the mathematical literature (Monaghan et al. (1998); Lagnese et al. (1994); Ciarlet et al. (1989)). We would like to point out the work of Blanco et al. (2011), where a systematic treatment for coupling linear models (without directors) of different dimensions is given. In particular, they provide existence and uniqueness of solutions for their coupled problems. To the knowledge of the authors, coupling conditions for a reduced model with director variables has only been treated in Sander (to appear). Additionally, in that work, an algorithm based on fixed-point iteration was proposed to numerically solve coupling problems of the type considered here. Our variational approach instead suggests to treat the problem as a global minimization problem with nonlinear constraints. A detailed treatment may appear in a separate article.

We proceed as follows. In Sections 2 and 3 we formally introduce the rod and continuum models, and state several assumptions. Then in Section 4 we propose two sets of coupling conditions for the displacement and director variables. In Section 5 we give existence results and optimality conditions for solutions for both these conditions. This leads to corresponding conditions for the dual variables, i.e., the coupling forces and moments. Due to limitations of space, we skip proofs throughout the text. They will be presented in a forthcoming publication (Sander and Schiela (in preparation)).

2. GEOMETRICALLY EXACT COSSERAT RODS

Cosserat rods model the large deformation behavior of long, slender objects. To each point of a one-dimensional parameter domain they associate a point in space and an orthonormal frame of director vectors, which is to be interpreted as the orientation of a rigid cross-section. For an in-depth presentation see the book by Antman (1991).

2.1 Rigid Body Motion

Let SO(3) be the special orthogonal group in three-dimensional space, that is the group of orthogonal 3×3-matrices with positive determinant. SO(3) has the structure of a three-dimensional compact manifold. Elements R of SO(3) act on $\mathbb{R}^3$ by rotation around the origin.

Consider the product space $\mathrm{SE}(3) = \mathbb{R}^3 \times \mathrm{SO}(3)$, known as the special Euclidean group. We denote elements of this space as tuples $\rho = (r, R)$, with $r \in \mathbb{R}^3$ and $R \in \mathrm{SO}(3)$. An element $\rho \in \mathrm{SE}(3)$ acts on $\mathbb{R}^3$ by a rigid body motion $\rho : x \mapsto Rx + r$.

For any $R \in \mathrm{SO}(3)$ the tangent space of SO(3) at R can be characterized in two different ways

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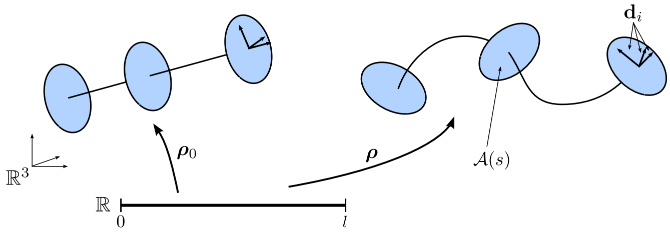

Fig. 1. Kinematics of Cosserat rods. Under deformation, rod cross-sections remain planar, but not necessarily orthogonal to the centerline.

$$T_R\mathrm{SO}(3) = \{\delta R \in \mathbb{R}^{3\times 3} \mid \delta R = \mathbf{M}R, \mathbf{M} \in \mathfrak{so}(3)\}$$
$$= \{\delta R \in \mathbb{R}^{3\times 3} \mid \delta R = R\mathsf{M}, \mathsf{M} \in \mathfrak{so}(3)\},$$

where $\mathfrak{so}(3)$, the Lie algebra of SO(3), is the space of skew-symmetric 3×3 matrices. The two representations are the spatial and body representation, respectively. Obviously, we have the relation

$$\mathbf{M} = R\mathsf{M}R^{-1},$$

and hence $\mathbf{M}$ arises from M by an orthogonal change of coordinates.

With each $M \in \mathfrak{so}(3)$ we can associate a vector u via the relation

$$Mv = u \times v \quad \forall v \in \mathbb{R}^3,$$

and we will denote this relation by $M = u^\times$, with the inverse $u = M_\times$. The vector u is called the axial vector of the skew-symmetric matrix M. Setting $\mathsf{u} = \mathsf{M}_\times$ and $\mathbf{u} = \mathbf{M}_\times$, and using the relation $R(w \times v) = Rw \times Rv$ we obtain

$$\mathbf{u} = R\mathsf{u}.$$

The structure of the tangent space of SE(3) can be inferred from the general rule about tangent spaces of product manifolds. For each $\rho = (r, R) \in \mathrm{SE}(3)$ we have

$$T_\rho\mathrm{SE}(3) = T_r\mathbb{R}^3 \times T_R\mathrm{SO}(3).$$

By the above identifications, each element $\delta\rho$ of $T_\rho\mathrm{SE}(3)$ can be represented as

$$\delta\rho = (\mathbf{v}, R\mathbf{u}^\times) = (R\mathsf{v}, \mathsf{u}^\times R).$$

Here $(\mathbf{v}, \mathbf{u}) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $(\mathsf{v}, \mathsf{u}) = (R^{-1}\mathbf{v}, R^{-1}\mathbf{u}) \in \mathbb{R}^3 \times \mathbb{R}^3$ are again representations in spatial and body coordinates, respectively.

2.2 Static Rod Model

Kinematics. The theory of Cosserat rods views a rod as a curve in space, which at each point has attached a rigid cross-section. The central assumption is that under load, cross-sections do not change shape. They may, however, change their orientations, and are in particular not restricted to remain normal to the curve tangent vector (Figure 1). Hence, configurations of Cosserat rods are continuous maps

$$\boldsymbol{\rho} : [0, l] \to \mathrm{SE}(3)$$
$$s \mapsto \boldsymbol{\rho}(s) = (r(s), R(s)),$$

for some $l > 0$. While the first component $r(s) \in \mathbb{R}^3$ of $\boldsymbol{\rho}$ determines the position of the centerline of the rod at s, the second component $R(s) \in \mathrm{SO}(3)$ determines the orientation of the cross-section $\mathcal{A}(s)$ (Figure 1).

Let $\mathbf{e}_i$ be the i-th canonical basis vector. The action of an element ρ of SE(3) on $\mathbf{e}_i$ produces a vector $\mathbf{d}_i$ at r, which is called a director. For any $\rho \in \mathrm{SE}(3)$, the three directors $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$ form an orthonormal frame. For any $s \in [0, l]$ we interpret $R(s)$ such that $\mathbf{d}_1$ and $\mathbf{d}_2$ span the plane of the cross-section $\mathcal{A}(s)$. There, the vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ introduce an orthonormal coordinate system with coordinates $\xi = (\xi_1, \xi_2)$. Under certain assumptions on the rod curvature, any point X in the rod described by the configuration function $\boldsymbol{\rho}$ can then be addressed as a triple (s, ξ_1, ξ_2) by

$$X(s, \xi_1, \xi_2) = r(s) + \xi_1 R(s)\mathbf{e}_1 + \xi_2 R(s)\mathbf{e}_2.$$

We single out one configuration function $\boldsymbol{\rho}_0 : [0, l] \to \mathrm{SE}(3)$ and call it the reference configuration. It will be convenient (but not necessary) to choose $\boldsymbol{\rho}_0$ to be the stress-free configuration.

Strains. Let $\boldsymbol{\rho} : [0, l] \to \mathrm{SE}(3)$ be a given rod configuration. We define the spatial derivative

$$\boldsymbol{\rho}' : [0, l] \to T\mathrm{SE}(3) \qquad s \mapsto (r'(s), R'(s)),$$

where the prime denotes derivation with respect to s.

Recall that any element $\delta R \in T_{\boldsymbol{\rho}(s)}\mathrm{SO}(3)$ can be represented as a product $\delta R = \mathbf{M}R$, where $\mathbf{M} \in \mathfrak{so}(3)$. This defines the vector

$$\mathbf{u} = \mathbf{M}_\times := (R'(s)R(s)^{-1})_\times.$$

Additionally, we define $\mathbf{v}(s) = r'(s)$. Then the corresponding strain in spatial variables is defined as a function $\epsilon(\boldsymbol{\rho}) : [0, l] \to \mathbb{R}^3 \times \mathbb{R}^3$

$$\epsilon(\boldsymbol{\rho})(s) := (\mathbf{v}, \mathbf{u})(s) - (RR_0^{-1}\mathbf{v}_0, RR_0^{-1}\mathbf{u}_0)(s).$$

In this expression, quantities with a subscript 0 refer to the reference configuration $\boldsymbol{\rho}_0$.

The strain measures become invariant under rigid-body motions $Q : \boldsymbol{\rho} \mapsto Q\boldsymbol{\rho}$ when expressed in body coordinates. There we have $(\mathsf{v}, \mathsf{u}) \in \mathbb{R}^3 \times \mathbb{R}^3$

$$(\mathsf{v}, \mathsf{u}) = (R^{-1}\mathbf{v}, R^{-1}\mathbf{u}) - (R_0^{-1}\mathbf{v}_0, R_0^{-1}\mathbf{u}_0)$$

as the strain in body coordinates. The coefficients of v and u with respect to the coordinate system spanned by the directors $\mathbf{d}_i$ can be interpreted in a natural way. In particular, $\mathsf{v}_1, \mathsf{v}_2$ are the shear strains, because they describe a displacement inside the cross section, while v_3—a displacement normal to the cross-section—is called the stretching strain. Further, the values $\mathsf{u}_1, \mathsf{u}_2$—infinitesimal rotations about an axis in the cross section—are the bending strains, and u_3 is the strain related to torsion.

Constitutive Laws. The stress variables in an elastic Cosserat rod model are the total resultant force and moment across a cross-section. Forces and moments are linked to the strain by constitutive relations which describe the properties of specific materials. We assume that the rod material is hyperelastic in the sense that there exists an energy functional $W(\mathsf{v}, \mathsf{u})$ with $\mathsf{v}, \mathsf{u} \in \mathbb{R}^3$ such that

$$\mathsf{n}^T := W_\mathsf{v}(\mathsf{v}, \mathsf{u}) = \frac{\partial W}{\partial \mathsf{v}}(\mathsf{v}, \mathsf{u}),$$
$$\mathsf{m}^T := W_\mathsf{u}(\mathsf{v}, \mathsf{u}) = \frac{\partial W}{\partial \mathsf{u}}(\mathsf{v}, \mathsf{u})$$

are the forces and moments in body coordinates. We refer to $\mathsf{m}_1, \mathsf{m}_2$ as the bending moments and to m_3 as the twisting moment. The values $\mathsf{n}_1, \mathsf{n}_2$ are shear forces and n_3 is the tension.

In spatial coordinates we obtain the forces $\mathbf{n} = R\mathsf{n}$ and moments $\mathbf{m} = R\mathsf{m}$. We note that due to $R^T = R^{-1}$ we get

$$\mathsf{n}^T \mathsf{v} = \mathsf{n}^T R^{-1} R \mathsf{v} = (R\mathsf{n})^T R\mathsf{v} = \mathbf{n}^T \mathbf{v}.$$

Absence of stress in the reference configuration $\boldsymbol{\rho}_0$ means that

$$W_{\mathsf{v}}(0,0) = W_{\mathsf{u}}(0,0) = 0.$$

We assume the strain-energy function W to be convex, Fréchet-differentiable, and coercive in the sense that

$$\frac{W(\mathsf{w},\mathsf{z})}{|\mathsf{w}|^2 + |\mathsf{z}|^2} \geq \alpha \quad \text{as} \quad |\mathsf{w}|^2 + |\mathsf{z}|^2 \to \infty$$

for some fixed $\alpha > 0$.

Formulation as a Minimization Problem. The stable equilibrium configurations of a Cosserat rod with a hyperelastic material law can be characterized as the minima of an energy functional

$$j : \boldsymbol{\rho} \mapsto \int_{[0,l]} W\big(\mathsf{v}(\boldsymbol{\rho}), \mathsf{u}(\boldsymbol{\rho})\big)\, ds. \tag{1}$$

To discuss the well-posedness of minimization problems for this functional we need to introduce certain function spaces. Note that SE(3) arises naturally as a submanifold of $\mathbb{R}^3 \times \mathbb{R}^{3\times 3} \simeq \mathbb{R}^{12}$. We define (see, e.g., Bethuel (1991))

$$H^1([0,l], \mathrm{SE}(3)) := \big\{ v \in H^1([0,l], \mathbb{R}^{12}) \mid v(x) \in \mathrm{SE}(3) \text{ a.e.} \big\}.$$

If we impose Dirichlet boundary conditions

$$\boldsymbol{\rho}(0) = (r_0, q_0) \quad \text{and} \quad \boldsymbol{\rho}(l) = (r_l, q_l), \tag{2}$$

then the problem of finding stable equilibrium configurations of Cosserat rods can be written as the optimization problem

$$\min j \text{ in } H^1([0,l], \mathrm{SE}(3)) \text{ subject to (2).}$$

Existence and regularity of solutions to this problem have been shown by Seidman and Wolfe (1988). The main difficulty is the additional condition

$$r'(s) \cdot \mathbf{d}_3 > 0 \qquad \text{for all } s \in [0,l] \tag{3}$$

used in that paper, which assures the preservation of orientation, since it is a strict inequality and hence the admissible set is open. Nevertheless, the following regularity result holds.

Theorem 2.1. (Seidman and Wolfe (1988), Thm. 4.24) Let $\boldsymbol{\rho}$ be a solution of the minimization problem (1), subject to the boundary conditions (2) and the orientation condition (3). Then $(\mathbf{u}(\boldsymbol{\rho}), \mathbf{v}(\boldsymbol{\rho}))$ is in $(C^1[0,l])^6$.

Seidman and Wolfe also showed that solutions are generally not unique.

2.3 Neumann-Type Boundary Conditions

To be able to understand the constraint forces for the coupling discussed in Section 4, we have to consider a special type of Neumann boundary conditions for Cosserat rods.

Let $\boldsymbol{\rho} : [0,l] \to \mathrm{SE}(3)$ be a configuration of a rod, fixed at $s = l$. An admissible variation $\delta\boldsymbol{\rho}$ is a map $\delta\boldsymbol{\rho} : [0,l] \to T\mathrm{SE}(3)$ with $\pi(\delta\boldsymbol{\rho}(s)) = \boldsymbol{\rho}(s)$, where $\pi : T\mathrm{SE}(3) \to \mathrm{SE}(3)$ is the canonical projection, and $\delta\boldsymbol{\rho}(l) = 0$. We want to investigate Neumann-type boundary conditions in a variational form. The case of pure Dirichlet conditions has been treated by Chouaïeb (2003). Assume therefore that $\boldsymbol{\rho}$ satisfies the following weak formulation

$$0 = \frac{d}{d\boldsymbol{\rho}} j(\boldsymbol{\rho})\delta\boldsymbol{\rho} - [h \cdot \delta\boldsymbol{\rho}]_0.$$

Here $h : T_{\boldsymbol{\rho}(0)}\mathrm{SE}(3) \to \mathbb{R}$ is a linear functional. We introduce a force field $H : \mathcal{A}(0) \to \mathbb{R}^3$ on the cross-section $\mathcal{A}(0)$ by requiring

$$h \cdot \delta\boldsymbol{\rho} := \int_{\mathcal{A}(0)} H(\xi) \cdot (\delta r + \delta R \xi)\, d\xi, \tag{4}$$

where ξ are the coordinates with respect to the directors $\mathbf{d}_1$ and $\mathbf{d}_2$.

Theorem 2.2. Let $h : T_{\boldsymbol{\rho}(0)}\mathrm{SE}(3) \to \mathbb{R}$ be a linear functional as in (4), and let $\boldsymbol{\rho} : [0,l] \to \mathrm{SE}(3)$ be a rod configuration such that

$$0 = \frac{d}{d\boldsymbol{\rho}} j(\boldsymbol{\rho})\delta\boldsymbol{\rho} - [h \cdot \delta\boldsymbol{\rho}]_0$$

for all admissible variations $\delta\boldsymbol{\rho} = (\delta r, \delta R)$. Then, in spatial coordinates the Euler–Lagrange equations of (1) read

$$\mathbf{n}' = 0, \qquad \text{on } [0,l],$$
$$\mathbf{m}' + r' \times \mathbf{n} = 0, \qquad \text{on } [0,l],$$

and at the boundary $s = 0$ we obtain Neumann-type conditions

$$\mathbf{n}(0) = \int_{\mathcal{A}(0)} H(\xi)\, d\xi, \quad \mathbf{m}(0) = \int_{\mathcal{A}(0)} R(0)\xi \times H(\xi)\, d\xi.$$

3. ELASTIC CONTINUA

We now describe our model of the elastic continuum. Let $\mathbb{E}^3$ be a three-dimensional Euclidean space, which we will use as the parameter space. A body is an open connected subset $\mathcal{B} \subset \mathbb{E}^3$. A configuration of $\mathcal{B}$ is a mapping $\boldsymbol{\phi} : \mathcal{B} \to \mathbb{R}^3$ such that $\boldsymbol{\phi}(\mathcal{B})$ is open and connected and $\boldsymbol{\phi}$ has an inverse $\boldsymbol{\phi}^{-1} : \boldsymbol{\phi}(\mathcal{B}) \to \mathcal{B}$.

For a given deformation $\boldsymbol{\phi}$ we define the deformation gradient

$$\mathbf{F} : T\mathcal{B} \to T\mathbb{R}^3, \qquad \mathbf{F} = \nabla\boldsymbol{\phi}.$$

For ease of notation we write $x_{\boldsymbol{\phi}} = \boldsymbol{\phi}(x)$. We single out one configuration $\boldsymbol{\phi}_0 : \mathcal{B} \to \mathbb{R}^3$ and call it the reference configuration. In principle it is possible to use any configuration as the reference configuration. We use the one obtained by identifying $\mathbb{E}$ with $\mathbb{R}^3$ and setting $\boldsymbol{\phi}_0 = \mathrm{Id}$. Then, for the reference configuration we have $\nabla\boldsymbol{\phi}_0 \equiv \mathrm{Id}$.

We assume the boundary $\partial\mathcal{B}$ to be Lipschitz continuous and to consist of two disjoint parts $\partial_D\mathcal{B}$ and $\partial_N\mathcal{B}$ such that $\partial\mathcal{B} = \overline{\partial_D\mathcal{B}} \cup \overline{\partial_N\mathcal{B}}$. The unit boundary normals in the undeformed and in the deformed region are denoted by $\boldsymbol{\nu}$ and $\boldsymbol{\nu}_{\boldsymbol{\phi}}$, respectively.

Let $\boldsymbol{\phi}_D : \partial_D\mathcal{B} \to \mathbb{R}^3$ be a prescribed displacement on the Dirichlet boundary $\partial_D\mathcal{B}$. By the Cauchy theorem, conservation of linear momentum yields the boundary value problem

$$\begin{aligned} -\operatorname{div} \boldsymbol{\sigma} &= f_{\boldsymbol{\phi}} && \text{in } \boldsymbol{\phi}(\mathcal{B}) \\ \boldsymbol{\sigma}\boldsymbol{\nu}_{\boldsymbol{\phi}} &= g_{\boldsymbol{\phi}} && \text{on } \boldsymbol{\phi}(\partial_N \mathcal{B}) \\ \boldsymbol{\phi} &= \boldsymbol{\phi}_D && \text{on } \partial_D \mathcal{B}. \end{aligned}$$

Here, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $f_{\boldsymbol{\phi}}$ is a volume force and $g_{\boldsymbol{\phi}}$ is a surface force, all on the deformed region.

By introducing the first Piola–Kirchhoff stress tensor

$$P(x) := \det \nabla \boldsymbol{\phi}(x) \boldsymbol{\sigma}(\boldsymbol{\phi}(x)) \nabla \boldsymbol{\phi}(x)^{-T}$$

we can reformulate the equilibrium conditions in the reference domain

$$\begin{aligned} -\operatorname{div} P &= f && \text{in } \mathcal{B}, \\ P\boldsymbol{\nu} &= g && \text{on } \partial_N \mathcal{B}, \\ \boldsymbol{\phi} &= \boldsymbol{\phi}_D && \text{on } \partial_D \mathcal{B}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} f(x) &= \det \nabla \boldsymbol{\phi}(x) f_{\boldsymbol{\phi}}(x_{\boldsymbol{\phi}}), \\ g(x) &= \det \nabla \boldsymbol{\phi}(x) \cdot \left| \nabla \boldsymbol{\phi}(x)^{-T} \boldsymbol{\nu}_{\boldsymbol{\phi}} \right| \cdot g_{\boldsymbol{\phi}}(x_{\boldsymbol{\phi}}). \end{aligned}$$

We cast the boundary value problem in a variational form. Let the space of admissible configurations be

$$\mathcal{C} := \{ \boldsymbol{\phi} \in \mathbf{H}^1(\mathcal{B}) \mid \boldsymbol{\phi} = \boldsymbol{\phi}_D \text{ on } \partial_D \mathcal{B} \}.$$

We consider hyperelastic continua, i.e., we assume the existence of a stored energy function $\hat{W}(X, \mathbf{F})$ which induces an energy functional

$$E(\boldsymbol{\phi}) = \int_{\mathcal{B}} \left[\hat{W}(x, \nabla \boldsymbol{\phi}) - V_f(\boldsymbol{\phi}) \right] dV - \int_{\partial_N \mathcal{B}} V_g(\boldsymbol{\phi}) \, dA. \tag{6}$$

We restrict our attention to dead loads $V_f(\boldsymbol{\phi}) = f \cdot \boldsymbol{\phi}$ and $V_g(\boldsymbol{\phi}) = g \cdot \boldsymbol{\phi}$.

Finally, we make two assumptions on the energy functional. The first guarantees existence of an energy minimizer.

Assumption 3.1. The energy functional E is weakly lower semi-continuous and coercive in the space $\mathbf{H}^1(\mathcal{B})$.

Large classes of stored energy functionals satisfy this condition, in particular, linearly elastic materials, and polyconvex materials, such as Mooney–Rivlin materials. Our restriction to the space $\mathbf{H}^1(\mathcal{B})$ is merely for simplicity. In the linear case, convexity and coercivity of E follow from Korn's inequality, and weak lower semi-continuity is a consequence of the convexity of E. In the polyconvex case, E is non-convex, and the proof of weak lower semi-continuity of E is involved (see, e.g., (Ciarlet, 1988, Chap. 7)).

The second assumption concerns the regularity at the minimizer.

Assumption 3.2. The energy functional E is differentiable at any local minimizer $\boldsymbol{\phi}_*$.

This is again clear in the case of linear elasticity. However, for realistic polyconvex energy functions it cannot be guaranteed a priori. If it does hold, we can derive the Euler–Lagrange-equation of (6)

$$\begin{aligned} 0 &= T_{\boldsymbol{\phi}_*} E(\delta \boldsymbol{\phi}) \\ &= \int_{\mathcal{B}} \left[P(x, \nabla \boldsymbol{\phi}_*) \delta \boldsymbol{\phi} - f \cdot \delta \boldsymbol{\phi} \right] dV - \int_{\partial_N \mathcal{B}} g \cdot \delta \boldsymbol{\phi} \, dA, \end{aligned} \tag{7}$$

for all $\delta \boldsymbol{\phi} \in C^\infty(\mathcal{B}) \cap \mathcal{C}$. It turns out (cf. (Ciarlet, 1988, Chap. 4)) that the first Piola–Kirchhoff tensor can be identified as

$$P(x) = P(x, \nabla \boldsymbol{\phi}) = \frac{\partial \hat{W}(x, \nabla \boldsymbol{\phi})}{\partial \nabla \boldsymbol{\phi}}.$$

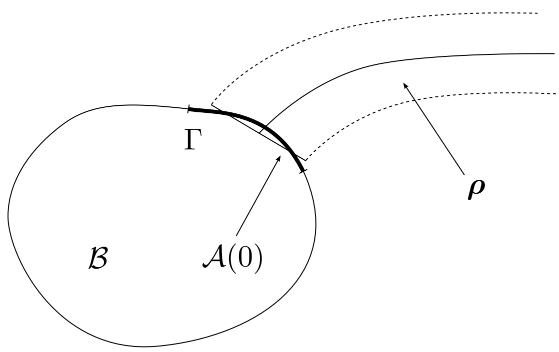

Fig. 2. Coupling between a two-dimensional domain and a rod

After formal integration by parts, the weak form (7) is formally equivalent to (5).

4. COUPLING CONDITIONS

We will now derive conditions for the coupling of an elastic three-dimensional object and a Cosserat rod. Let $\mathcal{B} \subset \mathbb{E}^3$ as in Section 3. However, the boundary $\partial \mathcal{B}$ is now supposed to consist of three disjoint parts Γ_D, Γ_N, and Γ such that $\partial \mathcal{B} = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}$. We assume that Γ_D and Γ have positive two-dimensional measure. The three-dimensional object represented by $\mathcal{B}$ will couple with the rod across Γ, and we call Γ the coupling boundary. Consider also a Cosserat rod defined on the interval $[0, l]$, with reference configuration $\boldsymbol{\rho}_0 : [0, l] \to \mathrm{SE}(3)$. The boundary of the one-dimensional parameter domain $[0, l]$ consists of the two points 0 and l. To be specific, we pick 0 as the coupling boundary (Figure 2).

Our coupling conditions involve the primal variables on the coupling boundaries. These are the restriction of the deformation $\boldsymbol{\phi}$ on Γ

$$\boldsymbol{\phi}|_\Gamma \in \mathbf{H}^{1/2}(\Gamma),$$

and the position and orientation of the rod cross-section at 0

$$\boldsymbol{\rho}(0) = (r(0), R(0)) \in \mathrm{SE}(3).$$

Since the two configuration spaces are not the same there are actually two classes of coupling conditions for the primal variables (cf. Blanco et al. (2011) for the linear case). One formulates the conditions in $\mathbf{H}^{1/2}(\Gamma)$, the configuration space of the coupling boundary of the continuum, and the other one formulates them on SE(3), the configuration space of the coupling boundary of the rod. Both choices have advantages and disadvantages. While $\mathbf{H}^{1/2}(\Gamma)$ is infinite-dimensional but linear, SE(3) is finite-dimensional but nonlinear.

4.1 A Variational Approach

In order to derive a model for the coupling between an elastic continuum and a Cosserat rod, we take a variational approach. This means that we formulate primal coupling conditions of the form

$$c(\boldsymbol{\phi}, \boldsymbol{\rho}) = 0$$

with a constraint mapping into a linear space V

$$c \;:\; \mathbf{H}^1(\mathcal{B}) \times H^1([0, l], \mathrm{SE}(3)) \to V,$$

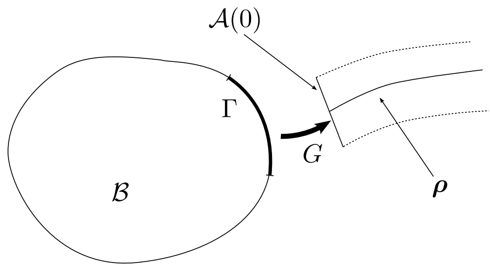

Fig. 3. Pointwise mapping G between the coupling boundary Γ and the rod coupling cross-section $\mathcal{A}(0)$

and consider the energy minimization problem

$$\min E(\boldsymbol{\phi}) + j(\boldsymbol{\rho}) \ \text{ s.t. } \ c(\boldsymbol{\phi}, \boldsymbol{\rho}) = 0.$$

At a minimizer $(\boldsymbol{\phi}_*, \boldsymbol{\rho}_*)$ we derive existence of Lagrange multipliers $\lambda \in V^*$ such that first-order optimality conditions are fulfilled in a function space sense

$$T_{\boldsymbol{\phi}_*} E + T_{\boldsymbol{\rho}_*} j + (T_{(\boldsymbol{\phi}_*, \boldsymbol{\rho}_*)} c)^* \lambda = 0. \tag{8}$$

Then we will interpret this system of equations as equilibrium conditions, where λ plays the role of a constraint force at the contact boundary. In this way, the so-called dual coupling conditions appear as a consequence of the primal coupling conditions and a variational principle. Since V depends on the type of coupling conditions, and $\lambda \in V^*$, we obtain different types of constraint forces for the two different coupling conditions.

In the remainder of this section we will introduce the two alternative coupling conditions. The discussion of existence of minimizers and the first-order optimality conditions is given in Section 5.

4.2 Rigid Coupling

Our first coupling condition mandates that the interface Γ be coupled pointwise to the cross-section $\mathcal{A}(0)$. To formulate this precisely we introduce the parameter domain

$$C := \Big\{ (\xi_1, \xi_2) \in \mathbb{R}^2 \ \big| \ r(0) + \sum_{i=1}^{2} \xi_i \mathbf{d}_i(0) \in \mathcal{A}(0) \Big\},$$

and a C^1-diffeomorphism

$$G : \Gamma \to C$$

(see Figure 3). Then we require

$$\boldsymbol{\phi}(x) = R(0)G(x) + r(0) \qquad \text{for almost all } x \in \Gamma.$$

This means that the rod and the continuum coincide at the interface. Our nonlinear operator c is then defined as

$$\begin{aligned} c \ &: \mathbf{H}^1(\mathcal{B}) \times H^1([0,l], \mathrm{SE}(3)) \to \mathbf{H}^{1/2}(\Gamma) \\ c(\boldsymbol{\phi}, \boldsymbol{\rho}) &= \tau\boldsymbol{\phi}(x) - (r(0) + R(0)G(x)) \end{aligned} \tag{9}$$

where $\tau : \mathbf{H}^1(\mathcal{B}) \to \mathbf{H}^{1/2}(\Gamma)$ is the trace mapping.

Proposition 4.1. Suppose that $(\boldsymbol{\phi}, \boldsymbol{\rho})$ satisfies the first order optimality conditions (8) with c given by (9). Then the following equilibrium of forces and moments holds

$$\begin{aligned} \mathbf{n}(0) &= \int_\Gamma P\boldsymbol{\nu}(x)\, dA \\ \mathbf{m}(0) &= \int_\Gamma (\boldsymbol{\phi}(x) - r(0)) \times P\boldsymbol{\nu}(x)\, dA, \end{aligned} \tag{10}$$

as long as $P\boldsymbol{\nu}(x) = P(x, \nabla\boldsymbol{\phi}(x))\boldsymbol{\nu}(x)$ is an integrable function.

The integrals in the balance equations (10) could equivalently be written on $\mathcal{A}(0)$ instead of Γ, which would fit to the Neumann boundary conditions in Section 2.3. Then, a factor $\det G'(x)^{-1}$ would appear.

4.3 Coupling on Average

Our second coupling condition is formulated in SE(3), the coupling trace space of the rod configuration space. We require that the equality of position and rotation only holds "on average"

$$0 = \int_\Gamma \boldsymbol{\phi}(x) - (r(0) + R(0)G(x))\, dA, \tag{11}$$

$$0 = \int_\Gamma (\boldsymbol{\phi}(x) - r(0)) \times R(0)G(x)\, dA. \tag{12}$$

Equation (11) states that on average $\boldsymbol{\phi}(x)$ coincides with $r(0) + R(0)G(x)$, while Eq. (12) states that on average the rod cross-section and the continuum boundary rotate the same way.

Proposition 4.2. For fixed $\boldsymbol{\phi}$ consider the minimization problem

$$\min_{(r,R) \in \mathrm{SE}(3)} \frac{1}{2} \big\| \boldsymbol{\phi}(x) - (r(0) + R(0)G(x)) \big\|^2_{L_2(\Gamma)}.$$

A minimizer (r, R) of this problem satisfies (11)–(12).

We can thus interpret these conditions in the following way: for given displacement $\boldsymbol{\phi} \in \mathbf{H}^{1/2}(\Gamma)$, find a rod position that fits best in the sense of least squares.

The coupling condition is cast in our variational framework by defining the operator

$$c \ : \mathbf{H}^1(\mathcal{B}) \times H^1([0,l], \mathrm{SE}(3)) \to \mathbb{R}^3 \times \mathbb{R}^3$$

$$c(\boldsymbol{\phi}, \boldsymbol{\rho}) = \begin{pmatrix} \displaystyle\int_\Gamma \boldsymbol{\phi}(x) - (r(0) + R(0)G(x))\, dA \\ \displaystyle\int_\Gamma (\boldsymbol{\phi}(x) - r(0)) \times R(0)G(x)\, dA \end{pmatrix}. \tag{13}$$

Proposition 4.3. Suppose that $(\boldsymbol{\phi}, \boldsymbol{\rho})$ satisfies the first-order optimality conditions (8) with c given by (13). Then the following equilibrium of forces and moments holds

$$\begin{aligned} \mathbf{n}(0) &= \int_\Gamma P\boldsymbol{\nu}(x)\, dA \\ \mathbf{m}(0) &= \int_\Gamma (\boldsymbol{\phi}(x) - r(0)) \times P\boldsymbol{\nu}(x)\, dA, \end{aligned} \tag{14}$$

where $P\boldsymbol{\nu}$ is of the form

$$P\boldsymbol{\nu}(x) = \eta + \Lambda R G(x) \tag{15}$$

for some $(\eta, \Lambda) \in \mathbb{R}^3 \times \mathfrak{so}(3)$.

Let us conclude this section with a comparison of the two sets of balance equations (10) and (14). Obviously, both have the same form, algebraically. However, while in the rigid coupling case $P\boldsymbol{\nu}$ is generically an element of $\mathbf{H}^{1/2}(\Gamma)^*$, in the averaged case $P\boldsymbol{\nu}$ defined via (15) as an element of a 6-dimensional space. This is a consequence of the face that the image space V of c is given by $V = \mathbf{H}^{1/2}(\Gamma)$ for rigid coupling and by $V = \mathbb{R}^3 \times \mathbb{R}^3$ for averaged coupling.

5. ENERGY MINIMIZERS OF THE COUPLED PROBLEM

Having introduced two alternative coupling conditions, together with formal balance equations on forces and moments, we now discuss the existence of energy minimizers for the coupled problems.

5.1 Existence of Minimizers

Recall the well-known definition of the indicator functional ι_S of a set S

$$\iota_S(x) = \begin{cases} 0 & x \in S, \\ \infty & x \notin S, \end{cases}$$

and define

$$F := \{(\boldsymbol{\phi}, \boldsymbol{\rho}) \in \mathbf{H}^1(\mathcal{B}) \times H^1([0,l], \mathrm{SE}(3)) \mid c(\boldsymbol{\phi}, \boldsymbol{\rho}) = 0\},$$

where c is given by either (9) or (13). Write down the total energy of the coupled system

$$\mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\rho}) = E(\boldsymbol{\phi}) + W(\boldsymbol{\rho}) + \iota_F(\boldsymbol{\phi}, \boldsymbol{\rho})$$

and note that this is a function

$$\mathcal{E} : \mathbf{H}^1(\mathcal{B}) \times H^1([0,l], \mathrm{SE}(3)) \to \mathbb{R}.$$

Remark 5.1. We may easily extend our analysis to an arbitrary number of finitely many rods, continua, and couplings. Then

$$\mathcal{E} = \sum_{k=1}^{n_E} E_k(\boldsymbol{\phi}_k) + \sum_{i=1}^{n_W} W_i(\boldsymbol{\rho}_i) + \sum_{j=1}^{n_C} \iota_{F_j}(\boldsymbol{\phi}_{i(j)}, \boldsymbol{\rho}_{k(j)}).$$

This would complicate the notation, without increasing the mathematical difficulty of the problem.

Theorem 5.1. The energy functional $\mathcal{E}$ has a global minimizer in among all $(\boldsymbol{\rho}, \boldsymbol{\phi}) \in \mathcal{C} \times H^1([0,l], \mathrm{SE}(3))$ such that $\boldsymbol{\rho}(l) = \boldsymbol{\rho}_l$.

For the proof of this theorem one essentially has to show weak lower semi-continuity of the energy functional $\mathcal{E}$. Then the standard proof on the existence of minimizers can be performed, cf. (Ekeland and Temam, 1999, Ch. II.1). Weak lower semi-continuity of $\mathcal{E}$ follows from weak lower semi-continuity of the addends, which is shown in Sander and Schiela (in preparation).

Remark 5.2. Theorem 5.1 asserts the existence of a global minimizer. Since the functional $\mathcal{E}$ is not convex, this minimizer may not be unique. The best one can hope for is local uniqueness, for which second-order sufficient optimality conditions have to be imposed.

5.2 First-Order Necessary Conditions

Finally we derive the first-order necessary conditions for energy minimizers of our coupled problem. To this end, let $(\boldsymbol{\phi}_*, \boldsymbol{\rho}_*)$ be a local minimizer of $\mathcal{E}$, again with c given by either (9) or (13). We impose the following additional local assumption on the energy functionals $E(\boldsymbol{\phi})$ and $j(\boldsymbol{\rho})$.

Assumption 5.1. Assume that E is Fréchet differentiable at $\boldsymbol{\phi}_*$ in $\mathbf{W}^{1,\infty}(\mathcal{B})$, with derivative $T_{\boldsymbol{\phi}_*}E(\boldsymbol{\phi}_*) \in \mathbf{H}^1(\mathcal{B})^*$. Assume further that j is Fréchet differentiable at $\boldsymbol{\rho}_*$ in $\mathbf{W}^{1,\infty}([0,l], \mathrm{SE}(3))$ with derivative $T_{\boldsymbol{\rho}_*}j \in H^1([0,l], \mathbb{R}^3 \times \mathfrak{so}(3))^*$.

Moreover, for the case of rigid coupling we have to assume that the trace mapping

$$\tau : \mathbf{H}^3(\mathcal{B}) \to \mathbf{H}^{5/2}(\Gamma) \tag{16}$$

is continuous and surjective. This is true for sufficiently regular $\mathcal{B}$ and Γ.

The restriction to the space $\mathbf{W}^{1,\infty}(\mathcal{B})$ is needed if we want to include polyconvex materials that avoid local self penetration. For such materials differentiability in weaker norms does not hold. Under these assumption the following theorem can be shown, which justifies our force and moment balance equations in Section 4.

Theorem 5.2. There exists a Lagrange multiplier $\lambda \in V^*$ such that the first order optimality conditions (8) have a solution. For the rigid coupling we have $V^* = \mathbf{H}^{1/2}(\Gamma)^*$, for the average coupling $V^* = \mathbb{R}^3 \times \mathbb{R}^3 \cong \mathbb{R}^3 \times \mathfrak{so}(3)$.

The proof is based on a combination of standard techniques. First, we show, using the implicit function theorem, that 0 minimizes a linearized problem on $\ker T_{(\boldsymbol{\phi}_*, \boldsymbol{\rho}_*)}c$. To this end, we need surjectivity of $T_{(\boldsymbol{\phi}_*, \boldsymbol{\rho}_*)}c$ in a sufficiently regular space, contained in $\mathbf{W}^{1,\infty}(\mathcal{B})$. This is always true for the case of averaged coupling, and in the case of rigid coupling, it holds by assumption due to (16). Then we show existence of a Lagrangian multiplier with the help of convex analysis, cf. (Ekeland and Temam, 1999, Ch. I.5).

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