A UNIQUE REPRESENTATION OF POLYHEDRAL TYPES

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Abstract. It is known that for each combinatorial type of convex 3-dimensional polyhedra, there is a representative with edges tangent to the unit sphere. This representative is unique up to projective transformations that fix the unit sphere. We show that there is a unique representative (up to congruence) with edges tangent to the unit sphere such that the origin is the barycenter of the points where the edges touch the sphere.

In today's language, Steinitz' *fundamental theorem of convex types* [11][12] states that the combinatorial types of convex 3-dimensional polyhedra correspond to the strongly regular cell decompositions of the 2-sphere. (A cell complex is *regular* if the closed cells are attached without identifications on the boundary. A regular cell complex is *strongly regular* if the intersection of two closed cells is a closed cell or empty.) Grünbaum and Shephard [6] asked whether for every combinatorial type there is a polyhedron with edges tangent to a sphere. This question has been answered affirmatively: There is always such a polyhedron. Furthermore, it is unique up to projective transformations of $\mathbb{R}\mathrm{P}^3$ which fix the sphere and do not make the polyhedron intersect the plane at infinity. For simplicial polyhedra, this result is contained in Thurston's notes [13]. The general case is equivalent to a theorem on circle packings, of which the first published proof seems to be by Brightwell and Scheinerman [3]. For another proof, which makes use of a variational principle, see [2][10]. Schramm [9] proves the much stronger theorem that for every combinatorial type T and for every smooth strictly convex body $K \subset \mathbb{R}^3$, there is a polyhedron of type T with edges tangent to K.

The purpose of this article is to provide a proof (taken from the author's doctoral dissertation [10]) for the following theorem, which singles out a unique representative for each convex type. The claim is not new (see Ziegler [14], p. 118, and the second edition of Grünbaum's classic [5], p. 296a) but this proof seems to be. I thank Alexander Bobenko and Günter Ziegler for making me familiar with the problem and Ulrich Pinkall who has provided an important insight for its solution.

Theorem. *For every combinatorial type of convex 3-dimensional polyhedra there is, among the representatives with edges tangent to the unit sphere $S^2 \subset \mathbb{R}^3$, a unique polyhedron (up to isometry) such that the origin $0 \in \mathbb{R}^3$ is the barycenter of the points where the edges touch the sphere.*

The following proof of the theorem is based on the fundamental relationship between projective, hyperbolic, and Möbius geometry. The equation

$$-x_0^2 + x_1^2 + x_2^2 + \ldots + x_{d+1}^2 = 0$$

represents the d-dimensional sphere S^d as a quadric in $(d+1)$-dimensional projective space $\mathbb{R}\mathrm{P}^{d+1}$. The group of projective transformations of $\mathbb{R}\mathrm{P}^{d+1}$ which fix S^d is $O(d+1,1)/\{\pm 1\}$, where the orthogonal group $O(d+1,1) \subset GL(d+2)$ acts linearly on the homogeneous coordinates. At the same time, $O(d+1,1)/\{\pm 1\}$ acts faithfully as the Möbius group on S^d, and as the isometry group of $(d+1)$-dimensional

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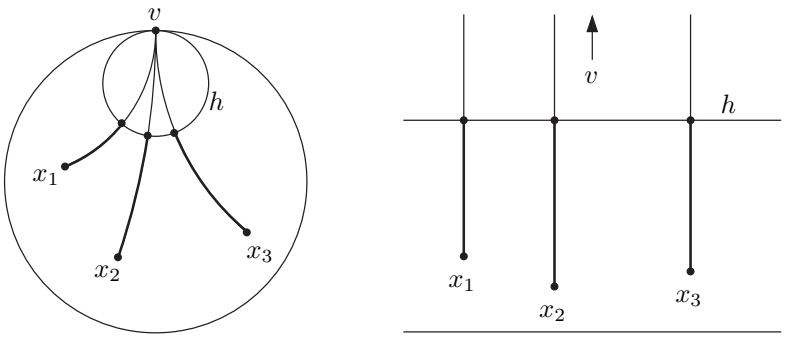

FIGURE. The 'distance' to an infinite point v is measured by cutting off at some horosphere through v. *Left:* Poincaré ball model. *Right:* half-space model.

hyperbolic space H^{d+1}, which is identified with the open ball bounded by S^d (the Klein model of hyperbolic space). For a detailed account of this classical material see, for example, Hertrich-Jeromin [7] and Kulkarni, Pinkall [8].

The same interplay of geometries led Bern and Eppstein [1][4] to another choice of a unique representative for each polyhedral type. For types with a symmetry group which is not just a cyclic group, their representative coincides with ours.

Since the projective transformations of $\mathbb{R}\mathrm{P}^{d+1}$ that fix S^d correspond to the Möbius transformations of S^d, the theorem follows from the following lemma.

Lemma 1. *Let $v_1, \dots, v_n$ be $n \geq 3$ distinct points in the d-dimensional unit sphere $S^d \subset \mathbb{R}^{d+1}$. There exits a Möbius transformation T of S^d, such that*

$$\sum_{j=1}^{n} Tv_j = 0.$$

If $\widetilde{T}$ is another such Möbius transformation, then $\widetilde{T} = RT$, where R is an isometry of S^d.

On the other hand, the Möbius transformations of S^d correspond to isometries of the hyperbolic space H^{d+1}, of which S^d is the infinite boundary. For $n \geq 3$ points $v_1, \dots, v_n \in S^d$, we are going show that there is a unique point $x \in H^{d+1}$ such that the sum of the 'distances' to $v_1, \dots, v_n$ is minimal. Of course, the distance to a point in the infinite boundary is infinite. The right quantity to use instead is the distance to a horosphere through the infinite point (see the figure).

Definition. For a horosphere h in H^{d+1}, define

$$\delta_h : H^{d+1} \to \mathbb{R},$$

$$\delta_h(x) = \begin{cases} -\operatorname{dist}(x,h) & \text{if } x \text{ is inside } h, \\ 0 & \text{if } x \in h, \\ \operatorname{dist}(x,h) & \text{if } x \text{ is outside } h, \end{cases}$$

where $\operatorname{dist}(x,h)$ is the distance from the point x to the horosphere h.

Suppose v is the infinite point of the horosphere h. Then the shortest path from x to h lies on the geodesic connecting x and v. If h' is another horosphere through v, then $\delta_h - \delta_{h'}$ is constant. If $g : \mathbb{R} \to H^{d+1}$ is an arc-length parametrized geodesic, then $\delta_h \circ g$ is a strictly convex function, unless v is an infinite endpoint of the geodesic g. In that case, $\delta_h \circ g(s) = \pm(s - s_0)$. These claims are straightforward

to prove using the Poincaré half-space model, where hyperbolic space is identified with the upper half space:

$$H^{d+1} = \big\{(x_0, \ldots, x_d) \in \mathbb{R}^{d+1} \;\big|\; x_0 > 0\big\},$$

and the metric is

$$ds^2 = \frac{1}{x_0^2}\big(dx_0^2 + dx_1^2 + \cdots + dx_d^2\big)$$

Also, one finds that, as $x \in H^{d+1}$ approaches the infinite boundary,

$$\lim_{x \to \infty} \sum_{j=1}^{n} \delta_{h_j}(x) = \infty,$$

where h_j are horospheres through different infinite points and $n \geq 3$. Thus, the following definition of the *point of minimal distance sum* is proper.

Lemma (and Definition) 2. *Let $v_1, \ldots, v_n$ be n points in the infinite boundary of H^{d+1}, where $n \geq 3$. Choose horospheres $h_1, \ldots, h_n$ through $v_1, \ldots, v_n$, respectively. There is a unique point $x \in H^{d+1}$ for which $\sum_{j=1}^{n} \delta_{h_j}(x)$ is minimal. This point x does not depend on the choice of horospheres. It is the* point of minimal distance sum *from the infinite points $v_1, \ldots, v_n$.*

In the Poincaré ball model, hyperbolic space is identified with the unit ball as in the Klein model, but the metric is $ds^2 = \frac{4}{(1-\sum x_j^2)^2} \sum dx_j^2$. (Since the Klein model and the Poincaré ball model agree on the infinite boundary and in the center of the sphere, one might as well use the Klein model in the following lemma.)

Lemma 3. *Let $v_1, \ldots, v_n$ be $n \geq 3$ different points in the infinite boundary of H^{d+1}. In the Poincaré ball model, $v_j \in S^d \subset \mathbb{R}^{d+1}$. The origin is the point of minimal distance sum, if and only if $\sum v_j = 0$.*

Proof. If h_j is a horosphere through v_j, then the gradient of δ_{h_j} at the origin is the unit vector $-\frac{1}{2}v_j$. □

Lemma 1 is now almost immediate. Let x be the point of minimal distance sum from the $v_1, \ldots, v_n$ in the Poincaré ball model. There is a hyperbolic isometry T which moves x into the origin. If $\widetilde{T}$ is another hyperbolic isometry which moves x into the origin, then $\widetilde{T} = RT$, with R is an orthogonal transformation of $\mathbb{R}^{d+1}$. Lemma 1 follows. This concludes the proof of the theorem.

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