NUMERICAL APPROXIMATION OF CAPILLARY SURFACES IN A NEGATIVE GRAVITATIONAL FIELD

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ABSTRACT. A capillary surface in a negative gravitational field describes the shape of the surface of a hanging drop in a capillary tube with wetting material on the bottom. Mathematical modeling leads to the volume- and obstacle-constrained minimization of a nonconvex nonlinear energy functional of mean curvature type which is unbounded from below. In 1984 Huisken proved the existence and regularity of local minimizers of this energy under the condition on gravitation being sufficiently weak. We prove convergence of a first order finite element approximation of these minimizers. Numerical results demonstrating the theoretic convergence order are given.

1. INTRODUCTION

By everyday experience hanging drops do not fall if they are small enough. By rescaling this condition is equivalent to gravitation being sufficiently weak. Mathematical modeling of this situation gives rise to the minimization of a nonlinear energy functional of mean curvature type. We refer to this minimization as the capillarity problem and to its solution as a capillary surface. It makes sense to consider the capillarity problem in positive and negative gravitational fields, i.e., sitting and hanging drops in capillary tubes. Considering positive gravitation leads to a convex energy functional, whereas negative gravitation leads to nonconvexity of the energy. In the case of negative gravitation we need to fix the volume of the liquid, whereas in positive gravitation this is optional. Furthermore, in modeling of closed tubes or multiphase fluids obstacle constraints arise.

Solvability of the capillarity problem in a positive gravitational field was shown in [Ger74], regularity of such a solution in [Ger76]. Obstacle and volume constraints were treated in [Hui85]. In [Hui84] it was shown that there exists a local minimizer to the capillarity problem with fixed volume and an obstacle constraint in a negative gravitational field for sufficiently weak gravitation.

Convergence of the approximation of a two-dimensional capillary surface in a positive gravitational field by first order finite elements was proven in [Mit77]. The nonconvex case of negative gravitation has not been treated so far. The inclusion of a volume constraint in the case of positive gravitation was briefly addressed in [Mit77]. Obstacle constraints have not been examined in this context.

In this work we will consider n-dimensional capillary surfaces of fixed volume over an obstacle in negative gravitation. We will show existence of local minimizers of a first order finite element discretization of the energy functional as well as convergence of the discrete to the continuous solution.

I would like to express my gratitude to Gerhard Huisken for initiating this work. Many thanks to Ralf Kornhuber, Oliver Sander, and Carsten Gräser for encouragement and enriching discussions.

Our method of proof provides the optimal order of approximation although the problem has global properties usually considered problematic, such as nonlinearity and nonconvexity. This is possible because we are only interested in the behavior of discrete solutions locally, i.e., near the continuous ones. We are using a concept of locality specific to the field of partial differential equations. By using Poincaré's inequality, we will exploit the higher order of the nonlinearity to establish regions of convexity for the problem where we can find unique minimizers. Modifications of the convergence proof of [Mit77] will show that these minimizers in the restricted regions converge to the continuous solution. Using the boundedness of the continuous solution established in [Hui84] we then show that we can thus obtain a discrete solution which lies in the interior of a region of convexity and therefore is a local minimizer of the energy functional. The convergence result then implies that this discrete solution converges to the continuous one in the $W^{1,2}$-norm with linear dependence on the mesh size.

After a mathematical description of the problem we will give a short overview of previous results leading to the work of this paper. In the following main part we show existence of solutions to a finite element discretization of the capillarity problem in a negative gravitational field as well as a convergence result. We will conclude this paper by giving some numerical results illustrating the theoretic convergence order.

2. The Continuous Problem

A capillary surface is a surface of prescribed mean curvature with Neumann boundary conditions. The name is derived from the typical example of the surface of a liquid rising in a capillary tube.

The principle of energy minimization states that the surface will be in a state of minimal potential energy. In order to obtain a mathematical expression for the energy we assume that the surface S of the liquid can be expressed as the graph of a function $u:\ \Omega \to \mathbb{R}$ over the cross section $\Omega \subseteq \mathbb{R}^n$ of the tube, where Ω is a connected and bounded domain. This parametrization of the surface $S = \operatorname{graph}(u)$ induces a metric g on S defined by

$$g_{ij}(x) = \delta_{ij} + D_i u(x) D_j u(x)\ ,$$

where

$$D_k u = \frac{\partial u}{\partial x^k}, \quad k = 1, \dots, n\ ,$$

and δ_{ij} denotes the Kronecker symbol.

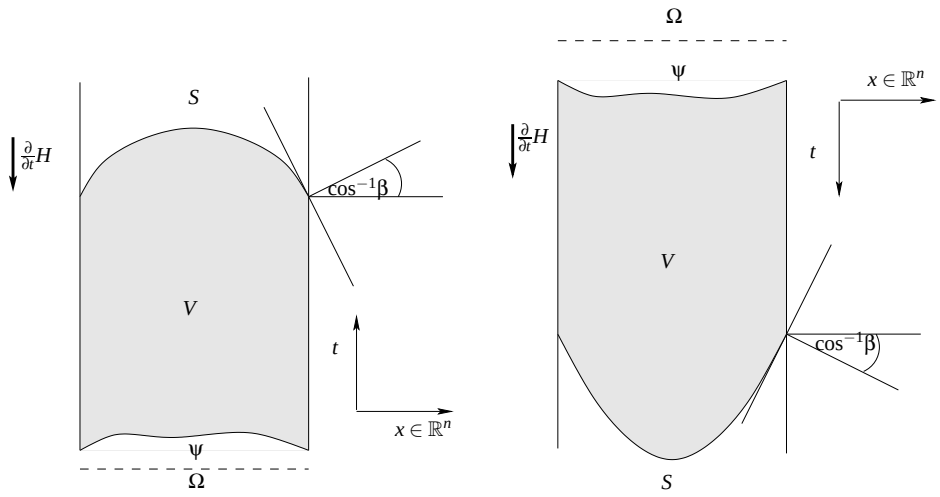
The shape of the surface is then given by a minimizer of the energy functional, i.e., the capillarity problem reads

$$u \in K:\ J(u) \le J(v) \quad \forall v \in K\ , \tag{1}$$

where K is a suitable function set and J is given by

$$J(v) = \int_\Omega \sqrt{1 + |Dv|^2}\ dx + \int_\Omega \int_0^{v(x)} H(x,t)\ dt\ dx + \int_{\partial\Omega} \beta v\ d\mathcal{H}_{n-1}\ . \tag{2}$$

Here the first term models the cohesive energy as proportional to the area of the surface. The second term describes the gravitational energy, where $H \in C^{0,1}(\mathbb{R}^n \times \mathbb{R})$ describes the gravitational potential. The third term is related to the adhesive energy at the boundary of the capillary tube.



(a) Positive gravitation (b) Negative gravitation

FIGURE 1. Liquid in a capillary tube

The Euler–Lagrange equation corresponding to (1) has the form

$$Au + H(x,u) = 0 \quad \text{in } \Omega \tag{3}$$

$$-\sum_{i=1}^{n} a^i(Du)\,\mu_i = \beta \quad \text{on } \partial\Omega\,, \tag{4}$$

where $a^i(Du) = \frac{D_i u}{\sqrt{1+|Du|^2}}$ and μ denotes the outer normal to $\partial\Omega$. Note that

$$Au = -\sum_{i=1}^{n} D_i\left(a^i(Du)\right)$$

is an expression for the mean curvature of S, and that the left hand side of the boundary condition (4) gives the cosine of the contact angle.

Following the work of Huisken [Hui84] we are concerned with the capillarity problem over an obstacle in a gravitational field. For modeling we consider the surface of a liquid of fixed volume V in a capillary tube. We assume that the bottom of the tube can be represented by an obstacle function ψ and is of a material which is perfectly wetting, i.e., it is completely covered by a thin film of the liquid and thus does not add to the energy functional. This situation is depicted on the left hand side of Figure 1. We may also consider the liquid being in an upside down capillary tube. This is depicted on the right hand side of Figure 1. In the latter case we will then reverse the coordinate system so that we are again in the setting of a capillary surface over an obstacle.

The capillarity problem is given by (1) with

$$K = W^{1,\infty}(\Omega) \cap \{v \geq \psi\} \cap \left\{ \int_\Omega (v-\psi)\,dx = V \right\}\,, \tag{5}$$

and a gravitational potential of the form

$$H(x,t) = -\kappa\, t\,, \tag{6}$$

where $-\kappa > 0$ in the case of a "sitting" liquid, and $-\kappa < 0$ in the case of a "hanging" liquid. The focus of this work is the setting in negative gravitation, i.e. $-\kappa < 0$. Note that in [Hui84] Huisken considered a more general gravitational potential of the form $H(x,t) = -\kappa\, t + \tilde{H}(x,t)$ with $\frac{\partial \tilde{H}}{\partial t} > 0$. The above approximation (6) of the gravitational field as constant is possible in many physically important cases like gravity on Earth. Generalizations of our method to other applications like centrifuges should be possible (using suitable cut-offs) but rather technical.

We assume furthermore $\beta \in C^{0,1}(\partial\Omega)$ with

$$|\beta| \leq 1 - a, \quad a > 0\,. \tag{7}$$

To motivate the last condition, note that $\beta \geq 1$ means that the liquid will be in a state of lesser energy if it pulls back from the tube (lotus effect), and $\beta \leq -1$ means that the material of the tube is perfectly wetting (just as the obstacles considered in this work). Our model will not account for such situations.

Because of the negative quadratic term the energy functional J defined by (2) with (6) may be neither convex nor bounded from below. This can be easily seen in the following example.

Example. Assume $\psi \equiv 0$ and $V = 1$. Consider the mollifier functions

$$\eta_\varepsilon := \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right)$$

for $\varepsilon > 0$ on the unit ball $\Omega = B_1(0) \subset \mathbb{R}^n$ for $n \geq 2$ (cf. [Eva98]). The standard mollifier η is defined by

$$\eta(x) := C \exp\left(\frac{1}{|x|^2 - 1}\right)\,,$$

where the constant $C > 0$ is chosen such that $\int_\Omega \eta \; dx = 1$. Hence, $\eta_\varepsilon \in K$, where K is defined by (5), and $J(\eta_\varepsilon) \to -\infty$ as $\varepsilon \to 0$.

Thus, we cannot generally expect the minimization problem (1) to have global minimizers. Nevertheless, we can study local minimizers which are solutions to the corresponding variational inequality

$$\int_\Omega \frac{Du \cdot D(v-u)}{\sqrt{1+|Du|^2}}\, dx - \kappa \int_\Omega u\,(v-u)\, dx + \int_{\partial\Omega} \beta\,(v-u)\, d\mathcal{H}_{n-1} \geq 0 \qquad \forall v \in K\,. \tag{8}$$

Even in the context of positive gravitational fields, i.e. $\frac{\partial H}{\partial t} > 0$, we cannot expect the existence of a bounded capillary surfaces if $\partial\Omega$ has vertices (cf. [Fin79]). Thus, we will assume that $\partial\Omega$ is of class $C^{2,\alpha}$.

Various results on the well-posedness of capillarity problems can be found in the literature. Relying solely on BV-techniques Gerhardt proved in [Ger74] existence and uniqueness of solutions to (1) without a volume or obstacle constraint in a positive gravitational field, i.e., under the assumption $\frac{\partial H}{\partial t} > 0$. Using a different approach, he showed the following global regularity result in [Ger76].

Theorem 2.1. *Let $\partial\Omega \in C^{2,\alpha}$, $H, \beta \in C^{1,\alpha}$ and $\frac{\partial H}{\partial t} > 0$. The capillarity problem* (1) *has a unique solution $u \in C^{2,\vartheta}(\Omega)$, where ϑ, $0 < \vartheta < 1$, is determined by H, β, and Ω.*

The proof relies on a rather technical a priori estimate for the gradient (cf. [Hui85, Section 2]). The proof itself is done by a method of continuity and uses standard theory of uniformly elliptic differential equations [LU68] (cf. [GT98, Thm. 17.30]).

Relying on Theorem 2.1 Husiken extended the theory to the capillarity problem with an obstacle constraint in a positive gravitational field [Hui85], and to the capillarity problem in a negative gravitational field [Hui84]. For the latter case he showed that for sufficiently weak gravity, i.e. sufficiently small $\kappa > 0$, the variational inequality (8) admits a solution.

Theorem 2.2. *Let $\partial\Omega$ be of the class $C^{2,\alpha}$, $\psi \in C^2(\overline{\Omega})$ and $H,\ \beta \in C^{1,\alpha}$ with the properties* (6) *and* (7)*. If $\kappa > 0$ is sufficiently small then:*

(1) *The capillarity problem in a negative gravitational field* (8) *admits a solution*

$$u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega) \cap W^{2,\infty}_{loc}(\Omega)$$

with continuous tangential derivatives at the boundary.

(2) *If $n = 2$ then $u \in C^1(\overline{\Omega})$.*

(3) *If we assume that $\partial\Omega$ is of class $C^{3,\alpha}$, $\beta \in C^{1,1}(\partial\Omega)$, and ψ satisfies*

$$-\frac{D^i\psi}{\sqrt{1+|D\psi|^2}}\,\mu_i \geq \beta \quad \text{on } \partial\Omega$$

then

$$u \in W^{2,\infty}(\Omega)\,.$$

3. Discretization

In order to give a numerical approximation of capillary surfaces we will employ a first order finite element method which is also used by Mittelmann for the case of positive gravitational fields [Mit77]. We will extend his results to the case of a negative gravitational field, where J is non-convex.

For each h, $0 < h < h_0$, let $\Omega_h = \bigcup_{j=1}^{L(h)} T_j$ be a finite collection of n-simplices with disjoint interiors such that each face of a simplex is either the face of another simplex or has its vertices on $\partial\Omega$. We assume the triangulation to be shape regular in the sense that each simplex is contained in a ball of radius h and contains a ball of radius γh for a fixed $0 < \gamma < 1$. Since in general $\Omega_h \not\subset \Omega$ we assume that any solution u of the capillarity problem (8) may be extended to a domain $\tilde{\Omega} \subset \mathbb{R}^n$ with $\overline{\Omega} \subset \tilde{\Omega}$ and $\Omega_h \subset \tilde{\Omega}$, such that the extension is of the same class as u, coincides with u in Ω, and the extension operator is continuous. Existence of such an operator was shown in [Neč67, Ch. 2]. For simplicity the extension will again be denoted by u.

Let S_h denote the space of linear finite elements on $\tilde{\Omega}$, i.e.,

$$S_h = \big\{ v \in C(\tilde{\Omega}) :\ v|_{T_j} \text{ is linear},\ j = 1,\dots,L(h), \text{ and piecewise linearly extended outside } \Omega_h \big\}\,.$$

The finite element space S_h is spanned by the nodal basis

$$\Lambda_h := \{\lambda_p \in S_h |\ p \in N_h\}\,, \qquad \lambda_p(q) = \delta_{pq}\ \forall p, q \in N_h\,,$$

where N_h denotes the set of all vertices corresponding to the triangulation Ω_h. For every continuous function $v : \Omega \to \mathbb{R}$ we define its interpolation $v_I \in S_h$ by

$$v_I(x) := \sum_{p \in N_h} v(p)\,\lambda_p(x)\,. \tag{9}$$

For the interpolation error the following estimates hold [Cia78].

Theorem 3.1. *Let h be sufficiently small, and $u \in W^{k+1,p}(\Omega)$, $k \geq 0$, $1 \leq p \leq \infty$ with $(k+1)p > n$. Then there exists a constant C_1 such that the linear interpolation error for $m = 0, 1$ can be estimated by*

$$|u - u_I|_{m,p,\Omega} \leq C_1 \gamma^{-m} h^{k+1-m} |u|_{k+1,p,\Omega} .$$

Here and below

$$|v|_{k,p,\Omega} = \left(\sum_{|\alpha|=k} \int_\Omega |D^\alpha v|^p \, dx \right)^{\frac{1}{p}}$$
$$\|v\|_{k,p,\Omega} = \sum_{l \leq k} |v|_{l,p,\Omega}$$

denote the Sobolev (semi) norms for $k \in \mathbb{N}$ and $1 \leq p \leq \infty$ with the usual modification when $p = \infty$. If $k = 0$ this index may be omitted.

In the following we will assume that the setting is such that Theorem 2.2 holds, i.e., that there exists a solution

$$u \in W^{1,\infty}(\tilde{\Omega}) \cap W^{2,2}(\tilde{\Omega})$$

to (8) where H is given by (6) and K is defined by (5). Note that Morrey's inequality (cf. e.g. [Eva98]) implies for $p > n$ that

$$W^{1,p}(\Omega) \subset C^{0,1-\frac{n}{p}}(\Omega) .$$

Therefore, interpolation is well defined for $u \in W^{1,\infty}(\Omega)$.

If $n \geq 4$ we will furthermore assume that $\partial\Omega$, β, and ψ fulfill the higher regularity assumptions of Theorem 2.2 (3) such that $u \in W^{2,\infty}(\tilde{\Omega})$. Theorem 3.1 then implies for the interpolation error

$$|u - u_I|_{0,2,\Omega_h} \leq C\, h^2 ,$$

where the constant depends on $|u|_{2,2,\tilde{\Omega}}$ for $n < 4$ and on $|u|_{2,\infty,\tilde{\Omega}}$ for $n \geq 4$.

Let ψ_I denote the linear interpolation of the obstacle function ψ. Let furthermore

$$V^h := V - \int_{\Omega_h} \psi_I \, dx + \int_\Omega \psi \, dx$$

denote the discrete prescribed volume. We approximate the function set K by

$$K_h := S_h \cap \{v_h \geq \psi_I\} \cap \left\{ \int_{\Omega_h} (v_h - \psi_I) \, dx = V^h \right\} . \tag{10}$$

The discrete energy functional is defined by

$$J_h(v_h) := -\frac{\kappa}{2} a_h(v_h, v_h) - l_h(v_h) + \phi_h(v_h) , \tag{11}$$

with

$$a_h(v_h, w_h) = \int_{\Omega_h} v_h \, w_h \, dx ,$$
$$l_h(v_h) = -\int_{\partial\Omega_h} \beta_I \, v_h \, ds_h ,$$
$$\phi_h(v_h) = \int_{\Omega_h} \sqrt{1 + |Dv_h|^2} \, dx ,$$

for $v_h, w_h \in K_h$. The choice of the sign of κ depends on the direction of the gravitational force, i.e., $\kappa > 0$ for negative gravitation, $\kappa < 0$ for positive gravitation. Note that we can

view $\partial\Omega_h$ as a triangulation of $\partial\Omega$ and that $\beta \in C^{0,1}(\partial\Omega)$ implies that the finite element interpolation β_I on the boundary is well-defined.

As in the continuous case, we cannot expect a solution to the global minimization problem

$$u_h \in K_h : \; J_h(u_h) \leq J_h(v_h) \quad \forall v_h \in K_h \; , \tag{12}$$

to exist for general $\kappa > 0$ because of the non-convex term $-\frac{\kappa}{2}\int_{\Omega_h} v_h^2 \, dx$. We consider instead the corresponding variational inequality for critical points (we concentrate on local minima)

$$\int_{\Omega_h} \frac{Du_h \cdot D(v_h - u_h)}{\sqrt{1+|Du_h|^2}} \, dx - \kappa a_h(u_h, v_h - u_h) - l_h(v_h - u_h) \geq 0 \qquad \forall v_h \in K_h \; . \tag{13}$$

In the case of positive gravitation, i.e., for the problem (12) with $\kappa < 0$ and $K_h = S_h$, uniqueness and existence of solutions follow by the direct method of the calculus of variations (cf., e.g., [KS80]). In [Mit77] the following convergence result for the difference of the discrete solution u_h and the interpolation u_I of the continuous solution is proven.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\partial\Omega \in C^2$. If the continuous problem* (1) *without an obstacle bound and $H(x,t) = \kappa t$ has a solution $u \in W^{2,2}(\tilde{\Omega}) \cap W^{1,\infty}(\tilde{\Omega})$, then*

$$\|u_h - u_I\|_{1,2,\Omega_h} \leq C_2 h \; , \tag{14}$$

for all $0 < h < h_0$, h_0 sufficiently small, where the constant C_2 is independent of h.

In the following we will need two lemmas proved in [Mit77] which contain general facts about finite element functions.

Lemma 3.3. *For sufficiently small h there exists a constant $C_3 = C_3(\gamma)$ such that for any function $v_h \in S_h$*

$$|v_h|_{1,1,\partial\Omega_h} \leq C_3 h^{-1} |v_h|_{1,1,B_h} \; , \tag{15}$$

where $B_h = \bigcup_{j=1}^{K} T_j$ is the union of all simplices T_j having a face t_j in common with $\partial\Omega_h$. Furthermore, we can estimate

$$\left| \int_{\partial\Omega_h} v_h \, ds_h - \int_{\partial\Omega} v_h \, ds \right| \leq C_4 h^2 |v_h|_{1,1,\partial\Omega_h} \; . \tag{16}$$

Lemma 3.4. *For sufficiently small h there exists a constant $C_5 = C_5(\Omega)$ such that we may estimate for $v_h \in S_h$*

$$\|v_h\|_{1,\Omega-\Omega_h} \leq C_5 h^2 \|v_h\|_{1,1,\Omega_h}$$

and

$$\|v_h\|_{1,\Omega_h-\Omega} \leq C_5 h^2 \|v_h\|_{1,1,\Omega_h} \; .$$

4. Discretization of Capillary Surfaces in a Negative Gravitational Field

Our main result is the following:

Theorem 4.1. *If κ is small enough there exists an h_0 such that for all $0 < h < h_0$ there exists a solution to the discrete capillarity problem in a negative gravitational field with an*

obstacle, i.e., a function $u_h \in K_h$ satisfying (13)*. Furthermore, the discretization error is bounded by*

$$\|u_h - u\|_{1,2,\Omega_h} \leq C\, h\,. \tag{17}$$

The theorem will be proven in several steps. First we treat a simplified problem where we omit the obstacle. We discuss the matter of existence in Theorem 4.2, and Theorem 4.4 addresses the convergence. The inclusion of the obstacle is then addressed in Theorem 4.5.

4.1. **Solvability without an Obstacle.** If no obstacle is given, the continuous solution u is of class $C^{2,\vartheta}$ [Hui84]. For simplicity we will begin by considering this case.

Theorem 4.2. *If $\kappa > 0$ is small enough there exists a solution to the discrete capillarity problem in a negative gravitational field without an obstacle, i.e., a function*

$$u_h \in \tilde{K}_h := S_h \cap \left\{ \int_{\Omega_h} v\, dx = V \right\}$$

satisfying (13)*. This function u_h is a local minimizer of the energy functional J_h defined by* (11)*.*

Before we can prove the theorem we will prove the following:

Lemma 4.3. *Let $M \in \mathbb{R}$. If $\kappa > 0$ is small enough then there exists a unique solution u_M to the discrete problem*

$$u_M \in K_M :\ J_h(u_M) \leq J_h(v) \quad \forall v \in K_M\,, \tag{18}$$

where J_h is defined by (11) *and*

$$K_M := \tilde{K}_h \cap \{|Dv|_{\infty,\Omega_h} \leq M\}\,.$$

Proof. Note that K_M is compact. If we can choose κ small enough such that the energy functional J_h is strictly convex we can apply the direct method of the calculus of variations (cf., e.g., [KS80]).

Let $v, w \in S_h$ be in the set of admissible functions K_M, i.e., we assume

$$\int_{\Omega_h} v\, dx = \int_{\Omega_h} w\, dx = V, \qquad \|Dv\|_{\infty,\Omega_h}, \|Dw\|_{\infty,\Omega_h} \leq M\,.$$

The nonlinearity ϕ_h can be written as

$$\phi_h(v) = \int_{\Omega_h} \psi(Dv)\, dx\,,$$
$$\psi(x) = \sqrt{1+|x|^2}\,.$$

ϕ_h is strongly convex, i.e., we have for $\omega \in (0,1)$

$$\phi_h(\omega v + (1-\omega)w) \leq \omega\phi_h(v) + (1-\omega)\phi_h(w) - \frac{1}{2} m\omega(1-\omega)\|D(v-w)\|^2_{2,\Omega_h}\,,$$

where the parameter m is a lower bound on the least eigenvalue of $D^2\psi(Dv)$. In particular we may set

$$m = \frac{1}{(1+M^2)^{\frac{3}{2}}}\,.$$

This yields the estimate

$$J_h(\omega v+(1-\omega)w)-\omega J_h(v)-(1-\omega)J_h(w)$$
$$\leq \frac{1}{2}\omega(1-\omega)\left(\kappa\|v-w\|^2_{2,\Omega_h}-\frac{1}{2(1+M^2)^{\frac{3}{2}}}\|D(v-w)\|^2_{2,\Omega_h}\right).$$

Using Poincaré's inequality

$$\left\|f-\frac{1}{|\Omega_h|}\int_{\Omega_h} f\,dx\right\|_{p,\Omega_h}\leq C_6(p,n,\Omega)\,\|Df\|_{p,\Omega_h}\,, \tag{19}$$

for $p=2$ and $f=v-w$, we see that J_h is strictly convex if we choose

$$\kappa<\kappa_1:=\frac{1}{2C_6^2(1+M^2)^{\frac{3}{2}}}\,. \tag{20}$$

For admissible functions $v\in K_M$ the bound on the gradient combined with (19) also implies

$$\|v\|_{p,\Omega_h}\leq C_6\|Dv\|_{p,\Omega_h}+\frac{V}{|\Omega_h|^{1-\frac{1}{p}}}\leq C_6|\Omega_h|^{\frac{1}{p}}\left(M+\frac{V}{|\Omega_h|}\right)$$

for any p.

Thus, the energy functional is bounded from below. Since we are considering the minimization on the compact set K_M this is enough to ensure the existence of a unique minimizer. □

Note that the restriction of the function set to K_M is similar to the approach in [Hui84] where the existence of a solution was then obtained by a fixed point argument combined with a priori bounds.

We will now prove Theorem 4.2. Note that we will assume convergence results which are shown in Section 4.2.

Proof of Theorem 4.2. For $M\in\mathbb{R}$ Lemma 4.3 provides the existence of a unique solution u_M to (18). If there exists an $\tilde{M}\in\mathbb{R}$ such that $|Du_{\tilde{M}}|<\tilde{M}$, we can find an $\varepsilon>0$ such that $u_{\tilde{M}+\varepsilon}=u_{\tilde{M}}$, i.e. $u_{\tilde{M}}$ is a local minimizer of J_h in K_h, and hence a solution to the variational inequality (13).

Such an $\tilde{M}$ exists if we have a bound on the gradient of a solution to the variational equation (as in the continuous case [Hui84]) which can be forced to be less than $\tilde{M}$, i.e., we need an estimate of the form

$$|Du_M|_{\infty,\Omega}\leq C\,,$$

where the constant may not depend on M. Since the bound on κ will depend on M, we need to make sure that the constant C does not depend on κ^{-1}, either.

For the desired estimate we refer to Theorem 4.4. □

4.2. **Convergence without Obstacle.** In this section we will extend the convergence proof for capillary surfaces in positive gravitational fields as stated in Theorem 3.2 and [Mit77] to the K_M-bounded solutions u_M (cf. Lemma 4.3).

Throughout the section we will use the abbreviations

$$W:=\sqrt{1+|Du|^2}\,,\quad\text{and}\quad W_M:=\sqrt{1+|Du_M|^2}\,.$$

We will need a priori bounds for u_M. However, we need to make sure that they do not depend on κ^{-1}. Following the approach of [Mit77] will not yield this independence. Instead, we impose the a priori bounds by force, i.e., we restrict the function set K_M further by setting

$$V_M := K_M \cap \left\{ \|v\|_{2,\Omega_h} \leq \|u_I\|_{2,\Omega_h} + 1 \right\} \cap \left\{ \|v\|_{1,1,\Omega_h} \leq \|u_I\|_{1,1,\Omega_h} + 1 \right\} .$$

Note that this will not influence the solvability result Theorem 4.2 since V_M is a closed convex subset of K_M.

The solution to the corresponding minimization problem, i.e., (18) with K_M replaced by V_M, will again be denoted by u_M.

Theorem 4.4. *Let $M \in \mathbb{R}$ be large and $\kappa > 0$ small enough. Then*

$$\|u_M - u_I\|_{1,2,\Omega_h} \leq C\, h \,, \tag{21}$$

for all $0 < h < h_0$, h_0 sufficiently small, where $u_M \in V_M$ is a solution to (18) *and u_I denotes the interpolation of the continuous solution. Furthermore, we have*

$$|u_M|_{1,\infty,\Omega_h} \leq C \,.$$

Note that, as indicated in the proof of Theorem 4.2, we then obtain the existence of a local solution to the problem without the additional constraints, i.e. a solution to (13) in the function set $\tilde{K}_h$, by choosing M large enough such that

$$\|Du_M\|_{\infty,\Omega_h} \leq C < M \,,$$

and h small enough such that

$$\|u_I - u_M\|_{1,2,\Omega_h} \leq C\, h < 1 \,.$$

Although the constants in Theorem 4.4 depend on the continuous solution u, which itself depends on κ this is not a circular argument, because we can assume that u is bounded by a constant for all κ smaller than some κ_0 as proved in [Hui84].

Theorem 4.1 (neglecting the obstacle bound) then follows by combining Theorem 4.2, Theorem 4.4, and Theorem 3.1.

Proof of Theorem 4.4. We will proceed similarly to the proof of Theorem 3.2 in [Mit77].

Set $e_h := u_I - u_M$ and consider

$$A^2 := \int_{\Omega_h} \frac{|De_h|^2}{W_M}\, dx \,.$$

Note that

$$\begin{aligned} A^2 = &\int_{\Omega} \frac{De_h \cdot Du}{W}\, dx - \int_{\Omega_h} \frac{De_h \cdot Du_M}{W_M}\, dx \\ &+ \int_{\Omega_h} \frac{De_h \cdot D(u_I - u)}{W_M}\, dx \\ &+ \int_{\Omega_h} De_h \cdot Du \left(\frac{1}{W_M} - \frac{1}{W} \right) dx \\ &+ \int_{\Omega_h - \Omega} \frac{De_h \cdot Du}{W}\, dx - \int_{\Omega - \Omega_h} \frac{De_h \cdot Du}{W}\, dx \,. \end{aligned}$$

We want to estimate all terms on the right hand side to obtain an inequality of the form $A^2 \leq C\, h^2$. To estimate the first two terms we will use the variational formulations of the continuous and the discrete problem. The main difference to [Mit77] is that instead of proving a priori estimates on the discrete solution we need to choose test functions

fulfilling the additional bounds on the function set. This will lead to extra terms which will turn out to be of order h^2 and thus do not alter the convergence result. For completeness we will also carry out the estimates for the remaining terms which can also be found, e.g. in [Mit77] and [Cia78].

To bound the first term we insert the test function $v := u - e_h + \theta_I + \theta_h$ into the continuous variational inequality (8), where the constants θ_I and θ_h are defined by

$$\theta_I := \frac{1}{|\Omega|} \int_\Omega u_I - u \, dx$$
$$\theta_h := \frac{1}{|\Omega|} \left(\int_{\Omega_h - \Omega} u_M \, dx - \int_{\Omega - \Omega_h} u_M \, dx \right) .$$

The constants are chosen such that v fulfills the volume constraint and thus is indeed an admissible test function. The variational inequality (8) then reads

$$\int_\Omega \frac{De_h \cdot Du}{W} \, dx \leq \kappa a(u, e_h) + l(e_h) - \kappa a(u, \theta_I + \theta_h) - l(\theta_I + \theta_h) , \tag{22}$$

where $a(\cdot,\cdot)$ and $l(\cdot)$ are the continuous analoga to $a_h(\cdot,\cdot)$ and $l_h(\cdot)$.

For the second term we insert the test function $v_h := u_I + \theta_{h,I} + \theta_{h,h}$ into the discrete variational inequality (13) with the constants $\theta_{h,I}$ and $\theta_{h,h}$ defined by

$$\theta_{h,I} := \frac{1}{|\Omega_h|} \int_{\Omega_h} u - u_I \, dx$$
$$\theta_{h,h} := \frac{1}{|\Omega_h|} \left(\int_{\Omega - \Omega_h} u \, dx - \int_{\Omega_h - \Omega} u \, dx \right) .$$

The constants ensure that v_h fulfills the volume constraint. For $M \geq |Du_I|_{\infty,\Omega}$, the function v_h fulfills the constraint

$$|Dv_h|_{\infty,\Omega} \leq M .$$

For the L^2- and the $W^{1,1}$-constraints note that Lemma 3.4 implies that $|\theta_{h,h}|$ is of order h^2. Theorem 3.1 yields the same for $|\theta_{h,I}|$. Thus in any Sobolev norm $\|\cdot\|$ we can estimate

$$\|v_h\| \leq \|u_I\| + C\, h^2 .$$

Choosing h small enough then yields $v_h \in V_M$.

The variational inequality (13) then reads

$$-\int_{\Omega_h} \frac{De_h \cdot Du_M}{W_M} \, dx \leq -\kappa a_h(u_M, e_h) - l_h(e_h) - \kappa a_h(u_M, \theta_{h,I} + \theta_{h,h}) - l_h(\theta_{h,I} + \theta_{h,h}) . \tag{23}$$

Adding (22) and (23), we obtain

$$\begin{aligned} \int_\Omega \frac{De_h \cdot Du}{W} \, dx - \int_{\Omega_h} \frac{De_h \cdot Du_M}{W_M} \, dx \leq{}& \kappa (a(u, e_h) - a_h(u_M, e_h)) + l(e_h) - l_h(e_h) \\ &- (\kappa a_h(u_M, \theta_{h,I} + \theta_{h,h}) + l_h(\theta_{h,I} + \theta_{h,h})) \\ &- (\kappa a(u, \theta_I + \theta_h) + l(\theta_I + \theta_h)) . \end{aligned}$$

The terms involving the constants can be estimated by

$$\begin{aligned} -(\kappa a_h(u_M, \theta_{h,I} + \theta_{h,h}) + l_h(\theta_{h,I} + \theta_{h,h})) - (\kappa a(u, \theta_I + \theta_h) + l(\theta_I + \theta_h)) & \\ \leq C(\kappa, V, \beta) (|\theta_I| + |\theta_h| + |\theta_{h,I}| + |\theta_{h,h}|) & \\ \leq C\, h^2 , & \end{aligned}$$

since Lemma 3.4 implies that $|\theta_h|$ and $|\theta_{h,h}|$ are of order h^2, and Theorem 3.1 yields the same for the other two constants.

To estimate the term $\kappa(a(u,e_h)-a_h(u_M,e_h))$ note that

$$a(u,e_h)-a_h(u_M,e_h)=a_h(e_h,e_h)+\int_{\Omega_h}(u-u_I)\,e_h\,dx+\int_{\Omega-\Omega_h}u\,e_h\,dx-\int_{\Omega_h-\Omega}u\,e_h\,dx$$

can be approximated using Lemma 3.4 and Theorem 3.1 by

$$a(u,e_h)-a_h(u_M,e_h)\le a_h(e_h,e_h)+C\,h^2\|e_h\|_{2,\Omega_h}+C\,h^2\|e_h\|_{1,1,\Omega_h}\,.$$

Lemma 3.3 yields

$$|l(e_h)-l_h(e_h)|\le C(\gamma,\beta)\,h|e_h|_{1,1,B_h}\,.$$

The first two terms of A^2 can thus be estimated by

$$\begin{aligned}\int_\Omega\frac{De_h\cdot Du}{W}\,dx-\int_{\Omega_h}\frac{De_h\cdot Du_M}{W_M}\,dx&\le\kappa a_h(e_h,e_h)+C\,h^2\left(1+\|e_h\|_{1,1,\Omega_h}+\|e_h\|_{2,\Omega_h}\right)\\&\quad+C\,h\,|e_h|_{1,1,B_h}\,.\end{aligned}\tag{24}$$

We now need to estimate the remaining terms of the decomposition of A^2. The third term can be estimated using Hölder's inequality

$$\begin{aligned}\left|\int_{\Omega_h}\frac{De_h\cdot D(u_I-u)}{W_M}\,dx\right|&\le\int_{\Omega_h}\frac{|De_h|}{\sqrt{W_M}}|D(u_I-u)|\,dx\\&\le A|u_I-u|_{1,2,\Omega_h}\,.\end{aligned}$$

Using Theorem 3.1, we obtain

$$\left|\int_{\Omega_h}\frac{De_h\cdot D(u_I-u)}{W_M}\,dx\right|\le C\,h\,A\,.\tag{25}$$

To estimate the fourth term note that

$$\begin{aligned}\left|\frac{1}{W_M}-\frac{1}{W}\right|&=\left|\frac{W^2-W_M^2}{W\,W_M\,(W+W_M)}\right|\\&=\left|\frac{D(u-u_M)\cdot D(u+u_M)}{W\,W_M\,(W+W_M)}\right|\\&\le\frac{|D(u-u_M)|}{W\,W_M}\,.\end{aligned}$$

Hence, we can estimate using Hölder's inequality

$$\begin{aligned}\left|\int_{\Omega_h}De_h\cdot Du\left(\frac{1}{W_M}-\frac{1}{W}\right)\,dx\right|&\le\int_{\Omega_h}\frac{|Du|}{W}\frac{|D(u-u_M)|\,|De_h|}{W_M}\,dx\\&\le\lambda\int_{\Omega_h}\frac{|De_h|^2+|D(u-u_I)|\,|De_h|}{W_M}\,dx\\&\le\lambda\left(A^2+A\left(\int_{\Omega_h}\frac{|D(u-u_I)|^2}{W_M}\,dx\right)^{\frac12}\right)\\&\le\lambda\,A\left(A+|u-u_I|_{1,2,\Omega_h}\right)\,,\end{aligned}$$

where $\lambda=\max_{\tilde\Omega}\frac{|Du|}{W}<1$. Taking Theorem 3.1 into account we can furthermore estimate

$$\left|\int_{\Omega_h}De_h\cdot Du\left(\frac{1}{W_M}-\frac{1}{W}\right)\,dx\right|\le\lambda\,A(A+C\,h)\,.\tag{26}$$

Since De_h is piecewise constant on $\Omega_h - \Omega$ and $\Omega - \Omega_h$, and the distance between $\partial\Omega$ and $\partial\Omega_h$ is in $O(h^2)$ we can estimate the last terms of A^2 by

$$\begin{aligned}\int_{\Omega_h-\Omega} \frac{De_h \cdot Du}{W}\, dx - \int_{\Omega-\Omega_h} \frac{De_h \cdot Du}{W}\, dx &\le 2\,\lambda\, C\, h^2\, |e_h|_{1,1,\partial\Omega_h} \\ &\le C\, h\, |e_h|_{1,1,B_h}, \end{aligned} \tag{27}$$

where we have used Lemma 3.3 to obtain the last inequality.

Combining (24), (25), (26), and (27) yields

$$A^2 - \kappa a_h(e_h,e_h) \le \lambda A^2 + C(1+\lambda)h\,A + C\,h^2\left(1 + \|e_h\|_{1,1,\Omega_h} + \|e_h\|_{2,\Omega_h}\right) + C\,h\,|e_h|_{1,1,B_h}\,. \tag{28}$$

$u_M \in V_M$ implies that $\|e_h\|_{1,1,\Omega_h}$ and $\|e_h\|_{2,\Omega_h}$ are bounded by a constant independent of h. Using Young's inequality we thus obtain

$$\frac{1-\lambda}{2} A^2 - \kappa a_h(e_h,e_h) \le C\,h^2 + C\,h\,|e_h|_{1,1,B_h}\,. \tag{29}$$

Note that the assumption $\|Du_M\|_{\infty,\Omega_h} \le M$ implies

$$\|De_h\|^2_{2,\Omega_h} \le \sqrt{1+M^2}A^2\,.$$

Using Poincaré's inequality (19) and the volume constraint we obtain

$$\begin{aligned}\kappa a_h(e_h,e_h) &\le 2\,\kappa \left(\left\| e_h - \frac{1}{|\Omega_h|}\int_{\Omega_h} e_h\, dx \right\|_2^2 + \left\| \frac{1}{|\Omega_h|}\int_{\Omega_h} e_h\, dx \right\|_2^2 \right) \\ &\le 2\,\kappa \left(C_6^2 \|De_h\|_2^2 + C\,h^2 \right) \\ &\le 2\,\kappa \left(C_6^2 \sqrt{1+M^2} A^2 + C\,h^2 \right) \\ &\le \frac{1-\lambda}{4} A^2 + 2\,\kappa\, C\, h^2 \end{aligned}$$

for κ small enough depending on M, i.e.,

$$\kappa \le \kappa_2 := \frac{1-\lambda}{8\, C_6^2 \sqrt{1+M^2}}\,. \tag{30}$$

Inserting this into (29) yields

$$A^2 \le C\,h^2 + C\,h\,|e_h|_{1,1,B_h}\,. \tag{31}$$

We will now use this to show that $|Du_M|_{\infty,\Omega_h}$ is bounded depending on h^{-1}. This will then lead to a first bound on $|e_h|_{1,2,\Omega_h}$ independent of h, which we will then improve until we obtain the desired estimate (21).

For any triangle T_j Theorem 3.1 and (31) yield

$$\begin{aligned}\int_{T_j} \frac{|Du_M|^2}{W_M}\, dx &\le 2\left(A^2 + \int_{T_j} \frac{|Du_I|^2}{W_M}\, dx \right) \\ &\le 2\left(A^2 + C|u|^2_{1,\infty,\Omega} |T_j| \right) \\ &\le C\,h + C|u|^2_{1,\infty,\Omega}\, h^2\,. \end{aligned}$$

Since Du_M is constant on each triangle T_j we additionally obtain, due to the shape regularity,

$$\int_{T_j} \frac{|Du_M|^2}{W_M}\,dx \geq \pi\gamma^2 \frac{|Du_{M|T_j}|^2}{\sqrt{1+|Du_{M|T_j}|^2}} h^2 \,.$$

Combining these implies

$$\frac{|Du_{M|T_j}|^2}{\sqrt{1+|Du_{M|T_j}|^2}} \leq C\, h^{-1}$$

for all T_j and hence

$$|Du_{M|T_j}| \leq W_{h|T_j} \leq \frac{|Du_{M|T_j}|^2}{\sqrt{1+|Du_{M|T_j}|^2}} + 1 \leq C\, h^{-1} \,.$$

From this and (31) it follows that $|e_h|_{1,2,\Omega_h}$ is bounded:

$$\begin{aligned} |e_h|^2_{1,2,\Omega_h} &= \int_{\Omega_h} \frac{|De_h|^2}{W_M} \sqrt{1+|Du_M|^2}\,dx \\ &\leq \sup_{\Omega_h} \sqrt{1+|Du_M|^2} A^2 \\ &\leq C \,. \end{aligned}$$

Since $|B_h| = O(h)$ we get by Hölder's inequality

$$|e_h|_{1,1,B_h} \leq |B_h|^{\frac{1}{2}}\, |e_h|_{1,2,B_h} \leq C\, h^{\frac{1}{2}} \,.$$

Inserting this back into (31) yields $A^2 \leq C\, h^{\frac{3}{2}}$. Repeating the arguments above we then obtain $|Du_M|_{\infty,\Omega_h} \leq C\, h^{-\frac{1}{2}}$, $|e_h|^2_{1,2,\Omega_h} \leq C\, h$, and thus $A^2 \leq C\, h^2$. Another repetition of the arguments yields the pointwise gradient estimate

$$|u_M|_{1,\infty,\Omega_h} \leq C$$

and thus the estimates

$$|e_h|_{1,2,\Omega_h} \leq C\, h \,, \quad \text{and} \quad |e_h|_{1,\infty,\Omega_h} \leq C \,.$$

The final estimate

$$\|e_h\|_{1,2,\Omega_h} \leq C\, h$$

then follows using Poincaré's inequality

$$\begin{aligned} \|e_h\|^2_{1,2,\Omega_h} &\leq 2\left(\|e_h\|^2_{2,\Omega_h} + |e_h|^2_{1,2,\Omega_h}\right) \\ &\leq C\, \|De_h\|^2_2 + C\, h^2 \\ &\leq C\, h^2 \,. \end{aligned}$$

□

4.3. **Convergence with an Obstacle.** We will now include the obstacle in our considerations, i.e. we look for solutions of (13) in K_h as defined in (10).

Analogously to the case without an obstacle we restrict the function set to

$$V_M^{obst} := K_h \cap \left\{ \|Dv_h\|_{\infty,\Omega_h} \leq M \right\} \cap \left\{ \|v_h\|_{2,\Omega_h} \leq \|u_I\|_{2,\Omega_h} + 1 \right\} \\ \cap \left\{ \|v_h\|_{1,1,\Omega_h} \leq \|u_I\|_{1,1,\Omega_h} + 1 \right\} .$$

The existence of a solution $u_M \in V_M^{obst}$ for a given $M \in \mathbb{R}$ follows by the same reasoning as in the case without an obstacle (Theorem 4.2). Thus, if we can prove a convergence result like Theorem 4.4, we obtain a local solution to (13) since we can then choose M large enough such that

$$\|Du_M\|_{\infty,\Omega_h} \leq C < M ,$$

and h small enough such that

$$\|u_I - u_M\|_{1,2,\Omega_h} \leq C\, h < 1 .$$

Hence, we just need to prove the following:

Theorem 4.5. *Let $M \in \mathbb{R}$ be large and κ small enough. Then*

$$\|u_M - u_I\|_{1,2,\Omega_h} \leq C\, h , \tag{32}$$

for all $0 < h < h_0$, h_0 sufficiently small, where $u_M \in V_M^{obst}$ is a solution to (18) *and u_I denotes the interpolation of the solution to the continuous problem* (8)*. Furthermore, we have*

$$|u_M|_{1,\infty,\Omega_h} \leq C .$$

Proof. We may essentially repeat the proof of Theorem 4.4. However, we need to choose different test functions to insert into the discrete and the continuous problem since the original ones might violate the obstacle constraint.

As a discrete test function we choose instead

$$v_h := u_I + C_{V^h}(u_I - \psi_h) ,$$

where

$$C_{V^h} := \frac{\theta_h}{V^h - \theta_h} ,$$
$$\theta_h := \int_{\Omega_h} u - u_I \, dx + \int_{\Omega - \Omega_h} u \, dx - \int_{\Omega_h - \Omega} u \, dx .$$

Note that u_I does not violate the discrete obstacle constraint. C_V enforces compliance with the volume constraint. In view of the approximating properties of u_I and Lemma 3.4, we have $|\theta_h| \leq C\, h^2$. Thus choosing h small enough implies $|C_{V^h}| \leq C\, h^2 \leq 1$. Therefore v_h does not violate the obstacle constraint. Note furthermore that

$$\|u_I - \psi_h\|_{1,\infty,\Omega_h} \leq C ,$$

where the constant depends on the continuous solution and the obstacle. $\|Dv_h\|_{\infty,\Omega_h}$ is then bounded. Thus, for M large enough and h small enough $v_h \in V_M^{obst}$. The variational inequality for v_h reads

$$-\int_{\Omega_h} \frac{Du_M \cdot De_h}{W_M} \, dx \leq -\kappa a_h(u_M, e_h) - l_h(e_h) \\ + C_{V^h} \left(\int_{\Omega_h} \frac{Du_M \cdot D(u_I - \psi_h)}{W_M} \, dx - \kappa a_h(u_M, u_I - \psi_h) - l_h(u_I - \psi_h) \right) ,$$

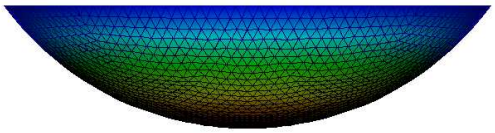


FIGURE 2. Discrete capillary surface for $-\kappa = -0.1$

which implies

$$-\int_{\Omega_h} \frac{Du_M \cdot De_h}{W_M}\, dx \leq -\kappa a_h(u_M, e_h) - l_h(e_h) + C\, h^2 \,.$$

As a continuous test function we choose

$$v := v_\psi + C_V(v_\psi - \psi) \,,$$

where

$$\begin{aligned} v_\psi &:= u + u_M - u_I + C_\psi \,, \\ C_\psi &:= \|\psi_h - \psi\|_{\infty,\tilde{\Omega}} + \|u - u_I\|_{\infty,\tilde{\Omega}} \,, \\ C_V &:= \frac{\theta}{V - \theta} \,, \\ \theta &:= \int_\Omega u_I - u\, dx + \int_{\Omega_h - \Omega} u_M\, dx - \int_{\Omega - \Omega_h} u_M\, dx - C_\psi |\Omega| \,. \end{aligned}$$

C_ψ ensures that v_ψ lies above the obstacle. C_V enforces the continuous volume constraint. Again $|\theta| \leq C\, h^2$ and thus $|C_V| \leq C\, h^2$ for h small enough. Therefore v is an admissible test function for the continuous problem. Furthermore

$$\begin{aligned} \|v_\psi - \psi\|_{2,\Omega_h} &\leq C \,, \\ \|v_\psi - \psi\|_{1,1,\Omega_h} &\leq C \,, \end{aligned}$$

since all parts of v_ψ fulfill these bounds. The variational inequality then reads

$$\int_\Omega \frac{Du \cdot De_h}{W}\, dx \leq \kappa a(u, e_h) + l(e_h) + C\, h^2 \,.$$

We are now in the same setting as before and can repeat the arguments of the proof of Theorem 4.4 to obtain the assertion. □

5. NUMERICAL EXPERIMENTS

Our aim in this section is to numerically illustrate the convergence result stated in Theorem 4.1. To this end we used a truncated nonsmooth Newton multigrid method [GSS09] to solve the capillarity problem. The implementation was done in C++ using the Distributed and Unified Numerics Environment (DUNE) [BBD+08].

We consider the discrete capillarity problem (13) for mesh sizes $h_k = \sqrt{2}\, 2^{-(k+1)}$ for $k = 0, \ldots, 8$ on a disc of diameter 1 with the parameters $-\kappa = -0.1$, $\beta = -0.8$, a prescribed volume $V = \pi$, and a constant obstacle at height 0. A reference solution u_f was computed with a mesh size of $h = \sqrt{2}\, 2^{-10}$. The graph of u_f can be observed in Figure 2.

The convergence result Theorem 4.1 essentially bounds the approximation error in the $W^{1,2}(\Omega_h)$-seminorm, and the full norm estimate comes from Poincaré's inequality. Since we want to observe the order of convergence we will monitor the errors $|u_h - u_f|_{0,2,\Omega_h}$ and $|u_h - u_f|_{1,2,\Omega_h}$ as functions of the mesh size parameter h.

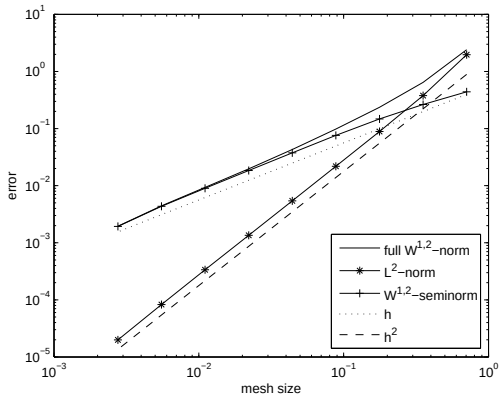


FIGURE 3. Doubly logarithmic plot of the $W^{1,2}$-error over the mesh size h for $-\kappa = -0.1$

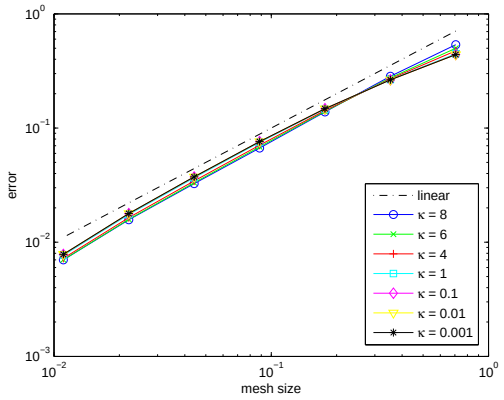


FIGURE 4. Doubly logarithmic plot of the error in the $W^{1,2}$-seminorm over the mesh size h for varying κ

The expected linear decay of the error in the $W^{1,2}$-seminorm can be observed in Figure 3. For the approximation error in the L^2-norm we observe quadratic decay. This corresponds to the well-known convergence behavior of minimal surfaces [Ran77].

An essential ingredient in the proof of Theorem 4.1 is the κ-independence of the error. We tested this by repeating the above test for varying κ. The reference solution was computed with $h = \sqrt{2}\,2^{-8}$ and h_k is as above with $k = 0,\dots,6$. In the numerical results the discretization error indeed does not appear to depend on κ as can be observed in Figure 4.

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