

NON-PERSISTENCE OF HETEROCLINIC ORBITS IN VISCOUS BALANCE LAWS FOR VANISHING VISCOSITY

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ABSTRACT. The article investigates the relation between global solutions of hyperbolic balance laws and viscous balance laws on the circle. It is thematically located at the crossroads of hyperbolic and parabolic partial differential equations with one-dimensional space variable and periodic boundary conditions. The two equations are given by:

$$u_t + f(u)_x = g(u).$$

and

$$u_t + f(u)_x = \varepsilon u_{xx} + g(u).$$

The main result of the paper corrects a result on the persistence of heteroclinic connections by Fan and Hale [FH95] for the case $\varepsilon \rightarrow 0$ (*Connection Lemma* 2.8). It states that a connection can only persist if the zero number of the source state is a multiple of the zero number of the target state. The *Cascading Theorem* 2.12 then yields convergence of heteroclinic connections to a sequence of heteroclinic connections and stationary solutions in case of non-persistence.

In addition a full description of the connection problem of rotating waves on the parabolic attractor is given.

1. INTRODUCTION

Parabolic differential equations with scalar spatial variable have been studied for a long time. In particular viscous balance laws can be described as exceptionally well understood: existence, uniqueness of solutions, long time behavior, global attractors, heteroclinic orbits etc. have been analyzed in detail for a range of boundary conditions.

The same is true for scalar hyperbolic partial differential equations. In particular for hyperbolic balance laws, where again questions of existence, uniqueness, the long time behavior, global attractors and heteroclinics have been studied thoroughly.

However, when the two fields, viscous balance laws and hyperbolic balance laws come together many question marks appear.

This article investigates the behavior of solutions on the global attractor \mathcal{A}^ε of the viscous balance law (P) for vanishing viscosity $\varepsilon \rightarrow 0$. It analyzes if and in which sense these solutions persist to solutions on the global attractor of the limiting equation - the hyperbolic balance law (H). The results have strong implications on the question whether the global attractor of (H) can be obtained as a limit of the global attractors of (P) when viscosity vanishes.

Before going into further details we set the formal stage that clarifies the framework we will be working in.

The viscous balance law is given by

$$(P) \quad u_t(x, t) + [f(u(x, t))]_x = \varepsilon u_{xx}(x, t) + g(u(x, t)).$$

The hyperbolic balance law is given by

$$(H) \quad u_t(x, t) + [f(u(x, t))]_x = g(u(x, t)).$$

The sub index denotes the partial derivative with respect to the index. We solve for $x \in S^1$ with $S^1 := \mathbb{R}/(2\pi\mathbb{Z})$. This is equivalent to imposing periodic boundary conditions on a domain of length 2π . If we scale $\tilde{x} = \frac{Lx}{2\pi}$ all our results remain true for the situation of periodic boundary conditions in a domain of size L for any bounded and fixed $L \in \mathbb{R}$. u is a function mapping from $S^1 \times \mathbb{R} \rightarrow \mathbb{R}$.

The non-linearities f, g map from $\mathbb{R} \rightarrow \mathbb{R}$. Furthermore we make additional hypotheses:

(H1) f is C^2 and strictly convex ($\exists \gamma \in \mathbb{R}$ s.t. $f'' > \gamma > 0$) and $f'(0) = 0$.

(H2) g is C^2 and dissipative, i.e. there exists a constant $M > 0$ such that

$$(1) \quad ug(u) < M$$

for all $|u| > M$.

(H3) g has three simple zeros at $u_- < u_0 < u_+$, and $u_0 = 0$.

The three hypotheses guarantee existence and uniqueness of solutions and existence of a global attractor for both the parabolic and hyperbolic equation (see next subsections).

The paper is organized as follows: the remainder of the introduction will provide the necessary background in the theory of hyperbolic (Section 1.1) and parabolic (Section 1.2) equations on the circle. The main results of the paper the *Connection Lemma* 2.8: necessary condition on persistence; and the *Cascading Theorem* 2.12 are stated and proved in Section 2. Theorems 2.9 and 2.10 then give the necessary tools to prove that convergence in fact fails. The proof of these theorems is presented in Section 3 where we investigate the structure of the global attractor of the parabolic equation (P) including a description of all rotating waves and the connection problem. We conclude with a brief discussion in the last section.

1.1. The hyperbolic equation. We obtain equation (H) by setting $\varepsilon = 0$ in the parabolic equation (P). In this sense (H) is the limiting equation of the viscous balance law (P) for vanishing viscosity. In the limit the type of the equations changes from parabolic to hyperbolic. This has many implications on the solution theory. In general there is no unique solution of the Cauchy problem of (H). However we will use the notion of entropy solutions introduced by Volpert [Vol67] and Kruzhkov [Kr70]:

Definition 1.1. We call $u \in BV([0, \infty) \times S^1, \mathbb{R})$ an entropy or admissible solution of equation (H) to the initial condition $u_0(x)$

- if $u(x, 0) = u_0(x)$;
- if it solves equation (H) in the weak sense:

$$(2) \quad \int_{S^1 \times \mathbb{R}^+} [u\varphi_t + f(u)\varphi_x - g(u)\varphi] dx dt = 0$$

for all $\varphi \in C_0^1(S^1 \times \mathbb{R}^+, \mathbb{R})$;

- and if the entropy condition

$$(3) \quad u(x+, t) \leq u(x-, t)$$

holds for all $t > 0$.

Here $u(x+, t)$ defines the right hand, $u(x-, t)$ the left hand limit of u in x at time t and $BV([0, \infty) \times S^1, \mathbb{R})$ denotes the space of functions with bounded variation mapping from $[0, \infty) \times S^1$ to \mathbb{R} .

In this framework equation (H) has a unique entropy solution and defines a semi-flow on $BV([0, \infty) \times S^1, \mathbb{R})$ which we denote by Φ^0 . Fan and Hale could prove in [FH95] that (H) possesses a global attractor defined as the maximal compact invariant subset \mathcal{A}^0 of the phase space BV that attracts all bounded subsets $\mathcal{B} \subset \mathcal{X}$. In our case this is equivalent to defining the global attractor as set of all global orbits, i.e. orbits that exist for all times $t \in \mathbb{R}$.

Proposition 1.2 ([FH95]). *Assume (H1), (H2) and (H3) hold. Then*

$$(4) \quad \mathcal{A}^0 := \{u_0 \in BV(S^1) : \Phi^0(u_0, t) \text{ exists for all } t \in \mathbb{R} \text{ and is bounded}\}$$

is the global attractor of (H) in $L^p(S^1)$, for any $p \in [1, \infty]$, i.e. it is invariant and attracts bounded sets in $L^p(S^1)$.

Regarding the structure of the global attractor several authors proved Poincaré Bendixson type results for the scalar balance laws. See for example Fan and Hale [FH93], Sinestrari [Sin97] or Lyberopoulos [Lyb94]:

Proposition 1.3. *For $t \rightarrow \infty$ any entropy solution of (H) tends either to a homogeneous solution $u \equiv u_-, u_0, u_+$ or it converges to a rotating wave solution*

$$u(x, t) = v(x - ct)$$

where the wave-speed c can only take the value $c = f'(u_0) = 0$.

Hence all waves are frozen waves in our case. Note that (H3) only simplifies the situation but is not a principle restriction. In case the hypothesis is not fulfilled and g has more than three zeros the wave speed of a wave is given by $c = f'(u_i)$ where the u_i is one of zeros of g with $g'(u_i) > 0$ (unstable zero).

For global solutions a theorem similar to 1.3 holds true in backward time. This leads to a description of the global attractor \mathcal{A}^0 as the unification of the homogeneous steady states, the frozen waves and heteroclinic connections between all these objects. This leads naturally to the definition of the following sets. Let

- \mathcal{E}^0 denote the set of homogeneous equilibria of (H);
- \mathcal{F}^0 denote the set of frozen waves of (H);

we then define a heteroclinic connection as a solution $u^0(x, t)$ of (H) that has the property that

$$(5) \quad \begin{aligned} \lim_{t \rightarrow +\infty} u^0(x, t) &\in \mathcal{E}^0 \cup \mathcal{F}^0 \\ \lim_{t \rightarrow -\infty} u^0(x, t) &\in \mathcal{E}^0 \cup \mathcal{F}^0. \end{aligned}$$

and denote the set of all heteroclinic connections with \mathcal{H}^0 . Then the global attractor \mathcal{A}^0 of (H) can be described as

$$(6) \quad \mathcal{A}^0 = \mathcal{E}^0 \cup \mathcal{F}^0 \cup \mathcal{H}^0.$$

In [Sin95] Sinestrari was able to settle the description of all rotating or frozen waves. He proved that for any closed set $Z \subseteq S^1$ there exists a unique wave u_Z with wave speed $c = 0$ and the property

$$Z = \{y \in S^1 : u_Z(y) = u_0 = 0\}.$$

The uniqueness automatically proves that these are all waves of (H).

The connection problem, i.e. the question which

$$u^0 \in \mathcal{E}^0 \cup \mathcal{F}^0$$

are connected to one another was solved by Sinestrari [Sin97] and Härterich [Haer99]. Sinestrari proved that if two rotating waves $u_{-\infty}^0, u_{\infty}^0$ are connected, then necessarily

$$(7) \quad \mathcal{Z}(u_{\infty}^0) \subset \mathcal{Z}(u_{-\infty}^0).$$

Here $\mathcal{Z}(\cdot)$ assigns each solution $u(\cdot, t) \in BV(S^1)$ its zero set:

$$(8) \quad \mathcal{Z}(u(\cdot, t)) := \{x \in S^1; u(x, t) = u_0 = 0\}.$$

Härterich proved that condition (7) was not only necessary but sufficient. His results are summarized in the three Theorems A, B and C in [Haer99]:

Theorem 1.4 (Theorem A). *For any rotating wave $u_{-\infty}$ there exist heteroclinic orbits which connect $u_{-\infty}$ to the homogeneous states $u \equiv u_-$ and $u \equiv u_+$.*

Theorem 1.5 (Theorem B). *For any rotating wave $u_{+\infty}$ there exist (several) heteroclinic orbits that connect the spatially homogeneous solution $u \equiv u_0 = 0$ to $u_{+\infty}$.*

Theorem 1.6 (Theorem C). *Suppose that for two rotating waves $u_{-\infty}$ and $u_{+\infty}$ the condition $\mathcal{Z}(u_{\infty}^0) \subset \mathcal{Z}(u_{-\infty}^0)$ holds. Then there is a heteroclinic solution that approaches $u_{\pm\infty}$ as the time t tends to $\pm\infty$.*

The three theorems allow a full description of the connection problem on the global attractor of equation (H) and give together with the result on rotating waves of Sinestrari [Sin97] a full description of the structure of the global attractor of equation (H). The remaining problem of the description of the dynamics on the attractor was solved by the author in [Ehrt2010/2] for all finite dimensional sub-attractors of \mathcal{A}^0 .

1.2. The parabolic equation. It is known that the initial value problem (Cauchy problem) of the PDE (P) together with Neumann, Dirichlet or periodic boundary conditions is well posed and has unique solutions for sufficiently regular initial conditions.

On the Sobolev space of twice weakly differentiable L^2 -functions

$$X = W^{2,2}([0, 2\pi], \mathbb{R}) = H^2([0, 2\pi], \mathbb{R})$$

that satisfy the boundary conditions, the PDE generates a C^1 semi flow. The books of Henry [Hen81] or Pazy [Pazy83] give a more detailed description for the semi group theory related to parabolic PDEs.

In the S^1 case dissipativity of the non-linearities is sufficient for the existence of a global attractor on $X = H^2$. This is ensured by our hypotheses (H2) and the fact that $f(u)_x$ grows sub-quadratically in u_x . For an explicit general definition of dissipativity in the S^1 case I refer to [MN97].

Angenent and Fielder [AF88] and Matano [Ma88] could show that, similar to the Neumann case, any solution $u^\varepsilon(\cdot, t)$ of (P) tends to a set of functions $\Gamma(v) :=$

$\{v(\cdot + \theta) : \theta \in S^1\}$ for $t \rightarrow +\infty$. Here $v(x)$ is given by a solution of the ordinary differential equation

$$(9) \quad v_{xx} - (f'(v) + c)v_x + g(v) = 0$$

for some value of $c \in \mathbb{R}$ and $x \in S^1$. The same holds true in backward time if the solution $u^\varepsilon(\cdot, t)$ stays bounded. Equation (9) is called rotating wave equation and can be obtained by plugging the rotating wave ansatz $u(x, t) := v(x - ct)$ into the PDE (P). Following the definitions in the previous paragraph we define the following sets:

Let

- \mathcal{E}^ε denote the set of homogeneous equilibria;
- \mathcal{F}^ε denote the set of frozen waves;
- \mathcal{R}^ε denote the set of rotating waves and
- \mathcal{H}^ε denote the set of heteroclinic connections.

Heteroclinic connections are similarly defined as in equation (5) as solutions $u^\varepsilon(x, t)$ of (P) with the property

$$\lim_{t \rightarrow \pm\infty} u^\varepsilon(\cdot, t) \in \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon.$$

It is important to note that in the parabolic case \mathcal{R}^ε is not necessarily empty and the wave speed c is neither a priori zero nor the same for all waves of the equation as it is the case in the hyperbolic equation.

The following theorem holds due to Angenent and Fiedler [AF88] or Mantano [Ma88]:

Theorem 1.7. *Let the non-linearity of equation (P) be dissipative and C^2 . Then the global attractor \mathcal{A}^ε of the PDE can be described as follows:*

$$(10) \quad \mathcal{A}^\varepsilon = \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon \cup \mathcal{H}^\varepsilon.$$

In particular, any time periodic orbit is a rotating wave and any orbit in $\mathcal{A}^\varepsilon \setminus (\mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon)$ is a heteroclinic connection connecting $u_1, u_2 \in \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ with $u_1 \neq u_2$.

In [FRW04] Fiedler, Rocha and Wolfrum were able to develop tools to resolve the connection problem for the periodic boundary conditions case. Their idea was to use homotopies, such that every solution of the S^1 case solves a Neumann problem and vice versa. Then they could apply their earlier results on the Neumann case [FR96].

The key ingredient is the concept of $k - (\mathcal{P})$ -adjacency (see Definition 3.10 in Section 3), that was developed and used for the Neumann case in [FR96] and later in [Wol02a] and [Wol02b]. The whole approach relies heavily on nodal properties that have their origin in the fact that the linearization of the PDE (P) is a Sturm-Liouville type problem. This goes back to Sturm [Stu1836]. A key observation is that the number of strict sign changes in a solution can only drop along trajectories, hence can be considered as a discrete Lyapunov function. This is also true for the difference of two solutions.

The connection problem will be solved in Section 3. The basis for this analysis is the complete description of all rotating waves of equation (P). The results will be used to prove that certain heteroclinic connections do not persist.

2. NON-PERSISTENCE OF HETEROCLINIC CONNECTIONS

In the following we will formulate and prove the two main results of the work: non-persistence of heteroclinic connections and the cascading result. Before that, we will investigate the result by Fan and Hale from 1995 [FH95] mentioned earlier where they address the question of viscous regularizations of the hyperbolic equation.

In the first part of the paper they investigate the connection problem of the global attractor of the hyperbolic equation (H). In the second part of the publication Fan and Hale investigate the regularized equation, which is precisely our equation (P).

In Theorem 4.7 they state a persistence result for heteroclinic connections within this framework:

Theorem 2.1. *If $B = \{u^\varepsilon(x, t), 0 < \varepsilon \leq \varepsilon_0\}$ is a set of connecting (heteroclinic) orbits of the parabolic equation (P), then there is a sub-sequence $\{u^{\varepsilon_n}(x, t)\}$ of B converging to $u^0(x, t)$ as $\varepsilon \rightarrow 0$ a.e. in $S^1 \times \mathbb{R}$ where $u^0(x, t)$ is a connecting orbit of the hyperbolic equation (H).*

Unfortunately this theorem is wrong. The claim that convergence is a.e. on $S^1 \times \mathbb{R}$ is not true. As a result of this the limiting solution u_0 is not necessarily a heteroclinic connection.

Taking a closer look at the proof of their theorem one realizes that it is almost completely correct. Only their conclusion using a diagonalizing sequence argument at the very end of the proof does not work. Hence a minor change corrects the result - however this has vast implications on the persistence of heteroclinic connections.

The corrected version of the Fan Hale theorem reads:

Theorem 2.2 (Global Solution). *Let $B := \{u^\varepsilon(x, t) \in \mathcal{H}^\varepsilon : 0 < \varepsilon < \varepsilon_0\}$ for some $\varepsilon_0 \ll 1$ and let $T \in \mathbb{R}^+$ be arbitrary. Then there exists a subset $\{u^{\varepsilon_n}(x, t)\}$ of B with the property that*

$$(11) \quad \lim_{n \rightarrow \infty} u^{\varepsilon_n}(x, t) = u^0(x, t)$$

a.e. on S^1 for all $t \in [-T, T]$. Moreover $u^0(x, t)$ is a global solution of equation (H).

I do not include a proof here. It would be an exact copy of the original proof found in [FH95] omitting the last few lines. In addition a more detailed version of the proof can as well be found in [Ehrt2010] in Chapter 3.2.

Certainly the question arises why “global solution” does not imply “heteroclinic connection” in this case. The main obstacle for this is the occurrence of additional equilibria.

In order to rigorously prove that non-persistence is possible this we need some results on the convergence of rotating waves:

Corollary 2.3. *Let $C := \{u^\varepsilon(x, t) \in \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon; 0 < \varepsilon < \varepsilon_0\}$ for some $0 < \varepsilon_0 \ll 1$. Then there exists a subset $\{u^{\varepsilon_n}(x, t)\}$ of C with the property that*

$$\lim_{n \rightarrow \infty} u^{\varepsilon_n}(x, t) = u^0(x, t)$$

a.e. on $S^1 \times [-T, T]$. Moreover $u^0(x, t)$ is a global solution of (H).

Secondly, we make a statement on all possible limits of solutions in the set B :

Corollary 2.4. *Let $T \in \mathbb{R}$ and B be defined as in Theorem 2.2. Let $u^0 \in BV(S^1 \times [-T, T], \mathbb{R})$ with*

$$(12) \quad u^0(x, t) := \lim_{n \rightarrow 0} u^{\varepsilon_n}(\cdot, \tau_n + t)$$

a.e. on $S^1 \times [-T, T]$ for sequences $\{\varepsilon_n\} \rightarrow 0$ and $\{\tau_n\}$ and all bounded $T \in \mathbb{R}$. Then

$$u(\cdot, t) \in \mathcal{A}^0.$$

Proof. Certainly u must be globally bounded and due to the convergence of the limit be a solution of the hyperbolic equation. Therefore it must be a global solution and hence

$$u(\cdot, t) \in \mathcal{A}^0. \quad \square$$

In the case of rotating waves Corollary 2.3 can be improved considerably. The limiting object is not only a global solution but a frozen wave of the hyperbolic equation. This is the content of the following Theorem. Note that we use ODE theory to obtain convergence for all ε . Before stating the theorem we introduce the zero-number of a function $u : S^1 \rightarrow \mathbb{R}$:

Let therefore $u : S^1 \rightarrow \mathbb{R}$ then we define

$$(13) \quad z(u) := \#\{x \in S^1; u(x) = 0\},$$

if the zero set of u is not countable we define $z(u) = \infty$.

Theorem 2.5 (Rotating Waves). *Let $a = 2\alpha$ for $\alpha \in \mathbb{N}$ and u_a^ε be the unique rotating or frozen wave of (P) with the property*

$$(14) \quad \begin{aligned} z(u_a^\varepsilon) &= a \\ u_a^\varepsilon(0, 0) &= 0 \quad \partial_x u_a^\varepsilon(0, 0) > 0. \end{aligned}$$

Then there exists a rotating wave $u_\alpha^0 \in \mathcal{F}^0$ such that

$$(15) \quad \lim_{\varepsilon \rightarrow 0} \|u_a^\varepsilon(x, t) - u_\alpha^0(x, t)\| = 0$$

holds for all $t \in \mathbb{R}$. Moreover we have

$$(16) \quad z(u_\alpha^0) = \alpha.$$

Proof. We perform the proof in several steps:

- (i) For the existence of the limit we assume $a = 2$, the other cases just work with the same argument.

We observe that, according to Lemma 3.6, the rotating wave profile v_a^ε associated to $u_a^\varepsilon(\cdot, t)$ and its derivative lie in a $O(\varepsilon)$ channel around $\phi(x)$ outside a $(\varepsilon \log \varepsilon)$ -neighborhood of some $x_2(\varepsilon)$.

Because $x_2(\varepsilon_1) - x_2(\varepsilon_2) < C|\varepsilon_1 - \varepsilon_2|$ for some constant C and $0 < \varepsilon_1, \varepsilon_2 \ll 1$, the limit of u^ε for $\varepsilon \rightarrow 0$ exists outside any open neighborhood of x_2 and is in fact uniform. This proves the existence of a solution u_α^0 such that equation (15) holds. It remains to prove that u_0 is a rotating wave.

- (ii) From Corollary 2.3 we obtain that $u_0(x, t)$ is a global solution and therefore lies on the attractor. Because it converges uniformly to ϕ outside a neighborhood of x_2 , the solution u_0 neither can be a homogeneous solution, nor a heteroclinic connection. From equation (6) follows that it must be a rotating wave which is unique up to shifts. This proves the claim.

- (iii) The relation between the zero-numbers of the parabolic wave and the hyperbolic wave is obvious. All frozen waves for $\varepsilon = 0$ have positive derivative in all their zeros [Ehrt2010/2]. From (i) it follows that they persist. For $\varepsilon > 0$ all rotating wave profiles are continuous and thus have alternating signs in the derivative. Together with the already proved persistence this yields equation (16).

Remark 2.6. *The relation of the zero-number between solutions on the parabolic attractor $u^\varepsilon \in \mathcal{A}^\varepsilon$ and their limits is true for all elements $u \in \mathcal{A}^\varepsilon$ with simple zeros. The zero-number drops by one half when taking the limit $\varepsilon \rightarrow 0$.*

The zero-number property is true because all solutions $u \in \mathcal{A}^0$ have the property that the derivative in isolated zeros is positive [Ehrt2010/2], whereas it has alternating sign for all $u^\varepsilon \in \mathcal{A}^\varepsilon$.

Coming back to rotating waves, we summarize that all rotating waves persist for $\varepsilon \rightarrow 0$. Moreover there is the relation between the zero-number of the rotating wave for $\varepsilon > 0$ and the number of zeros of the limiting frozen wave, this allows the following definition.

Definition 2.7. *Let $a := 2\alpha$ for some $\alpha \in \mathbb{N}$ be given and let ε_0 be sufficiently small. Then $u_a^\varepsilon(\cdot, \cdot)$ denotes the up to rotation unique rotating wave with zero-numbers $z = a$ for all $0 < \varepsilon < \varepsilon_0$.*

A set of rotating and frozen waves u_a^ε with a given zero-number $z(u_a^\varepsilon) = a$ shall be denoted by

$$B_a := \{u_a^\varepsilon \in \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon : 0 < \varepsilon < \varepsilon_0\}.$$

Moreover we fix the notation of Theorem 2.5 by defining

$$u_\alpha^0(\cdot, t) := \lim_{\varepsilon \rightarrow 0} u_a^\varepsilon(\cdot, t)$$

for all $t \in \mathbb{R}$ after potentially taking a sub sequence in ε .

As mentioned above, the persistence result that is valid for This implies that no U_{ab}^j can contain a full heteroclinic connection of the hyperbolic problem and hence the rotating waves, is not true for heteroclinic orbits although Theorem 2.2 yields convergence to a global solution. The next Lemma will provide a criterion when heteroclinic orbits cannot persist. In order to prove this criterion we define the set of heteroclinic orbits connecting two rotating waves u_a^ε and u_b^ε with zero-number a and b by

$$(17) \quad B_{ab} := \{u^\varepsilon \in \mathcal{H}^\varepsilon : \begin{aligned} \lim_{t \rightarrow -\infty} \|u^\varepsilon(\cdot, t) - u_a^\varepsilon\|_{L^1} &= 0 \\ \lim_{t \rightarrow \infty} \|u^\varepsilon(\cdot, t) - u_b^\varepsilon\|_{L^1} &= 0, 0 < \varepsilon < \varepsilon_0 \end{aligned}\}.$$

The rotating wave u_a^ε is called the source and u_b^ε the target.

Lemma 2.8 (Connection Lemma). *Let B_a, B_b and B_{ab} be defined as above with $a = 2\alpha$ and $b = 2\beta$ for $\alpha, \beta \in \mathbb{N}$, $\alpha > \beta$. We assume there exists $u^0 \in BV$ and a sub sequence $\{\varepsilon_n\}$ such that*

$$(18) \quad \lim_{n \rightarrow \infty} \|u^{\varepsilon_n}(x, t) - u^0(x, t)\|_{L^1} = 0$$

for $t \in \mathbb{R}$ and u_α^0, u_β^0 are the limiting waves of B_a and B_b respectively. Then there exists $k \in \mathbb{N}$ such that

$$(19) \quad a = kb.$$

In other words, if $a \neq kb$ for all $k \in \mathbb{N}$ then the limit of the heteroclinic orbits connecting the rotating waves u_a^ε and u_b^ε does not persist to a heteroclinic connection connecting the limits of source and target u_a^0, u_b^0 .

Proof. For all $0 < \varepsilon < \varepsilon_0$ the rotating waves u_a^ε and u_b^ε are periodic solutions of the rotating wave equation. Their period is given by $T_a = \frac{2\pi}{\alpha}$ and $T_b = \frac{2\pi}{\beta}$.

If the heteroclinic connection $u^\varepsilon(\cdot, t)$ persists, u_α^0 and u_β^0 have to be connected by a heteroclinic orbit. According to condition (7) this implies

$$(20) \quad \mathcal{Z}(u_\beta^0) \subset \mathcal{Z}(u_\alpha^0).$$

Taking the limit $\varepsilon \rightarrow 0$ for the rotating waves, we obtain that the zeros of u_α^0 and u_β^0 must be periodic in x and the distance of neighboring zeros is given by T_a and T_b respectively.

Then equation (20) implies

$$T_b = kT_a$$

for some $k \in \mathbb{N}$.

Hence

$$\alpha = k\beta$$

which implies

$$a = kb$$

just as desired. \square

Certainly the result of the *Connection Lemma* is void if all connections a priori fulfill the $a = kb$ condition. A consequence of the two following theorems is that this is not the case and hence there are more connections than those with $a = kb$. First Theorem 2.9 states that if $\varepsilon > 0$ is small enough there exists a up to rotation unique rotating wave for each given zero number $a \in 2\mathbb{N}$:

Theorem 2.9. *For every $n \in 2\mathbb{N}$ there exists $0 < \varepsilon_n \in \mathbb{R}$ such that for each $0 < \varepsilon < \varepsilon_n$ there exists a solution v_n^ε of the rotating wave equation (9) with*

$$z(v_n^\varepsilon) = n.$$

Let $\tilde{u} \in \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ with $z(\tilde{u}^\varepsilon) = n$. Then $\tilde{u} \in \{v_n^\varepsilon(\cdot + \theta) | \theta \in S^1\}$ hence \tilde{u} is a shifted copy of v_n^ε . For $n = 0$ there exist two unique stationary solutions $u_0^\varepsilon \equiv u_\pm$.

In addition we have

- (i) *The morse index (see Definition 3.1) of the wave \tilde{u} associated to the profile v_n^ε is given by*

$$(21) \quad i(u_n^\varepsilon) = n - 1.$$

For $u^\varepsilon \equiv u_\pm$ we have

$$(22) \quad i(u^\varepsilon \equiv u_\pm) = 0.$$

- (ii) *The maximum/minimum of the profile v_n^ε decreases/increases with increasing number of zeros. In other words for $m > n$ we have*

$$u_+ > \max_{x \in S^1} v_n^\varepsilon > \max_{x \in S^1} v_m^\varepsilon > 0.$$

and

$$u_- < \min_{x \in S^1} v_n^\varepsilon < \min_{x \in S^1} v_m^\varepsilon < 0.$$

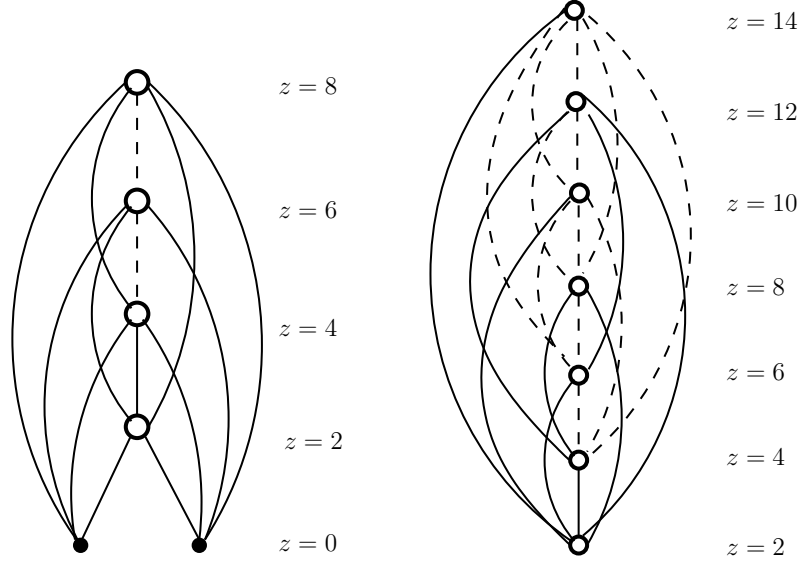


FIGURE 1. Structure of connections between rotating and frozen waves and homogeneous equilibria with zero number $z \leq 8$ and $z \leq 14$.

Theorem 2.10 then states that there exists a heteroclinic connection between rotating waves u_a^ε and u_b^ε if and only if the zero number of the target wave is strictly smaller than the zero number of the source:

Theorem 2.10. *Let $u_a^\varepsilon, u_b^\varepsilon \in \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ with zero numbers $z(u_a^\varepsilon) = a$ and $z(u_b^\varepsilon) = b$. Then there exists a heteroclinic orbit connecting u_a^ε and u_b^ε , i.e. a heteroclinic orbit with source u_a^ε and target u_b^ε if, and only if, $a > b$.*

Hence the structure of the global attractors of equation (P) is Chafee-Infante like [CI74] which implies that the condition in the connection Lemma 2.8 is not empty and not all heteroclinic connections of the parabolic equation persist. In fact persistence becomes less likely for connections between waves with increasing zero number. Figure 1 shows the structure of connections up to zero number $z = 8$ and $z = 14$. The dashed connections are the ones which cannot persist according to the *Connection Lemma*. In the right figure we have omitted connections to the homogeneous states with $z = 0$. Note that the Lemma does not imply persistence of the other connections. The proofs of the two theorems can be found in the following section.

The remainder of this section is devoted to the question of what happens in the limit of vanishing viscosity if a connection does not persist. The main difficulty lies in the fact that nothing is known about the geometric structure of a heteroclinic connection except for their target and source. However any set of heteroclinic connections B_{ab} of the parabolic equation where $a \neq kb$ cannot converge to one global solution of the hyperbolic equation, but, somehow has to converge to several. The question which solution the a set of heteroclinics B_{ab} converges to depends on the time parametrization of the $u^\varepsilon \in B_{ab}$ (every heteroclinic connection can be

shifted by a arbitrary time $\sigma \in \mathbb{R}$!) and in addition depends on the choice of the sub sequence $\{\varepsilon_n\}$.

The *Cascading Theorem* 2.12 yields that the number of different heteroclinic connections that can occur between a source u_a^ε and u_b^ε in the limit in fact is bounded by $\frac{a-b}{2} = \alpha - \beta$. This is true because of nodal properties which translate from the parabolic to the hyperbolic equation. We recall that the zero number along a heteroclinic orbit u_{ab}^ε drops strictly with time. In the limit the sources have zero number α the targets have zero number β . As the zero number has to drop strictly in every heteroclinic connection of the limiting equation (H) at most $\alpha - \beta$ different heteroclinic connections can be contained in the limit. This constitutes the key observation leading to the *Cascading Theorem* 2.12.

In order to prove this theorem we first fix one sub sequence $\{\varepsilon_n\}$. Then we will investigate the set of all time shifted connections $u_{ab}^{\varepsilon_n}(\cdot, t + \sigma_n)$ where the time shifts σ_n are chosen such that the $u_{ab}^{\varepsilon_n}$ still converge for $n \rightarrow \infty$. We will denote this set by U_{ab} : Let therefor B_{ab} be defined as usual, and $T > 0$. Let $\{\varepsilon_n\}$ be a sequence such that there exists $u^0 \in BV$ such that

$$\lim_{n \rightarrow \infty} \|u^{\varepsilon_n}(x, t) - u^0(x, t)\|_{L^1} = 0$$

for all $t \in [-T, T]$. Now let $\{\sigma_n\}$ be a sequence such that there exists $u_{\{\sigma_n\}} \in BV(S^1 \times [-T, T])$

$$(23) \quad \lim_{n \rightarrow \infty} \|u^{\varepsilon_n}(x, t - \sigma_n) - u_{\{\sigma_n\}}(x, t)\|_{L^1} = 0$$

for all $t \in [-T, T]$. In general there will be many sequences $\{\tilde{\sigma}_n\}$ with property (23). We call two sequences $\{\sigma_n\}_{1,2}$ equivalent if and only if $u_{\{\sigma_n\}_1} = u_{\{\sigma_n\}_2}$ in (23) a.e.. We denote the representative of all equivalent sequences by $\{\sigma_n\}$ and denote its equivalent class by $[\{\sigma_n\}]$

Then we define the set

$$U_{ab} := \{\tilde{u} \in BV(S^1) \mid \exists u_{\{\sigma_n\}} \in BV(S^1 \times [-T, T]) \text{ with the property (23), then} \\ \tilde{u} := u_{\{\sigma_n\}}(\cdot, 0)\}$$

The set U_{ab} is well defined: Let $u_1, u_2 \in U_{ab}$ with sequences $\{\sigma_n\}_1$ and $\{\sigma_n\}_2$ then

$$u_1 = u_2 \iff [\{\sigma_n\}_1] = [\{\sigma_n\}_2]$$

It is important to note that the sequences $\{\sigma_n\}$ in the above definition are not necessarily bounded. The set U_{ab} consist of all possible time shifted limits of a set of heteroclinic connections indexed by ε_n .

Lemma 2.11. *The set U_{ab} defined above has the following properties:*

- (i) U_{ab} is not empty
- (ii) $U_{ab} \subset \mathcal{A}^0$
- (iii) $u_\alpha^0 \in U_{ab}$ and $u_\beta^0 \in U_{ab}$, where $u_{\alpha,\beta}^0$ are the limits of the sources and targets for $\varepsilon_n \rightarrow 0$
- (iv) For all $u \in U_{ab}$ we have $\alpha \geq z(u) \geq \beta$

Proof. (i) is obvious as $u^{\varepsilon_n}(x, t - \sigma_n)$ converges by definition if we set all $\sigma_n = 0$

Property (ii) is a direct consequence of the global solution Theorem 2.2 and Corollary 2.4.

For (iii) we just prove $u_\beta^0 \in U_{ab}$: Due to the fact that all u^{ε_n} are heteroclinic connections with target $u_b^{\varepsilon_n} \in \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ we have $\lim_{t \rightarrow \infty} \|u^{\varepsilon_n}(\cdot, t) - u_b^{\varepsilon_n}\| = 0$ for all ε_n . Hence for every ε_n there exists σ_n such that

$$\|u^{\varepsilon_n}(\cdot, \sigma_n) - u_a^{\varepsilon_n}(\cdot)\|_{L^2} < \varepsilon_n.$$

In addition we know that $\|u_a^{\varepsilon_n} - u_a^0\|_{L^2} = \delta_n$ for some vanishing sequence δ_n . We conclude that

$$\|u^{\varepsilon_n}(\sigma_n) - u_a^0\| < \|u^{\varepsilon_n}(\sigma_n) - u_a^{\varepsilon_n}\| + \|u_a^{\varepsilon_n} - u_a^0\| \leq \varepsilon_n + \delta_n$$

converges to zero for $n \rightarrow \infty$ and hence yields the result.

(iv) is a direct consequence of Remark 2.6 and the fact that $a \geq z(u^{\varepsilon_n}(t)) \geq b$ for all $t \in \mathbb{R}$. \square

Theorem 2.12 (Cascading). *Let $a = 2\alpha$ and $b = 2\beta$ be given. Let B_{ab} and U_{ab} be defined as above. Then there are at most $\alpha - \beta$ different heteroclinic orbits contained in U_{ab} .*

Proof. Let $m := \alpha - \beta$. Then we define

$$\tau_j^n := \inf_{\tau \in \mathbb{R}} \{z(u^{\varepsilon_n}(\cdot, \tau)) > a - 2j\}$$

for $j \in 1 \dots m$. Hence at time τ_j^n the zero number of u^{ε_n} drops from $a - 2j + 2$ to $a - 2j$.

Now we define

$$U_{ab}^j := \{u_{\{\sigma_n\}} \in U_{ab} | \tau_n^j < \sigma_n \leq \tau_n^{j+1} \text{ for all } n > N_0 \in \mathbb{N}\}$$

for $1 \leq j \leq m - 1$,

$$U_{ab}^0 := \{u_{\{\sigma_n\}} \in U_{ab} | \sigma_n \leq \tau_n^1 \text{ for all } n > N_0 \text{ for some } N_0 \in \mathbb{N}\}$$

and

$$U_{ab}^m := \{u_{\{\sigma_n\}} \in U_{ab} | \tau_n^m \leq \sigma_n \text{ for all } n > N_0 \text{ for some } N_0 \in \mathbb{N}\}$$

We first observe that clearly

$$\bigcup_{j=0}^m U_{ab}^j = U_{ab}$$

by definition of the U_{ab}^j . Moreover all $\tilde{u} \in U_{ab}^j$ have the same zero number, namely $z(\tilde{u}) = a - 2j$ for all $0 \leq j \leq m$.

This implies that no U_{ab}^j can contain a full heteroclinic connection of the hyperbolic problem and hence the Theorem follows. \square

As a Corollary to Theorem 2.12 we obtain two necessary conditions on the persistence of a heteroclinic orbits.

Corollary 2.13 (Persistence). *Let $u^\varepsilon(x, t)$ be a heteroclinic orbit connecting u_a^ε with u_b^ε . Then the following statements are true:*

- (i) *Let the set U_{ab} as defined contain above at least one solution $u^0(x, t)$ that is not stationary. If*

$$\lim_{t \rightarrow -\infty} u^0(\cdot, t) = u_\alpha^0(\cdot)$$

and

$$\lim_{t \rightarrow \infty} u^0(\cdot, t) = u_\beta^0(\cdot)$$

then the heteroclinic connection $u^\varepsilon(x, t)$ persists.

(ii) If $U_{ab} \cap \mathcal{F}^0 = \{u_\alpha^0, u_\beta^0\}$ then the heteroclinic orbit persists.

Corollary 2.13 yields two independent sufficient conditions for the persistence of a heteroclinic orbit.

In addition Theorem 2.12 gives a result on the structure of the limit of heteroclinic connections in case of non-persistence: a cascade of heteroclinic connections in \mathcal{A}^0 .

3. ROTATING WAVES AND HETEROCLINIC CONNECTIONS FOR THE PARABOLIC EQUATION

This section is devoted to the proof of Theorems 2.9 and 2.10. The methods and ideas used in the proofs go back to results of Fiedler, Rocha and Wolfrum in [FRW04] which allow to decide whether two rotating waves are connected or not. However this requires information on all existing rotating waves which makes a complete description of all rotating waves necessary to tackle the connection problem. Although not easy this is a huge simplification as for the rotating waves classification we only have to solve an ODE problem.

Let therefor $L(u)$ define the linear operator obtained when the PDE (P) is linearized in the solution $u \in \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$, and let $\sigma(L(u))$ denote the spectrum of $L(u)$. We follow the definition given in [MN97] for the Morse index $i(u)$.

Definition 3.1. For each $u \in \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ we define the Morse index $i(u)$ and the generalized Morse index $i_0(u)$ by

$$i(u) := \#\{\lambda \in \sigma(L(u)); \operatorname{Re}(\lambda) > 0\}$$

and

$$i_0(u) := \#\{\lambda \in \sigma(L(u)); \operatorname{Re}(\lambda) \geq 0\}.$$

Here $\#$ counts eigenvalues repeatedly according to their multiplicity.

In terms of the Morse index we call a homogeneous stationary solution u hyperbolic, if

$$i_0(u) = i(u).$$

We call a rotating wave u hyperbolic, if

$$i_0(u) = i(u) + 1.$$

Note that u_x is always an eigenfunction of $L(u)$ to $\lambda = 0$. Hence the wave is called hyperbolic, if zero is a simple eigenvalue, and u_x is the only eigenvector to $\lambda = 0$.

Remark 3.2. The Morse index i corresponds to the number of strong unstable eigendirections of the solution $u^\varepsilon \in \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$, hence equals the dimension of the strong unstable manifold of u^ε in case of fixed points. For rotating waves u^ε the dimension of the strong unstable manifold is given by the Morse index $+1$.

The proof of Theorem 2.9 is accomplished by a series of Lemmata. We begin by rewriting the rotating wave equation in Lienard coordinates:

$$(24) \quad \begin{aligned} \varepsilon v_x &= f(v) - c(\varepsilon)v + p \\ p_x &= -g(v). \end{aligned}$$

These coordinates are adapted to the geometry of the problem. However, sometimes it is more convenient to work with standard phase plane coordinates:

$$(25) \quad \begin{aligned} \varepsilon w_x &= q \\ q_x &= \frac{(f'(w) - c(\varepsilon))q}{\varepsilon} - g(w) \end{aligned}$$

We will use both sets of coordinates as each one has its advantages. We will always use (v, p) when referring to the Lienard version and (w, q) when utilizing phase plane coordinates.

The coordinates can be transformed into each other by the transformation:

$$(26) \quad w(v, p) = v \quad v(w, q) = w$$

$$(27) \quad q(v, p) = f(v) - p \quad p(w, q) = f(w) - q.$$

In phase plane coordinates the system (25) is a **rotated vector field (mod $q = 0$)** with respect to the parameter c .

The notion of rotated vector fields was introduced by Duff [Duf53] and refined by Perko [Per75, Per93]. For exact definitions I refer to their papers or to Definition 4.1 in [Haer03].

The geometric interpretation of this is that the whole vector field rotates in the same direction when changing the parameter c except on the curve $q = 0$.

We now define the cyclicity set \mathcal{C}_p :

Definition 3.3. *The cyclicity set \mathcal{C}_p consists of all points $(w, q) \in \mathbb{R}^2$ that lie on a periodic orbit of equation (25) for some value of c or correspond to homogeneous equilibria $(e, 0)$ of (P) that undergo a Hopf bifurcation for some value of c .*

We immediately observe that in our situation \mathcal{C}_p is non-empty because the homogeneous solution associated with the middle equilibrium $w \equiv 0$ undergoes a Hopf bifurcation at $c = 0$.

According to Lemma 4.2 in [FRW04] the cyclicity set has in this case the following properties

Lemma 3.4. *The cyclicity set \mathcal{C}_p is bounded and open. There exist C^2 -functions*

$$(28) \quad \mathbf{c}, \mathcal{T} : \mathcal{C}_p \rightarrow \mathbb{R}$$

with the properties:

- (i) *For each non-stationary point $(w, q) \in \mathcal{C}_p$ the value $\mathbf{c}(w, q)$ defines the unique wave speed for which (w, q) lies on a periodic orbit of (25). Similarly, $\mathcal{T}(w, q)$ defines the minimal period of this orbit.*
- (ii) *The wave speeds \mathbf{c} are uniformly bounded.*
- (iii) *The minimal periods \mathcal{T} tend to infinity at the boundary $\partial\mathcal{C}_p$ of \mathcal{C}_p .*
- (iv) *$\partial\mathcal{C}_p$ consists of saddles and of points which are homoclinic or heteroclinic to saddles for some parameter value of c .*

We do not give a proof here but refer the reader to the paper quoted above. We now prove three Lemmata that will finally yield the proof of Theorem 2.9.

Lemma 3.5. *Let $\varepsilon > 0$ be arbitrary. Then the following is true:*

- a) *The cyclicity set \mathcal{C}_p is homeomorphic to a disc, i.e. it consists of one connected component and has no holes.*
- b) *Let $(w(x), q(x)) \neq (0, 0)$ be a periodic orbit. Then $w(x) \neq 0$ except at two points x_1, x_2 .*
- c) *All periodic orbits can be uniquely parametrized by their maxima $(\alpha, 0)$ and $\alpha > 0$.*

Proof. We first prove c): we assume that $v_1 \neq v_2$ are two rotating waves with

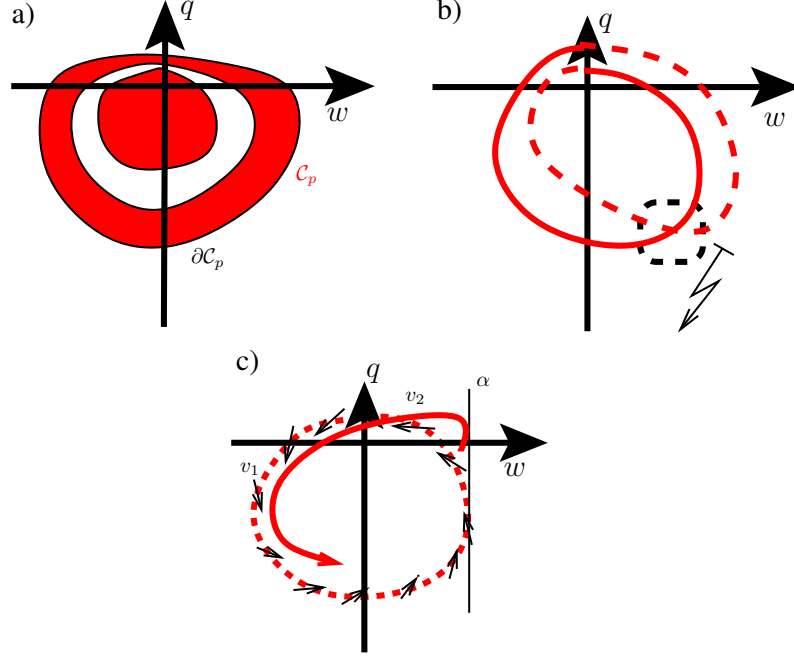


FIGURE 2. Illustration for the proof of Lemma 3.5.

wave speeds c_1 and c_2 and identical maximum

$$(29) \quad \alpha = \max_{x \in S^1} \{v_1(x)\} = \max_{x \in S^1} \{v_2(x)\}.$$

We observe that the origin $(0,0)$ has to lie in the interior of the area encircled by v_1 and v_2 respectively. This is a direct consequence of the persistence of the slow manifold $q = 0$ in phase plane coordinates outside a neighborhood of $(0,0)$ due to its normal hyperbolicity (see [Fen79]).

If v_1 does not intersect or touch v_2 , then necessarily either

$$v_1(x) < \max_{x \in S^1} \{v_2(x)\}$$

or vice versa. This contradicts equation (29). The curves therefore have to touch or intersect at least once.

We now distinguish two cases:

- (i) Assume $c_1 = c_2$. In this case the two curves have at least one point in the phase plane in common and solve the same equation. Hence they are the same. This violates $v_1 \neq v_2$
- (ii) Assume $c_1 \neq c_2$. We investigate the vector field of (25) for $c = c_2$ on the curve defined by v_1 . Due to the fact that (25) is a rotated vector field with respect to c we obtain, that the vector field has to either point strictly to the outside or strictly to the inside of the area encircled by v_1 . Assume the vector field points inwards, then the area encircled by v_1 is positive invariant. See panel c) for illustration.

Therefore v_1 enters at the intersection point but cannot intersect twice due

to the positive invariance of the area encircled by v_1 – and thus cannot be closed. This contradicts that v_1 is a periodic orbit.

If the vector field points to the outside, the same argument holds (just reverse the "time direction" x).

This proves c).

For b) we observe that the number of zeros is necessarily even. The fact that $(0, 0)$ lies in the area encircled by the periodic orbit excludes the no-zero case. The fact that the periodic orbit cannot intersect itself excludes the case of more than two zeros (see panel b) in Figure 2). This proves b).

For a) we observe that the nesting property of the periodic orbits in c) excludes holes in \mathcal{C}_p . Hence $\partial\mathcal{C}_p$ must consist of nested closed curves. Due to the boundedness of \mathcal{C}_p and the fact that $(0, 0) \in \mathcal{C}_p$, there must be a minimum of three curves. See Panel a) for such a situation.

According to 3.4 (iv) these curves must consist of saddles, homoclinic and heteroclinic connections. Hence all curves that form the boundary $\partial\mathcal{C}_p$ must start and end in the two saddles $(u_-, 0)$ or $(u_+, 0)$. Due to the rotated vector field property there is at most one homoclinic orbit at each saddle and at most one heteroclinic orbit connecting $(u_-, 0)$ and $(u_+, 0)$ and one connecting $(u_+, 0)$ with $(u_-, 0)$. Due to the persistence of the slow manifold $q = 0$ outside a neighborhood of $(0, 0)$ neither $(u_-, 0)$ can lie inside the homoclinic loop of $(u_+, 0)$ nor vice versa. This completes a). \square

The next Lemma gives a first-order description of all rotating waves.

Lemma 3.6. *Let $T > 0$ be given. Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ there exists a rotating wave w with minimal period T .*

Through a shift we can assume that $w(0) = 0$ and $w_x(0) > 0$, then $w(x)$ can be written in the following way:

$$(30) \quad w(x) = \phi(x) + O(\varepsilon) \quad \text{for } x \in [0, x_2 - \varepsilon \log \varepsilon] \cup [x_2 + \varepsilon \log \varepsilon, T]$$

$$(31) \quad w(x) = \psi\left(\frac{x - x_2}{\varepsilon}, x_2\right) + O(\varepsilon) \quad \text{for } x \in [x_2 - \varepsilon \log \varepsilon, x_2 + \varepsilon \log \varepsilon]$$

where x_2 is the second zero of w .

$\phi(\cdot)$ is a solution of

$$(32) \quad \phi_x = \frac{g(\phi)}{f'(\phi)} \quad \phi(0) = 0$$

and $\psi(\cdot, x_2)$ is a solution of

$$(33) \quad \psi_{\frac{x}{\varepsilon}} = f\left(\psi\left(\frac{x}{\varepsilon}\right)\right) - \phi(x_2) \quad \psi(0) = 0$$

Proof. For existence we observe that the centre in the origin $(0, 0)$ undergoes a Hopf bifurcation. A straight forward calculation shows that the imaginary parts of the hopf eigenvalues are given by

$$\nu := \text{Im}(\lambda_{1/2}) = \pm \frac{\sqrt{g'(0)}}{\sqrt{\varepsilon}}.$$

Thus, the limiting period of the at the Hopf bifurcation emerging limiting cycle is given by

$$T_{Hopf} = \frac{2\pi}{\nu} = \frac{2\pi\sqrt{\varepsilon}}{\sqrt{g(0)}}.$$

This yields existence of rotating waves for $T \rightarrow 0$. For $T \rightarrow \infty$ existence follows from Lemma 3.4 (iv). Continuity of \mathcal{T} yields existence by virtue of the intermediate value theorem for all T if we choose ε_0 such that

$$\varepsilon < \varepsilon_0 := T^2 \frac{g'(0)}{4\pi^2}$$

holds.

It remains to prove equations (30),(31). From Lemma 3.5 c) we know that the periodic orbits can be parametrized by their maxima, moreover, Lemma 3.5 b) proves that w_ε has exactly two zeros. Without loss of generality we shift the zero with positive slope to $x_1 = 0$. We denote the other zero with x_2 and note that $w'(x)|_{x=x_2} < 0$ necessarily. We assume that the wave-speed $c(\varepsilon) = 0$ and prove equations (30,31). Then we will argue that the correct wave-speed is in fact small and hence does not destroy the approximation.

We start computing the trajectory of $(w(x), w_x(x))$ in $x = x_2$ and assume that

$$(34) \quad |w_x(x)|_{x=x_2} | >> \varepsilon_0.$$

This is always possible because we are free in the choice of ε_0 . We use phase plane coordinates.

Due to equation (34) we can use the fast vector field to describe the solution up to the first order. In forward time direction the solution will converge exponentially to a ε -neighborhood of the unstable manifold of $(u_-, 0)$. In backward time direction the solution will converge exponentially to a ε -neighborhood of the stable manifold of $(u_+, 0)$. This part can be described due to Fenichel [Fen79] by the fast equations given by (33). This proves equation in (31).

The unstable manifold of $(u_-, 0)$ is transversely stable in forward time direction. So is the stable manifold of $(u_+, 0)$ in negative time direction. Hence for $T > x > x_2 + C\varepsilon \log \varepsilon$ the periodic solution $w(x)$ converges exponentially to the unstable manifold of $(u_-, 0)$, the same is true for $0 < x < x_2 - C\varepsilon \log \varepsilon$ and the stable manifold of $(u_+, 0)$. Because $w(x)$ is periodic we have $w(T) = w(0)$ or in other words in the phase plane the trajectories given by $\gamma(x) := (w(x), v(x))$ for $x \rightarrow T$ and for $x \rightarrow 0$ intersect.

Now we quote a result by Haerterich stating that the wave-speed c^* for which the unstable manifold of $(u_-, 0)$ and the stable manifold of $(u_+, 0)$ are connected (i.e. the heteroclinic connection between the two equilibria persists) obeys

$$|c^*(\varepsilon)| < \sigma\varepsilon,$$

see Lemma 4.3 in [Haer03]. By virtue of the same argument that essentially makes use of the fact that our system is a rotated vector field we obtain the same bound must hold in our case for the wave speed of our periodic solution w . Because the wave speed enters the equation for $c = 0$ as a regular perturbation our approximation stays valid for $|c| \leq \sigma\varepsilon$. \square

The next Lemma uses the above descriptions to prove monotonicity of $\mathcal{T}(w, q)$. This result forms the basis of a relation between the zeros of a solution and the number of its unstable eigenvalues.

Lemma 3.7. *Let T be arbitrary but fixed. Then there exists a $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the minimal period $\mathcal{T}(w, q)$ grows monotone with the maxima of the periodic orbits.*

Proof. We use the formula of the periodic orbit $w(x)$ obtained in equation (30). Let us assume we have two periodic orbits w_1 and w_2 with period T_1 and T_2 and the property that

$$\max_{x \in [0, T_1]} w_1 =: \alpha_1 < \alpha_1 + \delta_1 := \alpha_2 := \max_{x \in [0, T_2]} w_2$$

for some $\delta_1 > 0$. It is sufficient to prove $T_1 < T_2 - \delta_2$ for sufficiently small ε and some $\delta_2 > 0$.

Due to the fact that the periodic orbits are nested (Lemma 3.5 c)) $\alpha_1 < \alpha_2$ implies immediately

$$0 > \min_{x \in [0, T_1]} w_1 =: \beta_1 > \beta_2 := \min_{x \in [0, T_2]} w_2.$$

The solution $\phi(x)$ is strict monotonically growing because

$$\phi_x = \frac{g(\phi)}{f'(\phi)} > 0$$

due to the convexity of f and the fact that the zero of f' and the zero of g at $u = 0$ are simple. This implies invertibility of ϕ and monotonicity of ϕ^{-1}

We now have

$$(35) \quad T_1 = \phi^{-1}(\alpha_1) - \phi^{-1}(\beta_1) + O(\varepsilon \log \varepsilon)$$

$$(36) \quad T_2 = \phi^{-1}(\alpha_2) - \phi^{-1}(\beta_2) + O(\varepsilon \log \varepsilon)$$

The monotonicity of ϕ implies

$$\phi^{-1}(\alpha_2) = \phi^{-1}(\alpha_1 + \delta_1) = \phi^{-1}(\alpha_1) + (\phi^{-1})'(\alpha_1)\delta_1 + O(\delta_1) > \phi^{-1}(\alpha_1) + \delta_2$$

for some $\delta_2 > 0$ and

$$-\phi^{-1}(\beta_1) < -\phi^{-1}(\beta_2).$$

For sufficiently small $0 < \varepsilon_0$ we obtain the desired inequality for some δ_2 for all $0 \leq \varepsilon \leq \varepsilon_0$. \square

Corollary 3.8. *Let $T \in \mathbb{R}^+$ be given. Then there is a unique periodic orbit with minimal period T , and it is hyperbolic as a rotating wave of (P).*

Proof. The uniqueness is a direct consequence of the monotonicity of the \mathcal{T} -map. The hyperbolicity is also a direct consequence of the monotonicity of the \mathcal{T} -map. A periodic orbit is non-hyperbolic if, and only if, the time \mathcal{T} map has a vanishing derivative. See for example Lemma 4.4 in [FRW04]. This would contradict monotonicity. \square

We are now set to construct rotating waves of the PDE (P) by using the periodic orbits constructed earlier in this section.

Proof of Theorem 2.9. Let $n \in 2\mathbb{N}$ be fixed.

Every rotating wave with n zeros corresponds to a periodic solution of the rotating wave equation (9) with period $T_n = \frac{2\pi}{n}$.

Corollary 3.8 provides for the unique existence of a periodic orbit of the rotating wave equation with period

$$T_n = \frac{2\pi}{n},$$

which we denote by v_n^ε . This settles the existence part.

Now assume there is a $\tilde{u}^\varepsilon \in \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ which $z(\tilde{u}) = n$. Then there must be a solution \tilde{v}_n^ε of equation (9) with $z(\tilde{v}_n^\varepsilon) = n$ and $\tilde{u}^\varepsilon(x, t) = \tilde{v}_n^\varepsilon(x - ct + \theta)$. This settles the first part of the Theorem.

(i) is a direct consequence of Lemma 5.3 in [FRW04] which states:

Lemma 3.9. *Let*

$$\dot{T} = \partial_\alpha T$$

be the derivative of the minimal period with respect to the maximum of the periodic orbits just as in Lemma 3.7. Then the Morse index of a rotating or frozen wave u is given by the following relations:

$$(37) \quad i(u) = z(u) - 1 \iff \dot{T} > 0$$

$$(38) \quad i(u) = z(u) \iff \dot{T} < 0.$$

Lemma 3.7 states monotonicity of the function T hence we have $\dot{T} = \partial_\alpha T > 0$ for all possible α which yields

$$i(u_n^\varepsilon) = z(u_n^\varepsilon) - 1 = n - 1$$

for all $u^\varepsilon \in \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$. For $u^\varepsilon \equiv u_\pm$ we use the fact that in Sturm-Liouville eigenvalue problems the eigenfunction to the leading eigenvalue λ_0 (eigenvalue with largest real part) has a sign, i.e. has no zeros. This can be found in [CL55] in Chapter 8, Theorem 3.1.

A small calculation shows that $\lambda_0 = g'(u_\pm) < 0$ with constant eigenfunction. Hence $i(u_\pm) = 0$.

(ii) is a direct consequence of the nested property of the rotating waves in Lemma 3.5 c). \square

We now turn to the proof of Theorem 2.10 which relies on the concept of $k - (\mathcal{P})$ -adjacency developed and used in [FR96] and later in [Wol02a] and [Wol02b] for the Neumann case. Fiedler, Rocha and Wolfrum presented in [FRW04] a version for the S^1 case which we will use:

Definition 3.10 ($k - (\mathcal{P})$ -adjacency). *Let $u_a^\varepsilon, u_b^\varepsilon \in \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$. Then u_a^ε and u_b^ε are called $k - (\mathcal{P})$ -adjacent if the following holds:*

$$z(u_a^\varepsilon - u_b^\varepsilon) = k$$

for some $k \in \mathbb{N}$ and there does not exist a solution $u_c^\varepsilon \in \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ with the property

$$(39) \quad z(u_a^\varepsilon - u_c^\varepsilon) = z(u_b^\varepsilon - u_c^\varepsilon) = k \text{ and}$$

$$(40) \quad \max_{x \in S^1} u_c^\varepsilon(x) \text{ is strictly between } \max_{x \in S^1} u_a^\varepsilon(x) \text{ and } \max_{x \in S^1} u_b^\varepsilon(x).$$

Theorem 1.3 in [FRW04] now states that two solutions $u_a^\varepsilon, u_b^\varepsilon \in \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ are connected by a heteroclinic if and only if u_a^ε and u_b^ε are $k - (\mathcal{P})$ -adjacent. The authors call a violation of $k - (\mathcal{P})$ -adjacency the blocking principle because in this case there is another rotating wave u_c^ε that blocks the connection.

Proof of Theorem 2.10. The “only if” has already been proven by Matano and Nakamura in [MN97]. The statement can be found in *Theorem C* on page 5. It is a direct consequence of the fact that due to the Sturm property of the problem the zero-number can only drop along trajectories and so does the Morse index.

For the “if” part we have to prove $k - (\mathcal{P})$ -adjacency of u_a^ε and u_b^ε . The key observation lies in the fact that the number of zeros of the difference of two rotating waves is given by the minimum of the zero-numbers individually. In other words, we have for $\tilde{u}, \hat{u} \in \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ the following relation:

$$(41) \quad z(\tilde{u} - \hat{u}) = \min\{z(\tilde{u}), z(\hat{u})\}.$$

This is not true in general, but a direct consequence of the fact that in our situation all periodic orbits of the rotating wave equation are nested (Lemma 3.5 c)).

Let $u_a^\varepsilon, u_b^\varepsilon \in \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ for some $a, b \in \mathbb{N}$. Now assume there exists $u_c^\varepsilon \in \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ with the property

$$(42) \quad z(u_a^\varepsilon - u_c^\varepsilon) = z(u_b^\varepsilon - u_c^\varepsilon) = k \text{ and}$$

$$(43) \quad \max_{x \in S^1} u_c^\varepsilon(x) \text{ is strictly between } \max_{x \in S^1} u_a^\varepsilon(x) \text{ and } \max_{x \in S^1} u_b^\varepsilon(x)$$

exists.

Due to Theorem 2.9 there is a unique rotating wave to each zero-number $k \in 2\mathbb{N}_0$ for fixed and sufficiently small $\varepsilon > 0$. From this we conclude that

$$a \neq k \neq b$$

otherwise $u_c^\varepsilon = u_a^\varepsilon$ or $u_c^\varepsilon = u_b^\varepsilon$.

In case $k > b$ equation (42) is violated due to (41). Hence, we have necessarily $k < b$. Due to the nested property of rotating waves this implies $\max u_c^\varepsilon > \max u_{a,b}^\varepsilon$, which violates (43). Thus u_a^ε and u_b^ε are $k - (\mathcal{P})$ -adjacent and therefor connected.

In case $u_a^\varepsilon \in \mathcal{E}^\varepsilon$ or $u_b^\varepsilon \in \mathcal{E}^\varepsilon$ the same argument works, the zero properties are obvious because in this case u_a or u_b is constant. \square

4. CONCLUSIONS

In this work we have investigated a key aspect of the question of convergence of attractors for viscous and inviscid balance laws on the circle: persistence of heteroclinic connections when viscosity vanishes. The *Connection Lemma* 2.8 provides a necessary condition for the persistence of heteroclinic connections between given rotating or frozen waves namely that connections can only persist if the zero numbers of target and source are multiples of each other. It shows together with the results of the section on the parabolic attractor that heteroclinic connections do not persist generally. In fact persistence or non-persistence of a heteroclinic connection exclusively depends on the structural relation between source and target of the connection. The non persistence has strong implications on the question whether the global attractor of the inviscid balance law (H) can be obtained as a limit of the global attractors of the parabolic equation (P) when viscosity vanishes. The result presented in this paper excludes a orbit equivalence relation even when we restrict the analysis to reasonable subsets of the full attractors \mathcal{A}^0 and \mathcal{A}^ε . However the corrected result of Fan and Hale [FH95] still yields a point wise convergence of global solutions for $\varepsilon \rightarrow 0$ which suggests a convergence of attractors in terms of sets, for example with respect to a Hausdorff norm on L^1 , at least for appropriately chosen subsets of the attractors. Unfortunately it remains rather unclear how to rigorously proof such a result. Related to this question is whether the purely algebraic condition (19) in the connection Lemma 2.8 is only necessary or whether it is sufficient. Results on the structure of certain sub-attractors of the hyperbolic

equation in [Ehrt2010/2] suggest that the condition is sufficient, however there is no rigorous proof in sight at the moment.

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