

# Error Estimates for a Quadrature Rule for Cauchy Principal Value Integrals<sup>1</sup>

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ABSTRACT. The Cauchy principal value integral  $\int_{-1}^1 \frac{f(x)}{x-\lambda} dx$  with Hölder continuous  $f$  and  $\lambda \in (-1, 1)$  is approximated using the method of “subtracting the singularity” and application of the classical trapezoidal formula with  $n$  nodes to the resulting integral. Using Peano kernel methods, we prove that, for such  $f$ , the error is  $O(n^{-1} \ln n)$  uniformly for  $\lambda \in (-1, 1)$ . Explicit upper and lower bounds for the Peano constants are given. Similar results hold if the trapezoidal rule is replaced by the midpoint rule.

## 1. Introduction and statement of the main results

We consider quadrature formulae  $Q$  for the numerical evaluation of the Cauchy principal value integral

$$(1) \quad I[f; \lambda] := \int_{-1}^1 \frac{f(x)}{x-\lambda} dx := \lim_{\varepsilon \rightarrow 0^+} \left( \int_{-1}^{\lambda-\varepsilon} \frac{f(x)}{x-\lambda} dx + \int_{\lambda+\varepsilon}^1 \frac{f(x)}{x-\lambda} dx \right),$$

where  $\lambda \in (-1, 1)$  and  $f \in H_1$ , the class of all functions which fulfill a Hölder condition of order 1. Under these assumptions,  $I[f; \lambda]$  exists.

Our approach is based on the representation (cf. Davis and Rabinowitz [1, p. 184])

$$(2) \quad I[f; \lambda] = \int_{-1}^1 \frac{f(x) - f(\lambda)}{x - \lambda} dx + f(\lambda) \ln \frac{1 - \lambda}{1 + \lambda}$$

and approximation of the proper Riemann integral on the right-hand side of (2) by a classical quadrature formula  $\tilde{Q}$ . Writing  $f_\lambda(x) := \frac{f(x) - f(\lambda)}{x - \lambda}$  for  $x \neq \lambda$  and  $f_\lambda(\lambda) := f'(\lambda)$ , we obtain a Cauchy principal value quadrature formula  $Q$  defined by

$$(3) \quad Q[f; \lambda] := \tilde{Q}[f_\lambda] + f(\lambda) \ln \frac{1 - \lambda}{1 + \lambda}.$$

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We denote the error by  $R[f; \lambda] := I[f; \lambda] - Q[f; \lambda]$ . This method has been considered frequently; see, e.g., [4, 5] and the literature cited therein.

Here, we will choose the classical quadrature formula to be either the mid-point formula  $Q_n^{\text{Mi}}$  or the trapezoidal formula  $Q_{n+1}^{\text{Tr}}$ . The resulting Cauchy principal value quadrature formulae will be denoted by  $Q_{n+1}^{\text{Mi}\dagger}$  and  $Q_{n+2}^{\text{Tr}\dagger}$ , respectively. The additional superscript ‘‘SS’’ indicates the method of ‘‘Subtraction of the Singularity’’.

In particular, we are interested in estimates of the type

$$(4) \quad |R[f; \lambda]| \leq c(\lambda) \|f'\|_\infty$$

with best possible constant  $c(\lambda)$ . The best possible constant  $c(\lambda)$  in inequality (4) is known as the first Peano constant of  $R[\cdot; \lambda]$ . In §2, we will give an exact expression for the Peano constants, which we use to obtain explicit upper and lower bounds for  $c(\lambda)$  differing essentially, for  $n \rightarrow \infty$ , only by a constant factor. The main results are stated in the following theorems and a straightforward corollary.

**THEOREM 1.** *Let  $x_\nu = -1 + 2\frac{\nu}{n}$ . For the formula  $Q_{n+2}^{\text{Tr}\dagger}$ , we have, if  $|\lambda| \in [x_\mu, x_{\mu+1})$  ( $\mu < n$ ),*

$$\frac{1}{2}n^{-1} \ln(\mu(n-\mu)) \leq c(\lambda) \leq \frac{3}{4}n^{-1} \ln\left(\left(\mu - \frac{1}{2}\right)\left(n - \mu - \frac{1}{2}\right)\right) + \frac{35}{2}n^{-1}.$$

**THEOREM 2.** *Let  $x_\nu = -1 + \frac{2\nu-1}{n}$ . For the formula  $Q_{n+1}^{\text{Mi}\dagger}$ , we have, if  $|\lambda| \in [x_\mu, x_{\mu+1})$  ( $\mu < n$ ),*

$$\begin{aligned} \frac{1}{2}n^{-1} \left(\frac{1}{2} - \ln 4 + \ln(\mu(n-\mu))\right) &\leq c(\lambda) \\ &\leq \frac{3}{4}n^{-1} \ln\left(\left(\mu - \frac{3}{2}\right)\left(n - \mu - \frac{1}{2}\right)\right) + 20n^{-1}. \end{aligned}$$

If  $|\lambda| \geq x_n$ , then

$$\frac{1}{2}n^{-1} \left(\frac{1}{4} - \ln 2 + \ln n\right) \leq c(\lambda) \leq \frac{3}{4}n^{-1} \ln n + \frac{61}{6}n^{-1}.$$

We will prove Theorem 1 in §3; Theorem 2 can be shown analogously.

**COROLLARY 3.** *For the formula  $Q_{n+2}^{\text{Tr}\dagger}$ , we have, for every  $\lambda \in (-1, 1)$ ,*

$$\frac{1}{2}n^{-1} \ln(n-1) \leq c(\lambda) \leq \frac{3}{2}n^{-1} \ln n + \left(\frac{35}{2} - \ln 2\right)n^{-1}.$$

For the formula  $Q_{n+1}^{\text{Mit}}$ , we have, for every  $\lambda \in (-1, 1)$ ,

$$\frac{1}{2}n^{-1} \left( \frac{1}{4} - \ln 2 + \ln(n-1) \right) \leq c(\lambda) \leq \frac{3}{2}n^{-1} \ln n + 20n^{-1}.$$

Thus, we obtain, for both these quadrature formulae and for every  $f \in H_1$ , that

$$(5) \quad R[f; \lambda] = O(n^{-1} \ln n)$$

uniformly for  $\lambda \in (-1, 1)$ . This substantially improves a result of Rabinowitz [5, Theorem 4]. Uniform convergence is an important feature when these formulae are used for the numerical solution of integral equations with Cauchy principal value integrals, which arise in fields such as aerodynamics or fluid mechanics [3, 5].

REMARK. In [2], we have shown that there does not exist a sequence of Cauchy principal value quadrature formulae for which  $c(\lambda) = o(n^{-1} \ln n)$  holds uniformly for  $|\lambda| < 1$ . Hence, the error of the rules considered here is of an optimal order.

## 2. Peano's theorem and the Peano kernel

The main tool which we will use to prove upper bounds for the Peano constants is the following theorem, which is simply a generalization of Peano's Theorem for classical quadrature formulae (cf. Davis and Rabinowitz [1, p. 286]):

THEOREM 4 (Peano). *Let  $Q[\cdot; \lambda]$  be a Cauchy principal value quadrature formula given by  $Q[f; \lambda] = \sum_{\nu=1}^N a_\nu f(x_\nu) + bf'(\lambda)$  which is exact for all polynomials of degree zero. Then, we have, for every  $f \in H_1$ ,*

$$(6) \quad R[f; \lambda] = \int_{-1}^1 K_\lambda(x) f'(x) dx,$$

where  $K_\lambda$  is the so-called (first) Peano kernel of  $R[\cdot; \lambda]$  given by

$$\begin{aligned} K_\lambda(x) &= \ln \left| \frac{1-\lambda}{x-\lambda} \right| - \sum_{\nu=1}^N a_\nu (x_\nu - x)_+^0 - b\delta(x-\lambda) \\ &= \ln \left| \frac{1+\lambda}{x-\lambda} \right| + \sum_{\nu=1}^N a_\nu (x - x_\nu)_+^0 - b\delta(x-\lambda). \end{aligned}$$

( $\delta$  denotes Dirac's distribution, and  $(\cdot)_+^0$  is the truncated power function [1, p. 286].)

This theorem can be shown simply by an explicit evaluation of the integral in equation (6) using either of the representations of  $K_\lambda$ .

As in the case of classical quadrature formulae [1, (4.3.11)], we can see that the Peano constant  $c(\lambda)$  is now given by

$$(7) \quad c(\lambda) = \int_{-1}^1 |K_\lambda(x)| dx.$$

Using the notation of Theorem 4, we can immediately deduce

COROLLARY 5. *Assuming that  $-1 \leq x_1 < x_2 < \dots < x_N \leq 1$ , we have:*

1. *If  $\lambda \geq x_{\nu+1}$ , then  $K_\lambda$  is strictly monotone increasing and convex on the interval  $(x_\nu, x_{\nu+1})$ .*
2. *If  $\lambda \leq x_\nu$ , then  $K_\lambda$  is strictly monotone decreasing and convex on  $(x_\nu, x_{\nu+1})$ .*
3. *If  $x_\nu < \lambda < x_{\nu+1}$ , then  $K_\lambda$  is strictly monotone increasing and convex on  $(x_\nu, \lambda)$  and strictly monotone decreasing and convex on  $(\lambda, x_{\nu+1})$ .*
4.  $\lim_{x \rightarrow \lambda} K_\lambda(x) = \infty$ .

### 3. Proof of Theorem 1

In this section, we are going to give a proof for the explicit upper bounds for  $c(\lambda)$  based on the representation from Theorem 4. This theorem is applicable to the quadrature formulae under consideration here because we have that, for all  $\lambda \in (-1, 1)$ ,  $R_{n+2}^{\text{Tr}\dagger}[f; \lambda] = R_{n+1}^{\text{Mi}\dagger}[f; \lambda] = 0$  whenever  $f$  is a polynomial of degree less than or equal to 2 (cf. Monegato [4]).

We will also prove the explicit lower bounds for  $c(\lambda)$ .

In both cases, we will only give proofs for the case  $R = R_{n+2}^{\text{Tr}\dagger}$ ; the case of the midpoint formulae can be treated in a very similar fashion.

Owing to symmetry, we can restrict our analysis to the case  $\lambda \geq 0$ .

**3.1. The lower bound for the Peano constant.** First, we show the validity of the lower bound in Theorem 1. The proof is based on the relation

$$(8) \quad c(\lambda) = \sup\{|R[f; \lambda]|; f \in H_1 \text{ and } \|f'\|_\infty \leq 1\}.$$

Defining

$$g_\lambda(x) := \begin{cases} x - x_\nu & \text{if } x_\nu \leq x < x_\nu + \frac{1}{n} \text{ and } \lambda \leq x_\nu, \\ x_{\nu+1} - x & \text{if } x_\nu + \frac{1}{n} \leq x < x_{\nu+1} \text{ and } \lambda \leq x_\nu, \\ x_\nu - x & \text{if } x_\nu \leq x < x_\nu + \frac{1}{n} \text{ and } \lambda \geq x_{\nu+1}, \\ x - x_{\nu+1} & \text{if } x_\nu + \frac{1}{n} \leq x < x_{\nu+1} \text{ and } \lambda \geq x_{\nu+1}, \\ 0 & \text{else,} \end{cases}$$

we can deduce, for  $\lambda \in [x_\mu, x_{\mu+1})$ ,

$$\begin{aligned} c(\lambda) &\geq \left| R_{n+2}^{\text{Tr}\dagger}[g_\lambda; \lambda] \right| = |I[g_\lambda; \lambda]| = \sum_{\nu=0}^{n-1} \left| \int_{x_\nu}^{x_{\nu+1}} \frac{g_\lambda(x)}{x - \lambda} dx \right| \\ &\geq \frac{1}{2n} \left( \sum_{\nu=1}^{\mu} \frac{1}{\nu} + \sum_{\nu=2}^{n-\mu} \frac{1}{\nu} \right) \geq \frac{1}{2} n^{-1} \ln(\mu(n - \mu)). \end{aligned}$$

**3.2. The upper bound for the Peano constant.** For the proof of the upper bound, we will use the representation of  $c(\lambda)$  via the Peano kernel as given by equation (7). Let  $K_\lambda$  denote the Peano kernel of  $R_{n+2}^{\text{Tr}}[\cdot; \lambda]$ . We will start by giving some elementary properties of  $K_\lambda$ :

$$(9) \quad K_\lambda(-1+) = \frac{1}{n(-1-\lambda)} < 0.$$

If  $x_\nu < \lambda$ , then

$$(10) \quad \begin{aligned} K_\lambda(x_\nu+) &= \ln \left| \frac{1+\lambda}{x_\nu-\lambda} \right| + \frac{2}{n} \sum_{\kappa=1}^{\nu} \frac{1}{x_\kappa-\lambda} + \frac{1}{n(-1-\lambda)} \\ &\leq \ln \left| \frac{1+\lambda}{x_\nu-\lambda} \right| + \frac{1}{n(-1-\lambda)} + \ln \left| \frac{\nu - \frac{n}{2} - \frac{n\lambda}{2}}{1 - \frac{n}{2} - \frac{n\lambda}{2}} \right| \\ &\quad + \frac{1}{2} \left( \frac{1}{1 - \frac{n}{2} - \frac{n\lambda}{2}} + \frac{1}{\nu - \frac{n}{2} - \frac{n\lambda}{2}} \right) \\ &\leq \frac{1}{n} \left( \frac{1}{\lambda - x_1} - \frac{1}{\lambda - x_\nu} \right) + \frac{1}{n(-1-\lambda)} < 0 \end{aligned}$$

and

$$(11) \quad |K_\lambda(x_\nu+)| = -K_\lambda(x_\nu+) \leq \frac{2}{n(\lambda - x_\nu)} + \frac{1}{n(x_\nu - \lambda - \frac{1}{n})}.$$

If  $x_{\nu+1} < \lambda$ , then

$$(12) \quad 0 \leq K_\lambda(x_{\nu+1}-) \leq \frac{1}{n} \left( \frac{1}{1+\lambda - \frac{2}{n}} - \frac{1}{1+\lambda} + \frac{1}{x_\nu - \lambda} - \frac{2}{x_{\nu+1} - \lambda} \right).$$

We note that the upper bound for  $|K_\lambda|$  given by inequality (12) is larger than the bound given by inequality (11). Now, the convexity of  $K_\lambda$  (Corollary 5) in connection with  $\frac{d}{dx}|K_\lambda(x)|_{x=x_\nu+} = \frac{1}{x_\nu-\lambda}$  for  $x_\nu < \lambda$ , yields for  $x_{\nu+1} < \lambda$ :

$$(13) \quad \int_{x_\nu}^{x_{\nu+1}} |K_\lambda(x)| dx \leq \frac{3}{2n^2} \left( \frac{1}{1+\lambda - \frac{2}{n}} - \frac{1}{1+\lambda} + \frac{1}{x_\nu - \lambda} - \frac{2}{x_{\nu+1} - \lambda} \right).$$

Analogous statements hold for the cases  $\lambda < x_\nu$  and  $\lambda < x_{\nu+1}$ , respectively.

If  $\lambda = x_\nu$  or  $\lambda = x_{\nu+1}$ , then, using the inequalities stated above, one can show

$$(14) \quad \int_{x_\nu+}^{x_{\nu+1}-} |K_\lambda(x)| dx \leq \frac{10}{3n},$$

and an additional summand  $\frac{2}{n}$  resulting from the Dirac distribution has to be taken into consideration. (The Dirac distribution only occurs if  $\lambda$  is equal to one of the nodes of  $Q_{n+1}^{\text{Tr}}$ .)

If  $\lambda$  is in an adjacent interval of  $[x_\nu, x_{\nu+1}]$ , then

$$(15) \quad \int_{x_\nu}^{x_{\nu+1}} |K_\lambda(x)| dx \leq \frac{18}{5n}.$$

Finally, if  $\lambda \in (x_\nu, x_{\nu+1})$ ,

$$(16) \quad \int_{x_\nu}^{x_{\nu+1}} |K_\lambda(x)| dx \leq \frac{16}{3n}.$$

Adding up these bounds for all the intervals and estimating carefully, the statement of the theorem follows.

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