

# Minimal Supersolutions of Convex BSDEs\*

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We study a nonlinear operator defined via minimal supersolutions of backward stochastic differential equations with generators that are monotone in  $y$ , convex in  $z$ , jointly lower semicontinuous, and bounded below by an affine function of the control variable. We show existence, uniqueness, monotone convergence, Fatou's Lemma and lower semicontinuity of this functional. We provide a comparison principle for the underlying minimal supersolutions of BSDEs, which we illustrate by maximizing expected exponential utility.

**Keywords:** Supersolutions of Backward Stochastic Differential Equations; Non-Linear Expectations; Supermartingales

## 1. Introduction

On a filtered probability space, where the filtration is generated by a  $d$ -dimensional Brownian motion  $W$ , we consider the process  $\hat{\mathcal{E}}^g(\xi)$  given by

$$\hat{\mathcal{E}}_t^g(\xi) = \text{ess inf} \{ Y_t \in L_t^0 : (Y, Z) \in \mathcal{A}(\xi, g) \}, \quad t \in [0, T],$$

where  $\mathcal{A}(\xi, g)$  is the set of all pairs of càdlàg *value processes*  $Y$  and *control processes*  $Z$  such that

$$Y_s - \int_s^t g_u(Y_u, Z_u) du + \int_s^t Z_u dW_u \geq Y_t \quad \text{and} \quad Y_T \geq \xi, \quad (1.1)$$

for all  $0 \leq s \leq t \leq T$ . Here the *terminal condition*  $\xi$  is a random variable, the *generator*  $g$  a measurable function of  $(y, z)$  and the pair  $(Y, Z)$  is a *supersolution* of the *backward stochastic differential equation (BSDE)* (1.1).

The main objective of this paper is to state conditions which guarantee that there exists a unique minimal supersolution. More precisely, we show that the process  $\mathcal{E}^g(\xi) = \lim_{s \downarrow \cdot, s \in \mathbb{Q}} \hat{\mathcal{E}}_s^g(\xi)$  is a modification

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of  $\hat{\mathcal{E}}^g(\xi)$  and equals the value process of the unique minimal supersolution, that is, there exists a unique control process  $\hat{Z}$  such that  $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$ . The existence theorem immediately yields a comparison theorem for minimal supersolutions. We also study the stability of the minimal supersolution with respect to perturbations of the terminal condition or the generator. Our results show that the mapping  $\xi \mapsto \hat{\mathcal{E}}_0^g(\xi)$ , which can be viewed as a nonlinear expectation, fulfills a monotone convergence theorem and Fatou's Lemma on the same domain as the expectation operator  $E[\cdot]$ . These properties allow us to conclude that  $\hat{\mathcal{E}}_0^g(\cdot)$  is  $L^1$ -lower semicontinuous.

Before we give further details on the specific choice of our spaces and assumptions, let us recall that related problems have been investigated throughout the literature before. Most notably, nonlinear expectations have been a prominent topic in mathematical economics since Allais famous paradox, see Föllmer and Schied [23, Section 2.2]. Typical examples are the monetary risk measures introduced by Artzner et al. [2] and Föllmer and Schied [22], Peng's  $g$  and  $G$ -expectations, see [34, 36, 37], the variational preferences by Maccheroni et al. [32], and the recursive utilities by Duffie and Epstein [17]. Especially the  $g$ -expectation, which is defined as the initial value of the solution of a BSDE, is closely related to  $\mathcal{E}_0^g(\cdot)$ , since each pair  $(Y, Z)$  that solves the BSDE corresponding to (1.1) is also a supersolution and hence an element of  $\mathcal{A}(\xi, g)$ . The concept of a supersolution of a BSDE appears already in El Karoui et al. [20, Section 2.2]. For further references see Peng [35], who derives monotonic limit theorems for supersolutions of BSDEs and proves the existence of a minimal constrained supersolution. Another related concept are stochastic target problems, which were introduced and studied by Soner and Touzi [42], by means of controlled stochastic differential equations and dynamic programming methods.

Our first contribution is to provide a setting where we relax the usual Lipschitz requirements for the generator  $g$ . Namely, we suppose that  $g$  is convex with respect to  $z$ , monotone in  $y$ , jointly lower semicontinuous, and bounded below by an affine function of the control variable  $z$ . In order to provide an intuition on how these assumptions contribute toward the existence and uniqueness of a control process  $\hat{Z}$  such that  $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$ , let us suppose for the moment that  $g$  is positive. Given an adequate space of control processes, the value process of each supersolution and the process  $\hat{\mathcal{E}}^g(\xi)$  are in fact supermartingales. By suitable pasting, we may now construct a decreasing sequence  $(Y^n)$  of supersolutions, whose pointwise limit is again a supermartingale and equal to  $\hat{\mathcal{E}}^g(\xi)$  on all dyadic rationals. Since the generator  $g$  is positive, it can be shown that  $\mathcal{E}^g(\xi)$  lies below  $\hat{\mathcal{E}}^g(\xi)$ ,  $P$ -almost surely, at any time. This suggests to consider the càdlàg supermartingale  $\mathcal{E}^g(\xi)$  as a candidate for the value process of the minimal supersolution. However, it is not clear a priori that the sequence  $(Y^n)$  converges to  $\mathcal{E}^g(\xi)$  in some suitable sense. Yet, taking into account the additional supermartingale structure we can prove, by using Helly's theorem, that  $(Y^n)$  converges  $P \otimes dt$ -almost surely to  $\mathcal{E}^g(\xi)$ . It remains to obtain a unique control process  $\hat{Z}$  such that  $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$ . To that end, we prove that, for monotone sequences of supersolutions, a positive generator yields, after suitable stopping, a uniform  $L^1$ -bound for the sequence of supremum processes of the associated sequence of stochastic integrals. This, along with a result by Delbaen and Schachermayer [11], and standard compactness arguments and diagonalization techniques yield the candidate control process  $\hat{Z}$  as the limit of a sequence of convex combinations. Now, joint lower semicontinuity of  $g$ , positivity, and convexity in  $z$  allow us to use Fatou's Lemma to verify that the candidate processes  $(\mathcal{E}^g(\xi), \hat{Z})$  are a supersolution of the BSDE. Thus,  $\mathcal{E}^g(\xi)$  is in fact the value process of the minimal supersolution and a modification of  $\hat{\mathcal{E}}^g(\xi)$ . Finally, the uniqueness of  $\hat{Z}$  follows from the uniqueness of the Doob-Meyer decomposition of the càdlàg supermartingale  $\mathcal{E}^g(\xi)$ .

Let us give further reference of related assumptions and methods in the existing literature. Delbaen et al. [14] consider superquadratic BSDEs with generators that are positive and convex in  $z$  but do not depend on  $y$ . However, their principal aim and their method differ from ours. Indeed, they primarily study the well-posedness of superquadratic BSDEs by establishing a dual link between cash additive

time-consistent dynamic utility functions and supersolutions of BSDEs. To view supersolutions as supermartingales is one of the key ideas in our approach and we make ample use of the rich structure supermartingales provide. Note that the idea to use classical limit theory of supermartingales in the theory of BSDEs appears already in El Karoui and Quenez [18], who study the problem of option pricing in incomplete financial markets. However, the analysis is done via dual formulations and only for linear generators that do not depend on  $y$ . The construction of solutions of BSDEs by monotone approximations is also a classical tool, see for example Kobylanski [30] for quadratic generators and Briand and Hu [5] for generators that are in addition convex in  $z$ . The application of compactness theorems such as Delbaen and Schachermayer [11, Lemma A1.1], or Delbaen and Schachermayer [12, Theorem A], in order to derive existence of BSDEs seems to be new to the best of our knowledge. Usually existence proofs rely on a priori estimates combined with a fixed point theorem, see for example [20], or on constructing Cauchy sequences in complete spaces, see for example Briand and Confortola [4] or Ankirchner et al. [1]. As already mentioned, Peng [35] studies the existence and uniqueness of minimal supersolutions. However, he assumes a Lipschitz generator, a square integrable terminal condition, and employs a very different approach. It is based on a monotonic limit theorem, [35, Theorem 2.4], the penalization method introduced in El Karoui et al. [19], and it leads to monotone increasing sequences of supersolutions. Parallel to us, Cheridito and Stadje [7] have investigated existence and stability of supersolutions of BSDEs. They consider generators that are convex in  $z$  and Lipschitz in  $y$ . However, their setting and methods are quite different from ours. Namely, they approximate by discrete time BSDEs and work with terminal conditions that are bounded lower semicontinuous functions of the Brownian motion. Finally, given our local  $L^1$ -bounds, the compactness underlying the construction of the candidate control process is a special case of results obtained by Delbaen and Schachermayer [12].

Our second contribution is to allow for *local supersolutions*. This happens to be particularly adequate to establish monotone continuity properties of the minimal supersolution with respect to the terminal condition or the generator. We call a supersolution  $(Y, Z)$  of the BSDE local, if the stochastic integral of  $Z$  is well defined and thus a continuous local martingale. In order to avoid so-called “doubling strategies”, present even for the most simple driver  $g \equiv 0$ , see Dudley [16] or Harrison and Pliska [24, Section 6.1], we require in addition that  $\int Z dW$  is a supermartingale. In this setting, the stochastic integral of the candidate control process is only a local martingale. However, we may once again use our assumptions on the generator to prove the supermartingale property. In addition, similar arguments allow us to formulate theorems such as monotone convergence and Fatou’s lemma for the non-linear operator  $\hat{\mathcal{E}}_0^g(\cdot)$  on the same domain as the standard expectation  $E[\cdot]$  and to obtain its  $L^1$ -lower semicontinuity. To complete the picture, we point out that our approach neither needs nor provides much integrability for the control processes. The underlying reason is that the compactness arguments in our proof are based on  $L^1$  rather than  $\mathcal{H}^1$ -bounds for the stochastic integrals. Yet, given some additional integrability on the terminal condition, we obtain a candidate control process, whose stochastic integral belongs to  $\mathcal{H}^1$  and therefore is a true martingale. However, monotone stability for an increasing sequence of terminal conditions does not hold without the additional assumption that  $\mathcal{A}(\xi, g)$ , where  $\xi$  is the limit terminal condition, is not empty. This guarantees the required integrability of the limit pair  $(\hat{Y}, \hat{Z})$ . In contrast, such an assumption is not necessary in our initial setting, where, in order to obtain suitable bounds and to construct the dominating candidate supermartingale, it is enough to know that the monotone limit of the minimal supersolutions at time zero is finite.

Dropping the positivity assumption, it is obvious that the value and control processes of our supersolutions are supermartingales under another measure closely linked to the generator  $g$ . In fact, for a positive generator we have supermartingales with respect to the initial probability measure  $P$ , while for a non-positive generator, which is bounded below by an affine function of the control variable, we con-

sider supermartingales under the measure given by the corresponding Girsanov transform. This yields a generator dependent concept of admissibility. The implication thereof is illustrated by giving a minimal supersolution based approach to the well known problem of exponential expected utility maximization, where this admissibility is related in a natural way to the market price of risk.

The paper is organized as follows. In Section 2 we fix our notations and the setting. A precise definition of minimal supersolutions, some of our main conditions and first structural properties of  $\hat{\mathcal{E}}^g(\xi)$  are given in Section 3. Our main results, that is, existence and stability theorems, are given in Section 4, which concludes with an example on maximizing expected exponential utility.

## 2. Setting and Notations

We consider a fixed time horizon  $T > 0$  and a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$ , where the filtration  $(\mathcal{F}_t)$  is generated by a  $d$ -dimensional Brownian motion  $W$  and fulfills the usual conditions. We further assume that  $\mathcal{F} = \mathcal{F}_T$ . The set of  $\mathcal{F}$ -measurable and  $\mathcal{F}_t$ -measurable random variables is denoted by  $L^0$  and  $L_t^0$ , respectively, where random variables are identified in the  $P$ -almost sure sense. The sets  $L^p$  and  $L_t^p$  denote the set of random variables in  $L^0$  and  $L_t^0$ , respectively, with finite  $p$ -norm, for  $p \in [1, +\infty]$ . Throughout this work, inequalities and strict inequalities between any two random variables or processes  $X^1, X^2$  are understood in the  $P$ -almost sure or in the  $P \otimes dt$ -almost sure sense, respectively, that is,  $X^1 \leq (<) X^2$  is equivalent to  $P[X^1 \leq (<) X^2] = 1$  or  $P \otimes dt[X^1 \leq (<) X^2] = 1$ , respectively. Given a process  $X$  and  $t \in [0, T]$  we denote  $X_t^* := \sup_{s \in [0, t]} |X_s|$ . We denote by  $\mathcal{T}$  the set of stopping times with values in  $[0, T]$  and hereby call an increasing sequence of stopping times  $(\tau^n)$ , such that  $P[\bigcup_n \{\tau^n = T\}] = 1$ , a *localising sequence of stopping times*. By  $\mathcal{S} := \mathcal{S}(\mathbb{R})$  we denote the set of all càdlàg progressively measurable processes  $Y$  with values in  $\mathbb{R}$  and further denote with  $Prog$  the  $\sigma$ -algebra on  $\Omega \times [0, T]$  generated by all progressively measurable processes. For  $p \in [1, +\infty[$ , we further denote by  $\mathcal{L}^p := \mathcal{L}^p(W)$  the set of progressively measurable processes  $Z$  with values in  $\mathbb{R}^{1 \times d}$ , such that  $\|Z\|_{\mathcal{L}^p} := E[(\int_0^T Z_s^2 ds)^{p/2}]^{1/p} < +\infty$ . For any  $Z \in \mathcal{L}^p$ , the stochastic integral  $(\int_0^t Z_s dW_s)_{t \in [0, T]}$  is well defined, see [38], and is by means of the Burkholder-Davis-Gundy inequality a continuous martingale. For the  $\mathcal{L}^p$ -norm, the set  $\mathcal{L}^p$  is a Banach space, see [38]. We further denote by  $\mathcal{L} := \mathcal{L}(W)$  the set of progressively measurable processes with values in  $\mathbb{R}^{1 \times d}$ , such that there exists a localising sequence of stopping times  $(\tau^n)$  with  $Z1_{[0, \tau^n]} \in \mathcal{L}^1$ , for all  $n \in \mathbb{N}$ . Here again, the stochastic integral  $\int Z dW$  is well defined and is a continuous local martingale.

For adequate integrands  $a, Z$ , we generically write  $\int a ds$  or  $\int Z dW$  for the respective integral processes  $(\int_0^t a_s ds)_{t \in [0, T]}$  and  $(\int_0^t Z_s dW_s)_{t \in [0, T]}$ . Finally, given a sequence  $(x_n)$  in some convex set, we say that a sequence  $(y_n)$  is in the *asymptotic convex hull* of  $(x_n)$ , if  $y_n \in conv\{x_n, x_{n+1}, \dots\}$ , for all  $n$ .

*Normal integrands* have been introduced by Rockafellar [40] and are particularly adequate to model integral problems with constraints. In our setting, a normal integrand is a function  $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow ]-\infty, +\infty]$ , such that

- $(y, z) \mapsto g(\omega, t, y, z)$  is lower semicontinuous for all  $(\omega, t) \in \Omega \times [0, T]$ ;
- $(\omega, t) \mapsto g(\omega, t, y, z)$  is progressively measurable for all  $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$ .

It is shown in [41, Chapter 14.F], that for all progressively measurable processes  $Y, Z$ , the process  $g(Y, Z)$  is itself progressively measurable and so, the integral  $\int g(Y, Z) ds$  is well defined  $P$ -almost surely under the assumption that  $+\infty - \infty = +\infty$ . The section theorem as well as the Fubini, Tonelli theorem [27, Lemma 1.26 and Theorem 1.27] extend to that context. Finally, the lower semicontinuity

yields an extended Fatou's lemma, that is,

$$\int \liminf_n g_s(Y_s^n, Z_s^n) ds \leq \liminf_n \int g_s(Y_s^n, Z_s^n) ds,$$

for any sequence  $Y^n, Z^n$  of progressively measurable processes, if  $g \geq 0$ .

### 3. Minimal Supersolutions of BSDEs

#### 3.1. Definitions

Given a normal integrand  $g$ , henceforth called *generator*, and a *terminal condition*  $\xi \in L^0$ , a pair  $(Y, Z) \in \mathcal{S} \times \mathcal{L}$  is a *supersolution* of a BSDE, if, for all  $s, t \in [0, T]$ , with  $s \leq t$ , holds

$$Y_s - \int_s^t g_u(Y_u, Z_u) du + \int_s^t Z_u dW_u \geq Y_t \quad \text{and} \quad Y_T \geq \xi. \quad (3.1)$$

For such a supersolution  $(Y, Z)$ , we call  $Y$  the *value process* and  $Z$  its *control process*. Due to the càdlàg property, Relation (3.1) holds for all stopping times  $0 \leq \sigma \leq \tau \leq T$ , in place of  $s$  and  $t$ , respectively. Note that the formulation in (3.1) is equivalent to the existence of a càdlàg increasing process  $K$ , with  $K_0 = 0$ , such that

$$Y_t = \xi + \int_t^T g_u(Y_u, Z_u) du + (K_T - K_t) - \int_t^T Z_u dW_u, \quad t \in [0, T]. \quad (3.2)$$

Although the notation in (3.2) is standard in the literature concerning supersolutions of BSDEs, see for example [20, 35], we will keep with (3.1) since the proofs of our main results exploit this structure. We consider only those supersolutions  $(Y, Z) \in \mathcal{S} \times \mathcal{L}$  of a BSDE where  $Z$  is *admissible*, that is, where the continuous local martingale  $\int Z dW$  is a supermartingale. We are then interested in the set

$$\mathcal{A}(\xi, g) = \{(Y, Z) \in \mathcal{S} \times \mathcal{L} : Z \text{ is admissible and (3.1) holds}\} \quad (3.3)$$

and the process

$$\hat{\mathcal{E}}_t^g(\xi) = \text{ess inf} \{Y_t \in L_t^0 : (Y, Z) \in \mathcal{A}(\xi, g)\}, \quad t \in [0, T]. \quad (3.4)$$

By  $\hat{\mathcal{E}}^g$  we mean the functional mapping terminal conditions  $\xi \in L^0$  to the process  $\hat{\mathcal{E}}^g(\xi)$ . Since the set  $\mathcal{A}(\xi, g)$  and therefore  $\hat{\mathcal{E}}^g(\xi)$  depends on the time horizon  $T$ , we indicate this by writing  $\mathcal{A}_T(\xi, g)$  and  $\hat{\mathcal{E}}_{\cdot, T}^g(\xi, g)$ , if necessary. Note that the essential infima in (3.4) can be taken over those  $(Y, Z) \in \mathcal{A}(\xi, g)$ , where  $Y_T = \xi$ . A pair  $(Y, Z)$  is called *minimal supersolution*, if  $(Y, Z) \in \mathcal{A}(\xi, g)$ , and if for any other supersolution  $(Y', Z') \in \mathcal{A}(\xi, g)$ , holds  $Y_t \leq Y'_t$ , for all  $t \in [0, T]$ .

#### 3.2. General Properties of $\mathcal{A}(\cdot, g)$ and $\mathcal{E}^g$

In this section we collect various statements regarding the properties of  $\mathcal{A}(\cdot, g)$  and  $\hat{\mathcal{E}}^g$ . The first lemma ensures that the set of admissible control processes is stable under pasting and that we may concatenate elements of  $\mathcal{A}(\xi, g)$  along stopping times and partitions of our probability space.

**Lemma 3.1.** *Fix a generator  $g$  and a terminal condition  $\xi \in L^0$ .*

1. Let  $Z^1, Z^2 \in \mathcal{L}$  be admissible and  $\sigma \in \mathcal{T}$ . Then the pasted control process  $\bar{Z} = Z^1 1_{[0, \sigma]} + Z^2 1_{] \sigma, T]}$  is admissible.

2. Let  $((Y^n, Z^n)) \subset \mathcal{A}(\xi, g)$ ,  $\sigma \in \mathcal{T}$ ,  $(B^n) \subset \mathcal{F}_\sigma$  be a partition of  $\Omega$ , and  $M$  a martingale. Suppose  $Y_\sigma^1 1_{B^n} \geq Y_\sigma^n 1_{B^n}$ ,  $Z^1 1_{[0, \sigma]} = Z^n 1_{[0, \sigma]}$ , and  $\int Z^n dW \geq M$ , for all  $n \in \mathbb{N}$ . Then  $(\bar{Y}, \bar{Z}) \in \mathcal{A}(\xi, g)$ , where

$$\bar{Y} = Y^1 1_{[0, \sigma[} + \sum_{n \geq 1} Y^n 1_{B^n} 1_{] \sigma, T]} \quad \text{and} \quad \bar{Z} = Z^1 1_{[0, \sigma]} + \sum_{n \geq 1} Z^n 1_{B^n} 1_{] \sigma, T]}. \quad (3.5)$$

*Proof.* 1. Let  $M^1, M^2, \bar{M}$  denote the stochastic integrals of  $Z^1, Z^2$  and  $\bar{Z}$ , respectively. From the admissibility of  $Z^1$  and  $Z^2$  follows

$$E[\bar{M}_t - \bar{M}_s | \mathcal{F}_s] = E[M_{(t \wedge \sigma) \vee s}^1 - M_s^1 | \mathcal{F}_s] + E[E[M_{t \vee \sigma}^2 - M_{s \vee \sigma}^2 | \mathcal{F}_{s \vee \sigma}] | \mathcal{F}_s] \leq 0,$$

for all  $0 \leq s \leq t \leq T$ . Thus  $\bar{Z}$  is admissible.

2. From  $(B^n)$  being a partition follows that  $P(\int_0^T \bar{Z}_s^2 ds < \infty) = 1$ , hence  $\bar{Z} \in \mathcal{L}$ , and  $\int_\sigma^T \bar{Z}_s dW_s = \sum 1_{B^n} \int_\sigma^T Z_s^n dW_s$ . Consequently,  $\int Z^n dW \geq M$ , for all  $n \in \mathbb{N}$ , implies

$$\int \bar{Z} dW = \sum_{n \geq 1} 1_{B^n} \int Z^n dW \geq M.$$

Since the continuous local martingale  $\int \bar{Z} dW$  is bounded from below by a martingale, by Fatou's lemma, it is a supermartingale. Thus  $\bar{Z}$  is admissible. Provided that  $Y_\sigma^1 1_{B^n} \geq Y_\sigma^n 1_{B^n}$ , for all  $n \in \mathbb{N}$ , it follows on the set  $\{s < \sigma \leq t\}$  that

$$\begin{aligned} Y_s^1 - \int_s^\sigma g_u(Y_u^1, Z_u^1) du + \int_s^\sigma Z_u^1 dW_u - \int_\sigma^t g_u(\bar{Y}_u, \bar{Z}_u) du + \int_\sigma^t \bar{Z}_u dW_u \\ \geq Y_\sigma^1 - \sum_{n \geq 1} 1_{B^n} \left( \int_\sigma^t g_u(Y_u^n, Z_u^n) du - \int_\sigma^t Z_u^n dW_u \right) \\ \geq \sum_{n \geq 1} 1_{B^n} \left( Y_\sigma^n - \int_\sigma^t g_u(Y_u^n, Z_u^n) du + \int_\sigma^t Z_u^n dW_u \right) \geq \sum_{n \geq 1} 1_{B^n} Y_t^n. \end{aligned}$$

Hence,

$$\bar{Y}_s - \int_s^t g_u(\bar{Y}_u, \bar{Z}_u) du + \int_s^t \bar{Z}_u dW_u \geq 1_{\{\sigma > t\}} Y_t^1 + \sum_{n \geq 1} 1_{B^n} (1_{\{\sigma \leq s\}} Y_t^n + 1_{\{s < \sigma \leq t\}} Y_t^n) = \bar{Y}_t$$

and thus  $(\bar{Y}, \bar{Z}) \in \mathcal{A}(\xi, g)$ . □

In the following, some properties of the generator are key, and therefore, we say that a generator  $g$  is

(POS) positive, if  $g(y, z) \geq 0$ , for all  $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$ .

(INC) increasing, if  $g(y, z) \geq g(y', z)$ , for all  $y, y' \in \mathbb{R}$  with  $y \geq y'$ , and all  $z \in \mathbb{R}^{1 \times d}$ .

(DEC) decreasing, if  $g(y, z) \leq g(y', z)$ , for all  $y, y' \in \mathbb{R}$  with  $y \geq y'$ , and all  $z \in \mathbb{R}^{1 \times d}$ .

The next proposition addresses how  $\mathcal{A}(\xi, g)$  depends on the terminal condition and the generator and which impact they have on  $\hat{\mathcal{E}}^g(\xi)$ . The first two properties are crucial in the proof of the existence and uniqueness theorem in Section 4. The third item concerns the monotonicity of  $\hat{\mathcal{E}}^g(\xi)$  with respect to  $\xi$  and  $g$ . Combined with the existence theorem, this yields in fact a comparison principle for minimal supersolutions of BSDEs. We will illustrate its scope in the proof of our stability results and in the example on utility maximization in Section 4.3. Finally, the last item concerns the cash (super/sub) additivity of the functional  $\hat{\mathcal{E}}^g(\xi)$ .

**Proposition 3.2.** *For  $t \in [0, T]$ , generators  $g, g'$  and terminal conditions  $\xi, \xi' \in L^0$ , it holds*

1. *the set  $\{Y_t : (Y, Z) \in \mathcal{A}(\xi, g)\}$  is directed downwards.*
2. *assuming (POS),  $\xi^- \in L^1$  and  $\mathcal{A}(\xi, g) \neq \emptyset$ , then for all  $\varepsilon > 0$ , there exists  $(Y^\varepsilon, Z^\varepsilon) \in \mathcal{A}(\xi, g)$  such that  $\hat{\mathcal{E}}_t^g(\xi) \geq Y_t^\varepsilon - \varepsilon$ .*
3. *(monotonicity) if  $\xi' \leq \xi$  and  $g'(y, z) \leq g(y, z)$ , for all  $y, z \in \mathbb{R} \times \mathbb{R}^{1 \times d}$ , then  $\mathcal{A}(\xi', g') \supset \mathcal{A}(\xi, g)$  and  $\hat{\mathcal{E}}_t^{g'}(\xi') \leq \hat{\mathcal{E}}_t^g(\xi)$ .*
4. *(convexity) if  $(y, z) \mapsto g(y, z)$  is jointly convex, then  $\mathcal{A}(\lambda\xi + (1 - \lambda)\xi', g) \supset \lambda\mathcal{A}(\xi, g) + (1 - \lambda)\mathcal{A}(\xi', g)$ , for all  $\lambda \in (0, 1)$ , and so*

$$\hat{\mathcal{E}}_t^g(\lambda\xi + (1 - \lambda)\xi') \leq \lambda\hat{\mathcal{E}}_t^g(\xi) + (1 - \lambda)\hat{\mathcal{E}}_t^g(\xi').$$

5. *for  $m \in L_t^0$ ,*

- *(cash superadditivity) assuming (INC) and  $m \geq 0$ , then  $\hat{\mathcal{E}}_t^g(\xi + m) \geq \hat{\mathcal{E}}_t^g(\xi) + m$ .*
- *(cash subadditivity) assuming (DEC),  $m \geq 0$ , and the existence of  $(Y, Z) \in \mathcal{A}(\xi, g)$ , such that  $\mathcal{A}_t(Y_t + m, g) \neq \emptyset$ , then  $\hat{\mathcal{E}}_t^g(\xi + m) \leq \hat{\mathcal{E}}_t^g(\xi) + m$ .*
- *(cash additivity) assuming that  $g$  does not depend on  $y$ , the existence of  $(Y, Z) \in \mathcal{A}(\xi, g)$ , such that  $\mathcal{A}_t(Y_t + m^+, g) \neq \emptyset$ , and the existence of  $(Y, Z) \in \mathcal{A}(\xi + m, g)$ , such that  $\mathcal{A}_t(Y_t + m^-, g) \neq \emptyset$ , then  $\hat{\mathcal{E}}_t^g(\xi + m) = \hat{\mathcal{E}}_t^g(\xi) + m$ .*

*Proof.* 1. Given  $(Y^i, Z^i) \in \mathcal{A}(\xi, g)$ , for  $i = 1, 2$ , we have to construct  $(\bar{Y}, \bar{Z}) \in \mathcal{A}(\xi, g)$ , such that  $\bar{Y}_t \leq \min\{Y_t^1, Y_t^2\}$ . To this end, we define the stopping time

$$\tau = \inf\{s > t : Y_s^1 > Y_s^2\} \wedge T$$

and set  $\bar{Y} = Y^1 1_{[0, \tau[} + Y^2 1_{[\tau, T]}$ ,  $\bar{Y}_T = \xi$ , and  $\bar{Z} = Z^1 1_{[0, \tau[} + Z^2 1_{[\tau, T]}$ . Since  $Y_\tau^1 \geq Y_\tau^2$ , Lemma 3.1 yields  $(\bar{Y}, \bar{Z}) \in \mathcal{A}(\xi, g)$  and by definition holds  $\bar{Y}_t = \min\{Y_t^1, Y_t^2\}$ .

2. In view of the first assertion, there exists a sequence  $((\tilde{Y}^n, \tilde{Z}^n)) \subset \mathcal{A}(\xi, g)$  such that  $(\tilde{Y}_t^n)$  decreases to  $\hat{\mathcal{E}}_t^g(\xi)$ . Set  $Y^n = \tilde{Y}^1 1_{[0, t)} + \tilde{Y}^n 1_{[t, T]}$  and  $Z^n = \tilde{Z}^1 1_{[0, t)} + \tilde{Z}^n 1_{[t, T]}$ . From Lemma 3.1 follows that  $((Y^n, Z^n)) \subset \mathcal{A}(\xi, g)$  and  $(Y_t^n)$  decreases to  $\hat{\mathcal{E}}_t^g(\xi)$  by construction. Relation (3.1),  $Y_0^n = Y_0^1$ , and  $g$  positive yield

$$\int_0^t Z_s^n dW_s \geq \xi - \int_t^T Z_s^n dW_s + \int_0^T g_s(Y_s^n, Z_s^n) ds - Y_0^n \geq -\xi^- - \int_t^T Z_s^n dW_s - Y_0^1.$$

Taking conditional expectation with respect to  $\mathcal{F}_t$  and using the supermartingal property of  $\int Z^n dW$  yield

$$\int_0^t Z_s^n dW_s \geq -E[\xi^- | \mathcal{F}_t] - Y_0^1.$$

Given  $\varepsilon > 0$ , since  $\mathcal{A}(\xi, g) \neq \emptyset$ , the sets  $B^n = A^n \setminus A^{n-1} \in \mathcal{F}_t$ , where  $A^n = \{\hat{\mathcal{E}}_t^g(\xi) \geq Y_t^n - \varepsilon\}$  and  $A^0 = \emptyset$ , form a partition of  $\Omega$ . Since  $(Y_t^n)$  is decreasing, it follows that  $Y_t^1 1_{B^n} \geq Y_t^n 1_{B^n}$ , for all  $n \in \mathbb{N}$ . Consequently, by means of Lemma 3.1,  $(\bar{Y}, \bar{Z})$ , defined as in (3.5), is an element of  $\mathcal{A}(\xi, g)$  and  $\hat{\mathcal{E}}_t^g(\xi) \geq \bar{Y}_t - \varepsilon$  by construction.

3. Straightforward inspection.

4. The joint convexity of  $g$  yields  $(\lambda Y + (1 - \lambda)Y', \lambda Z + (1 - \lambda)Z') \in \mathcal{A}(\lambda\xi + (1 - \lambda)\xi', g)$ , for all  $(Y, Z) \in \mathcal{A}(\xi, g)$ , all  $(Y', Z') \in \mathcal{A}(\xi', g)$ , and all  $\lambda \in (0, 1)$ . Hence,  $\lambda\mathcal{A}(\xi, g) + (1 - \lambda)\mathcal{A}(\xi', g) \subset \mathcal{A}(\lambda\xi + (1 - \lambda)\xi', g)$  and in particular,  $\hat{\mathcal{E}}_t^g(\lambda\xi + (1 - \lambda)\xi') \leq \lambda\hat{\mathcal{E}}_t^g(\xi) + (1 - \lambda)\hat{\mathcal{E}}_t^g(\xi')$ .

5. Let us show the cash superadditivity. For  $m \in L_0^0$  with  $m \geq 0$ , given  $(Y, Z) \in \mathcal{A}(\xi + m, g)$ , and  $0 \leq s \leq t' \leq T$ , it follows from (3.1) and (INC) that

$$\begin{aligned} Y_s - m1_{[t, T]}(s) - \int_s^{t'} g_u(Y_u - m1_{[t, T]}(u), Z_u) du + \int_s^{t'} Z_u dW_u \\ \geq Y_s - m1_{[t, T]}(s) - \int_s^{t'} g_u(Y_u, Z_u) du + \int_s^{t'} Z_u dW_u \geq Y_{t'} - m1_{[t, T]}(t'). \end{aligned}$$

Hence,  $(Y - m1_{[t, T]}, Z) \in \mathcal{A}(\xi, g)$  and thus  $\hat{\mathcal{E}}_t^g(\xi + m) - m \geq \hat{\mathcal{E}}_t^g(\xi)$ . For the cash subadditivity the same argument yields

$$Y_s + m1_{[t, T]}(s) - \int_s^{t'} g_u(Y_u + m1_{[t, T]}(u), Z_u) du + \int_s^{t'} Z_u dW_u \geq Y_{t'} + m1_{[t, T]}(t'),$$

for all  $t \leq s \leq t' \leq T$ , and all  $(Y, Z) \in \mathcal{A}(\xi, g)$ . Given  $(\tilde{Y}, \tilde{Z}) \in \mathcal{A}_t(Y_t + m, g)$  our usual pasting argument yields  $(\bar{Y}, \bar{Z}) \in \mathcal{A}(\xi + m, g)$ , with  $Y_t + m = \bar{Y}_t$  and thus  $\hat{\mathcal{E}}_t^g(\xi) + m \geq \hat{\mathcal{E}}_t^g(\xi + m)$ . The cash additivity in case where  $g$  is independent of  $y$  follows from  $\hat{\mathcal{E}}_t^g(\xi) + m = \hat{\mathcal{E}}_t^g(\xi + m^+) - m^- = \hat{\mathcal{E}}_t^g(\xi + m + m^-) - m^- = \hat{\mathcal{E}}_t^g(\xi + m)$ , since (DEC) and (INC) are simultaneously fulfilled.  $\square$

The following Lemma states that under the assumption of a positive generator the value process of a supersolution is a supermartingale. To view supersolutions as supermartingales is one of the key ideas in our approach. We will make ample use of the rich structure supermartingales provide throughout this work.

**Lemma 3.3.** *Let  $g$  be a generator fulfilling (POS), and  $\xi \in L^0$  be a terminal condition such that  $\xi^- \in L^1$ . Let  $(Y, Z) \in \mathcal{A}(\xi, g)$ . Then  $\xi \in L^1$ ,  $Y$  is a supermartingale,  $Z$  is unique and  $Y$  has the unique decomposition*

$$Y = Y_0 - A + M, \tag{3.6}$$

where  $M$  denotes the supermartingale  $\int Z dW$  and  $A$  is a predictable, increasing, càdlàg process with  $A_0 = 0$ .



*Proof.* Relation (3.1), positivity of  $g$ , admissibility of  $Z$  and  $\xi^- \in L^1$  imply  $E[|Y_t|] < +\infty$ , for all  $t \in [0, T]$ . Since  $-\xi^- \leq \xi \leq Y_T$ , we deduce that  $\xi \in L^1$ . Again, from (3.1), admissibility of  $Z$  and positivity of  $g$  we derive by taking conditional expectation, that  $Y_s \geq E[Y_t | \mathcal{F}_s]$ . Thus  $Y$  is a supermartingale with  $Y_t \geq E[\xi | \mathcal{F}_t]$ . Relation (3.1) implies further that there exist an increasing and càdlàg process  $K$ , with  $K_0 = 0$ , such that (3.6) holds with  $A = \int g(Y, Z)ds + K$ . Note that  $A$  is optional and therefore predictable due to the Brownian filtration, see [39, Corollary V.3.3]. Since  $Y$  is a càdlàg supermartingale the Doob-Meyer theorem, see [38, Theorem III.3.13], implies the unique decomposition  $Y = Y_0 + \tilde{M} - \tilde{A}$ , where  $\tilde{M}$  is a local martingale and  $\tilde{A}$  is an increasing process which is predictable, and  $\tilde{M}_0 = \tilde{A}_0 = 0$ . In our filtration every local martingale is continuous, see [38, Corollary IV.3.1, p. 187] and thus  $\tilde{A}$  is càdlàg. Hence  $A$  and  $\tilde{A}$  and  $M$  and  $\tilde{M}$  are indistinguishable. Moreover, from the predictable representation property of local martingales and from  $P(\bigcup_n \{\tau_n = T\}) = 1$ , for  $\tau^n = \inf\{t \geq 0 \mid |M_t| \geq n\} \wedge T$ , we obtain the  $P \otimes dt$ -almost sure uniqueness of  $Z$ .  $\square$

We now prove that for a positive generator  $\hat{\mathcal{E}}^g(\xi)$  is in fact a supermartingale, which, in addition, dominates its right hand limit process. This is crucial for the proof of the existence and uniqueness theorem.

**Proposition 3.4.** *Let  $g$  be a generator fulfilling (POS), and  $\xi \in L^0$  be a terminal condition such that  $\xi^- \in L^1$ . Suppose that  $\mathcal{A}(\xi, g) \neq \emptyset$ , then  $\hat{\mathcal{E}}^g(\xi)$  is a supermartingale,*

$$\mathcal{E}_s^g(\xi) := \lim_{t \downarrow s, t \in \mathbb{Q}} \hat{\mathcal{E}}_t^g(\xi), \quad \text{for all } s \in [0, T], \quad \mathcal{E}_T^g(\xi) := \xi,$$

is a well-defined càdlàg supermartingale, and

$$\hat{\mathcal{E}}_s^g(\xi) \geq \mathcal{E}_s^g(\xi), \quad \text{for all } s \in [0, T]. \quad (3.7)$$

*Proof.* Note first that  $\hat{\mathcal{E}}^g(\xi)$  is adapted by definition. Furthermore, given  $(Y, Z) \in \mathcal{A}(\xi, g) \neq \emptyset$ , Lemma 3.3 implies  $\xi \in L^1$  and  $Y_t \geq E[\xi | \mathcal{F}_t]$ . Hence  $Y_t \geq \hat{\mathcal{E}}_t^g(\xi) \geq E[\xi | \mathcal{F}_t]$  and  $\hat{\mathcal{E}}_t^g(\xi) \in L^1$ . As for the supermartingale property and (3.7), fix  $0 \leq s \leq t \leq T$ . In view of item 2 of Proposition 3.2, for all  $\varepsilon > 0$ , there exists  $(Y^\varepsilon, Z^\varepsilon) \in \mathcal{A}(\xi, g)$  such that  $\hat{\mathcal{E}}_s^g(\xi) \geq Y_s^\varepsilon - \varepsilon$ . Due to (3.1) it follows

$$\begin{aligned} \hat{\mathcal{E}}_t^g(\xi) &\leq Y_t^\varepsilon \leq Y_s^\varepsilon - \int_s^t g_u(Y_u^\varepsilon, Z_u^\varepsilon)du + \int_s^t Z_u^\varepsilon dW_u \\ &\leq \hat{\mathcal{E}}_s^g(\xi) - \int_s^t g_u(Y_u^\varepsilon, Z_u^\varepsilon)du + \int_s^t Z_u^\varepsilon dW_u + \varepsilon \leq \hat{\mathcal{E}}_s^g(\xi) + \int_s^t Z_u^\varepsilon dW_u + \varepsilon. \end{aligned} \quad (3.8)$$

Taking conditional expectation on both sides with respect to  $\mathcal{F}_s$  and the supermartingale property of  $\int Z^\varepsilon dW$  yields  $\hat{\mathcal{E}}_s^g(\xi) \geq E[\hat{\mathcal{E}}_t^g(\xi) | \mathcal{F}_s]$ , and so  $\hat{\mathcal{E}}^g(\xi)$  is a supermartingale. That  $\mathcal{E}^g(\xi)$  is well-defined càdlàg supermartingale follows from Karatzas and Shreve [28, Proposition 1.3.14]. Finally, (3.7) follows directly from (3.8) and the definition of  $\mathcal{E}^g(\xi)$ .  $\square$

*Remark 3.5.* The previous proposition suggests to consider the càdlàg supermartingale  $\mathcal{E}^g(\xi)$  as a candidate for the value process of the minimal supersolution. Note further that, if  $\mathcal{E}^g(\xi)$  is the value process of the minimal supersolution it is a modification of  $\hat{\mathcal{E}}^g(\xi)$  by definition. Hence, in the following, whenever we assume that  $\mathcal{E}^g(\xi)$  is minimal, if we make  $P$ -almost sure statements with regard to fixed deterministic times, we may write either of them.  $\blacklozenge$

The final result of this Section shows that our setup allows to derive various properties that are important in the context of non-linear expectations and dynamic risk measures. In particular, we prove that  $\mathcal{E}^g(\xi)$ , if it is the value process of the minimal supersolution, fulfills the flow-property and, under the additional assumption  $g(y, 0) = 0$ , for all  $y \in \mathbb{R}$ , we show projectivity, with time-consistency as a special case. In the context of BSDE solutions such properties were first established in [34], for the case of Lipschitz generators. For dynamic risk measures the (strong) time-consistency has been investigated in discrete time in [8, 21] as well as in continuous time in [3, 10], for instance.

**Proposition 3.6.** *For  $t \in [0, T]$ , generator  $g$  and terminal condition  $\xi \in L^0$ , it holds*

1.  $\hat{\mathcal{E}}_{s,T}^g(\xi) \leq \hat{\mathcal{E}}_{s,t}^g(\hat{\mathcal{E}}_{t,T}^g(\xi))$ , for all  $0 \leq s \leq t$ . Suppose that  $\mathcal{E}^g(\xi)$  is a minimal supersolution, then the flow-property holds, that is

$$\mathcal{E}_{s,T}^g(\xi) = \mathcal{E}_{s,t}^g(\mathcal{E}_{t,T}^g(\xi)), \quad \text{for all } 0 \leq s \leq t. \quad (3.9)$$

2. if  $g(y, 0) = 0$ , for all  $y \in \mathbb{R}$ , then  $\hat{\mathcal{E}}_s^g(\hat{\mathcal{E}}_t^g(\xi)) \leq \hat{\mathcal{E}}_s^g(\xi)$ , for all  $0 \leq s \leq t$ . Assuming (POS), and supposing that  $\mathcal{E}^g(\xi)$  is a minimal supersolution, then  $\mathcal{E}^g(\xi)$  is time-consistent, that is

$$\mathcal{E}_s^g(\mathcal{E}_t^g(\xi)) = \mathcal{E}_s^g(\xi), \quad \text{for all } 0 \leq s \leq t. \quad (3.10)$$

3. assuming (POS),  $g(y, 0) = 0$ , for all  $y \in \mathbb{R}$ , and  $\mathcal{E}^g(\xi)$  is a minimal supersolution, then the projectivity holds, that is

$$\mathcal{E}_s^g(1_A \mathcal{E}_t^g(\xi)) = \mathcal{E}_s^g(1_A \xi), \quad \text{for all } 0 \leq s \leq t \text{ and } A \in \mathcal{F}_t. \quad (3.11)$$

*Proof.* 1. Fix  $0 \leq s \leq t$ . Obviously,  $(Y_s, Z_s)_{s \in [0,t]} \in \mathcal{A}_t(\hat{\mathcal{E}}_{t,T}^g(\xi), g)$ , for all  $(Y, Z) \in \mathcal{A}_T(\xi, g)$ . Hence  $\hat{\mathcal{E}}_{s,t}^g(\hat{\mathcal{E}}_{t,T}^g(\xi)) \leq \hat{\mathcal{E}}_{s,T}^g(\xi)$ . Suppose now that  $\mathcal{E}_{s,T}^g(\xi)$  is a minimal supersolution with corresponding admissible control process  $\hat{Z} \in \mathcal{L}$ . For all  $(Y, Z) \in \mathcal{A}_t(\mathcal{E}_{t,T}^g(\xi), g)$ , holds  $Y_t \geq \mathcal{E}_{t,T}^g(\xi)$  and, with the same argumentation as in Lemma 3.1, we can paste in a monotone way to show that  $(\bar{Y}, \bar{Z}) \in \mathcal{A}_T(\xi, g)$ , where  $\bar{Y} = Y1_{[0,t]} + \mathcal{E}_{s,T}^g(\xi)1_{[t,T]}$  and  $\bar{Z} = Z1_{[0,t]} + \hat{Z}1_{[t,T]}$ . Thus, by definition,  $\mathcal{E}_{s,t}^g(\mathcal{E}_{t,T}^g(\xi)) \geq \mathcal{E}_{s,T}^g(\xi)$ .

2. Given  $(Y, Z) \in \mathcal{A}(\xi, g)$ , we define  $\bar{Y} = Y1_{[0,t]} + \hat{\mathcal{E}}_t^g(\xi)1_{[t,T]}$  and  $\bar{Z} = Z1_{[0,t]}$ . Since  $Y_t \geq \hat{\mathcal{E}}_t^g(\xi)$  and  $g(y, 0) = 0$ , it is straightforward to verify that  $(\bar{Y}, \bar{Z}) \in \mathcal{A}(\hat{\mathcal{E}}_t^g(\xi), g)$ . From  $Y_s \geq \bar{Y}_s$ , for all  $s \in [0, t]$ , follows that  $\hat{\mathcal{E}}_s^g(\hat{\mathcal{E}}_t^g(\xi)) \leq \hat{\mathcal{E}}_s^g(\xi)$ , for all  $s \in [0, t]$ . The case where  $\mathcal{E}^g(\xi)$  is a minimal supersolution and Assumption (POS) holds, follows from (3.11) for  $A = \Omega$ .

3. Fix  $A \in \mathcal{F}_t$ . Suppose that  $\mathcal{E}^g(\xi)$  is a minimal supersolution with corresponding control process  $\hat{Z}$ . Given  $(Y, Z) \in \mathcal{A}(1_A \mathcal{E}_t^g(\xi), g)$ , it follows from Proposition 3.4 below that  $Y_t \geq E[1_A \mathcal{E}_t^g(\xi) | \mathcal{F}_t] = 1_A \mathcal{E}_t^g(\xi)$ . Since  $g(y, 0) = 0$ , it is straightforward to check that  $\bar{Y} = Y1_{[0,t]} + \mathcal{E}_t^g(\xi)1_A 1_{[t,T]}$ , and  $\bar{Z} = Z1_{[0,t]}$  is such that  $(\bar{Y}, \bar{Z}) \in \mathcal{A}(1_A \mathcal{E}_t^g(\xi), g)$ . We can henceforth assume that  $Y_s = 1_A \mathcal{E}_t^g(\xi)$ , for all  $s \geq t$ . Now, we define  $\bar{Y} = Y1_{[0,t]} + \mathcal{E}^g(\xi)1_A 1_{[t,T]}$  and  $\bar{Z} = Z1_{[0,t]} + \hat{Z}1_A 1_{[t,T]}$ . For  $0 \leq s < t \leq t' \leq T$  holds

$$\begin{aligned} \bar{Y}_s - \int_s^{t'} g(\bar{Y}_u, \bar{Z}_u) du + \int_s^{t'} \bar{Z}_u dW_u &\geq Y_t + \left( - \int_t^{t'} g_u(\mathcal{E}_u^g(\xi), \hat{Z}_u) du + \int_t^{t'} \hat{Z}_u dW_u \right) 1_A \\ &\geq \left( \mathcal{E}_t^g(\xi) - \int_t^{t'} g_u(\mathcal{E}_u^g(\xi), \hat{Z}_u) du + \int_t^{t'} \hat{Z}_u dW_u \right) 1_A \geq \mathcal{E}_{t'}^g(\xi) 1_A. \end{aligned}$$

Hence, for all  $0 \leq s \leq t' \leq T$ , holds

$$\bar{Y}_s - \int_s^{t'} g(\bar{Y}_u, \bar{Z}_u) du + \int_s^{t'} \bar{Z}_u dW_u \geq Y_{t'} 1_{\{t' < t\}} + \mathcal{E}_{t'}^g(\xi) 1_A 1_{\{t \leq t'\}} = \bar{Y}_{t'}$$

and  $\bar{Y}_T = 1_A \xi$ , which implies that  $(\bar{Y}, \bar{Z}) \in \mathcal{A}(1_A \xi, g)$ . Since  $\bar{Y}_s = Y_s$ , for all  $s \leq t$ , we deduce  $\mathcal{E}_s^g(1_A \xi) \leq \mathcal{E}_s^g(1_A \mathcal{E}_t(\xi))$ .

On the other hand, consider  $(Y, Z) \in \mathcal{A}(1_A \xi, g)$ . From  $Y_t \geq E[1_A \xi | \mathcal{F}_t] = 1_A E[\xi | \mathcal{F}_t]$ , we obtain  $Y_t 1_{A^c} \geq 0$ . Since  $\mathcal{E}^g(\xi)$  is a minimal supersolution, it follows that  $Y_t \geq \mathcal{E}_t^g(\xi) 1_A$ . Indeed, let  $B = \{Y_t < \mathcal{E}_t^g(\xi) 1_A\}$ , then  $Y_t 1_{A^c} \geq 0$  implies  $B \subset A$ . Consequently, by similar arguments as in Lemma 3.1, the processes  $\tilde{Y} = \mathcal{E}^g(\xi)(1_{[0,t[} + 1_{B^c} 1_{[t,T]}) + Y 1_B 1_{[t,T]}$  and  $\tilde{Z} = \hat{Z}(1_{[0,t[} + 1_{B^c} 1_{[t,T]}) + Z 1_B 1_{[t,T]}$  are such that  $(\tilde{Y}, \tilde{Z}) \in \mathcal{A}(\xi, g)$ , which implies  $P[B] = 0$ . It is also straightforward to check that  $\tilde{Y} = Y 1_{[0,t[} + \mathcal{E}^g(\xi) 1_A 1_{[t,T]}$  and  $\tilde{Z} = Z 1_{[0,t[} + \hat{Z} 1_{[t,T]} 1_A$  are such that  $(\tilde{Y}, \tilde{Z}) \in \mathcal{A}(1_A \xi, g)$ . Thus we can assume that  $Y_t = 1_A \mathcal{E}_t^g(\xi)$ . Defining  $\bar{Y} = Y 1_{[0,t[} + \mathcal{E}_t^g(\xi) 1_A 1_{[t,T]}$  and  $\bar{Z} = Z 1_{[0,t[}$ , it holds  $(\bar{Y}, \bar{Z}) \in \mathcal{A}(1_A \mathcal{E}_t^g(\xi), g)$ . Thus  $\mathcal{E}_s^g(1_A \mathcal{E}_t^g(\xi)) \leq \mathcal{E}_s^g(1_A \xi)$ , since  $\bar{Y}_s = Y_s$ , for all  $s \leq t$ .  $\square$

## 4. Existence, Uniqueness and Stability

In this section, we give conditions, which guarantee the existence and uniqueness of a minimal supersolution. We show that the corresponding value process is given by  $\mathcal{E}^g(\xi)$ . Moreover, we analyze the stability of  $\hat{\mathcal{E}}^g(\xi)$  with respect to perturbations of the terminal condition or the generator. In addition to the assumptions (POS) and (INC) or (DEC) introduced above, we require convexity of  $g$  in the control variable. To that end, we say that a generator  $g$  is

(CON) convex, if  $g(y, \lambda z + (1 - \lambda) z') \leq \lambda g(y, z) + (1 - \lambda) g(y, z')$ , for all  $y \in \mathbb{R}$ , all  $z, z' \in \mathbb{R}^{1 \times d}$  and all  $\lambda \in (0, 1)$ .

### 4.1. Existence and Uniqueness of Minimal Supersolutions

The following theorem on existence and uniqueness of a minimal supersolution is the first main result of this paper. Note, that we require that  $\mathcal{A}(\xi, g) \neq \emptyset$ . In the context of finding minimal elements in some set this is quite a standard assumption, see [35] for an example in the context of minimal supersolutions. However, let us point out that in many applications  $\mathcal{A}(\xi, g) \neq \emptyset$  might be guaranteed by specific model assumptions, see for instance the example on utility maximization in Section 4.3. It might also be automatically granted under further assumptions, see Cheridito and Stadje [7], or for instance if the BSDE  $Y_t - \int_t^T g_s(Y_s, Z_s) ds + \int_t^T Z_s dW_s = \hat{\xi}$  has a solution  $(Y, Z) \in \mathcal{S} \times \mathcal{L}$ , such that  $Z$  is admissible. In the latter case,  $\mathcal{A}(\xi, g) \neq \emptyset$ , for all  $\xi \in L^0$  such that  $\xi^- \in L^1$ , with  $\hat{\xi} \geq \xi$ .

**Theorem 4.1.** *Let  $g$  be a generator fulfilling (POS), (CON) and either (INC) or (DEC) and  $\xi \in L^0$  be a terminal condition, such that  $\xi^- \in L^1$ . If  $\mathcal{A}(\xi, g) \neq \emptyset$ , then there exists a unique minimal supersolution  $(\hat{Y}, \hat{Z}) \in \mathcal{A}(\xi, g)$ . Moreover,  $\mathcal{E}^g(\xi)$  is the value process of the minimal supersolution, that is  $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$ .*

Note that Remark 3.5 implies that under the assumptions of Theorem 4.1 the process  $\mathcal{E}^g(\xi)$  is in fact a modification of  $\hat{\mathcal{E}}^g(\xi)$ .

*Proof. Step 1: Uniqueness.* Given  $\hat{Z} \in \mathcal{L}$  such that  $(\mathcal{E}^g(\xi), Z) \in \mathcal{A}(\xi, g)$ , the definition of  $\mathcal{E}^g(\xi)$  implies that for any other supersolution  $(Y, Z') \in \mathcal{A}(\xi, g)$  holds  $\mathcal{E}_t^g(\xi) \leq Y_t$ , for all  $t \in [0, T]$ . The uniqueness of  $\hat{Z}$  follows as in Lemma 3.3.

The remainder of the proof provides existence of  $\hat{Z} \in \mathcal{L}$  such that  $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$ .

*Step 2: Construction of an approximating sequence.* For any  $n, i \in \mathbb{N}$ , let  $t_i^n = iT/2^n$ . There exist  $((Y^n, Z^n)) \subset \mathcal{A}(\xi, g)$  such that

$$\hat{\mathcal{E}}_{t_i^n}^g(\xi) \geq Y_{t_i^n}^n - 1/n, \quad \text{for all } n \in \mathbb{N}, \text{ and all } i = 0, \dots, 2^n - 1, \quad (4.1)$$

and

$$Y_t^n \geq Y_t^{n+1}, \quad \text{for all } t \in [0, T], \text{ and all } n \in \mathbb{N}. \quad (4.2)$$

Indeed, by means of Proposition 3.2.2, for each  $n \in \mathbb{N}$ , we may select a family  $((Y^{n,i}, Z^{n,i}))_{i=0, \dots, 2^n-1}$  in  $\mathcal{A}(\xi, g)$ , such that  $\hat{\mathcal{E}}_{t_i^n}^g(\xi) \geq Y_{t_i^n}^{n,i} - 1/n$ ,  $i = 0, \dots, 2^n - 1$ . We suitably paste this family in order to obtain (4.1). We start with

$$\bar{Y}^{n,0} = Y^{n,0}, \quad \bar{Z}^{n,0} = Z^{n,0},$$

and continue by recursively setting, for  $i = 1, \dots, 2^n - 1$ ,

$$\begin{aligned} \bar{Y}^{n,i} &= \bar{Y}^{n,i-1} 1_{[0, \tau_i^n]} + Y^{n,i} 1_{[\tau_i^n, T]}, & \bar{Y}_T^{n,i} &= \xi, \\ \bar{Z}^{n,i} &= \bar{Z}^{n,i-1} 1_{[0, \tau_i^n]} + Z^{n,i} 1_{[\tau_i^n, T]}, \end{aligned}$$

where  $\tau_i^n$  are stopping times given by  $\tau_i^n = \inf\{t > t_i^n : \bar{Y}_t^{n,i-1} > Y_t^{n,i}\} \wedge T$ . From the definition of these stopping times and Lemma 3.1 follows that the pairs  $(\bar{Y}^{n,i}, \bar{Z}^{n,i})$ ,  $i = 0, \dots, 2^n - 1$ , are elements of  $\mathcal{A}(\xi, g)$ . Hence, the sequence

$$((Y^n, Z^n) := (\bar{Y}^{n,2^n-1}, \bar{Z}^{n,2^n-1}))$$

fulfills (4.1) by construction. Note that  $((Y^n, Z^n))$  is not necessarily monotone in the sense of (4.2). However, this can be achieved by pasting similarly. More precisely, we choose

$$\bar{Y}^1 = Y^1, \quad \bar{Z}^1 = Z^1,$$

and continue by recursively setting, for  $n \in \mathbb{N}$ ,

$$\begin{aligned} \bar{Y}^n &= \sum_{i=0}^{2^n-1} \left( Y^n 1_{[t_i^n, \tau_i^n]} + \bar{Y}^{n-1} 1_{[\tau_i^n, t_{i+1}^n]} \right), & \bar{Y}_T^n &= \xi, \\ \bar{Z}^n &= \sum_{i=0}^{2^n-1} \left( Z^n 1_{[t_i^n, \tau_i^n]} + \bar{Z}^{n-1} 1_{[\tau_i^n, t_{i+1}^n]} \right), \end{aligned}$$

where  $\tau_i^n$  are stopping times given by  $\tau_i^n = \inf\{t > t_i^n : Y_t^n > \bar{Y}_t^{n-1}\} \wedge t_{i+1}^n$ , for  $i = 0, \dots, 2^n - 1$ . By construction  $((\bar{Y}^n, \bar{Z}^n))$  fulfills both (4.1) and (4.2), and  $((\bar{Y}^n, \bar{Z}^n)) \subset \mathcal{A}(\xi, g)$  with Lemma 3.1.

*Step 3: Bound on  $\int Z^n dW$ .* We now take the sequence  $((Y^n, Z^n))$  fulfilling (4.1) and (4.2) and provide an inequality which will enable us to use compactness arguments for  $(Z^n)$  later in the proof. More precisely, we argue that, for all  $n \in \mathbb{N}$ , holds

$$\left| \int_0^t Z_s^n dW_s \right| \leq B_t^n := |Y_t^1| + E[\xi^- | \mathcal{F}_t] + E[\xi^-] + |Y_0^1| + A_t^n, \quad (4.3)$$

for all  $t \in [0, T]$ , where  $A_t^n$  is the positive increasing process defined in Lemma 3.3. Moreover, it holds

$$E[A_T^n] \leq Y_0^1 - E[\xi].$$

Indeed, by the same arguments as in the proof of Lemma 3.2.2, recall  $Y_0^n \leq Y_0^1$ , follows

$$\int_0^t Z_s^n dW_s \geq -E[\xi^- | \mathcal{F}_t] - Y_0^1. \quad (4.4)$$

On the other hand, from  $Y_t^n \leq Y_t^1$  and  $-Y_0^n \leq E[\xi^-]$ , recall Lemma 3.3, it follows

$$\int_0^t Z_s^n dW_s \leq Y_t^1 + A_t^n - Y_0^n \leq Y_t^1 + A_t^n + E[\xi^-]. \quad (4.5)$$

Combining (4.5) and (4.4) yields (4.3). The  $L^1$  bound on  $A^n$  follows from  $Y_0^n - A_T^n + \int_0^T Z_s^n dW_s = \xi$ ,  $Y_0^1 \geq Y_0^n$ , and the supermartingale property of  $\int Z^n dW$ .

Note that if  $(B_T^{n,*})$  in (4.3) were bounded in  $L^1$ , then, by means of the Burkholder-Davis-Gundy inequality,  $(Z^n)$  would be a bounded sequence in  $\mathcal{L}^1$  and we could apply [12, Theorem A] to find a sequence in the asymptotic convex hull of  $(Z^n)$ , which converges in  $\mathcal{L}^1$  and  $P \otimes dt$ -almost surely along some localizing sequence of stopping times to some limit  $Z \in \mathcal{L}^1$ . Here, even if  $(A_T^{n,*}) = (A_T^n)$  is uniformly bounded, this is however not necessarily the case for  $Y_T^{1,*}$  and  $(E[\xi^- | \mathcal{F}])_T^*$ , and this is the reason why we introduce the following localization.

*Step 4: First localization.* Due to our Brownian setting and since  $\xi^- \in L^1$ , we know that the martingale  $E[\xi^- | \mathcal{F}]$ , has a continuous version, see [39, Theorem V.3.5]. Moreover,  $Y^1$  is a càdlàg supermartingale and thus we may take the localising sequence

$$\sigma_k = \inf \{t > 0 : |Y_t^1| + E[\xi^- | \mathcal{F}_t] > k\} \wedge T, \quad k \in \mathbb{N}, \quad (4.6)$$

which is independent of  $n \in \mathbb{N}$ . For a fixed  $k \in \mathbb{N}$ , Inequality (4.3) yields

$$\left( \int Z^n 1_{[0, \sigma_k]} dW \right)_T^* \leq B^{k,n}, \quad \text{for all } n \in \mathbb{N}, \quad (4.7)$$

where  $B^{k,n} = |Y_0^1| + E[\xi^-] + k + A_T^n$ . Due to  $E[A_T^n] \leq Y_0^1 - E[\xi]$  we have

$$\sup_{n \in \mathbb{N}} E[B^{k,n}] < \infty. \quad (4.8)$$

Since  $(B^{k,n})_{n \in \mathbb{N}}$  is a sequence of positive random variables we may apply [11, Lemma A1.1]. It provides a sequence  $(\tilde{B}^{k,n})_{n \in \mathbb{N}}$  in the asymptotic convex hull of  $(B^{k,n})_{n \in \mathbb{N}}$ , which converges almost surely to a random variable  $\tilde{B}^k \geq 0$ . The  $\tilde{B}^{k,n}$  inherit the integrability of the  $B^{k,n}$  and we can conclude with Fatou's lemma that

$$E[\tilde{B}^k] < \infty. \quad (4.9)$$

Let  $\tilde{Z}^{k,n}$  be the convex combination of  $(Z^n)$  corresponding to  $\tilde{B}^{k,n}$  so that

$$\left( \int \tilde{Z}^{k,n} 1_{[0, \sigma_k]} dW \right)_T^* \leq \tilde{B}^{k,n}, \quad \text{for all } n \in \mathbb{N}. \quad (4.10)$$

*Step 5: Second localization.* The next two steps follow some known compactness arguments, which, in the case of  $\mathcal{L}^1$ , can be found in [12]. For the sake of completeness we develop the argumentation. Given an  $m \in \mathbb{N}$ , we start by taking a fast subsequence  $(\tilde{B}^{k,m,n})_{n \in \mathbb{N}}$  of  $(\tilde{B}^{k,n})_{n \in \mathbb{N}}$  converging in probability to  $\tilde{B}^k$ . More precisely, we choose  $(\tilde{B}^{k,m,n})_{n \in \mathbb{N}}$  such that

$$P \left[ \left| \tilde{B}^{k,m,n} - \tilde{B}^k \right| \geq 1 \right] \leq \frac{2^{-n}}{m}. \quad (4.11)$$

Consider now the stopping time  $\tau^{k,m}$  given by

$$\tau^{k,m} = \inf \left\{ t \geq 0 : \left( \int \tilde{Z}^{k,m,n} 1_{[0,\sigma_k]} dW \right)_t^* \geq m, \text{ for some } n \in \mathbb{N} \right\} \wedge T, \quad (4.12)$$

where the sequence  $(\tilde{Z}^{k,m,n} 1_{[0,\sigma_k]})_{n \in \mathbb{N}}$  is the subsequence of  $(\tilde{Z}^{k,n} 1_{[0,\sigma_k]})_{n \in \mathbb{N}}$  corresponding to the fast subsequence  $(\tilde{B}^{k,m,n})_{n \in \mathbb{N}}$ . The definition of  $\tau^{k,m}$  as well as the Burkholder-Davis-Gundy inequality imply that the sequence of processes

$(\tilde{Z}^{k,m,n} 1_{[0,\sigma_k]} 1_{[0,\tau^{k,m}]})_{n \in \mathbb{N}}$  is bounded in  $\mathcal{L}^2$ . The Alaoglu-Bourbaki theorem and the Eberlein-Šmulian theorem in the Banach space  $\mathcal{L}^2$  imply the existence of  $\hat{Z}^{k,m} \in \mathcal{L}^2$ , such that, up to a subsequence,  $(\tilde{Z}^{k,m,n} 1_{[0,\sigma_k]} 1_{[0,\tau^{k,m}]})_{n \in \mathbb{N}}$  converges weakly to  $\hat{Z}^{k,m}$ . As a consequence of the Hahn-Banach theorem, there exists a sequence in the asymptotic convex hull of  $(\tilde{Z}^{k,m,n} 1_{[0,\sigma_k]} 1_{[0,\tau^{k,m}]})_{n \in \mathbb{N}}$ , again denoted with  $(\tilde{Z}^{k,m,n} 1_{[0,\sigma_k]} 1_{[0,\tau^{k,m}]})_{n \in \mathbb{N}}$ , which converges in  $\mathcal{L}^2$  to  $\hat{Z}^{k,m}$ . By taking another subsequence we also have the  $P \otimes dt$ -almost sure convergence.

*Step 6:*  $(\tau^{k,m})_{m \in \mathbb{N}}$  is a localizing sequence of stopping times. We estimate as follows

$$\begin{aligned} P[\tau^{k,m} = T] &= P \left[ \left( \int \tilde{Z}^{k,m,n} 1_{[0,\sigma_k]} dW \right)_T^* < m, \text{ for all } n \in \mathbb{N} \right] \\ &\geq 1 - P \left[ \tilde{B}^{k,m,n} \geq m, \text{ for some } n \in \mathbb{N} \right] \\ &\geq 1 - P \left[ \left\{ \left| \tilde{B}^{k,m,n} - \tilde{B}^k \right| \geq 1, \text{ for some } n \in \mathbb{N} \right\} \cup \left\{ \tilde{B}^k > m - 1 \right\} \right] \\ &\geq 1 - \sum_n P \left[ \left| \tilde{B}^{k,m,n} - \tilde{B}^k \right| \geq 1 \right] - P \left[ \tilde{B}^k > m - 1 \right] \\ &\geq 1 - \frac{1}{m} - \frac{E \left[ \tilde{B}^k + 1 \right]}{m} \xrightarrow{m \rightarrow \infty} 1, \end{aligned}$$

where we used (4.10) in the second line and (4.11), the Markov inequality and the fact that  $E[\tilde{B}^k] < \infty$  in the last one.

*Step 7: Construction of the candidate  $\hat{Z}$ .* For given  $k, m > 0$ , we constructed in Step 5 the process  $\hat{Z}^{k,m}$  as the  $\mathcal{L}^2$  and  $P \otimes dt$ -almost sure limit of a sequence in the asymptotic convex hull of  $(\tilde{Z}^{k,m,n} 1_{[0,\sigma_k]} 1_{[0,\tau^{k,m}]})_{n \in \mathbb{N}}$ . With  $(\tilde{B}^{k,m,n})_{n \in \mathbb{N}}$  we denote the corresponding subsequence of convex combinations of  $(\tilde{B}^{k,m,n})_{n \in \mathbb{N}}$  and note that  $(\int \tilde{Z}^{k,m,n} 1_{[0,\sigma_k]} dW)_T^* \leq \tilde{B}^{k,m,n}$ , for all  $n \in \mathbb{N}$ , as in (4.10). Hence, by the same procedure as in Step 5, we can find, for  $m' > m$ , a fast subsequence  $(\tilde{Z}^{k,m',n} 1_{[0,\sigma_k]})_{n \in \mathbb{N}}$  in the asymptotic convex hull of  $(\tilde{Z}^{k,m,n} 1_{[0,\sigma_k]})_{n \in \mathbb{N}}$  and a  $\hat{Z}^{k,m'} \in \mathcal{L}^2$  such that  $(\tilde{Z}^{k,m',n} 1_{[0,\sigma_k]} 1_{[0,\tau^{k,m'}]})_{n \in \mathbb{N}}$  converges in  $\mathcal{L}^2$  and  $P \otimes dt$ -almost surely to  $\hat{Z}^{k,m'}$ . We iterate this procedure and define  $(\tilde{Z}^{k,n})_{n \in \mathbb{N}}$  as the diagonal sequence  $\tilde{Z}^{k,n} = \tilde{Z}^{k,n,n}$  and  $\hat{Z}^k$  as

$$Z_0^k = 0, \quad \hat{Z}^k = \sum_{m=1}^{\infty} \hat{Z}^{k,m} 1_{[\tau^{k,m-1}, \tau^{k,m}]}. \quad (4.13)$$

From  $\hat{Z}^{k,m'} 1_{[0,\tau^{k,m}]} = \hat{Z}^{k,m}$ , for  $m' > m$ , follows that  $(\tilde{Z}^{k,n} 1_{[0,\sigma_k]} 1_{[0,\tau^{k,n}]})_{n \in \mathbb{N}}$  converges in  $\mathcal{L}^2$  and  $P \otimes dt$ -almost surely to  $\hat{Z}^k$ . With the sequence  $(\tilde{Z}^{k,n})_{n \in \mathbb{N}}$  and the process  $\tilde{Z}^k$  at hand, we now diagonalize our program above with respect to  $k$  and  $n$ . As before, we get a diagonal sequence  $\tilde{Z}^n = \tilde{Z}^{n,n}$ , and a process  $\hat{Z}$  given by

$$\hat{Z}_0 = 0, \quad \hat{Z} = \sum_{k=1}^{\infty} 1_{[\sigma_{k-1}, \sigma_k]} \hat{Z}^k, \quad (4.14)$$

such that

$$\tilde{Z}^n 1_{[0,\tau_n]} \xrightarrow[n \rightarrow \infty]{P \otimes dt\text{-almost surely}} \hat{Z}, \quad (4.15)$$

for  $\tau_n = \sigma_n \wedge \tau^{n,n}$ , where  $\sigma_n$  and  $\tau^{n,n}$  are as in (4.6) and (4.12), respectively. For later reference, note that by construction holds  $\tilde{Z}^{k',m} 1_{[0,\sigma_k]} 1_{[0,\tau^{k,m}]} = \tilde{Z}^{k,m}$ , as soon as  $k' \geq k$  and also  $\tilde{Z} 1_{[0,\sigma_k]} 1_{[0,\tau^{k,m}]} = \tilde{Z}^{k,m}$ . Likewise  $(\tilde{Z}^n 1_{[0,\tau_l]})_{n \in \mathbb{N}}$  converges in  $\mathcal{L}^2$  and  $P \otimes dt$ -almost surely to  $\hat{Z}^{l,l}$ . This yields, via the Burkholder-Davis-Gundy inequality, up to a subsequence,

$$\int_0^{t \wedge \tau_l} \tilde{Z}_s^n dW_s \xrightarrow[n \rightarrow +\infty]{} \int_0^{t \wedge \tau_l} \hat{Z}_s dW_s, \quad \text{for all } t \in [0, T], P\text{-almost surely.} \quad (4.16)$$

Hence, diagonalizing yields

$$\int_0^t \tilde{Z}_s^n dW_s \xrightarrow[n \rightarrow +\infty]{} \int_0^t \hat{Z}_s dW_s, \quad \text{for all } t \in [0, T], P\text{-almost surely.} \quad (4.17)$$

*Step 8: Monotone convergence to  $\mathcal{E}^g(\xi)$ .* Let  $\tilde{Y}_t = \lim_n Y_t^n$ , for  $t \in [0, T]$ , be the pointwise monotone limit of the sequence  $(Y^n)$ . By monotone convergence  $\tilde{Y}$  is a supermartingale and, since our filtration is right-continuous, by standard arguments we may define the càdlàg supermartingale  $\hat{Y}$  by setting  $\hat{Y}_t = \lim_{s \downarrow t, s \in \mathbb{Q}} \tilde{Y}_s$ , for all  $t \in [0, T]$ , and  $\hat{Y}_T = \xi$ . By construction  $\hat{Y}_{t_n} = \hat{\mathcal{E}}_{t_n}^g(\xi)$ . Hence,  $\hat{Y}_t = \mathcal{E}_t^g(\xi)$ , for all  $t \in [0, T]$ , and

$$Y_t^n \geq \tilde{Y}_t \geq \hat{\mathcal{E}}_t^g(\xi) \geq \mathcal{E}_t^g(\xi) \geq E[\xi | \mathcal{F}_t], \quad (4.18)$$

where the third inequality follows from Proposition 3.4. Now, the process  $\mathcal{E}^g(\xi)$  is the natural candidate for the value process of the minimal supersolution for two reasons. It is càdlàg and it is dominated by  $\hat{\mathcal{E}}^g(\xi)$  as (4.18) shows. However, it is not clear a priori that the sequence  $(Y^n)$  converges to  $\mathcal{E}^g(\xi)$  in some suitable sense. Taking into account the additional structure provided by the supermartingale property of the  $Y^n$  we can prove nonetheless

$$\mathcal{E}^g(\xi) = \hat{Y} = \lim_{n \rightarrow \infty} Y^n, \quad P \otimes dt\text{-almost surely.} \quad (4.19)$$

Indeed, recall the decomposition  $Y_t^n = Y_0^n - A_t^n + M_t^n$ , for all  $t \in [0, T]$ , and the  $L^1$ -bound on  $(A_T^n)$  given in Step 3. We now consider the sequence  $((\tilde{Y}^n, \tilde{Z}^n))$  in the asymptotic convex hull of  $(Y^n, Z^n)$ , which corresponds to the sequence  $(\tilde{Z}^n)$  constructed in Step 7. From the decomposition of the  $Y^n$ , see Lemma 3.3, we obtain that  $\tilde{Y}_t^n = \tilde{Y}_0^n - \tilde{A}_t^n + \tilde{M}_t^n$ , for all  $t \in [0, T]$ . Moreover, the  $(\tilde{A}_T^n)$  inherit the  $L^1$ -bound given in Step 3. By means of Helly's theorem, see Lemma A.1, we obtain the existence of a subsequence  $(\tilde{A}^n)$  in the asymptotic convex hull of  $(\tilde{A}^n)$  and the existence of an increasing positive integrable process  $\tilde{A}$ , such that  $\lim_{k \rightarrow \infty} \tilde{A}_t^{n_k} = \tilde{A}_t$ , for all  $t \in [0, T]$ ,  $P$ -almost surely. Consequently, by

monotonicity of  $(Y^n)$  and (4.17),  $\tilde{Y}_t = \tilde{Y}_0 - \tilde{A}_t + \tilde{M}_t$ , for all  $t \in [0, T]$ . Hence, the jumps of  $\tilde{Y}$  are given by the countably many jumps of the increasing process  $\tilde{A}$ , which implies

$$\hat{Y}_t = \tilde{Y}_0 - \lim_{s \downarrow t, s \in \mathbb{Q}} \tilde{A}_s + \tilde{M}_t, \quad \text{for all } t \in [0, T], \quad \hat{Y}_T = \xi.$$

Moreover, the jumptimes of the càdlàg process  $\hat{Y}$  are exhausted by a sequence of stopping times  $(\rho^j) \subset \mathcal{T}$ , which coincide with the jumptimes of  $\tilde{A}$ . Therefore,  $\hat{Y} = \tilde{Y}$ ,  $P \otimes dt$ -almost surely, which implies (4.19).

*Step 9: Verification.* Let us now show that  $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$ , which, by means of (4.18), would end the proof. We start with the verification of (3.1) under the Assumption (INC). Due to (4.19) there exists a Lebesgue nullset  $\mathcal{I} \subset [0, T]$ , such that  $\mathcal{E}_t^g(\xi) = \lim_{n \rightarrow \infty} Y_t^n$ ,  $P$ -almost surely, for all  $t \in \mathcal{I}^c$ . Let  $s, t \in \mathcal{I}^c$  with  $s \leq t$ . By using (4.17), the  $P \otimes dt$ -almost sure convergence of  $\tilde{Z}^n 1_{[0, \tau^n]}$  to  $\hat{Z}$ , and Fatou's lemma we obtain

$$\begin{aligned} \mathcal{E}_s^g - \int_s^t g_u(\mathcal{E}_u^g, \hat{Z}_u) du + \int_s^t \hat{Z}_u dW_u \\ \geq \limsup_n \left( \tilde{Y}_s^n - \int_s^t g_u(\mathcal{E}_u^g, \tilde{Z}_u^n 1_{[0, \tau^n]}(u)) du + \int_s^t \tilde{Z}_u^n dW_u \right), \end{aligned} \quad (4.20)$$

where  $\tilde{Y}^n$  denotes the convex combination of  $(Y^n)$  corresponding to  $\tilde{Z}^n$ . We denote by  $\lambda_i^{(n)}$ ,  $n \leq i \leq M^{(n)}$ ,  $\lambda_i^{(n)} \geq 0$ ,  $\sum_i \lambda_i^{(n)} = 1$  the convex weights of  $\tilde{Z}^n$ . Since our generator fullfills (CON), and since, for  $n$  large enough, we have  $\tilde{Z}_u^n 1_{[0, \tau^n]}(u) = \tilde{Z}_u^n$ , for all  $s \leq u \leq t$ , we may further estimate the above by

$$\begin{aligned} \mathcal{E}_s^g - \int_s^t g_u(\mathcal{E}_u^g, \hat{Z}_u) du + \int_s^t \hat{Z}_u dW_u \\ \geq \limsup_n \sum_{i=n}^{M^{(n)}} \lambda_i^{(n)} \left( Y_s^i - \int_s^t g_u(\mathcal{E}_u^g, Z_u^i) du + \int_s^t Z_u^i dW_u \right). \end{aligned}$$

Since  $Y_t^i \geq \hat{\mathcal{E}}_t^g(\xi) \geq \mathcal{E}_t^g(\xi)$ , for all  $t \in [0, T]$ , and  $i \in \mathbb{N}$ , we use (INC) and the fact that the  $(Y^n, Z^n)$  are supersolutions to conclude

$$\begin{aligned} \mathcal{E}_s^g - \int_s^t g_u(\mathcal{E}_u^g, \hat{Z}_u) du + \int_s^t \hat{Z}_u dW_u \\ \geq \limsup_n \sum_{i=n}^{M^{(n)}} \lambda_i^{(n)} \left( Y_s^i - \int_s^t g_u(Y_u^i, Z_u^i) du + \int_s^t Z_u^i dW_u \right) \\ \geq \limsup_n \sum_{i=n}^{M^{(n)}} \lambda_i^{(n)} Y_t^i = \limsup_n \tilde{Y}_t^n = \limsup_n Y_t^n = \mathcal{E}_t^g. \end{aligned} \quad (4.21)$$

As for the case of  $s, t \in \mathcal{I}$ , with  $s \leq t$ , we approximate them both from the right by some sequences  $(s^n) \subset \mathcal{I}^c$  and  $(t^n) \subset \mathcal{I}^c$ , such that  $s^n \downarrow s$ ,  $t^n \downarrow t$ ,  $s^n \leq t^n$ . For each  $s^n$  and  $t^n$  holds (4.21). Passing to



the limit by using the right-continuity of  $\mathcal{E}^g$  and the continuity of  $-\int g(\mathcal{E}^g, \hat{Z})du + \int \hat{Z}dW$  we deduce that (4.21), holds for all  $s, t \in [0, T]$  with  $s \leq t$ .

It remains to show admissibility of  $\hat{Z}$ . By means of (4.21), (4.18), and positivity of  $g$  it holds

$$\int_0^t \hat{Z}_s dW_s \geq E[\xi | \mathcal{F}_t] - \mathcal{E}_0. \quad (4.22)$$

Being bounded from below by a martingale, the continuous local martingale  $\int \hat{Z}dW$  is by Fatou's lemma a supermartingale and thus  $\hat{Z}$  is admissible. Hence, the proof under Assumptions (POS), (CON) and (INC) is completed.

The proof under (DEC) replacing (INC) only differs in the verification of (3.1). Indeed, instead of only approximating  $\hat{Z}$  in the Lebesgue integral we approximate  $\mathcal{E}^g(\xi) P \otimes dt$ -almost surely with the sequence  $(Y^n)$  as well, that is (4.20) becomes, by means of (4.19) and Fatou's lemma,

$$\begin{aligned} \mathcal{E}_s^g - \int_s^t g_u(\mathcal{E}_u^g, \hat{Z}_u)du + \int_s^t \hat{Z}_u dW_u \\ \geq \limsup_n \left( \tilde{Y}_s^n - \int_s^t g_u(Y_u^n, \tilde{Z}_u^n 1_{[0, \tau^n]}(u))du + \int_s^t \tilde{Z}_u^n dW_u \right). \end{aligned}$$

This entails, by monotonicity of the sequence  $(Y^n)$  and the fact that the convex combinations in  $\tilde{Z}^n$  consist of elements of  $(Z^i)$  with index greater or equal than  $n$ , that we may write  $-\int_s^t g_u(Y_u^n, Z_u^i)du \geq -\int_s^t g_u(Y_u^i, Z_u^i)du$  in (4.21) and this ends the proof.  $\square$

*Remark 4.2.* Note that the existence theorem also holds if we additionally take into account a volatility process in the stochastic integral. More precisely, consider a progressively measurable process  $h : \Omega \times [0, T] \rightarrow \mathbb{S}_d^{>0}$ , where  $\mathbb{S}_d^{>0}$  denotes the set of strictly positive definite  $d \times d$  matrices and define  $\mathcal{L}^h$  as the set of progressively measurable processes  $Z : \Omega \times [0, T] \rightarrow \mathbb{R}^{1 \times d}$  such that  $Zh^{1/2} \in \mathcal{L}$ . Analogously to the previous setting, given a generator  $g$  and a terminal condition  $\xi \in L^0$ , we say that  $(Y, Z) \in \mathcal{S} \times \mathcal{L}^h$  is a supersolution of the BSDE under volatility  $h$ , if

$$Y_s - \int_s^t g_u(Y_u, Z_u) du + \int_s^t Z_u h_u^{1/2} dW_u \geq Y_t, \quad \text{and} \quad Y_T \geq \xi, \quad (4.23)$$

for all  $0 \leq s \leq t \leq T$ . We say that the control process is admissible, if  $\int Zh^{1/2}dW$  is a supermartingale, and define

$$\mathcal{A}(\xi, g, h) = \{(Y, Z) \in \mathcal{S} \times \mathcal{L}^h : Z \text{ is admissible and (4.23) holds}\}, \quad (4.24)$$

as well as

$$\hat{\mathcal{E}}_t^{g,h}(\xi) = \text{ess inf} \{Y_t : (Y, Z) \in \mathcal{A}(\xi, g, h)\}, \quad t \in [0, T]. \quad (4.25)$$

We can formulate the following existence theorem.

**Theorem 4.3.** *Let  $g$  be a generator fulfilling (POS), (CON) and either (INC) or (DEC) and  $\xi \in L^0$  be a terminal condition, such that  $\xi^- \in L^1$ . If  $\mathcal{A}(\xi, g, h) \neq \emptyset$ , then there exists a unique minimal supersolution  $(\hat{Y}, \hat{Z}) \in \mathcal{A}(\xi, g, h)$ . Moreover,  $\mathcal{E}^g(\xi)$  is the value process of the minimal supersolution, that is  $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g, h)$ .*

The proof follows exactly the same scheme as the proof of Theorem 4.1 with a compactness argument in the Hilbert space  $\mathcal{L}^{2,h}$ , the set of processes in  $\mathcal{L}^h$  such that  $E[\int_0^T Z_u^2 h_u du] < +\infty$ , instead of  $\mathcal{L}^2$ .  $\blacklozenge$

Theorem 4.1 ensures the existence and uniqueness of the minimal supersolution which is càdlàg. The following proposition provides a condition under which  $\mathcal{E}^g(\xi)$  is in fact continuous.

**Proposition 4.4.** *Let  $g$  be a generator fulfilling (POS), (CON) and either (INC) or (DEC) and  $\xi \in L^0$  be a terminal condition, such that  $\xi^- \in L^1$ . Suppose that  $\mathcal{A}(\xi, g) \neq \emptyset$ . Assume that for any  $\zeta \in L^\infty(\mathcal{F}_\tau)$ ,  $\tau \in \mathcal{T}$ , there exist  $Y \in \mathcal{S}$  and an admissible  $Z \in \mathcal{L}$ , which solve the backward stochastic differential equation*

$$Y_t - \int_t^\tau g_s(Y_s, Z_s) ds + \int_t^\tau Z_s dW_s = \zeta, \quad \text{for all } t \in [0, \tau].$$

Then  $\mathcal{E}^g(\xi)$  is continuous.

*Proof.* In view of Theorem 4.1, there exists  $\hat{Z} \in \mathcal{L}$  such that  $(\mathcal{E}^g, \hat{Z}) \in \mathcal{A}(\xi, g)$ . Hence,  $\mathcal{E}^g$  can only have negative jumps. Assume that  $\mathcal{E}^g$  has a negative jump, that is  $P[\tau \leq T] > 0$ , for the stopping time  $\tau = \inf\{t > 0 : \Delta \mathcal{E}_t^g < 0\}$ . We then fix  $m$  big enough such that the stopping time  $\tau^m = \inf\{t > 0 : |\mathcal{E}_t^g| > m\} \wedge \tau$  satisfies  $P[\{-m < \Delta \mathcal{E}_{\tau^m}^g < 0\} \cap \{\tau^m = \tau\}] > 0$ . Since  $\mathcal{E}^g$  is continuous on  $[0, \tau[$  and  $\mathcal{E}^g$  has only negative jumps,  $\mathcal{E}_{\tau^m}^g \vee -m \in L^\infty(\mathcal{F}_{\tau^m})$ . By assumption there exist  $\bar{Y} \in \mathcal{S}$  and an admissible  $\bar{Z} \in \mathcal{L}$  such that

$$\bar{Y}_s + \int_s^{\tau^m} g_u(\bar{Y}_u, \bar{Z}_u) - \int_s^{\tau^m} \bar{Z}_u dW_u = \mathcal{E}_{\tau^m}^g \vee -m, \quad \text{for all } s \in [0, \tau^m].$$

Similar to Lemma 3.1, we derive  $(\bar{Y}1_{[0, \tau^m[} + \mathcal{E}^g 1_{[\tau^m, T]}, \bar{Z}1_{[0, \tau^m]} + \hat{Z}1_{[\tau^m, T]}) \in \mathcal{A}(\xi, g)$ . Hence, by optimality of  $\mathcal{E}^g$  in  $\mathcal{A}(\xi, g)$  holds  $\mathcal{E}^g \leq \bar{Y}1_{[0, \tau^m[} + \mathcal{E}^g 1_{[\tau^m, T]}$ . Moreover, we have

$$\mathcal{E}_{\tau^m-}^g > \mathcal{E}_{\tau^m}^g \vee -m = \bar{Y}_{\tau^m} = \bar{Y}_{\tau^m-} \quad \text{on the set } \{-m < \Delta \mathcal{E}_{\tau^m}^g < 0\} \cap \{\tau^m = \tau\}.$$

Hence, for the stopping time  $\hat{\tau} = \inf\{t > 0 : \mathcal{E}_t^g > \bar{Y}_t\} \wedge \tau^m$  we deduce  $P[\hat{\tau} < \tau^m] > 0$ , since the processes  $\mathcal{E}^g$  and  $\bar{Y}$  are continuous on  $[0, \tau^m[$ . But then  $\mathcal{E}^g \not\leq \bar{Y}$  on  $[0, \tau^m[$ , which is a contradiction.  $\square$

Under the assumptions of Theorem 4.1,  $\mathcal{E}^g$  is the value process of the minimal supersolution with a control process  $\hat{Z}$  in  $\mathcal{L}$  which defines a supermartingale. Next we address the question whether and under which conditions some stronger assumptions on the control process can be obtained. More precisely, that the corresponding control process  $\hat{Z}$  belongs to some  $\mathcal{L}^p$ , for  $p \geq 1$ , and therefore defines a true martingale instead of a supermartingale. Defining

$$\mathcal{A}^p(\xi, g) := \{(Y, Z) \in \mathcal{A}(\xi, g) : Z \in \mathcal{L}^p\}, \quad (4.26)$$

this means that  $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}^p(\xi, g)$ . Peng [35] provides a positive answer to that question in the case where  $p = 2$ , the terminal condition  $\xi \in L^2$  and the generator is not necessarily positive but Lipschitz. Compare also with Cheridito and Stadje [7] for supersolutions of BSDEs where the control process is in BMO, if the terminal condition is a bounded lower semicontinuous function of the Brownian motion and the generator is convex in  $z$  and Lipschitz and increasing in  $y$ . Here, we provide an answer to the case where  $p = 1$  in the context of Section 3. Given a terminal condition  $\xi$ , obtaining  $\mathcal{E}^g(\xi)$  as a minimal solution with a control process within  $\mathcal{L}^1$  comes at two costs. Indeed, a stronger integrability condition on the terminal value is required, that is, we impose that  $(E[\xi^- | \mathcal{F}_T])^* \in L^1$ . As for the second cost,  $\mathcal{A}^1(\xi, g) \neq \emptyset$  is also required, which, in view of  $\mathcal{A}^1(\xi, g) \subset \mathcal{A}(\xi, g)$ , is also a stronger assumption.

**Theorem 4.5.** *Suppose that the generator  $g$  fulfills (POS), (CON) and either (INC) or (DEC). Let  $\xi \in L^0$  be a terminal condition, such that  $(E[\xi^- | \mathcal{F}])_T^* \in L^1$ . If  $\mathcal{A}^1(\xi, g) \neq \emptyset$ , then there exists a unique minimal supersolution  $(\hat{Y}, \hat{Z}) \in \mathcal{A}^1(\xi, g)$ . Moreover,  $\mathcal{E}^g(\xi)$  is the value process of the minimal supersolution, that is  $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}^1(\xi, g)$ .*

*Remark 4.6.* As in Section 3, note that for  $(Y, Z) \in \mathcal{A}^1(\xi, g)$ , the value process  $Y$  is a supermartingale with terminal value greater or equal than  $\xi$ . Moreover, we have  $Y_T^* \in L^1$ . Indeed, by using the decomposition (3.6), we derive  $Y_t^* \leq |Y_0| + A_T + (\int Z dW)_T^*$ . We further have  $A_T \leq Y_0 + \int_0^T Z_s dW_s - \xi$  and thus  $E[|A_T|] \leq Y_0 + E[\xi^-]$ . Consequently

$$E[Y_T^*] \leq |Y_0| + E[\xi^-] + Y_0 + E\left[\left(\int Z dW\right)_T^*\right] < \infty. \quad \blacklozenge$$

*Proof (of Theorem 4.5).* Since  $\mathcal{A}^1(\xi, g) \subset \mathcal{A}(\xi, g)$ , the assumption  $\mathcal{A}^1(\xi, g) \neq \emptyset$  implies the existence of  $\hat{Z} \in \mathcal{L}$  such that  $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$ . We are left to show that  $\hat{Z} \in \mathcal{L}^1$ . Since  $\mathcal{A}^1(\xi, g) \neq \emptyset$ , we can suppose in the proof of Theorem 4.1 that  $(Y^1, Z^1) \in \mathcal{A}^1(\xi, g)$ . Since (4.3) holds for  $(\mathcal{E}^g(\xi), \hat{Z})$ , instead of  $(Y^n, Z^n)$ , we have

$$\left(\int \hat{Z} dW\right)_T^* \leq |Y_0^1| + E[\xi^-] + \hat{A}_T + (Y^1)_T^* + (E[\xi^- | \mathcal{F}])_T^*, \quad \text{for all } n \in \mathbb{N}, \quad (4.27)$$

where  $0 \leq E[\hat{A}_T] \leq E[\xi] - Y_0^1$ . Since  $(E[\xi^- | \mathcal{F}])_T^* \in L^1$ , by means of Remark 4.6, the right hand side of (4.27), is in  $L^1$ . Thus, by means of the Burkholder-Davis-Gundy inequality,  $\hat{Z}$  belongs to  $\mathcal{L}^1$ .  $\square$

## 4.2. Stability Results

In this section we address the stability of  $\hat{\mathcal{E}}^g(\cdot)$  with respect to perturbations of the terminal condition or the generator. First we show that the functional  $\hat{\mathcal{E}}_0^g$  is not only defined on the same domain as the usual expectation, but also shares some of its main properties, such as Fatou's lemma as well as a monotone convergence theorem.

**Theorem 4.7.** *Suppose that the generator  $g$  fulfills (POS), (CON) and either (INC) or (DEC). Let  $(\xi^n)$  be a sequence in  $L^0$ , such that  $\xi^n \geq \eta$ , for all  $n \in \mathbb{N}$ , where  $\eta \in L^1$ .*

- *Monotone convergence:* If  $(\xi^n)$  is increasing  $P$ -almost surely to  $\xi \in L^0$ , then  $\hat{\mathcal{E}}_0^g(\xi) = \lim_n \hat{\mathcal{E}}_0^g(\xi^n)$ .
- *Fatou's lemma:*  $\hat{\mathcal{E}}_0^g(\liminf_n \xi^n) \leq \liminf_n \hat{\mathcal{E}}_0^g(\xi^n)$ .

*Proof. Monotone convergence:* From Proposition 3.2 and by monotonicity, it follows that  $\hat{\mathcal{E}}^g(\xi^n) \leq \hat{\mathcal{E}}^g(\xi^{n+1}) \leq \dots \leq \hat{\mathcal{E}}^g(\xi)$ . Hence, we may define  $\hat{Y}_0 = \lim_n \hat{\mathcal{E}}_0^g(\xi^n)$ . Note that  $\hat{Y}_0 \leq \hat{\mathcal{E}}_0^g(\xi)$ . If  $\hat{Y}_0 = +\infty$ , then also  $\hat{\mathcal{E}}_0^g(\xi) = +\infty$  and there is nothing to prove. Suppose now that  $\hat{Y}_0 < \infty$ . This implies that  $\mathcal{A}(\xi^n, g) \neq \emptyset$ , for all  $n \in \mathbb{N}$ . Since  $\xi^n \geq \eta$ , Proposition 3.4 yields  $(\xi^n) \subset L^1$  and  $(\mathcal{E}^g(\xi^n))$  is a well-defined increasing sequence of càdlàg supermartingales. We define  $Y_t = \lim_n \mathcal{E}_t^g(\xi^n)$ , for all  $t \in [0, T]$ . Note that  $Y_0 = \hat{Y}_0$ . We show that  $Y$  is a càdlàg supermartingale.

To this end, note that the sequence  $(\mathcal{E}^g(\xi^n) - \mathcal{E}^g(\xi^1))$  is positive and increases to  $Y - \mathcal{E}^g(\xi^1)$ . Therefore monotone convergence yields

$$0 \leq E[Y_t - \mathcal{E}_t^g(\xi^1)] = \lim_n E[\mathcal{E}_t^g(\xi^n) - \mathcal{E}_t^g(\xi^1)].$$

The supermartingale property of  $\mathcal{E}^g(\xi^n)$  implies that  $E[\mathcal{E}_t^g(\xi^n)] \leq \mathcal{E}_0^g(\xi^n) \leq Y_0$ . Furthermore,  $E[\xi^1] \leq E[\mathcal{E}_t^g(\xi^1)] \leq Y_0$  and thus

$$0 \leq E[Y_t - \mathcal{E}_t^g(\xi^1)] \leq -E[\xi^1] + Y_0 < +\infty.$$

From  $\mathcal{E}_t^g(\xi^1) \in L^1$ , we deduce that  $Y_t \in L^1$ . Since  $\xi = Y_T$ , this implies in particular that  $\xi \in L^1$ . The supermartingale property follows by a similar argument. Moreover, [15, Theorem VI.18] implies that  $Y$  is indistinguishable from a càdlàg process. Hence,  $Y$  is a càdlàg supermartingale.

Theorem 4.1 provides a sequence of optimal controls  $(Z^n)$  such that  $(\mathcal{E}^g(\xi^n), Z^n) \in \mathcal{A}(\xi^n, g)$ , for all  $n \in \mathbb{N}$ . Now we apply the procedure introduced in the proof of Theorem 4.1 and obtain a candidate control process  $\hat{Z}$ . The only notable difference in the proof, except for the fact that  $Y$  is already càdlàg, is that, here, the sequence  $(\mathcal{E}^g(\xi^n))$  is increasing instead of decreasing. Thus, the càdlàg supermartingales  $Y$  and  $\mathcal{E}^g(\xi^1)$  serve as upper and lower bound, respectively. Consequently, we replace  $Y^1$  by  $Y$  and  $E[\xi^- | \mathcal{F}_t]$  by  $\mathcal{E}^g(\xi^1)$  in the key Inequality (4.3). The verification follows exactly the same argumentation as in the proof of Theorem 4.1 for both monotonicity Assumptions (INC) and (DEC). Finally, to get the admissibility of  $\hat{Z}$  we denote with  $(\tilde{\xi}^n)$  the sequence of convex combinations of  $(\xi^n)$  corresponding to  $(\tilde{Z}^n)$ . Monotonicity of the sequence  $(\xi^n)$  implies  $\xi^1 \leq \tilde{\xi}^n \leq \xi$ , for all  $n \in \mathbb{N}$ . We may and do switch to a subsequence such that  $(\tilde{\xi}^n)$  is increasing as well. Now, fix an arbitrary  $t \in [0, T]$ . Dominated convergence implies the  $L^1$ -convergence  $\lim_n E[\tilde{\xi}^n | \mathcal{F}_t] = E[\xi | \mathcal{F}_t]$ . Hence, we may select a subsequence such that we have  $P$ -almost sure convergence. Similar to (4.22) this implies

$$Y_0 - \int_0^t g_u(Y_u, \hat{Z}_u) du + \int_0^t \hat{Z}_u dW_u \geq \limsup_n E[\tilde{\xi}^n | \mathcal{F}_t] = E[\xi | \mathcal{F}_t].$$

As before, this entails that  $(Y, \hat{Z}) \in \mathcal{A}(\xi, g)$ . Hence, from  $\mathcal{A}(\xi, g) \neq \emptyset$  and  $\xi^- \in L^1$  we derive by Theorem 4.1 that there exists a control process  $Z$  such that  $(\mathcal{E}^g(\xi), Z) \in \mathcal{A}(\xi, g)$ . In particular this yields  $Y_0 = \mathcal{E}_0^g(\xi)$ , that is  $\lim_n \mathcal{E}_0^g(\xi^n) = \mathcal{E}_0^g(\xi)$ , since otherwise  $\mathcal{E}_0^g(\xi)$  were not optimal.

*Fatou's lemma:* The result follows by applying monotone convergence. Indeed, denote by  $\zeta^n$  the random variables  $\zeta^n = \inf_{k \geq n} \xi^k$ . Then from  $\liminf_n \xi^n = \lim_n \zeta^n$ ,  $\zeta^n \geq \eta$ ,  $\zeta^n \leq \xi^n$ , for all  $n \in \mathbb{N}$ , and monotone convergence follows

$$\hat{\mathcal{E}}_0^g(\liminf_n \xi^n) = \hat{\mathcal{E}}_0^g(\lim_n \zeta^n) = \lim_n \hat{\mathcal{E}}_0^g(\zeta^n) \leq \liminf_n \hat{\mathcal{E}}_0^g(\xi^n). \quad \square$$

*Remark 4.8.* An inspection of the proof of Theorem 4.7 shows that under the assumptions implying monotone convergence, if  $\lim_n \hat{\mathcal{E}}_0^g(\xi^n) < +\infty$ , then  $\mathcal{A}(\xi, g) \neq \emptyset$  and  $\mathcal{E}_t^g(\xi^n)$  converges  $P$ -almost surely to  $\mathcal{E}_t^g(\xi)$ , for all  $t \in [0, T]$ .

Similarly, given a sequence  $((Y^n, Z^n)) \subset \mathcal{A}(\xi, g)$  such that  $(Y^n)$  is increasing and  $\lim_n Y_0^n < \infty$ , then there exists a control process  $Z \in \mathcal{L}$  such that  $(Y, Z) \in \mathcal{A}(\xi, g)$ , where  $Y_t$  is the  $P$ -almost sure limit of  $(Y_t^n)$ , for all  $t \in [0, T]$ .  $\blacklozenge$

A consequence of the preceding theorem is the following result on  $L^1$ -lower semicontinuity.

**Theorem 4.9.** *Let  $g$  be a generator fulfilling (POS), (CON) and either (INC) or (DEC). Then  $\hat{\mathcal{E}}_0^g$  is  $L^1$ -lower semicontinuous.*

*Proof.* Let  $(\xi^n)$  be a sequence of terminal conditions, which converges in  $L^1$  to a random variable  $\xi$ . Suppose that there exists a subsequence  $(\tilde{\xi}^n) \subset (\xi^n)$  such that  $(\hat{\mathcal{E}}_0^g(\tilde{\xi}^n))$  converges to some real  $a <$

$\hat{\mathcal{E}}_0^g(\xi)$ . We can assume, up to another fast subsequence, that  $\|\tilde{\xi}^n - \xi\|_{L^1} \leq 2^{-n}$ , for all  $n \in \mathbb{N}$ . Consider now the sequence  $(\zeta^n)$ , with  $\zeta^n$  given by

$$\zeta^n = \xi - \sum_{k \geq n} (\tilde{\xi}^k - \xi)^-.$$

Clearly,  $\zeta^n \in L^1$  and  $\zeta^n \leq \zeta^{n+1} \leq \dots \leq \xi$ . Moreover,  $(\zeta^n)$  converges in  $L^1$  to  $\xi$ , and, since it is increasing, it converges also  $P$ -almost surely. Thus, from Theorem 4.7, we get  $\lim_n \hat{\mathcal{E}}_0^g(\zeta^n) = \hat{\mathcal{E}}_0^g(\xi)$ . Now,  $\zeta^n \leq \xi - (\tilde{\xi}^n - \xi)^- + (\tilde{\xi}^n - \xi)^+ \leq \xi^n$  and monotony of the functional  $\hat{\mathcal{E}}_0^g$  imply  $a = \lim_n \hat{\mathcal{E}}_0^g(\xi^n) \geq \lim_n \hat{\mathcal{E}}_0^g(\zeta^n) = \hat{\mathcal{E}}_0^g(\xi)$ , which is a contradiction. Hence,  $\liminf_n \hat{\mathcal{E}}_0^g(\xi^n) \geq \hat{\mathcal{E}}_0^g(\xi)$ .  $\square$

The preceding results allows to derive a dual representation, by means of the Fenchel-Moreau theorem, of the functional  $\hat{\mathcal{E}}^g(\cdot)$  at time zero.

**Corollary 4.10.** *Let  $g$  be a generator fulfilling (POS) and either (INC) or (DEC). Assume that  $g$  is jointly convex in  $y$  and  $z$ . Then, either  $\hat{\mathcal{E}}_0^g \equiv +\infty$  or*

$$\hat{\mathcal{E}}_0^g(\xi) = \mathcal{E}_0^g(\xi) = \inf_{\nu \in L_+^\infty} \left\{ E[\nu \xi] - \left( \hat{\mathcal{E}}_0^g \right)^* (\nu) \right\}, \quad \xi \in L^1, \quad (4.28)$$

for the conjugate  $(\hat{\mathcal{E}}_0^g)^*(\nu) = \inf_{\xi \in L^1} \left\{ E[\nu \xi] - \hat{\mathcal{E}}_0^g(\xi) \right\}$ , where  $\nu \in L^\infty$ .

*Proof.* Since  $\hat{\mathcal{E}}_0^g > +\infty$  on  $L^1$ , either  $\hat{\mathcal{E}}_0^g \equiv +\infty$  or  $\hat{\mathcal{E}}_0^g$  is proper. In the latter case, in view of Proposition 3.2 and Theorem 4.9, the function  $\hat{\mathcal{E}}_0^g$  is convex and  $\sigma(L^1, L^\infty)$ -lower semicontinuous on  $L^1$ . Hence, the Fenchel Moreau theorem yields the dual representation (4.28). That the domain of  $(\hat{\mathcal{E}}_0^g)^*$  is concentrated on  $L_+^\infty$  follows from the monotonicity of  $\hat{\mathcal{E}}_0^g$ , see Proposition 3.2.  $\square$

*Remark 4.11.* Notice that, if the generator in Corollary 4.10 does not depend on  $y$ , then by Item 5 of Proposition 3.2 the operator  $\hat{\mathcal{E}}_0^g(\cdot)$  is translation invariant. Therefore, it is a lower semicontinuous, convex risk measure and the Representation (4.28) corresponds to the robust representation of lower semicontinuous, convex risk measures; see Föllmer and Schied [23].  $\blacklozenge$

Under additional integrability assumptions on the terminal condition we may also formulate stability results for supersolutions in the set  $\mathcal{A}^1(\xi, g)$  introduced in (4.26).

**Theorem 4.12.** *Suppose that the generator  $g$  fulfills (POS), (CON) and either (DEC) or (INC). Let  $(\xi^n)$  be a sequence in  $L^0$ , such that  $\xi^n \geq \eta$ , for all  $n \in \mathbb{N}$ , where  $(E[\eta | \mathcal{F}])_T^* \in L^1$ .*

- *Suppose  $(\xi^n)$  is increasing  $P$ -almost surely to  $\xi \in L^0$ , where  $(E[\xi^- | \mathcal{F}])_T^* \in L^1$  and  $\mathcal{A}^1(\xi, g) \neq \emptyset$ . Then  $\mathcal{E}_t^g(\xi) = \lim_n \mathcal{E}_t^g(\xi^n)$ ,  $P$ -almost surely, for all  $t \in [0, T]$ .*
- *Suppose  $(E[(\liminf_n \xi^n)^- | \mathcal{F}])_T^* \in L^1$  and  $\mathcal{A}^1(\liminf_n \xi^n, g) \neq \emptyset$ . Then  $\mathcal{E}_t^g(\liminf_n \xi^n) \leq \liminf_n \mathcal{E}_t^g(\xi^n)$ ,  $P$ -almost surely, for all  $t \in [0, T]$ .*

We omit the proof of the preceding theorem, as it is a simple adaptation of the proofs of Theorems 4.5 and 4.7. Note that Theorem 4.12 is a weaker version of Theorem 4.7. Indeed, here, given a sequence  $(\xi^n)$  increasing to  $\xi$ , we need to assume that  $\mathcal{A}^1(\xi, g)$  is not empty. The underlying reason being the lack of knowledge whether the limit process  $Y$ , defined in the proof of Theorem 4.7, fulfills  $Y_T^* \in L^1$ .

The theorem above allows to state the following result on  $\|\cdot\|_{L^1}$ -lower semicontinuity of  $\hat{\mathcal{E}}^g$ . Its proof is virtually the same as the proof of Theorem 4.9.

**Theorem 4.13.** *Suppose that the generator  $g$  fulfills (POS), (CON) and either (DEC) or (INC). Then  $\xi \mapsto \hat{\mathcal{E}}_0^g(\xi)$  is  $\|\cdot\|_{L^1}$ -lower semicontinuous on its domain, that is on*

$$\{\xi \in L^0 : (E[\xi^- | \mathcal{F}])_T^* \in L^1 \text{ and } \mathcal{A}^1(\xi, g) \neq \emptyset\}. \quad (4.29)$$

We conclude this section with a theorem on monotone stability with respect to the generator.

**Theorem 4.14.** *Let  $\xi \in L^0$  be a terminal condition, such that  $\xi^- \in L^1$ , and let  $(g^n)$  be an increasing sequence of generators, which converge pointwise to a generator  $g$ . Suppose that each generator fulfills (POS), (CON) and either (INC) or (DEC). Then  $\lim_n \hat{\mathcal{E}}_0^{g^n}(\xi) = \hat{\mathcal{E}}_0^g(\xi)$ . If, in addition,  $\lim_n \hat{\mathcal{E}}_0^{g^n}(\xi) < \infty$ , then  $\mathcal{A}(\xi, g) \neq \emptyset$  and  $\mathcal{E}_t^{g^n}(\xi)$  converges  $P$ -almost surely to  $\mathcal{E}_t^g(\xi)$ , for all  $t \in [0, T]$ .*

*Proof.* Note that from Proposition 3.2, we have  $\hat{\mathcal{E}}^{g^n}(\xi) \leq \hat{\mathcal{E}}^{g^{n+1}}(\xi) \leq \dots \leq \hat{\mathcal{E}}^g(\xi)$ . Hence, we may set  $\hat{Y}_0 = \lim_n \hat{\mathcal{E}}_0^{g^n}(\xi)$ . If  $\hat{Y}_0 = \infty$ , then also  $\hat{\mathcal{E}}_0^g(\xi) = \infty$  and we are done. Suppose that  $\hat{Y}_0 < \infty$ . By the same arguments as in the proof of Theorem 4.7, we construct a càdlàg supermartingale  $Y$ . With the same procedure as in Theorem 4.7, we construct the candidate  $\hat{Z}$ . It remains to show  $(Y, \hat{Z}) \in \mathcal{A}(\xi, g)$ . However, this can be done similarly as in the proof of Theorem 4.1. We only show how to obtain the analogue of (4.21). Note first that the pointwise convergence of the generators implies that  $(g^k(Y, \hat{Z}))$  converges  $P \otimes dt$ -almost surely to  $g(Y, \hat{Z})$ . Hence, Fatou's lemma yields

$$Y_s - \int_s^t g_u(Y_u, \hat{Z}_u) du + \int_s^t \hat{Z}_u dW_u \geq \limsup_k \left( Y_s - \int_s^t g_u^k(Y_u, \hat{Z}_u) du + \int_s^t \hat{Z}_u dW_u \right). \quad (4.30)$$

As in the previous proof, we use the expression in the bracket on the right hand side to obtain

$$\begin{aligned} Y_s - \int_s^t g_u^k(Y_u, \hat{Z}_u) du + \int_s^t \hat{Z}_u dW_u \\ \geq \limsup_n \sum_{i=n}^{M^{(n)}} \lambda_i^{(n)} \left( Y_s^i - \int_s^t g_u^k(Y_u^i, Z_u^i) du + \int_s^t Z_u^i dW_u \right). \end{aligned}$$

Since on the right hand side we consider the lim sup with respect to  $n$  and  $k$  being fixed for the moment, we may assume  $k \leq n$ , which entails by monotonicity of the sequence of generators

$$\begin{aligned} Y_s - \int_s^t g_u^k(Y_u, \hat{Z}_u) du + \int_s^t \hat{Z}_u dW_u \\ \geq \limsup_n \sum_{i=n}^{M^{(n)}} \lambda_i^{(n)} \left( Y_s^i - \int_s^t g_u^i(Y_u^i, Z_u^i) du + \int_s^t Z_u^i dW_u \right). \end{aligned}$$

From here, we obtain as before  $Y_s - \int_s^t g_u^k(Y_u, \hat{Z}_u) du + \int_s^t \hat{Z}_u dW_u \geq Y_t$ , where the right hand side does not depend on  $k$  anymore. Combined with (4.30), this yields the analogue of (4.21).  $\square$

### 4.3. Non positive generators

In this section we extend our results to generators that are not necessarily positive. This is important with regards to applications in mathematical finance, where the generators are quite often of the linear-quadratic type. It turns out that we can extend the scope of our theorems to cover precisely some of these

situations; see the end of this section for such an example. Using some measure change, the positivity assumption on the generator  $g$  can be relaxed to a linear bound below. This leads to optimal solutions under  $P$ , where the admissibility is required with respect to the related equivalent probability measure. More precisely, we say in the following that a generator  $g$  is

- (LB) linearly bounded from below, if there exist adapted measurable  $\mathbb{R}^{1 \times d}$  and  $\mathbb{R}$ -valued processes  $a$  and  $b$ , respectively, such that  $g(y, z) \geq az^\top + b$ , for all  $y, z \in \mathbb{R} \times \mathbb{R}^{1 \times d}$ . Furthermore,  $\int_0^t b_s ds \in L^1(P^a)$ , for all  $t \in [0, T]$  and

$$\frac{dP^a}{dP} = \mathcal{E} \left( \int adW \right)_T,$$

defines an equivalent probability measure  $P^a$ .

**Example 4.15.** For instance, given a generator  $g$ , assume that there exists a generator  $\hat{g}$  independent of  $y$  fulfilling (CON) and such that  $g \geq \hat{g}$ . Then, there exists an  $\mathbb{R}^{1 \times d}$ -valued adapted measurable process  $a$  such that  $g(y, z) \geq az^\top - \hat{g}^*(a)$ , for all  $y, z \in \mathbb{R} \times \mathbb{R}^{1 \times d}$ , where  $\hat{g}^*$  denotes the convex conjugate of  $g$ .  $\diamond$

In the following, we say that  $Z$  is  $a$ -admissible, if  $\int Z dW^a$  is a  $P^a$ -supermartingale, where  $W^a = (W^1 - \int a^1 ds, \dots, W^d - \int a^d ds)^\top$  is the respective Brownian motion under  $P^a$ . We are interested in the sets

$$\mathcal{A}^a(\xi, g) = \{(Y, Z) \in \mathcal{S} \times \mathcal{L} : Z \text{ is } a\text{-admissible and (3.1) holds}\}, \quad (4.31)$$

and define the random process

$$\hat{\mathcal{E}}_t^{g,a}(\xi) = \text{ess inf} \{Y_t \in L^0(\mathcal{F}_t) : (Y, Z) \in \mathcal{A}^a(\xi, g)\}, \quad t \in [0, T]. \quad (4.32)$$

The analogue of Theorem 4.1 is given as follows

**Theorem 4.16.** *Let  $g$  be a generator fulfilling (LB), (CON) and either (INC) or (DEC) and  $\xi \in L^0$  be a terminal condition, such that  $\xi^- \in L^1(P^a)$ . If  $\mathcal{A}^a(\xi, g) \neq \emptyset$ , then there exists a unique minimal supersolution  $(\hat{Y}, \hat{Z}) \in \mathcal{A}^a(\xi, g)$ . Moreover,  $\mathcal{E}^g(\xi)$  is the value process of the minimal supersolution, that is  $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}^a(\xi, g)$ .*

The analogues of Theorem 4.7 and Theorem 4.9 read as follows.

**Theorem 4.17.** *Suppose that the generator  $g$  fulfills (LB), (CON) and either (INC) or (DEC). Let  $(\xi^n)$  be a sequence in  $L^0$ , such that  $\xi^n \geq \eta$ , for all  $n \in \mathbb{N}$ , where  $\eta \in L^1(P^a)$ .*

- *Monotone convergence:* *If  $(\xi^n)$  is increasing  $P$ -almost surely to  $\xi \in L^0$ , then  $\hat{\mathcal{E}}_0^{g,a}(\xi) = \lim_n \hat{\mathcal{E}}_0^{g,a}(\xi^n)$ .*
- *Fatou's lemma:*  $\hat{\mathcal{E}}_0^{g,a}(\liminf_n \xi^n) \leq \liminf_n \hat{\mathcal{E}}_0^{g,a}(\xi^n)$ .

*In particular,  $\hat{\mathcal{E}}_0^{g,a}$  is  $L^1(P^a)$ -lower semicontinuous.*

We only prove the first theorem.

*Proof (of Theorem 4.16).* In the setting of Section 4.1, given a positive generator  $\bar{g}$  and a random variable  $\zeta$ , let us denote by  $\mathcal{A}(\zeta, \bar{g}, W^a)$  the set defined in (3.3) to indicate the dependence of this set on the Brownian motion  $W^a$  and the respective probability measure  $P^a$ . Let us now define the generator  $\bar{g}$  as

$$\bar{g}(y, z) = g \left( y + \int_0^\cdot b_s ds, z \right) - az^\top - b, \quad \text{for all } (y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}. \quad (4.33)$$

By Assumption (LB), this generator fulfills (POS), (CON) and either (INC) or (DEC). Since  $\int Z dW^a$  is a  $P^a$ -supermartingale, a simple inspection shows that the affine transformation  $\bar{Y} = Y - \int b ds$  and  $\bar{Z} = Z$  yields a one-to-one relation between  $\mathcal{A}^a(\xi, g)$  and  $\mathcal{A}(\xi - \int_0^T b_s ds, \bar{g}, W^a)$ . Hence, the assumptions of Theorem 4.1 are fulfilled for  $\bar{g}$  and  $\mathcal{A}(\xi - \int_0^T b_s ds, \bar{g}, W^a)$ , and thus its application ends the proof.  $\square$

*Remark 4.18.* Note that if  $(E^a[(\xi - \int_0^T b_s ds)^- | \mathcal{F}])^* \in L^1(P^a)$ , then Theorem 4.5 applies in the same way, that is, under the assumptions of Theorem 4.16, if

$$\mathcal{A}^{1,a}(\xi, g) := \{(Y, Z) \in \mathcal{A}^a(\xi, g) : Z \in \mathcal{L}^1(P^a)\} \neq \emptyset,$$

then  $\mathcal{E}^{g,a}(\xi)$  is the value process of the minimal supersolution with unique control process  $Z \in \mathcal{L}^1(P^a)$ .  $\blacklozenge$

In the remainder of this section, we illustrate our results, in particular our set based comparison principle and the existence theorem, by maximizing exponential expected utility. For an introduction to this well-known problem and for similar statements, see for instance Delbaen et al. [13], Hu et al. [26], Mania and Schweizer [33], and the references therein.

Consider a financial market where the discounted stock prices are modelled by a  $n$ -dimensional process  $S$  satisfying the differential equation

$$dS_u^i = S_u^i (\mu_u^i du + \sigma_u^i dW_u), \quad S_0^i > 0, \quad i = 1, \dots, n, \quad (4.34)$$

where  $n \leq d$ , and the drift  $\mu = (\mu^1, \dots, \mu^n)^\top$  and the volatility row-vectors  $\sigma^i = (\sigma^{i1}, \dots, \sigma^{id})$  are progressive processes. We suppose that  $\sigma\sigma^\top$  is invertible,  $P \otimes dt$ -almost surely. Hence, without loss of generality, we may assume that  $\sigma^{ij} = 0$ , for all  $j = n+1, \dots, d$ , and that the  $n \times n$ -matrix  $\tilde{\sigma}^{ij} = \sigma^{ij}$ ,  $i, j = 1, \dots, n$ , is invertible  $P \otimes dt$ -almost surely. By use of the  $d$ -dimensional market price of risk vector  $\theta = (\tilde{\sigma}^{-1}\mu, 0, \dots, 0)^\top$ , the dynamics in (4.34) are equivalent to  $dS_u^i = S_u^i \sigma_u^i (\theta_u du + dW_u)$ , for  $i = 1, \dots, n$ . We further assume that  $\int \theta^\top dW \in BMO(P)$ . Therefore,  $\mathcal{E}(-\int \theta^\top dW)_T$  defines an equivalent probability measure  $P^{-\theta}$ . According to Girsanov's change of measure theorem, the discounted price process  $S$  is then a local martingale under  $P^{-\theta}$ . We consider the problem

$$U_t^\vartheta = E \left[ \exp \left( -\xi - \int_0^T \vartheta_u dS_u \right) \mid \mathcal{F}_t \right] \rightarrow \min, \quad (4.35)$$

where  $\xi \in L^0(\mathcal{F}_T)$  is a random endowment and  $\vartheta$  an  $\mathbb{R}^{1 \times n}$ -valued admissible strategy. Here, the optimization takes place over the set  $\Theta$  of those strategies  $\vartheta$  such that  $\vartheta$  is progressive,  $\int \vartheta dS$  is a  $P^{-\theta}$ -supermartingale and  $\exp(-\int_0^\tau \vartheta dS)_{\tau \in \mathcal{T}}$  is  $P$ -uniformly integrable.

In principle, the following proposition is well-known, see the references mentioned above. However, our approach relies exclusively on the theory of supersolutions developed in the previous sections and extends the method of pointwise optimization of generators used in Horst et al. [25] to our setting of an incomplete market.

**Proposition 4.19.** *Suppose that  $\int \theta^\top dW \in BMO(P)$  and  $\xi \in L^\infty$ . Then,  $\vartheta^*$  defined by the following componentwise multiplication between  $\vartheta^*$  and  $S$ ,*

$$\vartheta_t^* S_t = (Z_t + \theta_t^\top) \begin{pmatrix} \tilde{\sigma}_t^{-1} \\ 0 \end{pmatrix} = (Z_t^1, \dots, Z_t^n) \tilde{\sigma}_t^{-1} + (\theta_t^1, \dots, \theta_t^n) \tilde{\sigma}_t^{-1}, \quad (4.36)$$

<sup>1</sup>For the definition and important results concerning the space  $BMO$ , we refer to Kazamaki [29].



is the optimal solution in  $\Theta$  of (4.35). Here,  $Z \in \mathcal{L}^1(P^{-\theta})$  is the control process corresponding to the minimal supersolution of the BSDE

$$Y_s - \int_s^t \left[ \frac{1}{2} \left( \sum_{k=n+1}^d (Z_u^k)^2 - \sum_{k=1}^n (\theta_u^k)^2 \right) - \sum_{k=1}^n \theta_u^k Z_u^k \right] du + \int_s^t Z_u dW_u \geq Y_t$$

and  $Y_T \geq -\xi$ . (4.37)

*Proof.* Let  $\vartheta \in \Theta^\infty$ , the set of those strategies in  $\Theta$  such that  $\int \vartheta dS$  is uniformly bounded. The general result follows by stopping any strategy  $\vartheta$  as soon as  $|\int \vartheta dS|$  is above  $n \in \mathbb{N}$  and using the uniform integrability to show that  $U_t^{\vartheta^n}$  converges  $P$ -almost surely to  $U_t^\vartheta$ , for all  $t \in [0, T]$ , where  $(\vartheta^n)$  is the stopped strategy. In view of Itô's lemma, the certainty equivalent  $Y_t^\vartheta = \log(U_t^\vartheta) + \int_0^t \vartheta_u dS_u$  satisfies the BSDE

$$Y_s^\vartheta - \int_s^t g_u(\vartheta_u, Z_u^\vartheta) du + \int_s^t Z_u^\vartheta dW_u = Y_t^\vartheta, \quad Y_T^\vartheta = -\xi, \quad (4.38)$$

where the control process is given by  $Z_t^\vartheta = V_t^\vartheta / U_t^\vartheta + \vartheta_t S_t \sigma_t$ , with  $V^\vartheta$  coming from the martingale representation of  $U^\vartheta$ . The generator is given by

$$g_u(\vartheta, z) = \frac{1}{2} (z - \vartheta_u S_u \sigma_u)^2 - \vartheta_u S_u \mu_u,$$

for all  $z \in \mathbb{R}^{1 \times d}$ , and, from our assumption on  $\theta$ , it follows that  $\int Z^\vartheta dW^{-\theta}$  is a  $P^{-\theta}$ -supermartingale. This implies that  $(Y^\vartheta, Z^\vartheta) \in \mathcal{A}^{-\theta}(-\xi, g(\vartheta, \cdot))$ . Pointwise optimization of  $\vartheta \mapsto \frac{1}{2} (z - \vartheta S_u \sigma_u)^2 - \vartheta S_u \mu_u$ , for  $\vartheta \in \mathbb{R}^{1 \times n}$ , yields a pointwise minimum  $\vartheta^*(z)$  such that

$$\vartheta_u^*(z) S_u = (z + \theta_t^\top) \begin{pmatrix} \tilde{\sigma}_t^{-1} \\ 0 \end{pmatrix}.$$

Plugged into the generator  $g_u(\vartheta, z)$ , it yields the optimized generator

$$g_u^*(z) = \frac{1}{2} \left( \sum_{k=n+1}^d (z^k)^2 - \sum_{k=1}^n (\theta_u^k)^2 \right) - \sum_{k=1}^n \theta_u^k z^k.$$

Now we use our comparison result, see Proposition 3.2, to obtain

$$\mathcal{A}^{-\theta}(-\xi, g(\vartheta, \cdot)) \subset \mathcal{A}^{-\theta}(-\xi, g^*), \quad \text{for all } \vartheta \in \Theta^\infty,$$

and

$$\hat{\mathcal{E}}^{g^*, -\theta}(-\xi) \leq Y^\vartheta,$$

for all  $\vartheta \in \Theta^\infty$ . Next we want to use existence and uniqueness of the minimal supersolution given by Theorem 4.16 to ensure that  $\hat{\mathcal{E}}^{g^*, -\theta}(-\xi)$  has indeed a modification  $\mathcal{E}^{g^*, -\theta}(-\xi)$  which is the value process of the minimal supersolution and to obtain the corresponding control process. To that end note the generator  $g^*$  fulfills

$$g_u^*(z) \geq -\frac{1}{2} \sum_{k=1}^n (\theta_u^k)^2 - \theta^\top z^\top$$

and that this is a valid lower bound in the sense of Assumption (LB) in Section 4.3. From our assumptions on  $\theta$  and  $\xi$  follows

$$(E^{-\theta}[\|\xi\| + \frac{1}{2} \int_0^T \sum_{k=1}^n (\theta_u^k)^2 du | \mathcal{F}])_T^* \in L^1(P^{-\theta}).$$

This and  $\xi$  bounded below imply that

$$(\|\xi^-\|_\infty - \frac{1}{2} \int_0^n (\theta^k)^2 du, 0) \in \mathcal{A}^{1,-\theta}(-\xi, g^*),$$

that is  $\mathcal{A}^{1,-\theta}(-\xi, g^*) \neq \emptyset$ . Hence, all the assumptions of Theorem 4.16 are fulfilled, which, together with Remark 4.18, yields that  $Y^* := \mathcal{E}^{g^*, -\theta}(-\xi)$  is the value process of the minimal supersolution of (4.37) with control process  $Z^* \in \mathcal{L}^1(P^{-\theta})$ . Defining  $\vartheta^* := \vartheta^*(Z^*)$ , yields

$$(Y^*, Z^*) = (Y^{\vartheta^*}, Z^*) \in \mathcal{A}^{1,-\theta}(-\xi, g(\vartheta^*, \cdot)).$$

We finally have to show that  $\vartheta^* \in \Theta$  and that  $U_0^{\vartheta^*}$  is minimal. From  $\int \theta^\top dW^{-\theta} \in BMO(P^{-\theta})$  it follows that  $\int \vartheta^* dS = \sum_{k=1}^n \int (Z^{*k} + \theta^k) dW^{-\theta, k} \in \mathcal{L}^1(P^{-\theta})$ . Thus,  $\int \vartheta^* dS$  is a  $P^{-\theta}$ -(super)martingale. Furthermore, by (4.37), for all stopping times  $\tau \in \mathcal{T}$ , holds

$$E^{-\theta} \left[ \int_\tau^T \sum_{k=n+1}^d (Z^{*k})^2 du | \mathcal{F}_\tau \right] \leq 2Y_\tau^* + 2E^{-\theta}[\xi | \mathcal{F}_\tau] + E^{-\theta} \left[ \int_\tau^T \sum_{k=n+1}^d (\theta^k)^2 du | \mathcal{F}_\tau \right].$$

From  $Y^* \leq \|\xi^-\|_\infty$  and  $\xi \leq \|\xi^+\|_\infty$ , it follows

$$\left( \int Z^{*n+1} dW^{-\theta, n+1}, \dots, \int Z^{*d} dW^{-\theta, d} \right) \in BMO(P^{-\theta}).$$

Thus, for  $\tilde{V}^* := Z^* - \vartheta^* S\sigma$ , it holds

$$\int \tilde{V}^* dW^{-\theta} = - \sum_{k=1}^n \int \theta^k dW^{-\theta, k} + \sum_{k=n+1}^d \int Z^{*k} dW^{-\theta, k} \in BMO(P^{-\theta}),$$

and therefore,  $\int \tilde{V}^* dW \in BMO(P)$ . Since  $\int \vartheta^* dS = \int \vartheta^* S\sigma dW^{-\theta}$  and  $\vartheta^* S\sigma = Z^* + \tilde{V}^*$ , the admissibility of  $Z^*$  yields that  $\int \vartheta^* dS$  is a  $P^{-\theta}$ -supermartingale. The process  $C^{\vartheta^*} = Y^* - \int \vartheta^* dS$  satisfies

$$C_s^{\vartheta^*} - \int_s^t \frac{1}{2} (\tilde{V}_u^*)^2 du + \int_s^t \tilde{V}_u^* dW_u \geq C_t^{\vartheta^*}, \quad C_T^{\vartheta^*} = -\xi - \int_0^T \vartheta_u^* dS_u. \quad (4.39)$$

Consequently,

$$\exp \left( - \int_0^t \vartheta^* dS \right) = \exp \left( C_t^{\vartheta^*} - Y_t^* \right) \leq \mathcal{E} \left( \int \tilde{V}^* dW \right)_t \exp \left( C_0^{\vartheta^*} - Y_t^* \right).$$

Hence, from  $\int \tilde{V}^* dW \in BMO(P)$  and  $Y^* \geq -\|\xi\|_\infty$ , it follows that the family  $(\exp(-\int_0^\tau \vartheta^* dS))_{\tau \in \mathcal{T}}$  is  $P$ -uniformly integrable. Thus,  $\vartheta^* \in \Theta$ . As for the optimality of  $\vartheta^*$ , note that,

for  $\vartheta \in \Theta^\infty$ , holds

$$\begin{aligned} E \left[ \exp \left( -\xi - \int_0^T \vartheta_u^* dS_u \right) \mid \mathcal{F}_t \right] &= E \left[ \exp \left( C_T^{\vartheta^*} \right) \mid \mathcal{F}_t \right] \\ &\leq \exp \left( C_t^{\vartheta^*} \right) = \exp \left( Y_t^* - \int_0^t \vartheta_u^* dS_u \right) \leq \exp \left( Y_t^\vartheta - \int_0^t \vartheta_u^* dS_u \right), \end{aligned}$$

which implies

$$E \left[ \exp \left( -\xi - \int_t^T \vartheta_u^* dS_u \right) \mid \mathcal{F}_t \right] \leq E \left[ \exp \left( -\xi - \int_t^T \vartheta_u dS_u \right) \mid \mathcal{F}_t \right]. \quad \square$$

## A. Helly's theorem

In principle the following result is well-known, see for example [6, 9, 31] for similar statements. We give a proof for sake of completeness.

**Lemma A.1.** *Let  $(A^n)$  be a sequence of increasing positive processes such that the sequence  $(A_T^n)$  is bounded in  $L^1$ . Then, there is a subsequence  $(\tilde{A}^{n_k})$  in the asymptotic convex hull of  $(A^n)$  and an increasing positive integrable process  $\tilde{A}$  such that*

$$\lim_{k \rightarrow \infty} \tilde{A}_t^{n_k} = \tilde{A}_t, \quad \text{for all } t \in [0, T], \quad P\text{-almost surely.} \quad (\text{A.1})$$

*Proof.* Let  $(t_j)$  be a sequence running through  $\mathcal{I} = ([0, T] \cap \mathbb{Q}) \cup \{T\}$ . Since  $(A_{t_j}^n)$  is an  $L^1$ -bounded sequence of positive random variables, due to [11, Lemma A1.1], there exists a subsequence  $(\tilde{A}_{t_1}^{1,k})$  in the asymptotic convex hull of  $(A^n)$  and a random variable  $\tilde{A}_{t_1}$  such that  $(\tilde{A}_{t_1}^{1,k})$  converges  $P$ -almost surely to  $\tilde{A}_{t_1}$ . Moreover, Fatou's lemma yields  $\tilde{A}_{t_1} \in L^1$ . Let  $(\tilde{A}^{2,k})$  be a subsequence in the asymptotic convex hull of  $(\tilde{A}^{1,k})$  such that  $(\tilde{A}_{t_2}^{2,k})$  converges  $P$ -almost surely to  $\tilde{A}_{t_2} \in L^1$  and so on. Then,  $\tilde{A}_t^{k,k} \rightarrow \tilde{A}_t$ , on a set  $\hat{\Omega} \subset \Omega$ , where  $P(\hat{\Omega}) = 1$ . The process  $\tilde{A}$  is positive, increasing and integrable on  $\mathcal{I}$ . Thus, we may define

$$\hat{A}_t = \lim_{r \downarrow t, r \in \mathcal{I}} \tilde{A}_r, \quad t \in [0, T), \quad \hat{A}_T = \tilde{A}_T.$$

We now show that  $(\tilde{A}^{k,k})$ , henceforth named  $(\tilde{A}^k)$ , converges  $P$ -almost surely on the continuity points of  $\hat{A}$ . Fix  $\omega \in \hat{\Omega}$  and a continuity point  $t \in [0, T)$  of  $\hat{A}(\omega)$ . We show that  $(\tilde{A}_t^k(\omega))$  is a Cauchy sequence in  $\mathbb{R}$ .

Fix  $\varepsilon > 0$  and set  $\delta = \varepsilon/11$ . Since  $t$  is a continuity point of  $\hat{A}(\omega)$ , we may choose  $p_1, p_2 \in \mathcal{I}$  such that  $p_1 < t < p_2$  and  $\hat{A}_{p_2}(\omega) - \hat{A}_{p_1}(\omega) < \delta$ . By definition of  $\hat{A}$ , we may choose  $r_1, r_2 \in \mathcal{I}$  such that  $p_1 < r_1 < t < p_2 < r_2$  and  $|\hat{A}_{p_2}(\omega) - \tilde{A}_{r_2}(\omega)| < \delta$  and  $|\hat{A}_{p_1}(\omega) - \tilde{A}_{r_1}(\omega)| < \delta$ . Now choose  $N \in \mathbb{N}$  such that  $|\tilde{A}_{r_1}^m(\omega) - \tilde{A}_{r_1}^n(\omega)| < \delta$ , for all  $m, n \in \mathbb{N}$  with  $m, n \geq N$ , and  $|\tilde{A}_{r_2}^j(\omega) - \tilde{A}_{r_2}(\omega)| < \delta$  and  $|\tilde{A}_{r_1}(\omega) - \tilde{A}_{r_1}^j(\omega)| < \delta$ , for  $j = m, n$ . We estimate

$$|\tilde{A}_t^m(\omega) - \tilde{A}_t^n(\omega)| \leq |\tilde{A}_t^m(\omega) - \tilde{A}_{r_1}^m(\omega)| + |\tilde{A}_{r_1}^m(\omega) - \tilde{A}_{r_1}^n(\omega)| + |\tilde{A}_{r_1}^n(\omega) - \tilde{A}_t^n(\omega)|.$$

For the first and the third term on the right hand side, since  $\tilde{A}^m$  and  $\tilde{A}^n$  are increasing, we deduce that  $|\tilde{A}_t^m(\omega) - \tilde{A}_{r_1}^m(\omega)| \leq |\tilde{A}_{r_2}^m(\omega) - \tilde{A}_{r_1}^m(\omega)|$  and  $|\tilde{A}_t^n(\omega) - \tilde{A}_{r_1}^n(\omega)| \leq |\tilde{A}_{r_2}^n(\omega) - \tilde{A}_{r_1}^n(\omega)|$ . Moreover

$$\begin{aligned} |\tilde{A}_{r_2}^j(\omega) - \tilde{A}_{r_1}^j(\omega)| &\leq |\tilde{A}_{r_2}^j(\omega) - \tilde{A}_{r_2}(\omega)| + |\tilde{A}_{r_2}(\omega) - \hat{A}_{p_2}(\omega)| \\ &\quad + |\hat{A}_{p_2}(\omega) - \hat{A}_{p_1}(\omega)| + |\hat{A}_{p_1}(\omega) - \tilde{A}_{r_1}(\omega)| + |\tilde{A}_{r_1}(\omega) - \tilde{A}_{r_1}^j(\omega)|, \end{aligned}$$

for  $j = m, n$ . Combining the previous inequalities yields  $|\tilde{A}_t^m(\omega) - \tilde{A}_t^n(\omega)| \leq \varepsilon$ , for all  $n, m \geq N$ . Hence,  $(\tilde{A}^k(\omega))$  converges for all continuity points  $t \in [0, T)$  of  $\hat{A}(\omega)$ , for all  $\omega \in \hat{\Omega}$ . We denote the limit with  $\tilde{A}$ .

It remains to see that  $(\tilde{A}^k)$  also converges for the discontinuity points of  $\hat{A}$ . To this end, note that  $\hat{A}$  is càdlàg and adapted to our filtration which fulfills the usual conditions. By a well-known result, see for example [28, Proposition 1.2.26], this implies that the jumps of  $\hat{A}$  may be exhausted by a sequence of stopping times  $(\rho^j)$ . Applying once more [11, Lemma A1.1] iteratively on the sequences  $(\tilde{A}_{\rho^j}^k)_{k \in \mathbb{N}}$ ,  $j = 1, 2, 3, \dots$ , and diagonalizing yields the result.  $\square$

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