

# TRADING

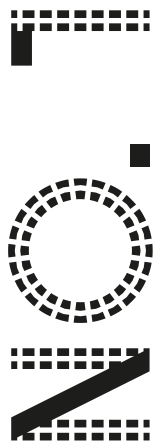
## INVESTING

### Investing as Random Trial

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## Preface

This is the first paper of a series about algorithmic trading, which provides a forum to openly discuss research results from the field of algorithmic trading theory. The author works in the field of signal processing and has avoided trading jargon. So, reading is not restricted to trading experts and economists. In order to develop a theory and confirm it by experimental findings, he developed a software research platform. A considerable amount of results have been produced with the help of this platform so far. An overview is given in the outlook. Due to the amount of data, the presentation of the results will be distributed over several papers. For the future, further research using the research platform is intended by the author and his partners.

To relate the results to existing literature turned out to be difficult, because the topic of trading is spread over a huge amount of scientific journals about econometrics and other specialist fields. Furthermore, the author was confronted with fee-based full access to articles, which sometimes appear secretive and incomprehensible. Publication in a journal proved to be not convenient, because a publication series was not provided. Huge waiting times till publishing, submission fees — combined with the possibility of a desk rejection —, strict demands with respect to size and styles of a paper, were other reasons to opt for a publication as an open access research report. The series does not focus on the software research platform, but on scientific results. What is intended by this series is to provide a theory about trading, which will be taken seriously by professionals in Trading.

The author thanks his staff member Patrick Blättermann for many helpful discussions and his comprehensive research findings in the existing literature universe. Thanks also to Mrs. Aust and Mrs. Regulski from the library of the *University of Applied Sciences, Düsseldorf* for their important support over the years, and Mrs. Söhnitz and Mrs. Geick for their assistance.

# Investing as Random Trial

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## Abstract

We introduce an investment algorithm for a market of individual securities. The investment algorithm is derived from constraints depending on investment parameters in order to limit the risk and to take into account an individual investor. One constraint is devoted to trading costs. Purchased securities are selected randomly among securities that meet the buy condition, making trading a random trial. Simulations with historical price data are demonstrated for a simple example: The buy condition is evaluated on the basis of the price relationship for two subsequent trading days and the sales condition is defined by holding securities only for one day. A trading expert evaluates the expected return for the investment algorithm with respect to the random selection. Thus, the expert informs precisely on how many market players perform using the same investment algorithm. Its findings are for a parametrized set of buy conditions simultaneously, which makes a trading expert a valuable tool for theorists as well as for practitioners. In our example, the trading expert demonstrated clearly a significant mean reversion effect for a horizon of one day.

**Keywords:** Investment constraints, investment algorithm, investing as random trial, trading expert, mean reversion

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# 1 Introduction

## 1.1 Methods of commercial practice

Methods of commercial practice have previously been scientifically tested for example in [1] (moving average, relative strength index, stochastic oscillator), [2] (moving average for comparison), [3] (moving average), [4] (formations like head- and shoulder), [5] (moving average and trading range break), [6] (bull and bear flag stock chart), [7] (universe of nearly 8000 trading rules). For example, a moving average averages the prices of the last 200 trading days. A purchase will be performed, if the actual price of the security exceeds the moving average by a certain amount (*buy condition*) and will be sold, if the actual price fall below another threshold (*sales condition*). With except of [1], the investigated methods have been shown to be effective and thus provide a scientific justification for algorithmic trading. They demonstrate that prices do not follow random walks [8], but have significant influence on future returns. This is not surprising under the premise that market players rely on the methods and thus follow them. A return of a security can be measured by the price relationship or *momentum*  $s(t_2)/s(t_1)$  according to  $R = s(t_2)/s(t_1) - 1$ , where  $s$  denotes the prices of the security, which are sampled at time  $t_1$  and  $t_2$  [8]. A momentum greater 1 yields a positive return and reflects raising prices while a momentum less than 1 yields a negative return, signaling falling prices within the period from  $t_1$  to  $t_2$ .

**Return continuation and mean reversion:** In the moving average example, we trust in *return continuation*, which means that rising prices in the past follow rising prices in the future. In [9], it is shown that it is profitable, to buy shares with high returns in the past  $J$  months and hold them for a certain period. The best return continuation is obtained from  $J = 12$  months and a holding period of 3 months. In contrast to return continuation, it has been recognized that prices quickly return to normality after rising or falling periods. This effect has been recognized in [10] for longer horizons of several years and was called *mean reversion*. In [11], mean reversion is pointed out for shorter horizons of one to three months, and is utilized for successful trading. Horizons of one week or lower are investigated for example in [12] and [13]. Since the autocorrelation of portfolio returns is positive in this case, mean reversion is not verified for short horizons by autocorrelation. In [14], positive autocorrelation is demonstrated for weekly returns as well, but autocorrelations of individual securities are generally negative.

Return continuation as well as mean reversion are parts of a professional trading system sketched in [15]. Return continuation and mean reversion are located in the Alpha model, belonging to the *pre-trade analysis* component of the trading system. The pre-trade analysis also includes the *risk model* and the *transaction cost model*. The pre-trade analysis offers recommendations for securities to buy or sell. This means that a buy condition and sales condition are evaluated by pre-trade analysis. It encompasses any system that uses financial data or news to analyze certain properties of an asset.

**Statistical evaluations:** The statistical evaluations of the return differ, depending on whether an arithmetic mean return is computed for an index or future as in [2], [5], [6], [7], or a mean return is determined for single shares first and averaged over the shares afterwards [1], [9]. In [3], a cumulative return is evaluated. In [4], conditional return pdfs are computed for single shares. In [10], [11], [12], [13] and [14], correlation coefficients are computed between past and future returns.

**Markets:** Algorithmic trading will be applied to a *market*, defining the securities to be traded. It may contain a single security, like an index, a future or share, or the market contains several individual securities. A real market consists of historical or real-time prices, as depicted in [15]. Using real markets comes along two difficulties. First, one is faced with the fact that historical prices are more or less incomplete and erroneous. This is an important subject of data analysis and data mining, e.g. [16]. Secondly, a market should be not too small in order to obtain statistically reliable results. On the other hand, markets should exclude securities with low-level data quality. For research purposes, securities and markets may be designed artificially, for example by a random process, e.g. a random

walk, as used in [2], [4]. In [5], the AR and GARCH models are used to explain empirical findings. An overview over random processes used in financial statistics can be found for example in [8].

### Risk:

Besides a market, it has to be decided, which securities should be selected for investments and in what quantities. According to [15], this is decided in the Portfolio Construction Model of an algorithmic trading system. It needs the pre-trade analysis component, which provides information about the return and the risk of investments to be expected. Thus, a risk model is included in pre-trade analysis as well [15], [17]. According to [17], risk management might recommend simple stop-loss values, or using a fixed amount of the capital on every trade. The risk model attempts to quantify both the risk associated with an individual security and with the portfolio, using the *Sharp Ratio* for example or a *Standard value at Risk* calculation (VAR) per security or for the portfolio [15]. Of course, all measures for risk are evaluated by past data and thus, only reflect historical risks.

Following the *mean variance portfolio theory*, expected return and risk of a portfolio are optimized together in order to determine weighting factors for the securities in the portfolio. The risk is defined by the variance of the portfolio return. Minimizing risk at a specified expected return (or maximizing the expected return at a specified risk) requires the covariance matrix of the returns *for all* securities in the portfolio [8]. The covariances must be determined by analyzing historical returns and the matrix must be inverted. No wonder, important objections against mean variance portfolio optimization can be found in the literature, e.g. [18]: Experienced investment professionals have noted that these methods often were unintuitive and without obvious investment value. Furthermore, optimized portfolios were generally found to be unmarketable and equal weighting is often significantly superior.

## 1.2 Our concepts

In [19], the conclusion about trading strategies ends with a citation of Roll (1994): "Many of these effects are surprisingly strong in reported empirical work, but I have never yet found one that worked in practice". We hope that our concepts, we start to explain, are simple enough to disprove this belief.

### Market of individual securities:

For theory and simulations we assume a market of individual securities, represented by a number of  $N$  time series

$$s(n, t), \quad 1 \leq n \leq N, \quad t_1(n) \leq t \leq T. \quad (1)$$

Here,  $n$  denotes the number of the security,  $t$  is a whole number representing a trading day and  $s(n, t)$  is the price of security  $n$  at trading day  $t$ .  $t_1(n)$  denotes the beginning of a security, which can vary over the securities. By (1), securities do not end before day  $T$ . We assume that trading ends at this day. By this assumption, we avoid that investments cannot be finished due to missing prices. For simplicity, we assume that trading starts at  $t = 1$ , i.e.  $t = 1$  is the first trading day.

### Trading formulas:

The conditions for purchases and sales are defined by trading formulas: A *buy formula* defining the buy condition and a *sales formula* for the sales condition. Both trading formulas use only past but no future prices (principle of causality). The trading formulas depend on properties  $c_1, c_2, \dots$ , given by real numbers, evaluating past data. One of the properties is the holding period, which only appears in the sales formula. Thus, a *trading state* is used, which stores past purchases in order to define the holding period, given by  $\tau_{\text{hold}} = t - \text{purchase date}$ . In this case, the sales formula may be composed of a holding period condition, e.g.  $\tau_{\text{hold}} \geq 30$ , and an additional condition depending on past prices of the security. If a security has been bought several times, only those investments will be finished, which have been started until  $t - \tau_{\text{hold}}$ . Thus, the security may not be completely sold. In contrast, without holding period condition, securities will be always sold completely. Properties are not restricted to prices of a security and the holding period, but can be related to arbitrary past data, quantifiable by real numbers.

**Statistics:**

Our statistics include different quantities like chance and risk and measures for volatility. In this paper, only the expected return is considered. For statistical evaluations, the buy formulas for trading can be used (and even sales formulas, which do not contain the holding period). They are evaluated for a certain horizon of one or several trading days and a certain trading period. The horizon corresponds to the holding period used for trading. Besides certain buy formulas, statistical evaluations are carried out by so-called *trading experts*. A trading expert computes statistical quantities for a parametrized set of buy conditions simultaneously. This means that the properties  $c_1, c_2 \dots$  used in a buy formula build up a property space, which is clustered by the trading expert. This paper provides a simple example with one property. As with buy formulas, statistical evaluations of a trading expert are performed for a certain horizon and a certain trading period.

**Risk and random selection:**

Our concept with respect to risk provides constraints for investments, which spread investments over time and over several securities. One constraint limits the ratio of investments to the capital. Concerning the risk associated with a single security, a measure for the historical risk of the security can be limited by means of a certain property, used in the buy formula. The buy formula provides *buy recommendations* and the purchased securities are selected randomly among them. By this concept, minimization of the (historical) risk of a portfolio is not carried out and a covariance matrix of past returns is not required. Furthermore, the random selection enables the computation of the expected return, carried out in this paper for a holding period of one day. Perhaps, the possible danger of algorithmic trading as reported in [17] can be reduced by random selection, which would be a further important advantage. Due to the random selection of securities, trading becomes a *random trial*. This means that each market player achieves a different return although using the same investment algorithm, introduced in this paper.

**1.3 Main Focus of this paper**

In our works, investments are based on a time-independent buy condition and constraints, depending on investment parameters, limiting risk and taken into account an individual investor. One constraint avoids that the investment in a security is too small so that trading costs are no longer proportional to the transaction volume and become more prevalent. The amount of new investment at a trading day is defined by maximizing it under these constraints. The purchased securities are selected randomly among the *buy recommendations*, making trading a random trial. The expectation value of the return with respect to this random trial defines the *expected return*. Thus, the expected return states the mean return, achieved by many market players, using all the same investment parameters. For a holding period of one day, we find a simple expression for the expected return under a weak condition. A trading expert evaluates this expression for a parametrized set of buy conditions simultaneously, which makes an expert a valuable tool in trading research.

**Organization of this paper:**

The rest of this paper is organized as follows. In section 2, the concept for the investment algorithm and the constraints for investing are introduced. A justification of the constraint concerning the investment ratio is given in appendix A. In section 3, the investment algorithm is specified by maximizing the amount of new investment under these constraints. Proofs are delegated to appendix B. Section 4 gives insights into simulation results for special investment parameters and trading formulas, in particular for a holding period of one day. In section 5, a formula for the expected return is given for a holding period of one day. The evaluations are delegated to appendix C. Experts for a holding periods of more than one day will be treated in a next paper. An overview about planned publications is given in the outlook. Appendix D gives an overview about notations and terms.

## 2 Investing under constraints

In section 2.1, we proceed to explain our trading concept sketched in the introduction. In section 2.2 and 2.3, the constraints for investing are introduced. In the last two sections, we discuss the problem arising from two constraints.

### 2.1 Investment algorithm (overview)

We assume a market of individual securities, a buy and sales condition and a trading state, which has to be initialized before the first trading day  $t = 1$ . For each trading day  $t \geq 1$ , the following steps have to be carried out:

**Step 1:** Perform all sales, using the sales condition and update the trading state.

**Step 2:** Evaluate all *buy recommendations*, defined by all securities of the market, which meet the buy condition. The number of buy recommendations is denoted  $N(t)$ .

**Step 3:** Determine the amount  $I_{new}(t)$  of new investment and the number  $N_{buy}(t)$  of purchases.

**Step 4:** Select  $N_{buy}(t)$  from the  $N(t)$  buy recommendations for purchase. For  $0 < N_{buy}(t) < N(t)$  select them randomly.

**Step 5:** For each selected security, invest the same amount according to

$$I_{buy}(t) = I_{new}(t)/N_{buy}(t) \quad (2)$$

and update the trading state.

#### Trading state:

The *trading state* includes past investments (purchases), which are not finished, defined by an identifier for the security, the amount of investment in the security and the purchase date. Besides past investments, the trading state includes the cash. Trading starts with an initial trading state, which stores past investments and the cash, denoted by  $C_{cash}(0)$ . The amount of investment at the beginning of trading is denoted by  $C_{inv}(0)$ . It is obtained by multiplying the quantity of a security with its price, accumulated over all securities initially bought. When starting trading without investments,  $C_{inv}(0) = 0$  holds. For  $C_{inv}(0) > 0$ , not only purchases but sales are possible at the first day as well. For each trading day, the trading state must be updated for each sale and purchase, to keep track on trading (steps 1 and 5). By our concept of trading, it is possible, to interrupt trading at the end of a certain trading day  $t_0$  and continue trading at the next day  $t_0 + 1$ , using the trading state achieved at the end of day  $t_0$ .

#### Coincidence of sale and purchase:

According to steps 1 and 5, sales are carried out before the purchases. In particular, the same security may be sold and purchased at the same trading day, if the sales and buy condition are satisfied simultaneously. At first glance, the coincidence of sale and purchase seems to be senseless, producing unnecessary trading costs. However, depending on the investments finished and started for a security, the amount of investment for the security may be changed. Depending on the investment algorithm, the amount of investments in other securities are affected as well. Thus, the coincidence of sale and purchase may be reasonable. For the investment algorithm, the coincidence of sale and purchase is even required in order to evaluate the expected return for a holding period of one day in section 5 precisely, i.e. trading and statistics are *consistent*. A better treatment of the situation, when sale and buy condition coincide for a security, will reduce trading costs, but consistency of trading and statistics is impaired and thus, is not considered in this paper.



**Capital:**

$C_{cash^-}(t)$  and  $C_{inv^-}(t)$  denote the amount of cash and investment achieved immediately after the sales in step 1, while  $C_{cash}(t)$  and  $C_{inv}(t)$  denote cash and investment at the end of day  $t$ . The sum

$$C_{tot}(t) = C_{cash}(t) + C_{inv}(t) > 0 \quad (3)$$

defines the (total) capital at the end of trading day  $t$ . For  $t = 0$ , we obtain  $C_{tot}(0) = C_{cash}(0) + C_{inv}(0)$ , which is the total capital at the beginning of trading.

The total capital plays an important role, because the amount  $I_{new}(t)$  of new investment at trading day  $t$  will depend on the total capital at this day. Hence, the total capital will be required after sales and before purchases, given by

$$C_{tot^-}(t) = C_{cash^-}(t) + C_{inv^-}(t) . \quad (4)$$

Since  $C_{cash^-}(t)$  is decreased in step 5 by the amount of new investments  $I_{new}(t)$  and  $C_{inv^-}(t)$  is increased by the same amount,

$$C_{cash}(t) = C_{cash^-}(t) - I_{new}(t) , \quad (5)$$

$$C_{inv}(t) = C_{inv^-}(t) + I_{new}(t) , \quad (6)$$

we obtain

$$\begin{aligned} C_{tot}(t) &= C_{cash}(t) + C_{inv}(t) = C_{cash^-}(t) + C_{inv^-}(t) \\ &= C_{tot^-}(t) , \end{aligned} \quad (7)$$

i.e. the total capital before purchases equals the total capital at the end of a trading day.

**Pre-assumption:**

A general *pre-assumption* consists in equal new investments for all purchases, denoted by  $I_{buy}(t)$  for trading day  $t$ , which results in (2). Since the quantity of purchased securities is an integer number, the new investment  $I_{buy}(t)$  in a security is an integer multiple of its price,

$$I_{buy}(t) = \text{quantity} \cdot \text{price} . \quad (8)$$

This quantization effect is clearly noticeable, if the price is high, but we will ignore this effect in theory. In simulations, the real-valued quantity  $I_{buy}/\text{price}$  will be evaluated first and then rounded down to obtain a whole-numbered quantity.

**2.2 Constraints for investing**

**Investment diversification:** The new investment in a security is limited to the proportion of  $1/U_{\min}$  of the total capital by the constraint

$$I_{buy}(t)/C_{tot}(t) \leq 1/U_{\min} , \quad (\text{constraint A})$$

where  $U_{\min}$  is assumed to be a positive integer. Hence, the total capital cannot be invested in less than  $U_{\min}$  securities, i.e. is spread over  $U_{\min}$  securities at least.

**Temporal investment diversification:** The new investment for all securities is limited to the proportion of  $1/\tau$  of the total capital,

$$I(t) := I_{new}(t)/C_{tot}(t) \leq 1/\tau, \quad (\text{constraint B})$$

where  $\tau$  is assumed to be a positive integer. Here,  $I(t)$  denotes the rate for new investments, which will be called *investment rate*. By this constraint, new investments should be temporally spread, which is called *dollar cost averaging* in [13]. The effect of constraint (B) will be demonstrated in the following example. We use a holding period  $\tau_{\text{hold}}$  and start trading without initial investments. Then, for the first trading days  $t \leq \tau_{\text{hold}}$ , sales do not appear. Hence, cash is steadily decreasing and the investment is steadily increasing by new investments, that is

$$C_{cash}(t) = C_{cash}(0) - \sum_{i=1}^t I_{new}(i), \quad (9)$$

$$C_{inv}(t) = \sum_{i=1}^t I_{new}(i), \quad t \leq \tau_{\text{hold}}. \quad (10)$$

In contrast, the total capital remains constant,

$$C_{tot}(t) = C_{cash}(t) + C_{inv}(t) = C_{cash}(0), \quad t \leq \tau_{\text{hold}}. \quad (11)$$

By constraint (B)

$$I_{new}(t) \leq C_{cash}(0)/\tau, \quad t \leq \tau_{\text{hold}}$$

holds. From (9), we obtain

$$C_{cash}(t) \geq C_{cash}(0) - C_{cash}(0) \cdot t/\tau = C_{cash}(0) \cdot [1 - t/\tau], \quad t \leq \tau_{\text{hold}}.$$

Thus,  $\tau$  trading days must pass at least, until the initial cash is completely exhausted. Hence,  $\tau$  serves as *investing period*, preventing that investments take place only at the first trading day (for  $\tau > 1$ ), but are temporally spread.

**Lower limitation of new investments:** In order to limit trading costs, the new investment in a security is limited by a real number  $\text{inv}_{\min} > 0$  according to the constraint

$$I_{buy}(t) \geq \text{inv}_{\min} \text{ for } N_{buy}(t) > 0. \quad (\text{constraint C})$$

From (2) we obtain

$$N_{buy}(t) = I_{new}(t)/I_{buy}(t) \leq I_{new}(t)/\text{inv}_{\min}. \quad (12)$$

Thus, the number of purchases is limited upwards due to constraint (C). Since costs for buying a single security in reality are not below a certain threshold, the lower bound  $\text{inv}_{\min}$  prevents high trading costs.

**Limitation of number of purchases:** Besides (12), the number of purchases is limited by the number of buy recommendations, denoted by  $N(t)$ ,

$$N_{buy}(t) \leq N(t). \quad (\text{constraint D})$$

If the number of purchases is less than then the number of buy recommendations, suggested by the buy formula,

$$0 < N_{buy}(t) < N(t),$$

purchased securities have to be selected from the recommended ones. To perform the selection, one could try to minimize the risk, as intended by the mean variance portfolio theory. Instead of

this concept, the securities are simply selected randomly. As a consequence, different trading players, although using the same investment algorithm with equal investment parameters and trading formulas, attain different trading results, if trading days with  $0 < N_{buy}(t) < N(t)$  exist. On this way trading becomes a random trial.

### Limitation of investment ratio:

By the next constraint, the investment ratio does not exceed a certain upper limit  $IR_{\max}$  with  $0 \leq IR_{\max} \leq 1$ . As for cash and investment, the investment ratio is related to the end of a trading day and thus is defined as

$$IR(t) := C_{inv}(t)/C_{tot}(t) . \quad (13)$$

Hence, the constraint is

$$IR(t) = C_{inv}(t)/C_{tot}(t) \leq IR_{\max} . \quad (\text{constraint E})$$

Constraint (E) limits new investments as is the case with the constraints (A) and (B). In order to quantify this limitation, (6) will be used. By this relation and (13) we obtain

$$\begin{aligned} IR(t) &= C_{inv}(t)/C_{tot}(t) \\ &= C_{inv^-}(t)/C_{tot}(t) + I_{new}(t)/C_{tot}(t) . \end{aligned}$$

Using the investment ratio after the sales and before the purchases at trading day  $t$ ,

$$IR^-(t) := C_{inv^-}(t)/C_{tot}(t) , \quad (14)$$

this relation rewrites

$$IR(t) = IR^-(t) + I(t) . \quad (15)$$

Thus, constraint (E) is equivalent to a limitation of new investments according to

$$I(t) \leq IR_{\max} - IR^-(t) . \quad (16)$$

## 2.3 The constraints at a glance

The constraints for determining new investments are

$$I_{buy}(t)/C_{tot}(t) \leq 1/U_{\min} \quad (17a)$$

$$I(t) = I_{new}(t)/C_{tot}(t) \leq 1/\tau \quad (17b)$$

$$I_{buy}(t) \geq \text{inv}_{\min} \text{ for } N_{buy}(t) > 0 \quad (17c)$$

$$N_{buy}(t) \leq N(t) \quad (17d)$$

$$I(t) \leq IR_{\max} - IR^-(t) . \quad (17e)$$

They depend on the following investment parameters:  $U_{\min}$ , defining the investment diversification over securities, the investment period  $\tau$  for temporal investment diversification, the lower bound  $\text{inv}_{\min}$  for investments in securities and the maximum investment ratio  $IR_{\max}$ .

## 2.4 The problem with constraint (A)

Constraint (A) avoids that the total capital cannot be invested in less than  $U_{\min}$  securities with respect to a certain trading day. However, it does not avoid that the same security may be bought on several days. This seems to be reasonable, since the buy condition is satisfied several times in this case. On the other hand, the amount of capital invested in a single security could exceed the threshold  $C_{tot}(t)/U_{\min}$  considerable, if investments are hold longer than one day. Only for selling after a holding period of one day, this problem does not occur. The effect will be particularly noticeable, if only a few securities are recommended by the buy condition. In this case, the investments become very risky. On the other hand, buy conditions with a small number of buy recommendations prove to be interesting. To limit the risk, the number of investments in each purchased security, which is stored in the trading state of past purchases, could be limited. This or other additional constraints are not considered in this paper, because the consistency of trading and statistics is impaired.

## 2.5 The problem with constraint (E)

Constraint (E) gives rise to the following question: Since new investments are non-negative, the investment rate  $I(t)$  must be non-negative as well. From (17e) we obtain the condition

$$IR^-(t) \leq IR_{\max} . \quad (18)$$

This means that the investment ratio after sales and before purchases may not exceed the maximum investment ratio  $IR_{\max}$ . Thus, the question rises, how to confirm this condition. A justification of the condition is given in the following lemma. The proof is found in appendix A.

**LEMMA 1 (Condition  $IR^-(t) \leq IR_{\max}$ )** *We assume that for each trading day the total capital is positive and the proceeds are non-negative. Furthermore, we assume that new investments carried out before trading day  $t$  ( $t \geq 1$ ) are non-negative and*

$$IR(t-1) \leq IR_{\max} \quad (19)$$

*holds. Then, (18) is satisfied for trading day  $t$ .*

**Discussion:** We assume a total capital

$$C_{tot}(t) > 0 \quad (20)$$

for all trading days. Thus, the investment ratios  $IR(t) = C_{inv}(t)/C_{tot}(t)$  and  $IR^-(t) = C_{inv}^-(t)/C_{tot}(t)$  are well-defined. Secondly, proceeds from the sales are assumed to be not negative. The proceeds from the sales at trading day  $t$  are denoted by  $sale(t)$ . Thus, we have for all trading days

$$sale(t) \geq 0 . \quad (21)$$

Consequently,  $sale(t) = 0$  holds in the worst case, being a total loss.

For the first trading day  $t = 1$ , (19) represents the trading start condition

$$IR(0) = C_{inv}(0)/C_{tot}(0) \leq IR_{\max} . \quad (\text{trading start condition}) \quad (22)$$

It means that the investment ratio at the beginning of the first trading day  $t = 1$  is upper bounded by  $IR_{\max}$ . From lemma 1 we obtain

$$IR^-(1) \leq IR_{\max} ,$$

which is the statement for  $t = 1$ . This inequality ensures a non-negative investment rate  $I(1)$  according to

$$0 \leq I(1) \leq IR_{\max} - IR^-(1) .$$

Thus, new investments before  $t = 2$  are non-negative and lemma 1 yields

$$IR^-(2) \leq IR_{\max} .$$

Continuing this process, we obtain the condition in (18) for all trading days. It requires investment rates before trading day  $t$  lying within the range

$$0 \leq I(t') \leq IR_{\max} - IR^-(t') , \quad 1 \leq t' < t . \quad (23)$$

### 3 Maximization of the investment rate

We first consider the case that investments are not carried out at any time. In particular, trading starts with  $C_{inv}(0) = 0$ . In this case, we get

$$N_{buy}(t) = 0, I_{buy}(t) = I_{new}(t) = I(t) = 0$$

and the constraints (A) – (D) are satisfied by triviality. Furthermore, since no investments are carried out,

$$IR(t) = IR^-(t) = 0$$

holds for all trading days  $t \geq 1$ . Constraint (E) rewrites  $0 \leq IR_{\max}$  and is satisfied as well.

#### 3.1 Investment algorithm (step 3)

In contrast to the former case, we claim a maximum amount of new investment, which means a maximum investment rate  $I(t)$  under the constraints (A) – (E). This optimization problem is studied in detail in appendix B. The main results are summarized in the following theorem.

**THEOREM 1 (Investment algorithm)** *Maximizing the investment rate under the constraints (A) – (E) yields the investment rate  $I(t)$  or*

$$y := I(t) \cdot U_{\min} \tag{24}$$

depending on the total capital  $C_{tot}(t)$  or

$$x := C_{tot}(t) / [\text{inv}_{\min} \cdot U_{\min}] > 0 \tag{25}$$

according to

$$x < 1 : y = 0 \quad (\text{no new investments}) \tag{26}$$

$$x \geq 1 : y = \begin{cases} y_1 : \text{RBF}(x, y_1) \geq 0 \\ \text{int}[y_1] : \text{RBF}(x, y_1) < 0 \end{cases} \tag{27}$$

with  $\text{int}(a)$  denoting rounding off a real number  $a$ . Here,

$$\text{RBF}(x, y) := \text{int}[x \cdot y] - y \tag{28}$$

denotes the range buy function, which has to be evaluated for  $x$  and

$$y_1 := I_1(t) \cdot U_{\min} \tag{29}$$

with

$$I(t) \leq I_1(t) := \min\{N(t)/U_{\min}, 1/\tau, IR_{\max} - IR^-(t)\}. \tag{30}$$

The minimum number of purchases is given by

$$N_{buymin}(t) := \text{int}^+(y) = \text{int}^+(I(t) \cdot U_{\min}) \tag{31}$$

with  $\text{int}^+(a)$  denoting rounding up a real number  $a$ , and the maximum number of purchases is

$$\begin{aligned} N_{buymax}(t) &:= \min\{\text{int}[x \cdot y], N(t)\} \\ &= \min\{\text{int}[I(t) \cdot C_{tot}(t) / \text{inv}_{\min}], N(t)\}. \end{aligned} \tag{32}$$

**Non-negative new investments:**

In order to ensure non-negative investment rates, the condition

$$IR^-(t) \leq IR_{\max}$$

is required. Otherwise, (30) yields a negative investment rate  $I(t)$ . The condition  $IR^-(t) \leq IR_{\max}$  has been confirmed for investment rates  $I(t')$  lying within the range given by (23),

$$0 \leq I(t') \leq IR_{\max} - IR^-(t') , \quad 1 \leq t' < t ,$$

assuming  $IR^-(t') \leq IR_{\max}$ . Confirmation of (23) under the assumption  $IR(t') \leq IR_{\max}$  is simply done: From (30) we obtain  $I(t') \leq IR_{\max} - IR^-(t')$ , which is the right side of (23). From (30) and  $IR^-(t') \leq IR_{\max}$  we further obtain  $I_1(t') \geq 0$ . Evaluating (26) and (27) for  $I_1(t') \geq 0$  results in  $I(t') \geq 0$ , which is the left side of (23). We conclude that all investment rates are non-negative,

$$I(t) \geq 0 , \quad t \geq 1 . \quad (33)$$

**The range buy function:**

The range buy function defines the range of the number of purchases, because by (B5)

$$y \leq N_{buy}(t) \leq \min\{y + \text{RBF}(x, y), N(t)\}$$

holds. Thus,  $\text{RBF}(x, y)$  must be non-negative, to ensure a non-empty range. From the two inequalities above, we obtain the limits for the number of purchases, given in (31) and (32): From the left inequality we obtain the minimum number of purchases in (31), because the number of purchases is whole numbered. By (28), we obtain from the right inequality

$$N_{buy}(t) \leq \min\{\text{int}[x \cdot y], N(t)\} ,$$

which confirms the maximum number of purchases in (32).

In order to achieve a maximum amount of new investment under the constraints (A) – (E), the investment rate  $I(t)$  or  $y = I(t) \cdot U_{\min}$  is maximized according to (B7),

$$y = y_{\max} = \max\{y \leq y_1 | \text{RBF}(x, y) \geq 0\} .$$

Here,  $y_1$  is a upper bound of  $y$ , defined by  $y_1 = I_1(t) \cdot U_{\min}$  and (30). Obviously, we obtain  $y = y_1$  for  $\text{RBF}(x, y_1) \geq 0$ , which confirms the upper part of (27). In this case, the investment rate  $I(t)$  coincides with the upper bound  $I_1(t)$ . For  $\text{RBF}(x, y_1) < 0$ ,  $y_1$  is rounded down by the lower part of (27). For  $\text{RBF}(x, y_1) < 0$ ,  $y$  must be smaller than  $y_1$ , i.e. the upper bound  $I_1(t)$  is not achieved. Hence, we have the following *coincidence criterion*:

$$I(t) = I_1(t) \iff \text{RBF}(x, y_1) \geq 0 , \quad (34)$$

holding for any  $x > 0$ . In order to decide, whether  $\text{RBF}(x, y_1)$  is negative or not, the criterion in (B8) will be used. For  $y = y_1$ , it rewrites

$$\text{RBF}(x, y_1) < 0 \iff y_1 > 0 , \quad x < \text{int}^+[y_1]/y_1 . \quad (35)$$

In particular,

$$x \geq 1 , \quad y_1 \geq 0 \text{ integer} \implies \text{RBF}(x, y_1) \geq 0 \quad (36)$$

holds according to (B9).

**No new investments:**

By (26), no investments take place for  $x < 1$ . This is possible for  $x \geq 1$  as well due to the criterion in (B14),

$$I(t) = 0 \iff x < 1 \text{ or } y_1 \cdot x < 1 , \quad (37)$$

which can be explained as follows.

$x < 1$ : By constraints (A) and (C), for  $N_{buy}(t) > 0$

$$\text{inv}_{\min} \leq I_{buy}(t) \leq C_{tot}(t)/U_{\min}$$

holds. Thus,  $x = C_{tot}(t)/[\text{inv}_{\min} \cdot U_{\min}]$  must be at least 1. By contrast,  $x < 1$  prevents any new investments.

$y_1 \cdot x < 1$ : The condition  $y_1 \cdot x < 1$  becomes clear by converting it as follows:

$$\begin{aligned} y_1 \cdot x < 1 &\iff [I_1(t) \cdot U_{\min}] \cdot C_{tot}(t)/[\text{inv}_{\min} \cdot U_{\min}] < 1 \\ &\iff I_1(t) \cdot C_{tot}(t) < \text{inv}_{\min} . \end{aligned}$$

Due to  $I(t) \leq I_1(t)$ , new investments  $I_{new}(t) = I(t) \cdot C_{tot}(t)$  are lying below  $\text{inv}_{\min}$ , presenting the minimum amount for a new investment. Thus, constraint (C) prevents any purchases, i.e no new investments are made.

### Two limits for the total capital:

The condition  $x \geq 1$  for investments and the condition  $\text{RBF}(x, y_1) \geq 0$  for coincidence of  $I(t)$  and  $I_1(t)$  imply two limits for the total capital.

The condition  $x \geq 1$  rewrites

$$C_{tot}(t) \geq \text{inv}_{\min} \cdot U_{\min} . \quad (38)$$

If the total capital is below the limit  $\text{inv}_{\min} \cdot U_{\min}$ , new investments are not possible.

By (35), the condition  $\text{RBF}(x, y_1) \geq 0$  for  $y_1 > 0$  rewrites

$$x \geq \text{int}^+[y_1]/y_1 ,$$

which is equivalent to

$$y_1 > 0 : C_{tot}(t) \geq \text{int}^+[y_1]/y_1 \cdot [\text{inv}_{\min} \cdot U_{\min}] . \quad (39)$$

Since  $\text{int}^+[y_1]/y_1 \geq 1$  holds, this limit is at least equal to the limit in (38).

## 3.2 Investing for a holding period of one day

In simulations shown in this paper, a holding period  $\tau_{\text{hold}} = 1$  is used. In this case, the investment rate  $I_1(t)$  can be substantially simplified, because all securities are sold, before new securities are bought. This simplifies the trading process, since no investments are hold after sales and before purchases, i.e.

$$IR^-(t) = 0 . \quad (40)$$

Thus, the relation between  $IR(t)$  and  $I(t)$  in (15) simplifies as

$$IR(t) = I(t) , \quad (41)$$

i.e. the investment ratio at the end of a trading day corresponds to the investment rate at this day. The investment rate  $I_1(t)$  in (30) becomes

$$I_1(t) = \min\{N(t)/U_{\min}, 1/\tau, \text{IR}_{\max}\} ,$$

i.e. the time-varying expression  $\text{IR}_{\max} - IR^-(t)$  is to be replaced by the constant  $\text{IR}_{\max}$ .

A further simplification consists in identifying the investment period  $\tau$  with the holding period  $\tau_{\text{hold}} = 1$ ,

$$\tau = \tau_{\text{hold}} = 1 . \quad (42)$$

This becomes clear, remembering the task of constraint (B) for temporal investment diversification, as explained in section 2.2. For a holding period of one day, this task becomes superfluous. Thus, for a holding period of one day, the investment period  $\tau$  is set to one. Due to  $\text{IR}_{\max} \leq 1$ ,  $I_1(t)$  becomes

$$I_1(t) = \min\{N(t)/U_{\min}, \text{IR}_{\max}\} \quad (43)$$

or

$$y_1 = U_{\min} \cdot I_1(t) = \min\{N(t), U_{\min} \cdot \text{IR}_{\max}\}. \quad (44)$$

This result will be applied to our criterion for a negative range buy function  $\text{RBF}(x, y_1) < 0$  in (35) for  $x \geq 1$ . In order to evaluate  $x < \text{int}^+[y_1]/y_1$ , the following two cases have to be distinguished:

1. For

$$N(t) \leq U_{\min} \cdot \text{IR}_{\max}, \quad (45)$$

$y_1 = N(t)$  holds and we obtain from (36)  $\text{RBF}(x, y_1) \geq 0$ .

2. For

$$N(t) > U_{\min} \cdot \text{IR}_{\max}, \quad (46)$$

$y_1 = U_{\min} \cdot \text{IR}_{\max} > 0$  holds and we obtain from (35)  $\text{RBF}(x, y_1) < 0$  for

$$x < \frac{\text{int}^+[U_{\min} \cdot \text{IR}_{\max}]}{U_{\min} \cdot \text{IR}_{\max}}.$$

Due to  $x \geq 1$ , this is only possible, if  $U_{\min} \cdot \text{IR}_{\max}$  is not whole-numbered.

Our results for a holding period of one day can be summarized as follows.

**COROLLARY 1 (Holding period one)** *If sales take place after a holding period of one day, no investments are hold after sales and before purchases, i.e.  $\text{IR}^-(t) = 0$  holds. Consequently, the investment ratio at the end of trading day  $t$  equals the investment rate  $I(t)$ . Setting the investment period  $\tau = 1$ , constraint (B) is dropped and the investment rate  $I_1(t)$  is given by (43). The criterion for a negative range buy function for  $x \geq 1$  becomes*

$$\text{RBF}(x, y_1) < 0 \iff N(t) > U_{\min} \cdot \text{IR}_{\max}, x < \frac{\text{int}^+[U_{\min} \cdot \text{IR}_{\max}]}{U_{\min} \cdot \text{IR}_{\max}}. \quad (47)$$

### 3.3 Investing without constraint (C)

In our investment algorithm, the new investment  $I_{\text{buy}}(t)$  in a single security is lower bounded by  $\text{inv}_{\min} > 0$  according to constraint (C) in order to take trading costs into account. When constraint (C) is dropped, the trading process simplifies according to the following corollary, which is proven in B.4.

**COROLLARY 2 (Investing without constraint (C))** *If constraint (C) is dropped, the investment rate is simply given by*

$$I(t) = I_1(t), \quad (48)$$

*and the maximum number of purchases is*

$$N_{\text{buymax}}(t) = N(t). \quad (49)$$

By (48),  $I_1(t)$  proves to be the investment rate, if constraint (C) is dropped. Dropping constraint (C) can be expressed setting  $\text{inv}_{\min} = 0$ . In this case, constraint (C) rewrites  $I_{\text{buy}}(t) \geq 0$ , which is automatically satisfied, because new investments are non-negative by (33).



## 4 Simulation results for a holding period of one day

**Definition of the market:** In the following, the investment algorithm developed so far will be applied to a real market. The securities in (1) represent historical share prices in the national currency, the EUR, provided by Lenz+Partner AG, Germany. The prices are opening prices from the Frankfurt stock exchange, adjusted to the event of capital measures like share splits, but not to the event of dividend distributions.

Among the large amount of shares, a selection is carried out, to improve data quality and tradeability. By this selection, we obtain 867 securities, which do not end before Dec 11, 2014, allowing simulations up to this date at least. At first, we claim that the shares are included in one of the following share indices: The German indices DAX, MDAX, TECDAX, SDAX, HDAX, CDAX, Technology All Share, Prime All Share and GEX as well as the American indices S&P500 and Nasdaq. Secondly, shares do not end before the Dec 30, 2013. Finally, the opening price at Dec 30, 2013 (not adjusted to capital measurements) is 1 EUR at least. By this rule we want to exclude *penny stocks*. Because we don't know before the Dec 30, 2013, whether a share will be a penny stock at this date, buy formulas use price relationships only.

**Adjustments for simulations:** Among the large variety of adjustments for the investment algorithm, the following investment parameters are selected:

$$\begin{aligned} \tau = \tau_{\text{hold}} &= 1, \quad U_{\text{min}} = 10, \\ \text{IR}_{\text{max}} &= 0.35, \quad \text{inv}_{\text{min}} = 7000, \\ N_{\text{buy}}(t) &= N_{\text{buymax}}(t). \end{aligned} \tag{50}$$

In particular, the number of purchases  $N_{\text{buy}}(t)$  is maximum as defined in (32). The buy conditions depend on the price relationship between two subsequent trading days

$$c_1 := s(t)/s(t-1) \tag{51}$$

according to the two buy formulas BF1 and BF2,

$$\text{BF1} \quad : \quad 1.0625 \leq c_1 < 1.125, \tag{52}$$

$$\text{BF2} \quad : \quad 0.875 \leq c_1 < 0.9375. \tag{53}$$

The two intervals are build up by interval division, starting with the interval  $0 \leq c_1 < 2$ . According to the definition of buy formula BF1, a security will be purchased at trading day  $t$ , if the (opening) price compared to the previous day  $t-1$  has increased significantly, i.e. we rely on continuation of high returns. In contrast, for buy formula BF2, we believe in price recovery after a slump. For simulations, the first trading day  $t=1$  is Jan 2, 2002 (number 3764), and the last trading day is Jan 2, 2003 (number 4017), resulting in 254 trading days.

### 4.1 Results for buy formula BF1

The plots in Fig. 1 and 2 are showing the number of buy recommendations  $N(t)$  and the number of purchases  $N_{\text{buy}}(t)$  for the given trading period. Since  $0 < N_{\text{buy}}(t) < N(t)$  holds on many days, the trading process is random. This means that the following plots vary from trial to trial. Only the first plot is fixed, because the number of buy recommendations does only depend on the buy formula. Fig. 3 shows the capital flows for buy formula 1 and Fig. 4 shows the investment ratio  $IR(t)$ . For a holding period  $\tau_{\text{hold}} = 1$ , the investment ratio equals the investment rate  $I(t)$ , as justified in (41).

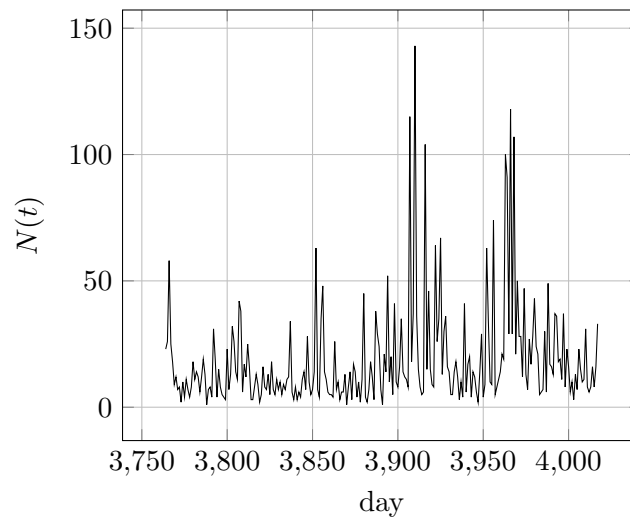


Figure 1: Number of buy recommendations for buy formula BF1:  $1.0625 \leq c_1 < 1.125$  depending on trading day  $t$  ranging from Jan. 2, 2002 – Jan. 2, 2003

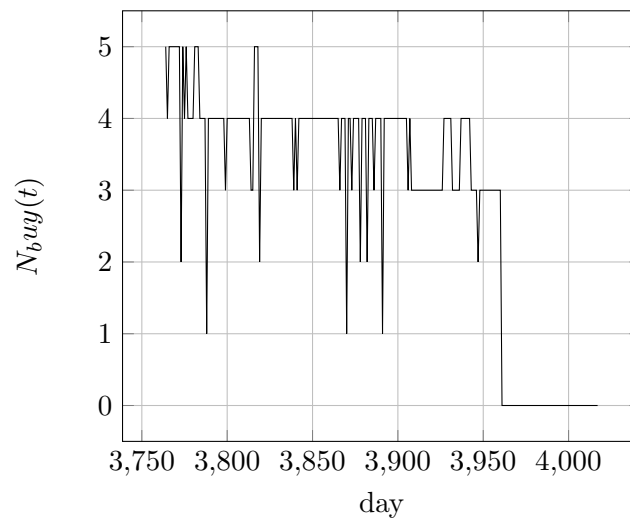


Figure 2: Number of purchases for a random trial. Buy formula is BF1:  $1.0625 \leq c_1 < 1.125$ . Trading period: Jan. 2, 2002 – Jan. 2, 2003

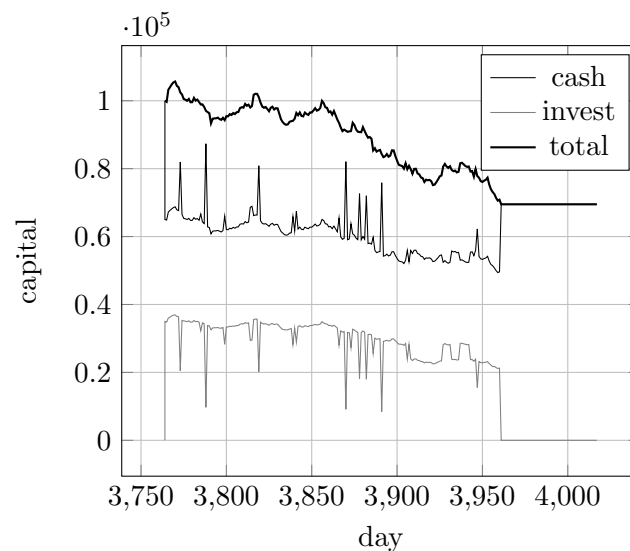


Figure 3: Cash, investment and total capital for a random trial. Buy formula is BF1:  $1.0625 \leq c_1 < 1.125$ . Trading period: Jan. 2, 2002 – Jan. 2, 2003

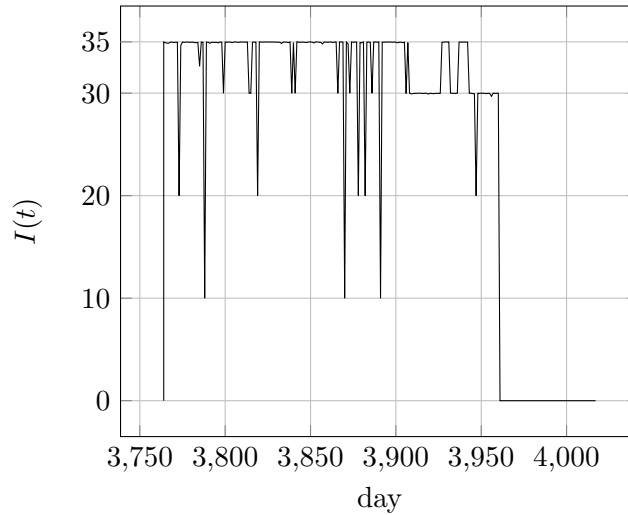


Figure 4: Investment ratio  $IR(t)$  and investment rate  $I(t) = IR(t)$  (in percents) for a random trial. Buy formula is BF1:  $1.0625 \leq c_1 < 1.125$ . Trading period: Jan. 2, 2002 – Jan. 2, 2003

Since the number of purchases is maximum, it is given by (32). Hence, the number of purchases is given by the quantity

$$N_{buy}(t) = \text{int}[I(t) \cdot C_{tot}(t)/\text{inv}_{\min}] ,$$

whenever the number of buy recommendations  $N(t)$  is not below this quantity. In order to evaluate this quantity, the investment rate  $I(t)$  is required. Due to  $I(t) = IR(t)$ , the investment rate is limited by  $IR_{\max} = 0.35$ . Fig. 4 shows that this value is achieved at many trading days. Furthermore, the total capital  $C_{tot}(t)$  exceeds 100,000 on several trading days at the beginning of trading, as Fig. 3 shows. Consequently, for these trading days, we get  $N_{buy}(t) = 0.35 \cdot 100,000/7000 = 5$ , whenever  $N(t)$  is not less than 5. Fig. 2 shows that  $N_{buy}(t) = 5$  is actually achieved several times.

**Trading abort:** In Fig. 2, purchases are not carried out from  $t_2 = \text{Oct. 9, 2002}$  (number 3961). At this day, only sales take place. After this day, neither purchases nor sales occur, i.e. trading is aborted at trading day  $t_2 + 1$ . This trading abort is also obvious in Fig. 3 and 4, because capital courses keep constant from  $t_2$  and in particular, investment and investment ratio are zero at the end of  $t_2$  and for the following trading days. The trading abort is caused by  $x = C_{tot}(t)/[\text{inv}_{\min} \cdot U_{\min}] < 1$ , preventing new investments according to (26), which is equivalent to

$$C_{tot}(t) < \text{inv}_{\min} \cdot U_{\min} . \quad (54)$$

From  $\text{inv}_{\min} = 7000$  and  $U_{\min} = 10$  we obtain  $C_{tot}(t) \leq 70,000$ . The total capital falls below this limit at  $t_2$  the first time, justifying the trading abort.

**Remark 1 (Trading abort)** In the simulations with  $\tau_{\text{hold}} = 1$ , the trading abort is caused by  $x < 1$ . By (37), a trading abort is also possible for  $y_1 \cdot x < 1$ . Assuming the trading abort is not caused by  $x < 1$  or by the absence of buy recommendations, we have  $x \geq 1$  and  $N(t) > 0$ . For  $x \geq 1$  the condition  $y_1 \cdot x < 1$  requires

$$y_1 = \min\{N(t) , U_{\min} \cdot IR_{\max}\} < 1 .$$

For  $N(t) > 0$ ,  $U_{\min} \cdot IR_{\max} < 1$  must be hold, which is not possible for  $U_{\min} = 10$  and  $IR_{\max} = 0.35$ , but only for smaller values  $U_{\min}$  or  $IR_{\max}$ . It is straight forward to show that the condition  $y_1 \cdot x < 1$  for  $x \geq 1$  and  $N(t) > 0$  rewrites

$$\text{inv}_{\min} \cdot U_{\min} \leq C_{tot}(t) < \text{inv}_{\min}/IR_{\max} .$$

**Investment rate and range buy function:** In the following, we consider only the trading process before the trading abort, i.e. for  $x \geq 1$ . Fig. 4 reveals that the investment rate assumes one of the following values  $I(t) = 0, 0.1, 0.2, 0.3$  or  $0.35$  or comes out somewhat lower, caused by the whole-numbered quantity of securities in (8). The piecewise constant course at  $0.35$  represents the trading

days, on which the maximum investment ratio  $\text{IR}_{\max} = 0.35$  is (approximately) achieved. The values for  $I(t)$ , which actually occur, can be explained by (44). From  $U_{\min} = 10$  and  $\text{IR}_{\max} = 0.35$  we obtain

$$y_1 = \min\{N(t), U_{\min} \cdot \text{IR}_{\max}\} = \begin{cases} N(t) : N(t) \leq 3 \\ 3.5 : N(t) \geq 4 \end{cases} \quad (55)$$

1.  $N(t) \leq 3$  :

This is the first case (45). Consequently, the range buy function  $\text{RBF}(x, y_1)$  is non-negative for  $x \geq 1$ . By (27),  $y$  equals  $y_1$ ,

$$N(t) \leq 3 : \text{RBF}(x, y_1) \geq 0, y = y_1 = N(t). \quad (56)$$

Hence, the investment rate becomes one of the following values:  $I(t) = I_1(t) = 0, 0.1, 0.2, 0.3$ .

2.  $N(t) \geq 4$  :

This is the second case (46). In this case,  $y$  depends on whether the range buy function  $\text{RBF}(x, y_1)$  is negative or not. For  $\text{RBF}(x, y_1) \geq 0$ , we obtain  $y = y_1 = 3.5$ , i.e.  $I(t) = 0.35$ . For  $\text{RBF}(x, y_1) < 0$ , we get  $y = \text{int}[y_1] = \text{int}[3.5] = 3$  and  $I(t) = 0.3$ .

In order to identify the trading days with  $\text{RBF}(x, y_1) < 0$ , we use the criterion in (47), according to which

$$\text{RBF}(x, y_1) < 0 \iff N(t) \geq 4, x < \frac{\text{int}^+[U_{\min} \cdot \text{IR}_{\max}]}{U_{\min} \cdot \text{IR}_{\max}}$$

holds for  $x \geq 1$ . From  $U_{\min} \cdot \text{IR}_{\max} = 3.5$ , we obtain  $x < 4/3.5 = 8/7$  and from  $x = C_{tot}(t)/[\text{inv}_{\min} \cdot U_{\min}]$

$$C_{tot}(t) < 8/7 \cdot [\text{inv}_{\min} \cdot U_{\min}]. \quad (57)$$

For  $\text{inv}_{\min} = 7000$  and  $U_{\min} = 10$  we get  $C_{tot}(t) < 80,000$ . Thus, the trading days with negative range buy function  $\text{RBF}(x, y_1)$  are identified as days with number of buy recommendations  $N(t)$  being 4 at least and a total capital falling below the limit 80,000. In the simulation for buy formula BF1, this situation occurs at Jul. 24, 2002 the first time (day number 3906). On most days afterwards, the condition is satisfied as well, resulting in a negative range buy function.

**Remark 2 (Minimum number of purchases)** *From  $y$ , we obtain the minimum number of purchases by rounding up  $y$  according to (31). Thus, the minimum number of purchases achieves one of the following values:  $N_{buymin}(t) = 0, 1, 2, 3$  or 4. In particular, the value  $N_{buy}(t) = 5$  as in Fig. 2 is not possible, if the number of purchases is chosen to be minimal.*

## 4.2 Results for buy formula BF2

Fig. 5 shows the capital development for buy formula BF2. In contrast to buy formula BF1 in Fig. 3, the total capital is increasing. This means that believing in price recovery after a slump is successful for the given trading period. For buy formula BF2, neither trading is aborted nor the range buy function  $\text{RBF}(x, y_1)$  becomes negative, because the total capital does not fall below the limits in (54) and (57).

In order to demonstrate the randomness of the trading process, the empirical distribution function of the total capital at the end of the trading period is shown in Fig. 6. The empirical distribution function  $F(c)$  is defined by dividing the number of trials with an outcome  $C_{tot}(T) \leq c$  through the total number of random trials (1000), for the end  $T = \text{Jan}, 2, 2003$  of the trading period. The total capital  $C_{tot}(T)$  is varying between 105,200 and 235,400, and is nearly normal distributed. The mean value amounts 170,000 and the standard deviation is 20,400. These values characterize the investment algorithm with investment parameters according to (50) and buy formula BF2, since they describe the return of a large set of market players, trading all in compliance with the same investment algorithm.

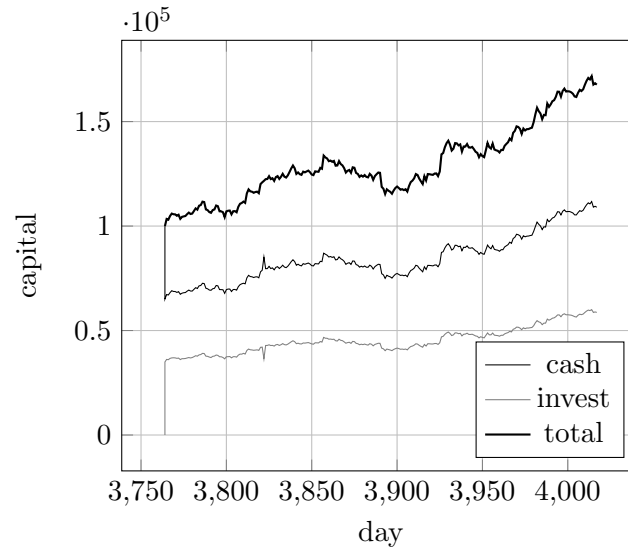


Figure 5: Cash, investment and total capital for a random trial. Buy formula is BF2:  $0.875 \leq c_1 < 0.9375$ . Trading period: Jan. 2, 2002 – Jan. 2, 2003

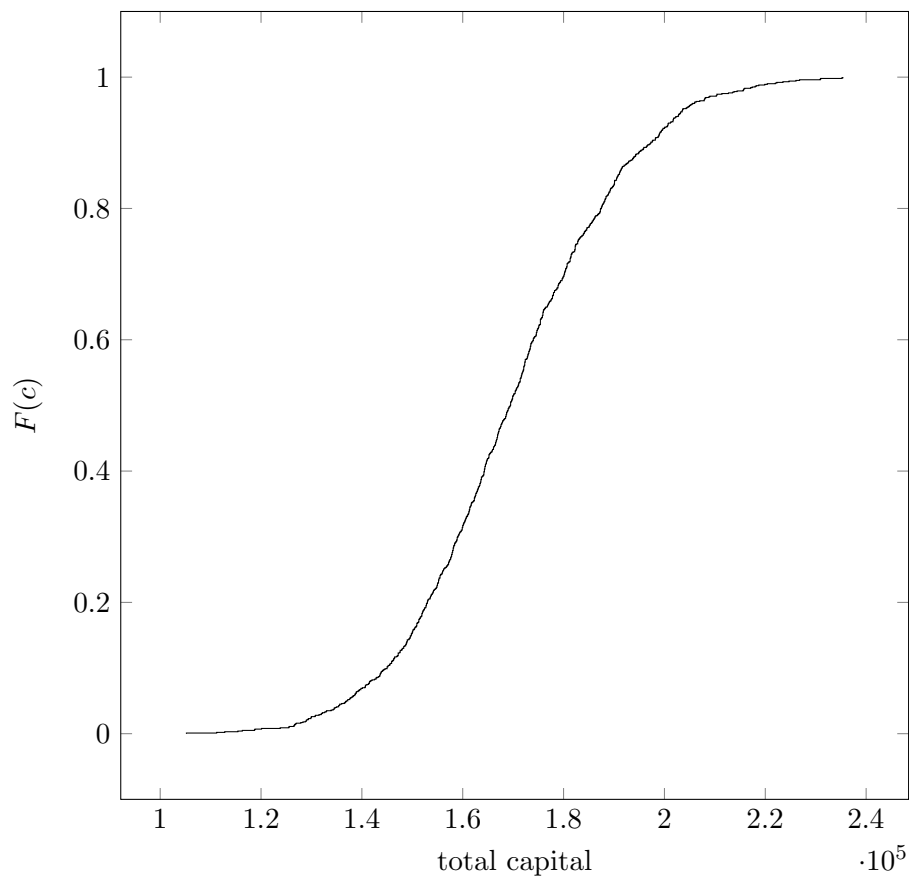


Figure 6: Empirical distribution function of the total capital  $C_{tot}(T)$  for a thousand random trials with  $inv_{\min} = 7000$ . Trading period: Jan. 2, 2002 – Jan. 2, 2003. Buy formula is BF2:  $0.875 \leq c_1 < 0.9375$ . The mean value is 170,000 and the standard deviation is 20,400.

## 5 Statistical evaluation for a holding period of one day

### 5.1 Return and gain factor

So far, simulation results have been shown for two contrasting buy formulas. In order to decide, whether they discover regularity of trading results depending on the buy formula, it would be evident, to perform simulations for different buy formulas. Since the trading process is a random trial, many trials would be required for each buy formula with subsequent averaging the total capital at the end of the trading period. In contrast, the author suggests a statistical evaluation of historical prices, providing the desired information. This evaluation is carried out by a *trading expert*.

In this paper, we only derive the expected return and no other statistical quantities. Furthermore, we only consider a holding period of one day. For a holding period of one day, the return at the end  $T$  of the trading period can be described analytically by (58). The proof is delegated to C.1. In order to find the expected return, (58) will be specialized to deterministic trading afterwards.

**LEMMA 2 (Return and gain factor)** *For a holding period of one day, the return at the end of the last trading day  $T$  is  $G - 1$  with the gain factor*

$$G := C_{tot}(T)/C_{tot}(1) = \prod_{t=1}^{T-1} [1 + I(t) \cdot (G(t) - 1)] . \quad (58)$$

Here,

$$G(t) := \frac{1}{N_{buy}(t)} \sum_{n \in buy(t)} s(n, t+1)/s(n, t) , \quad N_{buy}(t) > 0 \quad (59)$$

is the gain factor for trading day  $t$  and  $buy(t)$  denotes the set of securities purchased on trading day  $t$ , identified by their number  $n$ . For a empty set  $buy(t)$ ,  $I(t) = 0$  holds and  $G(t)$  is not defined.

#### Discussion:

By (58), the gain factor  $G$  is simply given by

$$G = \prod_{t=1}^{T-1} g(t) , \quad g(t) := [1 - I(t)] \cdot 1 + I(t) \cdot G(t) . \quad (60)$$

Here, the gain factor  $G(t)$  defined in (59) is the price relationship  $s(t+1)/s(t)$ , averaged over all purchased securities. This is caused by our assumption in (2) that the amounts for new investments are identical for all purchases. In contrast to the gain factor  $G(t)$  in (59), the gain factor  $g(t)$  considers the investment rate  $I(t)$ . The gain factor  $g(t)$  is the convex combination of 1 and  $G(t)$  with  $1 - I(t)$  as weighting factor for 1 and  $I(t)$  as weighting factor for  $G(t)$ . The weighting factor  $1 - I(t)$  can be interpreted as the share of not invested capital at trading day  $t$ , remaining unchanged, and  $I(t)$  is the share of invested capital, to be multiplied with  $G(t)$ .

By (58), the total capital  $C_{tot}(T)$  at the end of the last trading day depends on the total capital  $C_{tot}(1)$  at the end of the first trading day. When we start trading without initial investments for example, we obtain from (11)

$$C_{tot}(1) = C_{cash}(0) , \quad (61)$$

i.e. the total capital at the first day equals the initial cash.

### 5.2 Deterministic trading

Starting point for deriving the expected return is the case when new investments are not lower bounded by  $inv_{\min}$ . Then, the number of purchases is only limited by the number  $N(t)$  of buy recommendations according to (49). Making use of the maximum number of purchases, we obtain

$$N_{buy}(t) = N(t) \quad (62)$$

and trading becomes deterministic, because a random selection of securities among the buy recommendations does not take place.

**Remark 3 (Small  $\text{inv}_{\min}$ )** *The condition in (62) is possible even if  $\text{inv}_{\min}$  is not set to zero, but is sufficiently small. In this case, (49) holds as well. For example, for the adjustments in (50), a deterministic trading process arrives, modifying  $\text{inv}_{\min}$  according to  $\text{inv}_{\min} \leq 180$  for buy formula BF1 and  $\text{inv}_{\min} \leq 525$  for buy formula BF2.*

The gain factor in (58) simplifies as follows: By (48), the investment rate  $I(t)$  equals  $I_1(t)$ . For a holding period of one day,  $I_1(t)$  is given by (43). Hence, we obtain

$$I(t) = I_1(t) = \min\{N(t)/U_{\min}, \text{IR}_{\max}\}. \quad (63)$$

Furthermore, the gain factor  $G(t)$  becomes

$$M_1(t) := \frac{1}{N(t)} \sum_{n \in \text{buy}(t)} s(n, t+1)/s(n, t), \quad N(t) > 0. \quad (64)$$

For  $N(t) = 0$ ,  $I(t) = I_1(t) = 0$  holds and  $M_1(t)$  is not defined. In contrast to (59), the averaging process extends over all  $N(t)$  buy recommendations and hence, the gain factors  $M_1(t)$  are deterministic, given by the *first momentum* of the price relationship  $s(n, t+1)/s(n, t)$ .

The gain factor  $G$  for deterministic trading becomes

$$G_e := \prod_{t=1}^{T-1} [1 + I_1(t) \cdot (M_1(t) - 1)]. \quad (65)$$

Since the investment rates  $I_1(t)$  and the gain factors  $M_1(t)$  are deterministic,  $G_e$  is deterministic as well. It is computed by a *trading expert* for a holding period of one day.

### 5.3 Expected return

The gain factor in (65) can be interpreted in two different ways. By definition, it represents the gain factor for  $\text{inv}_{\min} = 0$  and  $N_{\text{buy}}(t) = N(t)$ . It depends on the investment parameters  $U_{\min}$  and  $\text{IR}_{\max}$ , and the buy condition defining the set  $\text{buy}(t)$ . In addition to this interpretation,  $G_e$  can be interpreted as expected gain factor for greater values  $\text{inv}_{\min}$  due to the following theorem, which is proven in C.2.

**THEOREM 2 (Expected return and gain factor)** *For a holding period of one day, the gain factor  $G_e$  in (65) represents the expected gain factor,*

$$E\{G\} = E\{C_{\text{tot}}(T)/C_{\text{tot}}(1)\} = \prod_{t=1}^{T-1} [1 + I_1(t) \cdot (M_1(t) - 1)], \quad (66)$$

for any investment parameters  $U_{\min}$ ,  $\text{IR}_{\max}$  and  $\text{inv}_{\min}$  under the assumption

$$I(t) = I_1(t). \quad (67)$$

*The expected return is given by  $E\{G\} - 1$ . The expectation relates to the random selection of  $N_{\text{buy}}(t)$  purchases from the  $N(t)$  buy recommendations in step 4 of the investment algorithm.*

#### Random trading process:

In (66), we assume a deterministic total capital  $C_{\text{tot}}(1)$  at the beginning of trading. The expectation relates to the randomness of the trading process. It is not associated with the securities, which are considered to be deterministic. The randomness of the trading process arises from the random selection of purchased securities, which meet the buy condition, during the trading days  $1 \leq t < T$ . Consequently, the randomly varying total capital  $C_{\text{tot}}(T)$ , caused by these random selections, averaged over a large number of random trials (simulations), is correctly reproduced by (66). Each random trial or simulation must use the same investment parameters  $U_{\min}$ ,  $\text{IR}_{\max}$  and  $\text{inv}_{\min}$  and the same buy condition.

**The assumption  $I(t) = I_1(t)$ :**

By the coincidence criterion in (34), the assumption  $I(t) = I_1(t)$  is equivalent to  $\text{RBF}(x, y_1) \geq 0$ . For the adjustments  $U_{\min} = 10$  and  $\text{IR}_{\max} = 0.35$  in (50), we obtain from (57) for  $x \geq 1$  a negative range buy function  $\text{RBF}(x, y_1) < 0$  only if the total capital falls below the limit  $8/7 \cdot [\text{inv}_{\min} \cdot U_{\min}]$ . Consequently,  $\text{RBF}(x, y_1) \geq 0$  holds for

$$C_{\text{tot}}(t) \geq 8/7 \cdot [\text{inv}_{\min} \cdot U_{\min}]. \quad (68)$$

Under this condition,  $x = C_{\text{tot}}(t)/[\text{inv}_{\min} \cdot U_{\min}] \geq 1$  is satisfied automatically. We conclude that the assumption  $I(t) = I_1(t)$  is true, if the total capital does not fall below the limit in (68).

For  $\text{inv}_{\min} = 7000$ , (68) becomes  $C_{\text{tot}}(t) \geq 80,000$ , which holds for buy formula BF2, but not for BF1 on several trading days. In order to cover BF1 as well,  $\text{inv}_{\min}$  will be adjusted somewhat lower. For  $\text{inv}_{\min} = 3500$  for example, (68) becomes  $C_{\text{tot}}(t) \geq 40,000$  and is satisfied for BF1 as well. This means that the total capital remains above this limit for all trading days and all random trials inspected so far. Moreover, (68) holds for all buy formulas used by the trading expert, presented in Fig. 7. Thus, this trading expert evaluates the expected gain factor for all buy formulas correctly, assuming  $\text{inv}_{\min} = 3500$  for example.

**Trading expert:**

The trading expert has expertise on future prices for a horizon of one trading day, interpreting past prices for a certain trading period, using the property  $c_1 = s(t)/s(t-1)$  in (51). Other experts, using longer horizons and several properties simultaneously, are not considered in this paper. Thus, the expert used in this paper is very elementary. For a single property, the expert evaluates (65) for all buy formulas, given by intervals  $c_{\min} \leq c_1 < c_{\max}$ , which are obtained by successive interval bisections, starting with the base interval  $0 \leq c_1 < 2$ . In this way, one get for example the two intervals  $1.0625 \leq c_1 < 1.125$  for buy formula BF1 in (52) and  $0.875 \leq c_1 < 0.9375$  for BF2 in (53) (see Fig. 7). Since the two intervals have the length  $0.0625 = 2/32$ , they belong to level 5, containing all intervals with 5 bisections.

In order to limit the number of intervals, only *significant* intervals are considered. They are defined by a minimum number of buy recommendations within the trading period. In Fig. 7, only intervals with level  $\geq 4$  are plotted, having similar investment ratios for the whole trading period. These investment ratios are measured by averaging the investment rates  $I_1(t) = \min\{N(t)/U_{\min}, \text{IR}_{\max}\}$  over the trading period according to

$$\bar{I}_1 := \frac{1}{T-1} \sum_{t=1}^{T-1} I_1(t).$$

For  $\bar{I}_1 \geq 0.3$ , this results in mean investment rates in the range of  $0.3 \leq \bar{I}_1 \leq \text{IR}_{\max} = 0.35$ . Since the mean investment rates are similar for all 125 intervals shown in Fig. 7, the gain factors for different intervals are fairly compared with one another.

In Fig. 7, intervals have different length. Short intervals only occur nearby the center point  $c_1 = 1$ . The reason is that the values of  $c_1$  are concentrated at the center. Thus, values  $c_1$  more distanced from the center occur less frequent. Consequently, intervals in the outer area must be longer in order to achieve a minimum number of buy recommendations or a minimum mean investment ratio  $\bar{I}_1$  for the trading period. On the other hand, values  $c_1$  lying far away from the center are of interest for trading, because the gain factors differ significantly from 1. In particular, the gain factor is high for intervals with small values  $c_1$  and conversely, the gain factor is low for intervals with high values  $c_1$ . The buy formulas BF1 and BF2, marked in Fig. 7, perfectly illustrate this.



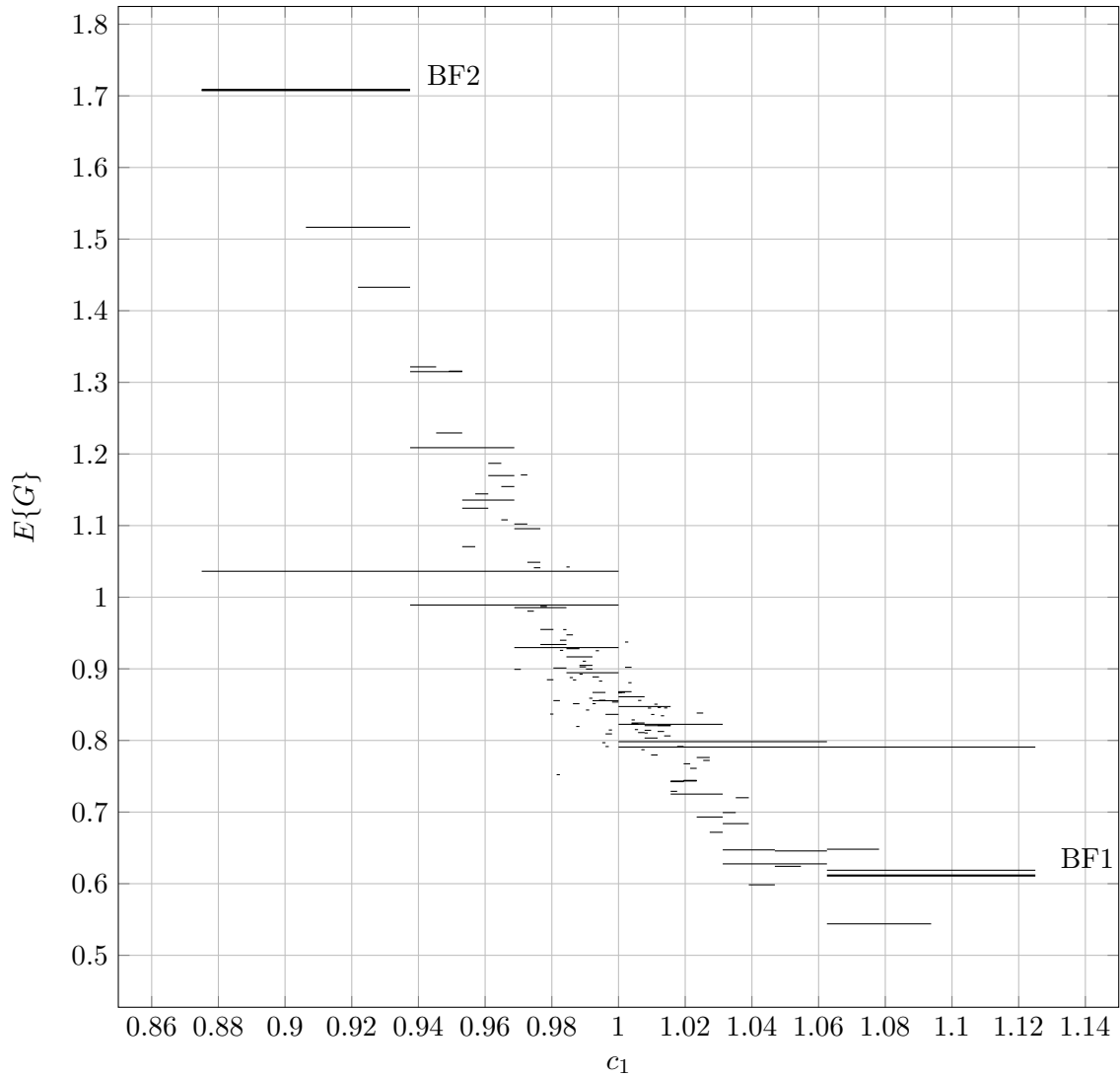


Figure 7: Expected gain factor  $E\{G\} = E\{C_{tot}(T)/C_{tot}(1)\}$  evaluated by a trading expert. Trading period: Jan. 2, 2002 – Jan. 2, 2003. Investment parameters:  $U_{\min} = 10$ ,  $IR_{\max} = 0.35$ ,  $\tau = \tau_{\text{hold}} = 1$ . In simulations,  $\text{inv}_{\min}$  must be sufficiently small to ensure  $I(t) = I_1(t)$ , which is satisfied for all buy formulas setting  $\text{inv}_{\min} = 3500$  for example. The buy formulas are given by intervals  $c_{\min} \leq c_1 < c_{\max}$  with length  $c_{\max} - c_{\min} \leq 0.125$ . Only buy formulas with mean investment rate  $\bar{I}_1$  lying within  $0.3 \leq \bar{I}_1 \leq 0.35 = IR_{\max}$  are plotted.

We obtain for buy formula BF1  $E\{G\} = 0.619$  and for BF2  $E\{G\} = 1.707$ , which coincides with our simulation results with  $\text{inv}_{\min} = 0$  (BF1; 0.620, BF2: 1.705) and with  $\text{inv}_{\min} = 3500$  for the mean values of a thousand random trials (BF1: 0.622, BF2: 1.71). The coincidence is not perfect due to the quantities in (8), which are rounded down in simulations. The standard deviation is 5300 for BF1 and 9500 for BF2. Concerning BF2, the standard deviation is significantly below the standard deviation 20,400 for our simulation results with  $\text{inv}_{\min} = 7000$ , shown in Fig. 6. This arises from the fact that more buy recommendations are bought for  $\text{inv}_{\min} = 3500$  and thus, the randomness of selecting buy recommendations is less prevalent.

## 5.4 Mean reversion for one day

Fig. 7 shows high returns, e.g. for buy formula BF2. The high returns simply consists in price recovery after a slump at a horizon of one trading day, revealing a strong mean reversion effect for a very short horizons of one day. Fig. 5 shows that the total capital increases considerable steady in 2002, although stock markets heavily decreased in this year. This result is confirmed for other trading periods as well. That is interesting, since the mean reversion effect depends on time only on a small extend and thus is valid in each phase of the stock market (bull and bear market, sideways market). So-called *swing trading* benefits from this, where only short-termed investments are entered. In Fig. 5, trading starts with 100,000 for cash and ends with a total capital of 167,600, representing a return of about 70 percents, which is caused by a price recovery after a slump in the range of  $0.875 \leq c_1 < 0.9375$ . This effect is significantly increased by a slump, which is greater, e.g. for  $c_1 \leq 0.9$ . On the other hand, several objections against these results can be argued:

### 1. Trading costs:

First, trading costs and taxes are not evaluated. Trading costs can be estimated using the simple trading cost model, assuming 0.1 percent of turnover for all purchases and sales. For our simulations shown in Fig. 5 for example, we obtain trading costs of 21,700, which is about a third of our earnings  $167,600 - 100,000 = 67,600$ . For the buy formula  $c_1 \leq 0.9$ , the share of trading costs on earnings is decreased significantly.

### 2. Noisy prices:

Secondly, high returns could be caused by noisy prices, pretending slumps at isolated trading days. Some comparisons for stock prices between different data sources were carried out. However, because this was done on a random basis, this effect was not completely ruled out.

### 3. Trading in reality:

Third, trading in reality might differ from our findings substantially. In our simulations of the trading process and statistical evaluations, the price relationship  $c_1 = s(t)/s(t-1)$  is evaluated, where  $s(t)$  and  $s(t-1)$  are historical opening prices. In reality, a purchase at the opening price  $s(t)$  after evaluating  $c_1$  cannot be realized in general. Furthermore, the execution price of an order must be considered, which is influenced by the bid-ask spread. The order even may not be executed, or is only executed with unexpected high trading costs. Besides order execution, a wrong decision is possible, caused by wrong price data due to a split at day  $t$ . If the price  $s(t-1)$  is not well adjusted,  $c_1$  pretends a slump. Thus, trading in reality might differ significantly from simulation results, presented in this paper. These problems seem to be complex and cannot be tested by the current version of our software for historical prices.

The problems of a very short horizon are well-known. For example, in [14] we find "daily sampling yields many observations, the biases associated with nontrading, the bid-ask spread, asynchronous prices, etc. are troublesome" and in [20] "daily data have potential biases associated with infrequent trading, the bid-ask effect etc." This is the reason that in both investigations a horizon of one week is used. According to [14], these effects cause wrong correlation, in particular for small stocks. On the other hand, the correlation strongly depends on the method used to evaluate correlation [12].

Despite all objections, statistical dependencies between past and future are expected to be maximum using the price relationship  $c_1 = s(t)/s(t-1)$  for evaluating the past and the return  $s(t+1)/s(t)$  for the future. Thus, the strong mean reversion effect for one day is not that surprising anymore. Furthermore, this effect is plausible as well: A price slump of a security makes the shareholder unhappy. It inflames his fear of a further decrease of prices and he will sell. On the other side, a clever market player will buy, making him often happy at the following trading day and at the end of the trading period. Thus, one man's joy is another man's sorrow.

## 6 Summary and outlook

In this paper, daily new investments are axiomatically determined by constraints, depending on several investment parameters. Simulation results with historical prices are shown for a well-defined set of shares, traded on the Frankfurt stock exchange. The buy condition is defined by two buy formulas and the sales condition is defined by a holding period of one trading day. The first buy formula relies on continuation of returns and fails. The second buy formula trusts in price recovery after a slump, yielding a high return. For the failing buy formula, the falling total capital even causes a trading abort, if the constraints cannot be satisfied any longer. The trading process is a random trial, because purchased securities are selected randomly among the buy recommendations, i.e. the securities, for which the buy condition is satisfied. In contrast, if all buy recommendations are purchased, the trading process becomes deterministic, which is possible for a minimum new investment ( $\text{inv}_{\min}$ ) being sufficiently small. In this case, an expert provides the return at the end of the trading period for a parametrized set of buy formulas simultaneously. For a random trading process, the expert reflects the expected return at the end of the trading period with respect to the random trial and in this sense is consistent with trading. Thus, the expert informs on how many market players perform, using the same formulas for purchases and sales and the same investment parameters. The expert confirms that the simulation results for the two buy formulas are no coincidence, but clearly reflect the effect of mean reversion. The expert described in this paper has expertise on future prices for a horizon of one trading day, interpreting past prices by a single property, which is the price relationship for two subsequent days. Extensions towards longer horizons, using several properties simultaneously and other statistical quantities besides the expected return, are addressed in the outlook.

### Outlook:

The following topics are subject to future research concerning the extension of the algorithmic trading theory.

**Extension towards greater horizons:** The product formula describing the expected return is extended to  $\tau_{\text{hold}} > 1$ . For this purpose, the trading period is divided into so-called *segments*, each comprising  $\tau_{\text{hold}}$  subsequent trading days, and the total capital at the end of each segment is described analytically. In order to keep a simple product formula as in this paper, without simulating the trading process, *ideal trading* is introduced, consisting of three idealizations. One idealization uses deterministic trading as in this paper. By another idealization, the amount of new investment is adjusted depending on the total capital at the end of a segment, yielding a non-causal trading process. Finally, the maximum investment ratio is incorporated differently into the trading process. For  $\tau_{\text{hold}} > 1$ , the trading expert does only reproduce the expected return approximately. This is analyzed by simulations with real stocks as in this paper and more abstractly, by recursive equations stimulated by two random processes, the first one for the gain factors  $M_1(t)$  and the second one for the number of buy recommendations  $N(t)$ .

**Complete definition of the statistics:** Besides the expected return, other statistical quantities are introduced like the arithmetic mean return, chance and risk and volatility measures, all depending on the investment parameters. The statistical quantities also depend on one or more real-valued properties  $c_1, c_2, \dots$  for parametrizing the buy condition. We investigate the influence of the maximum investment ratio  $\text{IR}_{\max}$  and the number of buy recommendations  $N(t)$  on the statistics. By this analysis, we obtain a type of liquidity condition for the statistics. Empirical results are shown for a horizon of one day. In this case, chance and risk clearly confirm the strong mean reversion effect demonstrated in this paper. Empirical results are also shown for a horizon of 30 days, demonstrating clearly a return continuation effect. In this case, the trading expert uses two parameters, defined by two moving averages for 30 days.

**Maximum return hypothesis:** A trading expert reveals buy conditions with maximum expected return within the parametrized set of buy conditions, for certain investment parameters and a holding period  $\tau_{\text{hold}}$ , and a certain trading period. Simulations with this buy condition indicate that it is difficult to further improve the return, using sales conditions different from the simple holding period rule. This gives rise to the following hypothesis:

*For a buy condition, which is optimized for a certain holding period, the expected return cannot be significantly increased, if selling is not after expiration of the holding period.* As a strange consequence, price data for trading days following the purchase date contain no valuable information for selling.

**Modified investment algorithm:** For  $\tau_{\text{hold}} > 1$ , constraint (A) does not avoid that the capital invested in a single security can exceed the threshold  $C_{\text{tot}}(t)/U_{\text{min}}$  considerable, resulting in higher risk than intended by constraint (A). This effect will be evident, if the number of buy recommendations is small. On the other hand, rare buy conditions prove to be valuable, as the example of buy formula BF2 has been shown. Thus, the investment algorithm proposed in this paper has to be modified, limiting the amount of investment in a single security by  $C_{\text{tot}}/U_{\text{min}}$ . For this modification, a replenishment algorithm is required in order to distribute the invested capital over securities uniformly. However, consistency of trading and statistics is impaired by the modified investment algorithm.

**Model generation:** We have established that a market of real securities can be statistically evaluated by a trading expert. The inverse problem starts with an expert and generates a random process, having similar statistics as the expert provides. Thus, the random process serves as a statistical model for the real market. The model can be confirmed experimentally, generating a market of artificial securities from the random process, and then evaluating this market statistically by the expert: Its statistics coincide with the statistics of the real market. The inverse problem is solved for a holding period of one day. A nonlinear feedback of order one, described by a difference equation, is stimulated randomly and controlled by an expert.

## 7 Appendix A. Proof of Lemma 1

In the following, the investment finished at day  $t$  due to the sales at this day is required, which will be denoted by  $I_{end}(t)$ . For example, for selling after expiration of a holding period  $\tau_{hold}$ ,  $I_{end}(t)$  is given by  $I_{new}(t - \tau_{hold})$ . By means of  $I_{end}(t)$ , the following two recursions can be formulated:

$$C_{inv^-}(t) = C_{inv}(t-1) - I_{end}(t), \quad (A1)$$

$$C_{tot}(t) = C_{tot}(t-1) + sale(t) - I_{end}(t). \quad (A2)$$

According to (A1), the investment at the end of day  $t-1$  is decreased by the investments finished at day  $t$ , yielding the investment at day  $t$  after the sales,  $C_{inv^-}(t)$ . According to (A2), the total capital at day  $t-1$  is increased by the proceeds at day  $t$ , denoted by  $sale(t)$ , and decreased by the investments finished at this day, yielding the total capital at day  $t$ .

From (A1), we get for  $t \geq 1$

$$IR^-(t) = \frac{C_{inv^-}(t)}{C_{tot}(t)} = \frac{C_{inv}(t-1) - I_{end}(t)}{C_{tot}(t)}$$

and for the difference

$$\begin{aligned} IR_{\max} - IR^-(t) &= IR_{\max} - \frac{C_{inv}(t-1) - I_{end}(t)}{C_{tot}(t)} \\ &= IR_{\max} - \frac{IR(t-1) \cdot C_{tot}(t-1) - I_{end}(t)}{C_{tot}(t)}. \end{aligned}$$

According to our premises,  $C_{tot}(t-1)$  and  $C_{tot}(t)$  are positive and  $IR(t-1) \leq IR_{\max}$  holds. Hence,

$$\begin{aligned} IR_{\max} - IR^- &\geq IR_{\max} - \frac{IR_{\max} \cdot C_{tot}(t-1) - I_{end}(t)}{C_{tot}(t)} \\ &= \frac{IR_{\max} \cdot [C_{tot}(t) - C_{tot}(t-1)] + I_{end}(t)}{C_{tot}(t)} \end{aligned}$$

and by (A2)

$$\begin{aligned} IR_{\max} - IR^-(t) &\geq \frac{IR_{\max} \cdot [sale(t) - I_{end}(t)] + I_{end}(t)}{C_{tot}(t)} \\ &= \frac{IR_{\max} \cdot sale(t) + [1 - IR_{\max}] \cdot I_{end}(t)}{C_{tot}(t)}. \end{aligned} \quad (A3)$$

The expression in (A3) is non-negative, since all of the quantities  $sale(t)$ ,  $I_{end}(t)$ ,  $C_{tot}(t)$  and  $IR_{\max}$  as well as  $1 - IR_{\max}$  are non-negative:

1. By our assumptions,  $sale(t)$  and  $C_{tot}(t)$  are non-negative.
2. The investment finished at day  $t$  only consists of new investments before  $t$ , which by our assumptions are non-negative. Thus,  $I_{end}(t)$  is not negative as well.
3. For the maximum investment ratio,  $0 \leq IR_{\max} \leq 1$  holds due to our premises.

q.e.d.

**Remark 4 (The case  $IR^-(t) = IR_{\max}$ )** By the estimation carried out in the proof, the case

$$IR_{\max} - IR^-(t) = 0$$

is possible for

$$IR(t-1) = IR_{\max}, \quad sale(t) = I_{end}(t) = 0.$$

In this case, the maximum investment ratio  $IR_{\max}$  is achieved on trading day  $t-1$ . Furthermore, on trading day  $t$  no investments will be finished and consequently, no proceeds occur at this day. Thus, the investment ratio immediately before purchases at day  $t$  equals the investment ratio at the beginning of day  $t$  and thus equals the investment ratio at the end of day  $t-1$ , which is  $IR_{\max}$ .

## 8 Appendix B. Solution of the optimization problem

### B.1 Conversion of the constraints

In the following, the investment rate  $I(t)$  is introduced to the constraints (A) and (C). The total capital  $C_{tot}(t)$  is assumed to be greater than 0.

**Constraint (A):** Constraint (A) is equivalent to

$$\begin{aligned} I(t) \cdot C_{tot}(t) &= I_{new}(t) = N_{buy}(t) \cdot I_{buy}(t) \\ &\leq N_{buy}(t) \cdot C_{tot}(t)/U_{\min} . \end{aligned}$$

Thus, we get

$$I(t) \leq N_{buy}(t)/U_{\min} . \quad (B1)$$

If the condition is satisfied with equality, then constraint (A) is satisfied with equality too.

**Constraint (C):** For  $N_{buy}(t) > 0$ , constraint (C) is equivalent to

$$\begin{aligned} I(t) \cdot C_{tot}(t) &= I_{new}(t) = N_{buy}(t) \cdot I_{buy}(t) \\ &\geq N_{buy}(t) \cdot \text{inv}_{\min} \end{aligned}$$

or

$$N_{buy}(t) \leq I(t) \cdot C_{tot}(t)/\text{inv}_{\min} . \quad (B2)$$

The inequality is also true for  $N_{buy}(t) = 0$  with  $I(t) = 0$ .

**Constraints (A), (C) and (D) :** Combining the constraint (A), (C) and (D), we obtain

$$\begin{aligned} I(t) \cdot U_{\min} &\leq N_{buy}(t) \\ &\leq \min\{I(t) \cdot C_{tot}(t)/\text{inv}_{\min}, N(t)\} . \end{aligned} \quad (B3)$$

With  $x = C_{tot}(t)/[\text{inv}_{\min} \cdot U_{\min}]$  and  $y = I(t) \cdot U_{\min}$  we get

$$y \leq N_{buy}(t) \leq \min\{x \cdot y, N(t)\} .$$

Since the number of purchases is whole-numbered, this condition is equivalent to

$$y \leq N_{buy}(t) \leq \min\{\text{int}[x \cdot y], N(t)\} \quad (B4)$$

with  $\text{int}[a]$  denoting rounding off a real number  $a$ . It defines the range for the number of purchases. With the *range buy function*  $\text{RBF}(x, y) = \text{int}[x \cdot y] - y$ , we obtain

$$y \leq N_{buy}(t) \leq \min\{y + \text{RBF}(x, y), N(t)\} . \quad (B5)$$

Thus, the range buy function must be non-negative, to obtain a non-empty range for the number of purchases.

### B.2 Definition of the investment rate

Concerning the constraints (B) and (E), we have

$$I(t) \leq 1/\tau , I(t) \leq \text{IR}_{\max} - \text{IR}^-(t) .$$

By (B3) we obtain the additional limitation

$$I(t) \leq N(t)/U_{\min} .$$

Thus, the investment rate is upper bounded by  $I_1(t)$  according to (30). Consequently,  $y = I(t) \cdot U_{\min}$  is upper bounded by

$$y \leq y_1 = I_1(t) \cdot U_{\min} . \quad (B6)$$

Since the range buy function must be non-negative, a maximum amount of new investment is achieved by the definition

$$y = y_{\max} := \max\{y \leq y_1 | \text{RBF}(x, y) \geq 0\} . \quad (B7)$$

### B.3 Evaluation of the investment rate (proof of theorem 1)

In the following, (B7) will be evaluated, to determine the maximum investment rate. For this purpose, three different cases are distinguished:  $x < 1$  (case 1) and  $x \geq 1$  with range buy function  $\text{RBF}(x, y_1)$  being non-negative (case 2) or negative (case 3). To decide, whether the range buy function is negative or not, we use the following criterion:

$$\begin{aligned} \text{RBF}(x, y) &= \text{int}[x \cdot y] - y < 0 \\ &\Leftrightarrow \text{int}[x \cdot y] < y \\ &\Leftrightarrow x \cdot y < \text{int}^+[y] \end{aligned}$$

with  $\text{int}^+[a]$  denoting rounding up a real number  $a$ . Since  $y = 0$  is excluded, we obtain

$$\text{RBF}(x, y) < 0 \iff y > 0, x < \text{int}^+[y]/y. \quad (\text{B8})$$

Thus, for a negative range buy function, the value of  $x$  is below the expression  $\text{int}^+[y]/y$  depending on  $y$ .

#### Case 1: $x < 1$

In this case, we obtain for  $y > 0$  a negative range buy function, since the criterion in (B8) is satisfied:

$$x \cdot y < y \leq \text{int}^+[y].$$

From (B7) we get  $y_{\max} = 0$ , i.e. no investments are made, which confirms (26).

#### Case 2: $x \geq 1$ and $\text{RBF}(x, y_1) \geq 0$

In this case, the solution of (B7) is  $y_{\max} = y_1$  by triviality, which confirms the upper part of (27).

#### Case 3: $x \geq 1$ and $\text{RBF}(x, y_1) < 0$

In the following, we discuss the criterion in (B8) for a negative range buy function  $\text{RBF}(x, y) < 0$ . For an integer value  $y > 0$ , we obtain  $x < \text{int}^+[y]/y = 1$ , which contradicts  $x \geq 1$ . Thus, for an integer  $y > 0$ ,  $\text{RBF}(x, y) \geq 0$  holds. For  $y = 0$ , we have  $\text{RBF}(x, y) = 0$ . Thus, we obtain

$$x \geq 1, y \geq 0 \text{ integer} \Rightarrow \text{RBF}(x, y) \geq 0. \quad (\text{B9})$$

This means that  $\text{RBF}(x, y) < 0$  is only possible for a non-integer  $y > 0$  with

$$y < \text{int}^+[y]/x \quad (\text{B10})$$

and thus holds in the following situations only:

$$\begin{aligned} 1.) \quad 0 < y < 1 & : y < 1/x \\ 2.) \quad 1 < y < 2 & : y < 2/x \\ 3.) \quad 2 < y < 3 & : y < 3/x \dots \end{aligned} \quad (\text{B11})$$

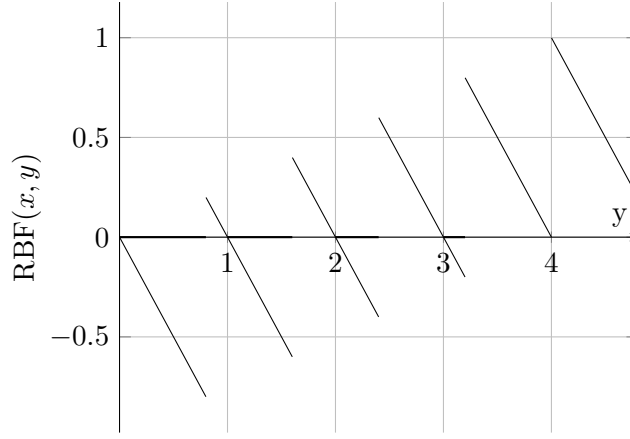
Hence,  $y$  lies in intervals of the form  $n < y < (n+1)/x$  with  $n$  denoting an integer value:

$$n < y < (n+1)/x, n = 0, 1, \dots \iff \text{RBF}(x, y) < 0. \quad (\text{B12})$$

In Fig. 8, these intervals are marked for the example  $x = 5/4$ . From (B9) and (B12) we conclude that the maximum value  $y \leq y_1$  with  $\text{RBF}(x, y) \geq 0$  is attained by rounding down  $y_1$ ,

$$y_{\max} = \text{int}[y_1], \quad (\text{B13})$$

which confirms the lower part of (27). Fig. 8 shows that for this value the range buy function has a zero.

Figure 8: Range buy function  $\text{RBF}(x, y)$  for  $x = 5/4$ **No investments:**

According to (26), new investments don't take place in the first case  $x < 1$ . This is not the only case where no investments take place, as discussed in the following.

**Case 2:  $x \geq 1$  and  $\text{RBF}(x, y_1) \geq 0$** 

By (27),  $y = y_1$  holds in this case. Hence, no investments take place for  $y_1 = 0$ . According to (30), either no buy recommendations exist ( $N(t) = 0$ ), or new investments are not possible, because the maximum investment ratio has been already achieved ( $IR^-(t) = IR_{\max}$ ).

**Case 3:  $x \geq 1$  and  $\text{RBF}(x, y_1) < 0$** 

By (B12),  $y_1$  is lying in one of the intervals  $n < y_1 < (n+1)/x$ ,  $n = 0, 1, \dots$ . Only for the first interval  $0 < y_1 < 1/x$ , rounding down  $y_1$  yields  $y = 0$ . For the example in Fig. 8, the first interval is given by  $0 < y_1 < 4/5$ .

**Remark 5 (The case  $x \geq 2$ )** For  $x \geq 2$  and  $\text{RBF}(x, y_1) < 0$ ,  $y_1$  is lying in the first interval, because only this interval is not empty. Consequently, for  $x \geq 2$  new investments are not possible in case 3.

To summarize, we obtain the following criterion for no investments:

$$I(t) = 0 \iff x < 1 \text{ or } y_1 \cdot x < 1. \quad (\text{B14})$$

**B.4 Investing without constraint (C) (proof of corollary 2)**

Without constraint (C), the inequalities in (B4) are to be replaced by

$$y \leq N_{\text{buy}}(t) \leq N(t) = N_{\text{buymax}}(t). \quad (\text{B15})$$

This means that the maximum number of purchases is simply given by  $N(t)$ , which confirms (49). The equation in (31) for the minimum number of purchases remains valid.

As another consequence, (B7) is simplified according to

$$y_{\max} = \max\{y \leq y_1 \mid y \leq N(t)\}.$$

By definition of  $I_1(t)$  in (30),

$$y_1 = I_1(t) \cdot U_{\min} \leq N(t)$$

holds. Thus, for  $y \leq y_1$ , the condition  $y \leq N(t)$  is satisfied automatically. We obtain  $y_{\max} = y_1$  and the identity in (48).



## 9 Appendix C. Evaluations for a holding period of one day

### C.1 Proof of lemma 2

The course of the total capital  $C_{tot}(t)$  depends on the securities  $s(n, t)$  purchased on each trading day  $t$ . These securities are identified by the set  $buy(t)$ . Then, the proceeds from sales at day  $t + 1$  are given by

$$sale(t + 1) = \sum_{n \in buy(t)} I_{buy}(t) \cdot s(n, t + 1) / s(n, t) .$$

$I_{buy}(t)$  is the amount of investment in each security  $n \in buy(t)$ , given by (2) according to  $I_{buy}(t) = I_{new}(t) / N_{buy}(t)$ . For  $N_{buy}(t) = 0$ , the set  $buy(t)$  is empty and new investments are not carried out, i.e.  $I_{new}(t) = sale(t + 1) = 0$  holds in this case. We obtain

$$sale(t + 1) = I_{new}(t) \cdot G(t) \tag{C1}$$

with the gain factor

$$G(t) = \frac{1}{N_{buy}(t)} \sum_{n \in buy(t)} s(n, t + 1) / s(n, t) , \tag{C2}$$

defined for  $N_{buy}(t) > 0$ .

By the recursion for the total capital in (A2),

$$C_{tot}(t + 1) = C_{tot}(t) + sale(t + 1) - I_{end}(t + 1) \tag{C3}$$

holds. For a holding period of one day, we have for each trading day  $t \geq 1$

$$I_{end}(t + 1) = I_{new}(t) = I(t) \cdot C_{tot}(t) .$$

Furthermore, from (C1) we get

$$sale(t + 1) = I(t) \cdot C_{tot}(t) \cdot G(t) .$$

Thus, we obtain the recursive relationship

$$C_{tot}(t + 1) = C_{tot}(t) \cdot [1 + I(t) \cdot G(t) - I(t)] .$$

This leads to the total capital at the end of the last trading day  $t = T$ ,

$$C_{tot}(T) = C_{tot}(1) \cdot \prod_{t=1}^{T-1} [1 + I(t) \cdot (G(t) - 1)] , \tag{C4}$$

from which we obtain (58).

### C.2 Proof of theorem 2

The total capital in (C4) is a random variable, caused by trading days  $t$  with  $0 < N_{buy}(t) < N(t)$ . On those days  $t$ , the securities to be purchased are selected randomly among the buy recommendations. This situation corresponds to a random draw, where  $m = N_{buy}(t)$  securities are drawn from  $N(t)$  securities (the buy recommendations) without returns. Introducing the random variable  $G_i$ ,  $i = 1, \dots, m$ , representing the gain factor  $s(n, t + 1) / s(n, t)$  for draw number  $i$ , the gain factor  $G(t)$  becomes

$$G(t) = \frac{G_1 + G_2 + \dots + G_m}{m} .$$

For the expectation of  $G(t)$  we obtain

$$E\{G(t)\} = \frac{E\{G_1\} + E\{G_2\} + \dots + E\{G_m\}}{m} .$$

The expected values of all  $G_i$ ,  $i = 1, \dots, m$  are given by  $M_1(t)$ , as defined in (64),

$$E\{G(t)\} = M_1(t) . \tag{C5}$$

This holds for all  $m$ , i.e. the number of purchases has no effect on the expectation of  $G(t)$ . In contrast,  $m$  effects the variance of the random variable  $G(t)$ , which decreases with  $m$ , vanishing for  $m = N(t)$ , but the variance will be not considered in this paper.

Inserting the assumption  $I(t) = I_1(t)$  into (C4) leads to

$$C_{tot}(T) = C_{tot}(1) \cdot \prod_{t=1}^{T-1} [1 + I_1(t) \cdot (G(t) - 1)] .$$

Since the draws take place independent from each other on different days, the gain factors  $G(t)$  are statistical independent random variables. Since the quantities  $I_1(t)$  are deterministic, the expectation of  $C_{tot}(T)$  is

$$E\{C_{tot}(T)\} = C_{tot}(1) \cdot \prod_{t=1}^{T-1} [1 + I_1(t) \cdot (E\{G(t)\} - 1)] .$$

Using the identity (C5) we obtain (66).

## 10 Appendix D. Notations and terms

### Notations:

#### Capital

$C_{cash}(t)$	Amount of cash at the end of trading day $t$ ( $1 \leq t \leq T$ )
$C_{cash}(0)$	Amount of cash at the beginning of trading
$C_{inv}(t)$	Amount of investment at the end of trading day $t$
$C_{inv}(0)$	Amount of investment at the beginning of trading
$C_{inv^-}(t)$	Amount of investment after the sales and before the purchases at trading day $t$
$C_{tot}(t)$	Amount of cash and investment at the end of trading day $t$
$C_{tot}(0)$	Amount of cash and investment at the beginning of trading

#### Investments

$I(t)$	Investment rate at trading day $t$ , given by $I(t) = I_{new}(t)/C_{tot}(t)$
$I_{buy}(t)$	Amount of new investment in a single security at trading day $t$
$I_{new}(t)$	Amount of new investment (in all securities) at trading day $t$
$IR(t)$	Investment ratio at the end of trading day $t$ , given by $C_{inv}(t)/C_{tot}(t)$
$IR^-(t)$	Investment ratio after the sales and before the purchases at trading day $t$ , given by $C_{inv^-}(t)/C_{tot}(t)$
$I_{end}(t)$	Amount of investment finished at trading day $t$
$sale(t)$	The proceeds from the sales at trading day $t$

#### Investment parameters

$inv_{\min}$	Lower bound for $I_{buy}(t)$
$IR_{\max}$	Maximum investment ratio
$\tau$	Temporal investment diversification
$U_{\min}$	Investment diversification over securities
$\tau_{\text{hold}}$	Holding period for purchase

#### Securities

$s(t)$	Price of a security at trading day $t$
$s(n, t)$	Price of security number $n$ at trading day $t$
$T$	Securities do not end before the last trading day $T$
$N(t)$	Number of buy recommendations at trading day $t$
$N_{buy}(t)$	Number of purchased securities at trading day $t$
$buy(t)$	Set of all securities, satisfying the buy condition at trading day $t$

## Investment algorithm

$x$	Input quantity, given by $x = C_{tot}(t)/[\text{inv}_{\min} \cdot U_{\min}]$
$I_1(t)$	Upper bound for $I(t)$ with $I_1(t) = \min\{N(t)/U_{\min}, 1/\tau, IR_{\max} - IR^-(t)\}$
$y, y_1$	Investment rates multiplied with $U_{\min}$ , i.e. $y = I(t) \cdot U_{\min}$ and $y_1 = I_1(t) \cdot U_{\min}$
$\text{RBF}(x, y_1)$	Range buy function, given by $\text{RBF}(x, y) = \text{int}[x \cdot y] - y$

## Gain factor and return

$G(t)$	Gain factor for trading day $t$ and a holding period of one day, given by (59)
$M_1(t)$	Gain factor for deterministic trading for trading day $t$ and a holding period of one day, given by (64)
$G$	Gain factor for the whole trading period and a holding period of one day, given by (58)
$G_e$	Gain factor for a trading expert, given by (65)
$G - 1$	Return for the trading period

**Terms:**

Buy condition	Must be satisfied to start a new investment in a security
Buy formula	Implements a buy condition
Gain factor	Ratio of total capital for two different trading days
Investment ratio	Ratio of invested capital to total capital
Investment rate	Ratio of new invested capital to total capital for a trading day
Range buy function	Defines the range of the number of purchases according to the investment algorithm
Return	Gain factor minus 1
Sales condition	Under the sales condition, an investment in a security is finished
Sales formula	Implements a sales condition
Total capital	Sum of cash capital and invested capital
Trading expert	Evaluates statistical quantities for a market, depending on investment parameters and a parametrized set of buy conditions
Trading state	Stores past investments (purchases), which are not finished, the amount of cash and investment

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