

Challenges in using the Kingman Equation for Prediction and Validation

Authors

Prof. Dr.-Ing. Andreas Rügauer (corresponding author)

Production and Industrial Management

Faculty of Economics and Business Administration

University of Applied Science Würzburg-Schweinfurt

Münzstr. 12, 97070 Würzburg, Germany

Email: andreas.ruekgauer@fhws.de

Prof. Dr. Gernot Fabeck

Mathematics and Physics

Faculty of Applied Natural Sciences and Humanities

University of Applied Science Würzburg-Schweinfurt

Ignaz-Schön-Straße 11, 97421 Schweinfurt, Germany

Email: gernot.fabeck@fhws.de

Abstract

This paper discusses the application of the Kingman formula as a means for verification and validation of a production simulation framework. It turns out that this approach generates more questions than answers. This leads to a more in-depth analysis of the Kingman equation properties and the need for refinement of the resulting quantities for the practitioner.

First, the one-stage queueing problem and its properties are analyzed. Then a discrete simulation model is implemented utilizing an industry standard factory simulation framework. A puzzling outcome is the observation that the simulation results match the Kingman approximation reasonably well at lower levels of utilization at which the Kingman equation itself is not very accurate, but they don't match well at high levels of utilization at which the equation is more accurate. Alternative implementation means lead to similar results. It turns out that only one approach chosen utilizing the Lindley equation generates reasonable results, leaving the suggestion of accumulation of rounding errors as a source of the mismatch with all other approaches. The Kingman formula is applicable to general distribution patterns. Four common such patterns are employed and the results are compared to the Kingman equation, proving the point.

The paper finishes with recommendations for simulation verification, it advises against the use of stochastic phenomena models like the Kingman formula, it does however strongly advise for verification means in general, based on the experienced gathered during the course of the studies discussed herein.

Keywords

Simulation, Production, Stochastics and Statistics

Section / Category

Simulation

1 Introduction

Alex Rogo has a simple piece of advice for his foreman on the search for his production plant's bottleneck: "‘Don't you see?’ I ask them. ‘If we've got a Herbie, it's probably going to have a huge pile of work-in-process sitting in front of it.’"¹ (Goldratt and Cox, 2004). What the authors of this novel about the daily challenges of production have their hero derive his advice from is Kingman's formula (Kingman, 1961) (though strangely enough they refrain from ever referring to it explicitly). In a nutshell, the advice given states that a bottleneck will result in an upstream material jam – which is almost a tautology, since a bottleneck is defined as the process with the lowest throughput rate in a connected chain of activities (Hopp and Spearman, 2011).

When dealing with the dynamic effects of production or of logistics in general, the two big challenges are impact and reproducibility. One cannot simply try out things as a real production environment is cost-intensive and thus permanently in need of operation to match customers' demands. For that reason, simulation is often considered a suitable alternative for the study of cause-and-effect relationships. However, simulation requires modeling, and modeling always requires validation (to ensure close enough similarity between model and reality, e.g. using a proper set of parameters) and verification (to show that the mathematical model behaves as expected) (Gutenschwager et al., 2017). This applies in particular when simple, trusted and tried concepts like Little's law (Kummer, Grün and Jammerneegg, 2013) or the above-mentioned Kingman's formula coming in handy as the logical outcome of a consistent mathematical model can be compared against simulated results.

That is how the study at hand had started: The production and logistics simulation system Plant Simulation (Siemens AG) had been introduced – and attempting to verify a simulation task based on the quantities related to Kingman's formula led to more questions than answers. An example from the above-mentioned novel is used to prove the point, see Figure 1: Five dice are lined up in a queue and matches are moved according to the dice scores rolled. Each die is considered a bottleneck and the authors of the novel expect infinite accumulation of inventory within the queue. In Figure 2, the inventory development for the whole queue over time is shown for 10.000 dice rolls. It turns out that the inventory is by far not growing monotonously; in fact, it is a matter of patience to generate (stochastic)

¹ Herbie is the slowest in a group of kids hiking in the novel and thus presents a bottleneck

simulation results where after 10.000 rolls the inventory will be either rather high or rather low.

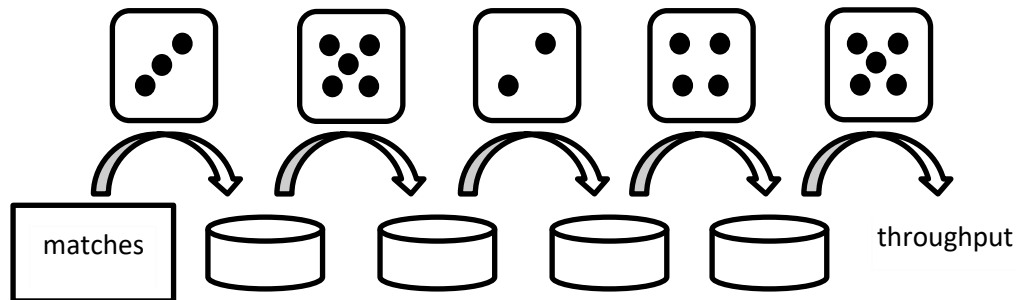


Figure 1: Dice Game as example of a bottleneck queue (Goldratt and Cox, 2004)

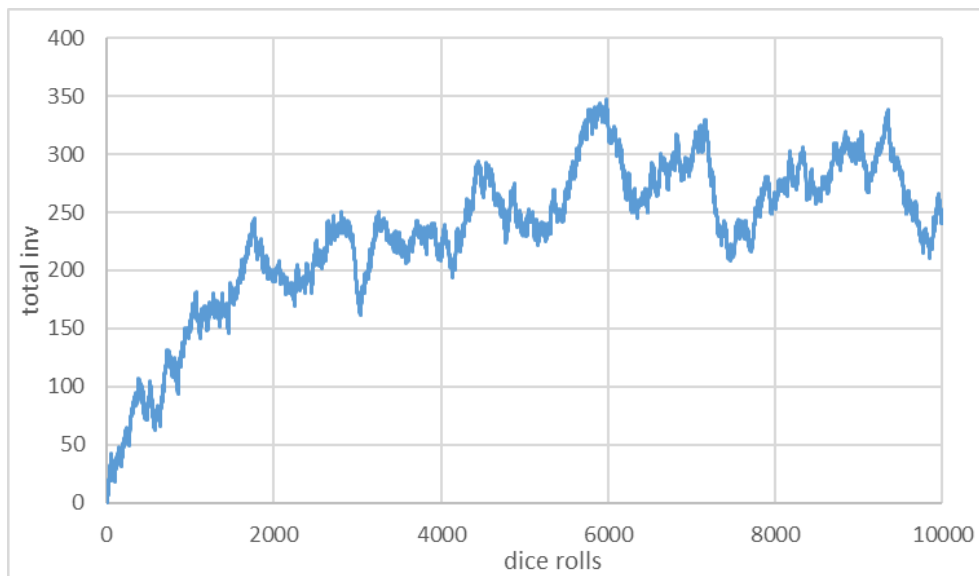


Figure 2: Inventory buildup, 10.000 dice rolls

These experiences led to the conclusion that the above statement is too simplistic and needs further clarification. The aim of this article is to give such clarification about the quantities related to Kingman's equation in order to make this almost 60 years old model even more applicable for the practitioner.

2 Nature of Kingman's Formula

2.1 Standard Model for Queueing Systems

The above example as shown in Figure 1 is built upon a chain of activities and thus inherently involved. In order to isolate relevant aspects, one should focus on one single incident as shown in Figure 3: There is only one die with an associated input buffer in front of it, which is fed by an arrival stream and is emptied by the die acting as a service. In queueing theory, this translates into the representation as shown in Figure 4: The arrival rate R_a is

defined by the mean interarrival time $E(T_a) = 1 / R_a$ between two arrivals, the mean service time is $E(T_s)$. Both for the interarrival time T_a and the service time T_s , statistical variations are assumed with the standard deviations σ_a and σ_s , respectively, here described by the variation coefficients $c_{a/s} := \sigma_{a/s} / E(T_{a/s})$. The utilization of the service is defined as $\rho := E(T_s) / E(T_a)$. As a result from the stochastic process dynamics, a random waiting time T_w might or might

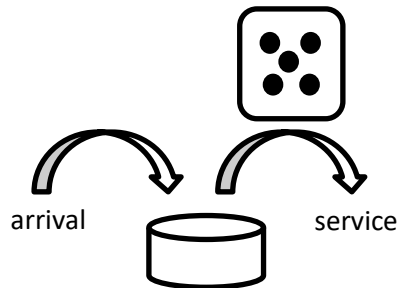


Figure 3: Single Process in a Queue

not build up within the input buffer. This model is widely used in literature (Baum, 2013; Medhi, 2003) to analyze the corresponding system behavior. For the remainder of this article,

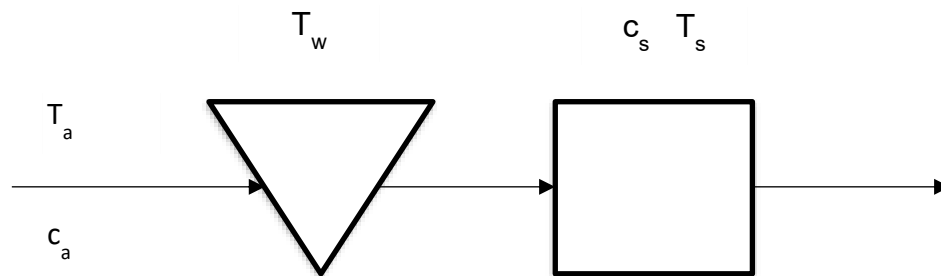


Figure 4: Mathematical process model for single process in a queue.

the same model is being used.

2.2 Mathematical Description and Interpretation

The mathematical properties of such a system have been described by (Kingman, 1961). The often referred to as “Kingman equation” for the mean waiting time reads

$$E(T_w) \approx E(T_s) \frac{\rho}{1-\rho} \frac{c_a^2 + c_s^2}{2}. \quad (1)$$

It is often referred to as VUT-formula (Hopp and Spearman, 2011) because the three defining terms can be attributed to (service-) time, utilization, and variability. It is also called the central limit theorem for G/G/1 systems (Medhi, 2003). Here, Kendall notation is being used (Baum, 2013). For the logistics practitioner, the inventory level I_w is more meaningful

than the waiting time. It can be derived from (1) utilizing Little's law translating time into inventory by way of the arrival rate R_a :

$$E(I_w) = R_a \times E(T_w). \quad (2)$$

It is crucial to select the valid time base. Counter to intuition, not $E(T_s)$ but $E(T_a)$ must be used, because $E(T_s) \leq E(T_a)$ and thus R_a limits the output rate of the overall system:

$$E(I_w) \approx \frac{\rho^2}{1-\rho} \frac{c_a^2 + c_s^2}{2}. \quad (3)$$

Both equations allow for a few important insights in addition to what is often described in literature:

- The mean inventory level is completely independent of any times or process rates and a result of variations and utilization only for any $E(T_s)$ and $E(T_a)$ for as long as the utilization is maintained.
- The Kingman formula is applicable to general stochastic variation patterns.

In addition, careful study of the literature (Medhi, 2003) reveals that the following assumptions of the Kingman equations must be taken into consideration also:

- It is exact for "heavy traffic", thus for $\rho \rightarrow 1$, in any other case it is only an approximation.
- Similar to Little's law (2) it assumes stationary conditions and is not applicable to transient phases.
- Both waiting time T_w and inventory level I_w are random variables, which are exponentially distributed for the case of heavy traffic in stationary conditions only (Kingman, 1961). What the equations express are expected values $E(\cdot)$ of random variables, not face value representations.

This last property indicates why Rogo's above advice is either too simplistic or must be read much more carefully (with reference to the use of the term "probably" in particular). On average, the bottleneck will build up a material jam upstream, but as shown in Figure 2 it is a matter of randomness what the actual current inventory level will be, and by no means will it grow monotonously.

The literature also introduces upper bounds for the expected value of the waiting time:

$$E(T_w) \leq \frac{1}{R_a} \frac{\rho}{1-\rho} \frac{\frac{c_a^2}{\rho^2} + c_s^2}{2}, \quad (4)$$

and

$$E(T_w) \leq \frac{1}{R_a} \frac{\rho}{1-\rho} \frac{\left[(2-\rho) \frac{c_a^2}{\rho} \right] + c_s^2}{2}, \quad (5)$$

respectively, the latter being a refined version (Daley, 1977) of the first (Kingman, 1962).

2.3 Numerical Analysis and Expected Values²

This paper discusses the application of the Kingman equation and the challenges associated with it. An expected value is a theoretical quantity and cannot be measured in experiments directly. Instead, time series and based on these, maximum values and averages can be recorded (Gutenschwager et al., 2017) as empirical data. Here, the expected values from theory are compared with averages from experiments. Thereby, the question of proper calculation of averages arises. There seems to be consensus in the simulation literature that a time-weighted approach is to be employed (Elizandro and Taha, 2012). During simulation studies, inventory levels must then be integrated with fixed step size for proper results, whereas waiting times can simply be accumulated on occurrence. If then subsequently average inventory levels are derived from the sum of waiting times, the results are automatically time-weighted.

3 Modeling and First Results

3.1 Discrete Event Simulation

3.1.1 Plant Simulation Model and Parameters

The original goal pursued was verification and validation of Plant Simulation models and therefore, the according implementation is being discussed first. The basic model is displayed in Figure 5. Arrival is implemented as a source block providing parts in a pre-

² Thanks to an undisclosed, anonymous reviewer of our working paper and the constructive feedback on this discussion!

defined fashion as constructor. These parts flow into the waiting line from which they are eliminated by a service block with pre-defined processing time. Finally, all parts are eliminated by a departure block acting as deconstructor.

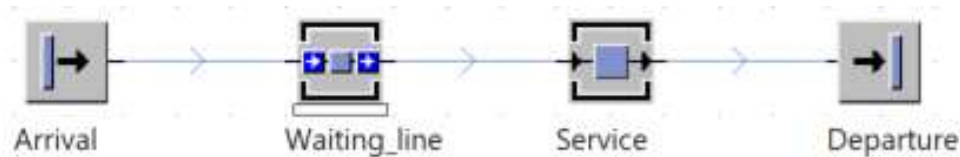


Figure 5: Basic Plant Simulation Model

For the remainder of this article, the following simplifying model assumptions have been made: The mean interarrival time is $E(T_a) = 10\text{min}$, and the coefficient of variation is $c_a = c_s = c$, to be defined further down. From these specified values, the dependent parameters can be calculated. Plant Simulation allows for the use of various different distribution functions for the specification of processing and waiting times. Initially, only the uniform distribution is being employed, though it is not a typical distribution pattern found in logistics.

Relevant output information is the average inventory level in the waiting line. In Plant Simulation, both state change weighted and time weighted analysis can be implemented by way of extending the basic blocks by appropriate algorithmic code and/or time triggered functions.

3.1.2 First Results

A first simulation result using this model is shown in Figure 6 for $\rho = 0,9$ and $c = 3^{-\frac{1}{2}}$. According to equation (3), the approximation of the expected value for I_W is $E(I_W) \approx 2,7$, with upper bounds on the expected value given by 3,334 and 3,512 according to equations (4) and (5). A first observation is easy – the inventory level fluctuates heavily as expected, and the average inventory level of 2,43 is well below the upper bounds, it is also below the expected value though. The relative error of the deviation of average from expected value is roughly 11%. Two possible explanations come to mind: either the simulation has not yet reached stationary level or the traffic is not “heavy” enough yet.

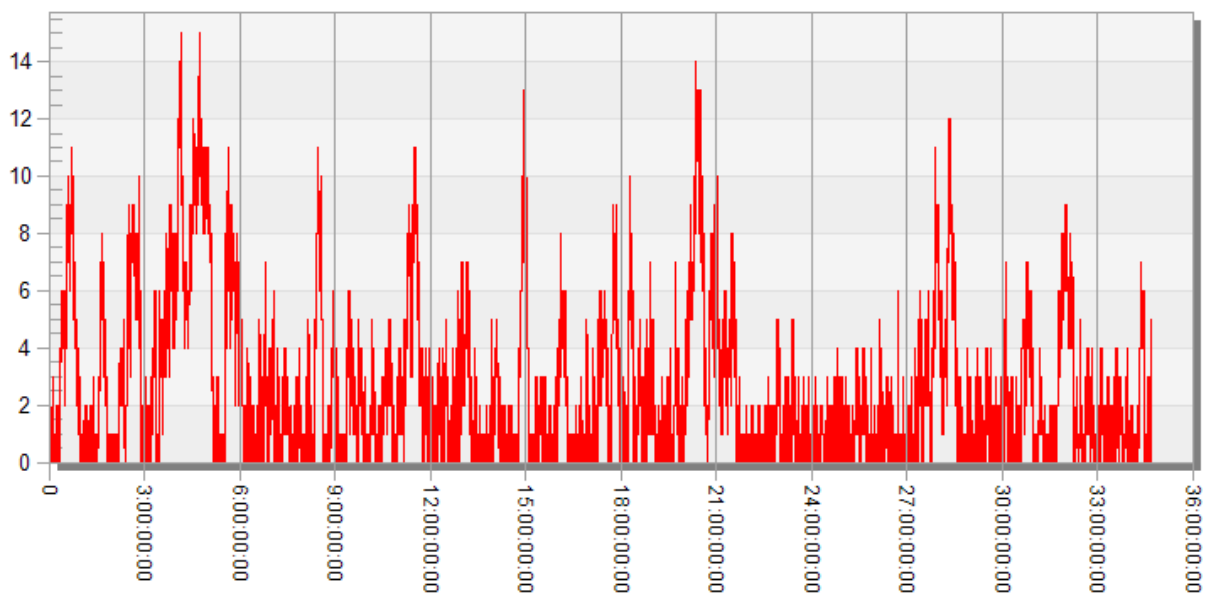


Figure 6: Simulation output inventory level over time with uniform distribution, $\rho = 0.9$ and $c = 3^{-\frac{1}{2}}$, duration = 5.000min

3.1.3 Variation of Duration

Therefore, a parameter variation is conducted with the duration ranging from 10.000min to 5 Mio. min. The results are shown in Figure 7. Very clearly, the transient phase is prolonged, even after a duration of 1 Mio. min the output variable, which is the overall mean of the fluctuating inventory level, still shows some measurable fluctuation, beyond a duration of approximately 200.000min this fluctuation becomes minor though. Nonetheless, even after a duration of 5 Mio. min the targets are not met, the mean inventory level with 2,617 being too low, the relative error is now down to 3% though.

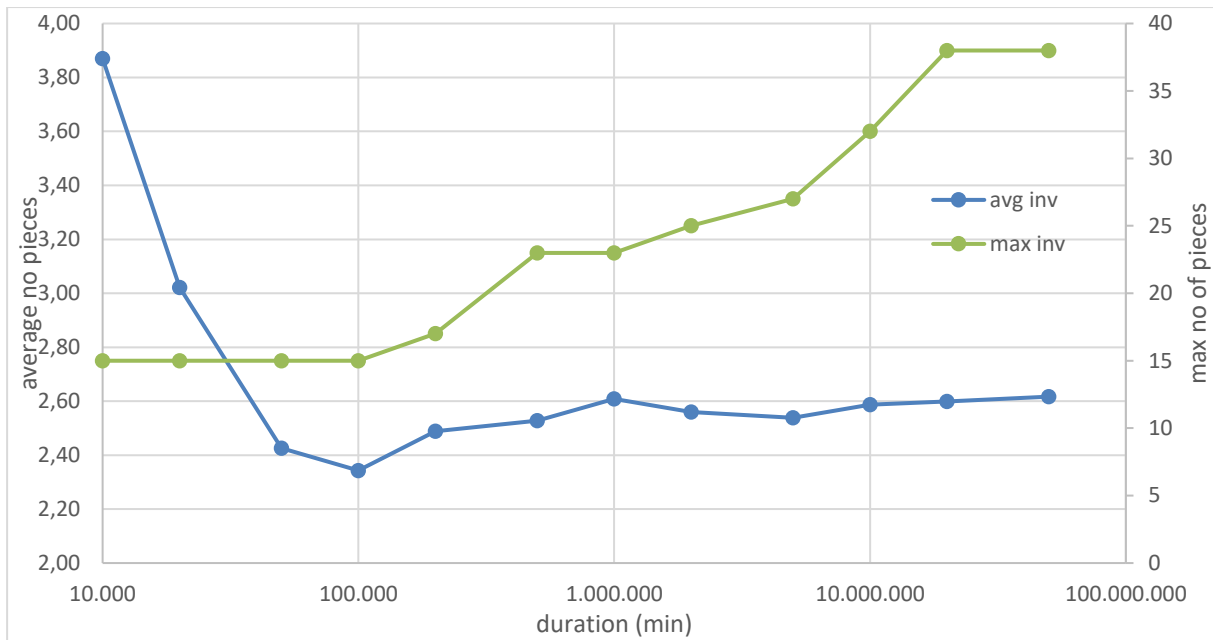


Figure 7: Inventory development over simulation duration

3.1.4 Variation of Utilization

Next the utilization ρ is being varied between 0,5 and 0,999 at a duration of 200.000min. The results are shown in Figure 8. The simulation results approximate the expected values reasonably well at low to medium utilization levels. However, even though the Kingman equation is accurate for high utilizations, the simulation results are much worse at a very high utilization level of 0,999. This is against all logic and reason.

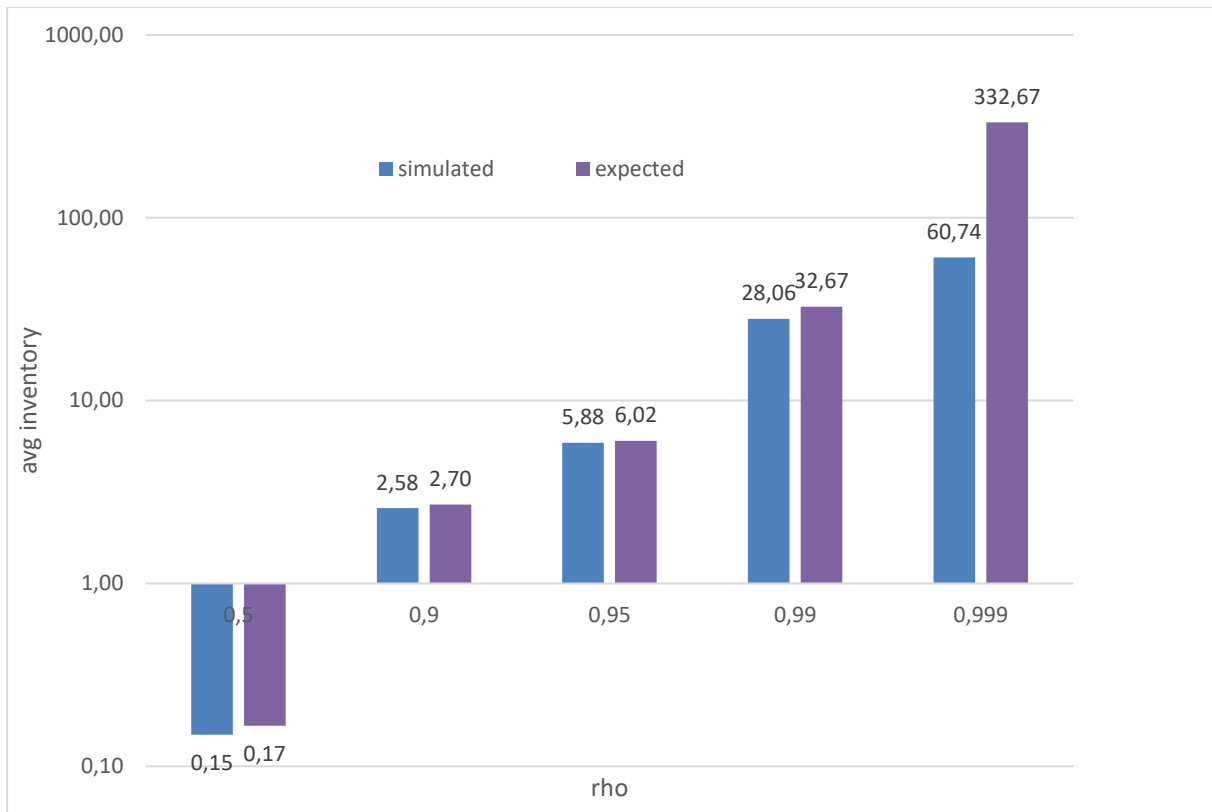


Figure 8: Inventory Development over Utilization with Plant Simulation

3.1.5 Alternative Simulation using Excel

Given the above results, other means of implementing the problem at hand for further analysis have been considered, though this is counter to the original intent. The same problem as shown in Figure 5 can with some effort been implemented using Excel. A random number represents the uniform distribution, the different states have to be tracked and depending on the state at hand inventory has be adjusted correspondingly. The averages can be calculated just similar as before.

The simulation results are shown in Figure 9 for 20.000 state events, which translates into approximately 100.000min. For the simulation study, each step was repeated 20 times for better averaging. The results are just equally wrong and very similar to the above results with plant simulation for low and high levels of utilization.

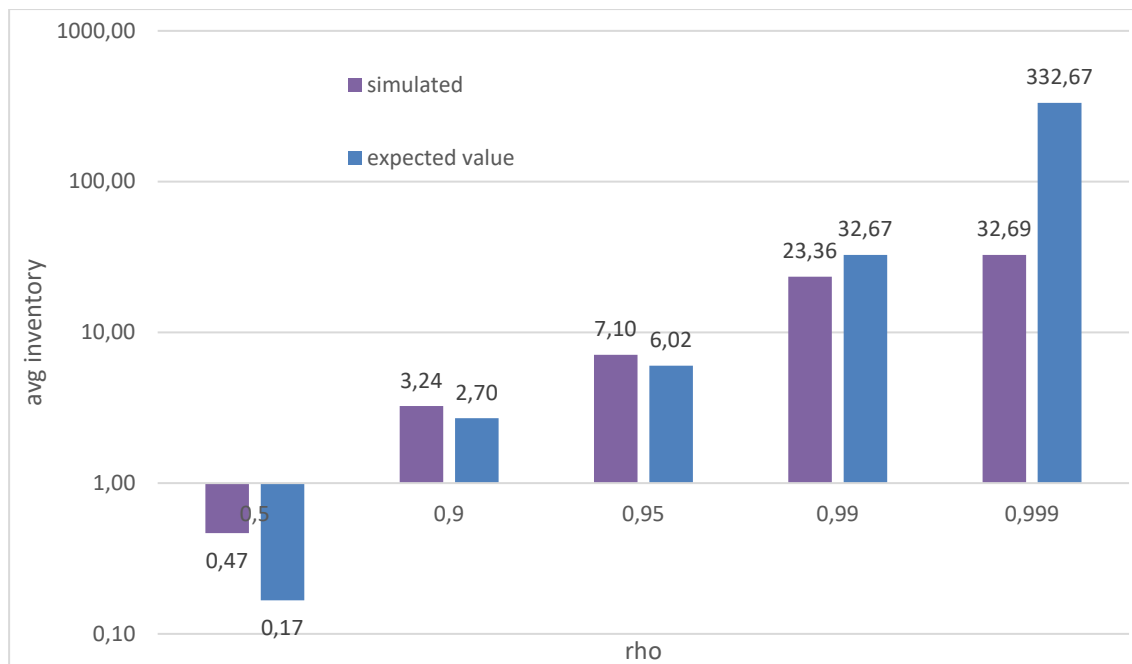


Figure 9: Inventory Development over Utilization with Excel

3.2 Queueing Simulation

An alternative approach for the study of queueing systems is described by the Lindley equation (Lindley, 1952):

$$T_{w,n+1} = \max(0, T_{w,n} + T_{s,n} - T_{a,n}). \quad (6)$$

The Lindley equation calculates waiting times. When the average inventory level is derived from the sum of all waiting times, it automatically represents a time-weighted solution, although technically the Lindley equation itself distinguishes between states only.

3.2.1 Implementation in Excel

The results from a simulation study again at different utilization levels with 20.000 steps and 20 repeats just as above are shown in Figure 10. Again, at heavy traffic the results show substantial deviation from the expected value. Though at low utilization the results seem to fit the theoretical model much better one has to keep in mind that here, the Kingman equation is only an approximation and becomes accurate for heavy traffic only.

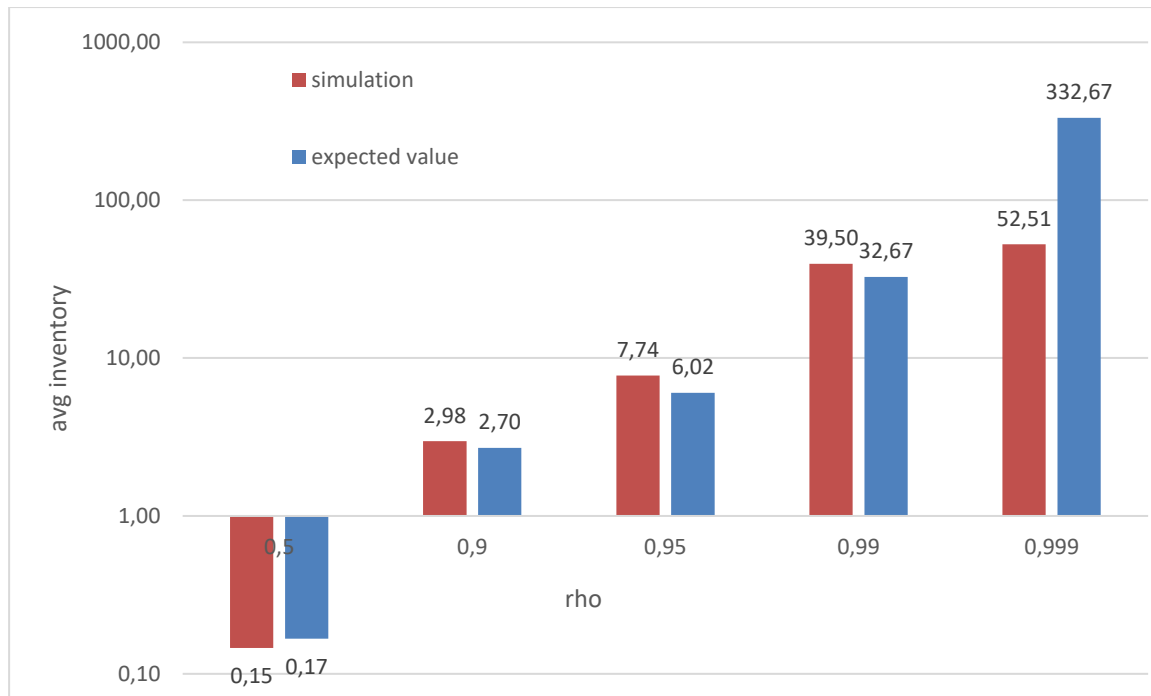


Figure 10: Inventory Development over Utilization with Excel using the Lindley Equation

3.2.2 Implementation in R and Conclusions

A final comparison is analyzed again based on the Lindley equation (6) but now with the statistical programming language R (Hellbrück, 2016). The results for a simulation study with 10 Mio. simulated waiting times are shown in Figure 11 for another study of utilization variation. Now at all levels of utilization the corresponding inventory levels calculated on the basis of the simulated waiting times meet the expected values from the modified Kingman equation (3) quite well!

Now, it could be argued from the results that this final study obtained its accuracy from the large number of repeats. This is however contrary to the findings from section 3.1.3 that showed that beyond a certain number of iterations the successive impact on the output variables is rather minor. In addition, another test in Excel based on the Lindley equation, now implemented within a Visual Basic macro utilizing high accuracy number representations where possible, instead of a worksheet implementation (with no control over the number representation) led to equally incorrect results as shown in sections 3.1.5 and 3.2.1 respectively.

Therefore, another suggestion is made from these observations: Both the implementation in Plant Simulation as well as the one based on Excel suffer from rounding errors at these high

number of repeats, in particular for heavy traffic situations. The statistical analysis language R seems to suffer to a much lesser extent from this effect.

This leaves one with a bitter preliminary result:

- For small utilization, the simulated results deviate because the Kingman equation is exact only for high levels of utilization and thus only a rough approximation at low utilization levels.
- For heavy traffic, the experimental results deviate considerably in some cases due to rounding errors.



Figure 11: Inventory Development over Utilization with R using the Lindley Equation

4 Study of General Distribution Patterns

4.1 Setup

4.1.1 Distribution Functions

So far, only the uniform distribution has been employed. The Kingman equation is applicable to general distribution patterns though and the uniform distribution is not well-suited to logistics problems. Often, logistical problem show Poisson-distributed occurrence pattern, because of which an exponential distribution could be deemed appropriate (Hopp and Spearman, 2011). As it is one-parametric it is however not considered here. Rather, the

following list of distribution patterns for T_s and T_a is being employed in addition to the uniform distribution as employed so far: normal distribution, logarithmic normal distribution, and Erlang distribution. Most distributions require certain balances for the defining parameters to be met, which had to be observed:

- The normal distribution allows for negative numbers, this must be prevented by way of a lower bound at 0. Then, the average will not meet the expected value; therefore, also an upper bound has to be introduced at 2 times the expected value. Then it is however technically far from a normal distribution and is for that reason referred to herein as “normal limited” distribution. It is important to note that though the average value can be conserved by cutting at lower and upper bound simultaneously, the standard deviation cannot: For example, a normal distribution with mean $\mu = 10$ and coefficient of variation $c = 0,4$, when limited to numbers between 0 and 20 results $c = 0,383$, a higher variation of $c = 0,8$ even drops down to $c = 0,519$. The total mass is conserved though, so technically, this “normal limited” distribution still represents a qualified stochastic pattern.
- The Erlang distribution is a collection of k independent exponentially distributed random variables. The variation coefficient c must be chosen such that the parameter $k = 1/c^2$ is always integer. Here, a range of $k = 1 \dots 20$ has been chosen arbitrarily, resulting in $c = 1,0 \dots 0,0025$. For $k = 1$, the Erlang distribution represents the exponential distribution.
- The uniform distribution must also not be negative, therefore c has to be limited to the range $c = 0,1 \dots 3^{-1/2}$ in 20 increments.

For the normal limited and logarithmic normal distributions, a range for c had been chosen arbitrarily as $c = 0,1 \dots 2,0$. Finally, given the above results, the range for the utilization was limited to $\rho = 0,9 \dots 0,99$ which marked the more accurate range of results in section 3.1.4.

4.1.2 Simulation and Experiment Parameters

Due to the results in section 3.1.3, the duration was set to 1,0 Mio min. The quite reduced range of usable utilizations, limited at the lower bound by the reduced applicability of the Kingman equation, and at the upper bound by the rounding errors observed, limits the use of the model quite a bit, alongside the applicability of the simulation approach for queueing problems in general!

At a selected scope of 20 iterations for either utilization ρ and variability c a total of 400 simulations per distribution is conducted. For better approximation, each simulation is

repeated three times with different random numbers, which results in a total of 6.400 simulation runs. Such a complex parameter study can be managed very easily with Plant Simulation experiment manager, and it would be almost impossible for practical reasons to conduct such study for instance with Excel. Each of the simulations consists of 100.000 state changes for either T_s and T_a on average. Overall, this complete parameter study then consists of 1,24 Bln. simulation steps!

4.2 Results

4.2.1 Qualitative Review

The results from the above parameter study are displayed in Figure 12 through Figure 15. Due to their special restrictions, the parameter ranges for the simulation runs of all four distribution forms are different and the graphical results cannot be compared.

What is to be expected and not thoroughly met though is smoothness of the surfaces. In particular, the results from the Erlang and normal limited distribution simulations show some strange irregularities. Also, given the G/G/1 nature of the Kingman equation, for identical combinations of utilization and variation, at least similar average inventory levels should be expected. This is, with some error margin, confirmed by the simulation results. For instance, all four parameter studies contained a run at $c \approx 0,58$ and $\varrho = 0,99$. The mean inventory levels obtained reach from roughly 29,29 for the Normal limited distribution to roughly 45,16, with the other two distribution forms resulting in roughly 32. A strange behavior is shown in Figure 14 for the normal limited distribution. It should be noted however that due to the above discussed boundary limitation, the resulting scale in c is not linear.

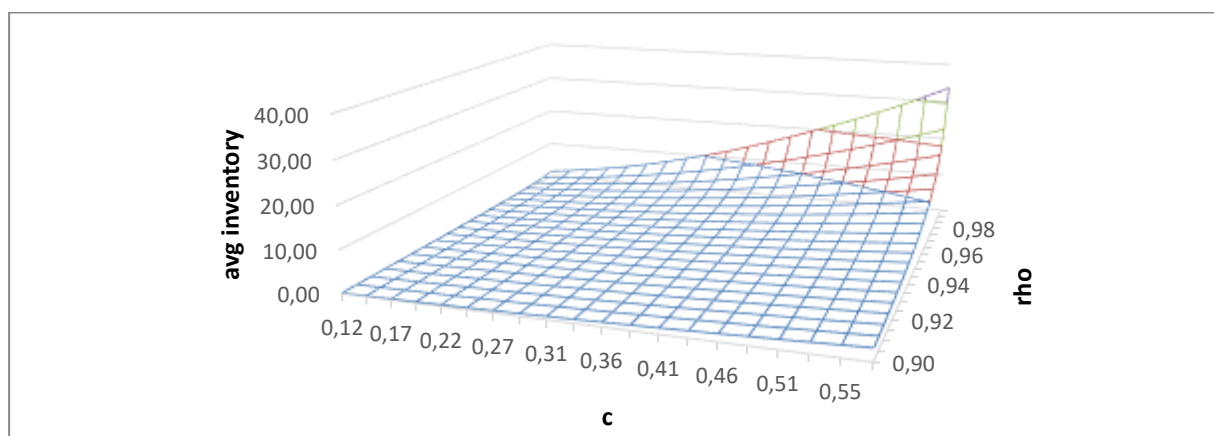


Figure 12: Uniform Distribution Parameter Study Results

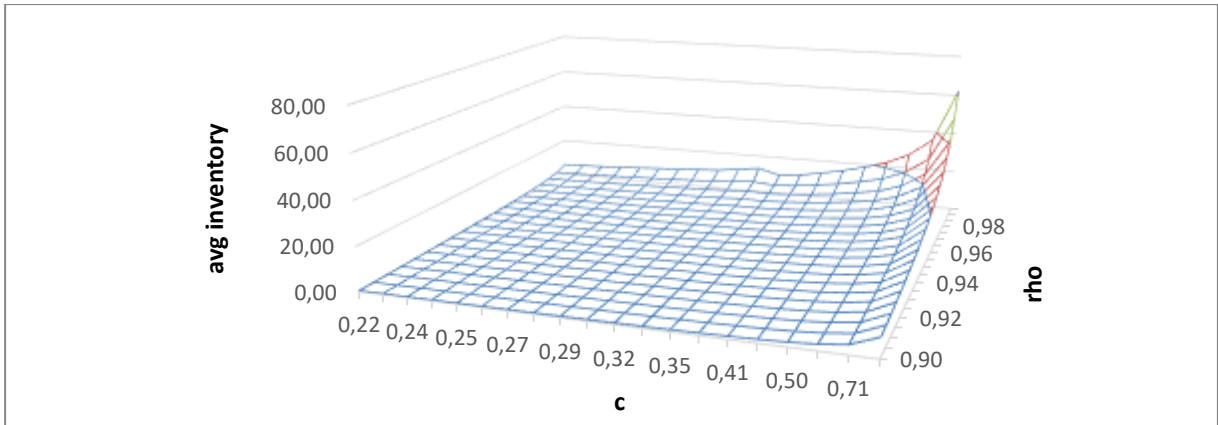


Figure 13: Erlang Distribution Parameter Study Results

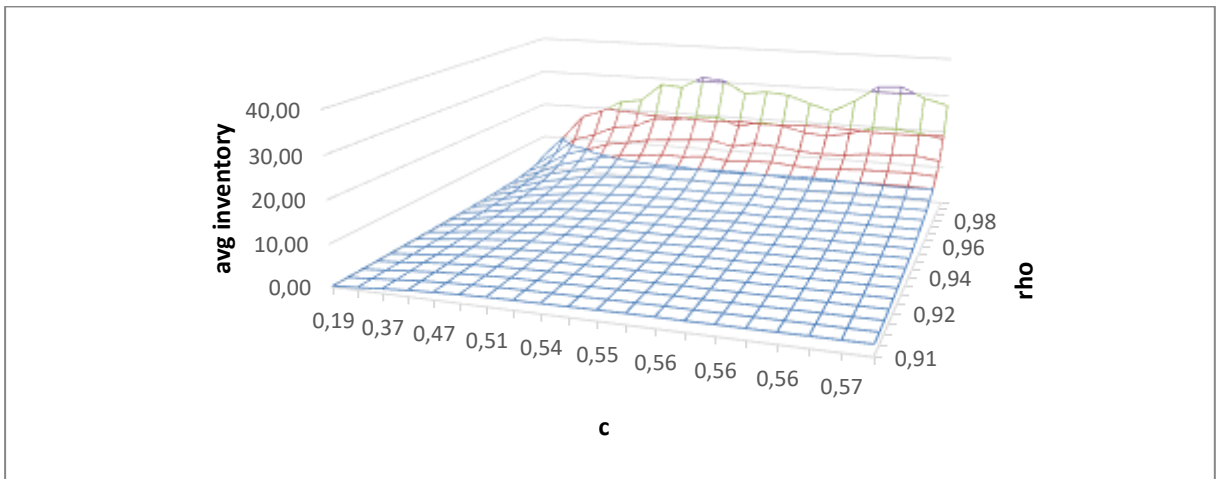


Figure 14: Normal Limited Distribution Parameter Study Results

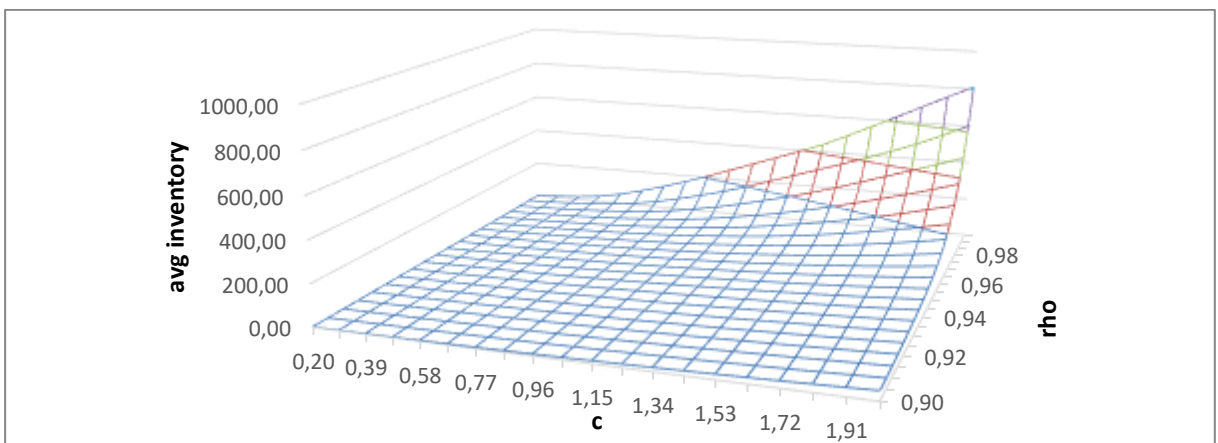


Figure 15: Logarithmic Normal Distribution Parameter Study Results

4.2.2 Quantitative Review

More important than the qualitative aspects of the results above are quantitative measures, in particular, the accuracy of the outcomes, relative to the expected values. Given the sheer amount of data, a fair comparison of simulation averages to expected values is only possible by way of a norm, which aggregates the deviation between the surfaces into one single number. Similar to typical distance norms as for instance RMSE or MAE (Gron, 2018) the following measure G is employed here to compare target and approximation matrices \underline{x} and \underline{y} , respectively:

$$G := \sqrt{\frac{1}{n \cdot m} \sum_{i=1}^n \sum_{j=1}^m \left(\frac{y_{ij} - x_{ij}}{x_{ij}} \right)^2}, \quad (7)$$

which can be interpreted as a relative normalized difference on a per element basis and thus be insensitive to different parameter magnitudes. Therefore, the results for all four distribution forms should be comparable. Table 1 shows the results. With reference to the findings in sections 3.1.3, and 3.1.4, the parameter selections in sections 4.1.1, and 4.1.2, respectively have been made very carefully to assure proper results. The deviation between expected value and simulated average spans from roughly 6,2% for the application of the Normal limited distribution to roughly 15% for the Logarithmic Normal distribution. Within the scope of accuracy obtained so far in this study this confirms the G/G/1 quality of the Kingman equation, it is roughly equally applicable to all distribution functions tested.

Distribution Function Employed	G
Uniform	13,0%
Normal Limited	6,2%
Logarithmic Normal	15,0%
Erlang	11,7%

Table 1 : Quantitative Comparison of the Simulation Results

5 Conclusion

The work presented herein had been started initially with the aim of verifying and validating simple Plant Simulation models using the Kingman equation. This approach led to more questions than answers. Therefore, first the nature of the Kingman equation has been described in detail. It provides an approximation for the expected value for the waiting time in

a queue, and becomes accurate only for heavy traffic. It's applicability for lower utilization levels is questionable.

Then a simple queueing model has been implemented in Plant Simulation by means of a discrete event simulation, and simulation results of inventory averages have been compared to expected values, employing a simple uniform distribution for all random variables. It turns out that the resulting approximation is overall not very good, and in particular, for very high levels of utilization, for which the Kingman equation would be precise, it proved outright incorrect. Comparison to other means of implementation using Excel and R, as well as an additional modeling approach using the Lindley equation indicates that the deviations from the expected outcomes might be due to an accumulation of rounding errors inside of the simulator³.

Regardless of the questionable performance of the simulation approach at this point, an in-depth study of the impact of different distribution functions was conducted within the range of parameters that could be considered reasonably applicable. The outcomes are questionable at best and pose further doubt about the applicability of the chosen simulation approach.

Overall, this leaves the authors with a few answers:

The Kingman equation is not a good choice for the validation of the given simulation framework. At low levels of utilization it provides only a rough approximation and at high levels of utilization it seems to pose a challenge in terms of accuracy.

Based on the study presented, it should be stressed that validation based on models with known behavior is quite important – what reads rather easy is the result of countless iterations which became necessary as the results would not add up to the expected values. Despite the fact that some deviations could not be explained this proved that model-based validation is important as the complexity of the models becomes quickly so complicated that one cannot challenge them without clear cut reference.

As a general recommendation, stochastic models that allow for stationary approximation only, and applicability only for heavy traffic cases, are not a good choice for validation. Rather, on a block level, very simple arithmetic operations should be verified in isolated

³ These observations have been reported to the supplier of the simulator more than six months ago with no outcome at all at the time of this writing.

fashion only, so that every single operation can be validated individually and regardless of long term accumulation effects like the Kingman equation.

Plant Simulation is by all means a quite powerful simulation framework and it certainly does answer a lot of practical questions. It is however maybe not the ideal choice when rather simple models with deep stochastic complexity are to be analyzed in depth. It should be pointed out that any such study requires utmost attention to detail and careful clarification of all parameters involved because of the high number of iterations involved which over time seems to systematically generate incorrect answers if not set up properly.

So, what is Alex Rogo's takeaway from all this? Maybe he (or his authors) should revise his advice to "If we've got a Herbie, it's probably going to have a huger pile of work-in-process sitting in front of it – on average."

References

Baum D. Grundlagen der Warteschlangentheorie. Springer Berlin Heidelberg: Berlin, Heidelberg; 2013.

Daley DJ. Inequalities for moments of tails of random variables, with a queueing application. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 1977; 139-143.

Elizandro D, Taha H. Performance Evaluation of Industrial Systems. Discrete Event Simulation in Using Excel/VBA, Second Edition. CRC Press: Hoboken; 2012.

Goldratt EM, Cox J. The goal. A process of ongoing improvement. North River Press: Great Barrington MA; 2004.

Gron A. Hands-on Machine Learning with Scikit-Learn, Keras, and Tensorflow. Concepts, tools, and techniques to build intelligent systems. Preprint. O'Reilly Media: Sebastopol; 2018.

Gutenschwager K, Rabe M, Spieckermann S, Wenzel S. Simulation in Produktion und Logistik. Grundlagen und Anwendungen. Springer Vieweg: Berlin; 2017.

Hellbrück R. Angewandte Statistik mit R. Eine Einführung für Ökonomen und Sozialwissenschaftler. Springer Fachmedien Wiesbaden GmbH; Springer Gabler: Wiesbaden; 2016.

Hopp WJ, Spearman ML. Factory physics. Waveland: Long Grove (Illinois); 2011.

Kingman JFC. The single server queue in heavy traffic. Mathematical Proceedings of the Cambridge Philosophical Society 1961;57; 902-904.

Kingman JFC. Inequalities for the queue GI/G/1. Biometrika 1962; 315-324.

Kummer S, Grün O, Jammerneegg W (Eds). Grundzüge der Beschaffung, Produktion und Logistik. Pearson: München, Harlow, Amsterdam, Madrid, Boston, San Francisco, Don Mills, Mexico City, Sydney; 2013.

Lindley DV. The theory of queues with a single server. Mathematical Proceedings of the Cambridge Philosophical Society 1952;48; 277-289.

Medhi J. Stochastic models in queueing theory. Acad. Press: Amsterdam; 2003.

Siemens AG. Simulation mit Plant Simulation; https://www.plant-simulation.de/?gclid=CjwKCAjwq-TmBRBdEiwAaO1en_QJrxTxCQhDLucMCfmgmc9MuO-394I0-kuNLxltv95_w2qxUAH7AhoCxMgQAvD_BwE (13.05.2019).