Continuous Equilibrium in Affine and Information-Based Capital Asset Pricing Models

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January 9, 2012

We consider a class of generalized capital asset pricing models in continuous time with a finite number of agents and tradable securities. The securities may not be sufficient to span all sources of uncertainty. If the agents have exponential utility functions and the individual endowments are spanned by the securities, an equilibrium exists and the agents' optimal trading strategies are constant. Affine processes, and the theory of information-based asset pricing are used to model the endogenous asset price dynamics and the terminal payoff. The derived semi-explicit pricing formulae are applied to numerically analyze the impact of the agents' risk aversion on the implied volatility of simultaneously-traded European-style options.[1]

Keywords: Continuous-time equilibrium, exponential utility, CAPM, affine processes, information-based asset pricing, implied volatility.

JEL Classification: C62, D52, D53

Introduction

In this paper we propose an analytically-tractable equilibrium model in continuous time, within which financial securities are priced in a generalized capital asset pricing model (CAPM).

It is well known that when markets are incomplete, competitive equilibria may fail to exist. Even if they exist, they may not be Pareto optimal, nor supportable as equilibria of a suitable representative agent economy. The equilibrium analysis of incomplete markets is therefore always confined to special cases,

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[1]The authors are indebted to Julio Backhoff for his assistance with the numerical illustrations; special thanks are due to Martin Keller-Ressel for helpful comments and fruitful discussions. U. Horst and M. Kupper acknowledge support from the DFG research center MATHEON; U. Horst and C. Mainberger acknowledge support from the SFB 649 "Economic Risk". A. Macrina acknowledges generous hospitality by the Humboldt Universität zu Berlin. This paper has been presented under the title "Continuous Equilibrium under Base Preferences and Attainable Initial Endowments" at the CRC 649 annual meeting (July 2011) and the 4th Int. Conference of Math. in Finance, Kruger National Park (August 2011). E-mail addresses for correspondence: horst@math.hu-berlin.de; kupper@math.hu-berlin.de; andrea.macrina@kcl.ac.uk; mainberg@math.hu-berlin.de

for instance to single agent models [16, 15], multiple agent models where markets are complete in equilibrium [10, 18, 21], or models with particular classes of goods [20] or preferences [5].

Cheridito et al. [6] recently established existence and uniqueness of equilibrium results for incomplete financial market models in discrete time when agents' preferences are translation invariant[2]. In the situation where uncertainty is spanned by finitely many random walks, they showed that the equilibrium dynamics can be described as the solution to a coupled system of forward-backward stochastic difference equations. The system is usually high-dimensional because one obtains one equation per security and market participant. This renders simulations and calibrations of the model cumbersome, if not impossible. Within the framework of generalized CAPMs, that is, if all agents share the same base preferences (as in the case of exponential utility functions) and the endowments lie in the span of the tradable assets, the system simplifies to a single equation representing the equilibrium utility of some representative agent. Furthermore, the equilibrium price process depends only on the aggregated endowment, the market risk aversion, and the flow of market information. It is in this sense that these three items fully characterize equilibrium prices in generalized CAPMs.

In this paper we extend the generalized CAPM analyzed in [6] to continuous time. In particular, the advantage of the herewith presented continuous-time framework is that we obtain (semi-)explicit formulae for equilibrium prices. If not explicitly computable, key equilibrium quantities can be computed using numerical integration only. In particular, no Monte Carlo methods are needed. Specifically, we consider a model with a finite number of agents, which are initially endowed with an attainable random payoff. They trade a finite number of securities so as to maximize expected exponential utility from terminal wealth. The financial securities are characterized by their terminal payoffs, which we assume to be functions of finitely many *market factors*. The market factors may or may not be observable to the agents. Affine processes, and the theory of information-based asset pricing are used to model the endogenous asset price dynamics and the terminal payoff.

Within our first approach, the dynamics of the market factors follows an affine process that generates the market filtration. Affine processes are widely used in mathematical finance (see for instance [11, 12, 22] and references therein), as they lend themselves to a transparent mathematical analysis and to the application of efficient numerical methods. We show that within an affine framework, equilibrium securities prices are given by the quotient of two integrals. Both integrals are the product of an exponential function evaluated at the current state of the factor process and the Fourier transform of a smooth function. Representing equilibrium prices in terms of deterministic integrals allows for a fast and efficient numerical analysis of other equilibrium quantities, such as option implied volatilities. We analyze implied volatilities for two single-security benchmark models: (i) an additive Heston stochastic volatility model, and (ii) a pure jump Ornstein-Uhlenbeck model. Both models reproduce the well-documented smile-effect of implied volatilities and identify investor risk aversion as a key determinant of implied volatilities.

The second approach to continuous equilibrium presented in this paper is based on the theory of information-based asset pricing; see Brody et al. [4] and Hoyle et al. [19]. Within this approach, the asset price dynamics is explicitly generated by taking the conditional expectation of the future cash flows, which are multiplied by the pricing rule, given the partial information about the market factors that is available to the agents. The filtration is modeled by stochastic processes, which (i) carry information about the a priori distribution of the market factors, and (ii) embody pure noise preventing market participants from accessing full knowledge as to what is the "true" value of the asset at any time before the cash flows occur. We use the information-based framework to show the dependence of the equilibrium prices of

[2]Exponential utility functions, for instance, are translation invariant after a logarithmic transformation.

credit-risky securities on information about the financial standing of a company.

The paper is structured as follows. A general existence result along with a discussion on the information-generating processes is given in Section 1. In Section 2 and 3, we present affine and information-based equilibrium pricing models, respectively. Proofs to the theorems are collected in the appendix.

1. A Generalized Capital Asset Pricing Model

We consider an equilibrium model in continuous time with a finite set $\mathbb{A}$ of economic agents. Uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, P)$ carrying a filtration $(\mathcal{F}_t)_{t\in[0,T]}$. The filtration captures the flow of information that is available to the agents over the trading period $[0, T]$. It is assumed to satisfy the usual assumptions of completion and right-continuity. In what follows, all equalities and inequalities are to be understood in the P-almost sure sense.

1.1. Existence of Equilibrium

The agents can lend to and borrow from the money market account at some exogenously given interest rate, and they can trade K securities. The securities are in net supply $n = (n^1, \cdots, n^K) \in \mathbb{R}^K$ and characterized by their terminal payoffs $S_T = (S_T^1, ..., S_T^K)$, which we assume to be $\mathcal{F}_T$-measurable random variables. Securities are priced to match demand and supply. Each agent $a \in \mathbb{A}$ is initially endowed with some $\mathcal{F}_T$-measurable random payoff H^a of the form

$$H^a = c^a + \eta^a \cdot S_T \,,$$

for constants $c^a \in \mathbb{R}$ and $\eta^a \in \mathbb{R}^K$. Furthermore, at each time $t \in [0, T]$ the agent's preferences can be described by the utility functional

$$U_t^a(X) = -\frac{1}{\gamma^a} \log\left(E\left[\mathrm{e}^{-\gamma^a X} \mid \mathcal{F}_t\right]\right) \,,$$

where $\gamma^a > 0$ is the risk aversion parameter. Thus at time $t \in [0, T]$, the agent faces the optimization problem

$$\sup_{\vartheta\in\Theta} U_t^a\left(H^a + \int_t^T \vartheta_u dS_u\right) \,,$$

where the set of admissible trading strategies Θ is given by

$$\Theta = \left\{\vartheta \in L(S) \,:\, G(\vartheta) \text{ is a } \tilde{Q}\text{-supermartingale, for all } \tilde{Q} \in \mathcal{P}\right\}.$$

Here, $L(S)$ and $G_t(\vartheta) := \int_0^t \vartheta_u dS_u$ denote the set of S-integrable predictable processes and the gains process, respectively, whereas $\mathcal{P}$ denotes the set of all equivalent martingale measures (EMM) for S.[3]

The goal is now to establish existence of a (discounted) equilibrium price process $(S_t)_{t\in[0,T]}$.[4] Since all agents share the same base preferences, and because all payoffs lie in the span of the tradable assets, our

[3] Note that in equilibrium, there is an EMM Q, that is, an equivalent probability measure Q under which the price process S will be a true martingale. In particular, S will be a P-semimartingale. For related discussions on suitable sets of admissible strategies see for instance [7], [8], or [2].

[4] For simplicity, we assume that the trading horizon T is short so that interest rate risk can be ignored.

model can be viewed as a generalized CAPM. Just like in the classical CAPM, in our incomplete market model existence of an equilibrium can be established using the standard representative agent approach that underlies equilibrium models of *complete* markets. Furthermore, all agents share the market portfolio according to their risk aversion in equilibrium. The equilibrium pricing kernel depends on the agents' preferences and endowments, however only through the *endowment- and supply-adjusted risk aversion*

$$\tilde{\gamma} := \gamma(\eta + n) \in \mathbb{R}^K . \tag{1.1}$$

Here, $\eta := \sum_a \eta^a$ denotes the aggregate endowment and $\gamma^{-1} := \sum_a \gamma_a^{-1}$ can be viewed as the market risk aversion. The following result can be proved using standard duality results for entropic utility functions; see [6, Theorem 5.1] for a related result in discrete time.

Theorem 1.1. *Suppose that the following integrability conditions hold:*

$$\exp\left(-\tilde{\gamma} \cdot S_T\right) \in L^1(P) \quad \textit{and} \quad S_T \in L^1(Q)^K , \tag{1.2}$$

where Q is an equivalent probability measure with density

$$\frac{dQ}{dP} = \frac{\exp(-\tilde{\gamma} \cdot S_T)}{E\left[\exp(-\tilde{\gamma} \cdot S_T)\right]} . \tag{1.3}$$

Then, the price process S defined by

$$S_t = E_Q\left[S_T \,|\, \mathcal{F}_t\right] , \quad t \in [0, T] , \tag{1.4}$$

together with the constant trading strategies

$$\hat{\vartheta}_t^a \equiv \frac{\gamma}{\gamma^a}(n + \eta) - \eta^a, \quad a \in \mathbb{A},$$

constitutes an equilibrium.

We notice that the equilibrium pricing kernel Q depends only on the terminal payoffs weighted by the endowment- and supply-adjusted risk aversion. In particular, if the k-th security is in zero *endowment-adjusted supply*, that is, if $\eta^k + n^k = 0$, then its payoff does not affect the equilibrium pricing kernel.

Furthermore, the integrability assumption on S under the pricing measure Q guarantees that equilibrium prices are Q-martingales. Hence they are, by (1.2) and (1.3), P-semimartingales and thus well defined as an integrator in the sense of [24, Chapter II and IV].

1.2. The Market Filtration

The previous theorem established existence of a continuous equilibrium under no assumptions on the underlying filtration $(\mathcal{F}_t)$. We emphasize that the construction of the filtration characterizes the dynamics of the derived price processes. In order to obtain (semi-)explicit equilibrium price processes, we assume that the terminal payoffs depend in a functional form on a vector X of market factors the distribution of which is known to the agents. We define the following:

$$S_T^k = f^k(X) .$$

We assume that the *market filtration* $(\mathcal{F}_t)_{t\in[0,T]}$, to which the equilibrium prices will be adapted, is generated by an observable stochastic process (ξ_t) such that, possibly up to a constant, $\xi_T = X$. Equilibrium dynamics will then be studied within an affine and an information-based framework. The former assumes that the dynamics of the (observable) market factors follows an affine process; in the latter the observables generating the market filtration are modeled by Brownian random bridges with drift from zero to X.

2. Affine Equilibrium Framework

In this section, we assume that the dynamics of the market factors ξ is observable and that it follows an affine process Y, that is $\xi = Y$. After the formulation of the setup and a brief introduction into the theory of affine processes, the results in Section 1 are used to derive equilibrium pricing formulae in Section 2.1. This is followed by an analysis of equilibrium option prices in Section 2.2 and equilibrium asset prices in a Heston stochastic volatility framework and an Ornstein-Uhlenbeck jump model in Sections 2.3 and 2.4, respectively.

2.1. Setup and Equilibrium Pricing Formulae

In this section, we consider the case where the payoff S_T is a functional of an *affine* observable factor process. To this end, we assume that the underlying probability space $(\Omega, \mathcal{F}, P)$ is rich enough to support an affine Markov process Y taking values in the state space $D := \mathbb{R}_+^m \times \mathbb{R}^n$.

We set $d = m + n$ and write $Y = (V, X)$. We interpret $X \in \mathbb{R}^n$ as the factor process that determines the payoff and $V \in \mathbb{R}_+^m$ as a process driving it; a typical example would be a stochastic volatility model. We assume that Y^T, the Markov process stopped at time T, is conservative, meaning that there are no explosions or absorbing states up to time T. The market filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is then chosen to be the one generated by Y:

$$\mathcal{F}_t = \sigma(Y_s\,, s \leq t), \quad t \in [0, T].$$

Usually, one associates with Y a family of probability measures $(P^y)_{y\in D}$, which represents the law of the process Y starting at $y \in D$. Since every affine process is a Feller process, the filtration $(\mathcal{F}_t)$ can be completed with respect to the family $(P^y)_{y\in D}$, so that the filtration is automatically right-continuous [25, Section III.2].

2.1.1. Affine processes

Before turning to the problem of equilibrium pricing, we recall some useful definitions and results on affine processes, the details of which can be found in Duffie et al. [12] or in Keller-Ressel [22].

Definition 2.1. An *affine process* is a stochastically continuous[5], time-homogeneous Markov process (Y, P^y) with state-space D, whose log-characteristic function is an affine function of the state vector. This means that there exist functions $\phi : \mathbb{R}_+ \times i\mathbb{R}^d \to \mathbb{C}$ and $\psi : \mathbb{R}_+ \times i\mathbb{R}^d \to \mathbb{C}^d$ such that

$$E^y\left[\exp\left(u \cdot Y_t\right)\right] = \exp\left[\phi(t,u) + \psi(t,u) \cdot y\right], \tag{2.1}$$

for all $y \in D$ and $(t, u) \in \mathbb{R}_+ \times i\mathbb{R}^d$. An affine process Y is called *regular*, if the derivatives

$$F(u) := \partial_t \phi(t,u)|_{t=0^+} \quad , \quad R(u) := \partial_t \psi(t,u)|_{t=0^+}$$

exist for all $u \in \mathcal{U} := \{u = (u_v, u_x) \in \mathbb{C}^m \times \mathbb{C}^n \,:\, \mathrm{Re}(u_v) \leq 0,\ \mathrm{Re}(u_x) = 0\}$ and are continuous in $u = 0$.[6]

[5] A stochastic process Y is stochastically continuous, if for any sequence $(t_m) \to t$ in $\mathbb{R}_+$, Y_{t_m} converges to Y_t in probability.

[6] In the recent work [23], the authors actually showed that each affine process as defined above is regular, whereas in [12] and [22] regularity was still an assumption on Y.

The definition of an affine process Y implies that the $\mathcal{F}_t$-conditional characteristic function of Y_T $(T \geq t)$ is an affine function of Y_t:

$$E\left[\exp\left(u \cdot Y_T\right) \mid \mathcal{F}_t\right] = \exp\left[\phi(\tau, u) + \psi(\tau, u) \cdot Y_t\right], \tag{2.2}$$

for all $(\tau, u) \in \mathbb{R}_+ \times i\mathbb{R}^d$, where $\tau := T - t$. The affine property will be used in this form throughout. The admissible parameters associated with an affine process Y determine its generator and its functional characteristics F and R. The functional characteristics completely determine a regular affine process, since the functions ϕ and ψ satisfy generalized Riccati equations of the form $\partial_t \phi(t, u) = F(\psi(t, u))$ and $\partial_t \psi(t, u) = R(\psi(t, u))$; see Appendix B for further details.

Although the special form of the log-characteristic function of an affine process perfectly lends itself to tractable computations, we need to consider a class of processes for which formulae (2.1) or (2.2) extend to a broader subspace of $\mathbb{C}^d$ than $i\mathbb{R}^d$.[7]

Definition 2.2. For Y a regular affine process, and for each $t \geq 0$, the following sets are considered:

$$\begin{aligned} \mathcal{D}_t &:= \left\{ z \in \mathbb{R}^d \,:\, \sup_{0 \leq s \leq t} E^y\left[\exp(z \cdot Y_s)\right] < \infty, \;\; \text{for all } \; y \in D \right\}, \\ \mathcal{D}_{t^+} &:= \bigcup_{s > t} \mathcal{D}_s \quad \text{and} \quad \mathcal{D} := (\mathcal{D}_{0^+})^\circ \cup \{0\}. \end{aligned} \tag{2.3}$$

$\mathcal{D}$ is called the *real domain* of Y, and Y is said to be *analytic*, if the interior $\mathcal{D}^\circ$ of $\mathcal{D}$ is non-empty.

Remark 2.3. It was shown in [22, Chapter 3] that the functions ϕ and ψ characterizing the process Y have unique extensions to analytic functions on the interior $\mathcal{E}_{\mathbb{C}}^\circ$ of the tube domain $\mathcal{E}_{\mathbb{C}}$ defined by

$$\mathcal{E}_{\mathbb{C}} := \left\{(t, u) \in \mathbb{R}_+ \times \mathbb{C}^d \,:\, (t, \mathrm{Re}(u)) \in \mathcal{E}\right\}, \quad \text{where} \;\; \mathcal{E} := \left\{(t, v) \in \mathbb{R}_+ \times \mathbb{R}^d \,:\, v \in \mathcal{D}_{t^+}\right\}.$$

The extensions still satisfy the aforementioned Riccati equations and (2.1) and (2.2) extend to $\mathcal{E}_{\mathbb{C}}$.[8] ♦

2.1.2. Equilibrium pricing formulae

We are now ready to state the main result of this section, that is a semi-explicit formula for the equilibrium price processes in an affine framework. For simplicity, we restrict our analysis to processes $Y = (V, X)$ with state space $D = \mathbb{R}_+ \times \mathbb{R}$, and we assume that the agents can trade K securities $S^1, ..., S^K$ with terminal payoffs

$$S_T^k = f^k(X_T), \tag{2.4}$$

for payoff functions $f^k : \mathbb{R} \to \mathbb{R}$. Under suitable integrability conditions our results carry over to more general payoff functions of the form $f^k(Y_T)$ and to affine processes on multi-dimensional state spaces. However, the resulting pricing formulae would be quite cumbersome and the Riccati equations that determine the processes' functional characteristics would no longer be solvable in closed form (the semi-explicit structure of the solution would be preserved, though). We define $f(x) := (f^1(x), ..., f^K(x))$.

[7] By extension it is meant that the functions ϕ and ψ can be uniquely analytically extended to a suitable subspace of $\mathbb{R}_+ \times \mathbb{C}^d$.

[8] More precisely, [22, Lemma 3.12] states that this holds on the set $\{(t, u) \in \mathcal{E}_{\mathbb{C}} \,:\, \left|E^0\left[\exp(u \cdot Y_s)\right]\right| \neq 0, \;\; \text{for all } s \in [0, t)\}$, whereas [22, Lemma 3.19] then yields that both sets coincide.

Theorem 2.4. *Let $Y = (V, X)$ be an analytic affine process on $\mathbb{R}_+ \times \mathbb{R}$, and suppose that the terminal payoffs of the securities are of the form (2.4). Suppose furthermore that there exists a vector of damping parameters $(\alpha^1, \ldots, \alpha^K, \beta) \in \mathbb{R}^{K+1}$ such that the functions*

$$g^k(x) := \exp\left(\alpha^k x\right) f^k(x) \exp\left(-\tilde{\gamma} \cdot f(x)\right) , \tag{2.5}$$

$$h(x) := \exp\left(\beta x\right) \exp\left(-\tilde{\gamma} \cdot f(x)\right) \tag{2.6}$$

as well as their respective Fourier transforms,

$$\hat{g}^k(s) = \int_{\mathbb{R}} \mathrm{e}^{-isy} g^k(y) dy \qquad \text{and} \qquad \hat{h}(s) = \int_{\mathbb{R}} \mathrm{e}^{-isy} h(y) dy \,,$$

are integrable, and that

$$\left(T, (0, -\alpha^k)\right) \in \mathcal{E} \,, \quad \text{for all } k, \quad \text{and} \quad (T, (0, -\beta)) \in \mathcal{E}. \tag{2.7}$$

Then the following holds:

- *The equilibrium price of S at time t is a function of $\tau := T - t$ and the current state of the process Y, and the price of the k-th security at time $t \in [0, T]$ is given by*

$$S_t^k = \frac{\int_{\mathbb{R}} \exp\left[\phi\big(\tau, (0, -\alpha^k + is)\big) + \psi\big(\tau, (0, -\alpha^k + is)\big) \cdot Y_t\right] \hat{g}^k(s)\, ds}{\int_{\mathbb{R}} \exp\left[\phi\big(\tau, (0, -\beta + is)\big) + \psi\big(\tau, (0, -\beta + is)\big) \cdot Y_t\right] \hat{h}(s)\, ds} \,. \tag{2.8}$$

  Here, ϕ and ψ denote the analytic extensions of the functions introduced in Definition 2.1.

- *The equilibrium price process of S at time t can alternatively be computed by*

$$S_t^k = -\frac{\partial}{\partial \zeta^k} H(\zeta) / H(\tilde{\gamma}) \Big|_{\zeta = \tilde{\gamma}} \,. \tag{2.9}$$

  Here, the function $H : \mathbb{R}^K \to \mathbb{R}$ is given by

$$H(\zeta) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left[\phi\Big(\tau, (0, -\beta(\zeta) + is)\Big) + \psi\Big(\tau, (0, -\beta(\zeta) + is)\Big) \cdot Y_t\right] \hat{h}(\zeta, s)\; ds \,,$$

  where we allowed for a functional dependence of $\beta \equiv \beta(\tilde{\gamma})$ on $\tilde{\gamma}$, and used the explicit dependence of the function $\hat{h}(s) \equiv \hat{h}(\tilde{\gamma}, s)$ on $\tilde{\gamma}$ through (2.6)*.*

The benchmark case where only one security is in non-zero endowment-adjusted supply and its payoff function is linear, and all other securities are in zero endowment-adjusted supply, does not require Fourier transform methods, as shown by the following corollary.

Corollary 2.5. *Let the process Y and the functions ϕ and ψ be as in Theorem 2.4. Let us further assume that there is only one security, denoted by S^1, in non-zero endowment-adjusted supply, that is*

$$\tilde{\gamma} = \left(\gamma(\eta^1 + n^1), 0, \ldots, 0\right) .$$

If furthermore $S_T^1 = X_T$ and $\tilde{\gamma}$ satisfies $\left(T, (0, -\tilde{\gamma}^1)\right) \in \mathcal{E}$, then the equilibrium price process of S^1 is given by

$$S_t^1 = \left[\partial_{u_x} \phi(\tau, u) + \partial_{u_x} \psi(\tau, u) \cdot Y_t\right] \Big|_{u = (0, -\tilde{\gamma}^1)} \,, \quad t \in [0, T] \,, \tag{2.10}$$

where $\tau := T - t$ and ∂_{u_x} denotes the partial derivative with respect to the second argument of the vector $u = (u_v, u_x)$. Furthermore, whenever the remaining securities $(S^2, \ldots, S^K)$ satisfy the assumptions of Theorem 2.4, their price processes equal

$$S_t^k = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left[\Delta_\tau^{\alpha^k, \tilde{\gamma}^1}(\phi) \;+\; \Delta_\tau^{\alpha^k, \tilde{\gamma}^1}(\psi) \cdot Y_t\right] \hat{g}^k(s)\, ds\,, \tag{2.11}$$

for $k = 2, \ldots, K$, and each $t \in [0, T]$. The functional shift operator $\Delta_t^{w,z}(\varphi)$ in (2.11) *is defined by*

$$\Delta_t^{w,z}(\varphi) := \varphi\big(t, (0, -w + is)\big) - \varphi\big(t, (0, -z)\big).$$

2.2. Pricing of Call Options

We are now going to establish semi-explicit pricing formulae for European call options. The main challenge will be to find suitable "damping" functions such that the Fourier methods of Theorem 2.4 can be applied. Specifically, we consider a market model with a single stock with terminal payoff $S_T = X_T$ and N call options on the stock with payoffs $C_T^i = (S_T - K_i)^+$, for $i = 1, ..., N$, and strike prices $K_1 < ... < K_N$. The stock and the options are traded simultaneously and hence collectively influence the equilibrium pricing kernel. The flattening functions for S and C^k are denoted α and α^k, respectively; the corresponding weighted payoff functions are denoted g and g^k, respectively. We first state the pricing formula for the most general case of multiple simultaneously traded options in non-zero endowment-adjusted supply. The formulae are a direct application of Theorem 2.4. Subsequently, we consider the particularly transparent cases where either a single option in non-zero endowment-adjusted supply or multiple options in zero endowment-adjusted supply are traded.

2.2.1. Multiple, simultaneously traded options

Let us first consider the general case where $N > 0$ call options and one stock in non-zero endowment-adjusted supply are traded. As an illustration, we assume that throughout Sections 2.2.1 and 2.2.2 all supply-adjusted risk aversion parameters satisfy

$$\tilde{\gamma}^1 = \ldots = \tilde{\gamma}^{N+1} = \gamma\,.$$

The pricing measure is then given by

$$\frac{dQ}{dP} = \frac{\exp\left(-\gamma\left(S_T + \sum_{i=1}^N (S_T - K_i)^+\right)\right)}{E\left[\exp\left(-\gamma\left(S_T + \sum_{i=1}^N (S_T - K_i)^+\right)\right)\right]}\,, \tag{2.12}$$

and the following result is an immediate consequence of Theorem 2.4.

Theorem 2.6. *Whenever α and β satisfy $\gamma < \alpha, \beta < (N+1)\gamma$ and* (2.7)*, the equilibrium price of the underlying security S at time $t \in [0, T]$ is given by*

$$S_t = \frac{\int_{\mathbb{R}} \exp\left[\phi\big(\tau, (0, -\alpha + is)\big) + \psi_1\big(\tau, (0, -\alpha + is)\big) V_t + \psi_2\big(\tau, (0, -\alpha + is)\big) X_t\right]\, \hat{g}(s) ds}{\int_{\mathbb{R}} \exp\left[\phi\big(\tau, (0, -\beta + is)\big) + \psi_1\big(\tau, (0, -\beta + is)\big) V_t + \psi_2\big(\tau, (0, -\beta + is)\big) X_t\right]\, \hat{h}(s) ds}\,,$$

and the price of the k-th call option is given by

$$C_t^k = \frac{\int_{\mathbb{R}} \exp\left[\phi\big(\tau, (0, is)\big) + \psi_1\big(\tau, (0, is)\big) V_t + \psi_2\big(\tau, (0, is)\big) X_t\right]\, \hat{g}^k(s) ds}{\int_{\mathbb{R}} \exp\left[\phi\big(\tau, (0, -\beta + is)\big) + \psi_1\big(\tau, (0, -\beta + is)\big) V_t + \psi_2\big(\tau, (0, -\beta + is)\big) X_t\right]\, \hat{h}(s) ds}\,,$$

for $k = 1, \dots, N$. Here the functions $\hat{g}$, $\hat{g}^k$ and $\hat{h}$ are given by

$$\hat{g}(s) = \sum_{j=1}^{N} \exp\left(\gamma \sum_{k=1}^{j-1} K_k\right) \exp\left[(-is + \alpha - j\gamma)K_j\right] \left[\left(\frac{-K_j\gamma}{(-is + \alpha - j\gamma)(-is + \alpha - (j+1)\gamma)}\right) + \left(\frac{1}{(-is + \alpha - (j+1)\gamma)^2} - \frac{1}{(-is + \alpha - j\gamma)^2}\right)\right],$$

$$\hat{h}(s) = \sum_{j=1}^{N} \exp\left(\gamma \sum_{k=1}^{j-1} K_k\right) \exp\left[(-is + \beta - j\gamma)K_j\right] \left[\frac{-\gamma}{(-is + \beta - j\gamma)(-is + \beta - (j+1)\gamma)}\right],$$

$$\begin{aligned}\hat{g}^k(s) = {} & \exp\left(\gamma \sum_{h=1}^{k-1} K_h\right) \exp\left[(-is - k\gamma)K_k\right] \left[\frac{1}{(-is - (k+1)\gamma)^2}\right] \\ & + \sum_{j=k+1}^{N} \exp\left(\gamma \sum_{h=1}^{j-1} K_h\right) \exp\left[(-is - j\gamma)K_j\right] \left[\left(\frac{-(K_j - K_k)\gamma}{(-is - j\gamma)(-is - (j+1)\gamma)}\right) \right. \\ & \left. + \left(\frac{1}{(-is - (j+1)\gamma)^2} - \frac{1}{(-is - j\gamma)^2}\right)\right].\end{aligned}$$

The assumption $\gamma < \alpha, \beta < (N+1)\gamma$ imposed on the damping factors ensures that the functions g, g^k, h of (2.5) and (2.6) allow for an integrable Fourier transform. In what follows, all model parameters have to be chosen such that (2.7) is satisfied and hence (2.2) applies. Further details will be discussed below.

2.2.2. A single option model

The pricing kernel (2.12) and the Fourier transforms from Theorem 2.6 simplify considerably when only one option with strike $K > 0$ is traded. In this case the price processes (S_t) and (C_t) can be computed analogously to Theorem 2.6 by

$$\begin{aligned}\hat{g}(s) &= \exp\left[(\alpha - \gamma - is)K\right] \left[\frac{-K\gamma}{(-is - \gamma + \alpha)(-is - 2\gamma + \alpha)} \right. \\ &\qquad \left. + \left(\frac{1}{(\alpha - 2\gamma - is)^2} - \frac{1}{(\alpha - \gamma - is)^2}\right)\right]. \\ \hat{h}(s) &= \exp\left[(\beta - \gamma - is)K\right] \left(\frac{-\gamma}{(\beta - \gamma - is)(\beta - 2\gamma - is)}\right). \\ \hat{g}^1(s) &= \exp\left[-(is + \gamma)K\right] \frac{1}{(-is - 2\gamma)^2}.\end{aligned}$$

2.2.3. Options in zero-endowment-adjusted supply

Let us finally consider the simplest case, where all options are in zero endowment-adjusted supply. In this case, the equilibrium prices are independent of option payoffs and one only needs to find a suitable α corresponding to the weighted payoff function (2.5) in Theorem 2.4. The simple choice $\alpha = 0$ already guarantees that the Fourier-transform

$$\hat{g}^1(s) = \frac{1}{(is + \tilde{\gamma}^1)^2} \exp\left[-K(is + \tilde{\gamma}^1)\right]$$

of the function $g^1(x) := \mathrm{e}^{-\tilde{\gamma}^1 x}(x-K)^+$ is integrable. In view of Corollary 2.5, the price process S is then given by (2.10), and we obtain that the price of the call option at time $t \in [0,T]$ is given by

$$C_t = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left[\Delta_\tau^{0,\tilde{\gamma}^1}(\phi) + \Delta_\tau^{0,\tilde{\gamma}^1}(\psi_1) V_t + \Delta_\tau^{0,\tilde{\gamma}^1}(\psi_2) X_t\right] \hat{g}^1(s) ds \,,$$

with $\tau := T - t$ and $\Delta_\tau^{0,\tilde{\gamma}^1}$ defined in Corollary 2.5.

2.3. Equilibrium Dynamics in a Heston Stochastic Volatility Model

By Choosing the dynamics of Y according to the well-known Heston stochastic volatility model [17], it is possible to derive explicit equilibrium stock price formulae. Specifically, suppose that $Y = (V, X)$ evolves according to

$$\begin{aligned} dV_t &= (\kappa - \lambda V_t)dt + \sigma\sqrt{V_t} dW_t^1 && V_0 = v_0 \,, \\ dX_t &= \mu dt + \sqrt{V_t} dW_t^2 && X_0 = x_0 \,, \end{aligned} \tag{2.13}$$

where $(\Omega, \mathcal{F}, P)$ is assumed to be rich enough to support the two-dimensional Brownian motion $W = (W^1, W^2)$.[9] The market filtration is the augmentation of the filtration generated by Y. The parameters $\mu, \kappa, \lambda, \sigma > 0$ will be chosen appropriately later on. We initially assume that the agents are trading a single security S in unit endowment-adjusted supply with payoff $S_T = X_T$. Since the additive Heston model is analytic affine and allows for explicit solutions of the functions ϕ and ψ, we apply the results obtained in Sections 1 and 2 to compute the equilibrium price S_t at time $t \in [0,T]$ in closed form as a function of Y_t.

Theorem 2.7. *Let $\theta(\gamma)$ be defined by*

$$\theta(\gamma) = \begin{cases} \sqrt{\lambda^2 - \sigma^2\gamma^2} & \text{if } \gamma < \frac{\lambda}{\sigma} \\ i\sqrt{\sigma^2\gamma^2 - \lambda^2} & \text{if } \gamma > \frac{\lambda}{\sigma} \end{cases} .$$

We suppose that γ is such that T satisfies

$$T < \begin{cases} +\infty & \gamma < \frac{\lambda}{\sigma} \\ \frac{2}{|\theta(\gamma)|}\left(\arctan\frac{|\theta(\gamma)|}{-\lambda} + \pi\right) & \gamma > \frac{\lambda}{\sigma} \end{cases} . \tag{2.14}$$

Then we have that, with $\tau := T - t$, $\theta := \theta(\gamma)$ and $\theta' := \frac{\partial}{\partial\gamma}\theta(\gamma)$, the equilibrium price process S is given by

$$S_t = T(\tau, \gamma) - \gamma\Gamma(\tau, \gamma)\, V_t + X_t \,, \tag{2.15}$$

for $t \in [0,T]$, and where

$$\begin{aligned} T(\tau,\gamma) = \frac{2\kappa}{\sigma^2\theta}\Big[\theta(\mathrm{e}^{\theta\tau}+1) + \lambda(\mathrm{e}^{\theta\tau}-1)\Big]^{-1} \Big[\Big(\theta(\mathrm{e}^{\theta\tau}+1) + \lambda(\mathrm{e}^{\theta\tau}-1)\Big)\Big(\theta' - \frac{1}{2}\sigma^2\gamma\tau\Big) \\ - \theta\Big(\theta'(\mathrm{e}^{\theta\tau}+1) + \tau\mathrm{e}^{\theta\tau}(\lambda\theta' - \gamma\sigma^2)\Big)\Big] \,, \end{aligned}$$

[9] The more general case of correlated Brownian motions could be included in (2.13) by considering $W^3 := \rho W^1 + \sqrt{1-\rho^2}W^2$ instead of W^2, and goes through analogously. We choose zero correlation in order to keep the notation tractable.

$$\Gamma(\tau,\gamma) = \left[\theta\left(\mathrm{e}^{\theta\tau}+1\right)+\lambda\left(\mathrm{e}^{\theta\tau}-1\right)\right]^{-1}\left[\left(2\left(\mathrm{e}^{\theta\tau}-1\right)-\gamma\tau\theta'\mathrm{e}^{\theta\tau}\right)\right.$$
$$\left.+\,\gamma\left(\mathrm{e}^{\theta\tau}-1\right)\left(\theta'\left(\mathrm{e}^{\theta\tau}+1\right)+\tau\mathrm{e}^{\theta\tau}\left(\lambda\theta'+\gamma\sigma^2\right)\right)\left(\theta\left(\mathrm{e}^{\theta\tau}+1\right)+\lambda\left(\mathrm{e}^{\theta\tau}-1\right)\right)^{-1}\right].$$

Note that (2.14) ensures that (2.7) in Theorem 2.4 is satisfied, which, in combination with the discussion in Section 2.2, allows us to study the impact of the model parameters in a framework comprising European options. More specifically, we illustrate within the Heston framework the effect of the parameters γ and σ on implied volatilities using the formulae obtained in Theorem 2.6. To this end, we consider a setting with one underlying asset and fifteen simultaneously traded call options written on it, all affecting the pricing density. In Figure 1, four different implied volatility curves are shown, corresponding to four different

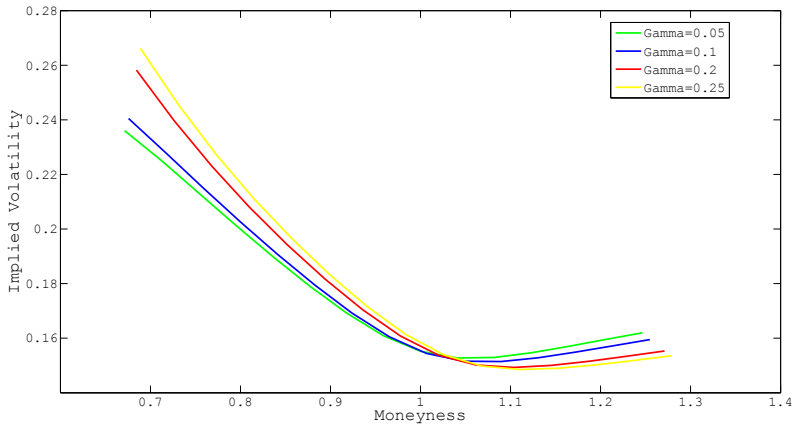

Figure 1: **Implied volatility curves with varying risk aversion**

values of the risk aversion γ. We see that, especially for in-the-money options, higher risk aversion yields a higher level of implied volatility. The more risk averse the representative agent is, the more in-the-money options are appreciated as good hedges against possibly low values of the underlying. The implied volatility curves for two different choices of the vol-of-vol parameter σ in (2.13) are shown in Figure 2. We observe a significant increase in implied volatility when changing from the low value (blue curve) to the higher one (red curve). That is due to the fact that a high value of σ increases the probability of S_T taking on extreme tail values and hence rendering even out-of-the-money options attractive instruments.[10]

2.4. Equilibrium Dynamics in a Pure Jump Ornstein-Uhlenbeck Setting

We consider now a single stock with terminal payoff $S_T = X_T$ where X is an Ornstein-Uhlenbeck process with a pure jump component as Lévy part[11]:

$$dX_t = -\lambda(X_t-\mu)dt + dJ_t \quad , \quad X_0 = x_0\,.$$

[10] For the Figures 1 and 2, the following parameters were used for the numerical computations: $\mu = 0.1$, $\kappa = 0.006$, $\lambda = 0.2$, $T = 0.5$, $t = 0$, $(x_0, v_0) = (1, 0.03)$. In Figure 1, we set $\sigma = 0.3$, whereas in Figure 2, $\gamma = 0.2$ was used.

[11] This is a specific subclass of basic affine processes, compare [11, Section A.2].

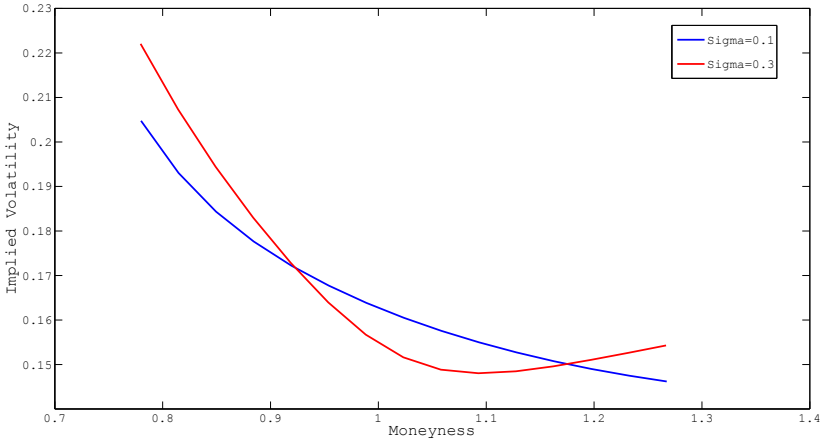

Figure 2: **Implied volatility curves with varying vol-of-vol**

Here, J is an adapted compound Poisson process with intensity $\kappa > 0$ and jump distribution $\nu(dx) = \frac{1}{2}\theta \exp(-\theta\,|x|)dx$.[12] The parameters μ and λ describe the long term mean and the mean reversion rate, respectively. In this one-dimensional setting the equations for the functional characteristics F and R are given by

$$F(u) = \lambda\mu u + \frac{\kappa\, u^2}{\theta^2 - u^2} \qquad \text{and} \qquad R(u) = -\lambda u\,, \tag{2.16}$$

see (B.1) and (B.2). Combining (2.16) with (B.4) and (B.5), we deduce that the functions ϕ and ψ satisfy the following system of Riccati equations

$$\begin{aligned} \partial_t\phi(t,u) &= \lambda\mu\psi(t,u) + \frac{\kappa\,\psi^2(t,u)}{\theta^2 - \psi^2(t,u)}\ , & \phi(0,u) &= 0 \\ \partial_t\psi(t,u) &= -\lambda\psi(t,u)\ , & \psi(0,u) &= u\ , \end{aligned}$$

which allows for the explicit solutions

$$\phi(t,u) = \frac{\kappa}{2\lambda}\log\left(\frac{\theta^2 - u^2\mathrm{e}^{-2\lambda t}}{\theta^2 - u^2}\right) + \mu u(1 - \mathrm{e}^{-\lambda t}) \qquad \text{and} \qquad \psi(t,u) = u\mathrm{e}^{-\lambda t}.$$

Thus, (2.2) holds, as long as $u \in \mathbb{R}\backslash\{-\theta,\theta\}$ and $T < t^*(u)$, with

$$t^*(u) = \begin{cases} +\infty & |u| < \theta \\ -\frac{1}{2\lambda}\log(\frac{\theta^2}{u^2}) & |u| > \theta \end{cases}\ . \tag{2.17}$$

This, together with Corollary 2.5, allows us to formulate the following:

Proposition 2.8. *If $|\tilde{\gamma}| \neq \theta$ and $T < t^*(-\tilde{\gamma})$, where t^* is as in* (2.17)*, then, with $\tau := T - t$, the equilibrium price process S is given by*

$$S_t = \left[\frac{\kappa\theta^2\tilde{\gamma}\left(\mathrm{e}^{-2\lambda\tau} - 1\right)}{\lambda(\theta^2 - \tilde{\gamma}^2)\left(\theta^2 - u^2\mathrm{e}^{-2\lambda\tau}\right)} + \mu(1 - \mathrm{e}^{-\lambda\tau})\right] + \mathrm{e}^{-\lambda\tau}X_t\,, \quad t \in [0,T]\,.$$

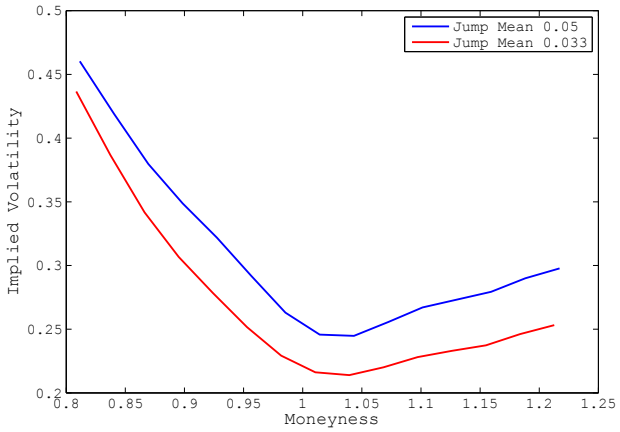

Figure 3: **Implied volatility curves with varying jump mean and intensity**

Figure 3 illustrates the dependence of option implied volatilities on the jump parameters. The red curve corresponds to smaller jumps arriving at a high frequency ($(\kappa, \frac{1}{\theta}) = (30, \frac{1}{30})$), whereas the blue one was obtained considering higher jumps at a lower frequency ($(\kappa, \frac{1}{\theta}) = (20, \frac{1}{20})$).[13] Increasing the mean jump height distinctly lifts the level of implied volatility, since the probability of S_T taking extreme values is higher that way. We further note that in general an affine model including jumps seems more suitable to reproduce the right-hand side smile observed in real market data.

Remark 2.9. Due to the linear payoff structure of the underlying asset $S_T = X_T$ in Sections 2.3 and 2.4, negative equilibrium prices can not be excluded a priori. However, this can be avoided by either considering only short trading horizons T as in the present work, or modeling the log-payoff of the underlying by $f(X)$, that is $S_T = \exp(f(X_T))$. Within the second approach, a verification of the integrability assumptions in Theorem 2.4 turns out to be extremely cumbersome due to the "double-exponential" structure, but once this is achieved, all our results can be adapted to that case. ⧫

3. Information-Based Equilibrium Pricing

In this section, we propose another method to model the market filtration based on the information-based asset pricing approach of Brody et al. [4] and Hoyle et al. [19]. They developed an asset pricing approach based on the modeling of cash flows and the explicit construction of market filtrations, which can be naturally embedded in the equilibrium pricing model considered in the present paper. The key idea is that, instead of assuming from the outset some abstract filtration representing the information available to the market, processes carrying market-relevant information are explicitly constructed, and a distinction between "genuine" information and market noise is made. The equilibrium dynamics is then computed by using the special form of the pricing measure obtained in Section 1, by assuming an *a priori* distribution of the market factor determining the terminal payoff, and by updating *a posteriori* distributions about the assets' payoffs obtained by a version of Bayes formula.

[12] More precisely, $J_t = \sum_{i=0}^{N_t} b_i D_i$, where N_t is a Poisson process with intensity κ, D_i are exponentially distributed i.i.d. random variables with jumps of mean $\frac{1}{\theta} > 0$, and b_i are i.i.d. Bernoulli random variables with $P[b_1 = 1] = P[b_1 = -1] = 0.5$.

[13] The remaining parameters were chosen as $(\mu, \lambda, \gamma, T, t, x_0) = (1, 2, 0.2, 0.1, 0, 1)$. As before, we considered 15 simultaneously traded call options.

3.1. Setup and Equilibrium Pricing Formula

We assume that the probability space $(\Omega, \mathcal{F}, P)$ supports a N-dimensional Brownian motion B together with N independent random market factors $(X_i)_{i=1}^N$, all independent of B such that $S_T^k = f^k(X_1, \ldots, X_N)$. The agents know the *a priori* distributions ν^i of all the factors X_i. With each market factor X_i we associate an observable process $(\xi_t)_{t\in[0,T]}$, the so-called *information process*. The information processes are defined by

$$\xi_t^i = \sigma_i X_i t + \beta_t^i\,, \quad t \in [0,T]\,, \tag{3.1}$$

where the independent standard Brownian bridges β^i on $[0,T]$ are defined in terms of B as solutions to the SDEs

$$d\beta_t^i = \frac{-\beta_t^i}{T-t}dt + dB_t^i\,, \quad \beta_0^i = 0\,, \tag{3.2}$$

for $t \in [0,T)$, and $\beta_T^i = 0$. Looking at the different components of the processes (3.1), we can distinguish between the part $\sigma_i X_i t$ containing real information about the realization of a market factor revealed over time, and the bridge part representing market noise. The speed at which the true outcome of X_i is revealed, is governed by the *information rate* σ_i. The information processes capture the flow of information available to the market agents, and thus generate the market filtration:

$$\mathcal{F}_t = \sigma\left(\xi_s^1, \ldots, \xi_s^N, s \le t\right)\,, \quad t \in [0,T]\,.$$

By construction, S_T is $\mathcal{F}_T$-measurable, and at each time $t \in [0,T]$, the equilibrium price S_t will be determined using the results of Section 1.

Theorem 3.1. *Assume that all a priori distributions ν^i allow for a density with respect to the Lebesgue-measure denoted by $v^i(x)$, respectively. If in addition the functions $(f^k)_{k=1}^K$ and the a priori densities $(v^i)_{i=1}^N$ are such that* (1.2) *is satisfied, then, for $t < T$, the equilibrium price process of the k-th security is given by*

$$S_t^k = \frac{\int_{\mathbb{R}^N} z(x_1,\ldots,x_N) f^k(x_1,\ldots,x_N)\pi_t^1(x_1)\cdots\pi_t^N(x_N)dx_1\ldots dx_N}{\int_{\mathbb{R}^N} z(x_1,\ldots,x_N)\pi_t^1(x_1)\cdots\pi_t^N(x_N)dx_1\ldots dx_N}\,, \tag{3.3}$$

where the function z is defined by

$$z(\cdot) = \exp\left[-\sum_{l=1}^K \tilde{\gamma}^l f^l(\cdot)\right]. \tag{3.4}$$

The regular conditional density function π_t^i associated with the i-th market factor is given by

$$\pi_t^i(x) = \frac{v^i(x)\exp\left[\frac{T}{T-t}\left(\sigma_i x\xi_t^i - \frac{1}{2}(\sigma_i x)^2 t\right)\right]}{\int_{\mathbb{R}} v^i(y)\exp\left[\frac{T}{T-t}\left(\sigma_i y\xi_t^i - \frac{1}{2}(\sigma_i y)^2 t\right)\right]dy}\,. \tag{3.5}$$

3.2. Innovation Processes and Equilibrium Market Price of Risk

Let us consider in the following the case $K = N = 1$, and in particular the case $S_T = X$ with corresponding information process given by $\xi_t = \sigma X t + \beta_t$, for $t \in [0,T]$. We assume that the market factor X is such that the conditions of Theorem 3.1 are satisfied. Formula (3.3) now reduces to

$$S_t = \frac{E\left[S_T \exp\left(-\tilde{\gamma}S_T\right) \mid \mathcal{F}_t\right]}{E\left[\exp\left(-\tilde{\gamma}S_T\right) \mid \mathcal{F}_t\right]} = \frac{\int x\exp\left(-\tilde{\gamma}x\right)\pi_t(x)dx}{\int \exp\left(-\tilde{\gamma}x\right)\pi_t(x)dx}\,. \tag{3.6}$$

Results from general filtering theory guarantee the existence of a P-Brownian motion W on $[0,T)$, *adapted* to the market filtration generated by ξ. Observe to this end that rearranging (3.2) leads to the following SDE satisfied by ξ on $[0,T)$

$$d\xi_t = \left[\frac{1}{T-t}\left(\sigma T X - \xi_t\right)\right] dt + dB_t\,, \quad \xi_0 = 0\,.$$

Hence, W is the innovations process associated with the information ξ given by

$$W_t = \xi_t - \int_0^t \left[\frac{1}{T-s}\left(\sigma T E\left[X \,|\, \mathcal{F}_s\right] - \xi_s\right)\right] ds\,, \quad t < T\,. \tag{3.7}$$

Thus, instead of having to assume the existence of Brownian motions as drivers for the prices, they rather emerge naturally from within the information-driven structure. By the Fujisaki-Kallianpur-Kunita Theorem, see [1, Proposition 2.31], both expressions appearing in (3.6) allow for a representation with respect to the innovations Brownian motion. Furthermore, we know the structure of the integrands in the above representations. For every function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(X) \in L^2(P)$ and for $t < T$, we obtain that

$$dE\left[\varphi(X) \,|\, \mathcal{F}_t\right] = \frac{\sigma T}{T-t} V_t^{\varphi} dW_t\,,$$

where

$$V_t^{\varphi} = E\left[\varphi(X)X \,|\, \mathcal{F}_t\right] - E\left[\varphi(X) \,|\, \mathcal{F}_t\right] E\left[X \,|\, \mathcal{F}_t\right] \tag{3.8}$$

is the conditional covariance of the market factor with the function φ. As long as square integrability is satisfied, we can apply the above to (3.6) with $g(x) = xe^{-\tilde{\gamma}x}$ and $h(x) = e^{-\tilde{\gamma}x}$ and use the Itô product rule to obtain

$$dS_t = \sigma^h \left[\sigma^h \left(S_t (V_t^h)^2 - V_t^g V_t^h\right) dt + \left(V_t^g - S_t V_t^h\right) dW_t\right]\,,$$

where

$$\sigma^h = \frac{\sigma T}{(T-t) E\left[h(X) \,|\, \mathcal{F}_t\right]}.$$

The expressions V_t^g, V_t^h and $E\left[h(X) \,|\, \mathcal{F}_t\right]$ can be worked out semi-explicitly by means of (3.8), the integral formula (3.6), and the regular conditional density $\pi(x)$ defined in (3.5). Each of them is a function of t and ξ_t due to the Markov property of the information process.

3.3. Pricing Credit-Risky Securities

In this section, we illustrate the impact of the "noisyness" of information on the equilibrium prices of a credit-sensitive security within a simple benchmark model, see [3], where the a-priori distribution of $S_T = X$ is discrete: $S_T \in \{0,1\}$. We denote by $p_0 := P[X = 0]$ the probability of default. Due to the discrete payoff structure, formula (3.3) simplifies, and allows us to examine the impact of model parameters such as the *information flow rate*, or the risk aversion and supply on the equilibrium price of S. The price of the security threatened by default can be obtained in closed form analogously to (3.3) and is given by

$$S_t = \frac{p_1 x_1 \exp\left(-\tilde{\gamma} x_1\right) \exp\left[\frac{T}{T-t}\left(\sigma x_1 \xi_t - \frac{1}{2}(\sigma x_1)^2 t\right)\right]}{\sum_{i=0,1} p_i \exp\left(-\tilde{\gamma} x_i\right) \exp\left[\frac{T}{T-t}\left(\sigma x_i \xi_t - \frac{1}{2}(\sigma x_i)^2 t\right)\right]}\,, \quad t < T\,. \tag{3.9}$$

Applying Itô's product rule to (3.9) together with (3.7) yields the following dynamics[14] for S

$$dS_t = \frac{\sigma T}{T-t}\mathrm{Var}_{Q,t}(X)\left[\frac{\sigma T}{T-t}\left(E\left[X \mid \mathcal{F}_t\right] - S_t\right)dt + dW_t\right], \quad t < T\,.$$

Due to the Markov property of the information process, the terms $E\left[X \mid \mathcal{F}_t\right]$ and $\mathrm{Var}_{Q,t}(X)$ are functions of the pair (t, ξ_t) and triplet $(t, \xi_t, \tilde{\gamma})$, respectively. Figure 4 shows the impact of σ on the price of a defaultable bond, where the probability of default was chosen to be $p_0 = 0.2$. On the left-hand side the bond does not default, whereas on the right-hand side we considered the situation of a default. In

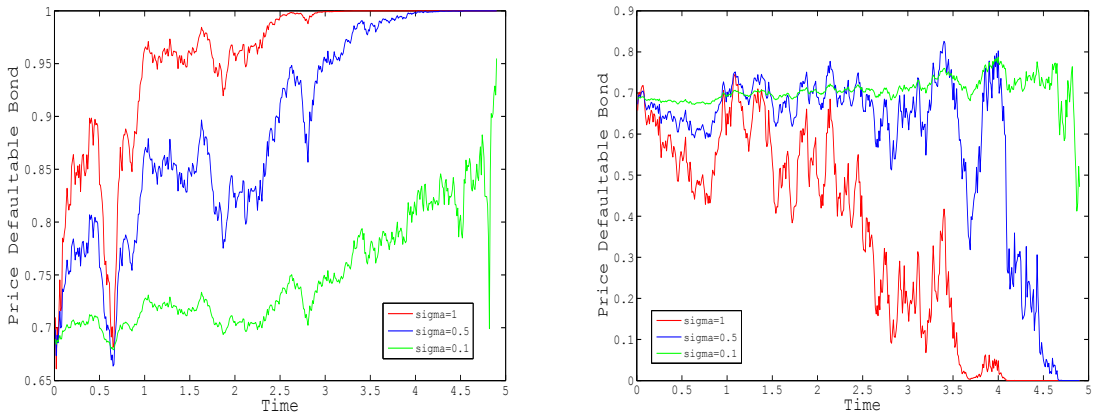

Figure 4: **Defaultable Bond Prices**

both cases, a low information flow rate (green curve, $\sigma = 0.1$) leads to a rather late adjustment of the equilibrium price process towards the prevailing terminal value, while the red curve ($\sigma = 1$) reacts earlier to the information about the outcome of X.[15]

3.4. One-Dimensional, Exponentially-Distributed Terminal Cash Flow

We illustrate how, for particular choices of v and f, the formulae (3.5) and (3.6) can be worked out explicitly. We assume $f(x) = x$, corresponding to the assets payoff itself being the market factor. Furthermore, the a priori distribution of S_T, the dividend at time T, is assumed to be exponential.

Corollary 3.2. *Assume that the a-priori distribution of $S_T = X$ is of the exponential form, that is, $v(x) = (1_{\{x\geq 0\}}/\kappa)\exp(-x/\kappa)$ for some $\kappa > 0$. If $\tilde{\gamma} > \kappa - 1$, then the equilibrium price at time $t < T$ is given by*

$$S_t = \left[\frac{\exp\left(-\frac{1}{2}B_t^2/A_t\right)}{\sqrt{2\pi A_t}\,\mathcal{N}(B_t/A_t)} + \frac{B_t}{A_t}\right], \tag{3.10}$$

where

$$A_t = \sigma^2 tT/(T-t) \quad \textit{and} \quad B_t = \sigma T\xi_t/(T-t) - \frac{\tilde{\gamma}\kappa + 1}{\kappa}\,. \tag{3.11}$$

[14] The expression $\mathrm{Var}_{Q,t}(X)$ denotes the conditional variance of X under Q, given $\mathcal{F}_t$.

[15] The following parameters were used for the simulations: $x_1 = 1$, $P[X = x_1] = 0.8$, $T = 5$, $\tilde{\gamma} = 0.6$. The price process is shown for $t \in [0, 4.9]$.

Since the pricing measure depends only on the terminal cash-flow as a consequence of the attainable endowments, changing from P to Q could be interpreted as a different view $\tilde{v}$ of the representative agent on the a-priori-distribution of S_T. More precisely, under Q the cash-flow S_T is exponentially distributed with new parameter $(\tilde{\gamma}\kappa + 1)/\kappa$, also appearing in (3.11), which can be seen by working out the adjusted density

$$\tilde{v}(x) = \frac{\exp(-\tilde{\gamma}x)v(x)}{\int \exp(-\tilde{\gamma}y)v(y)dy}\,. \tag{3.12}$$

A. Proofs

Proof of Theorem 1.1
Due to the time-consistency and strict monotonicity of the entropic preferences, it suffices to show that the strategies $\hat{\vartheta}^a$ are optimal for the utility maximization in $t = 0$. Note first that (1.2) ensures that (1.3) and (1.4) are well-defined. In particular, the price process S is a Q-martingale, and thus $Q \in \mathcal{P}$. Furthermore, the constant strategies $\hat{\vartheta}^a$ lie in Θ, since for any $\tilde{Q} \in \mathcal{P}$, the process $G_t(\hat{\vartheta}^a) = \hat{\vartheta}^a \cdot (S_t - S_0)$ is by assumption a $\tilde{Q}$-martingale, and hence in particular a $\tilde{Q}$-supermartingale.

We now show that the quantity γ introduced in (1.1) can be seen as the risk aversion of some *representative agent* whose optimal utility is attained at the constant strategy $\vartheta^* \equiv n$. Indeed, since S is a Q-martingale, and $n \cdot S_T \in L^1(Q)$, the utility maximization of the representative agent can be formulated as follows[16]:

$$\begin{aligned}
&\sup_{\vartheta\in\Theta, E_Q[G_T(\vartheta)]\le E_Q[n\cdot S_T]} \{U_0^\gamma\left(\eta\cdot S_T + G_T(\vartheta)\right)\} \\
=\;& \sup_{\vartheta\in\Theta} \left\{U_0^\gamma\big(\eta\cdot S_T + G_T(\vartheta) - E_Q\left[G_T(\vartheta)\right] + E_Q\left[n\cdot S_T\right]\big)\right\} \\
=\;& \sup_{\vartheta\in\Theta} \left\{U_0^\gamma\big(G_T(\vartheta)\big) - E_Q\left[G_T(\vartheta)\right]\right\} + E_Q\left[(n+\eta)\cdot S_T\right] \\
\le\;& \frac{1}{\gamma}H(Q|P) + E_Q\left[(n+\eta)\cdot S_T\right]\,.
\end{aligned} \tag{A.1}$$

The last equality is derived from the dual representation of U_0^γ, where the relative entropy is given by $H(Q\,|\,P) = E[\frac{dQ}{dP}\log(\frac{dQ}{dP})]$. But $G_T(\vartheta^*)$ with $\vartheta^* \equiv n$ plugged into the representative agent's utility $U_0^\gamma(\eta\cdot S_T + \cdot)$ yields

$$U_0^\gamma\big((n+\eta)\cdot S_T\big) = \frac{1}{\gamma}H(Q|P) + E_Q\left[(n+\eta)\cdot S_T\right]\,.$$

Comparing this with (A.1) shows that $\vartheta^* \equiv n$ is indeed optimal for the representative agent when the price process S is given by (1.4). Individual optimality of $\hat{\vartheta}^a$ for the single agents now follows by a scaling argument and the specific form of the aggregated endowment. Note that, for all $a \in \mathbb{A}$,

$$\vartheta^* + \eta = \arg\max_{\vartheta\in\Theta}\{U_0^\gamma(G_T(\vartheta))\}$$

is equivalent to

$$\frac{\gamma}{\gamma^a}(\vartheta^* + \eta) = \arg\max_{\vartheta\in\Theta}\{U_0^a(G_T(\vartheta))\}\,,$$

[16] Note that the first expression in (A.1) is equivalent to the representative agent's utility maximization of terminal wealth against both, the aggregated initial endowments η *and* aggregated net supply n, over all admissible strategies.

which in turn is equivalent to

$$\frac{\gamma}{\gamma^a}(\vartheta^* - \eta) - \eta^a = \arg\max_{\vartheta\in\Theta} \{U_0^a(H^a + G_T(\vartheta))\} \ .$$

This shows that $\hat{\vartheta}^a$ is the optimal strategy for agent $a \in \mathbb{A}$. Since the strategies $(\hat{\vartheta}^a)_{a\in\mathbb{A}}$ add up to n, the market clears at any time, and hence $((S_t)_{t\in[0,T]}, (\hat{\vartheta}^a)_{a\in\mathbb{A}})$ forms an equilibrium. □

Proof of Theorem 2.4

Part 1: Pricing Formula (2.8). By assumption, the process $Y = (V, X)$ is analytic affine and hence we know from Section 2 that its conditional characteristic function allows for the representation

$$E\left[\exp\left(u \cdot Y_T\right) \mid \mathcal{F}_t\right] = \exp\left[\phi(\tau, u) + \psi(\tau, u) \cdot Y_t\right] \ , \tag{A.2}$$

for all $u = (u_v, u_x) \in \mathbb{C}^2$ such that $(T, u) \in \mathcal{E}_{\mathbb{C}}$. This holds, since the fact that $\mathcal{D}_t \supseteq \mathcal{D}_T$, whenever $t \leq T$, and $(T, u) \in \mathcal{E}_{\mathbb{C}}$ imply that formula (2.1) holds for t, whenever it holds for T, and hence (2.2) as well.

Let us assume for the moment that (1.2) holds. This will be verified later. We then know from (1.3) that the equilibrium pricing measure Q is given by its Radon-Nikodym-density

$$\frac{dQ}{dP} = \frac{\exp\left(-\tilde{\gamma} \cdot S_T\right)}{E\left[\exp\left(-\tilde{\gamma} \cdot S_T\right)\right]} = \frac{\exp\left(-\tilde{\gamma} \cdot f(X_T)\right)}{E\left[\exp\left(-\tilde{\gamma} \cdot f(X_T)\right)\right]} \, .$$

Hence, by applying Bayes formula and following (1.4), we obtain

$$S_t^k = E_Q\left[S_T^k \,|\, \mathcal{F}_t\right] = \frac{E\left[f^k(X_T) \exp\left(-\tilde{\gamma} \cdot f(X_T)\right) \mid \mathcal{F}_t\right]}{E\left[\exp\left(-\tilde{\gamma} \cdot f(X_T)\right) \mid \mathcal{F}_t\right]} \tag{A.3}$$

for the equilibrium price of the k-th security. The Fourier transforms $\hat{g}^k$ and $\hat{h}$ defined in (2.5) and (2.6), respectively, exist and are integrable by assumption. Hence we apply the Fourier inversion formula[17] to obtain

$$g^k(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{isx} \hat{g}^k(s) ds \qquad \text{and} \qquad h(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{isx} \hat{h}(s) ds \, ,$$

dx-almost surely. With this at hand, (A.3) transforms to

$$S_t^k = \frac{E\left[\exp\left(-\alpha^k X_T\right) g^k(X_T) \,|\, \mathcal{F}_t\right]}{E\left[\exp\left(-\beta X_T\right) h(X_T) \,|\, \mathcal{F}_t\right]} = \frac{E\left[\int_{\mathbb{R}} \exp\left[(-\alpha^k + is)X_T\right] \hat{g}^k(s) ds \,|\, \mathcal{F}_t\right]}{E\left[\int_{\mathbb{R}} \exp\left[(-\beta + is)X_T\right] \hat{h}(s) ds \,|\, \mathcal{F}_t\right]} \, . \tag{A.4}$$

Now we observe that

$$E\left[\left| \int_{\mathbb{R}} \exp\left[(-\alpha^k + is)X_T\right] \hat{g}^k(s) ds \right| \mid \mathcal{F}_t\right]$$

$$< E\left[\exp\left(-\alpha^k X_T\right) \int_{\mathbb{R}} \left|\hat{g}^k(s)\right| ds \mid \mathcal{F}_t\right] < \infty \, , \tag{A.5}$$

[17] See [9, Theorem 9.5.4].

since $\big(T,(0,-\alpha^k)\big) \in \mathcal{E} \subseteq \mathcal{E}_{\mathbb{C}}$ and $\hat{g}^k$ is integrable. The same holds analogously for the denominator in (A.4). In particular, we have

$$0 < E\left[\exp\left(-\tilde{\gamma}\cdot F(X_T)\right) \mid \mathcal{F}_t\right] < \infty\,, \quad \text{for all } \; t \in [0,T]\,, \tag{A.6}$$

since we required Y^T to be conservative and $(T,(0,-\beta))$ to lie in $\mathcal{E}$. Thus, (A.5) and (A.6), in combination with (A.3), yield that (1.2) is indeed satisfied. We may now apply Fubini's Theorem to exchange the order of integration, and we get that

$$\begin{aligned} E\left[\int_{\mathbb{R}} \exp\left[(-\alpha^k+is)X_T\right]\hat{g}^k(s)ds \,|\, \mathcal{F}_t\right] &= \int_{\mathbb{R}} E\left[\exp\left[(-\alpha^k+is)X_T\right] \mid \mathcal{F}_t\right]\hat{g}^k(s)ds \\ &= \int_{\mathbb{R}} \exp\left[\phi\big(\tau,(0,-\alpha^k+is)\big)+\psi\big(\tau,(0,-\alpha^k+is)\big)\cdot Y_t\right]\hat{g}^k(s)ds\,. \end{aligned} \tag{A.7}$$

The affine transformation formula (A.2) holds, since $\big(T,(0,-\alpha^k)\big) \in \mathcal{E}$. Applying the same arguments to the denominator in (A.4) combined with (A.7) yields the desired form of S_t^k in (2.8).

Part 2: Pricing Formula (2.9). We outline the details for $K = 1$, the rest follows by repeating the arguments for the partial derivative with respect to each ζ^k. So we assume we only have one security S with corresponding $\tilde{\gamma} \in \mathbb{R}$ affecting the density of the pricing measure Q. It follows that

$$\frac{dQ}{dP} = \frac{\exp(-\tilde{\gamma}S_T)}{E\left[\exp(-\tilde{\gamma}S_T)\right]} = \frac{\exp(-\tilde{\gamma}f(X_T))}{E\left[\exp(-\tilde{\gamma}f(X_T))\right]}$$

and the equilibrium price of S at time t can be obtained again by computing

$$S_t = \frac{E\left[f(X_T)\exp(-\tilde{\gamma}f(X_T)) \,|\, \mathcal{F}_t\right]}{E\left[\exp(-\tilde{\gamma}f(X_T)) \,|\, \mathcal{F}_t\right]}\,. \tag{A.8}$$

Recall from Part 1 that

$$\exp\left(-\tilde{\gamma}f(X_T)\right) \in L^1(P) \quad \text{and} \quad f(X_T)\exp\left(-\tilde{\gamma}f(X_T)\right) \in L^1(P)\,, \tag{A.9}$$

due to the assumption of $(T,(0,-\alpha))$ and $(T,(0,-\beta))$ lying in $\mathcal{E}$. Since the set $\mathcal{E}_{\mathbb{C}}$ is open, compare [22, Lemmata 3.12 and 3.19], the first integrability in (A.9) even holds in some neighbourhood of $\tilde{\gamma}$, allowing us to differentiate the function $\zeta \mapsto E[\exp(-\zeta f(X_T))|\mathcal{F}_t]$ at $\zeta = \tilde{\gamma}$.[18] Indeed, by the smoothness of the mapping $\zeta \mapsto \exp(-\zeta f(X_T))$ and the integrability of the second term in (A.9), we obtain

$$E\left[f(X_T)\exp\left(-\tilde{\gamma}f(X_T)\right) \mid \mathcal{F}_t\right] = -\,\frac{\partial}{\partial\zeta}E\left[\exp\left(-\zeta f(X_T)\right) \mid \mathcal{F}_t\right]\Big|_{\zeta=\tilde{\gamma}}\,, \tag{A.10}$$

as an application of the triangular inequality and dominated convergence. On the other hand we know from an analogue of (A.4) and (A.7) that the denominator in (A.8) can be computed by

$$\begin{aligned} &E\left[\exp\left(-\tilde{\gamma}f(X_T)\right) \mid \mathcal{F}_t\right] \\ &\qquad = \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left[\phi\big(\tau,(0,-\beta(\tilde{\gamma})+is)\big)+\psi\big(\tau,(0,-\beta(\tilde{\gamma})+is)\big)\cdot Y_t\right]\,\hat{h}(\tilde{\gamma},s)\;ds\,, \end{aligned} \tag{A.11}$$

[18] Here, we may assume without loss of generality that the model dependent "damping factor" β is chosen according to a mapping $\tilde{\gamma} \mapsto \beta(\tilde{\gamma})$ that is continuous in sufficiently small neighbourhood of $\tilde{\gamma}$.

where we need the dependence of $\beta = \beta(\tilde{\gamma})$ and $\hat{h}(s) = \hat{h}(\tilde{\gamma}, s)$ on $\tilde{\gamma}$. Combining (A.10) and (A.11) yields

$$E\left[f(X_T)\exp\left(-\tilde{\gamma}f(X_T)\right) \mid \mathcal{F}_t\right] = -\frac{\partial}{\partial\zeta}\left(\frac{1}{2\pi}\int_{\mathbb{R}} \exp\Big[\phi\big(\tau,(0,-\beta(\zeta)+is)\big) + \psi\big(\tau,(0,-\beta(\zeta)+is)\big)\cdot Y_t\Big]\,\hat{h}(\zeta,s)\,ds\right)\Bigg|_{\zeta=\tilde{\gamma}}. \quad \square$$

Proof of Corollary 2.5
Expression (2.10) is an immediate consequence of (2.9) in Theorem 2.4 with $f(x) = x$, and the fact that there is no need of Fourier methods to compute the denominator $H(\tilde{\gamma})$ in the analogue to (A.8)

$$S_t^1 = \frac{E\left[X_T\exp(-\tilde{\gamma}^1 X_T)\,|\,\mathcal{F}_t\right]}{E\left[\exp(-\tilde{\gamma}^1 X_T)\,|\,\mathcal{F}_t\right]}, \tag{A.12}$$

since the affine transformation formula directly applies to the denominator in (A.12). We recall that $\big(T,(0,-\tilde{\gamma}^1)\big) \in \mathcal{E}$. Now we only need to compute $\frac{\partial}{\partial\zeta}E\left[\mathrm{e}^{-\zeta X_T}\,|\,\mathcal{F}_t\right]$, the actual derivative in formula (2.9). However, from (2.2) it follows that

$$-\frac{\partial}{\partial\zeta}E\left[\exp\left(-\zeta X_T\right)\,|\,\mathcal{F}_t\right] = \exp\big[\phi\big(\tau,u\big) + \psi\big(\tau,u\big)\cdot Y_t\big]\,\big[\partial_{u_x}\phi(\tau,u) + \partial_{u_x}\psi(\tau,u)\cdot Y_t\big]\big|_{u=(0,-\zeta)}\,.$$

Combining the above with (A.12) yields

$$S_t^1 = \big[\partial_{u_x}\phi(\tau,u) + \partial_{u_x}\psi(\tau,u)\cdot Y_t\big]\big|_{u=(0,-\tilde{\gamma}^1)}\,.$$

As to the remaining securities $S^2,\ldots,S^K$, their price processes given in (2.11) directly follow from formula (2.8) in Theorem 2.4 and the discussion above. $\square$

Proof of Theorem 2.6
An application of Theorem 2.4 with $\alpha^k = 0$, for all $k = 1,\ldots,N$, in addition to the observation that the Fourier transforms are all integrable functions yields the desired result. As to the second claim of integrability, straightforward calculations show that there exist constants $\hat{M}, \hat{z} > 0$, just depending on the model parameters, which give

$$\max_{f\in\{\hat{g},\hat{h},(\hat{g}^k)_{k=1}^N\}}\int_{\mathbb{R}}|f(s)|ds < \hat{M}\int_{\mathbb{R}}\frac{1}{s^2+\hat{z}}ds < \infty\,. \quad \square$$

Proof of Theorem 2.7
The process $Y = (V,X)$ belongs to a subclass of affine processes, namely to the $\mathbb{R}^2$-valued affine diffusions.[19] That is, Y is a solution to the stochastic differential equation $dY_t = \mu(Y_t)dt + \rho(Y_t)dW_t$, with $Y_0 = y_0$, for a continuous function $b : D \to \mathbb{R}^2$ and a measurable function $\rho : D \to \mathbb{R}^{2\times 2}$ such that $y \mapsto \rho(y)\rho(y)^T$ is continuous. In particular, Y is analytic, since the set $\mathcal{D}$ from (2.3) is non-empty. See

[19] We emphasize that we would not have needed the complete theory on general affine processes including various possible behavior of jumps, had we only considered pure diffusion processes, since it was shown in [13, Theorem 10.1] that every diffusion Markov process with continuous diffusion matrix is affine, if and only if the functions b and $\rho\rho^T$ are affine in the state variable and the solutions ϕ and ψ of the Riccati equations satisfy $\mathrm{Re}(\phi(t,u)+\psi(t,u)\cdot y) \le 0$, for all $y \in D$ and $(t,u) \in \mathbb{R}_+ \times i\mathbb{R}^d$. Our equilibrium approach can cover more sophisticated models than pure diffusions though.

for instance the discussion on explosion times of the Heston model in [14]. Furthermore, the process Y is conservative and, hence, so is the stopped process Y^T. Combining (B.3) with the fact that the generator of (V, X) is determined by its diffusion matrix $\rho\rho^T$ and its drift vector b, we identify the admissible parameters in (B.1), (B.2) and (B.3), where the parts connected with jumps do not play a role here. Hence we conclude that the conditional characteristic function of Y allows a representation as follows

$$E\left[\exp\left(u \cdot Y_T\right) \mid \mathcal{F}_t\right] = \exp\left[\phi(\tau, u) + \psi(\tau, u) \cdot Y_t\right] , \tag{A.13}$$

whenever $(T, u) = (T, (u_v, u_x)) \in \mathcal{E}_{\mathbb{C}}$, so in particular for $(T, (u_v, u_x)) \in \mathcal{E}$. The functions ϕ and ψ satisfy the following system of Riccati equations

$$\begin{aligned}
\partial_t\phi(t,u) &= \kappa\,\psi_1(t,u) + \mu\,\psi_2(t,u)\,, && \phi(0,u) = 0\\
\partial_t\psi_1(t,u) &= \frac{1}{2}\sigma^2\psi_1(t,u)^2 - \lambda\psi_1(t,u) + \frac{1}{2}\psi_2(t,u)^2\,, && \psi_1(0,u) = u_v\\
\partial_t\psi_2(t,u) &= 0\,, && \psi_2(0,u) = u_x\,.
\end{aligned} \tag{R}$$

A solution to the above system (R), evaluated at the vector $u = (0, u_x)$, is given by[20]

$$\begin{aligned}
\phi\big(t,(0,u_x)\big) &= \frac{2\kappa}{\sigma^2}\log\left(\frac{2\theta(u_x)\exp\left(\frac{\theta(u_x)+\lambda}{2}t\right)}{\theta(u_x)(\mathrm{e}^{\theta(u_x)t}+1)+\lambda(\mathrm{e}^{\theta(u_x)t}-1)}\right) + \mu u_x t\,,\\
\psi_1\big(t,(0,u_x)\big) &= \frac{u_x^2(\mathrm{e}^{\theta(u_x)t}-1)}{\theta(u_x)(\mathrm{e}^{\theta(u_x)t}+1)+\lambda(\mathrm{e}^{\theta(u_x)t}-1)}\,,\\
\psi_2\big(t,(0,u_x)\big) &= u_x\,.
\end{aligned}$$

where

$$\theta(u_x) = \begin{cases} \sqrt{\lambda^2 - \sigma^2\,u_x^2} & \text{if} \quad |u_x| < \frac{\lambda}{\sigma} \\ i\sqrt{\sigma^2\,u_x^2 - \lambda^2} & \text{if} \quad |u_x| > \frac{\lambda}{\sigma} \end{cases} .$$

Following Friz and Keller-Ressel [14] and recalling that $\lambda > 0$, we distinguish two different cases

$$t^+(u_x) = \begin{cases} +\infty & |u_x| < \frac{\lambda}{\sigma} \\ \frac{2}{|\theta(u_x)|}\left(\arctan\frac{|\theta(u_x)|}{-\lambda} + \pi\right) & |u_x| > \frac{\lambda}{\sigma} \end{cases}$$

such that $(T, (0, u_x)) \in \mathcal{E} \subseteq \mathcal{E}_{\mathbb{C}}$, for all $T \leq t^+(u_x)$.[21] Hence, as long as $T < t^+(u_x)$, formula (A.13) holds for all $u = (0, u_x)$, where $u_x \in \mathbb{R}$. It now follows from (2.10) in Corollary 2.5 that

$$S_t = \left[\partial_{u_x}\phi(\tau,u) + \partial_{u_x}\psi_1(\tau,u)V_t + \partial_{u_x}\psi_2(\tau,u)X_t\right]\Big|_{u=(0,-\gamma)} \quad , \quad \text{for all } \ t \in [0,T]\,. \tag{A.14}$$

Next we need to compute the derivatives of $\phi(t, u)$ and $\psi(t, u)$ with respect to u_x. Of course we have $\partial_{u_x}\psi_2(\tau, u) \equiv 1$ and a straightforward calculation yields, with $\theta := \theta(-\gamma)$ and $\theta' := [\partial_{u_x}\theta](-\gamma)$,

$$\partial_{u_x}\phi(\tau,(0,-\gamma)) = T(\tau,\gamma) \quad \text{and} \quad \partial_{u_x}\psi_1(\tau,(0-\gamma)) = -\gamma\Gamma(\tau,\gamma)\,.$$

This, together with (A.14), is (2.15), the proof is complete. □

[20] Compare [13, Lemma 10.12]. For $u_x = \lambda/\sigma$ we set $\psi_1(t,(0,\frac{\lambda}{\sigma})) = t/(2+\lambda t)$, resembling the limit and still satisfying $\psi_1(0,(0,\lambda/\sigma)) = 0$.

[21] Basically, this is exactly the time interval on which the solutions of the Riccati equations do not explode.

Proof of Theorem 3.1
By assumption, the conditions of Theorem 1.1 are satisfied. Recall that the equilibrium price is obtained by the change of measure from P to Q, that is:

$$S_t^k = E_Q\left[S_T^k \,|\, \mathcal{F}_t\right] = E_Q\left[f^k(X_1,\dots,X_N) \,|\, \mathcal{F}_t\right] = E\left[\frac{dQ}{dP} f^k(X_1,\dots,X_N) \,|\, \mathcal{F}_t\right] E\left[\frac{dQ}{dP} \,|\, \mathcal{F}_t\right]^{-1}.$$

By (1.3), we know that $\frac{dQ}{dP}$ is a function of S_T and hence of $X_1,\dots,X_N$, which is given in (3.4). Then we compute the regular conditional distribution of $(X_1,\dots,X_N)$ given $(\xi_t^1,\dots,\xi_t^N)$. Using the independence of the market factors, the Markov property of ξ, the Bayes formula, and observing that, given $(X_1,\dots,X_N) = (x_1,\dots,x_N)$, ξ_t^i is Gaussian with mean $\sigma_i x_i t$ and variance $\frac{tT}{T-t}$, yields (3.5). □

Proof of Corollary 3.2
The relation $\tilde{\gamma} > \kappa - 1$ ensures that the assumptions of Theorem 1.1 are met. It remains to apply Theorem 3.1 and explicitly work out the integrals, which is done by combining [4, Section VII] and (3.12), resulting in formulae (3.10) and (3.11). □

B. Addendum to Section 2: Regular Affine Processes

This proposition concerning the characterization of a regular affine process by its admissible parameters is stated without proof and we refer to [12, Theorem 2.7] or [22, Theorem 2.6 and Equations (2.2a),(2.2b)] for two different approaches to prove it.

Proposition B.1. *Let Y be a regular affine process with state space D. Let F and R be as in Definition 2.1. Then there exists a set of admissible parameters $(A, A^i, b, b^i, c, c^i, m, \mu^i)_{i\in\{1,\dots,d\}}$ such that F and R are of the Lévy-Khintchine form.*

$$F(u) = \frac{1}{2}\langle u, Au\rangle + \langle b, u\rangle - c + \int_{\mathbb{R}^d\setminus\{0\}} \left(\mathrm{e}^{\langle \xi, u\rangle} - 1 - \langle h(\xi), u\rangle\right) m(d\xi) \tag{B.1}$$

$$R_i(u) = \frac{1}{2}\langle u, A^i u\rangle + \langle b^i, u\rangle - c^i + \int_{\mathbb{R}^d\setminus\{0\}} \left(\mathrm{e}^{\langle \xi, u\rangle} - 1 - \langle \chi^i(\xi), u\rangle\right) \mu^i(d\xi)\,, \tag{B.2}$$

where $A, A^1,\dots,A^d$ are positive semi-definite real $d\times d$-matrices; $b, b^1,\dots,b^d$ are $\mathbb{R}^d$-valued vectors; $c, c^1,\dots,c^d$ are positive non-negative numbers; $m, \mu^1,\dots,\mu^d$ are Lévy measures on $\mathbb{R}^d$ and $h, \chi^1,\dots,\chi^d$ are suitably chosen truncation functions for the respective Lévy measures. Furthermore, the generator $\mathcal{A}$ of Y is given by

$$\begin{aligned}\mathcal{A}\varphi(x) = \;& \frac{1}{2}\sum_{k,l=1}^{d}\left(A_{kl} + \sum_{i\in I} A_{kl}^i x_i\right)\frac{\partial^2\varphi(x)}{\partial x_k \partial x_l} + \langle b + \sum_{i=1}^{d} b^i x_i, \nabla\varphi(x)\rangle - \left(c + \sum_{i\in I} c^i x_i\right)\varphi(x) \\ & + \int_{D\setminus\{0\}} (\varphi(c+\xi) - \varphi(x) - \langle h(\xi), \nabla\varphi(x)\rangle) m(d\xi) \\ & + \sum_{i\in I}\int_{D\setminus\{0\}} (\varphi(c+\xi) - \varphi(x) - \langle \chi^i(\xi), \nabla\varphi(x)\rangle) x_i \mu^i(d\xi)\,,\end{aligned} \tag{B.3}$$

and ϕ, ψ satisfy the following system of ODEs

$$\partial_t \phi(t,u) = F(\psi(t,u)), \quad \phi(0,u) = 0 \tag{B.4}$$

$$\partial_t \psi(t,u) = R(\psi(t,u)), \quad \psi(0,u) = u. \tag{B.5}$$

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