Novel Variants of Diffusive Representation of Fractional Integrals: Construction and Numerical Computation ⋆

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Abstract: In this paper, we revisit the diffusive representations of fractional integrals established in Diethelm (2023a) to explore novel variants of such representations which provide highly efficient numerical algorithms for the approximate numerical evaluation of fractional integrals.

Keywords: Diffusive Representation; Fractional Integrals; Quadrature Formula; Numerical Approximation, Exponential Sum Approximation.

1. INTRODUCTION

Fractional calculus, nowadays an established mathematical discipline, plays a pivotal role in understanding complex physical and mathematical phenomena. The application of fractional calculus extends to various fields such as physics, engineering, finance, and signal processing, among others. For a comprehensive introduction and exploration of its applications, we refer to Podlubny (1999), Kilbas et al. (2006) and Diethelm (2010). Fractional integrals, a fundamental concept within this domain, have been extensively studied and applied for solving real-world problems.

In certain scenarios, discovering analytical solutions for problems involving fractional order operators can be challenging or even practically impossible. Hence a large number of different approaches have been proposed for numerical approximation of fractional order differential and integral operators. Due to the intrinsic non-local nature of fractional order differential and integral operators, achieving their numerical evaluation in the conventional representation presents a significantly heightened computational challenge compared to assessing their integer order counterparts. This challenge pertains to both runtime efficiency and memory demands, particularly when evaluating at multiple points. For instance, conventional algorithms such as fractional linear multi-step methods in Lubich (1985) and Adams method in Diethelm et al. (2002, 2004) may pose computational challenges due to their reliance on numerical approximations incurring a computational cost of $\mathcal{O}(P^2)$ and a storage cost of $\mathcal{O}(P)$, where P represents the number of evaluation points.

In this case, employing diffusive representations of fractional operators presents a dynamic and sustainable solution to address the issue at hand. Utilizing diffusive representations of fractional differential and integral operators offers a practical avenue for crafting highly efficient numerical algorithms aimed at their approximate assessment with less memory footprint $\mathcal{O}(1)$ and arithmetic complexity $\mathcal{O}(P)$, which aligns with the techniques commonly employed in solving problems with integer order operators. For more comprehensive information on diffusive representation, please refer to Montseny (1998), Diethelm (2022) and Diethelm (2023b).

In this research, we delve into the realm of fractional integrals, focusing on their diffusive representations. These representations serve as a bridge between theoretical understanding and practical applications. Building upon the foundational work by Diethelm (2023a), we present novel variants of these diffusive representations, pushing the boundaries of efficiency and accuracy in numerical computation of fractional integrals.

Our objective is to enhance the existing diffusive representations, offering improved algorithms for approximating fractional integrals numerically. By doing so, we aim to provide researchers and practitioners with powerful tools to handle the intricate calculations associated with fractional integrals. These enhanced numerical methods open doors to solving complex problems with a higher degree of precision and efficiency, thus advancing the applicability and impact of fractional calculus in diverse domains. In the subsequent sections, we elucidate the existing diffusive representations, introduce our novel variant, and showcase how these variants pave the way for highly efficient numerical algorithms. Additionally, we outline potential future directions in this exciting area of research.

2. MATHEMATICAL BACKGROUND

Our aim is to explore novel variants of diffusive representation for Riemann-Liouville fractional integrals of order $\alpha \in (0,1)$ for functions $f \in C[a,b]$ (where $a,b \in \mathbb{R}$ such that $a < b$) with a starting point at a represented as

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau. \tag{1}$$

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As discussed in Diethelm (2023a), the diffusive representation of (1) can be formulated as an integral involving an auxiliary bivariate function such that the integration is performed with respect to a single variable, within a fixed range. This characteristic allows for numerical computation using a fixed quadrature formula whose arithmetic complexity does not increase over time. Additionally, the auxiliary function can be determined as the unique solution to an initial value problem for a first-order differential equation, obviating the need to consider memory effects.

Definition 1. Let Ω be a non-empty open interval. A strictly monotonically increasing function $\psi : \Omega \to (0, \infty)$ is called an admissible transformation if it possesses the following two properties:

(a) $\psi \in C^1(\Omega)$,
(b) $\lim_{r\to\inf\Omega} \psi(r) = 0$ and $\lim_{r\to\sup\Omega} \psi(r) = +\infty$.

Theorem 1. For a given admissible transformation $\psi : \Omega \to (0, \infty)$, the Riemann-Liouville integral of order $\alpha \in (0, 1)$ for the function $f \in C[a, b]$, where $a, b \in \mathbb{R}$, can be articulated in the following diffusive representation

$$I_a^\alpha f(t) = \int_\Omega \phi(t, r)\,\mathrm{d}r, \tag{2}$$

where

$$\phi(t, r) = c_\alpha \psi'(r)(\psi(r))^{-\alpha} \int_a^t \mathrm{e}^{-(t-\tau)\psi(r)} f(\tau)\,\mathrm{d}\tau, \tag{3}$$

with constant $c_\alpha = \pi^{-1} \sin \pi\alpha$.

Moreover, for a fixed value of $r \in \Omega$, the function $\phi(\cdot, r)$ is characterized as the unique solution to the following initial value problem for first-order differential equation on the interval $[a, b]$

$$\begin{aligned} \frac{\partial \phi(t, r)}{\partial t} &= -\psi(r)\phi(t, r) + c_\alpha \psi'(r)(\psi(r))^{-\alpha} f(t), \\ \phi(a, r) &= 0. \end{aligned} \tag{4}$$

Proof. The proof of the theorem can be straightforwardly deduced from the proof presented in Diethelm (2023a, Theorems 1 and 2), specifically for the case $n = 1$. □

In literature, numerous approaches have been devised to articulate the fractional integral of a provided function. For instance, in the work by Li (2010), the author computed the fractional integral by employing an integral representation of the convolution kernel. She proceeded to devise an effective quadrature for this integral representation and incorporated it into a fast time-stepping technique. In Beylkin and Monzón (2010), the authors applied the Poisson's summation formula to discretize the integral representations of power functions to obtain their approximations by exponential functions. Later on, in McLean (2018), the author extended the idea of Beylkin and Monzón (2010) by introducing an alternative integral representation of the power function. In the study Baffet (2019), the Gauss-Jacobi quadrature rule was utilized to approximate the kernel of the fractional integral through a linear combination of exponentials.

The main objective of this paper is to devise a diffusive representation that facilitates more efficient numerical handling of the integral (2) than traditional approaches. For this, we have examined an important special case of admissible transformation $\psi(\cdot)$ such that the corresponding bivariate function $\phi(t, \cdot)$ demonstrates an exponential decay at the endpoints of the domain Ω. This behavior holds significant value from an approximation theoretic perspective which allows us to use truncated trapezoidal scheme or the Gauss-Laguerre numerical quadrature scheme for integrating the function ϕ. Moreover, it is imperative to have knowledge of the integrand $\phi(t, \cdot)$, which we have achieved by numerically solving the associated first-order differential equation (4) through Euler's Backward method or the trapezoidal method.

3. NOVEL VARIANTS OF DIFFUSIVE REPRESENTATION OF FRACTIONAL INTEGRALS

In this section, we discuss a diffusive representation to the integral (2) using the well-known exponential function as an admissible transformation. We prove that in this new representation the integrand $\phi(t, r)$ showcases exponential decay as r tends towards $\pm\infty$. This characteristic, coupled with a smoothness result, allows a highly efficient numerical integration.

Theorem 2. Consider a function $f \in C[a, b]$, where $a, b \in \mathbb{R}$ such that $a < b$, and $\alpha \in (0, 1)$. Let us define the function $\phi(t, r)$ as

$$\phi(t, r) = c_\alpha \mathrm{e}^{(1-\alpha)r} \int_a^t \mathrm{e}^{-(t-\tau)\mathrm{e}^r} f(\tau)\,\mathrm{d}\tau. \tag{5}$$

for all $r \in \mathbb{R}$ and $t \in [a, b]$. The following properties can then be observed:

(i) The function $\phi(\cdot, r)$ is characterized as the unique solution to the following initial value problem for first-order differential equation on the interval $[a, b]$

$$\begin{aligned} \frac{\partial \phi(t, r)}{\partial t} &= -\mathrm{e}^r \phi(t, r) + c_\alpha \mathrm{e}^{(1-\alpha)r} f(t), \\ \phi(a, r) &= 0. \end{aligned} \tag{6}$$

(ii) For any $t \in [a, b]$,

$$I_a^\alpha f(t) = \int_{-\infty}^{\infty} \phi(t, r)\,\mathrm{d}r. \tag{7}$$

(iii) For any $t \in [a, b]$, the integrand $\phi(t, \cdot) \in C^\infty(\mathbb{R})$.
(iv) For any t belonging to the interval $[a, b]$, there exists a real constant C such that:

$$|\phi(t, r)| \le C\mathrm{e}^{-\alpha r} \quad as\ r \to \infty, \tag{8}$$

and

$$|\phi(t, r)| \le C\mathrm{e}^{(1-\alpha)r} \quad as\ r \to -\infty. \tag{9}$$

Proof. By selecting $\psi(r) = \mathrm{e}^r$ with $\Omega = (-\infty, \infty)$ in (2) and (3), the proof of the Theorem can be directly deduced from the results presented in Diethelm (2023a). □

As emphasized in Diethelm (2023a), the accuracy of the numerical approximation for the integral (7) is significantly influenced by two key aspects of the integrand. The first crucial aspect is the smoothness of the integrand, while the second pertains to the asymptotic behavior of the integrand as the integration variable approaches the integration limit. In the context of a diffusive representation, these two critical aspects of the integrand are profoundly influenced by the choice of the admissible transformation, which in this case is given by the exponential function.

While the general representation (2) and its special case (5) have proven to be very useful in analytical considerations (Montseny, 1998), it has been observed (McLean, 2018) that there are certain advantages from the numerical approximation perspective in handling the operators in a slightly different way. Specifically, for $t \geq a+h$ with some $h > 0$ it is useful to split up the hereditary integral from (1) into a local part $L^\alpha_{a,h} f(t)$ and a history part $H^\alpha_{a,h} f(t)$ according to

$$I^\alpha_a f(t) = L^\alpha_{a,h} f(t) + H^\alpha_{a,h} f(t)$$

where

$$L^\alpha_{a,h} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t-h}^{t} (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau \tag{10}$$

and

$$H^\alpha_{a,h} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t-h} (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau. \tag{11}$$

The local part $L^\alpha_{a,h} f$ can usually be handled directly, but the history part $H^\alpha_{a,h} f(t)$ benefits from a representation that is analog to the diffusive form for the full Riemann-Liouville integral discussed above. Specifically, the following relationships hold.

Theorem 3. Let $f \in C[a,b]$ with some real numbers $a < b$, and let $\alpha \in (0,1)$. Moreover, assume that $\psi : \Omega \to (0,\infty)$ is an admissible transformation and that $0 < h < b-a$. Then, for every $t \in [a+h, b]$, the history part of the Riemann-Liouville integral of order α of the function f can be expressed in the form of the diffusive representation

$$H^\alpha_{a,h} f(t) = \int_\Omega \mu(t,h,r)\,\mathrm{d}r \tag{12a}$$

with

$$\begin{aligned}\mu(t,h,r) &= c_\alpha \psi'(r)(\psi(r))^{-\alpha} \\ &\quad \times \int_a^{t-h} \exp(-(t-\tau)\psi(r)) f(\tau)\,\mathrm{d}\tau.\end{aligned} \tag{12b}$$

The function μ from (12b) has the following properties:

(a) For any $r \in \Omega$, the function $\mu(\cdot,h,r)$ is the unique solution on the interval $[a+h,b]$ to the first order differential equation

$$\begin{aligned}\frac{\partial \mu}{\partial t}(t,h,r) &= -\psi(r)\mu(t,h,r) \\ &\quad + c_\alpha \psi'(r)(\psi(r))^{-\alpha} \mathrm{e}^{-h\psi(r)} f(t-h)\end{aligned} \tag{13a}$$

subject to the initial condition

$$\mu(a+h,h,r) = 0. \tag{13b}$$

(b) If the transformation function ψ satisfies $\psi \in C^k(\Omega)$ with some $k \in \mathbb{N}$ then, for any $t \in [a+h,b]$, we have $\mu(t,h,\cdot) \in C^{k-1}(\Omega)$.

(c) If the transformation function ψ satisfies $\psi \in C^\infty(\Omega)$ then, for any $t \in [a+h,b]$, we have $\mu(t,h,\cdot) \in C^\infty(\Omega)$.

(d) If $f \in C^\ell[a,b]$ with some $\ell \in \mathbb{N}_0$ then, for any $r \in \Omega$, $\mu(\cdot,h,r) \in C^{\ell+1}[a+h,b]$.

(e) For any fixed $t \in [a+h,b]$, there exist some constants $C_1, C_2 > 0$ such that

$$|\mu(t,h,r)| \leq C_1 \psi'(r)(\psi(r))^{-\alpha-1} \mathrm{e}^{-h\psi(r)} \quad \text{for } r \to \sup\Omega \tag{14}$$

and

$$|\mu(t,h,r)| \leq C_2 \psi'(r)(\psi(r))^{-\alpha} \quad \text{for } r \to \inf\Omega. \tag{15}$$

Remark 4. With respect to Theorem 3 (e), we note that the asymptotic estimate (15) for $r \to \inf\Omega$ shows the same rate of decay for this history term as the corresponding result for the full Riemann-Liouville integral (Diethelm, 2023a, part 2 of Theorem 5). Our estimate (14) for $r \to \sup\Omega$, however, shows a much faster decay than the corresponding result for the full integral from Diethelm (2023a, part 1 of Theorem 5). To the best of our knowledge, this fact has not been fully exploited in the construction of numerical methods yet.

Proof. The proof of the representation (12) for the history part of the Riemann-Liouville integral is completely analog to the proof of the corresponding relation for the complete Riemann-Liouville integral provided in Diethelm (2023a, Section V.A).

To show the properties of the function μ, we proceed as follows: The differential equation in (a) can be verified by a direct differentiation of the function μ with respect to the first argument; the initial condition is an immediate consequence of the representation (12b).

Items (b) and (c) also follow from (12b).

For part (d), we use the notation $M(t) = \mu(t,h,r)$ for some arbitrary but fixed $r \in \Omega$ and $h > 0$. Then, the differential equation (13a) can be rewritten as $M'(t) = G(t,M(t))$ where $G(t,z) = -\psi(r)z + c_\alpha \psi'(r)(\psi(r))^{-\alpha} \mathrm{e}^{-h\psi(r)} f(t-h)$. Clearly, under the assumption of (d), $G \in C^\ell([a+h,b] \times \mathbb{R})$. and therefore a standard argument from the theory of differential equations (Coddington and Levinson, 1955, Chapter 1, Theorem 1.2) yields the claim.

Finally, for (e) we see that (12b) implies

$$|\mu(t,h,r)| \leq c_\alpha \sup_{\tau \in [a,b]} |f(\tau)| \cdot \psi'(r)(\psi(r))^{-\alpha} J(r)$$

with

$$J(r) = \int_a^{t-h} \mathrm{e}^{-(t-\tau)\psi(r)}\,\mathrm{d}\tau = \frac{1}{\psi(r)} \int_{h\psi(r)}^{(t-a)\psi(r)} \mathrm{e}^{-\rho}\,\mathrm{d}\rho.$$

Thus, when $r \to \inf\Omega$ and hence $\psi(r) \to 0$, then (since the integrand is always less than 1)

$$J(r) \leq \frac{1}{\psi(r)} \int_{h\psi(r)}^{(t-a)\psi(r)} \mathrm{d}\rho = t-a-h,$$

and so (15) follows. For $r \to \sup\Omega$ on the other hand, we find that

$$J(r) \leq \frac{1}{\psi(r)} \int_{h\psi(r)}^{\infty} \mathrm{e}^{-\rho}\,\mathrm{d}\rho = \frac{1}{\psi(r)} \mathrm{e}^{-h\psi(r)}$$

which proves (14). □

The result of Theorem 3 can be extended to the case of general $\alpha > 0$, $\alpha \notin \mathbb{N}$. We shall discuss this extension in a separate paper in the future.

3.1 Kernel Function Approximation by Exponential Sums

Now we derive the exponential sum approximation to the kernel function of the integral represented by (7). Here we utilize the well established trapezoidal rule to discretize the integral. This rule provides an explicit discretization of the integral over the unbounded domain, expressed as a sum of exponentials. Our integrand in (5) exhibits the

crucial property of exponential decay, as outlined in Theorem 2(iv). This property permits the effective utilization of the trapezoidal rule. For a deeper understanding of the exponential sum approximation, one may refer to the work of Beylkin and Monzón (2010) and McLean (2018).

Utilizing the proof presented in Diethelm (2023a, Theorem 1), we can express the value of $I_a^\alpha f(t)$ as

$$I_a^\alpha f(t) = c_\alpha \int_a^t \int_0^\infty \left(\frac{u}{t-\tau}\right)^{(1-\alpha)} \mathrm{e}^{-u} u^{-1} f(\tau)\,\mathrm{d}u\,\mathrm{d}\tau.$$

By making the substitution $u = (t-\tau)\mathrm{e}^r$, we arrive at

$$I_a^\alpha f(t) = c_\alpha \int_a^t \int_{-\infty}^\infty \mathrm{e}^{(1-\alpha)r} \exp[-(t-\tau)\mathrm{e}^r] f(\tau)\,\mathrm{d}r\,\mathrm{d}\tau$$

which can be further written as

$$I_a^\alpha f(t) = c_\alpha \int_a^t K(t-\tau) f(\tau)\,\mathrm{d}\tau,$$

where the kernel $K(t)$ is given by

$$K(t) = \int_{-\infty}^\infty \mathrm{e}^{(1-\alpha)r} \mathrm{e}^{-t\mathrm{e}^r}\,\mathrm{d}r \quad \text{for} \quad t \in [0, b-a]. \tag{16}$$

Now applying the trapezoidal rule to discretize the infinite integral with step size $h > 0$, we obtain

$$\begin{aligned} K(t) &\approx h \sum_{n=-\infty}^\infty \mathrm{e}^{(1-\alpha)nh} \mathrm{e}^{-t\mathrm{e}^{nh}} \\ &= \sum_{n=-\infty}^\infty \tilde{w}_n \mathrm{e}^{-\tilde{\beta}_n t} \end{aligned}$$

where

$$\tilde{w}_n = h\mathrm{e}^{(1-\alpha)nh} \quad \text{and} \quad \tilde{\beta}_n = \mathrm{e}^{nh}. \tag{17}$$

If now t is restricted to a compact interval $[\delta, b-a]$ with $0 < \delta < b-a < \infty$, we can proceed in this approach to obtain a finite exponential sum approximation

$$K(t) \approx \sum_{n=-M}^N \tilde{w}_n \mathrm{e}^{-\tilde{\beta}_n t} \quad \text{for} \quad t \in [\delta, b-a] \tag{18}$$

for the kernel with suitably chosen $M, N \in \mathbb{N}$.

Beylkin and Monzón (2010) discussed an innovative reduction algorithm based on Prony's method, specifically designed for cases with excessively many terms and small exponents in the exponential sum approximation to a kernel function. This suggests that employing Prony's method allows for a substantial reduction in the number of terms in the exponential sum in (18) without compromising accuracy. Consequently, numerical algorithms designed using this exponential sum approximation for approximating the fractional integral demonstrate markedly enhanced efficiency when compared to existing numerical methods.

3.2 Truncation Error Estimation of the Kernel Function

When conducting practical computations, particularly in numerical analysis and real-world applications, finite computational resources necessitate the use of truncated sums. In such cases, we need to approximate the truncation error for the exponential sum defined in (18). In this section, we assess the estimation of the two components of the truncation error:

Theorem 5. If $t \in [0, b-a]$, the exponents and weights are given by (17) and

$$t\mathrm{e}^{Nh} \geq 1-\alpha \geq t\mathrm{e}^{-Mh},$$

then

$$\sum_{n=N+1}^\infty \tilde{w}_n \mathrm{e}^{-\tilde{\beta}_n t} \leq \frac{1}{t^{(1-\alpha)}} \Gamma(1-\alpha, t\mathrm{e}^{Nh})$$

and

$$\sum_{n=-\infty}^{-M-1} \tilde{w}_n \mathrm{e}^{-\tilde{\beta}_n t} \leq \frac{1}{t^{(1-\alpha)}} \Big(\Gamma(1-\alpha) - \Gamma(1-\alpha, t\mathrm{e}^{-Mh})\Big)$$

where $\Gamma(\cdot,\cdot)$ denotes the upper incomplete Gamma function.

Proof. The proof directly follows from McLean (2018, Theorem 2) with appropriate modifications to the conditions and parameters. □

4. EFFICIENT NUMERICAL ALGORITHMS FOR APPROXIMATE EVALUATION OF FRACTIONAL INTEGRAL

In this section, we establish two distinct efficient numerical algorithms for the approximate evaluation of the fractional integral represented in (7). The first approach involves utilizing the exponential sum approximation to the kernel function, as defined in (18). The second approach employs the Gauss-Laguerre quadrature formula.

4.1 Numerical Algorithm for Approximating Fractional Integral using Exponential Sum Approximation of the Kernel Function

For the numerical computation of $I_a^\alpha f(t)$ at the points of the grid $a = t_0 < t_1 < \cdots < t_P = b$, let $f(t_n) \approx F^n$ and define a piecewise constant interpolant $\tilde{F}(t) = F^n$ for $t \in J_n = (t_{n-1}, t_n)$, then

$$I_a^\alpha f(t_n) = c_\alpha \int_a^{t_n} K(t_n - \tau) f(\tau)\,\mathrm{d}\tau \approx \sum_{j=1}^n z_{nj} F^j \tag{19}$$

where

$$z_{nj} = c_\alpha \int_{J_j} K(t_n - \tau)\,\mathrm{d}\tau. \tag{20}$$

The computational complexity for calculating this sum within the range of $1 \leq n \leq P$ scales proportionally to P^2. Such quadratic growth can pose challenges, particularly in scenarios where each F^j represents a large vector rather than a scalar. In particular, storing F^j in active memory for all time levels j may not be feasible.

Here we define an efficient algorithm to avoid these challenges based on the exponential sum approximation to the kernel K such as

$$K(t) \approx \sum_{l=1}^\Lambda w_l \mathrm{e}^{\beta_l t} \quad \text{for} \quad \delta \leq t \leq b-a, \tag{21}$$

provided that the moderate number of terms Λ can achieve adequate accuracy for a choice of $\delta > 0$ that is smaller than the time step $\Delta t_n = t_n - t_{n-1}$ for all n. Certainly, if $\Delta t_n \geq \delta$ then $\delta \leq t_n - \tau \leq b-a$ for $0 \leq \tau \leq t_{n-1}$; therefore

$$\sum_{j=1}^{n-1} z_{nj}F^j = c_\alpha \int_0^{t_{n-1}} K(t_n-\tau)\tilde{F}(\tau)\,\mathrm{d}\tau$$
$$\approx c_\alpha \int_0^{t_{n-1}} \sum_{l=1}^{\Lambda} w_l \mathrm{e}^{\beta_l(t_n-\tau)}\tilde{F}(\tau)\,\mathrm{d}\tau$$
$$= \sum_{l=1}^{\Lambda} \Phi_l^n$$

where

$$\Phi_l^n = c_\alpha w_l \int_0^{t_{n-1}} \mathrm{e}^{\beta_l(t_n-\tau)}\tilde{F}(\tau)\,\mathrm{d}\tau = \sum_{j=1}^{n-1} K_{lnj}F^j$$

with

$$K_{lnj} = c_\alpha w_l \int_{J_j} \mathrm{e}^{\beta_l(t_n-\tau)}\,\mathrm{d}\tau. \tag{22}$$

Hence we express the approximated value of the integral

$$I_a^\alpha \tilde{F}(t_n) \approx z_{nn}F^n + \sum_{l=1}^{\Lambda} \Phi_l^n, \tag{23}$$

and utilizing the recursive relation

$$\Phi_l^1 = 0, \tag{24}$$
$$\Phi_l^n = K_{ln,n-1}F^{n-1} + \mathrm{e}^{\beta l \Delta t_n}\Phi_l^{n-1} \quad (n=2,3,\ldots,P), \tag{25}$$

we obtain an evaluation scheme that, at time t_n, only needs to exploit information about the immediately preceding time t_{n-1} but not about any earlier point in history.

Thus we can efficiently compute $I_a^\alpha \tilde{F}(t_n)$ for $n = 1,2,\ldots,P$ with a satisfactory level of accuracy using a number of operations proportional to $\Lambda \cdot P$. This approach yields substantial computational savings compared to the common $\mathcal{O}(P^2)$ operation count for traditional approaches, particularly when $\Lambda \ll P$. Moreover, we have the flexibility to overwrite Φ_l^{n-1} with Φ_l^n, and F^{n-1} with F^n, effectively reducing the active storage requirement from being proportional to P to being proportional to Λ.

4.2 Numerical Algorithm for Approximating Fractional Integral using Gauss-Laguerre Quadrature Formula

Next, we approximate the integral (7) using the Λ-point Gauss-Laguerre quadrature formula. This quadrature method is an open-type Gaussian quadrature rule designed for evaluating integrals, specifically with the weight function e^{-u}, over the interval $[0,\infty)$. We write

$$I_a^\alpha f(t) = \int_{-\infty}^{\infty} \phi(t,r)\,\mathrm{d}r$$
$$= \int_{-\infty}^{0} \phi(t,r)dr + \int_0^{\infty} \phi(t,r)\,\mathrm{d}r$$
$$= J_1 + J_2.$$

Substituting $r = \frac{-s}{1-\alpha}$ in J_1 and $r = \frac{s}{\alpha}$ in J_2, we obtain

$$I_a^\alpha f(t) = \frac{1}{1-\alpha}\int_0^{\infty} \mathrm{e}^{-s}\mathrm{e}^{s}\phi(t,-s/(1-\alpha))\,\mathrm{d}s + \frac{1}{\alpha}\int_0^{\infty} \mathrm{e}^{-s}\mathrm{e}^{s}\phi(t,s/\alpha)\,\mathrm{d}s.$$

Thus, employing

$$\hat{\phi}(t,s) = \mathrm{e}^{s}\left(\frac{1}{1-\alpha}\phi(t,-s/(1-\alpha)) + \frac{1}{\alpha}\phi(t,s/\alpha)\right),$$

we obtain that

$$I_a^\alpha f(t) = \int_0^{\infty} \mathrm{e}^{-s}\hat{\phi}(t,s)\,\mathrm{d}s \approx Q_\Lambda^{GLa}[\hat{\phi}(t,\cdot)],$$

where the expression

$$Q_\Lambda^{GLa}[\hat{\phi}] = \sum_{l=1}^{\Lambda} w_l^{GLa}\hat{\phi}(x_l^{GLa})$$

represents the Λ-point Gauss-Laguerre quadrature formula with weights w_l^{GLa} and nodes x_l^{GLa}.

To determine the Gauss-Laguerre nodes x_l^{GLa} and the corresponding weights w_l^{GLa}, we need to find the zeros of the Laguerre polynomial L_Λ of order Λ and calculate the weights using a specific formula:

- Solve $L_\Lambda(x_l^{GLa}) = 0$, for $l = 1,2,\ldots,\Lambda$ to obtain the Gauss-Laguerre nodes x_l^{GLa}.
- Use the provided formula to calculate the weights:

$$w_l^{GLa} = \frac{x_l^{GLa}}{[L_{\Lambda+1}(x_l)^2]}.$$

For a more comprehensive understanding of Gauss-Laguerre quadrature formulas, refer to Davis and Rabinowitz (2007).

Theorem 6. Under the assumptions of Theorem 2,

$$\lim_{\Lambda\to\infty} Q_\Lambda^{GLa}[\hat{\phi}(t,\cdot)] = I_a^\alpha f(t)$$

for all $t \in [a,b]$.

Proof. By Theorem 2(iii), the function $\phi(t,\cdot)$ possesses multiple differentiability. Combined with the decay properties demonstrated in Theorem 2(iv), this enables us to apply a standard convergence result pertaining to the Gauss-Laguerre quadrature formula (Davis and Rabinowitz, 2007). Consequently, we derive the desired result.

4.3 Implementation and Computational Details

We are now ready to outline the method we propose for numerically computing $I_a^\alpha f(t_n)$, where $n = 1,2,\cdots,P$. In this algorithm, the symbol $\hat{\phi}_l$ is utilized to represent the approximate value of $\hat{\phi}(x_l^{GLa}, t_n)$ for the current time step, corresponding to the presently evaluated value of n.

For given the initial point $a \in \mathbb{R}$, the order $\alpha \in (0,1)$, the grid points t_n, $n = 1,2,\ldots,P$ and the number of quadrature nodes $\Lambda \in \mathbb{N}$,

(1) For $l = 1,2,\ldots,\Lambda$:
  a. Compute the Gauss-Laguerre nodes x_l^{GLa} and the associated weights w_l^{GLa}.
  b. Define the auxiliary terms $r_l \leftarrow \frac{-x_l^{GLa}}{1-\alpha}$ and $\tilde{r}_l \leftarrow \frac{x_l^{GLa}}{\alpha}$.
  c. Initialize $\phi_l \leftarrow 0$ and $\tilde{\phi}_l \leftarrow 0$ to denote the initial condition for the differential equation (4).
(2) For $n = 1,2,\ldots,P$:
  a. set $h \leftarrow t_n - t_{n-1}$.
  b. For $l = 1,2,\ldots,\Lambda$:

(i) Update the value of ϕ_l by solving the corresponding differential equation (4) using the backward Euler method

$$\phi_l \leftarrow \frac{1}{1+he^{r_l}}[\phi_l + hc_\alpha e^{(1-\alpha)r_l} f(t_n)]. \quad (26)$$

(ii) Similarly, update the value of $\tilde{\phi}_n$ by

$$\tilde{\phi}_l \leftarrow \frac{1}{1+he^{\tilde{r}_l}}[\tilde{\phi}_l + hc_\alpha e^{(1-\alpha)\tilde{r}_l} f(t_n)]. \quad (27)$$

c. Calculate the desired approximate value for $I_a^\alpha f(t_n)$ using the formula

$$I_a^\alpha f(t_n) \approx \sum_{l=1}^{\Lambda} w_l^{GLa} \exp(x_l^{GLa}) \left[\frac{1}{1-\alpha}\phi_l + \frac{1}{\alpha}\tilde{\phi}_l\right].$$

Here we select the backward Euler method, considering the constant factor that multiplies the unknown function $\phi(\cdot, r)$ in (4), making it necessary to use an A-stable method. This choice is the simplest among such methods. Alternatively, one could opt for the trapezoidal method, another A-stable approach, necessitating adjustments to the formulas outlined in (26) and (27) as follows:

$$\phi_l \leftarrow \frac{1}{1+\frac{h}{2}e^{r_l}}\left[\left(1-\frac{h}{2}e^{r_l}\right)\phi_l + \frac{h}{2}c_\alpha e^{(1-\alpha)r_l}[f(t_n)+f(t_{n-1})]\right], \quad (28)$$

and

$$\tilde{\phi}_l \leftarrow \frac{1}{1+\frac{h}{2}e^{\tilde{r}_l}}\left[\left(1-\frac{h}{2}e^{\tilde{r}_l}\right)\phi_l + \frac{h}{2}c_\alpha e^{(1-\alpha)\tilde{r}_l}[f(t_n)+f(t_{n-1})]\right], \quad (29)$$

respectively. Note that the effective evaluation of the formulas (26)–(29) in the given form may lead to overflows in some intermediate results. As shown in Diethelm (2023b), this can be avoided by simple reformulations.

5. CONCLUSION AND FUTURE DIRECTIONS

In this paper, our primary aim has been to craft highly efficient numerical techniques aimed at the approximate assessment of Riemann-Liouville fractional integrals with less computational complexity and memory footprint. For this, we have embarked on an exploration of innovative variations in diffusive representations tailored for fractional integrals. We have approximated the kernel function to the fractional integral by representing it as an exponential sum. This representation can be further optimized by leveraging Prony's method to curtail the number of terms involved. Subsequently, we have harnessed this refined approximation to compute an estimate for the fractional integral. This approach yields a notable reduction in both computational intricacy and memory usage, offering an enticing prospect for practical implementations. In addition to the exponential sum approximation, we have enriched our computational toolkit by developing the Gauss-Laguerre formula as an alternative method for approximating fractional integrals.

This research opens several promising avenues for future investigations and applications such as extending the exploration of diffusive representations to other integral transforms and fractional operators to obtain valuable insights. Investigating more general forms of admissible transformations and their impact on the efficiency of numerical algorithms is an interesting direction. Furthermore, a comprehensive analysis of the error bounds and convergence properties of the numerical algorithms presented in this paper is essential.

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