

FACE NUMBERS OF CENTRALLY SYMMETRIC POLYTOPES PRODUCED FROM SPLIT GRAPHS

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ABSTRACT. We analyze a remarkable class of centrally symmetric polytopes, the Hansen polytopes of split graphs. We confirm Kalai's 3^d -conjecture for such polytopes (they all have at least 3^d nonempty faces) and show that the Hanner polytopes among them (which have exactly 3^d nonempty faces) correspond to threshold graphs. Our study produces a new family of Hansen polytopes that have only $3^d + 16$ nonempty faces.

1. INTRODUCTION

A convex polytope P is *centrally symmetric* if $P = -P$. In 1989 Gil Kalai [5] posed three conjectures on the numbers of faces and flags of centrally symmetric polytopes, which he named conjectures A, B and C. Two of these, conjectures B and C, were refuted by Sanyal et al. in 2009 [7]. However, conjecture A, known as the 3^d -conjecture, was confirmed for dimension $d \leq 4$ and remains open for $d > 4$:

Conjecture (3^d -conjecture). *Every centrally symmetric convex polytope of dimension d has at least 3^d nonempty faces.*

As a contribution to the quest for settling this conjecture, we investigate a special class of centrally symmetric polytopes, namely Hansen polytopes, as introduced by Hansen in 1977 [4]. Hansen polytopes of split graphs served as counter-examples to conjectures B and C, so it seems natural to analyze this subclass more thoroughly. As our main result we express the total number of nonempty faces of such a polytope in terms of certain partitions of the node set of the underlying split graph. In particular, we confirm the 3^d -conjecture for Hansen polytopes of split graphs, and show that equality in this class corresponds to threshold graphs.

In Section 2 we define Hansen polytopes, which are derived from perfect graphs such as, for example, split graphs. In Section 3 we analyze the Hansen polytopes of threshold graphs, which are special split graphs. It turns out that a Hansen polytope is a Hanner polytope if and only if the underlying graph is threshold. In Section 4 we describe the Hansen polytopes of general split graphs and prove the main result mentioned above. Our study also produces examples of centrally symmetric polytopes that are not Hanner polytopes and have a total number of nonempty faces very close to the conjectured lower bound of 3^d .

General assumptions. All our graphs are finite and simple. The vertex set of a graph G is denoted by $V(G)$, and similarly the edge set is $E(G)$ if no other notation is specified. The complement of G is \overline{G} . The complete graph on n nodes is K_n . All polytopes are convex. We denote the polar of a polytope P by P^* . For details on graph theory we refer to Diestel [1], for polytope theory to [9].

Key words and phrases. Hansen polytopes, Hanner polytopes, split graphs, threshold graphs, 3^d -conjecture.

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2. HANSEN POLYTOPES

Hansen polytopes were introduced by Hansen [4] in 1977. Some of these centrally symmetric polytopes turn out to have “few faces”. One constructs them from the stable set structure of a (perfect) graph G by applying the twisted prism operation to the stable set polytope. Let us define these terms.

Definition 2.1 (Twisted prism). Let $P \subseteq \mathbb{R}^d$ be a polytope and $Q := \{1\} \times P$ its embedding in \mathbb{R}^{d+1} . The *twisted prism* of P is $\text{tp}(P) := \text{conv}(Q \cup -Q)$.

Twisted prisms are centrally symmetric by construction. We are interested in twisted prisms of stable set polytopes, which we introduce next; see also Schrijver [8, Sec. 64.4]. By e_i we denote the i th coordinate unit vector.

Definition 2.2. Let G be a graph. The *stable set polytope* of G is

$$\text{stab}(G) := \text{conv} \left\{ \sum_{i \in I} e_i : I \subseteq V(G) \text{ stable} \right\}.$$

Now we can define the main object of our studies.

Definition 2.3 (Hansen polytope). The *Hansen polytope* of a graph G is defined as $\mathbf{H}(G) := \text{tp}(\text{stab}(G))$.

Examples of Hansen polytopes are cubes (produced from empty graphs) and crosspolytopes (from complete graphs). Recall that a graph G is *perfect* if the size of the largest clique of any induced subgraph H of G equals the chromatic number of H . In the rest of the paper we need the following properties of Hansen polytopes:

Lemma 2.4. Let $G = (V, E)$ be a graph.

- (i) The vertex set of $\mathbf{H}(G)$ is $\text{vert}(\mathbf{H}(G)) = \{\pm(e_0 + \sum_{i \in I} e_i) : I \subseteq V \text{ stable}\}$.
- (ii) If G is perfect, $\{-1 \leq -x_0 + 2 \sum_{i \in C} x_i \leq 1 : C \subseteq V \text{ clique}\}$ is an irredundant facet description of $\mathbf{H}(G)$.
- (iii) If G is perfect, then the polar of the Hansen polytope of G is affinely equivalent to the Hansen polytope of \bar{G} , in symbols $\mathbf{H}(G)^* \cong \mathbf{H}(\bar{G})$.

Proof. Part (i) is obvious, a proof of (ii) can be found in Hansen’s paper [4] and (iii) follows from (ii). \square

From now on for the rest of the article we assume all graphs to be perfect.

3. HANSEN POLYTOPES OF THRESHOLD GRAPHS

An important class of polytopes that attain the conjectured lower bound of the 3^d -conjecture are the so-called Hanner polytopes. These polytopes were introduced by Hanner [3] in 1956 and are recursively defined as follows.

Definition 3.1 (Hanner polytope). A polytope $P \subseteq \mathbb{R}^d$ is a *Hanner polytope* if it is either a centrally symmetric line segment or, for $d \geq 2$, the direct product of two Hanner polytopes or the polar of a Hanner polytope.

It is neither the case that all Hanner polytopes are Hansen polytopes nor vice versa. A characterization of their relation is our first result. Before we can state it, we need to introduce threshold graphs, a subclass of perfect graphs; an extensive treatment is Mahadev & Peled [6]. The definition involves the notions of dominating and isolated nodes: A node in a graph is *dominating* if it is adjacent to all other nodes; it is *isolated* if it is not adjacent to any other node.

Definition 3.2 (Threshold graph). A graph $G = (V, E)$ is a *threshold graph* if it can be constructed from the empty graph by repeatedly adding either an isolated node or a dominating node.

This class of graphs is closed under taking complements.

Theorem 3.3. *The Hansen polytope $H(G)$ is a Hanner polytope if and only if G is a threshold graph.*

Proof. (\Leftarrow) We use induction of the number of nodes. If $G = \emptyset$, then $H(G)$ is just a centrally symmetric segment, and therefore a Hanner polytope. Now assume that G has $n + 1$ nodes. Since the class of Hanner polytopes is closed under taking polars and $H(G)^* \cong H(\overline{G})$, we can assume $G = T \cup \{v\}$ with T being threshold. Here \cup denotes the usual disjoint union of graphs and v is a single node with $v \notin T$. The stable sets of G are exactly the stable sets of T , with and without the new node v . Given a stable set S of T the vertices of $H(G)$ are of the form $\pm(e_0 + \sum_{i \in S} e_i)$ and $\pm(e_0 + \sum_{i \in S} e_i + e_{n+1})$, where we assign v the label $n + 1$. By the linear transformation defined by $e_0 \mapsto e_0 - e_{n+1}$, $e_{n+1} \mapsto 2e_{n+1}$, and $e_i \mapsto e_i$ for $i = 1, \dots, n$, we get $H(G) = H(T \cup v) \cong H(G) \times [-1, 1]$, which means that $H(G)$ is Hanner.

(\Rightarrow) Assume that $H(G)$ is Hanner. Again it is enough to cover just one case, namely $H(G) = P \times P'$ with P, P' being lower-dimensional Hanner polytopes. The stable set polytope $\text{stab}(G)$ is a facet of $H(G)$ and can therefore be written as $\text{stab}(G) = Q \times Q'$ with Q, Q' being faces of P, P' , respectively. Since we have $\dim(Q) + \dim(Q') = \dim(\text{stab}(G)) = \dim(P) + \dim(P') - 1$, we can further assume that $Q = P$ and Q' is a facet of P' . Let $q := \dim(Q)$ and $q' := \dim(Q')$.

We now construct a threshold graph H' on q' nodes such that $G = \overline{K}_q \cup H'$. This of course shows that G is threshold as well. Since $\text{stab}(G)$ is a product, we have $\text{vert}(\text{stab}(G)) = \text{vert}(Q) \times \text{vert}(Q')$. Each coordinate of a vertex of $\text{stab}(G)$ corresponds to a node in G . Let $V_1 \subseteq V(G)$ be the node set defined by the first q coordinates and $V_2 \subseteq V(G)$ the set defined by the last q' coordinates. Then

$$\begin{aligned} \text{vert}(\text{stab}(G)) &= \left\{ \sum_{i \in I} e_i : I \subseteq V(G) \text{ stable set of } G \right\} \\ &= \left\{ \sum_{i \in I} e_i : I \subseteq V_1 \text{ stable set of } G[V_1] \text{ and } N(I) \cap V_2 = \emptyset \right\} \\ &\quad \times \left\{ \sum_{i \in I} e_i : I \subseteq V_2 \text{ stable set of } G[V_2] \text{ and } N(I) \cap V_1 = \emptyset \right\}, \end{aligned}$$

where $N(I)$ is the set of nodes adjacent to some node in I and $G[V_j]$ is the subgraph of G induced by $V_j, j = 1, 2$. In particular, we have $e_i \in \text{vert}(\text{stab}(G))$ for all $i = 1, \dots, q + q'$. From this and the right-hand side of the equality above, we can deduce that there are no edges between V_1 and V_2 . By setting $H' := G[V_2]$ we get $G = G[V_1] \cup H'$. So what is left to show is that $G[V_1]$ is an edgeless graph and that H' is threshold.

Let us first see that H' is threshold. Since $P \cong \text{stab}(G[V_1])$ is at least one-dimensional, $G[V_1]$ has one node minimum, i.e., $|V(H')| < |V(G)|$. From [3, Corollary 3.4 and Theorem 7.4], we know that Hanner polytopes are twisted prisms over any of their facets, which means for us that $P' \cong \text{tp}(Q') \cong H(H')$. Thus, by induction, H' is threshold.

As $P \cong \text{stab}(G[V_1])$ is Hanner, it has a center of symmetry. So there exists a vector $c \in \mathbb{R}^q$ such that $\text{stab}(G[V_1]) = -\text{stab}(G[V_1]) + 2c$. The origin and all unit vectors e_i for $1 \leq i \leq q$ are vertices of $\text{stab}(G[V_1])$, thus we must have $c = (\frac{1}{2}, \dots, \frac{1}{2})$. This means $\text{stab}(G[V_1]) = [0, 1]^q$, which in turn yields $G[V_1] \cong \overline{K}_q$. \square

Corollary 3.4. *If G is a threshold graph, then $H(G)$ satisfies the 3^d -conjecture with equality.*

Kalai suggests in [5] that the Hanner polytopes should be the *only* polytopes that satisfy the 3^d -conjecture with equality. We will see below that other polytopes at least get close.

We also note that not all Hanner polytopes can be represented as Hansen polytopes of perfect graphs. For example, the product of two octahedra $O_3 \times O_3$ is a Hanner polytope but not a Hansen polytope.

4. HANSEN POLYTOPES OF SPLIT GRAPHS

Now we will analyze the Hansen polytopes of split graphs. It is easy to verify and well-known that all threshold graphs are split and that all split graphs are perfect.

Definition 4.1 (Split graph). A graph G is called *split graph* if the node set can be partitioned into a clique C and a stable set S .

The main result of our paper appears in this section as Theorem 4.6. We will prove it with a partitioning technique for the faces of Hansen polytopes of split graphs. This partitioning will be described first.

4.1. Partitioning the faces of Hansen polytopes of split graphs. Let $G = C \cup S$ be a split graph with $C = \{c_1, \dots, c_k\}$ and $S = \{s_1, \dots, s_\ell\}$. A stable set of G is either of the form A or $A \cup \{c_i\}$ for $A \subseteq S$. Similarly, a clique of G must be either of the form A or $A \cup \{s_j\}$ for $A \subseteq C$. Thanks to the simple composition of stable sets and cliques of split graphs, we can give a complete description of the vertices and facets of $H(G)$. In the following we omit set parentheses of singletons in order to enhance readability.

- The vertices of $H(G)$ will be denoted by
 - (1) (ε, A) with $\varepsilon = \pm$ and $A \subseteq S$,
 - (2) $(\varepsilon, A \cup c_i)$ with $\varepsilon = \pm$, $A \subseteq S$ and $A \cup c_i$ stable.
- The facets of $H(G)$ will be denoted by
 - (1) $[\varepsilon, A]$ with $\varepsilon = \pm$ and $A \subseteq C$,
 - (2) $[\varepsilon, A \cup s_j]$ with $\varepsilon = \pm$, $A \subseteq C$ and $C \cup s_j$ being a clique.

We will refer to the different kinds of vertices and facets as *type-(1)-vertices/-facets* and *type-(2)-vertices/-facets* according to the enumeration above. In the next step we discuss the vertex-facet incidences. By Lemma 2.4 a vertex of $H(G)$ is contained in a facet if and only if they have the same sign and their defining subsets of $V(G)$ meet or if they have different signs and the defining subsets are disjoint.

Type-(1)-facets:

- $(\varepsilon, A) \in [\varepsilon', B] \iff \varepsilon = -\varepsilon'$
- $(\varepsilon, A \cup c_i) \in [\varepsilon', B] \iff (c_i \in B \text{ and } \varepsilon = \varepsilon') \text{ or } (c_i \notin B \text{ and } \varepsilon = -\varepsilon')$

Type-(2)-facets:

- $(\varepsilon, A) \in [\varepsilon', B \cup s_j] \iff (s_j \in A \text{ and } \varepsilon = \varepsilon') \text{ or } (s_j \notin A \text{ and } \varepsilon = -\varepsilon')$
- $(\varepsilon, A \cup c_i) \in [\varepsilon', B \cup s_j] \iff$
 $(\varepsilon' = \varepsilon \text{ and } (c_i \in B) \text{ or } (s_j \in A)) \text{ or } (\varepsilon' = -\varepsilon \text{ and } c_i \notin B \text{ and } s_j \notin A)$

Observe that the events $c_i \in B$ and $s_j \in A$ are mutually exclusive if $A \cup c_i$ is stable and $B \cup s_j$ is a clique. The next two lemmas will be of good use later on.

Lemma 4.2. *Let $G = C \cup S$ be a split graph. Choose $A, B \subseteq C$ and $U \subseteq S$ such that $A \cup U$ and $B \cup U$ are cliques. Then we have*

- (i) $[\varepsilon, A \cup U] \cap [\varepsilon, B \cup U] = [\varepsilon, (A \cap B) \cup U] \cap [\varepsilon, A \cup B \cup U]$
- (ii) $[\varepsilon, A \cup U] \cap [-\varepsilon, B \cup U] \subseteq [\varepsilon, A] \cap [-\varepsilon, B]$

Proof. We skip the proof, which easily follows from the vertex-facet-incidences. \square

In particular, part (i) shows that every face can be written using at most two type-(1)-facets of each sign. Indeed, for $A_1, \dots, A_t \subseteq C$, we get inductively $\bigcap_{i=1}^t [\varepsilon, A_i] = [\varepsilon, \bigcap_{i=1}^t A_i] \cap [\varepsilon, \bigcup_{i=1}^t A_i]$. The next definition relies on this fact and will be essential for arguments in the upcoming parts.

Definition 4.3. For a split graph $G = C \cup S$ we define the following four classes of faces of $H(G)$:

- *Primitive* faces F , that are not contained in any type-(1)-facet.
- *Positive* faces $[+, A] \cap [+, B] \cap F$, with $A \subseteq B$ and F primitive.
- *Negative* faces $[-, A] \cap [-, B] \cap F$, with $A \subseteq B$ and F primitive.
- *Small* faces G , that are contained in type-(1)-facets of both signs.

This definition gives a partition of the faces of $H(G)$. For the primitive faces we get a nice characterization with respect to the containment of special vertices.

Lemma 4.4. *Let $G = C \cup S$ be a split graph. A face F of $H(G)$ is primitive if and only if it contains type-(1)-vertices of both signs.*

Proof. (\Rightarrow) Assume F is primitive, i.e., we can write it as

$$F = \bigcap_{i \in I} [+, A_i \cup s_i] \cap \bigcap_{j \in J} [-, B_j \cup s_j]$$

for some multisets I and J . If we had $\{s_i : i \in I\} \cap \{s_j : j \in J\} \neq \emptyset$, then Lemma 4.2 (ii) would yield a contradiction to primitivity. Thus, these two multisets must be disjoint. We get the vertex-facet incidences

- $(+, A) \in F \iff \{s_i : i \in I\} \subseteq A \subseteq S \setminus \{s_j : j \in J\}$,
- $(-, A) \in F \iff \{s_j : j \in J\} \subseteq A \subseteq S \setminus \{s_i : i \in I\}$.

This means we can always find positive and negative type-(1)-vertices of F .

(\Leftarrow) If F has a vertex (ε, A) it cannot be contained in a facet $[\varepsilon, B]$ for any $B \subseteq C$ according to the rules above. So if F contains type-(1)-vertices of both signs, it cannot be contained in any type-(1)-facet. This means F is primitive. \square

4.2. The number of nonempty faces of Hansen polytopes of split graphs.

We need the following definition to state the main theorem.

Definition 4.5. Let $G = C \cup S$ be a split graph. Then we denote by $p_G(C, S)$ the number of partitions of the form $(C^+, C^-, C^0, S^+, S^-, S^0)$ with $C = C^+ \cup C^- \cup C^0$ and $S = S^+ \cup S^- \cup S^0$ such that either $C^+ \cup C^- \neq \emptyset$ or $S^+ \cup S^- \neq \emptyset$, and the following hold:

- (A) Every element of $C^+ \cup C^-$ has a neighbor in $S^+ \cup S^-$.
- (B) Every element of $S^+ \cup S^-$ has a nonneighbor in $C^+ \cup C^-$.

In the case of Hansen polytopes of split graphs it turns out that $p_G(C, S)$ is exactly the number of faces that we have additionally to 3^d . By $s(P)$ we denote the number of nonempty faces of the polytope P .

Theorem 4.6. *Let $G = C \cup S$ be a split graph on $d - 1$ nodes. Then*

$$s(H(G)) = 3^d + p_G(C, S).$$

In particular, Hansen polytopes of split graphs satisfy the 3^d -conjecture.

Proof. Let Π be the set of all partitions and $\Pi_A, \Pi_B \subseteq \Pi$ be the subsets for which (A) and (B) hold, respectively. Observe that if (A) fails for a partition, that there must be a node in $C^+ \cup C^-$ which is not adjacent to any node in $S^+ \cup S^-$. Thus, this partition fulfills (B). From this we get $\Pi_A^c \subseteq \Pi_B$, where Π_A^c is the complement of Π_A in Π . Analogously, $\Pi_B^c \subseteq \Pi_A$ holds. This yields by some simple counting and inclusion-exclusion

$$3^{d-1} = |\Pi| = |\Pi_A| + |\Pi_B| - |\Pi_A \cap \Pi_B|.$$

Since $p_G(C, S) = |\Pi_A \cap \Pi_B| - 1$, we thus need to show that

$$s(\mathbf{H}(G)) = 3^d + |\Pi_A \cap \Pi_B| - 1 = 2 \cdot 3^{d-1} + |\Pi_A| + |\Pi_B| - 1.$$

For this we are going to use the partitioning of the face lattice of $\mathbf{H}(G)$, that was introduced in Definition 4.3. Let $f_p(G)$ be the number of primitive faces of $\mathbf{H}(G)$, $f_+(G)$ be the number of positive, and $f_-(G)$ be the number of negative ones. Regarding the small faces, one observes the following: If F is small, then by definition it is contained in type-(1)-facets of both signs. Type-(1)-facets correspond to type-(1)-vertices of the same sign of the polar polytope (via the usual bijection $F \mapsto F^*$ between the face lattice of a polytope and its polar). Lemma 4.4 yields that F^* must be a primitive face of $\mathbf{H}(G)^* \cong \mathbf{H}(\overline{G})$. Hence,

$$s(\mathbf{H}(G)) = f_p(G) + f_+(G) + f_-(G) + f_p(\overline{G}) - 1.$$

All we need in order to finish this proof is the following lemma.

Lemma 4.7. *In the setting above we have*

- (i) $f_+(G) = f_-(G) = 3^{d-1}$
- (ii) $f_p(G) = |\Pi_A|$ and $f_p(\overline{G}) = |\Pi_B|$

From this lemma the theorem obviously follows. \square

Proof of Lemma 4.7. For this proof we need to refine the notion of a primitive face. Given multisets $S^+ := \{s_i : i \in I\}$ and $S^- := \{s_j : j \in J\}$, a primitive face of the form

$$\bigcap_{i \in I} [+, A_i \cup s_i] \cap \bigcap_{j \in J} [-, B_j \cup s_j]$$

will be called (S^+, S^-) -primitive.

(i) It is clear that a facet $[\varepsilon, A]$ gets mapped to $[-\varepsilon, A]$ by the bijection $x \mapsto -x$. We therefore have $f_+(G) = f_-(G)$, and showing $f_+(G) = 3^{d-1}$ will finish this part of the proof. So let us consider a positive face $P = [+, A'] \cap [+, A] \cap F$, where $A' \subseteq A \subseteq C$ and

$$F = \bigcap_{i \in I} [+, A_i \cup s_i] \cap \bigcap_{j \in J} [-, B_j \cup s_j]$$

being primitive. As noted in the proof of Lemma 4.4, the multisets $\{s_i : i \in I\}$ and $\{s_j : j \in J\}$ are disjoint and P contains a vertex $(-, X)$ if and only if it is contained in F , i.e., if and only if $\{s_j : j \in J\} \subseteq X \subseteq S \setminus \{s_i : i \in I\}$. Since there are $3^{|S|}$ many ways to choose two disjoint subsets from S , it suffices to show that for fixed $\{s_i : i \in I\}$ and $\{s_j : j \in J\}$, we have $3^{|C|}$ many positive faces of the above form. To this end let F be a fixed $(\{s_i : i \in I\}, \{s_j : j \in J\})$ -primitive face. For such a face the type-(1)-vertices are determined as just explained, thus it is enough to find out which type-(2)-vertices belong to P . We can describe them precisely as

$$(+, X \cup z) \in P \Leftrightarrow z \in A' \text{ and } z \notin \bigcup_{j \in J} B_j \text{ and } \forall i \in I : (\{z, s_i\} \in E(G) \Rightarrow z \in A_i)$$

and similarly

$$(-, X \cup z) \in P \Leftrightarrow z \notin A \text{ and } z \notin \bigcup_{i \in I} A_i \text{ and } \forall j \in J : (\{z, s_j\} \in E(G) \Rightarrow z \in B_j).$$

These conditions tell us that for each $z \in C$, either there is an $X \subseteq S$ such that $(+, X \cup z) \in P$ or there is an X such that $(-, X \cup z) \in P$, or none of these is true. Furthermore, all three cases can be controlled independently and so we get the desired $3^{|C|}$ positive faces for fixed $\{s_i : i \in I\}$ and $\{s_j : j \in J\}$.

(ii) Each partition of G that satisfies (A), automatically satisfies (B) for \overline{G} , and the other way around. It is therefore enough to prove $f_p(G) = |\Pi_A|$. This will be

done by constructing a bijection $\mathcal{P} \rightarrow \Pi_A$, where \mathcal{P} is the set of all primitive faces of $H(G)$. For this purpose, we partition the domain and range as follows:

- Denote by $\mathcal{P}(S^+, S^-)$ the set of all (S^+, S^-) -primitive faces. Then

$$\mathcal{P} = \bigcup \{ \mathcal{P}(S^+, S^-) : S^+, S^- \subseteq S \text{ disjoint and } S^+ \cup S^- \neq \emptyset \}$$

is a partition of \mathcal{P} .

- Let $\Pi_A(S^+, S^-)$ be the set of all partitions of G that satisfy (A) and have S^+, S^- fixed (so only C^+, C^- vary). Then

$$\Pi_A = \bigcup \{ \Pi_A(S^+, S^-) : S^+, S^- \subseteq S \text{ disjoint and } S^+ \cup S^- \neq \emptyset \}$$

is a partition of Π_A .

From now on let $S^+, S^- \subseteq S$ be disjoint and $S^+ \cup S^- \neq \emptyset$. We will describe mappings

$$\Psi_{(S^+, S^-)} : \mathcal{P}(S^+, S^-) \rightarrow \Pi_A(S^+, S^-)$$

and

$$\Phi_{(S^+, S^-)} : \Pi_A(S^+, S^-) \rightarrow \mathcal{P}(S^+, S^-),$$

that will turn out to be inverse to each other. This of course shows that there exists a bijective correspondence between different parts of the partitions, which allows us to conclude the existence of a bijection $\mathcal{P} \rightarrow \Pi_A$. Define $\Psi_{(S^+, S^-)}$ to be

$$\Psi_{(S^+, S^-)} : F \mapsto (C^+, C^-, C^0, S^+, S^-, S^0),$$

and for $\varepsilon = \pm$ let

$$(1) \quad C^\varepsilon := \{ c \in C : (\varepsilon, (S^\varepsilon \setminus N(c)) \cup c) \in F \text{ and } \forall J \subseteq S : (-\varepsilon, J \cup c) \notin F \}.$$

Here $N(c)$ again stands for the neighborhood of c in G . On the other hand define $\Phi_{(S^+, S^-)}$ to be

$$\begin{aligned} \Phi_{(S^+, S^-)} : (C^+, C^-, C^0, S^+, S^-, S^0) \mapsto \\ \bigcap_{s \in S^+} [+, A'_s \cup s] \cap [+, A_s \cup s] \cap \bigcap_{s \in S^-} [-, B'_s \cup s] \cap [-, B_s \cup s], \end{aligned}$$

where $A'_s := C^+ \cap N(s)$, $A_s := N(s) \setminus C^-$, $B'_s := C^- \cap N(s)$ and $B_s = N(s) \setminus C^+$. Let us use the abbreviations $\psi := \Psi_{(S^+, S^-)}$ and $\phi := \Phi_{(S^+, S^-)}$ for the rest of this proof.

Then we have $\psi \circ \phi = \text{id}_{\mathcal{P}(S^+, S^-)}$: Given a partition $\pi = (C^+, C^-, C^0, S^+, S^-, S^0)$ it is sufficient to prove $\pi \subseteq \psi \circ \phi(\pi)$, where inclusion is to be understood componentwise. This is because both π and its image are partitions by construction. Let us denote the first component of the image by D^+ , the second by D^- , and the third by D^0 . We begin by explaining why $C^+ \subseteq D^+$. If $c \in C^+$, then by definition $c \in D^+$ only if

- the vertex $v = (+, (S^+ \setminus N(c)) \cup c) \in \phi(\pi)$ and
- for all $J \subseteq S$ the vertex $w_J = (-, J \cup c) \notin \phi(\pi)$.

Concerning the first item, one observes that the stable set $(S^+ \setminus N(c)) \cup c$ does not hit any of the $B_s \cup s$, so v is contained in all of the facets with a negative sign. For the facets with a positive sign the containment is clear if $c \in A'_s$, and in case $c \notin A'_s$, we have $c \notin N(s)$, i.e., $s \in S^+ \setminus N(c)$. Regarding the second item, by (A) there exists a neighbor $s \in S^+ \cup S^-$ of c . If $s \in S^+$, then $c \in C^+ \cap N(s) = A'_s$ and therefore $c \in A'_s \cup s$ which rules out that $(-, J \cup c) \in \phi(\pi)$. If $s \in S^-$, then $c \notin B'_s$ by construction. So if $w_J \in \phi(\pi)$, we must have $s \in J$ which contradicts $J \cup c$ being stable. These observations about the two items above yield $c \in D^+$. The inclusion $C^- \subseteq D^-$ can be proved similarly.

We continue by explaining $C^0 \subseteq D^0$, so assume $c \in C^0$. If $c \notin N(S^+ \cup S^-)$, then $N(c) \cap S \subseteq S^0$ and we get by the vertex-facet incidences $(+, S^+ \cup c)$, $(-, S^- \cup c) \in$

$\phi(\pi)$, and in addition $c \in D^0$. If $c \in N(S^+ \cup S^-)$, we can assume w.l.o.g. that $\{c, s\} \in E(G)$ for some $s \in S^+$. Then $c \in C^0 \cap N(s) \subseteq A_s$, which means $(-, J \cup c) \notin \phi(\pi)$. But we also must have $c \notin A'_s$, from which we get $(+, J \cup c) \notin \phi(\pi)$, since $s \notin J$ if $J \cup \{c\}$ is stable. This shows $c \in D^0$, and we therefore have $\psi \circ \phi = \text{id}_{\mathcal{P}(S^+, S^-)}$.

Furthermore, we can deduce $\phi \circ \psi = \text{id}_{\Pi_A(S^+, S^-)}$: Given a primitive face

$$F = \bigcap_{s \in S^+} [+, A'_s \cup s] \cap [+, A_s \cup s] \cap \bigcap_{s \in S^-} [-, B'_s \cup s] \cap [-, B_s \cup s],$$

we need to show $\phi \circ \psi(F) = F$. Both F and its image are (S^+, S^-) -primitive faces. Such faces contain type-(1)-vertices (ε, J) if and only if $S^\varepsilon \subseteq J \subseteq S \setminus S^{-\varepsilon}$; as usual this follows from the vertex-facet incidences. So F and $\phi \circ \psi(F)$ contain the same type-(1)-vertices and thus we only need to show that they also contain the same type-(2)-vertices.

We will begin by showing that if $(\varepsilon, J \cup c) \in F$, then $(\varepsilon, J \cup c) \in \phi \circ \psi(F)$. To this end we distinguish two cases.

1) Assume there exists $K \subseteq S$ such that $(-\varepsilon, K \cup c) \in F$. This means that c cannot be in A_s or B_s for $s \in S^+$ or $s \in S^-$, respectively. So because of our assumptions, we must have $S^{-\varepsilon} \subseteq K$ and $S^\varepsilon \subseteq J \subseteq S \setminus S^{-\varepsilon}$. From this we get that c has no neighbor in $S^{-\varepsilon}$. Altogether, this yields $(\varepsilon, J \cup c) \in [-\varepsilon, (C^{-\varepsilon} \cap N(s)) \cup s]$ for all $s \in S^{-\varepsilon}$, and $(\varepsilon, J \cup c) \in [\varepsilon, (C^\varepsilon \cap N(s)) \cup s]$ for all $s \in S^\varepsilon$. Hence, $(\varepsilon, J \cup c) \in \phi \circ \psi(F)$.

2) The other case is $(-\varepsilon, K \cup c) \notin F$ for all $K \subseteq S$. If $s \in S^\varepsilon$ is not adjacent to c , we must have $s \in J$, i.e., $S^\varepsilon \setminus N(c) \subseteq J$. According to (1) we also have $c \in C^\varepsilon$. So for every $s \in S^\varepsilon$, either $s \in J$ or $c \in C^\varepsilon \cap N(s)$. From this we get that $(\varepsilon, J \cup c)$ is contained in every facet defining $\phi \circ \psi(F)$ of sign ε . Since $J \cap S^{-\varepsilon} = \emptyset$, we conclude that $(\varepsilon, J \cup c)$ is also contained in every facet of sign $-\varepsilon$. This proves $(\varepsilon, J \cup c) \in \phi \circ \psi(F)$.

Finally, we need to prove that if $(\varepsilon, J \cup c) \in \phi \circ \psi(F)$, then $(\varepsilon, J \cup c) \in F$. Again, we distinguish between two cases. We know from the vertex-facet incidences that $J \subseteq S \setminus S^{-\varepsilon}$ for all $(\varepsilon, J \cup c) \in \phi \circ \psi(F)$.

1) Let $S^\varepsilon \subseteq J$. For the sake of contradiction assume $(\varepsilon, J \cup c) \notin F$. Then it is easy yet tedious to show that one must have $(\varepsilon, (S^\varepsilon \setminus N(c)) \cup c) \notin F$. (For this recall that $J \cup c$ is stable and that the facets defining F are induced by cliques, and then prove the contrapositive statement.) This means that $c \in D^\varepsilon$, where D^ε is again a component of $\psi(F)$. From this in turn we can conclude that $c \notin D^\varepsilon \cup N(s)$ and $c \notin N(s) \setminus D^\varepsilon$ for some $s \in S^{-\varepsilon}$, i.e., in particular $c \in N(s)$. Therefore $c \in (D^0 \cup D^{-\varepsilon}) \cap N(s)$, so $(\varepsilon, J \cup c) \notin [-\varepsilon, (N(s) \setminus D^\varepsilon) \cup s]$. But this contradicts $(\varepsilon, J \cup c) \in \phi(\psi(F))$.

2) If on the other hand $S^\varepsilon \not\subseteq J$, then there exists $s \in S^\varepsilon$ with $s \notin J$. Because $(\varepsilon, J \cup c) \in \phi(\psi(F))$ we then must have $c \in D^\varepsilon \cap N(s)$, where D^ε is a component of $\phi(F)$. So in particular $c \in D^\varepsilon$, which of course means $(\varepsilon, (S^\varepsilon \setminus N(c)) \cup c) \in F$. Now it can be easily (but again tediously) deduced that $(\varepsilon, J \cup c) \in F$.

This shows $\psi \circ \phi = \text{id}_{\Pi_A(S^+, S^-)}$, and therefore establishes the bijection and finishes the proof. \square

In particular this theorem says that the partition of the split graph does not play any role in the number of vertices of the corresponding Hansen polytope. So instead of $p_G(C, S)$ we will write p_G from now on. What we know about this function is summarized by the following corollary.

Corollary 4.8. *Let $G = C \cup S$ be a split graph on $d - 1$ nodes. Then*

$$s(\text{H}(G)) = 3^d + 16 \cdot \ell, \quad \text{for some } \ell \in \mathbb{N},$$

with $\ell = 0$ if and only if G is threshold.

Proof. Let us first establish that $p_G = 16 \cdot \ell$. Assume that $C = C^+ \cup C^- \cup C^0$ and $S = S^+ \cup S^- \cup S^0$ is given. If $C^+ \cup C^- = \emptyset$, then (B) is only satisfied if $S^+ \cup S^- = \emptyset$. Similarly, if $S^+ \cup S^- = \emptyset$, we have $C^+ \cup C^- = \emptyset$ because of (A). In both cases we deal with the trivial partition $C^0 = C$, $S^0 = S$ that is not counted by p_G , and thus can be ignored. If $C^+ \cup C^- = \{c\}$, then by (A) there exists a neighbor of c in $S^+ \cup S^-$. By (B) again, this neighbor must have a nonneighbor in $C^+ \cup C^-$, which clearly cannot be. So also this case is not counted by p_G and can be ignored as well. By similar reasoning, we can disregard the case $S^+ \cup S^- = \{s\}$. Therefore, we must have $|C^+ \cup C^-| \geq 2$ and $|S^+ \cup S^-| \geq 2$. Since we can assign the elements to C^+ , C^- or S^+ , S^- in an arbitrary way, we must have $p_G = 16 \cdot \ell$.

Now $\ell = 0$ if and only if $p_G = 0$. But if G has a path on four nodes P_4 as an induced subgraph, then the partition where C^+ is the two middle nodes of P_4 , S^+ is the two endpoints and $C^- = S^- = \emptyset$, satisfies the conditions (A) and (B). So if $\ell = 0$, then G is a split graph with no induced path of four nodes. But by Theorem 1.2.4 in [6], this happens exactly when G is threshold. On the other hand, if G is threshold then $H(G)$ is a Hanner polytope by Theorem 3.3, so $\ell = 0$. \square

4.3. High-dimensional Hansen polytopes with few faces. In the rest of this section we will study a construction that leads us to high-dimensional Hansen polytopes with few faces. To this end, consider a threshold graph T on m nodes and a split graph $G = C \cup S$ on n nodes. We construct a new graph $G \times T$ by taking the union of G and T and adding edges between every node of C and every node of T . Figure 1 is an illustration of our construction with G being the path on four nodes.

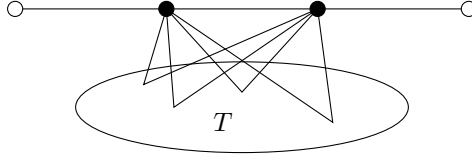


FIGURE 1. Appending a threshold graph to a split graph

It is clear that the resulting graph is again a split graph and therefore perfect.

Proposition 4.9. *Let $G = C \cup S$ be a split graph on n nodes. Then, for any given threshold graph T on m nodes, we have*

$$s(H(G \times T)) = 3^{m+n+1} + p_G.$$

This means $p_{G \times T} = p_G$, so $p_{G \times T}$ is independent of T .

Proof. By definition the threshold graph T can be built by successive adding of isolated and dominating nodes. This induces an ordering on the nodes v_1, \dots, v_m of T . Let $C_T := \{v_i : v_i \text{ dominating at step } i\}$ and $S_T := \{v_i : v_i \text{ isolated at step } i\}$. This splits T into a clique C_T and a stable set S_T , which in turn splits $G \times T$ into $C \cup C_T$ and $S \cup S_T$. By construction any node in C_T and S_T is connected to all nodes in C and none in S . Now consider a partition $(C^+ \cup C^- \cup C^0, S^+ \cup S^- \cup S^0)$ of $G \times T$ that is counted by $p_{G \times T}(C \cup C_T, S \cup S_T)$. By (A) for all $x \in (C^+ \cup C^-) \cap C_T$ there exists a neighbor $y \in (S^+ \cup S^-) \cap S_T$, which means that in T the once isolated node y was inserted before the once dominating node x . On the other hand, by (B), any given node $y \in (S^+ \cup S^-) \cap S_T$ has to have a nonneighbor $z \in (C^+ \cup C^-) \cap C_T$. Such a node z was used before y in the construction of T . These two observations can only hold in the case $(C^+ \cup C^-) \cap C_T = \emptyset = (S^+ \cup S^-) \cap S_T$. Therefore, for this partition we have $C_T \subseteq C^0$ and $S_T \subseteq S^0$, which implies that $p_{G \times T}(C \cup C_T, S \cup S_T) = p_G$. \square

This finally yields a series of high-dimensional Hansen polytopes with very few faces.

Corollary 4.10. *Let P_4 be a path on four nodes and T be an arbitrary threshold graph on m nodes. Then*

$$s(\mathbb{H}(P_4 \times T)) = 3^{m+5} + 16.$$

Proof. Determining $p_{P_4 \times T} = p_{P_4} = 16$ is an easy counting exercise. \square

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