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Convergence Analysis of Smoothing Methods for Optimal Control of Stationary Variational Inequalities

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Abstract. In the article an optimal control problem subject to a stationary variational inequality is investigated. The optimal control problem is complemented with pointwise control constraints. The convergence of a smoothing scheme is analyzed. There, the variational inequality is replaced by a semilinear elliptic equation. It is shown that solutions of the regularized optimal control problem converge to solutions of the original one. Passing to the limit in the optimality system of the regularized problem allows to prove C-stationarity of local solutions of the original problem. Moreover, convergence rates with respect to the regularization parameter for the error in the control are obtained. These rates coincide with rates obtained by numerical experiments, which are included in the paper.

Keywords. Variational inequalities, optimal control, control constraints, regularization, C-stationarity, path-following.
MSC subject qualification. 49K20, 65K15, 49M20, 90C33

1 Introduction, problem statement, regularization

In this article we analyze a regularization algorithm to solve the following non-smooth optimal control problem: Minimize the function J given by

$$J(y,u) = g(y) + j(u) \tag{P}$$

over all $(y,u) \in K \times L^2(\Omega)$ subject to the elliptic variational inequality

$$\langle Ay,\, v-y\rangle \geq (u,\, v-y) \quad \forall v \in K \tag{1.1}$$

and the control constraints

$$u \in U_{ad} := \{u \in L^2(\Omega) :\ u_a \leq u \leq u_b \text{ a.e. on } \Omega\}. \tag{1.2}$$

Here, $\Omega \subset \mathbb{R}^n$, $n = 2,3$, is a bounded Lipschitz domain. The pairings $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$ are short-hand notations for the duality pairing in $H^{-1}(\Omega) \times H_0^1(\Omega)$ and the scalar product in $L^2(\Omega)$, respectively. The operator A is an elliptic second-order differential operator. The set K is given by

$$K = \{v \in H_0^1(\Omega) : v \leq \psi\}. \tag{1.3}$$

Hence, the variational inequality is a classical obstacle problem. Here, the function $\psi \in H^1(\Omega) \cap L^\infty(\Omega)$ represents the obstacle. The assumptions on the various ingredients of the optimization problem (P) will be made precise below.

The optimization problem (P) is a non-smooth and non-convex optimization problem. As we shall argue below, the non-smoothness arises due to the variational inequality constraint (1.1). This non-smoothness makes it challenging to prove sharp first-order necessary optimality conditions. In addition, it is difficult to develop fast solution algorithms to solve (P). In the present paper, we investigate a smoothing scheme for (P) to cope with both issues.

Let us show briefly how the variational inequality (1.1) makes the problem non-smooth. If the operator A satisfies the classical assumptions

$$\nu_1\|v\|^2_{H^1_0} \leq \langle Av, v\rangle, \text{ and } \langle Av, w\rangle \leq \nu_2\|v\|_{H^1_0}\|w\|_{H^1_0}, \tag{1.4}$$

with $0 < \nu_1 \leq \nu_2$ then the variational inequality (1.1) is uniquely solvable, see [12]. Introducing a multiplier λ, the variational inequality can be written equivalently as

$$Ay + \lambda = u, \quad y \leq \psi,\ \lambda \geq 0,\ \langle \lambda, y - \psi\rangle = 0, \tag{1.5}$$

where $\lambda \in H^{-1}$, and $\lambda \geq 0$ is short for $\langle \lambda, v\rangle \geq 0$ for all $v \in H^1_0(\Omega)$, with $v \geq 0$. With the reformulation (1.5), the problem (P) is an optimization problem subject to a complementary condition constraint.

Under rather mild conditions [4], one obtains $\lambda \in L^2(\Omega)$. Then the complementarity condition in (1.5) can equivalently be expressed as

$$\lambda = \max(0, \lambda + c(y - \psi)), \tag{1.6}$$

for any $c > 0$, which emphasizes the fact that the constraint (1.1) makes the optimal control problem (P) non-smooth.

Optimal control problems constrained by a variational inequality were studied over the last decades. Existence results and different kind of necessary optimality conditions were obtained Barbu [1], Bergounioux [2], Hintermüller and Kopacka [7], Hintermüller and Surowiec [9], Ito and Kunisch [10], Mignot [15], Mignot and Puel [16]. Sufficient optimality conditions were studied by Kunisch and Wachsmuth [13].

The first-order necessary optimality conditions derived in these papers correspond to different kind of stationarity concepts used in the context of finite-dimensional optimization problems subject to complementarity constraints [17], see also [7]. Let us recall the definitions of C- and strong stationarity.

Definition 1.1. *The point* $(y^*, u^*, \lambda^*) \in K \times U_{ad} \times L^2(\Omega)$ *is called* **C-stationary** *for* (P) *if it satisfies* (1.5) *and if there exist* $(p^*, \mu^*) \in (H^1_0(\Omega) \cap L^\infty(\Omega)) \times (H^{-1}(\Omega) \cap L^\infty(\Omega)^*)$ *such that the following system is fulfilled:*

$$A^*p^* + \mu^* = g'(y^*), \tag{1.7a}$$

$$(j'(u^*) + p^*,\ u - u^*) \geq 0 \quad \forall u \in U_{ad}, \tag{1.7b}$$

$$p^* = 0 \quad \text{a.e. on } \{x \in \Omega :\ \lambda^* > 0\}, \tag{1.7c}$$

$$\langle \mu^*,\, \phi\rangle_{L^\infty(\Omega)^*, L^\infty(\Omega)} = 0\ \forall \phi \in C(\bar{\Omega}) :\ \phi = 0 \text{ a.e. on } \{x \in \Omega :\ y^* = \psi\} \tag{1.7d}$$

$$\langle \mu^*,\, p^*\rangle \geq 0. \tag{1.7e}$$

The point (y^*, u^*, λ^*) *is called* **strongly stationary** *if in addition*

$$p^* \geq 0 \quad \text{a.e. on } B, \tag{1.8a}$$

$$\langle \mu^*, \phi\rangle \geq 0 \text{ for all } \phi \in H^1_0(\Omega) : (\lambda^*,\, \phi) = 0,\ \phi \geq 0 \text{ a.e. on } B, \tag{1.8b}$$

is satisfied, where B *is the biactive set*

$$B := \{x \in \Omega :\ y^* = \psi,\ \lambda^* = 0\}.$$

Let us remark that C-stationarity gives $p^* = 0$ on the strongly active set $\{x \in \Omega : \lambda^* > 0\}$. Moreover, the support of μ^* is contained in the active set by (1.7d).

It is an open problem whether strong stationarity is a necessary optimality condition for problem (P). In fact, only in the special case $U_{ad} = L^2(\Omega)$ such a proof can be found in [16], see also [9]. For more general situations, it is difficult to prove the relations (1.8a) and (1.8b) on the biactive set. In general, only C-stationarity can be proven [1, 2, 10], [7] prove an even weaker variant, which they call ϵ-almost C-stationarity.

Due to the appearance of multipliers, which are only measures, and as a consequence of the complementarity conditions (1.5) and (1.7c)–(1.8b), stationarity systems are not well-suited for numerical realization. Algorithms to solve (P) are then based on a suitable smoothing of the underlying constraints. Hintermüller and Kopacka [7] used a relaxation scheme, where $\langle \lambda, y - \psi \rangle = 0$ is replaced by $\langle \lambda, y - \psi \rangle \leq \alpha$, coupled with a Moreau-Yosida regularization of the resulting state-constrained problem. Another approach is based on smoothing the max-function in (1.6), which is followed in Hintermüller and Kopacka [8], Ito and Kunisch [10], and Kunisch and Wachsmuth [14].

In this paper, we will use this smoothing approach and apply it to the original problem. That is, we replace the non-smooth condition $\lambda = \max(0, \lambda + c(y - \psi))$ by

$$\lambda_c = \max_c(0, \bar{\lambda} + c(y - \psi)).$$

Here, c is the regularization and smoothing parameter. Moreover, $\bar{\lambda} \in L^\infty(\Omega)$ is a given non-negative function, and $\max_c$ is a C^2-approximation of $x \to \max(0, x)$, which is made precise below. For properly chosen $\bar{\lambda}$ the solutions y_c of the regularized equation are feasible, i.e. $y_c \leq \psi$, see [10, 13, 14].

The resulting family of regularized problems is then given by

$$\left\{ \begin{array}{c} \min J(y, u) \\ \text{over } (y, u) \in H_0^1(\Omega) \times U_{ad} \text{ subject to} \\ Ay + \max_c(0, \bar{\lambda} + c(y - \psi)) = u. \end{array} \right. \tag{P_c}$$

An important observation is that this problem does not incorporate any inequality constraints on y_c, λ_c, and $\lambda_c(y_c - \psi)$.

If g and j are C^1-regular, then the first order optimality system for (P_c) is given by

$$Ay_c + \max_c\left(\bar{\lambda} + c(y_c - \psi)\right) = u_c, \tag{1.9a}$$

$$A^* p_c + c \max_c'\left(\bar{\lambda} + c(y_c - \psi)\right) p_c + g'(y_c) = 0, \tag{1.9b}$$

$$(j'(u_c) - p_c,\, u - u_c) \geq 0 \quad \forall u \in U_{ad}. \tag{1.9c}$$

In [11] and [13] existence of solutions $(y_c, u_c) \in H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega)$ to (1.9) for the case $U_{ad} = L^2(\Omega)$ was established and subsequential convergence for $c \to \infty$ to a solution (y^*, u^*) of the unregularized problem was argued. For the same subsequence we have

$$\lambda_c \to \lambda^* \text{ and } \mu_c \rightharpoonup \mu^* \text{ in } H^{-1}(\Omega),\ p_c \rightharpoonup p^* \text{ in } H_0^1(\Omega),$$

where $\to$ and $\rightharpoonup$ denote strong and weak convergence respectively, and the multiplier approximations λ_c and μ_c are defined by

$$\lambda_c = \max_c(0,\, \bar{\lambda} + c(y_c - \psi)), \qquad \mu_c = c \max_c'(\bar{\lambda} + c(y_c - \psi)).$$

Moreover, one can show that each strict local minimum of (P) is the (weak) limit of a sequence of solutions of the regularized problem [13]. Hence, one can use these sequences

to derive optimality conditions for the original problem. In this paper, we will extend these results to the control constrained case and prove C-stationarity of local solutions of (P).

This regularization concept then can be used to devise numerically implementable algorithms [14], that generate a sequence of solutions $\{(y_c, u_c)\}$. In [14] it was reported that the sequence converges as

$$\|u_c - u^*\|_{L^2(\Omega)} = O(c^{-1/2}).$$

We will prove this rate under assumptions on second-order information, see Theorem 4.5. Moreover, we prove for the value function

$$V(c) := J(y_c, u_c)$$

the estimate

$$\left|\frac{d}{dc}V(c)\right| = O(c^{-2}),$$

which can be found in Theorem 4.3 below. These findings improve the theoretical results and explain the convergence rates obtained in numerical experiments in [14]. In section 5, we report about our computational experiments, which confirm the convergence rates obtained in our analysis.

Standing assumptions

Throughout the paper we rely on the following regularity assumptions.

(A.i) The domain $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$ is a bounded Lipschitz domain.

(A.ii) The operator A is an elliptic differential operator defined by

$$(Ay)(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}y(x)\right) + \sum_{j=1}^{n} a_j(x)\frac{\partial}{\partial x_j}y(x) + a_0(x)y(x)$$

with functions $a_{ij} \in C^{0,1}(\bar{\Omega})$, $a_j, \frac{\partial}{\partial x_j}a_j, a_0 \in L^\infty(\Omega)$ satisfying the conditions $a_{ij}(x) = a_{ji}(x)$ and

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \delta_0|\xi|^2 \qquad \text{a.e. on } \Omega \text{ for all } \xi \in \mathbb{R}^n$$

with some $\delta_0 > 0$. Additionally, we require $a_0(x) \geq \delta_1 \geq 0$ with δ_1 sufficiently large such that A fulfills the coercivity condition (1.4).

(A.iii) The obstacle $\psi \in H^1(\Omega) \cap L^\infty(\Omega)$ fulfills $A\psi \in L^\infty(\Omega)$ and $\psi \geq 0$ on Γ.

The functions g, j satisfy:

(A.iv) $g : L^2(\Omega) \to \mathbb{R}$ is twice continuously Fréchet-differentiable,

(A.v) $j : L^2(\Omega) \to \mathbb{R}$ is twice continuously Fréchet-differentiable and weakly lower semicontinuous.

The function $\bar{\lambda}$ appearing in the regularization satisfies

(A.vi) $\bar{\lambda} \in L^\infty(\Omega)$, $\bar{\lambda} \geq 0$ a.e. on Ω.

Let us introduce the adjoint operator A^* to A by

$$(A^*p)(x) = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{n} a_{ij}(x)\frac{\partial}{\partial x_i}p(x) + a_j(x)p(x)\right) + a_0(x)p(x).$$

Assumptions on the smooth approximation of max

We assume that the function $\max_c$ admits the following properties:

(B.i) $\max_c : (c,x) \mapsto \max_c(x)$, $(c,x) \in (0,+\infty) \times \mathbb{R}$, is twice continuously differentiable with respect to (c,x),

(B.ii) $\max_c(x) = \max(0,x)$ for all x with $|x| \geq 1/2c$, $\max_c(x) \geq \max(0,x)$ for all x with $|x| \leq 1/2c$.

We will denote the derivatives with respect to x by $\max_c'$, $\max_c''$, whereas the derivatives with respect to c are denoted by $\frac{\partial}{\partial c}\max_c$.

In addition we assume that there is a constant $M > 0$ such that the following inequalities are satisfied for all x, x':

(B.iii) $0 \leq \max_c'(x) \leq 1$,

(B.iv) $0 \leq \max_c''(x)$,

(B.v) $\left|\frac{\partial}{\partial c}\max_c(x)\right| \leq \frac{M}{c^2}$,

Note, that the function

$$m_c(x) := \begin{cases} \max(0,x) & \text{if } |x| \geq \frac{1}{2c}, \\ \frac{c^3}{2}\left(x+\frac{1}{2c}\right)^3\left(\frac{3}{2c}-x\right) & \text{if } |x| < \frac{1}{2c}, \end{cases} \tag{1.10}$$

satisfies the requirements above, see [14]. Different kind of smooth approximations of $\max(0,\cdot)$ were used by Hintermüller and Kopacka [8].

A simple consequence of the monotonicity of $\max_c'$ provided by (B.iv) is the following inequality

$$\max_c(x) \leq \left(x + \frac{1}{2c}\right)\max_c'(x) \quad \forall x \in \mathbb{R}. \tag{1.11}$$

Notational convention

We will use in several places generic constants, all denoted by K. These constants are independent of c and sequences $\{(y_c, u_c)\}$ of local solutions of (P_c).

2 Regularization of the obstacle problem

Let us first recall the well-known existence and regularity results for the variational inequality (1.1), for the proofs we refer to [4, 12]. Then we proof additional results for the regularized obstacle problem that we will need later.

Proposition 2.1. *Let $u \in L^2(\Omega)$ be given. Then the variational inequality* (1.1) *admits a unique solution $y \in H_0^1(\Omega) \cap L^\infty(\Omega)$. The mapping $u \mapsto y$ is Lipschitz continuous. Moreover, it holds $\lambda, Ay \in L^2(\Omega)$.*

In addition, the mapping $u \mapsto y$ is directionally differentiable in a certain sense, see e.g. [15], but not Gâteaux differentiable.

2.1 Uniform boundedness of solutions

Let us now study the regularized equation

$$Ay + \max_c\left(\bar\lambda + c(y-\psi)\right) = u, \tag{2.1}$$

for fixed c and given $u \in L^2(\Omega)$. Since the function $\max_c$ is monotone, we have the following existence and uniqueness result.

Proposition 2.2. *Let $u \in L^2(\Omega)$ be given. Then (2.1) admits a unique solution $y \in H_0^1(\Omega)\cap L^\infty(\Omega)$. There is a constant $K > 0$ independent of c such that $\|y_c\|_{H_0^1} + \|y_c\|_{L^\infty} \le K\|u\|_{L^2}$.*

Proof. Existence and uniqueness of solutions follow from standard arguments. To obtain uniform bounds, one tests (2.1) by y_c and uses the uniform boundedness of $\max_c(\bar\lambda - c\psi)$. □

We are interested in the asymptotic behavior of the solutions of (2.1) for $c \to \infty$. In the next lemma we will show that the constraint violation $(y_c - \psi)_+$ tends to zero for $c \to \infty$. This improves earlier results of [10, 13]. Here, $(v)_+$ refers to the positive part of v, i.e. $(v)_+(x) = \max(0, v(x))$.

Lemma 2.3. *Let $u \in L^2(\Omega)$ be given. Let $y_{c,u}$ denote the corresponding solution of the regularized equation (2.1).*

If for some $2 \le q \le \infty$ it holds $u + \max(0, -A\psi - \bar\lambda) \in L^q(\Omega)$, then we have the estimates

$$\|(y_{c,u} - \psi)_+\|_{L^q} \le c^{-1}\|u + \max(0, -A\psi - \bar\lambda)\|_{L^q}, \tag{2.2}$$

$$\|(y_{c,u} - \psi)_+\|_{H_0^1} \le Kc^{-1/2},$$

and

$$\|Ay_{c,u}\|_{L^q} + \|\max_c\left(\bar\lambda + c(y_{c,u} - \psi)\right)\|_{L^q} \le K.$$

Proof. The proof follows to some extent the lines of the proof of [12, Lemma 5.1]. Clearly, the function $(y_{c,u} - \psi)_+^q$ belongs to $H_0^1(\Omega)$ for all $q \ge 2$, which follows from the chain rule taking into account the boundedness of $y_{c,u}$ and ψ. Hence, we can test the equation

$$A(y_{c,u} - \psi) + \max_c\left(\bar\lambda + c(y - \psi)\right) = u - A\psi,$$

which is equivalent to (2.1), by $(y_{c,u} - \psi)_+^{q-1}$.

By Assumption (A.ii), we have for every $z \in L^\infty \cap H_0^1(\Omega)$

$$\langle Az, z^{q-1}\rangle \ge \frac{2}{q}\langle Az^{q/2}, z^{q/2}\rangle + \left(1 - \frac{2}{q}\right)\int_\Omega a_0|z|^q \ge \frac{2\nu_1}{q}\,\|z^{q/2}\|_{H^1}^2.$$

Using the fact that $\max_c(x) \ge \max(0, x) \ge x$ for all $x \in \mathbb{R}$, cf. (B.ii), we obtain

$$\int_\Omega \max_c\left(\bar\lambda + c(y_{c,u} - \psi)\right)(y_{c,u} - \psi)_+^{q-1} \ge c\|(y_{c,u} - \psi)_+\|_{L^q}^q + \int_\Omega \bar\lambda(y_{c,u} - \psi)_+^{q-1}.$$

Combining these results yields

$$\begin{aligned} c\|(y_{c,u} - \psi)_+\|_{L^q}^q &\le (u - A\psi - \bar\lambda,\, (y_{c,u} - \psi)_+^{q-1}) \\ &\le \|u + \max(0, -A\psi - \bar\lambda)\|_{L^q}\|(y_{c,u} - \psi)_+\|_{L^q}^{q-1}, \end{aligned}$$

which gives

$$c\|(y_{c,u}-\psi)_+\|_{L^q} \le \|u+\max(0,-A\psi-\bar\lambda)\|_{L^q}. \tag{2.3}$$

This proves the first claim for $2 \le q < \infty$. If $u+\max(0,-A\psi-\bar\lambda) \in L^\infty(\Omega)$ then we can pass to the limit $q \to \infty$ in (2.3), which gives the wanted $L^\infty(\Omega)$-estimate, and (2.2) is proven. The estimate for the H^1-norm is an easy consequence of the estimates for $q = 2$.

Next, we write

$$Ay = u - \max_c\left(\bar\lambda + c(y_{c,u}-\psi)\right).$$

By (1.11) we have the estimate

$$\max_c\left(\bar\lambda + c(y_{c,u}-\psi)\right)(x) \le \frac{1}{2c} + \bar\lambda(x) + c(y_{c,u}(x)-\psi(x)).$$

Hence it holds $\|\max_c\left(\bar\lambda + c(y_{c,u}-\psi)\right)\|_{L^q} \le K$ by (2.2), with a constant $K > 0$ independent of q and $c > 2$. This implies $\|Ay_{c,u}\|_{L^q} \le K$. □

2.2 The linearized equation

Now let us consider the linearized version of (2.1)

$$Aw_c + c\max_c'\left(\bar\lambda + c(y_c-\psi)\right)w_c = r_c, \tag{2.4}$$

where y_c is a given solution of the regularized equation (2.1), and $r_c \in L^2(\Omega)$ is a given right-hand side. Regarding existence and boundedness of solutions we have the following:

Theorem 2.4. *Let $\{y_c\}_{c>0}$ and $\{r_c\}_{c>0}$ be bounded in $H_0^1(\Omega)$ and $L^2(\Omega)$, respectively. Then equation (2.4) admits for each $c > 0$ a unique solution $w_c \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and there is a constant $K > 0$ independent of c such that*

$$\begin{aligned}\|w_c\|_{H^1} + \|\sqrt{c\max_c'}w_c\|_{L^2} &\le K\|\left(1+c\max_c'\right)^{-1/2} r_c\|_{L^2},\\ \|c\max_c' w_c\|_{L^1} &\le K\|r_c\|_{L^1},\\ \|w_c\|_{L^\infty} &\le K\|r_c\|_{L^2}.\end{aligned}$$

Moreover, if $(c\max_c')^{-1}r_c \in L^\infty(\Omega)$, then we obtain

$$\|w_c\|_{L^\infty} \le K\|(c\max_c')^{-1}r_c\|_{L^\infty}$$

Here, we abbreviated $c\max_c' := c\max_c'\left(\bar\lambda + c(y_c-\psi)\right)$.

Proof. For our first result, we test (2.4) by w_c, and divide by the square-root of the left-hand side. The second assertion follows by testing with a smoothed-sign function similarly as in [10, Thm. 5.1]. The $L^\infty(\Omega)$-bound follows from the result of Stampacchia [18].

For our last result, we use a duality technique. For each $s \in L^2(\Omega)$ let $z_s = (A + c\max_c')^{-1}s$

$$\begin{aligned}\|w_c\|_{L^\infty} &= \sup_{s\in L^2,\|s\|_{L^1}\le 1}\langle w_c, s\rangle_{L^2}\\ &= \sup_s\langle (A+c\max_c')^{-1}r_c, s\rangle = \sup_s\langle r_c, z_s\rangle\\ &\le \sup_s\|(c\max_c')^{-1}r_c\|_{L^\infty}\|c\max_c' z_s\|_{L^1} \le K\|(c\max_c')^{-1}r_c\|_{L^\infty},\end{aligned}$$

which finishes the proof. □

We state a simple consequence of our previous result:

Corollary 2.5. *The mapping $u \to y_{c,u}$ is Lipschitz continuous in the following sense:*

$$\|A(y_{c,u_1} - y_{c,u_2})\|_{L^1} + \|y_{c,u_1} - y_{c,u_2}\|_{H^1} + \|y_{c,u_1} - y_{c,u_2}\|_{L^\infty} \le L\|u_1 - u_2\|_{L^2}$$

Moreover, we have the c-dependent Lipschitz estimate:

$$\|A(y_{c,u_1} - y_{c,u_2})\|_{L^2} \le K\sqrt{c}\|u_1 - u_2\|_{L^2}$$

Proof. This follows from the implicit function theorem and the mean value theorem. □

Remark 2.6. *Our estimates indicate that the mapping $u \to y_{c,u}$ cannot be expected to be Lipschitz-continuous from $L^2(\Omega)$ to $H^2(\Omega)$, uniformly in c. Still, by interpolation, we obtain the following uniform result on Hölder continuity:*

$$\|A(y_{c,u_1} - y_{c,u_2})\|_{L^2} \le K\|A(y_{c,u_1} - y_{c,u_2})\|_{L^\infty}^{1/2}\|A(y_{c,u_1} - y_{c,u_2})\|_{L^1}^{1/2} \le K\|u_1 - u_2\|_{L^2}^{1/2}.$$

2.3 Convergence results for fixed u

Let now $u \in L^\infty(\Omega)$ be fixed and $y_{c,u}$ the solution of the regularized obstacle problem (2.1) for parameter c. We consider the limiting behaviour of $J(y_{c,u}, u)$ for $c \to \infty$. To this end, we differentiate the equation

$$Ay_{c,u} + \max_c\left(\bar\lambda + c(y_{c,u} - \psi)\right) = u$$

with respect to c, and obtain

$$\begin{aligned} 0 &= A\dot y + c\max_c'\left(\bar\lambda + c(y_{c,u} - \psi)\right)\dot y \\ &\quad + \frac{\partial}{\partial c}\max_c(\bar\lambda + c(y_{c,u} - \psi)) + \max_c'(\bar\lambda + c(y_{c,u} - \psi))(y_{c,u} - \psi) = 0. \end{aligned} \tag{2.5}$$

Hence, the derivative $\dot y$ of $y_{c,u}$ with respect to c is the solution of a linearized equation.

Let us define the following family of sets that will play an important role in the subsequent analysis

$$\mathcal{P}_{c,u} = \left\{x \in \Omega :\ \bar\lambda(x) + c(y_{c,u}(x) - \psi(x)) > -\frac{1}{2c}\right\}. \tag{2.6}$$

That is, $\mathcal{P}_{c,u}$ contains the points, where $\max_c\left(\bar\lambda + c(y_{c,u} - \psi)\right) > 0$. Obviously $\max_c\left(\bar\lambda + c(y_{c,u} - \psi)\right) = 0$ on $\Omega \setminus \mathcal{P}_{c,u}$, and thus also $\max_c' = 0$ there.

Lemma 2.7. *It holds*

$$\|y_{c,u} - \psi\|_{L^\infty(\mathcal{P}_{c,u})} \le Kc^{-1}$$

with a constant $K > 0$ independent of c.

Proof. By definition of $\mathcal{P}_{c,u}$ it follows

$$\frac{1}{c}\left(-\bar\lambda - \frac{1}{2c}\right) \le y_{c,u} - \psi \le (y_{c,u} - \psi)_+.$$

Since $\|(y_{c,u} - \psi)_+\|_{L^\infty} \le Kc^{-1}$ by Lemma 2.3 and Assumption (A.iii), the desired result follows. □

Remark 2.8. *Let us remark that the bound* $\|y_{c,u} - \psi\|_{L^\infty(\mathcal{P}_{c,u})} = O(c^{-1})$ *is sharp in the following sense. Suppose it holds* $\|y_{c,u} - \psi\|_{L^q(\mathcal{P}_{c,u})} = o(c^{-1})$*. This implies on one hand the convergence of* $\lambda_{c,u} := \max_c\left(\bar\lambda + c(y_c - \psi)\right) \to \bar\lambda$ *in* L^q*. On the other hand,* $\lambda_{c,u}$ *converges to* λ_u *in* $H^{-1}(\Omega)$*, where* λ_u *is the multiplier in the obstacle problem associated to* u*, see Theorem 2.10 below. Hence,* $\bar\lambda = \lambda_u$ *is necessary to get* $\|y_{c,u} - \psi\|_{L^q(\mathcal{P}_{c,u})} = o(c^{-1})$*, which is unlikely, as this requires that the solution of the obstacle problem is known.*

Similarly, the estimate $\|(y_{c,u} - \psi)_+\|_{L^q} = O(c^{-1})$ *of Lemma 2.3 is sharp. The relation* $\|(y_{c,u} - \psi)_+\|_{L^q} = o(c^{-1})$ *implies* $0 \le \lambda_u \le \bar\lambda$ *a.e. on* Ω*. That is, if* $\lambda_u > \bar\lambda$ *on a set of positive measure, then the estimate* (2.2) *is sharp.*

The estimate of Lemma 2.7 will turn out to be essential for our final convergence estimate Theorem 4.5, as it allows to prove convergence rates of norms of $\dot y_c$ as well as of the value function V, see below Theorem 4.3.

Proposition 2.9. *Let* $\dot y$ *be the solution of* (2.5)*. Then we have the estimates*

$$\|\dot y\|_{L^\infty(\Omega)} \le Kc^{-2}, \tag{2.7}$$

$$\|\dot y\|_{H^1_0(\Omega)} \le Kc^{-3/2}. \tag{2.8}$$

Proof. By Theorem 2.4 we get

$$\|\dot y\|_{L^\infty} \le K\left(\|(c\max_c')^{-1}\max_c'\cdot(y_{c,u} - \psi)\|_{L^\infty} + \left\|\frac{\partial}{\partial c}\max_c\right\|_{L^2}\right),$$

where we omitted the argument $(\bar\lambda + c(y_{c,u} - \psi))$ in the derivatives of $\max_c$. By (B.v) the second addend of the right hand side is of order c^{-2}. Further, by definition of $\mathcal{P}_{c,u}$ and Lemma 2.7 we have

$$\|(c\max_c')^{-1}\max_c'\cdot(y_{c,u} - \psi)\|_{L^\infty(\Omega)} = \|c^{-1}\cdot(y_{c,u} - \psi)\|_{L^\infty(\mathcal{P}_{c,u})}.$$

Thus, we obtain

$$\|\dot y\|_{L^\infty(\Omega)} \le Kc^{-2}.$$

Similarly, Theorem 2.4 also yields

$$\|\dot y\|_{H^1} \le K\left(\|(c\max_c')^{-1/2}\max_c'\cdot(y_{c,u} - \psi)\|_{L^2} + \left\|\frac{\partial}{\partial c}\max_c\right\|_{L^2}\right),$$

which implies with analogous arguments as above

$$\|\dot y\|_{H^1} \le Kc^{-3/2}.$$

□

With the help of estimates on $\dot y$ we can now study the convergence of $y_{c,u}$ for $c \to \infty$ and fixed control u.

Theorem 2.10. *Let* $u \in L^2(\Omega)$ *be given. Let* y_u *and* $y_{c,u}$ *denote the corresponding solutions of the variational inequality* (1.1) *and the regularized equation* (2.1)*, respectively. Let* λ_u *be the corresponding multiplier in* (1.1) *and set* $\lambda_{c,u} := \max_c\left(\bar\lambda + c(y_{c,u} - \psi)\right)$*.*

Then there is a constant $K > 0$ *independent of* c *such that*

$$\|y_u - y_{c,u}\|_{L^\infty} \le Kc^{-1},$$

$$\|y_u - y_{c,u}\|_{H^1} + \|\lambda_u - \lambda_{c,u}\|_{H^{-1}} \le Kc^{-1/2}.$$

Moreover, we have an estimate on the function values:

$$|J(y_u, u) - J(y_{c,u}, u)| \le Kc^{-1}. \tag{2.9}$$

Proof. Our convergence results on y follow from integration of (2.7) and (2.8) with respect to c. The estimate on λ follows from inserting the estimates on y into the difference of (1.1) and (2.1).

As for (2.9) we compute by the chain rule:

$$\frac{d}{dc}J(y_{c,u},u) = g'(y_{c,u})\dot{y}$$

Inserting (2.7) we obtain

$$|\frac{d}{dc}J(y_{c,u},u)| \leq \|g'(y_{c,u})\|_{L^1(\Omega)}\|\dot{y}\|_{L^\infty(\Omega)} \leq Kc^{-2}.$$

Integration of this inequality with respect to c yields the assertion of this theorem. □

3 C-stationarity of solutions of (P)

Let now (y^*,u^*) be a strict local optimal solution of the original problem (P). Then owing to the following result there exist a sequence of solutions of the regularized problem (P_c).

Proposition 3.1. *Let (y^*,u^*) be a strict local optimal solution of the original problem (P). Then there exists a sequence $\{y_c,u_c\}$ of local solutions of the regularized problem with (y_c,u_c) with*

$$\begin{aligned} y_c &\to y^* \text{ in } H_0^1(\Omega),\\ y_c &\rightharpoonup^* y^* \text{ in } L^\infty(\Omega),\\ u_c &\rightharpoonup^* u^* \text{ in } L^\infty(\Omega). \end{aligned}$$

Moreover, for the associated multiplier λ_c we have $\lambda_c \to \lambda^*$ in $H^{-1}(\Omega)$.

Proof. The proof follows the lines of a similar result [13]. Here, weak lower semi-continuity of j is needed. The uniform boundedness of $\{u_c\}$ in $L^\infty(\Omega)$ is a result of the control constraints. □

In particular, $u_c \rightharpoonup u^*$ in $L_2(\Omega)$, which can also be shown in the absence of control constraints and which is all we need in the following.

Corollary 3.2. *There is a subsequence such that $y_c \to y^*$ in $C(\bar{\Omega})$.*

Proof. By Theorem 2.10, we have $y_{c,u^*} \to y^*$ strongly in $L^\infty(\Omega)$ as $c \to \infty$. Due to compact embeddings we also have $u_c \to u^*$ in $W^{-1,4}(\Omega)$. Using the result of [18], this implies $\|y_{c,u^*} - y_c\|_{L^\infty(\Omega)} \to 0$. This proves the claim. □

As observed in [13] the convergence of u_c to u^* is strong in $L^2(\Omega)$ if $j(u) = \frac{\alpha}{2}\|u\|^2$. Alternatively, strong convergence can be achieved by adding a penalty term $\|u^* - u_c\|^2$ to the functional, see [16] and the remarks at the end of this section.

Since the problem (P_c) is smooth, there exists an adjoint state p_c such that the first-order necessary optimality system (1.9) is satisfied. Now, let us prove the uniform boundedness of the dual quantities in the regularized problem.

Lemma 3.3. *There is a constant K, such that*

$$\|p_c\|_{H^1} + \|p_c\|_{L^\infty} + \|c\max_c'(\bar{\lambda} + (y_c - \psi))p_c\|_{L^1} \leq K.$$

Proof. Since $\{y_c\}$ is bounded in $L^2(\Omega)$, we have boundedness of $\{g'(y_c)\}$ in $L^2(\Omega)$ as well by Assumption (A.iv), and we can conclude from Theorem 2.4 the asserted $L^1(\Omega)$ and $L^\infty(\Omega)$ bounds for p_c. □

See also related results in [10, 13]. Let us remark, that the previous lemma implies uniform boundedness of the sequence $\{\mu_c\}$ in $L^1(\Omega)$.

Summarizing these results, we obtain the following.

Proposition 3.4. *Let (y^*, u^*) be a strict local optimal solution of the original problem* (P)*. Then there exists a subsequence of $\{(y_c, u_c, \lambda_c, p_c, \mu_c)\}$ of stationary points of* (P_c)*, which satisfy* (1.9)*, such that*

$$\begin{aligned}
y_c &\to y^* \ \textit{in}\ H_0^1(\Omega) \cap L^\infty(\Omega),\\
u_c &\rightharpoonup^* u^* \ \textit{in}\ L^\infty(\Omega),\\
\lambda_c &\to \lambda^* \ \textit{in}\ H^{-1}(\Omega),\\
\lambda_c &\rightharpoonup^* \lambda^* \ \textit{in}\ L^\infty(\Omega),\\
p_c &\rightharpoonup^* p^* \ \textit{in}\ H_0^1(\Omega) \cap L^\infty(\Omega),\\
p_c &\to p^* \ \textit{in}\ L^q(\Omega)\ \forall q < \infty,\\
\mu_c &\rightharpoonup^* \mu^* \ \textit{in}\ H^{-1}(\Omega) \cap L^\infty(\Omega)^*.
\end{aligned}$$

It remains to prove that the limit point $(y^*, u^*, \lambda^*, p^*, \mu^*)$ is a C-stationary point for (P).

Proposition 3.5. *It holds*

$$\|\mu_c(y_c - \psi)\|_{L^1} = O(c^{-1}),$$

$$\|\lambda_c p_c\|_{L^1} = O(c^{-1}).$$

Moreover, we have for the limit

$$\mu^*(y^* - \psi) = 0 \ \textit{in}\ L^\infty(\Omega)^*.$$

$$\lambda^* p^* = 0 \ \textit{a.e. in}\ \Omega.$$

Proof. For the first claim, we obtain using the result of Lemma 2.7

$$\|\mu_c(y_c - \psi)\|_{L^1} = \int_\Omega |\mu_c(y_c - \psi)| \le \|\mu_c\|_{L^1} \|y_c - \psi\|_{L^\infty(\mathcal{P}_{c,u_c})} \le K c^{-1}.$$

To prove the second assertion, we use the inequality $\max_c(x) \le (x + \frac{1}{2c}) \max_c'(x)$, cf. (1.11). Hence by Lemma 3.3

$$\begin{aligned}
\int_\Omega |\lambda_c p_c| &\le \int_{\mathcal{P}_{c,u_c}} \left| \left(\frac{1}{2c} + \bar\lambda + c(y_c - \psi) \right) \max_c' \left(\bar\lambda + c(y_c - \psi) \right) p_c \right| \\
&\le K c^{-1} \| c \max_c' \left(\bar\lambda + c(y_c - \psi) \right) p_c \|_{L^1} \le K c^{-1}.
\end{aligned}$$

By Proposition 3.4, we have that $\mu_c(y_c - \psi) \rightharpoonup^* \mu^*(y^* - \psi)$ in $L^\infty(\Omega)^*$. Moreover, for smooth test functions ϕ we have $\langle \lambda_c, \phi p_c \rangle \to \langle \lambda^*, \phi p^* \rangle$. Since $\lambda^*, p^* \in L^\infty(\Omega)$ this proves $\lambda^* p^* = 0$ a.e. on Ω. □

Lemma 3.6. *We have*

$$A^*p^* + \mu^* = g'(y_*)$$

and

$$\langle \mu^*, \zeta^2 p^* \rangle \geq \limsup_{c\to\infty} \langle \mu_c, \zeta^2 p_c \rangle \geq 0$$

for all functions $\zeta \in W^{1,\infty}(\bar{\Omega})$.

Proof. Our first assertion follows from the fact that Ap_c and μ_c converge weakly in H^{-1}. Testing

$$A^*(p_c - p^*) + (\mu_c - \mu^*) = g'(y_c) - g'(y_*)$$

with $\zeta^2(p_c - p^*)$ yields

$$\langle A^*(p_c - p^*), \zeta^2(p_c - p^*)\rangle + \langle \mu_c - \mu^*, \zeta^2(p_c - p^*)\rangle = (g'(y_c) - g'(y_*))(\zeta^2(p_c - p^*)).$$

Since $g'(y_c) \to g'(y^*)$ strongly the right-hand side tends to zero for $c \to \infty$. For the first addend we have due to the properties of A^*, see Assumption (A.ii),

$$\langle A^*(p_c - p^*), \zeta^2(p_c - p^*)\rangle = \langle A^*\zeta(p_c - p^*), \zeta(p_c - p^*)\rangle + \int_\Omega \pi_c \zeta(p_c - p^*)$$

with $\pi_c \in L^2(\Omega)$ given by

$$\pi_c = 2\sum_{i,j=1}^n a_{ij}(x)\frac{\partial(p_c - p^*)}{\partial x_i}\frac{\partial \zeta}{\partial x_j} + (p_c - p^*)\sum_{j=1}^n a_j \frac{\partial \zeta}{\partial x_j}.$$

Since $p_c \rightharpoonup p^*$ in $H_0^1(\Omega)$ and $p_c \to p^*$ in $L^2(\Omega)$, we find

$$\int_\Omega \pi_c \zeta(p_c - p^*) \to 0$$

for $c \to \infty$. Hence we obtain

$$\begin{aligned} 0 &\geq \limsup \langle \mu_c - \mu^*, \zeta^2(p_c - p^*)\rangle \\ &= \limsup(\langle \mu_c, \zeta^2 p_c\rangle + \langle \mu^*, \zeta^2 p^*\rangle - \langle \mu_c, \zeta^2 p^*\rangle - \langle \mu^*, \zeta^2 p_c\rangle) \\ &= \limsup(\langle \mu_c, \zeta^2 p_c\rangle - \langle \mu^*, \zeta^2 p^*\rangle), \end{aligned}$$

where we have used $\langle \mu_c, \zeta^2 p^*\rangle \to \langle \mu^*, \zeta^2 p^*\rangle$ and $\langle \mu^*, \zeta^2 p_c\rangle \to \langle \mu^*, \zeta^2 p^*\rangle$. □

Proposition 3.7. *The mapping* $v \to \langle vp^*, \mu^*\rangle_{L^\infty,(L^\infty)^*}$ *has a representation as a positive measure.*

Proof. Let v be any positive continuous function, and $v_k \in C^1$ an approximation sequence. Then setting $\hat{v}_k := \max(v_k, 0)$ we obtain $\hat{v}_k \in W^{1,\infty}(\Omega)$ and also $\|\hat{v}_k - v\|_\infty \to 0$. Moreover, since $\mu^* \in (L^\infty)^*$, and $p^* \in L^\infty$ the expression

$$\langle vp^*, \mu^*\rangle_{L^\infty,(L^\infty)^*}$$

is well defined for all $v \in C(\overline{\Omega})$, and continuous. This implies

$$\langle vp^*, \mu^*\rangle_{L^\infty,(L^\infty)^*} = \lim_{k\to\infty} \langle \hat{v}_k p^*, \mu^*\rangle_{L^\infty,(L^\infty)^*} \geq 0.$$

Thus, the linear functional

$$v \to \langle vp^*, \mu^*\rangle_{L^\infty,(L^\infty)^*}$$

is continuous on $C(\overline{\Omega})$ and positive, hence has a representation as a positive measure. □

Proposition 3.8. *Let $\zeta \in C(\bar{\Omega})$ with $\zeta = 0$ on $\{y^* = \psi\}$. Then it holds*

$$\langle \mu^*, \zeta \rangle_{(L^\infty)^*, L^\infty} = 0.$$

Proof. Let us define $I_\sigma := \{y^* < \psi - \sigma\}$ for $\sigma \geq 0$.

Let $\sigma > 0$ be given. By strong convergence $y_c \to y^*$ in $L^\infty(\Omega)$ it follows that $\max_c{}'(\bar{\lambda} + c(y_c - \psi)) = 0$ on the set $\{y^* \leq \psi - \sigma\} = I_\sigma$ for all $c > C_\sigma$, C_σ sufficiently large. This implies $\mu_c = 0$ on I_σ for $c > C_\sigma$.

Let us take a nonnegative function $\zeta \in C(\bar{\Omega})$ with $\zeta = 0$ on $\{y^* = \psi\} = I_0$. Then the function

$$\zeta_\sigma := \max(\zeta - \|\zeta\|_{L^\infty(I_0 \setminus I_\sigma)}, 0)$$

is continuous with support contained in I_σ. Hence $\langle \mu_c, \zeta_\sigma \rangle_{(L^\infty)^*, L^\infty} = 0$ for c sufficiently large. Due to $\mu_c \rightharpoonup^* \mu^*$ in $L^\infty(\Omega)^*$ it holds $\langle \mu^*, \zeta_\sigma \rangle_{(L^\infty)^*, L^\infty} = 0$ for all $\sigma > 0$.

By construction we have $\|\zeta - \zeta_\sigma\|_{L^\infty} \leq \|\zeta\|_{L^\infty(I_0 \setminus I_\sigma)}$. Now let us argue that $\lim_{\sigma \searrow 0} \|\zeta\|_{I_0 \setminus I_\sigma} = 0$. If this would not be true, then there would exist $\epsilon > 0$ and sequences $\sigma_n \searrow 0$, $x_{\sigma_n} \in I_0 \setminus I_{\sigma_n}$ with $\zeta(x_{\sigma_n}) > \epsilon$. Since Ω is bounded, the sequence $\{x_{\sigma_n}\}$ admits an accumulation point $\tilde{x}$. By definition of I_σ it follows that $\tilde{x}$ belongs to $\{y^* = \psi\}$. Hence $\zeta(x^*) = 0$, which leads to the contradiction.

This implies $\zeta_\sigma \to \zeta$ in $L^\infty(\Omega)$, which allows to pass to the limit to show $\langle \mu^*, \zeta \rangle_{(L^\infty)^*, L^\infty} = 0$. □

Theorem 3.9. *Let (y^*, u^*) be a strict local optimal solution of the original problem* (P). *Let $\{(y_c, u_c, \lambda_c, p_c, \mu_c)\}$ be the sequence given by Proposition 3.4, which converges to $(y^*, u^*, \lambda^*, p^*, \mu^*)$.*

Assume that for a subsequence $j'(u_c) \to j'(u^*)$ in $L^2(\Omega)$ holds.

Then $(y^*, u^*, \lambda^*, p^*, \mu^*)$ is a C-stationary point of (P) *that is it satisfies* (1.7).

Proof. The claim is a direct consequence of the previous results. The assumption on the convergence $j'(u_c) \to j'(u^*)$ then allows to pass to the limit in (1.9c). □

In order to prove C-stationarity of all local solutions of (P), we have to drop the assumption on strong convergence of $j'(u_c)$. This can be done by considering a penalized problem as follows

$$\begin{cases} \min J(y,u) + \|u - u^*\|_{L^2}^2 \\ \text{over } (y,u) \in H_0^1(\Omega) \times U_{ad} \text{ subject to} \\ Ay + \max_c(0, \bar{\lambda} + c(y - \psi)) = u. \end{cases} \tag{$\tilde{\mathrm{P}}_c$}$$

With the same arguments as in [13, 16], we obtain the existence of a sequence $\{(\tilde{y}_c, \tilde{u}_c)\}$ of solutions of ($\tilde{\mathrm{P}}_c$) with $(\tilde{y}_c, \tilde{u}_c) \to (y^*, u^*)$ strongly in $(H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)$. Besides an obvious modification of the necessary condition given in (1.9), all the results of this section remain valid. And we obtain

Theorem 3.10. *Let (y^*, u^*) be a local optimal solution of the original problem* (P). *Then $(y^*, u^*, \lambda^*, p^*, \mu^*)$ is a C-stationary point of* (P).

4 Convergence estimates of the path

In the following, we will prove convergence rates for $c \to \infty$. Here we will rely on the following assumption on the path:

Assumption 4.1. *Let us impose the following assumptions:*

(i) The pair (y^*, u^*) is a local solution of (P) *that satisfies a quadratic growth condition*

$$\|u - u_*\|^2_{L^2(\Omega)} \leq \alpha(J(y_*, u_*) - J(y(u), u)) \tag{4.1}$$

for all u in a neighourhood of u_*.

(ii) There is a sequence $\{(y_c, u_c)\}_{c>0} \in H^1_0(\Omega) \times L^2(\Omega)$ of local solutions of (P_c) *converging strongly to (y^*, u^*).*

(iii) The optimality system (1.9) *for (y_c, u_c) is strongly regular for all $c > C_0$ in the sense of [3] for some $C_0 < \infty$.*

Some comments are in order. The quadratic growth condition on (y^*, u^*) is fulfilled if (y^*, u^*) fulfills a second-order sufficient optimality condition. Sufficient optimality conditions were investigated in [13] for the special case $U_{ad} = L^2(\Omega)$. They can be transferred to the control-constrained case using the ideas of [5]. Then (y^*, u^*) satisfies a quadratic growth condition, if (y^*, u^*) is strongly stationary and the second-order derivative of the Lagrangian is positive definite on a certain cone.

The existence of the path was discussed in [13], see also the discussion in Section 2. The strong regularity of (1.9) is connected to the solvability of the linearization of (1.9). If the linearized system is uniquely solvable, and the solutions depend continuously on the data, then (1.9) is strongly regular, see e.g. [3, 6].

Let us define the optimal value function as the function value of J along the path,

$$V(c) := J(y_c, u_c).$$

Lemma 4.2. *The value function V is continuously differentiable from $(C_0, +\infty)$ to $\mathbb{R}$. Moreover we have*

$$\begin{aligned} \dot{V}(c) &= \left(\frac{\partial}{\partial c} \max_c \left(\bar{\lambda} + c(y_c - \psi) \right) + \max_c'(\bar{\lambda} + c(y_c - \psi))(y_c - \psi),\, p_c \right) \\ &= \left(\frac{\partial}{\partial c} \max_c \left(\bar{\lambda} + c(y_c - \psi) \right),\, p_c \right) + c^{-1}(\mu_c,\, y_c - \psi). \end{aligned} \tag{4.2}$$

Proof. The differentiability follows from the strong regularity of (1.9), see [3] and also [14]. The expression for $\dot{V}$ follows from the definition of μ_c. □

Theorem 4.3. *Under the assumption imposed in this section it holds*

$$\dot{V}(c) = O(c^{-2}).$$

Proof. We will estimate the two summands in the right hand side of (4.2). For the first, let us define the set

$$\mathcal{N}_c = \left\{ x : |\bar{\lambda}(x) + c(y_c(x) - \psi(x))| < \frac{1}{2c} \right\}.$$

On $\Omega \setminus \mathcal{N}_c$ we have $\frac{\partial}{\partial c}\max_c\left(\bar\lambda + c(y_c - \psi)\right) = 0$. On $\mathcal{N}_c$ it holds by (B.v)

$$\left|\frac{\partial}{\partial c}\max_c\left(\bar\lambda + c(y_c - \psi)\right)\right| \le K\, c^{-2}.$$

Uniform boundedness of p_c in $L^q(\Omega)$ yields an $O(c^{-2})$ bound for the first summand.

Consider the second summand. Due to Lemma 3.3, the functions μ_c are bounded in $L^1(\Omega)$. By Lemma 2.7, we have $\|y_c - \psi\|_{L^\infty(\mathcal{P}_{c,u_c})} \le Kc^{-1}$, where $\mathcal{P}_{c,u_c} \supset \operatorname{supp}\mu_c$. Hence,

$$|(\mu_c,\, y_c - \psi)| \le c^{-1}\|\mu_c\|_{L^1(\Omega)}\|y_c - \psi\|_{L^\infty(\mathcal{P}_{c,u_c})} \le Kc^{-2}$$

with a constant K independent of c. □

Remark 4.4. *The result of Theorem 4.3 improves the estimate $\dot V(c) = o(1/c)$ of [14]. Moreover, it explains the numerically observed convergence rate of $\dot V(c) = O(c^{-2})$ of [14].*

Let us now state and prove the main result of this section.

Theorem 4.5. *Under the assumption imposed in this section it holds*

$$\begin{aligned} |J(y_*, u_*) - J(y_c, u_c)| &\le Kc^{-1},\\ |J(y_*, u_*) - J(y(u_c), u_c)| &\le Kc^{-1},\\ \|y_* - y_c\|_{H^1} + \|y_* - y_c\|_{L^\infty} + \|u_* - u_c\|_{L^2} &\le Kc^{-1/2}. \end{aligned}$$

Proof. Integration of the estimate of $\dot V(c)$ in Theorem 4.3 with respect to c yields the first estimate. The second estimate follows from (2.9) via

$$J(y_*, u_*) - J(y(u_c), u_c) = J(y_*, u_*) - J(y_c, u_c) + J(y_c, u_c) - J(y(u_c), u_c).$$

The third inequality for $u_c - u_*$ follows from the growth condition (4.1):

$$\|u_c - u_*\|^2 \le \alpha(J(y_*, u_*) - J(y(u_c), u_c)) \le Kc^{-1}.$$

Finally, we can apply the triangle inequality:

$$\begin{aligned} \|y_c - y_*\|_{L^\infty} &\le \|y_c - y_{c,u_*}\|_{L^\infty} + \|y_{c,u_*} - y_*\|_{L^\infty} \le L\|u - u_*\|_{L^2} + Kc^{-1},\\ \|y_c - y_*\|_{H^1} &\le \|y_c - y_{c,u_*}\|_{H^1} + \|y_{c,u_*} - y_*\|_{H^1} \le L\|u - u_*\|_{L^2} + Kc^{-1/2}. \end{aligned}$$

Here we used Theorem 2.10 and Corollary 2.5. □

Remark 4.6. *The convergence rate $\|u_* - u_c\|_{L^2} \le Kc^{-1/2}$ provided by Theorem 4.5 coincides with the rates observed in the numerical computations of [14]. In this sense, the results of Theorem 4.5 is sharp. See also our numerical results in the next section.*

5 Numerical experiments

Let us report on the numerical results for the solution of the following problem: Minimize

$$J(y, u) = \frac{1}{2}\|y - y_d\|_{L^2}^2 + \frac{\alpha}{2}\|u\|_{L^2}^2$$

subject to the variational inequality

$$(\nabla y,\ \nabla v - \nabla y) \ge (u, v - y) \quad \forall v \in K$$

and the control constraints

$$u \in U_{ad}$$

with K and U_{ad} as in (1.3) and (1.2), respectively. As domain we choose $\Omega = (0,1)^2$. The data for our example are given by

$$y_d(x) = 5x_1 + x_2 - 1, \quad \psi(x) = 4(x_1(x_1-1) + x_2(x_2-1)) + 1.5$$

and

$$u_a = 0.2, \quad u_b = 3.$$

Furthermore, we set $\alpha = 0.01$. With these choices, all the standing assumptions are satisfied.

We applied the inexact path-following strategy from [14] with tolerances $\epsilon_{\text{cc}} = \epsilon_{\text{newt}} = 10^{-8}$. We started the path-following algorithm with $c_0 = 100$, and set the parameter $\theta = 0.5$.

The underlying partial differential equation was discretized by finite elements. We used $P1$-elements for state, adjoint, and control discretization. The computational mesh consisted of $80,000$ triangles with maximal diameter $h = 0.0071$.

The numerical solutions $(y_{c,h}, p_{c,h}, \lambda_{c,h}, \mu_{c,h})$ for $c = c_N \approx 10^9$ are depicted in Figures 1 and 2. Both control and state constraints are active. As one can see, the adjoint state is zero on the active set. As shown in figure 2, the multipliers $\lambda_{c,h}$ and $\mu_{c,h}$ only have low regularity. Moreover, the support of their irregular part is concentrated on the boundary of the active set $\{y = \psi\}$.

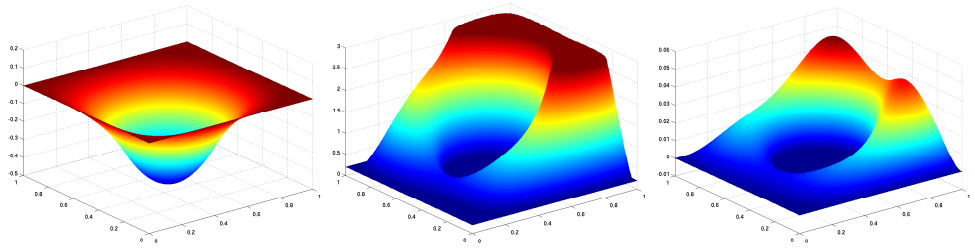

Figure 1: Numerical solution: $y_{c,h}$, $u_{c,h}$, $p_{c,h}$

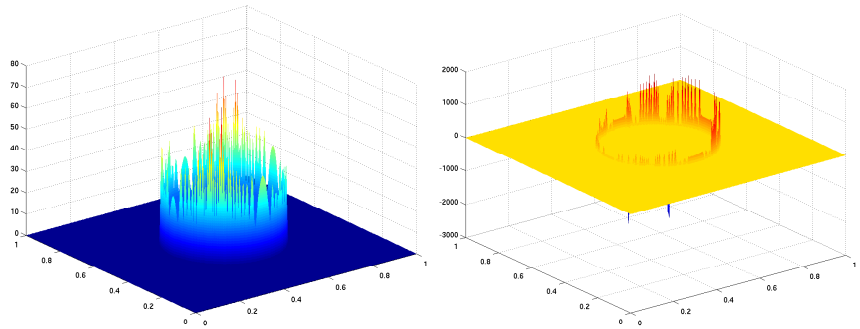

Figure 2: Numerical solution: $\lambda_{c,h}$, $\mu_{c,h}$

Let us now comment on the convergence rates for this example. The development of the convergence rates of $\|p_c\lambda_c\|_{L^1}$, $\|\mu_c(y_c - \psi)\|_{L^1}$, $\|u_c - u^*\|_{L^2}$, and $|\dot{V}(c)|$ are depicted in Figure

3. As the solution of the continuous problem u^* is unknown, we took the solution of the regularized problem for the largest value of $c_N \approx 10^9$ as reference solution, i.e. $u^* := u_{c_N}$.

We observe the same convergence rates as predicted by the theory in Proposition 3.5, Theorem 4.3, and Theorem 4.5.

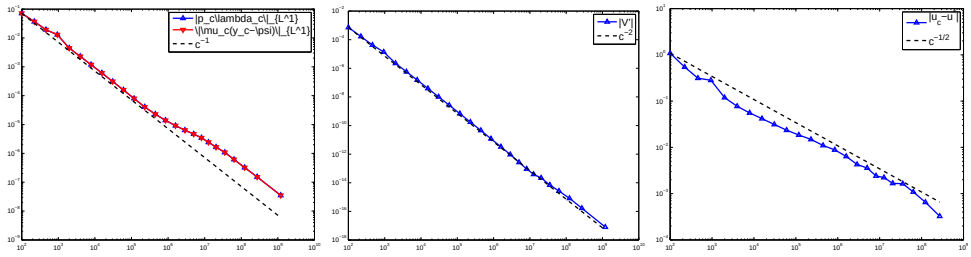

Figure 3: Convergence rates for $\|p_c\lambda_c\|_{L^1}$, $\|\mu_c(y_c - \psi)\|_{L^1}$, $|\dot{V}(c)|$, and $\|u_c - u^*\|_{L^2}$

References

[1] V. Barbu. *Optimal control of variational inequalities*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), London, 1984.

[2] M. Bergounioux and F. Mignot. Optimal control of obstacle problems: existence of Lagrange multipliers. *ESAIM Control Optim. Calc. Var.*, 5:45–70, 2000.

[3] J. F. Bonnans and A. Shapiro. *Perturbation analysis of optimization problems*. Springer, New York, 2000.

[4] H. Brezis and G. Stampacchia. Sur la régularité de la solution d'inéquations elliptiques. *Bull. Soc. Math. France*, 96:153–180, 1968.

[5] E. Casas, F. Tröltzsch, and A. Unger. Second-order sufficient optimality conditions for a nonlinear elliptic control problem. *J. for Analysis and its Applications*, 15:687–707, 1996.

[6] A. L. Dontchev. Implicit function theorems for generalized equations. *Math. Program.*, 70 A:91–106, 1995.

[7] M. Hintermüller and I. Kopacka. Mathematical programs with complementarity constraints in function space: C- and strong stationarity and a path-following algorithm. *SIAM J. Optim.*, 20(2):868–902, 2009.

[8] M. Hintermüller and I. Kopacka. A smooth penalty approach and a nonlinear multigrid algorithm for elliptic MPECs. *Comp. Opt. Appl.*, 2009. To appear.

[9] M. Hintermüller and T. Surowiec. First order optimality conditions for elliptic mathematical programs with equilibrium constraints via variational analysis. 2010.

[10] K. Ito and K. Kunisch. Optimal control of elliptic variational inequalities. *Appl. Math. Optim.*, 41(3):343–364, 2000.

[11] K. Ito and K. Kunisch. *On the Lagrange multiplier approach to variational problems and applications*, volume 24 of *Monographs and Studies in Mathematics*. SIAM,, Philadelphia, 2008.

[12] D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and their applications*. Academic Press, New York, 1980.

[13] K. Kunisch and D. Wachsmuth. Sufficient optimality conditions and semi-smooth Newton methods for optimal control of stationary variational inequalities. *ESAIM Control Optim. Calc. Var.*, 2010. To appear.

[14] K. Kunisch and D. Wachsmuth. Path-following for optimal control of stationary variational inequalities. *Comp. Opt. Appl.*, 2011. To appear.

[15] F. Mignot. Contrôle dans les inéquations variationelles elliptiques. *J. Functional Analysis*, 22(2):130–185, 1976.

[16] F. Mignot and J.-P. Puel. Optimal control in some variational inequalities. *SIAM J. Control Optim.*, 22(3):466–476, 1984.

[17] H. Scheel and S. Scholtes. Mathematical programs with complementarity constraints: stationarity, optimality, and sensitivity. *Math. Oper. Res.*, 25(1):1–22, 2000.

[18] G. Stampacchia. Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier (Grenoble)*, 15(fasc. 1):189–258, 1965.