

SOME MOMENTUM POLYTOPES FOR MULTIPLICITY FREE QUASI-HAMILTONIAN
MANIFOLDS

EINIGE IMPULSPOLYTOPE FÜR MULTIPLIZITÄTENFREIE QUASI-HAMILTONISCHE
MANNIGFALTIGKEITEN

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ABSTRACT. We determine some particularly interesting momentum polytopes for multiplicity free quasi-Hamiltonian manifolds by using the methods developed in [Kno16]. This leads to lots of new examples of multiplicity free quasi-Hamiltonian manifolds or equivalently, Hamiltonian loop group actions.

Precisely, we determine all convex quasi-Hamiltonian G -manifolds for G simple and simply connected where the momentum polytope is of rank one for compact Hamiltonian and quasi-Hamiltonian manifolds and we classify all Hamiltonian and q-Hamiltonian manifolds with a surjective moment map.

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Zusammenfassung

Spätestens seit Hamiltons Formulierung der klassischen Mechanik interessiert sich die Mathematik für die nach ihm benannten Hamiltonischen Mannigfaltigkeiten, die in der Mechanik Hamiltons als sogenannte Phasenräume auftauchen. Grob gesprochen handelt es sich bei einer Hamiltonischen K -Mannigfaltigkeit für eine einfach zusammenhängende kompakte Lie-Gruppe K um eine symplektische K -Mannigfaltigkeit mit einer exakten symplektischen Form und einer Abbildung von der Mannigfaltigkeit M in das Duale der Lie-Algebra von K , die sogenannte Impulsabbildung, die den Impuls und Drehimpuls der Mechanik verallgemeinert.

Es ist möglich, symplektische Mannigfaltigkeiten zu reduzieren und dadurch zu Unterstrukturen überzugehen, die wieder diese Struktur tragen. Die in einem gewissen Sinn einfachsten Hamiltonischen Mannigfaltigkeiten sind die, bei denen diese Reduktionen nur Punkte sind, man bezeichnet sie als multiplizitätenfrei. Für den Rest dieser Inhaltsangabe sollen alle Hamiltonischen oder quasi-Hamiltonischen Mannigfaltigkeiten multiplizitätenfrei sein.

Eine natürliche Verallgemeinerung im mathematischen Sinn ist der Übergang von K zu einer Schleifengruppe. In [AMM98] wird hierzu das Konzept einer quasi-Hamiltonischen Mannigfaltigkeit eingeführt, das äquivalent zur Operation einer Schleifengruppe ist, aber einige technische Vorteile bietet. Leicht vereinfacht dargestellt wird hierbei im Vergleich zur klassischen Hamiltonischen Mannigfaltigkeit die Operation durch Linksmultiplikation durch getwistete Konjugation ersetzt und eine Momentabbildung definiert, die nicht ins Duale der Lie-Algebra geht, sondern in die Gruppe selbst.

Knop hat diese Objekte in [Kno16] näher untersucht und dabei beweisen können, dass man sich eine quasi-Hamiltonische Mannigfaltigkeit lokal aus Hamiltonischen zusammengesetzt vorstellen kann. Seit [Sja98] weiß man, dass glatte affine sphärische Varietäten lokale Modelle für Hamiltonische Mannigfaltigkeiten sind. Eine normale G -Varietät heißt hierbei sphärisch, wenn die Borel-Untergruppe eine offene Bahn hat. Knop hat in seiner Arbeit [Kno11] Hamiltonische Mannigfaltigkeiten folgendermaßen klassifiziert:

Eine Hamiltonische Mannigfaltigkeit ist charakterisiert durch zwei Invarianten, ein konvexes Polytop \mathcal{P} und ein Gitter Λ , welches die allgemeine Standgruppe codiert. Seit Losev [Los09] weiß man, dass eine glatte affine sphärische Varietät durch ihr Gewichtsmonoid vollständig charakterisiert wird. Knop konnte nun beweisen, dass ein Paar (\mathcal{P}, Λ) genau dann zu einer Hamiltonischen Mannigfaltigkeit gehört, wenn gilt: In jeder Ecke des Impulspolytops muss das Monoid, das entsteht, wenn man den zu einer Ecke des Polytops \mathcal{P} gehörenden Kegel mit dem Gitter Λ schneidet, das Gewichtsmonoid einer affinen glatten sphärischen Varietät sein.

Da eine quasi-Hamiltonische Mannigfaltigkeit lokal aus Hamiltonischen “zusammengesetzt” gedacht werden kann, überträgt sich das entsprechende Kriterium ziemlich analog, nur dass das Impulspolytop nun nicht mehr in der dominanten Kammer von

K lebt, sondern in der Alkove eines affinen Wurzelsystems. Leider trägt die quasi-Hamiltonische Kategorie wesentlich weniger Struktur, sodass bisher nur sehr wenige Beispiele für quasi-Hamiltonische Mannigfaltigkeiten bekannt sind. Hier setzt diese Arbeit an.

Wir klassifizieren hier zwei Familien von quasi-Hamiltonischen Mannigfaltigkeiten, zum einen diejenigen (mit einer kleinen technischen Zusatzbedingung) mit einer Strecke als Impulspolytop. Hierzu muss man an beiden Enden des Impulspolytops eine sphärische $L_{\mathbb{C}}$ -Varietät von Rang eins angeben, deren Gewichtsmonoid zum gleichen (in diesem Fall etwas degenerierten) Gitter Λ gehört. Wir benutzen dabei die bekannte Klassifikation glatter affiner sphärischer Varietäten von Rang eins. Es handelt sich also im Wesentlichen um ein Problem in der Kombinatorik affiner Wurzelsysteme. Wir entwickeln Techniken, die uns starke notwendige Kriterien für die Existenz solcher sphärischer Paare geben, und untersuchen die übrigen Fälle Case by Case.

Zum anderen klassifizieren wir hier quasi-Hamiltonische Mannigfaltigkeiten mit surjektiver Momentabbildung, von diesen erwähnte Knop in seiner Arbeit [Kno16] bereits einige Fälle. Wir benutzen hierfür ein kombinatorisches Glattheits-Kriterium von Pezzini und Van Steirteghem aus [PVS15] und Erkenntnisse aus einem gemeinsamen Artikel mit den beiden letztgenannten Autoren aus [PPVS17].

Part 1. Introduction

Let K be a simple compact Lie group with Lie algebra \mathfrak{k} and complexification G , $T_{\mathbb{R}}$ a maximal torus for K and T its complexification. Let us briefly recall the definition of a classical Hamiltonian manifold:

Definition 1.1. A Hamiltonian K -manifold is a K -manifold M' with a K -invariant symplectic form ω on M' and a K -equivariant smooth map $m : M' \rightarrow \mathfrak{k}^*$, the so-called moment map, such that the following compatibility condition holds:

$$\omega(\xi x, \eta) = (\xi, m_*\eta) \text{ for all } \xi \in \mathfrak{k}, \eta \in T_x M'$$

Hamiltonian manifolds have been a subject for studies for quite a long time, as the phase space in Hamilton's formulation of classical mechanics carries this structure. This is also where the term "moment map" comes from, it generalizes the linear and angular momentum from Hamiltonian mechanics. We are particularly interested in so-called multiplicity free manifolds, meaning those where all symplectic reductions are just points.

In the late 1980ies, T. Delzant conjectured in [Del90] (based on partial results on torus actions in [Del88] and his results on rank two in [Del90]) that any compact multiplicity free space is uniquely determined by its momentum polytope \mathcal{P} (the image of the moment map m in the dominant chamber of K) and its principal isotropy group L_S . The information about the principal isotropy can be encoded in a sublattice Λ_S of the lattice of dominant weights.

The classification of multiplicity free Hamiltonian manifolds was finished by Knop in [Kno11]. He was able to prove that \mathcal{P} must locally look like the weight monoid of a smooth affine spherical variety. That means that the lattice Λ_S intersected with the tangent cone in $x \in \mathcal{P}$ looks like the weight monoid of a smooth affine spherical variety for every minimal face of \mathcal{P} , we will define this more precisely in [definition 2.10](#).

Note that the classification of smooth affine spherical varieties (in the language of so-called primitive spherical triples) had been finished before by Knop and Van Steirteghem [KVS06], after partial results have been known long before, e.g. the classification of homogenous smooth affine spherical varieties by Krämer in [Kra79]. But still the classification of multiplicity free Hamiltonian manifolds is not very explicit.

Another question about this classification of multiplicity free Hamiltonian manifolds is the question for generalizations. One possible generalization is to "exchange" the action of K with the Hamiltonian action of a loop group.

A quasi-Hamiltonian manifold M is a concept introduced by Alekseev, Malkin and Meinrenken in [AMM98] as a setting equivalent to Hamiltonian loop group actions. This setting avoids many technical issues, it reduces Hamiltonian loop group actions to actions of finite-dimensional groups on finite-dimensional manifolds.

Roughly speaking, a quasi-Hamiltonian manifold is something similar to a classical Hamiltonian manifold, but the moment map $m : M \rightarrow K$ takes values in K itself and not in the dual of the Lie algebra. Knop found out that it is even more natural to consider a K -action by twisted conjugation which leads to so-called twisted

quasi-Hamiltonian manifolds [Kno16]. Similar ideas were discussed independently in [Mei15].

The aim of this thesis is to classify two special families of quasi-Hamiltonian manifolds, those where $\dim \mathcal{P} = 1$ for multiplicity free Hamiltonian and quasi-Hamiltonian manifolds and second, those that have a surjective moment map. This gives a lot of new examples of multiplicity free (quasi)-Hamiltonian manifolds.

In [AMM98], it is shown that there is a natural 1:1-correspondence between compact q -Hamiltonian G -spaces and Hamiltonian loop group spaces with proper moment maps. Usual Hamiltonian reduction is still possible in the q -Hamiltonian context, so we can keep the idea of multiplicity freeness and call a quasi-Hamiltonian manifold **multiplicity free** if its symplectic reductions are discrete. For this thesis, a q -Hamiltonian manifold is always assumed to be multiplicity free and compact, unless otherwise stated. This is the setting of [Kno16].

Knop pointed out that his paper gives the possibility to construct many new quasi-Hamiltonian manifolds, for only very few examples were known in the literature. That is the point of this thesis.

It is well known that the general case of a twisted q -Hamiltonian manifold can be reduced to the untwisted case, and the theory of twisted q -Hamiltonian manifolds depends only on the class of τ in the outer automorphism group of K .

Knop also showed, in [Kno16], Theorem 6.7, a q -Hamiltonian analogon of his classification theorem for Hamiltonian manifolds: a twisted quasi-Hamiltonian manifold M is uniquely determined by the same two invariants, a polytope \mathcal{P} that determines the image of the moment map, and a lattice Λ_S determining the principal isotropy group.

Theorem 1.2. ([Kno16], *thm. 6.7*) *Let K be a simply connected compact Lie group with twist τ . Then the map $M \rightarrow (\mathcal{P}_M, \Lambda_M)$ furnishes a bijection between*

- *isomorphism classes of convex multiplicity free quasi-Hamiltonian $K\tau$ -manifolds and*
- *spherical pairs (\mathcal{P}, Λ_M) .*

Under this correspondence, M is compact if and only if \mathcal{P}_M is closed in \mathcal{A} .

We can naturally assume the momentum image living in the fundamental alcove \mathcal{A} of some affine root system. For most of this thesis, we want the momentum polytope to touch every wall of the alcove to make sure that the corresponding manifold is *genuine*.

Definition 1.3. We call a q -Hamiltonian **genuine** if the moment polytope touches every wall of the fundamental alcove at least once. That means that genuine q -Hamiltonian manifolds are exactly those that do not carry a Hamiltonian structure.

The chapters 3 and 4 of this thesis classify the genuine multiplicity free quasi-Hamiltonian $K\tau$ -manifolds of rank one for K simple.

Second, [Kno16] mentions some examples of q -Hamiltonian manifolds that have surjective moment maps, meaning spherical pairs (\mathcal{A}, Λ_S) . We classify all q -Hamiltonian

group actions with surjective moment map in terms of smooth affine spherical varieties, meaning all spherical pairs (\mathcal{A}, Λ_S) .

In **case of rank one**, the momentum polytope is a line segment $[X_1 X_2]$, so it hits the walls of \mathcal{A} in (at least) two points X_1 and X_2 . The next step is: both points give different local models for the q -Hamiltonian manifold which are smooth affine spherical varieties of rank one, uniquely determined by their highest weight ω by [Los09]. Our task is to check in which cases they “fit together”. That means: if ω generates the weight monoid of the local model in X_1 , then $-\omega$ does so in X_2 .

In the **case of surjective moment maps**, the problems change a bit. The vertices of \mathcal{P} and the vertices of the alcove are identical, so we “only” have to check if the weight lattice generated by the weight monoid of a “ G -saturated” smooth affine spherical variety in one vertex does also correspond to a smooth affine spherical variety in every other vertex.

We shall give the list of all spherical pairs (\mathcal{A}, Λ_S) in the last part of this thesis.

While this thesis was work in progress, the author came to know that Guido Pezzini and Bart Van Steirteghem had worked on some related problem, [PVS15], where they discuss a combinatorial algorithm to determine whether a given weight monoid is smooth, meaning that it is the weight monoid of a smooth affine spherical variety. We saw that, at least in the case of G -saturated monoids of full rank, it is possible to “invert” this algorithm: it can be used to classify some families of weight monoids of smooth affine spherical varieties. This and many other applications of this algorithm is discussed in [PPVS17]. We shall use of the techniques developed in [PVS15] and some results from [PPVS17] in this thesis.

And there is one more thing we should think about: In [Kno16], part 11, Knop pointed to a series of papers by Matsuki ([Mat95], [Mat97] and [Mat02]) where the latter investigated a certain class of double coset spaces for simply connected compact Lie groups. In [Kno16], these *disymmetric manifolds* are re-interpreted as multiplicity free q -Hamiltonian manifolds. So the list of rank-one-cases from Matsuki’s list should be and is a subset of our list of rank one q -Hamiltonians. We shall also answer the question if every bi-homogeneous multiplicity free quasi-Hamiltonian manifold is disymmetric. It turns out that this is not the case, there are q -Hamiltonian manifolds of rank one that are not disymmetric.

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Part 2. Generalities

1. AFFINE ROOT SYSTEMS

First we recall some well known facts about affine root systems that are needed for the rest of the thesis. We follow the introduction to affine root systems in [Kno16] that mainly used [Mac72] and [Mac03].

Let $\bar{\mathfrak{a}}$ be an Euclidean vector space (as an example we could think of \mathbb{R}^n equipped with the standard scalar product $\langle \cdot, \cdot \rangle$) with an associated affine space \mathfrak{a} . We denote by $A(\mathfrak{a})$ the set of affine linear functions on \mathfrak{a} , the gradient of $\alpha \in A(\mathfrak{a})$ is denoted by $\bar{\alpha} \in \bar{\mathfrak{a}}$ and it has the property

$$(1) \quad \alpha(x + t) = \alpha(x) + \langle \bar{\alpha}, t \rangle, \quad x \in \mathfrak{a}, \quad t \in \bar{\mathfrak{a}}$$

Let $O(\bar{\mathfrak{a}})$ be the orthogonal group of the vector space $\bar{\mathfrak{a}}$. Let $M(\mathfrak{a})$ denote the isometries of the affine space, and let $M(\mathfrak{a}) \rightarrow O(\bar{\mathfrak{a}}), w \mapsto \bar{w}$ be the natural projection between these two sets. It fulfills

$$w(x + t) = w(x) + \bar{w}(t).$$

A motion $s \in M(\mathfrak{a})$ is called a reflection if its fixed point set is an affine hyperplane H . We can write $s_\alpha(x) = x - \alpha(x)\bar{\alpha}^\vee$ with the usual $\bar{\alpha}^\vee = \frac{2\bar{\alpha}}{\|\bar{\alpha}\|^2}$. Its action on an affine linear function $\beta \in A(\mathfrak{a})$ is:

$$s_\alpha(\beta) = \beta - \langle \bar{\beta}, \bar{\alpha}^\vee \rangle \alpha$$

Now we have gathered all necessary notation to state the definition of an *affine root system*:

Definition 2.1. An affine root system on \mathfrak{a} is a subset $\Phi \subset A(\mathfrak{a})$ such that:

- (1) $\mathbb{R}1 \cap \Phi = \emptyset$
- (2) $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$
- (3) $\langle \bar{\beta}, \bar{\alpha}^\vee \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$
- (4) the Weyl group $W_\Phi := \langle s_\alpha, \alpha \in \Phi \rangle \subset M(\mathfrak{a})$ is an Euclidean reflection group, which means in particular that W_Φ acts properly discontinuously.
- (5) The affine root system Φ is called reduced if $\mathbb{R}\alpha \cap \Phi = \{+\alpha, -\alpha\}$ for all $\alpha \in \Phi$.

Note that this definition even allows Φ to be empty, we do not ask Φ to generate the affine space. All finite (classical) root systems are affine in this definition.

We can consider the operation of the Weyl group W_Φ on A modulo the kernel of this operation and denote the resulting group \bar{W}_Φ .

Definition 2.2. The set of non-decomposable roots of Φ (with respect to a fixed torus) is called the set of **simple roots**. We denote this set of all simple roots by S . A root is called positive if it is a linear combination of simple roots with non-negative coefficients.

Definition 2.3 ([Kno16], 3.3). Let $\Phi \subset A(\mathfrak{a})$ be an affine root system.

- (1) A *weight lattice* for Φ is a lattice $\Lambda \subseteq \mathfrak{a}$ with $\bar{\Phi} \subset \Lambda$ and $\bar{\Phi}^\vee \subseteq \Lambda^\vee$ where $\Lambda^\vee = \{t \in \bar{\mathfrak{a}} \mid \langle t, \Lambda \rangle \subseteq \mathbb{Z}\}$ is the dual lattice for Φ .
- (2) An *integral root system* is a pair (Φ, Λ) where $\Phi \subset A(\mathfrak{a})$ is an affine root system and $\Lambda \subseteq \mathfrak{a}$ is a weight lattice for Φ .

The connected components of the complement of the union of all reflection hyperplanes in \mathfrak{a} are called the **chambers** of Φ (or W_Φ). The closure \mathcal{A} of a single chamber is called an **alcove**.

There is a well known theory of Euclidean reflection groups stating that W_Φ acts simply transitively on the set of alcoves, that each alcove is a fundamental domain for the action of W_Φ and that the group W_Φ is generated by the finitely many reflections about the faces of codimension one of any alcove. This can be reviewed in more detail in [Mac72] and [Mac03], as mentioned at the beginning of this section, albeit the definition of an affine root system there is a little bit different.

We finish this section with the classification of all connected affine root systems that are not classical root systems, as it is found e.g. in [Kac90].

$\begin{array}{c} 1 \rightleftarrows 1 \\ A_1^{(1)} \end{array}$	$\begin{array}{c} 1 \\ / \quad \backslash \\ 1 \quad 1 \quad \dots \quad 1 \quad 1 \\ A_n^{(1)}, n \geq 2 \end{array}$	$\begin{array}{c} 1 \quad 2 \quad 2 \quad \dots \quad 2 \quad 2 \quad 2 \\ \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \dots \quad \alpha_{n-2} \quad \alpha_{n-1} \quad \alpha_n \\ \\ 1 \\ \alpha_0 \\ B_n^{(1)}, n \geq 3 \end{array}$
$\begin{array}{c} 1 \quad 2 \quad \dots \quad 2 \quad 1 \\ \alpha_0 \quad \alpha_1 \quad \dots \quad \alpha_{n-1} \quad \alpha_n \\ C_n^{(1)}, n \geq 2 \end{array}$	$\begin{array}{c} 1 \\ \\ 1 \\ \alpha_0 \\ 1 \quad 2 \quad 2 \quad \dots \quad 2 \quad 2 \quad 1 \\ \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \dots \quad \alpha_{n-3} \quad \alpha_{n-2} \quad \alpha_{n-1} \\ \\ 1 \\ \alpha_n \\ D_n^{(1)}, n \geq 4 \end{array}$	$\begin{array}{c} 1 \quad 2 \quad 3 \quad 2 \quad 1 \\ \alpha_0 \quad \alpha_2 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \\ \\ 2 \\ \alpha_3 \\ \\ 1 \\ \alpha_1 \\ E_6(1) \end{array}$
$\begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \quad 3 \quad 2 \quad 1 \\ \alpha_0 \quad \alpha_1 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \quad \alpha_7 \\ \\ \alpha_2 \\ E_7^{(1)} \end{array}$	$\begin{array}{c} 2 \quad 4 \quad 6 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1 \\ \alpha_1 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \quad \alpha_7 \quad \alpha_8 \quad \alpha_0 \\ \\ 3 \\ \alpha_2 \\ E_8^{(1)} \end{array}$	$\begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \quad 2 \\ \alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \\ F_4^{(1)} \end{array}$
$\begin{array}{c} 1 \quad 2 \quad 3 \\ \alpha_0 \quad \alpha_2 \quad \alpha_1 \\ G_2^{(1)} \end{array}$	$\begin{array}{c} 2 \quad 1 \\ \alpha_0 \quad \alpha_1 \\ A_2^{(2)} \end{array}$	$\begin{array}{c} 2 \quad 2 \quad \dots \quad 2 \quad 1 \\ \alpha_0 \quad \alpha_1 \quad \dots \quad \alpha_{n-1} \quad \alpha_n \\ A_{2n}^{(2)}, n \geq 2 \end{array}$
$\begin{array}{c} 1 \quad 2 \quad 2 \quad \dots \quad 2 \quad 2 \quad 1 \\ \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \dots \quad \alpha_{n-2} \quad \alpha_{n-1} \quad \alpha_n \\ \\ 1 \\ \alpha_0 \\ A_{2n-1}^{(2)}, n \geq 3 \end{array}$	$\begin{array}{c} 1 \quad 1 \quad \dots \quad 1 \quad 1 \\ \alpha_0 \quad \alpha_1 \quad \dots \quad \alpha_{n-1} \quad \alpha_n \\ D_{n+1}^{(2)}, n \geq 2 \end{array}$	$\begin{array}{c} 1 \quad 2 \quad 3 \quad 2 \quad 1 \\ \alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \\ E_6^{(2)} \end{array}$
	$\begin{array}{c} 1 \quad 2 \quad 1 \\ \alpha_0 \quad \alpha_1 \quad \alpha_2 \\ D_4^{(3)} \end{array}$	

When we need coordinates, we shall use [Bou81] for the classical root systems and calculate α_0 by using the relation between the simple roots given by the Dynkin labels. For an orthonormal basis $\epsilon_1, \dots, \epsilon_n$ of \mathbb{R}^n , the realization we use is:

System	Vector Space	Simple Roots
$A_n^{(1)}$	Hyperplane of coordinate sum 0 in \mathbb{R}^{n+1}	$\alpha_0 = 1 - \epsilon_1 + \epsilon_{n+1}, \alpha_i = \epsilon_i - \epsilon_{i+1} \forall i = 1 \dots n$
$B_n^{(1)}$	\mathbb{R}^n	$\alpha_0 = 1 - \epsilon_1 - \epsilon_2, \alpha_i = \epsilon_i - \epsilon_{i+1} \forall i = 1, \dots, n-1, \alpha_n = \epsilon_n$
$C_n^{(1)}$	\mathbb{R}^n	$\alpha_0 = 1 - 2\epsilon_1, \alpha_i = \epsilon_i - \epsilon_{i+1} \forall i = 1, \dots, n-1, \alpha_n = 2\epsilon_n$
$D_n^{(1)}$	\mathbb{R}^n	$\alpha_0 = 1 - \epsilon_1 - \epsilon_2, \alpha_i = \epsilon_i - \epsilon_{i+1} \forall i = 1, \dots, n-1, \alpha_n = \epsilon_{n-1} + \epsilon_n$
$E_6^{(1)}$	subspace of \mathbb{R}^8 with $x_6 = x_7 = -x_8$	$\alpha_0 = 1 - \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8),$ $\alpha_1 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8),$ $\alpha_2 = \epsilon_1 + \epsilon_2, \alpha_3 = \epsilon_2 - \epsilon_1, \alpha_4 = \epsilon_3 - \epsilon_2,$ $\alpha_5 = \epsilon_4 - \epsilon_3, \alpha_6 = \epsilon_5 - \epsilon_4$
$E_7^{(1)}$	subspace of \mathbb{R}^8 orthogonal to $\epsilon_7 + \epsilon_8$	$\alpha_0 = 1 + \epsilon_7 - \epsilon_8, \alpha_1 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8),$ $\alpha_2 = \epsilon_1 + \epsilon_2, \alpha_3 = \epsilon_2 - \epsilon_1, \alpha_4 = \epsilon_3 - \epsilon_2,$ $\alpha_5 = \epsilon_4 - \epsilon_3, \alpha_6 = \epsilon_5 - \epsilon_4, \alpha_7 = \epsilon_6 - \epsilon_5$
$E_8^{(1)}$	\mathbb{R}^8	$\alpha_0 = 1 - \epsilon_7 - \epsilon_8, \alpha_1 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8),$ $\alpha_2 = \epsilon_1 + \epsilon_2, \alpha_3 = \epsilon_2 - \epsilon_1, \alpha_4 = \epsilon_3 - \epsilon_2,$ $\alpha_5 = \epsilon_4 - \epsilon_3, \alpha_6 = \epsilon_5 - \epsilon_4, \alpha_7 = \epsilon_6 - \epsilon_5, \alpha_8 = \epsilon_7 - \epsilon_6$
$F_4^{(1)}$	\mathbb{R}^4	$\alpha_0 = 1 - \epsilon_1 - \epsilon_2, \alpha_1 = \epsilon_2 - \epsilon_3, \alpha_2 = \epsilon_3 - \epsilon_4,$ $\alpha_3 = \epsilon_4, \alpha_4 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)$
$G_2^{(1)}$	Hyperplane of coordinate sum 0 in \mathbb{R}^3	$\alpha_0 = 1 + \epsilon_1 + \epsilon_2 - 2\epsilon_3, \alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = -2\epsilon_1 + \epsilon_2 + \epsilon_3$
$A_2^{(2)}$	\mathbb{R}^1	$\alpha_0 = 1 - \frac{1}{2}\epsilon_1, \alpha_1 = \epsilon_1$
$A_{2n}^{(2)}$	\mathbb{R}^n	$\alpha_0 = 1 - \epsilon_1, \alpha_i = \epsilon_i - \epsilon_{i+1} \forall i = 1, \dots, n-1, \alpha_n = 2\epsilon_n$
$A_{2n-1}^{(2)}$	\mathbb{R}^n	$\alpha_0 = 1 - \epsilon_1 - \epsilon_2, \alpha_i = \epsilon_i - \epsilon_{i+1} \forall i = 1, \dots, n-1, \alpha_n = 2\epsilon_n$
$D_{n+1}^{(2)}$	\mathbb{R}^n	$\alpha_0 = 1 - \epsilon_1, \alpha_i = \epsilon_i - \epsilon_{i+1} \forall i = 1, \dots, n-1, \alpha_n = \epsilon_n$
$E_6^{(2)}$	\mathbb{R}^4	$\alpha_0 = 1 - \epsilon_1, \alpha_1 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4), \alpha_2 = \epsilon_4, \alpha_3 = \epsilon_3 - \epsilon_4, \alpha_4 = \epsilon_2 - \epsilon_3$
$D_4^{(3)}$	Hyperplane of coordinate sum 0 in \mathbb{R}^3	$\alpha_0 = 1 + \epsilon_2 - \epsilon_3, \alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = -2\epsilon_1 + \epsilon_2 + \epsilon_3$

Note that the normalization in the last 6 entries differs slightly from [MW04].

2. BASIC COMBINATORIAL INVARIANTS OF SPHERICAL VARIETIES

In this section, we shall recall some basic facts, notations and invariants from the (combinatorial) theory of spherical varieties which we will use in this thesis. For a more detailed reference, we refer to [Kno91] or [Lun01]. The author also recommends [BL10] as an introduction to the combinatorial theory of spherical varieties containing numerous instructive examples. We are interested in smooth affine spherical varieties because they are local models for (quasi-)Hamiltonian manifolds.

Remember that an irreducible G -variety (for reductive G) is called **spherical** if it is normal and has an open B -orbit (where B denotes the Borel subgroup of G). A closed subgroup H is called spherical if G/H is spherical.

The **weight monoid** of a complex affine algebraic G -variety is the set of isomorphism classes of irreducible representations of G in its coordinate ring. We identify the weight monoid $\Gamma(Z)$ of an affine variety Z with the following submonoid of Λ^+ :

$$\Gamma(Z) = \{\lambda \in \Lambda : \text{Hom}_G(V(\lambda), \mathbb{C}[Z]) \neq 0\}$$

Smooth affine spherical varieties are classified by their weight monoid, cf. [Los09].

The following enumeration of the most important data of spherical varieties is mostly taken from part 2.2 in [PVS15]:

Definition 2.4. Let Z be a spherical G -variety with open orbit G/H . The basic invariants that the theory of spherical varieties uses about Z are:

- (1) The **(weight) lattice** of Z , called $\Lambda(Z)$, is the subgroup of the lattice generated by dominant weights Λ that consists of the B -weights of B -eigenvectors in the field of rational functions $\mathbb{C}(Z)$.
- (2) The rank of the weight lattice $\Lambda(Z)$ is called the **rank** of Z .
- (3) Let $\nu : \mathbb{C}(Z)^\times \rightarrow \mathbb{Q}$ be a G -invariant discrete valuation. Then ν induces an element $\rho_Z(\nu)$ of $\text{Hom}_{\mathbb{Z}}(\Lambda(Z), \mathbb{Q})$ by

$$\langle \rho_Z(\nu), \gamma \rangle = \nu(f_\gamma)$$

and $f_\gamma \in \mathbb{C}(Z)$ is a B -eigenvector of B -eigenvalue $\gamma \in \Lambda(Z)$. If $D \subset Z$ is a B -invariant prime divisor, we denote by ν_D the associated discrete valuation and by $\rho_Z(D)$ the element $\rho_Z(\nu_D)$ of $\text{Hom}_{\mathbb{Z}}(\Lambda(Z), \mathbb{Q})$.

- (4) A **color** of Z is a B -stable, but not G -stable prime divisor of Z . The set of colors of Z is called $\Delta(Z)$.
- (5) The **Cartan pairing** of Z is the bilinear map

$$C_Z : \mathbb{Z}\Delta(Z) \times \Lambda(Z) \rightarrow \mathbb{Z}$$

given by extending by linearity the elements $\rho_Z(D) \in \text{Hom}_{\mathbb{Z}}(\Lambda(Z), \mathbb{Q})$ with $D \in \Delta(Z)$. In particular, for $D \in \Delta(Z)$ and $\gamma \in \Lambda(Z)$:

$$C_Z(D, \gamma) = \langle \rho_Z(D), \gamma \rangle$$

- (6) Let Γ be a set of dominant weights of G . The set of **simple roots orthogonal to Γ** is

$$S^p(\Gamma) := \{\alpha \in S : \langle \lambda, \bar{\alpha}^\vee \rangle = 0 \quad \forall \lambda \in \Gamma\}$$

- (7) We use $\mathcal{V}(Z)$ for the set of G -invariant \mathbb{Q} -valued discrete valuations of $\mathbb{C}(Z)$ and identify $\mathcal{V}(Z)$ with its image in $\text{Hom}_{\mathbb{Z}}(\Lambda(Z), \mathbb{Q})$ via the map ρ_Z . We call $\mathcal{V}(Z)$ the **valuation cone** of Z .
- (8) By [Bri90], $\mathcal{V}(Z)$ is a co-simplicial cone. The set of **spherical roots** $\Sigma(Z)$ of Z is the minimal set of primitive elements of $\Lambda(Z)$ such that

$$\mathcal{V}(Z) = \{\eta \in \text{Hom}_{\mathbb{Z}}(\Lambda(Z), \mathbb{Q}) \mid \langle \eta, \sigma \rangle \leq 0 \quad \forall \sigma \in \Sigma(Z)\}$$

- (9) A color D of Z is **moved by a simple root** $\alpha \in S$ if D is not stable under the minimal parabolic subgroup of G containing B and associated with the simple root α . If $\bar{\alpha} \in S \cap \Sigma(Z)$, we denote by $\mathbf{A}(Z, \bar{\alpha})$ the set of colors of Z moved by $\bar{\alpha}$, and we set $\mathbf{A}(Z) := \cup_{\bar{\alpha} \in S \cap \Sigma(Z)} \mathbf{A}(Z, \bar{\alpha})$.

Note that all these invariants only depend on the open orbit G/H of Z .

Let us recall the list of spherical roots of smooth affine spherical varieties. It is a subset of Akhiezer's classification of smooth spherical varieties of rank one in [Akh83], we basically used the list in the second chapter of [Was96]. By [Bri89] part 2, the elements of the following list are known to be exactly the weights of homogeneous smooth affine spherical varieties of rank one.

List 2.5. Spherical roots of smooth affine spherical varieties

G	spherical root	optional factor
SL_2	α_1	
$\text{SL}_{n+1}, n \geq 2$	$\alpha_1 + \cdots + \alpha_n$	
SL_2	$2\alpha_1$	
$\text{SL}_2 \times \text{SL}_2$	$\alpha_1 + \alpha'_1$	$\frac{1}{2}$
SL_4	$\alpha_1 + 2\alpha_2 + \alpha_3$	$\frac{1}{2}$
$\text{Spin}_{2n+1}, n \geq 2$	$\alpha_1 + \cdots + \alpha_n$	
$\text{Spin}_{2n+1}, n \geq 2$	$2\alpha_1 + \cdots + 2\alpha_n$	
Spin_7	$\alpha_1 + 2\alpha_2 + 3\alpha_3$	$\frac{1}{2}$
$\text{Sp}_{2n}, n \geq 2$	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$	
$\text{Spin}_{2n}, n \geq 3$	$2\alpha_1 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$	$\frac{1}{2}$
F_4	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$	
G_2	$2\alpha_1 + \alpha_2$	2

Definition 2.6. We will use the abbreviation $\bar{\alpha}_{i,j}$ for $\bar{\alpha}_i + \bar{\alpha}_{i+1} + \cdots + \bar{\alpha}_j$ where $\bar{\alpha}$ denotes the gradient of α .

In [Lun01], D. Luna described combinatorial axioms for the spherical systems of spherical varieties, mostly just referred as “Luna’s axioms”. Note that spherical varieties are classified by their spherically closed spherical systems, cf. [Tim11], chapter 30.

Definition 2.7. ([PVS15], Def. 2.10) Let $(S^p, \Sigma, \mathbf{A})$ be a triple where S^p is a subset of S , Σ is a subset of $\Sigma^{sc}(G)$ and \mathbf{A} is a finite set endowed with a \mathbb{Z} -bilinear pairing $c : \mathbb{Z}\mathbf{A} \times \mathbb{Z}\Sigma \rightarrow \mathbb{Z}$. For every $\bar{\alpha} \in \Sigma \cap S$, let $\mathbf{A}(\bar{\alpha})$ denote the set $\{D \in \mathbf{A} \mid c(D, \bar{\alpha}) = 1\}$. Such a triple is called a *spherically closed spherical system* if all the following axioms hold:

- A1) for every $D \in \mathbf{A}$ and every $\sigma \in \Sigma$, we have that $c(D, \sigma) \leq 1$ and that if $c(D, \sigma) = 1$ then $\sigma \in S$.
- A2) for every $\bar{\alpha} \in \Sigma \cap S$, $\mathbf{A}(\bar{\alpha})$ contains exactly two elements which we denote by $D_{\bar{\alpha}}^+$ and $D_{\bar{\alpha}}^-$, and for all $\sigma \in \Sigma$ we have $c(D_{\bar{\alpha}}^+, \sigma) + c(D_{\bar{\alpha}}^-, \sigma) = \langle \bar{\alpha}^\vee, \sigma \rangle$.
- A3) the set \mathbf{A} is the union of $\mathbf{A}(\bar{\alpha})$ for all $\bar{\alpha} \in \Sigma \cap S$.
- $\Sigma 1$) if $2\bar{\alpha} \in \Sigma \cap 2S$ then $\frac{1}{2}\langle \bar{\alpha}^\vee, \sigma \rangle$ is a non-positive integer for all $\sigma \in \Sigma \setminus \{2\bar{\alpha}\}$.
- $\Sigma 2$) if $\bar{\alpha}, \bar{\beta} \in S$ are orthogonal and $\bar{\alpha} + \bar{\beta}$ belongs to Σ then $\langle \bar{\alpha}^\vee, \sigma \rangle = \langle \bar{\beta}^\vee, \sigma \rangle$ for all $\sigma \in \Sigma$.
- S) every $\sigma \in \Sigma$ is compatible with S^p , that is, for every $\sigma \in \Sigma$ there exists a spherically closed spherical subgroup K of G with $S^p(G/K) = S^p$ and $\Sigma(G/K) = \{\sigma\}$.

Luna also introduced a very useful method to quickly describe the spherical systems of spherical varieties, the so-called Luna diagrams. They extend the well-known Dynkin diagrams of simple groups by adding information about the spherical systems, especially spherical roots and the set S^p can be read out immediately, but they also allow to reconstruct the Cartan pairing:

List 2.8. Diagrams of spherical roots

Spherical root	Diagram
α_1	
$2\alpha_1$	
$\alpha + \alpha'$	
$\alpha_1 + \dots + \alpha_r$	
$\alpha_1 + 2\alpha_2 + \alpha_3$	
$\alpha_1 + \dots + \alpha_r$	
$2\alpha_1 + \dots + 2\alpha_r$	
$\alpha_1 + 2\alpha_2 + 3\alpha_3$	
$\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{r-1} + \alpha_r$	
$2\alpha_1 + \dots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$	
$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$	
$2\alpha_1 + \alpha_2$	
$4\alpha_1 + 2\alpha_2$	

We again point to [BL10] for a *lot* of explicit examples and a step-by-step guide on how to read Luna diagrams.

3. LOCAL MODELS AND TWISTED CONJUGACY CLASSES

We start this section with a precise definition of the actual subject of our studies:

We choose a scalar product $\langle \cdot, \cdot \rangle$ on \mathfrak{k} . Let τ be an automorphism of the simple compact Lie group K leaving the scalar product on \mathfrak{k} invariant. There are two canonical 1-forms θ and $\bar{\theta}$ on \mathfrak{k} defined by $\theta(k\xi) = \xi = \bar{\theta}(\xi k)$ with $k \in K$ and ξ as above. Then we take $\Theta_\tau = \frac{1}{2}(\bar{\theta} + \tau^{-1}\theta)$ with $\Theta_\tau(k\xi) = \frac{1}{2}(\text{Ad}(k)\xi + \tau^{-1}\xi)$ where τk denotes the operation of τ on $k \in K$. We use the scalar product on \mathfrak{k} to define the canonical bi-invariant closed 3-form

$$\chi := \frac{1}{12} \langle \theta, [\theta, \theta] \rangle = \frac{1}{12} \langle \bar{\theta}, [\bar{\theta}, \bar{\theta}] \rangle$$

We write $K\tau$ for the twisted K -action to avoid confusions with the adjoint action of K .

Definition 2.9 ([Kno16], Definition 2.3.). A quasi-Hamiltonian $K\tau$ -manifold is a smooth manifold M equipped with a K -action, a 2-form w and a smooth map $m : M \rightarrow K\tau$, called the (group valued)moment map which have the following properties:

- a) m is K -equivariant
- b) the form w is K -invariant and satisfies $dw = -m^*\chi$
- c) $w(\xi x, \eta) = \langle \xi, m^*\Theta_\tau(\eta) \rangle$ for all $\xi \in \mathfrak{k}, \eta \in T_x M$
- d) $\ker w_x = \{\xi x \in T_x M \mid \xi \in \mathfrak{k} \text{ with } {}^{m(x)\tau}\xi + \xi = 0\}$

Every twisted q-Hamiltonian manifold locally looks like a Hamiltonian one ([Kno16], part 5, especially theorem 5.4 for details) and Sjamaar [Sja98] showed that smooth affine spherical varieties are local models for (multiplicity free) Hamiltonian manifolds.

Let Z be a smooth affine spherical variety. We denote by $\mathfrak{a}_\mathcal{P} \subseteq \mathfrak{t}$ the affine space spanned by \mathcal{P} . Any $a \in \mathcal{P}$ gives rise to an isotropy group $K(a)$ with respect to the twisted action. We denote by C_Z the convex cone generated by the weight monoid of Z and by $C_a\mathcal{P}$ the tangent cone of \mathcal{P} in $a \in \mathcal{A}$.

The principal isotropy can be encoded in a lattice as follows: Let $L_\mathbb{R} \subset K$ be the centralizer of \mathcal{P} seen as a subset of \mathfrak{k}^* , then $L_\mathbb{R}$ is a Levi subgroup of K for which $L_\mathbb{R}/L_S$ is a torus. Hence the principal isotropy L_S is determined by the character group $\Lambda_S := \text{Hom}(L_\mathbb{R}/L_S, \mathbb{C}^\times)$, a subgroup of Λ (cf. introduction of [PPVS17] for more details).

Definition 2.10. ([Kno16], Definition 6.6) Let K be a simply connected compact Lie group with automorphism τ and fundamental alcove \mathcal{A} . Let $\mathcal{P} \subseteq \mathcal{A}$ be a locally convex subset and $\Lambda_S \subseteq \bar{\mathfrak{a}}_\mathcal{P}$ a lattice. Then (\mathcal{P}, Λ_S) is called **spherical in** $a \in \mathcal{P}$ if

- (1) \mathcal{P} is polyhedral in a , i.e.

$$\mathcal{P} \cap U = (a + C_a\mathcal{P}) \cap U$$

for a neighborhood U of a in \mathcal{A} , and

- (2) there is a smooth affine spherical $K(a)_{\mathbb{C}}$ -variety Z with weight monoid Γ_Z such that

$$C_a\mathcal{P} \cap \Lambda_S = \Gamma_Z$$

The pair (\mathcal{P}, Λ_S) is called a **spherical pair** if it is spherical for all $a \in \mathcal{P}$.

The variety Z is called a **local model** in a . By abuse of notation we also call the corresponding weight monoid and the set of its generators a local model, as a smooth affine spherical variety is uniquely determined by its weight monoid, cf. [Los09].

The general structure of smooth affine spherical varieties is:

Theorem 2.11 ([KVS06], corollary 2.2). *Let Z be a smooth affine spherical G -variety. Then $Z \cong G \times^H V$ where H is a reductive subgroup of G such that G/H is spherical and V is a spherical H -module.*

A smooth affine spherical variety corresponds to a triple (G, H, V) . Hence a complete classification of smooth affine spherical varieties would require giving all these triples. Unfortunately, this is impossible, the most difficult issues come from \mathbb{C}^\times -factors in G or H . We refer to [KVS06], examples 2.3, 2.4, 2.5 and 2.6 as an illustration of which problems appear.

A doable approach that avoids these problems is to list so-called **primitive spherical triples** $(\mathfrak{g}', \mathfrak{h}', V)$ that contain the semisimple parts of the corresponding Lie algebras. The actual objects of the classification are:

Definition 2.12 ([KVS06], Def. 2.7.).

- (1) Let $\mathfrak{h} \subseteq \mathfrak{g}$ be semisimple Lie algebras and let V be a representation of \mathfrak{h} . For \mathfrak{s} , a Cartan subalgebra of the centralizer $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$ of \mathfrak{h} , put $\bar{\mathfrak{h}} := \mathfrak{h} \oplus \mathfrak{s}$, a maximal central extension of \mathfrak{h} in \mathfrak{g} . Let \mathfrak{z} be a Cartan subalgebra of $\mathfrak{gl}(V)^{\mathfrak{h}}$, the centralizer of \mathfrak{h} in $\mathfrak{gl}(V)$. We call $(\mathfrak{g}, \mathfrak{h}, V)$ a **spherical triple** if there exists a Borel subalgebra \mathfrak{b} of \mathfrak{g} and a vector $v \in V$ such that:
 - (a) $\mathfrak{b} + \bar{\mathfrak{h}} = \mathfrak{g}$ and
 - (b) $[(\mathfrak{b} \cap \bar{\mathfrak{h}}) + \mathfrak{z}]v = V$ where \mathfrak{s} acts via any homomorphism $\mathfrak{s} \rightarrow \mathfrak{z}$ on V .
- (2) Two triples $(\mathfrak{g}_i, \mathfrak{h}_i, V_i), i = 1, 2$ are *isomorphic* if there exist linear bijections $\alpha : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ and $\beta : V_1 \rightarrow V_2$ such that:
 - (a) α is a Lie algebra homomorphism
 - (b) $\alpha(\mathfrak{h}_1) = \mathfrak{h}_2$
 - (c) $\beta(\xi v) = \alpha(\xi)\beta(v)$ for all $\xi \in \mathfrak{h}_1$ and $v \in V_1$.
- (3) Triples of the form $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{h}_1 \oplus \mathfrak{h}_2, V_1 \oplus V_2)$ with $(\mathfrak{g}_i, \mathfrak{h}_i, V_i) \neq (0, 0, 0)$ are called *decomposable*.
- (4) Triples of the form $(\mathfrak{k}, \mathfrak{k}, 0)$ and $(0, 0, V)$ are said to be *trivial*.
- (5) A pair $(\mathfrak{g}, \mathfrak{h})$ of semisimple Lie algebras is called *spherical* if $(\mathfrak{g}, \mathfrak{h}, 0)$ is a spherical triple.
- (6) A spherical triple (or pair) is *primitive* if it is non-trivial and indecomposable.

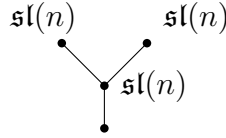
The use of these spherical triples is justified by the following theorem:

Theorem 2.13 ([KVS06], theorem 2.9.). *If $G \times^H V$ is a smooth affine spherical variety then $(\mathfrak{g}', \mathfrak{h}', V)$ is a spherical triple. Moreover, it follows from the classification in [KVS06] that every spherical triple arises this way.*

We shall make use of the diagram notation developed in [KVS06], section 3 that bases upon an idea of Mikityuk [Mik87]. We represent a triple by a three-layered graph: Let $\mathfrak{g}_1 + \cdots + \mathfrak{g}_r$, $\mathfrak{h}_1 + \cdots + \mathfrak{h}_s$, $V_1 \oplus \cdots \oplus V_t$ be the simple factors of \mathfrak{g} , \mathfrak{h} , V . The vertices of the graph symbolize all \mathfrak{g}_i , \mathfrak{h}_j and V_k . We draw an edge between \mathfrak{h}_j and \mathfrak{g}_i if the map $\mathfrak{h}_j \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_i$ is non-zero and an edge between V_k and \mathfrak{h}_j if V_k is a non-trivial \mathfrak{h}_j -module. We never draw an edge between V_k and \mathfrak{g}_i . We will write the graph in three rows, the \mathfrak{g}_i in the first, \mathfrak{h}_j in the second and V_k in the last. We should mark all the embeddings of the \mathfrak{h}_j into the \mathfrak{g}_i or how each \mathfrak{h}_j acts on the V_k , but we will omit these labels if these are the natural embeddings respectively actions.

Note that those \mathfrak{g}_i that are contained in \mathfrak{h} can be recovered from the graph as they correspond to those \mathfrak{h}_j that are not connected to any \mathfrak{g} , so we will never draw an extra vertex for them.

Example 2.14. The graph



represents the triple $(\mathfrak{sl}(n) + \mathfrak{sl}(n), \mathfrak{sl}(n), \omega_1)$.

We shall use the same conventions as in [KVS06] considering borderline cases: when \mathfrak{h}_j is 0, e.g. $\mathfrak{sl}(1)$ or $\mathfrak{sp}(0)$, then the corresponding vertex and everything connected to it has to be omitted.

For the **case of surjective moment maps**, all spherical varieties that correspond to a weight lattice Λ such that $\mathbb{Q}\Lambda \cap \mathcal{A} = \mathcal{A}$ are the candidates for local models. They were determined in [PPVS17].

The local models are closely related to the local root systems from [Kno16]. We shall give a brief overview how they are constructed.

We fix once and for all an alcove of this root system, the fundamental alcove of Φ^τ , by

$$\mathcal{A} := \{\xi \in \mathfrak{t}^\tau : \alpha(\xi) \geq 0 \ \forall \alpha \text{ simple root in } \Phi^\tau\}.$$

Recall in particular that \mathcal{A} is a fundamental domain for W^τ , the Weyl group of Φ^τ .

The alcove and the momentum image of a q-Hamiltonian manifold M have the following relation: As \mathcal{A} is a fundamental domain for the Weyl group W^τ , the map $\mathcal{A} \rightarrow \mathfrak{t}/W^\tau$ is a homeomorphism. Having a moment map $m : M \rightarrow K$, we can introduce the invariant moment map $m_+ : M \rightarrow \mathcal{A}$ with the property that it makes the following diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{m} & K \\ m_+ \downarrow & & \downarrow \\ \mathcal{A} & \xrightarrow{\text{bij}} & K/\tau K \end{array}$$

The bijection between the alcove and the set of twisted conjugacy classes was studied in more detail in [MW04].

The image of m_+ , called $\mathcal{P}_M = m_+(M) \subseteq \mathcal{A}$ determines the actual image of m via $m(M) = K \cdot \exp(\mathcal{P}_M)$. It follows that \mathcal{P}_M is a convex polytope lying in \mathcal{A} and all fibers of m_+ (and m) are connected ([Kno16], 4.4).

Another important definition is the one of a local root system:

Definition 2.15 ([Kno16], definition 7.2.). A local system of roots (or local root system) $\Phi(*)$ on \mathcal{P} is a family $(\Phi(X))_{X \in \mathcal{P}}$ of root systems on \mathfrak{a} such that for each $X \in \mathcal{P}$:

- (1) $\Phi_Y = \Phi(X)_Y$ for all Y in a sufficiently small neighborhood of X in \mathcal{P} .
- (2) Every root $\alpha \in \Phi(X)$ is either non-negative or non-positive on \mathcal{P} .

An **integral local root system on \mathcal{P}** is a pair $(\Phi(*), \Lambda)$ such that $(\Phi(X), \Lambda)$ is an integral root system for every $X \in \mathcal{P}$.

Remark 2.16.

- (1) We observe that setting $X = Y$ in the definition leads to $\Phi(X) = \{\alpha \in \Phi : \alpha(X) = 0\}$.
- (2) Part 1 of the above definition justifies only looking at the vertices of the momentum polytope when thinking about local models, for we can extend them for a neighborhood of the vertex.
- (3) The local root systems are in general not connected.

Definition 2.17. We denote $S(X) = \{\alpha \in S : \alpha(X) = 0\}$.

Part 3. Genuine Quasi-Hamiltonian Manifolds of Rank one

This part is devoted to the classification of all compact genuine quasi-Hamiltonian K -manifolds of rank one for a simple and simply connected Lie group K . This is equivalent to:

- $S = S(X_1) \cup S(X_2)$
- We find a triple (X_1, X_2, ω) such that $\mathbb{N}\omega$ is the weight monoid of a smooth affine spherical $K_{X_1}^{\mathbb{C}}$ -variety and $-\mathbb{N}\omega$ is the weight monoid of a smooth affine spherical $K_{X_2}^{\mathbb{C}}$ -variety.

Of course, the triples (X_1, X_2, ω) and $(X_2, X_1, -\omega)$ are isomorphic.

For the rest of this part, we assume our manifolds to have this property.

Proposition 3.1. Let $\mathbb{N}\omega$ be the weight monoid of a local model for a genuine quasi-Hamiltonian manifold M of rank one in X . Then the moment polytope can be written as $\{X + \mathbb{R}\omega\} \cap \mathcal{A}$.

Proof. The cone spanned by ω is the cone of the corresponding spherical variety via [definition 2.10](#). It is the line segment defined by $\{X + \mathbb{R}\omega \cap \mathcal{A}\}$. \square

4. LOCAL MODELS OF RANK ONE

We shall collect some information about the local models of rank one here. Recall the definition of a local model from [definition 2.10](#).

For rank reasons, it follows that for **smooth affine spherical varieties Z of rank one**, only the two extreme cases of $Z := G \times^H V$ can show up:

Lemma 3.2. *Let G, H be reductive. A smooth affine spherical G -variety Z of rank one is of exactly one of the following types:*

- (1) $Z = G/H, V = 0$. We will call this the “homogeneous case”, as it means that Z is a homogeneous spherical variety.
- (2) $Z = V = \mathbb{C}^n, H = G = \mathrm{SL}(n, \mathbb{C}), n \geq 1$ or $V = \mathbb{C}^{2n}, H = G = \mathrm{Sp}(2n, \mathbb{C}), n \geq 2$. We call this the “inhomogeneous case”.

Proof. As the homogeneous space G/H is the image of $Z = G \times^H V$ under the projection $Z \rightarrow G/H$, the rank of the homogeneous space G/H is at most the rank of Z , so either 0 or 1, cf. [\[KVS06\]](#).

If it is 0, then G/H is projective by [\[Tim11\]](#), Proposition 10.1, but, as it is also affine, it is a single point, i.e. $G = H$. We deduce $Z = V$, so Z is a spherical module of rank 1. By the classification of spherical modules (cf. [\[Kno98\]](#)), the module V and the group G must be of the form in statement (2) of the lemma.

Suppose now G/H has rank 1, that means Z and G/H have the same rank. Let $K(a) \subseteq G$ be the stabilizer of a point a in the open G -orbit of Z (which exists because Z is spherical) such that $K \subseteq H$. By [\[Gan10\]](#), Lemma 2.4, the quotient K/H is finite. This implies that the projection $Z \rightarrow G/H$ has finite fibers, i.e. V contains only finitely many points. Hence $V = 0$ and Z is of the form in statement (1) of the lemma. \square

That means the possible weights of smooth affine spherical homogeneous varieties of rank one are exactly the spherical roots we introduced in [list 2.5](#).

Inhomogeneous models can only occur if the local root systems have components of type A or C . In this case, the weight (that gives the direction of the momentum polytope) is:

Lemma 3.3. *Fix some root system of type A_n or C_n , then the corresponding inhomogeneous $\omega \in \mathcal{A}$ for A_l is defined by the condition $\langle \omega, \bar{\alpha}_i^\vee \rangle = \delta_{i,1}$.*

$$\begin{array}{ccccccc} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \hline \end{array}$$

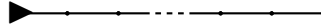
If the local root system is C_l , the situation is:

$$\begin{array}{ccccccc} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \hline \end{array}$$

Proof. We already saw that inhomogeneous smooth affine spherical varieties of rank one are spherical modules. The classification of spherical modules in [\[Kac80\]](#) resp. [\[Lea98\]](#) gives the desired result. \square

We introduce a diagram notation similar to Luna diagrams for these two cases:

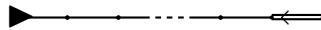
Definition 3.4. For the weight ω of the spherical module of rank one in $A_l, l \geq 1$ we shall use the following diagram, the root whose coroot pairs one with ω is the one marked with the triangle.



For A_1 , the diagram is



For the weight ω of the spherical module of rank one in C_l we shall use the diagram:



Note that there is a certain analogy to Luna's diagrams. Again, the simple roots in the support of ω whose coroots pair zero with ω are the ones that are not marked in the diagram. In that sense, the triangles in this diagrams have an analogous meaning to the circles in Luna's diagrams.

Note that in some local root systems, [lemma 3.3](#) does not give enough conditions for ω to be completely determined, as ω lives in a n -dimensional space where n is the rank of Φ . If the local root system contains less than n simple roots, some parameters remain for ω (we shall see later that $n = l$ for bi-inhomogeneous polytopes).

Let me introduce some notation to simplify denoting these homogeneous and inhomogeneous models of rank one. We will only use this in case of rank one. We want a notation for a local root system without having to know the actual subset of \mathcal{A} where it is realized.

As said, $\Phi(X)$ is a subset of Φ , the **local root system** associated to any point $X \in \mathcal{A}$ by the fact that $\Phi(X) = \{\alpha \in \Phi : \alpha(X) = 0\}$.

Definition 3.5. Let $X \in \mathcal{A}$ and suppose $\Phi(X) := S \setminus \{\alpha_{n_1}, \alpha_{n_2}, \dots, \alpha_{n_m}\} := S_{n_1, \dots, n_m}$. We denote the **set of all homogeneous local models in X** by

$$H(X) := H(n_1, \dots, n_m)$$

We also can define the **set of inhomogeneous local models w fulfilling $\langle \omega, \bar{\alpha}_k^\vee \rangle = 1$ in X** as

$$I_k(X) := I_k(n_1, \dots, n_m)$$

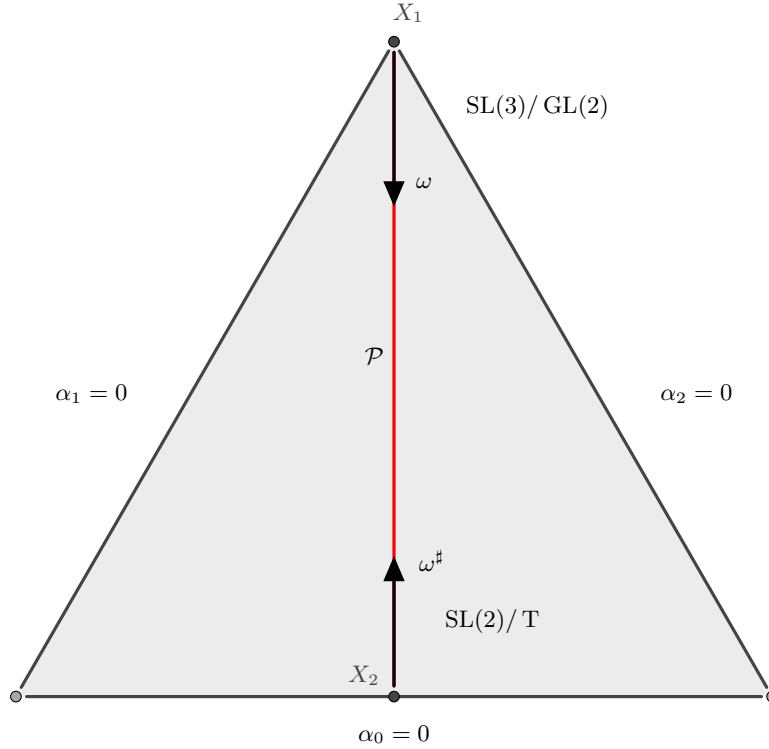
For example, by stating $\omega \in H(0)$ we say that ω is (the weight of) a homogeneous local model for $\Phi(X) = S \setminus \{\alpha_0\}$. Saying $\omega \in I_1(0)$ means that ω is an inhomogeneous local model on $\Phi(X) := S \setminus \{\alpha_0\}$ and that $\langle \omega, \bar{\alpha}_1^\vee \rangle = 1$, (cf. [lemma 3.3](#)).

4.1. Examples.

Let us assume that ω is the weight of a **homogeneous smooth affine spherical variety** of rank one. As the whole momentum polytope has to lie in the same fundamental alcove, this ω can't be associated to a local model in X_2 . But probably, $\omega^\sharp := t\delta - \omega$ is, as ω and ω^\sharp generate the same weight lattice. As a momentum polytope parallel to ω is also parallel to ω^\sharp , it gives rise to the same \mathcal{P} , so \mathcal{P} would be a valid polytope for the whole closed alcove \mathcal{A} , as both weights generate the same lattice. In this case, the two local models can be glued to one model of the whole manifold M by the above mentioned theorems. We call this a “bi-homogeneous” manifold (or momentum polytope), as both local models are of homogeneous type. Let us look at an example.

Example 3.6. We fix the affine root system $A_n^{(1)}$. We take $X_1 = (0, 0, 0, \dots, 0)$, the local root system is $S(X_1) = S \setminus \{\alpha_0\}$. We choose the local model given by $\omega = \bar{\alpha}_1 + \bar{\alpha}_2 + \dots + \bar{\alpha}_n = (1, 0, \dots, 0, -1)$ which corresponds to the variety $\mathrm{SL}(n+1)/\mathrm{GL}(n)$. We want our polytope to touch every wall of the alcove, hence we demand $\alpha_0 \in S(X_2)$. The other point where the moment polytope hits the walls of the alcove is $X_2 = (1, 0, \dots, 0, -1)$ with local root system $\Phi \setminus \{\alpha_1, \alpha_n\}$. Because $\omega^\sharp = -\bar{\alpha}_1 - \bar{\alpha}_2 - \dots - \bar{\alpha}_n = \bar{\alpha}_0$ is a spherical root of rank one in this local root system (corresponding to $\mathrm{SL}(2)/T$), we have found a bi-homogeneous variety.

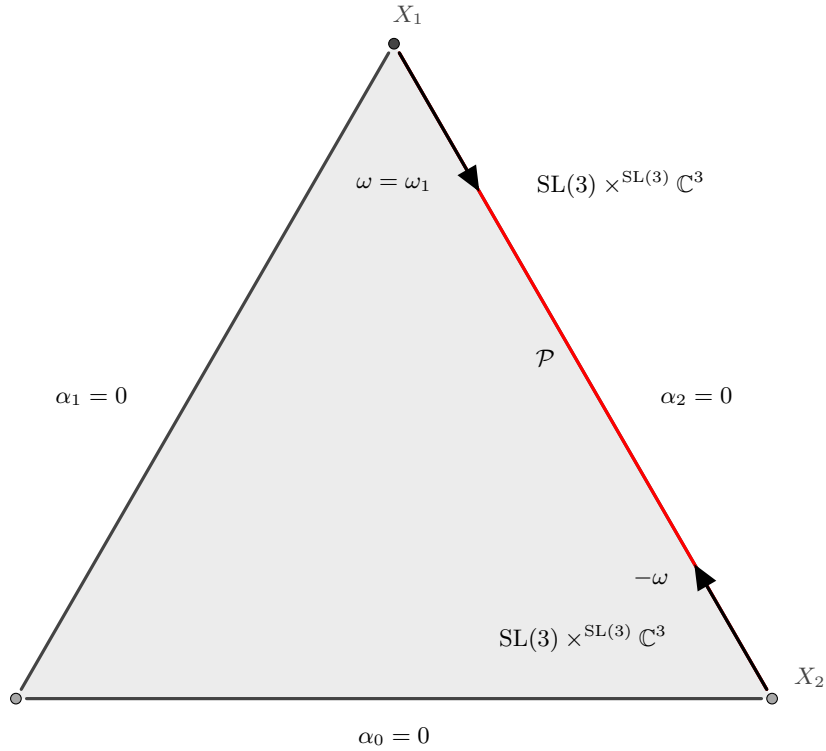
The picture shows the situation in $A_2^{(1)}$.



A bi-homogeneous polytope for $A_2^{(1)}$

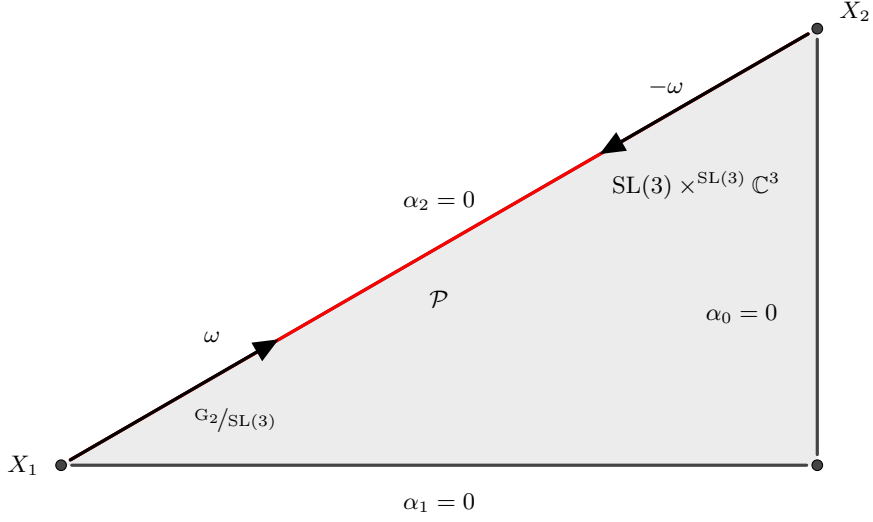
The next example is of the *bi-inhomogeneous* type, meaning ω and $-\omega$ are the weights of inhomogeneous smooth affine spherical varieties of rank one.

Example 3.7. We look at $A_n^{(1)}$ and again, $X_1 = (0, 0, 0, \dots, 0)$ with local root system $\Phi(X_1) = S \setminus \{\alpha_0\}$. The first fundamental weight ω_1 of A_n is an inhomogeneous local model here, corresponding to $\mathrm{SL}(n+1) \times^{\mathrm{SL}(n+1)} \mathbb{C}^{n+1}$. The local root system for X_2 is $S \setminus \{\alpha_1\}$ (we shall have a general theorem for that later: [theorem 3.19](#)) which also permits an inhomogeneous local model. We verify by elementary calculations that $-\omega$ is an inhomogeneous local model there.



A bi-inhomogeneous polytope for $A_2^{(1)}$

Example 3.8. We consider $G_2^{(1)}$. Then $\omega = \bar{\alpha}_2 + 2\bar{\alpha}_1$ is a homogeneous model for $X_1 = (0, 0, 0)$, the corresponding variety is $G_2/\mathrm{SL}(3)$. Hence $S(X_2) = \{\alpha_0, \alpha_2\}$, and $-\omega$ is an inhomogeneous model corresponding to $\mathrm{SL}(3) \times^{\mathrm{SL}(3)} \mathbb{C}^3$ there.



A mixed moment polytope for $G_2^{(1)}$

This leads to the following definition:

Definition 3.9. Let the line segment $[X_1 X_2] = X_1 + \mathbb{R}\omega \cap \mathcal{A}$ be the moment polytope of a genuine quasi-Hamiltonian manifold of rank one.

- (1) We call this moment polytope **bi-homogeneous** if ω is the generator of the weight monoid of a homogeneous smooth affine spherical variety in X_1 and $\omega^\sharp := t\delta - \omega$ for $t \in \mathbb{R}$ is the generator of the weight monoid of a homogeneous smooth affine spherical variety in X_2 .
- (2) We call this moment polytope **bi-inhomogeneous** if ω is the generator of the weight monoid of an inhomogeneous smooth affine spherical variety in X_1 and $-\omega$ is the generator of the weight monoid of an inhomogeneous smooth affine spherical variety in X_2 .
- (3) We call this moment polytope **mixed** if ω is the generator of the weight monoid of a homogeneous smooth affine spherical variety in X_1 and $-\omega$ is the generator of the weight monoid of an inhomogeneous smooth affine spherical variety in X_2 .

5. STRUCTURE OF BI-HOMOGENEOUS POLYTOPES

We first think about the structure of genuine bi-homogeneous moment polytopes. We want to prove:

Theorem 3.10. *For all genuine bi-homogeneous moment polytopes of quasi-Hamiltonian $K\tau$ -manifolds of rank one, $S(X_1)$ and $S(X_2)$ contain at least $n-1$ simple roots each, where n is the rank of K .*

The rest of this section is to prove this statement.

Definition 3.11.

- (1) Two simple roots are called neighbors if their scalar product is not zero. This is equivalent to being joined by an edge in the affine Dynkin diagram.
- (2) We call a simple root *at the end* of a diagram if it is only connected to one other root.
- (3) If we say “left” or “right”, we always mean left or right in the corresponding affine Dynkin diagram.
- (4) We call the root given by the Dynkin labels of an affine root system δ , e.g. for $A_n^{(1)}$, we have $\delta = \sum_{i=0}^n \bar{\alpha}_i$.
- (5) We say a simple root α_i is in the support of ω if its coefficient k_i in $\omega = \sum k_i \bar{\alpha}_i$ is nonzero.

We remember that we look for genuine polytopes which means $S(X_1) \cup S(X_2) = S$. Let us first note a lemma that helps us to see which simple roots of $S(X_1)$ are also in $S(X_2)$:

Lemma 3.12. *Let α be a simple root, ω a dominant weight of a smooth affine spherical variety of rank one that is a local model in X_1 . Let $X_2 = X_1 + c \cdot \omega$ for $c \in \mathbb{R}_{>0}$. Then:*

- (1) *Let $\alpha \in \Phi(X_1)$ and $\alpha \in S^p(\omega)$. Then $\alpha \in \Phi(X_2)$.*
- (2) *Let $\alpha \in \Phi(X_1)$ and $\alpha \notin S^p(\omega)$. Then $\alpha \notin \Phi(X_2)$.*
- (3) *Let $\alpha \notin \Phi(X_1)$ and $\alpha \in S^p(\omega)$. Then $\alpha \notin \Phi(X_2)$.*
- (4) *Let $\alpha \notin \Phi(X_1)$ and $\alpha \notin S^p(\omega)$. Then $\alpha \in \Phi(X_2)$ if and only if $c = -\frac{\alpha(X_1)}{\langle \omega, \bar{\alpha}^\vee \rangle}$*

Proof.

We use the arithmetic of affine combinations, namely equation (1) from [section 1](#).

- (1) Remember that $\alpha \in S^p(\omega)$ means that $\langle \omega, \bar{\alpha}^\vee \rangle = 0$. Having $\alpha \in \Phi(X_1)$ means $\alpha(X_1) = 0$. Hence

$$\alpha(X_2) = \underbrace{\alpha(X_1)}_{=0} + c \cdot \underbrace{\langle \omega, \bar{\alpha}^\vee \rangle}_{=0} = 0$$

so α is in $\Phi(X_2)$.

- (2) In this case, we have

$$\alpha(X_2) = \underbrace{\alpha(X_1)}_{=0} + c \cdot \underbrace{\langle \omega, \bar{\alpha}^\vee \rangle}_{\neq 0} \neq 0$$

so α is not in $\Phi(X_2)$.

(3) In this case, we have

$$\alpha(X_2) = \underbrace{\alpha(X_1)}_{\neq 0} + c \cdot \underbrace{\langle \omega, \bar{\alpha}^\vee \rangle}_{=0} \neq 0$$

so α is not in $\Phi(X_2)$.

(4) In this case, we have

$$\alpha(X_2) = \underbrace{\alpha(X_1)}_{\neq 0} + c \cdot \underbrace{\langle \omega, \bar{\alpha}^\vee \rangle}_{\neq 0}$$

We have $\alpha \in \Phi(X_2)$ if and only if this equation is zero. Hence $c = -\frac{\alpha(X_1)}{\langle \omega, \bar{\alpha}^\vee \rangle}$. □

Let us now discuss which $t \in \mathbb{R}$ can appear in $\omega^\sharp = t\delta - \omega$. As spherical roots are linear combinations of simple roots and the coefficients in these linear combinations are in $\mathbb{N}[\frac{1}{2}]$ and ≤ 3 , there are not many choices for this parameter t , namely t could only be chosen such that all coefficients of simple roots in δ are in $\mathbb{N}[\frac{1}{2}]$ and no coefficient of $t\delta$ is bigger than twice the biggest coefficient of a spherical root possible in any subsystem of the affine root system (e.g. 6 in root systems with subsystems of type B_3 or F).

So appropriate choices for this t are rare. In particular, the choice of t for a particular local model may not lead to a ω^\sharp where ω^\sharp supports all roots, as there is no point in the alcove of an affine root system where all roots are zero simultaneously.

Remember from [definition 3.5](#) that $H(X)$ denotes the set of weights of all homogeneous smooth affine spherical $K_X^{\mathbb{C}}$ -varieties of rank one.

Lemma 3.13. *Let \mathcal{A} be the fundamental alcove of an affine root system, and $\omega \in H(X_1)$. Then the simple roots not supported in ω must all be contained in one connected component of $S(X_2)$, or they must form $A_1 \times A_1$ in $S(X_2)$.*

Proof. It is well known that a homogeneous smooth affine spherical variety of rank one has a weight whose support has one connected component or is of type $A_1 \times A_1$. This is because the weights of homogeneous smooth affine spherical varieties are exactly the spherical roots of rank one. Inspecting [list 2.5](#) gives the desired result. □

We shall use this to prove the following theorem:

Theorem 3.14. *Let Φ be an affine root system not of type $A_n^{(1)}$, $E_i^{(1)}$, $i \in \{6, 7, 8\}$. If $|S(X_1)| < n - 1$, meaning at least three roots are nonzero in X_1 , no bi-homogeneous polytopes of rank one are possible.*

Proof. We first state that for $D_4^{(3)}$ and $G_2^{(1)}$, the theorem is trivially true. So let us assume the local root system contains more than three simple roots.

Remark 3.15. The arguments in the following proof do not see the length of the simple roots. So we only use the corresponding simply laced diagrams for instructive drawings.

We assume that three roots are missing in $S(X_1)$. The following things can happen:

- All three missing roots are neighbors. In root systems that do not have an ending of type D , the middle root of the missing roots is always in S^p for any spherical root we could choose, so it is never in $S(X_2)$.
If the local root system had one ending of type D and consisted of 4 roots (which is the case for $B_3^{(1)}$ or $A_3^{(2)}$), we could have $\{\alpha_0, \alpha_1, \alpha_3\} \notin S(X_1)$. But then these roots have to be in $S(X_2)$ and form three connected components. Impossible.
If the local root system in X_1 had an ending of type D at the (without loss of generality) left end and we had chosen $\{\alpha_2, \alpha_1, \alpha_0\} \notin S(X_1)$, then α_1 and α_0 are in S^p for any w we could choose on $S(X_1)$ and hence not in $S(X_2)$ by [lemma 3.12](#).
If we chose X_1 such that $\{\alpha_3, \alpha_2, \alpha_0\} \notin S(X_1)$, we had $\alpha_0 \in S^p$ for every ω we could choose and hence $\alpha_0 \notin S(X_2)$.
- Two of the missing roots are neighbors at the, without loss of generality, right end of the root system, meaning α_{n-1}, α_n for ending B and C and w.l.o.g α_{n-2}, α_n for type D . For every ω we could chose for this local root system, the missing root α_n can never be in $S(X_2)$ by [lemma 3.12](#), as it is in S^p .
- Two are neighbors not at the end of $S(X_1)$. Then there is no bi-homogeneous polytope for the following reasons: First assume the missing roots $\alpha_{n-2}, \alpha_{n-1}$ are surrounded by $A_1 \times A_1$, the situation looks like this:

We draw an excerpt of the affine root system where we mark the simple roots that are not in $S(X_1)$ (equivalent to being nonzero on X_1) with a cross:

$$\dots \xrightarrow{\quad \times \quad \times \quad \times \quad} \dots$$

$n-2 \quad n-1$

Both roots $\alpha_{n-2}, \alpha_{n-1} \notin S^p$ and hence could be in $S(X_2)$. But the roots $\bar{\alpha}_{n-4}, \bar{\alpha}_{n-2}, \bar{\alpha}_{n-1}$ needed to be supported in ω^\sharp and are neither connected nor $A_1 \times A_1$, as $\alpha_{n-3} \notin S(X_2)$, contradiction.

Assume now the two missing roots α_d, α_{d+1} are not neighbored by A_1 on both sides. For a spherical root ω supported on the component left of α_d , we get $\alpha_{d+1} \in S^p$, so α_{d+1} will never be in $S(X_2)$ by [lemma 3.12](#). The same argument for a spherical root on the component right of α_{d+1} .

$$\dots \xrightarrow{\quad \times \quad \times \quad} \dots$$

$d \quad d+1$

- None of the missing roots are neighbors.
If no ending of Φ is of type D , the three roots $\notin S(X_1)$ could cut out an $A_1 \times A_1$:

$$\xrightarrow{\quad \times \quad \times \quad \times \quad}$$

$\alpha \quad \alpha'$

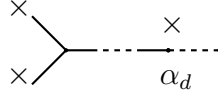
If there are exactly five roots in Φ , like in the instructive drawing above, the roots not supported in ω form an $A_1 \times A_1 \times A_1$ or $A_1 \times A_3$, contradicting [lemma 3.13](#).

If Φ contains more than five roots and no ending is of type D , $S(X_1)$ resp. $S(X_2)$ has too many components for a bihomogeneous solution:

$$\begin{array}{ccccccc} & \text{I} \times & \text{II} \times & \text{III} \times & \text{IV} & & \\ \cdots & \xrightarrow{d} & \xrightarrow{e} & \xrightarrow{f} & \cdots & & \end{array}$$

We number the connected components from left to right. A local model ω for component I or I+II or II has $\alpha_f \in S^p$ and hence $\alpha_f \notin S(X_2)$. Analogous for an ω for component III, III+IV or IV and α_d . If component I and IV form $A_1 \times A_1$, then $\alpha_e \notin S(X_2)$ for ω of type D_2 . If component I and II form $A_1 \times A_1$, then $\alpha_f \notin S(X_2)$ for such an ω . If component I and III form $A_1 \times A_1$, the roots not supported in ω of type D_2 supported on these two components are distributed over two connected components in $S(X_2)$ that are not of type A_1 , contradicting [lemma 3.13](#).

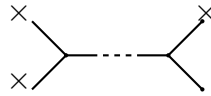
If there is an ending of type D on the w.l.o.g. left, it could happen that $S(X_1)$ has two components. Let us assume $\alpha_0, \alpha_1, \alpha_d$ with $2 < d < n - 1$ are $\notin S(X_1)$.



A weight ω on the component of $S(X_1)$ right of α_d leads to $\alpha_0, \alpha_1 \notin S(X_2)$ by [lemma 3.12](#). The other possibility is something supporting $\{\alpha_2, \dots, \alpha_{d-1}\}$. But then $\delta - \omega$ consists of three connected components of type either $A_1 \times A_1 \times$ something each containing one missing root if we had $\omega = \bar{\alpha}_2 + \dots + \bar{\alpha}_{d-1}$ or $A_3 \times$ something each containing a missing root if we had $\omega = [\frac{1}{2}]\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4$ or $\omega = [2]\bar{\alpha}_2$. All oppose [lemma 3.13](#).

If $\{\alpha_0, \alpha_1, \alpha_n\} \notin S(X_1)$ in this setting, the roots not supported in ω will be contained in two different connected components of $S(X_2)$ that are not $A_1 \times A_1$. Contradiction to [lemma 3.13](#).

Let us now assume that we are in $D_n^{(1)}$ and without loss of generality $\alpha_0, \alpha_1, \alpha_n \notin S(X_1)$:



If we chose $\omega = \bar{\alpha}_2 + \dots + \bar{\alpha}_{n-1}$, we had $\alpha_2, \alpha_{n-1} \notin S(X_2)$ by [lemma 3.12](#), and hence, the roots not supported in ω are in three connected components of $S(X_2)$. Contradiction to [lemma 3.13](#). If we chose $[\frac{1}{2}]\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4$ in $D_5^{(1)}$, we had $S(X_2)$ of type $A_3 \times A_1 \times A_1$, where the missing roots are distributed over one A_1 and A_3 , so there is no admissible ω^\sharp via [lemma 3.13](#).

This finishes the proof. □

Let us now settle the remaining cases. The following lemma is more general than needed to prove [theorem 3.10](#) above, but we shall profit later on.

Lemma 3.16. *Let Φ be an affine root system of type $E_i^{(1)}$, $i \in \{6, 7, 8\}$. Then there are no bi-homogeneous manifolds of rank one.*

Proof. We remember that for a bi-homogeneous polytope generated by a weight ω , we have $\omega + (\omega^\sharp) = t \cdot \delta$ for some coefficient $t \in \mathbb{R}^\times$. The highest coefficient of any simple root that would be possible by adding two spherical roots of subsystems of $E_n^{(1)}$ is 4. As coefficients of simple roots in spherical roots are $\frac{1}{2}, 1, 2, 3$, the parameter t must be chosen such that all coefficients of $t\delta$ are in $\mathbb{N}[\frac{1}{2}]$ and ≤ 4 .

Definition 3.17. Let $k(\bar{\alpha}_i)$ be the coefficient of $\bar{\alpha}_i$ in δ . Let ω be the weight of a homogeneous variety of rank one. Then $k_\omega(\bar{\alpha}_i)$ denotes the coefficient of $\bar{\alpha}_i$ in ω , that is $\omega = \sum k_\omega(\bar{\alpha}_i) \cdot \bar{\alpha}_i$.

Note in particular that the coefficient of $\bar{\alpha}$ in the support of ω must not be bigger than the corresponding coefficient of this root in $t\delta$. If it were, we could subtract $t\delta$ and $\omega - t\delta$ were not dominant. We do not consider such ω . We also do not have to consider any ω where $t\delta - \omega$ supported all simple roots.

- $E_6^{(1)}$. The coefficients of simple roots in δ are 1, 2, 3. That means $t \in \{\frac{1}{2}, 1\}$.
 - $t = \frac{1}{2}$. Then $k(\bar{\alpha}_4) = \frac{3}{2}$. As $E_6^{(1)}$ does not contain a subsystem of type B_3 , it follows that w.l.o.g. $k_\omega(\bar{\alpha}_4) = 1$ and $k_{\omega^\sharp}(\bar{\alpha}_4) = \frac{1}{2}$. That means that α_4 must be contained in $S(X_1)$ and $S(X_2)$, hence α_4 is in S^p by [lemma 3.12](#). The remaining possibilities for ω are the following (up to diagram automorphisms):

- * $\omega = \bar{\alpha}_1 + 2\bar{\alpha}_3 + \bar{\alpha}_4$.
- * $\omega = \bar{\alpha}_3 + \bar{\alpha}_4 + \bar{\alpha}_5$.
- * $\omega = \frac{1}{2}(\bar{\alpha}_3 + \bar{\alpha}_5 + 2\bar{\alpha}_4 + 2\bar{\alpha}_2)$

All are in contradiction to [lemma 3.13](#)

- $t = 1$. Then w.l.o.g. $k_\omega(\bar{\alpha}_4) = 2$ and $k_{\omega^\sharp}(\bar{\alpha}_4) = 1$. This again forces $\alpha_4 \in S^p$. The only possibility for ω is $\omega = \bar{\alpha}_3 + \bar{\alpha}_5 + 2\bar{\alpha}_4 + 2\bar{\alpha}_2$ is in contradiction to [lemma 3.13](#). So there are no bi-homogeneous polytopes for $E_6^{(1)}$.
- $E_7^{(1)}$ Here, $k(\bar{\alpha}_i) \in \{1, 2, 3, 4\}$ and hence, $t \in \{\frac{1}{2}, 1\}$ again.
 - Let $t = \frac{1}{2}$. Then $k(\alpha_4) = 2$. So either $\bar{\alpha}_4$ is in the support of both ω and ω^\sharp (which puts $\alpha_4 \in S^p$) or $2\bar{\alpha}_4$ is in ω and $\bar{\alpha}_4$ not in ω^\sharp , hence $\alpha_4 \notin S^p$ in this case. Let us start with $\bar{\alpha}_4 \in \omega, \alpha_4 \in S^p$
 - * $\omega = \bar{\alpha}_{2,4}$. Then $\omega^\sharp = \frac{1}{2}\bar{\alpha}_0 + \bar{\alpha}_1 + \frac{3}{2}\bar{\alpha}_3 + \frac{1}{2}\bar{\alpha}_4 + \frac{1}{2}\bar{\alpha}_5 + \bar{\alpha}_6 + \frac{1}{2}\bar{\alpha}_7$. This is impossible as this is not a spherical root.
 - * $\omega = \bar{\alpha}_1 + \bar{\alpha}_3 + \bar{\alpha}_4 + \bar{\alpha}_5$. This opposes [lemma 3.13](#)
 - * $\omega = \bar{\alpha}_2 + \bar{\alpha}_4 + \bar{\alpha}_3 + \bar{\alpha}_1$. Then ω^\sharp were $\frac{1}{2}\bar{\alpha}_0 + \frac{1}{2}\bar{\alpha}_3 + \frac{1}{2}\bar{\alpha}_4 + \frac{1}{2}\bar{\alpha}_5 + \bar{\alpha}_6 + \frac{1}{2}\bar{\alpha}_7$ which is not a spherical root.
 - * $\omega = \bar{\alpha}_1 + \bar{\alpha}_{3,6}$. This opposes [lemma 3.13](#)
 - * $\omega = \frac{1}{2}(2\bar{\alpha}_2 + 2\bar{\alpha}_4 + \bar{\alpha}_3 + \bar{\alpha}_5)$. Then ω^\sharp has to contain all simple roots, but there is no local root system that does so.
 - * $\omega = \frac{1}{2}(2\bar{\alpha}_3 + 2\bar{\alpha}_4 + \bar{\alpha}_2 + \bar{\alpha}_5)$. Then ω^\sharp were $\frac{1}{2}\bar{\alpha}_0 + \bar{\alpha}_1 + \bar{\alpha}_4 + \bar{\alpha}_5 + \bar{\alpha}_6 + \frac{1}{2}\bar{\alpha}_7$ which is not a spherical root.
 - * $\omega = \frac{1}{2}(2\bar{\alpha}_1 + \bar{\alpha}_3 + 2\bar{\alpha}_4 + \bar{\alpha}_2 + \bar{\alpha}_5)$. Then ω^\sharp were $\frac{1}{2}\bar{\alpha}_0 + \frac{1}{2}\bar{\alpha}_3 + \frac{1}{2}\bar{\alpha}_4 + \bar{\alpha}_5 + \bar{\alpha}_6 + \frac{1}{2}\bar{\alpha}_7$ which is not a spherical root.
 - Now let us look at cases with $k_\omega(\bar{\alpha}_4) = 2$ and $\alpha_4 \notin S^p$
 - * $\omega = 2\bar{\alpha}_4$. Opposed by [lemma 3.13](#)
 - * $\omega = \bar{\alpha}_3 + 2\bar{\alpha}_4 + \bar{\alpha}_5$. Opposed by [lemma 3.13](#)

- * $\omega = \bar{\alpha}_2 + 2\bar{\alpha}_4 + \bar{\alpha}_5$. Then ω^\sharp would be $\frac{1}{2}\bar{\alpha}_0 + \bar{\alpha}_1 + \frac{1}{2}\bar{\alpha}_3 + \frac{3}{2}\bar{\alpha}_5 + \bar{\alpha}_6 + \frac{1}{2}\bar{\alpha}_7$ which is not a spherical root.
- Let $t = 1$. Then $k_\omega(\bar{\alpha}_4) = k_{\omega^\sharp}(\bar{\alpha}_4) = 2$, so $\alpha_4 \in S^p$.
 - * $\omega = 2\bar{\alpha}_5 + 2\bar{\alpha}_4 + \bar{\alpha}_2 + \bar{\alpha}_3$. Then ω^\sharp contained all simple roots, which is impossible.
 - * $\omega = 2\bar{\alpha}_6 + 2\bar{\alpha}_5 + 2\bar{\alpha}_4 + \bar{\alpha}_2 + \bar{\alpha}_3$. Then ω^\sharp was $\bar{\alpha}_0 + 2\bar{\alpha}_1 + 2\bar{\alpha}_3 + \bar{\alpha}_2 + 2\bar{\alpha}_4 + \bar{\alpha}_5 + \bar{\alpha}_7$, but this is not a spherical root.
 - * $\omega = 2\bar{\alpha}_2 + 2\bar{\alpha}_4 + \bar{\alpha}_3 + \bar{\alpha}_5$. Then $\omega^\sharp = \bar{\alpha}_0 + 2\bar{\alpha}_1 + 2\bar{\alpha}_3 + 2\bar{\alpha}_4 + 2\bar{\alpha}_5 + 2\bar{\alpha}_6 + \bar{\alpha}_7$ is not a spherical root.

So there are no bi-homogeneous polytopes for $E_7^{(1)}$.

- $E_8^{(1)}$ only allows $t = \frac{1}{2}$. Then $k(\bar{\alpha}_4) = 3$ and w.l.o.g. $k_\omega(\bar{\alpha}_4) = 2$, $k_{\omega^\sharp}(\bar{\alpha}_4) = 1$ which again puts $\alpha_4 \in S^p$. All possible (no coefficient bigger than a Dynkin-label) ω with $k_\omega(\bar{\alpha}_4) = 2$ are:

- $\omega = 2\bar{\alpha}_3 + 2\bar{\alpha}_4 + \bar{\alpha}_2 + \bar{\alpha}_5$. Then ω^\sharp was $\frac{1}{2}\bar{\alpha}_2 + \bar{\alpha}_4 + \frac{3}{2}\bar{\alpha}_5 + 2\bar{\alpha}_6 + \frac{3}{2}\bar{\alpha}_7 + \bar{\alpha}_8 + \frac{1}{2}\bar{\alpha}_0$
- $\omega = 2\bar{\alpha}_5 + 2\bar{\alpha}_4 + \bar{\alpha}_2 + \bar{\alpha}_3$. Then ω^\sharp was $\frac{1}{2}\bar{\alpha}_2 + \bar{\alpha}_4 + \frac{1}{2}\bar{\alpha}_5 + 2\bar{\alpha}_6 + \frac{3}{2}\bar{\alpha}_7 + \bar{\alpha}_8 + \frac{1}{2}\bar{\alpha}_0$
- $\omega = 2\bar{\alpha}_6 + 2\bar{\alpha}_5 + 2\bar{\alpha}_4 + \bar{\alpha}_2 + \bar{\alpha}_3$. Then ω^\sharp was $\frac{1}{2}\bar{\alpha}_2 + \bar{\alpha}_4 + \frac{1}{2}\bar{\alpha}_5 + \frac{3}{2}\bar{\alpha}_7 + \bar{\alpha}_8 + \frac{1}{2}\bar{\alpha}_0$

In none of these cases, ω^\sharp is a spherical root. So there are no bi-homogeneous polytopes for $E_8^{(1)}$.

This proves the lemma. □

Finally, it remains to prove the theorem for $A_n^{(1)}$:

Lemma 3.18. *We consider an affine root system of type $A_n^{(1)}$. For all bi-homogeneous moment polytopes of rank one, we have $|S|(X_i)| \geq n - 1$, $i = 1, 2$.*

Proof. We distinguish the following cases:

- (1) Assume there are three missing roots that are neighbors, without loss of generality we assume $\alpha_k, \alpha_{k+1}, \alpha_{k+2} \notin S(X_1)$. But then $\alpha_{k+1} \in S^p(\omega)$, $\alpha_{k+1}(X_1) > 0$. Hence $\alpha_{k+1} \notin S(X_2)$ by [lemma 3.12](#).
- (2) Assume that three roots are missing, two of the missing roots α_k, α_{k+1} are neighbors, and a third one, α_l is not α_{k-1} or α_{k+2} . Then we have to consider:
 - ω has $\bar{\alpha}_{k+2}, \dots, \bar{\alpha}_{l-1}$ in its support. But then $\alpha_k \notin S(X_2)$ because $\alpha_k(X_1) \neq 0$, $\alpha_k \in S^p$ and [lemma 3.12](#).
 - ω has $\bar{\alpha}_{l+1}, \dots, \bar{\alpha}_{k-1}$ in its support. But then $\alpha_{k+1} \notin S(X_2)$ because $\alpha_{k+1}(X_1) \neq 0$, $\alpha_{k+1} \in S^p$ and [lemma 3.12](#).
- (3) Assume no pair of the three missing roots are neighbors (and hence $n \geq 5$), we call them $\alpha_k, \alpha_l, \alpha_m$ clockwise. We assume ω has $\bar{\alpha}_{k+1}, \dots, \bar{\alpha}_{l-1}$ in its support. But then $\alpha_m \notin S(X_2)$ because $\alpha_l(X_1) \neq 0$, $\alpha_l \in S^p$ and [lemma 3.12](#). If the three roots $\notin S(X_1)$ are w.l.o.g. $\alpha_0, \alpha_2, \alpha_4$ and $n \geq 5$, then the roots not in the support of ω are in two components of support $A_1 \times A_{\geq 3}$ in $S(X_2)$, and there is no spherical root with this support via [list 2.5](#).

□

This finishes the proof of [theorem 3.10](#).

6. STRUCTURE OF BI-INHOMOGENEOUS AND MIXED POLYTOPES

Fortunately we can find an even stronger necessary condition for the cases of bi-inhomogeneous and mixed polytopes.

We want to prove the following theorem which is an inhomogeneous analogon to lemma 3.12:

Theorem 3.19 (Structure of inhomogeneous polytopes). *Let $\omega \in I_k(X_1)$, this means it is an inhomogeneous local model for a genuine quasi-Hamiltonian manifold of rank one for $X_1 \in \mathcal{A}$ with $\langle \omega, \bar{\alpha}_k^\vee \rangle = 1$. Then:*

The local root system $S(X_2) := \Phi(X_1 + c\omega)$ is

$$\Phi(X_2) = S \setminus \{\alpha_k\}$$

Proof. After renumbering we can assume that $S(X_1) = \{\alpha_0, \dots, \alpha_p\}$ for some $p < n$. We know from lemma 3.3 that exactly one $\langle \omega, \bar{\alpha}_k^\vee \rangle > 0$, we can assume $k = p$. So, $\langle \omega, \bar{\alpha}_i^\vee \rangle = 0$ for $i = 0, \dots, p-1$ and $\langle \omega, \bar{\alpha}_p^\vee \rangle > 0$. We consider now the point $X_2 = X_1 + c\omega$ for some positive real number c , the second point where \mathcal{P} intersects the walls of the alcove. We have by eq. (1):

$$\alpha_i(X_2) = \alpha_i(X_1) + c \cdot \langle \omega, \bar{\alpha}_i^\vee \rangle \quad \forall i$$

By construction, $\alpha_i(X_1) = 0$ for $i \in \{0, \dots, p\}$. But $\langle \omega, \bar{\alpha}_i^\vee \rangle$ is also zero for $i = 0, \dots, p-1$. Furthermore, $\langle \omega, \bar{\alpha}_p^\vee \rangle > 0$. It follows:

$$(2) \quad \alpha_0, \dots, \alpha_{p-1} \in S(X_2), \alpha_p \notin S(X_2)$$

□

Geometrically seen, this theorem states the following: If we start at $X_1 \in \mathcal{A}$ with local root system $S(X_1)$ and find an ω that is an inhomogeneous local model here, then X_2 will be a vertex of the fundamental alcove.

This theorem has some remarkable consequences that make the finding of bi-inhomogeneous and mixed momentum polytopes very manageable:

Corollary 3.20 (bi-inhomogeneous polytopes). *Bi-inhomogeneous polytopes are only possible if $S(X_1) = S \setminus \{\alpha_j\}$ and $S(X_2) := S \setminus \{\alpha_k\}$, in particular, each contain n roots. If $\omega \in I_k(j)$, meaning $\langle \omega, \bar{\alpha}_k^\vee \rangle = 1$ for $\alpha_k \in S(X_1)$, then $S(X_2) = S \setminus \{\alpha_k\}$ and there we must have $-\omega \in I_j(k)$, and vice versa.*

Proof. This is an immediate consequence of theorem 3.19. Starting in X_1 with an inhomogeneous ω , X_2 is n -elemental (with n is the rank of K) and by the proof of theorem 3.19, α_k is not in $\Phi(X_2)$. As ω takes the value zero on all coroots of $S(X_2)$ but $\bar{\alpha}_j^\vee$, a bi-inhomogeneous polytope has only this one chance to take value 1 there. As the problem is symmetric in swapping X_1 and X_2 , we have shown everything. □

Remark 3.21. The geometric meaning is: if $\mathcal{P} = [X_1 X_2]$ is a bi-inhomogeneous momentum polytope, then X_1 and X_2 are vertices of \mathcal{A} .

Lemma 3.22. *Let \mathcal{P} be a bi-inhomogeneous moment polytope of rank one for a genuine quasi-Hamiltonian manifold.*

Then: ω is completely determined by the root α_k with $\langle \omega, \bar{\alpha}_k^\vee \rangle = 1$ and the one simple root not contained in the local root system $S(X_1)$.

Remark 3.23. That means: instead of writing $\omega \in I_k(j)$, we can and will write $\omega = I_k(j)$.

Proof. We use the same notation as in the proof of [theorem 3.19](#). Recall that, by using the relations of coordinates given by $\alpha_0, \dots, \alpha_p$, we defined $p + 1$ coordinates of X_1 , $n - p - 1$ coordinates of X_1 are still free. So we have $n - p - 1 + 1 = n - p$ free coordinates in X_2 , because by going from X_1 to $X_2 = X_1 + c\omega$, we get one additional parameter c . We have $n + 1 - p + 1 = n - p$ equations to solve, coming from the roots $\alpha_{p+1}, \dots, \alpha_n$. So, in general, we face an (in general inhomogeneous) system of $n - p$ equations in $n - p$ unknowns. Because the simple roots are linearly independent, this system has full rank and it is well known that it is always solvable with exactly one solution. \square

The fact $|S(X_1)| = |S(X_2)| = n$ leads to a strong tool for finding bi-inhomogeneous polytopes: It is necessary that the root system Φ has two n -elemental subsets that allow the existence of inhomogeneous solutions, meaning that there are components supporting n roots and have one connected component of type A or C . We also know from [corollary 3.20](#) which roots have to be or cannot be in the one or the other local root system. That reduces dramatically the number of cases that still have to be checked.

We also find a necessary condition for mixed polytopes:

Corollary 3.24 (mixed momentum polytopes). *Mixed momentum polytopes (homogeneous in X_1 , inhomogeneous in X_2) can only exist if $|S(X_1)| = n$.*

Proof. Let ω be an inhomogeneous local model in X_2 . Then $|S(X_1)| = n$ by [theorem 3.19](#). \square

7. EXPLICIT DETERMINATION OF MOMENT POLYTOPES OF RANK ONE

We first sum up the last three sections:

Theorem 3.25. *For every bi-homogeneous polytope $[X_1X_2]$ of any compact genuine quasi-Hamiltonian $K\tau$ -manifold of rank one, $S(X_1)$ and $S(X_2)$ contain at least $n-1$ roots (n denotes the rank of K). For every bi-inhomogeneous momentum polytope, $|S(X_1)| = |S(X_2)| = n$. For every mixed momentum polytope with homogeneous local model in X_1 , the local root system $S(X_1)$ contains n roots.*

Proof. This is the combination of [theorem 3.10](#), [corollary 3.20](#) and [corollary 3.24](#). \square

Which local models are candidates in $X \in \mathcal{A}$ is determined by the local root system and the known classification of smooth affine spherical varieties, we discussed this in detail in [section 4](#).

Remember that we write $\omega^\sharp = t\delta - \omega$ for some positive real parameter t .

We shall establish a short notation for going through our algorithm. We will use the following table:

$$\left| S(X_1) \right| \text{ choice of } \omega \left| S(X_2) \right| \omega^\sharp \text{ hom.?} \left| -w \text{ inhom.?} \right|$$

The first column contains the subsystem we start with as a local root system in X_1 . Then we go through all possible choices for the local model in column two where we use the notations from [definition 3.5](#). The next column gives the local root system at X_2 which is easy to see via [lemma 3.12](#), [corollary 3.20](#) or elementary calculations.

Definition 3.26. We denote $C(j)$ for the connected component of the local root system that contains the root α_j . We use $\omega \in C(j)$ to express that ω is a the weight of a **homogeneous** variety on $C(j)$.

Example 3.27. Let $\Phi = C_3^{(1)}$ and $S(X) = S \setminus \{\alpha_2\}$. Then $C(0) = \{\alpha_0, \alpha_1\}$.

$$\begin{array}{c} \times \\ \longleftrightarrow \end{array}$$

Remember that the \times indicates that α_2 is not in $S(X)$.

In columns 4 and 5, we check if $\omega^\sharp = t\delta - \omega$ resp. $-\omega$ gives rise to a smooth affine spherical variety in X_2 . If not, we briefly sketch why.

Of course, a weight cannot correspond to a homogeneous and an inhomogeneous model at the same time, hence, if one is okay, the other one is not possible and we will leave the entry empty. As mixed polytopes show up as counterparts of homogeneous polytopes where the local root system corresponding to the homogeneous local model supports n simple roots, we will find all of them by looking at the counterparts of those. Hence we will not look for homogeneous ω^\sharp when we started with ω inhomogeneous or for an inhomogeneous ω^\sharp if $S(X_1)$ supports $n-1$ roots.

We will not explicitly mention cases that only differ by a diagram automorphism. We recall the short notation $\bar{\alpha}_{i,j}$ for $\bar{\alpha}_i + \bar{\alpha}_{i+1} + \dots + \bar{\alpha}_j$.

Let us recall quickly the tools we will use:

- lemma 3.2, the types of spherical varieties of rank one.
- lemma 3.3, the shapes of weights of inhomogeneous varieties
- lemma 3.12, the formula for $S(X_2)$
- lemma 3.13, $S(X_2)$ is connected or $A_1 \times A_1$ for bi-homogeneous polytopes.

7.1. $A_n^{(1)}$. Note that in $A_n^{(1)}$, all subsystems of n roots only differ by a diagram automorphism. We will choose $S \setminus \{\alpha_0\}$ as a representative for our examination.

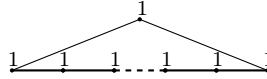
As this root system has the structure of a circle, all sums below have to be read “modulo $n + 1$ ”.

Let us start with $A_1^{(1)}$:



$S(X_1)$	choice of ω	$S(X_2)$	$\omega^\#$ hom.?	$-\omega$ inhom.?
α_1	I_1	α_0		$-\omega = I_0 \Rightarrow \mathbf{A5}$
α_1	$\bar{\alpha}_1$	α_0	$\omega^\# = \bar{\alpha}_0 \Rightarrow \mathbf{A4}$	
α_1	$2\bar{\alpha}_1$	α_0	$\omega^\# = 2\bar{\alpha}_0 \Rightarrow \mathbf{A1}$	

Let $n \geq 2$:



$S(X_1)$	choice of ω	$S(X_2)$	$\omega^\#$ hom.?	$-\omega$ inhom.?
$S \setminus \{\alpha_0\}$	$\bar{\alpha}_{1,n}$	$S \setminus \{\alpha_1, \alpha_n\}$	$\omega^\# = \bar{\alpha}_0 \Rightarrow \mathbf{A4}$	
	I_1	$S \setminus \{\alpha_1\}$		$-\omega = I_0 \Rightarrow \mathbf{A5}$
$S \setminus \{\alpha_0\}, n = 3$	$[\frac{1}{2}](\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3)$	$S \setminus \{\alpha_2\}$	$\omega^\# = [\frac{1}{2}]\bar{\alpha}_3 + 2\bar{\alpha}_0 + \bar{\alpha}_1 \Rightarrow \mathbf{A3}$	
$S \setminus \{\alpha_1, \alpha_n\}$	$2\bar{\alpha}_0$	$S \setminus \{\alpha_0\}$	no: $\omega^\# = 2\bar{\alpha}_{1,n}$, list 2.5	no: $\langle -\omega, \bar{\alpha}_1^\vee \rangle = \langle -\omega, \bar{\alpha}_n^\vee \rangle > 0$, lemma 3.3
$S \setminus \{\alpha_d, \alpha_e\}; d \leq e$	$\bar{\alpha}_{d+1, e-1}$	$S \setminus \{\alpha_{d+1}, \alpha_{e-1}\}$	$\omega^\# = \bar{\alpha}_{e,d} \Rightarrow \mathbf{A4}$	
$S \setminus \{\alpha_0, \alpha_4\}, n \geq 4$	$[\frac{1}{2}]\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3$	$S \setminus \{\alpha_2\}$	no: $\omega^\# = [\frac{1}{2}]2\bar{\alpha}_0 + \bar{\alpha}_1 + \bar{\alpha}_3 + 2\bar{\alpha}_4 + \dots + 2\bar{\alpha}_n$, list 2.5	no: $\langle -\omega, \bar{\alpha}_3^\vee \rangle = 0$, $\langle -\omega, \bar{\alpha}_4^\vee \rangle = [\frac{1}{2}]1$, lemma 3.3
$S \setminus \{\alpha_1, \alpha_3\}, n = 3$	$[\frac{1}{2}]\bar{\alpha}_1 + \bar{\alpha}_3$	$\{\alpha_2, \alpha_0\}$	$\omega^\# = [\frac{1}{2}](\bar{\alpha}_2 + \bar{\alpha}_0) \Rightarrow \mathbf{A2}$	
$S \setminus \{\alpha_1, \alpha_3\}, n > 3$	$[\frac{1}{2}]\bar{\alpha}_1 + \bar{\alpha}_3$	$S \setminus \{\alpha_1, \alpha_3\}$	no: $\omega^\# = [\frac{1}{2}](\bar{\alpha}_0 + \bar{\alpha}_{2,n})$, list 2.5	no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle > 0, \langle -\omega, \bar{\alpha}_3^\vee \rangle > 0$, lemma 3.3

7.2. $B_n^{(1)}$, $n \geq 3$.

$$\begin{array}{ccccccc} 1 & 2 & 2 & \cdots & 2 & 2 & 2 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-2} & \alpha_{n-1} & \alpha_n \\ & | & & & & & \\ & 1 & & & & & \\ & \alpha_0 & & & & & \end{array}$$

For the following reasons, there are no bi-inhomogeneous polytopes for $B_n^{(1)}$:

- $S(X_1) = S \setminus \{\alpha_0\}$. Then $S(X_1)$ is of type B , where no inhomogeneous models are possible.
- $S(X_1) = S \setminus \{\alpha_2\}$ for $n > 3$. Then $\omega = I_1$ is possible, but then, $S(X_2) = S \setminus \{\alpha_1\}$ is of type B .
- $S(X_1) = S \setminus \{\alpha_2\}$ for $n = 3$. Then we could choose $\omega = I_3$ and $S(X_2) = S \setminus \{\alpha_3\}$ is of type A . But $\langle -\omega, \bar{\alpha}_0^\vee \rangle = \langle -\omega, \bar{\alpha}_1^\vee \rangle = 0$, $\langle -\omega, \bar{\alpha}_2^\vee \rangle = 1$ is not possible by [lemma 3.3](#).
- $S(X_1) = S \setminus \{\alpha_d\}$, $2 < d < n - 2$. Then $S(X_1)$ is of type $D \times B$ and we don't find inhomogeneous models on any component.
- $S(X_1) = S \setminus \{\alpha_{n-2}\}$. Then $\omega = I_n$ is possible. But then $S(X_2)$ is of type D .
- $S(X_1) = S \setminus \{\alpha_{n-1}\}$ for $n > 3$. Then still $\omega = I_n$ is possible, but then $S(X_2)$ is of type D .
- $S(X_1) = S \setminus \{\alpha_n\}$. Then $S(X_1)$ is of type D .

As mixed polytopes are found while investigating models for $S(X_1)$ supporting n roots, we don't have to look at inhomogeneous models in $S(X_1)$.

$S(X_1)$	choice of ω	$S(X_2)$	ω^\sharp hom.?	$-\omega$ inhom.?
$S \setminus \{\alpha_0\}$	$[2]\bar{\alpha}_{1,n}$	$S \setminus \{\alpha_1\}$	$\omega^\sharp = [2]\bar{\alpha}_0 + \bar{\alpha}_{2,n} \Rightarrow \mathbf{B3}$	
$n = 3$	$\frac{1}{2}(\bar{\alpha}_1 + 2\bar{\alpha}_2 + 3\bar{\alpha}_3)$	$S \setminus \{\alpha_3\}$		$-\omega = I_0 \Rightarrow \mathbf{B1}$
$n = 3$	$\bar{\alpha}_1 + 2\bar{\alpha}_2 + 3\bar{\alpha}_3$	$S \setminus \{\alpha_3\}$	no: $\omega^\sharp = \frac{3}{2}\bar{\alpha}_0 + \frac{1}{2}\bar{\alpha}_1 + \bar{\alpha}_2$, list 2.5	no: $\langle -\omega, \bar{\alpha}_0^\vee \rangle = 2$, lemma 3.3
$S \setminus \{\alpha_0, \alpha_1\}$	$[2]\bar{\alpha}_{2,n}$	$S \setminus \{\alpha_2\}$	$\omega^\sharp = [2]\frac{1}{2}(\bar{\alpha}_0 + \bar{\alpha}_1) \Rightarrow \mathbf{B4}$	
$S \setminus \{\alpha_0, \alpha_1\}$ for $n = 4$	$[\frac{1}{2}]\bar{\alpha}_2 + 2\bar{\alpha}_3 + 3\bar{\alpha}_4$	$S \setminus \{\alpha_n\}$	no: $\omega^\sharp = [\frac{1}{2}](\frac{3}{2}\bar{\alpha}_0 + \frac{3}{2}\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3)$, list 2.5	no: type D , lemma 3.3
$S \setminus \{\alpha_0, \alpha_2\}$	$\omega \in C(1)$	never contains α_0 and α_2 , lemma 3.12		
$n = 3$	$[\frac{1}{2}]\bar{\alpha}_1 + \bar{\alpha}_3$	$\alpha_0 \notin S(X_2)$, lemma 3.12		
$S \setminus \{\alpha_0, \alpha_d\}$, $2 < d < n$	$\omega \in C(2)$	$S \setminus \{\alpha_1, \alpha_{d-1}\}$	no: $\omega^\sharp = 2\bar{\alpha}_0 + \bar{\alpha}_{2,d-1} + 2\bar{\alpha}_{d,n}$, list 2.5	no: corollary 3.24
	$\omega \in C(n)$	$\alpha_0 \notin S(X_2)$, lemma 3.12		
$S \setminus \{\alpha_0, \alpha_n\}$	$\omega \in C(2)$	elementary: never contains α_0 and α_n		
$S \setminus \{\alpha_{d+1}\}$, $1 \leq d+1 \leq n-1$	$[\frac{1}{2}]\bar{\alpha}_0 + \bar{\alpha}_1 + 2\bar{\alpha}_{2,d}$	$S \setminus \{\alpha_d\}$	$\omega^\sharp = [\frac{1}{2}]2\bar{\alpha}_{d+1,n} \Rightarrow \mathbf{B4}$	
$d = n - 3$	$[\frac{1}{2}]\bar{\alpha}_{n-2} + 2\bar{\alpha}_{n-1} + 3\bar{\alpha}_n$	$S \setminus \{\alpha_n\}$	no: $\omega^\sharp = [\frac{1}{2}]2\bar{\alpha}_0 + 2\bar{\alpha}_1 + 4\bar{\alpha}_{2,n-3} + 3\bar{\alpha}_{n-2} + 2\bar{\alpha}_{n1} + \bar{\alpha}_n$	no: Type D , lemma 3.3
$S \setminus \{\alpha_2\}$, $n = 3$	$\frac{1}{2}\bar{\alpha}_1 + \bar{\alpha}_3$	$S \setminus \{\alpha_1, \alpha_3\}$		$-\omega = I_2 \Rightarrow \mathbf{B2}$
	$\bar{\alpha}_1 + \bar{\alpha}_3$	$S \setminus \{\alpha_1, \alpha_3\}$	no: $\omega^\sharp = \bar{\alpha}_0 + 2\bar{\alpha}_2 + \bar{\alpha}_3$	no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle = 2$, lemma 3.3
$d = 2$	$[2]\bar{\alpha}_0$	$S \setminus \{\alpha_0\}$	no: $\omega^\sharp = \bar{\alpha}_1 + 2\bar{\alpha}_{2,n}$, list 2.5	no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle > 0$, lemma 3.3

$S(X_1)$	choice of ω	$S(X_2)$	$\omega^\#$ hom.?	$-w$ inhom.?
$S \setminus \{\alpha_d, \alpha_{d+1}\}$	$\omega \in C(0)$	$\alpha_{d+1} \notin S(X_2)$ lemma 3.12		
	$\omega \in C(n)$	$\alpha_d \notin S(X_2)$ lemma 3.12		
$S \setminus \{\alpha_d, \alpha_e\}$ $e \neq d+1$	$\omega \in C(0)$	$\alpha_e \notin S(X_2)$, lemma 3.12		
	$\omega \in C(d+1)$		no: lemma 3.13	no: corollary 3.24
	$\omega \in C(n)$	$\alpha_d \notin S(X_2)$, lemma 3.12		
$S \setminus \{a_d, \alpha_n\}$	$\omega \in C(0)$ or $C(0) + C(1)$	$\alpha_n \notin S(X_1)$, lemma 3.12		
	$\omega \in C(d+1)$		no, lemma 3.13	no, corollary 3.24
$S \setminus \{\alpha_n\}$	$[\frac{1}{2}]\bar{\alpha}_0 + \bar{\alpha}_1 + 2\bar{\alpha}_{2,n-1}$	$S \setminus \{a_{n-1}\}$	$\omega^\# = [2]\bar{\alpha}_n \Rightarrow B4$	
$n = 3$	$\bar{\alpha}_{0,2}$	$S \setminus \{\alpha_0, \alpha_2\}$	no: $\omega^\# = \bar{\alpha}_2 + 2\bar{\alpha}_3$, list 2.5	no: $\langle -\omega, \bar{\alpha}_3^\vee \rangle = 2$, lemma 3.3

7.3. $C_n^{(1)}, n \geq 2$.

$$\begin{array}{c} \xrightarrow{1} \xrightarrow{2} \dots \xrightarrow{2} \xrightarrow{1} \\ \alpha_0 \quad \alpha_1 \quad \quad \quad \alpha_{n-1} \quad \alpha_n \end{array}$$

$S(X_1)$	choice of ω	$S(X_2)$	$\omega^\#$ hom.?	$-w$ inhom.?
$S \setminus \{\alpha_0\}$	$\bar{\alpha}_1 + 2\bar{\alpha}_{2,n-1} + \bar{\alpha}_n$	$S \setminus \{\alpha_2\}$	$\omega^\# = \bar{\alpha}_0 + \bar{\alpha}_1 \Rightarrow C4$	
	I_1	$S \setminus \{\alpha_1\}$		$-\omega = I_0 \Rightarrow C5$
$n = 2$	$[2]\bar{\alpha}_1 + \bar{\alpha}_2$	$S \setminus \{\alpha_2\}$	$\omega^\# = [2]\bar{\alpha}_0 + \bar{\alpha}_1 \Rightarrow C2$	
$S \setminus \{\alpha_0, \alpha_1\}$	$\omega \in C(2)$	$\alpha_0 \notin S(X_1)$, lemma 3.12		
$S \setminus \{\alpha_0, \alpha_2\}$, $n > 2$	$[2]\bar{\alpha}_1$	$S \setminus \{\alpha_1\}$	no: lemma 3.13	no, $\langle -\omega, \bar{\alpha}_0^\vee \rangle = \langle -\omega, \bar{\alpha}_2^\vee \rangle = 1$, corollary 3.20
$S \setminus \{\alpha_0, \alpha_2\}$, $n = 2$	$[2]\bar{\alpha}_1$	$S \setminus \{\alpha_1\}$	$\omega^\# = [2]\frac{1}{2}(\bar{\alpha}_0 + \bar{\alpha}_2) \Rightarrow C1$	
$S \setminus \{\alpha_0, \alpha_2\}$, $n = 4$	$\bar{\alpha}_1 + \bar{\alpha}_3$	$S \setminus \{a_1, \alpha_3\}$	no: $\omega^\# = \bar{\alpha}_0 + \bar{\alpha}_1 + 2\bar{\alpha}_2$	no: corollary 3.24
$S \setminus \{\alpha_0, \alpha_d\}$, $2 < d < n$	$\omega \in C(1)$		no: lemma 3.13	no, corollary 3.24
	$\omega \in C(n)$	$\alpha_0 \notin S(X_1)$, lemma 3.12		
$S \setminus \{\alpha_0, \alpha_n\}$	$\bar{\alpha}_{1,n-1}$	$S \setminus \{\alpha_1, \alpha_{n-1}\}$	$\omega^\# = \frac{1}{2}(\bar{\alpha}_0 + \bar{\alpha}_n) \Rightarrow C3$	
$S \setminus \{\alpha_0, \alpha_n\}$, $n = 4$	$[\frac{1}{2}]\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3$	$S \setminus \{\alpha_2\}$	no: $\omega^\# = \bar{\alpha}_0 + \bar{\alpha}_1 + \bar{\alpha}_3 + \bar{\alpha}_4$, list 2.5	no: $\langle -\omega, \bar{\alpha}_0^\vee \rangle > 0, \langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$, lemma 3.3
$S \setminus \{\alpha_1\}$	$[2]\bar{\alpha}_0$	$S \setminus \{\alpha_1\}$	no: $\omega^\# = 2\bar{\alpha}_{1,n-1} + \bar{\alpha}_n$ list 2.5	no: $\langle -\omega, \bar{\alpha}_1^\vee \rangle = 2$, lemma 3.3

$S(X_1)$	choice of ω	$S(X_2)$	$\omega^\#$ hom.?	$-w$ inhom.?
$S \setminus \{\alpha_2\}$	$[\frac{1}{2}]\bar{\alpha}_0 + \bar{\alpha}_1$	$S \setminus \{\alpha_0, \alpha_1\}$	already seen: $\omega^\# = [\frac{1}{2}]\bar{\alpha}_4 + 2\bar{\alpha}_2 + \bar{\alpha}_3 \Rightarrow D2$	
	$[2]\bar{\alpha}_1$	$S \setminus \{\alpha_1\}$	no: $\omega^\# = [2]\bar{\alpha}_0 + 2\bar{\alpha}_2 + \bar{\alpha}_3 + \bar{\alpha}_4$, list 2.5	no: System of type D , lemma 3.3

Now let us consider $D_n^{(1)}$, $n > 4$

$S(X_1)$	choice of ω	$S(X_2)$	$\omega^\#$ hom.?	$-w$ inhom.?
$S \setminus \{\alpha_0\}$	$[\frac{1}{2}]2\bar{\alpha}_{1,n-2} + \bar{\alpha}_{n-1} + \bar{\alpha}_n$	$S \setminus \{\alpha_1\}$	$\omega^\# = [\frac{1}{2}]2\bar{\alpha}_0 + 2\bar{\alpha}_{2,n-2} + \bar{\alpha}_{n-1} + \bar{\alpha}_n \Rightarrow D1$	
$S \setminus \{\alpha_0, \alpha_1\}$	$[\frac{1}{2}]2\bar{\alpha}_{2,n-2} + \bar{\alpha}_{n-1} + \bar{\alpha}_n$	$S \setminus \{\alpha_2\}$	$\omega^\# = [\frac{1}{2}]\bar{\alpha}_0 + \bar{\alpha}_2 \Rightarrow D2$	
$S \setminus \{\alpha_0, \alpha_2\}$	$[2]\bar{\alpha}_1$	$\alpha_0 \notin S(X_2)$, lemma 3.12		
	$\omega \in C(n)$	$\alpha_0 \notin S(X_2)$, lemma 3.12		
$S \setminus \{\alpha_0, \alpha_d\}$, $2 < d < n-1$	$\bar{\alpha}_{1,d-1}$	$S \setminus \{\alpha_1, \alpha_{d-1}\}$	no: $\omega^\# = \bar{\alpha}_0 + 2\bar{\alpha}_{2,n-2} + \bar{\alpha}_{n-1} + \bar{\alpha}_n$, list 2.5	no: $\langle -\omega, \bar{\alpha}_0^\vee \rangle > 0$, $\langle -\omega, \bar{\alpha}_d^\vee \rangle > 0$, corollary 3.24
$d = 4$	$[\frac{1}{2}]\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3$	$S \setminus \{\alpha_2\}$	no: $S(X_2)$ neither connected nor $A_1 \times A_1$, lemma 3.13	no: $\langle -\omega, \bar{\alpha}_0^\vee \rangle > 0$, $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$, corollary 3.24
	$\omega \in C(n)$	$\alpha_0 \notin S(X_2)$ lemma 3.12		
$S \setminus \{\alpha_0, \alpha_n\}$	$\bar{\alpha}_{1,n-1}$	$S \setminus \{\alpha_1, \alpha_{n-1}\}$	$\omega^\# = \bar{\alpha}_0 + \bar{\alpha}_{2,n-2} + \bar{\alpha}_n \Rightarrow D3$	
$S \setminus \{\alpha_d\}$ with w.l.o.g $2 \leq d \leq n/2$	$[\frac{1}{2}]\bar{\alpha}_0 + \bar{\alpha}_1 + 2\bar{\alpha}_{2,d-1}$	$S \setminus \{\alpha_{d-1}\}$	$\omega^\# = [\frac{1}{2}]2\bar{\alpha}_{d,n-2} + \bar{\alpha}_{n-1} + \bar{\alpha}_n \Rightarrow D2$	
	$[\frac{1}{2}]2\bar{\alpha}_{d+1,n-2} + \bar{\alpha}_{n-1} + \bar{\alpha}_n$	$S \setminus \{\alpha_{d+1}\}$	$\omega^\# = [\frac{1}{2}]\bar{\alpha}_0 + \bar{\alpha}_1 + 2\bar{\alpha}_{2,d} \Rightarrow D3$	
$d = 2$	$[2]\bar{\alpha}_1$	$S \setminus \{\alpha_1\}$	no: $\omega^\# = [2]\bar{\alpha}_0 + 2\bar{\alpha}_{2,n-2} + \bar{\alpha}_{n-1} + \bar{\alpha}_n$, list 2.5	no: $\langle -\omega, \bar{\alpha}_3^\vee \rangle = 1$, lemma 3.3
$S \setminus \{\alpha_d, \alpha_e\}$ $1 < d, d+1 < e < n-1$	$\omega \in C(0)$	$\alpha_e \notin S(X_2)$		
	$\omega \in C(n)$	$\alpha_d \notin S(X_2)$ lemma 3.12		
	$\omega \in C(d+1)$		no: lemma 3.13	no: corollary 3.24
$S \setminus \{\alpha_d, \alpha_{d+1}\}$, $1 < d < n-2$	$\omega \in C(0)$	$\alpha_{d+1} \notin S(X_2)$, lemma 3.12		
	$\omega \in C(n)$	$\alpha_d \notin S(X_2)$, lemma 3.12		
$S \setminus \{\alpha_2, \alpha_n\}$	$\omega \in C(0)$ or $\omega \in C(0) + C(1)$	$\alpha_n \notin S(X_2)$		
	$\omega \in C(3)$		no: list 2.5	no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle > 0$, $\langle -\omega, \bar{\alpha}_n^\vee \rangle > 0$, corollary 3.24

7.5. $E_n^{(1)}$, $n \in \{6, 7, 8\}$. We saw in [lemma 3.16](#) that there are no bi-homogeneous polytopes for $E_n^{(1)}$. So it just remains to check for mixed and bi-inhomogeneous polytopes. That means we only have to check local root systems with n roots.

Let us start with $E_6^{(1)}$.

$$\begin{array}{cccccc} 1 & 2 & 3 & 2 & 1 & \\ \hline \alpha_0 & \alpha_2 & & \alpha_4 & \alpha_5 & \alpha_6 \\ & & 2 & & & \\ & & | & & & \\ & & \alpha_3 & & & \\ & & | & & & \\ & & 1 & & & \\ & & \alpha_1 & & & \end{array}$$

$S(X_1)$	choice of ω	$S(X_2)$	$\omega^\#$ hom.?	$-w$ inhom.?
$S \setminus \{\alpha_0\}$	nothing possible via list 2.5 and lemma 3.3			
$S \setminus \{\alpha_2\}$	$\bar{\alpha}_0$	$S \setminus \{\alpha_0\}$	done above	nothing possible lemma 3.3
	$\omega = I_0$	$S \setminus \{\alpha_0\}$		nothing possible lemma 3.3
	$\bar{\alpha}_1 + \bar{\alpha}_{3,6}$	$S \setminus \{\alpha_1, \alpha_6\}$	done above	not possible lemma 3.3
	I_1	$S \setminus \{\alpha_1\}$		nothing possible lemma 3.3
$S \setminus \{\alpha_4\}$	$\bar{\alpha}_1 + \bar{\alpha}_3$	$S \setminus \{\alpha_1, \alpha_3\}$	done above	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle = 1$, lemma 3.3
	I_1	$S \setminus \{\alpha_1\}$		nothing: lemma 3.3
	I_3	$S \setminus \{\alpha_3\}$		no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle = 1$, lemma 3.3

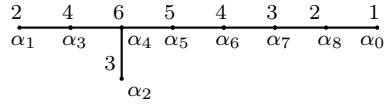
$E_7^{(1)}$:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 3 & 2 & 1 \\ \hline \alpha_0 & \alpha_1 & \alpha_3 & & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\ & & & & | & & & \\ & & & & \alpha_2 & & & \end{array}$$

$S(X_1)$	choice of ω	$S(X_2)$	$\omega^\#$ hom.?	$-w$ inhom.?
$S \setminus \{\alpha_0\}$	nothing possible via list 2.5 and lemma 3.3			
$S \setminus \{\alpha_1\}$	$[2]\bar{\alpha}_0$	$S \setminus \{\alpha_0\}$	done	nothing possible, lemma 3.3
	I_0	$S \setminus \{\alpha_0\}$	nothing: list 2.5	nothing: lemma 3.3
	$[\frac{1}{2}]\bar{\alpha}_3 + \bar{\alpha}_2 + 2\bar{\alpha}_{4,7}$	$S \setminus \{\alpha_7\}$	nothing: list 2.5	nothing: lemma 3.3
$S \setminus \{\alpha_2\}$	$\bar{\alpha}_0 + \bar{\alpha}_1 + \bar{\alpha}_{3,7}$	$S \setminus \{\alpha_0, \alpha_7\}$	nothing: list 2.5	nothing: lemma 3.3
	I_0	$S \setminus \{\alpha_0\}$	nothing: list 2.5	nothing: lemma 3.3
$S \setminus \{\alpha_3\}$	$\bar{\alpha}_{0,1}$	$S \setminus \{\alpha_0, \alpha_1\}$	done	nothing: lemma 3.3
	$\bar{\alpha}_2 + \bar{\alpha}_{4,7}$	$S \setminus \{\alpha_2, \alpha_7\}$	done	no: $\langle -\omega, \bar{\alpha}_3^\vee \rangle = 1$, lemma 3.3
	I_0	$S \setminus \{\alpha_0\}$		nothing: lemma 3.3
	I_1	$S \setminus \{\alpha_1\}$		$\langle -\omega, \bar{\alpha}_0^\vee \rangle = 0 \Rightarrow$ nothing: lemma 3.3
	I_2	$S \setminus \{\alpha_2\}$		no: $\langle -\omega, \bar{\alpha}_3^\vee \rangle > 0$, lemma 3.3
	I_7	$S \setminus \{\alpha_7\}$		nothing: lemma 3.3
$S \setminus \{\alpha_4\}$	$\bar{\alpha}_0 + \bar{\alpha}_1 + \bar{\alpha}_3$	$S \setminus \{\alpha_0, \alpha_3\}$	done	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle = 1$, lemma 3.3

$S(X_1)$	choice of ω	$S(X_2)$	$\omega^\#$ hom.?	$-\omega$ inhom.?
	$[\frac{1}{2}]\bar{\alpha}_0 + 2\bar{\alpha}_1 + \bar{\alpha}_3$	$S \setminus \{\alpha_1\}$	done	no: $\langle -\omega, \bar{\alpha}_0^\vee \rangle = \langle -\omega, \bar{\alpha}_3^\vee \rangle = 0, \langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$, lemma 3.3
	$[2]\bar{\alpha}_2$	$S \setminus \{\alpha_2\}$	done	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$, lemma 3.3
	I_0	$S \setminus \{\alpha_0\}$		nothing possible lemma 3.3
	I_3	$S \setminus \{\alpha_3\}$		no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle = 0, \langle -\omega, \bar{\alpha}_4^\vee \rangle = 0$ lemma 3.3
	I_2	$S \setminus \{\alpha_2\}$		no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle = 1$, lemma 3.3

$E_8^{(1)}$:



$S(X_1)$	choice of ω	$S(X_2)$	$\omega^\#$ hom.?	$-\omega$ inhom.?
$S \setminus \{\alpha_0\}$	nothing possible via list 2.5 and lemma 3.3			
$S \setminus \{\alpha_1\}$	$[\frac{1}{2}]\bar{\alpha}_2 + \bar{\alpha}_3 + 2\bar{\alpha}_{4,8} + 2\bar{\alpha}_0$	$S \setminus \{\alpha_0\}$	done	nothing, lemma 3.3
$S \setminus \{\alpha_2\}$	$\bar{\alpha}_1 + \bar{\alpha}_3 + \bar{\alpha}_{4,8} + \bar{\alpha}_0$	$S \setminus \{\alpha_1, \alpha_0\}$	done	nothing, lemma 3.3
	I_1	$S \setminus \{\alpha_1\}$		nothing: lemma 3.3
	I_0	$S \setminus \{\alpha_0\}$		nothing: lemma 3.3
$S \setminus \{\alpha_3\}$	$[2]\bar{\alpha}_1$	$S \setminus \{\alpha_1\}$	done	nothing: lemma 3.3
	$\bar{\alpha}_2 + \bar{\alpha}_{4,8} + \bar{\alpha}_0$	$S \setminus \{\alpha_2, \alpha_0\}$	done	no: $\langle -\omega, \bar{\alpha}_1^\vee \rangle = \langle -\omega, \bar{\alpha}_8^\vee \rangle = 0$, lemma 3.3
	I_1	$S \setminus \{\alpha_1\}$		nothing: lemma 3.3
	I_2	$S \setminus \{\alpha_2\}$		no: $\langle -\omega, \bar{\alpha}_3^\vee \rangle > 0$, lemma 3.3
	I_0	$S \setminus \{\alpha_0\}$		nothing: lemma 3.3
$S \setminus \{\alpha_4\}$	$\bar{\alpha}_1 + \bar{\alpha}_3$	$S \setminus \{\alpha_1, \alpha_3\}$	done	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$, lemma 3.3
	I_1	$S \setminus \{\alpha_1\}$		no: lemma 3.3
	I_3	$S \setminus \{\alpha_3\}$		no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$, lemma 3.3
	$[2]\alpha_2$	$S \setminus \{\alpha_2\}$	done	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$, lemma 3.3
	I_2	$S \setminus \{\alpha_2\}$		no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$, lemma 3.3
	$\bar{\alpha}_{5,8} + \bar{\alpha}_0$	$S \setminus \{\alpha_5, \alpha_0\}$	done	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$, lemma 3.3
	I_5	$S \setminus \{\alpha_5\}$		no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$, lemma 3.3
	I_0	$S \setminus \{\alpha_0\}$		nothing possible: lemma 3.3
$S \setminus \{\alpha_5\}$	$\bar{\alpha}_{1,4}$	$S \setminus \{\alpha_1, \alpha_2\}$	done	no: $\langle -\omega, \bar{\alpha}_5^\vee \rangle > 0$, lemma 3.3
	I_1	$S \setminus \{\alpha_1\}$		nothing: lemma 3.3
	I_2	$S \setminus \{\alpha_2\}$		no: $\langle -\omega, \bar{\alpha}_5^\vee \rangle > 0$, lemma 3.3
	$\bar{\alpha}_{6,8} + \bar{\alpha}_0$	$S \setminus \{\alpha_6, \alpha_0\}$	done	nothing lemma 3.3
	I_6	$S \setminus \{\alpha_6\}$		nothing lemma 3.3
	I_0	$S \setminus \{\alpha_0\}$		nothing lemma 3.3
$S \setminus \{\alpha_6\}$	$2(\bar{\alpha}_1 + \bar{\alpha}_{3,4}) + \bar{\alpha}_2 + \bar{\alpha}_5$	$S \setminus \{\alpha_1\}$	done	nothing: lemma 3.3

$S(X_1)$	choice of ω	$S(X_2)$	ω^\sharp hom.?	$-w$ inhom.?
	$\omega \in C(8)$		done	nothing: lemma 3.3
	I_7	$S \setminus \{\alpha_7\}$		nothing: lemma 3.3
	I_0	$S \setminus \{\alpha_0\}$		nothing: lemma 3.3
$S \setminus \{\alpha_7\}$	$\bar{\alpha}_8 + \bar{\alpha}_0$	$S \setminus \{\alpha_8, \alpha_0\}$	done	nothing lemma 3.3
	I_8	$S \setminus \{\alpha_8\}$		no: $\langle -\omega, \bar{\alpha}_7^\vee \rangle > 0$, lemma 3.3
	I_0	$S \setminus \{\alpha_0\}$		nothing: lemma 3.3
$S \setminus \{\alpha_8\}$	$[2]\bar{\alpha}_0$ or I_0	$S \setminus \{\alpha_0\}$	nothing: list 2.5	nothing: lemma 3.3

We have shown that there are no momentum polytopes of rank one for $E_n^{(1)}$.

7.6. $F_4^{(1)}$.

$$\begin{array}{ccccccc} & 1 & 2 & 3 & 4 & 2 & \\ & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \end{array}$$

Let us start to determine the bi-homogeneous polytopes. The Dynkin labels give a slightly shorter way to do that: Recall that $\omega^\sharp = t\delta - \omega$ for some real number t , we discussed the conditions for that parameter t in section 5. We have $\delta = \bar{\alpha}_0 + 2\bar{\alpha}_1 + 3\bar{\alpha}_2 + 4\bar{\alpha}_3 + 2\bar{\alpha}_4$ which leads to $t \in \{\frac{1}{2}, 1\}$.

Let us assume $t = 1$ first. Then there are two cases:

- (1) Assume $k_\omega(\bar{\alpha}_3) = 3$ (where $k_w(\bar{\alpha}_3)$ denotes the coefficient of $\bar{\alpha}_3$ in ω) and $\alpha_3 \in S^p$:
 - (a) Only $\omega = \bar{\alpha}_1 + 2\bar{\alpha}_2 + 3\bar{\alpha}_3 + 2\bar{\alpha}_4$ is possible. Then $\omega^\sharp = \bar{\alpha}_{1,3}$ which is an admissible spherical root. This corresponds to the local root systems $S(X_1) = S \setminus \{\alpha_0\}, S(X_2) = S \setminus \{\alpha_4\} \Rightarrow \mathbf{F1}$
- (2) $k_\omega(\bar{\alpha}_3) = 2$ and $\alpha_3 \in S^p$.
 - (a) $\omega = 2\bar{\alpha}_{1,3}$. This leads to $\omega^\sharp = \bar{\alpha}_0 + \bar{\alpha}_2 + 2\bar{\alpha}_3 + 2\bar{\alpha}_4$ which is not possible via list 2.5
 - (b) $\omega = 2\bar{\alpha}_{2,3}$, but then ω^\sharp supported all simple roots, which is impossible.

So now suppose $t = \frac{1}{2}$

- (1) $k_\omega(\bar{\alpha}_3) = 2, \alpha_3 \notin S^p$
 - (a) $\omega = 2\bar{\alpha}_3$. Then ω^\sharp opposes lemma 3.13
 - (b) $\omega = \bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4$. Then $-w = \frac{1}{2}\bar{\alpha}_0 + \bar{\alpha}_1 + \frac{1}{2}\bar{\alpha}_2$ which is a spherical root and admissible. This corresponds to $S(X_1) = S \setminus \{\alpha_1\}, S(X_2) = S \setminus \{\alpha_3\} \Rightarrow \mathbf{F2}$.
- (2) $k_\omega(\bar{\alpha}_3) = \frac{3}{2}, \alpha_3 \in S^p$. Not possible
- (3) $k_\omega(\bar{\alpha}_3) = 1, \alpha_3 \in S^p$
 - (a) $\omega = \bar{\alpha}_{1,3}$. Then $\omega^\sharp = \frac{1}{2}\bar{\alpha}_0 + \frac{1}{2}\bar{\alpha}_2 + \bar{\alpha}_3 + \bar{\alpha}_4$ which is impossible by list 2.5.
 - (b) $\omega = \bar{\alpha}_{2,3}$. Then ω^\sharp supported all simple roots, which is impossible.
 - (c) $\omega = \bar{\alpha}_1 + \bar{\alpha}_3$, but this opposes lemma 3.13

Now let us go on with finding mixed and bi-inhomogeneous polytopes.

$S(X_1)$	choice of ω	$S(X_2)$	$-\omega$ hom.?	$-\omega$ inhom.?
$S \setminus \{\alpha_0\}$	$\bar{\alpha}_1 + 2\bar{\alpha}_2 + 3\bar{\alpha}_3 + 2\bar{\alpha}_4$	$S \setminus \{\alpha_4\}$	already seen \Rightarrow F1	
$S \setminus \{\alpha_1\}$	$[2]\bar{\alpha}_0$	$S \setminus \{\alpha_0\}$		nothing: lemma 3.3
	I_0	$S \setminus \{\alpha_0\}$		nothing: lemma 3.3
	$\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4$	$S \setminus \{\alpha_3\}$	already seen \Rightarrow F2	
	I_4	$S \setminus \{\alpha_4\}$		nothing: lemma 3.3
$S \setminus \{\alpha_2\}$	$\bar{\alpha}_0 + \bar{\alpha}_1$	$S \setminus \{\alpha_0, \alpha_1\}$	done	no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle > 0$, lemma 3.3
	I_0	$S \setminus \{\alpha_0\}$		no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle > 0$, lemma 3.3
	I_1	$S \setminus \{\alpha_1\}$		no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle > 0$, lemma 3.3
	$\bar{\alpha}_3 + \bar{\alpha}_4$	$S \setminus \{\alpha_3, \alpha_4\}$	done	$-\omega = I_2 \Rightarrow$ F3
	I_3	$S \setminus \{\alpha_3\}$		no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle = \frac{2}{3}$, lemma 3.3
	I_4	$S \setminus \{\alpha_4\}$		nothing: lemma 3.3
$S \setminus \{\alpha_3\}$	$\bar{\alpha}_{0,2}$	$S \setminus \{\alpha_0, \alpha_2\}$	done	no: $\langle -\omega, \bar{\alpha}_3^\vee \rangle = 2$, lemma 3.3
	$\frac{1}{2}\bar{\alpha}_0 + 2\bar{\alpha}_1 + \bar{\alpha}_2$	$S \setminus \{\alpha_1\}$	done	no: $\langle -\omega, \bar{\alpha}_3^\vee \rangle > 0$, lemma 3.3
	I_0	$S \setminus \{\alpha_0\}$		no: $\langle -\omega, \bar{\alpha}_3^\vee \rangle > 0$, lemma 3.3
	I_2	$S \setminus \{\alpha_2\}$		no: $\langle -\omega, \bar{\alpha}_3^\vee \rangle = \frac{3}{2}$, lemma 3.3
	$[2]\bar{\alpha}_4$	$S \setminus \{\alpha_4\}$	done	nothing: lemma 3.3
	I_4	$S \setminus \{\alpha_4\}$	done	nothing: lemma 3.3
$S \setminus \{\alpha_4\}$	$[2]\bar{\alpha}_{0,3}$	$S \setminus \{\alpha_0\}$	done	nothing: lemma 3.3

7.7. $G_2^{(1)}$.

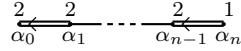
$$\frac{1}{\alpha_0} \xrightarrow{2} \frac{3}{\alpha_1}$$

$S(X_1)$	choice of ω	$S(X_2)$	$\omega^\#$ hom.?	$-\omega$ inhom.?
$S \setminus \{\alpha_0\}$	$\bar{\alpha}_2 + 2\bar{\alpha}_1$	$S \setminus \{\alpha_1\}$		$-\omega = I_0 \Rightarrow$ G1
	$2\bar{\alpha}_2 + 4\bar{\alpha}_1$	$S \setminus \{\alpha_1\}$	no: $\omega^\#$ supported all simple roots	$\langle -\omega, \bar{\alpha}_0^\vee \rangle = 2$
$S \setminus \{\alpha_1\}$	$\bar{\alpha}_0 + \bar{\alpha}_2$	$\{\alpha_1\}$	no: $\omega^\#$ supported $\bar{\alpha}_2$	no: $\langle -\omega, \bar{\alpha}_1^\vee \rangle = 3$
	I_0	$S \setminus \{\alpha_0\}$		nothing possible, lemma 3.3
	I_2	$S \setminus \{\alpha_2\}$		no: $\langle -\omega, \bar{\alpha}_1^\vee \rangle = 2$, lemma 3.3
$S \setminus \{\alpha_2\}$	$[2]\bar{\alpha}_0$	$S \setminus \{\alpha_0\}$	no: $\omega^\# = [2]2\bar{\alpha}_2 + 3\bar{\alpha}_1$, list 2.5	nothing possible, lemma 3.3
	$\bar{\alpha}_1$	$S \setminus \{\alpha_1\}$		$-\omega = I_2 \Rightarrow$ G2
	$2\bar{\alpha}_1$	$S \setminus \{\alpha_1\}$	no: $\omega^\#$ contained all simple roots	no: $\langle -\omega, \bar{\alpha}_1^\vee \rangle = 2$
	$\frac{1}{2}(\bar{\alpha}_0 + \bar{\alpha}_1)$	$\{\alpha_2\}$		$-\omega \in I_2 \Rightarrow$ G3
	$\bar{\alpha}_0 + \bar{\alpha}_1$	$\{\alpha_2\}$	no: $\omega^\#$ contained $\bar{\alpha}_1$	no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle = 2$

$S(X_1)$	choice of ω	$S(X_2)$	ω^\sharp hom.?	$-w$ inhom.?
	I_0	$S \setminus \{\alpha_0\}$		nothing possible lemma 3.3
	I_1	$S \setminus \{\alpha_1\}$		no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle = \frac{1}{2}$, lemma 3.3

7.8. $A_2^{(2)}$.

$S(X_1)$	choice of ω	$S(X_2)$	ω^\sharp hom.?	$-w$ inhom.?
$\{\alpha_0\}$	$\omega = I_0$	$\{\alpha_1\}$		no: $\langle -\omega, \bar{\alpha}_1^\vee \rangle = \frac{1}{2}$
	$\omega = \bar{\alpha}_0$	$\{\alpha_1\}$	no: $\omega^\sharp = \bar{\alpha}_0 + \bar{\alpha}_1$	$-\omega = I_1 \Rightarrow$ A7
	$\omega = 2\bar{\alpha}_0$	$\{\alpha_1\}$	$\omega^\sharp = \bar{\alpha}_1 \Rightarrow$ A6	

7.9. $A_{2n}^{(2)}, n \geq 2$.

$S(X_1)$	choice of ω	$S(X_2)$	ω^\sharp hom.?	$-w$ inhom.?
$S \setminus \{\alpha_0\}$	$\bar{\alpha}_1 + 2\bar{\alpha}_{2,n-1} + \bar{\alpha}_n$	$S \setminus \{\alpha_2\}$	no: $\omega^\sharp = 2\bar{\alpha}_0 + \bar{\alpha}_1$, list 2.5	no: $\langle -\omega, \bar{\alpha}_0^\vee \rangle = 2$, lemma 3.3
	I_1	$S \setminus \{\alpha_1\}$		no: $\langle -\omega, \bar{\alpha}_0^\vee \rangle = 2$, lemma 3.3
$S \setminus \{\alpha_0, \alpha_1\}$	$\omega \in C(n)$ or I_2	$\alpha_0 \notin S(X_2)$		
$S \setminus \{\alpha_0, \alpha_d\}$ $2 \leq d \leq n-1$	$\bar{\alpha}_{1,d-1}$	$S \setminus \{\alpha_1, \alpha_{d-1}\}$	no: $\omega^\sharp = \bar{\alpha}_0 + \bar{\alpha}_{d,n-1} + \frac{1}{2}\bar{\alpha}_n$, list 2.5	no: corollary 3.24
$d = 4$	$[\frac{1}{2}]\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3$	$S \setminus \{\alpha_2\}$	no: $\omega^\sharp = [\frac{1}{2}]2\bar{\alpha}_0 + 2\bar{\alpha}_1 + 2\bar{\alpha}_3 + \dots$, list 2.5	no: corollary 3.24
	$\omega \in C(n)$	$\alpha_0 \notin S(X_2)$ lemma 3.12		
$S \setminus \{\alpha_0, \alpha_n\}$	$\bar{\alpha}_{0,n}$	$S \setminus \{\alpha_1, \alpha_{n-1}\}$	no: $\omega^\sharp = \bar{\alpha}_0 + \frac{1}{2}\bar{\alpha}_n$, list 2.5	no: corollary 3.24
$n = 2$	$\omega = 2\bar{\alpha}_1$	$S \setminus \{\alpha_1\}$	no: $\omega^\sharp = 2\bar{\alpha}_0 + \bar{\alpha}_2$, list 2.5	no: corollary 3.24
$n = 4$	$[\frac{1}{2}]\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3$	$S \setminus \{\alpha_2\}$	no: $\omega^\sharp = [\frac{1}{2}]2\bar{\alpha}_0 + \bar{\alpha}_1 + \bar{\alpha}_3 + \bar{\alpha}_4$, lemma 3.13	no: $\langle -\omega, \bar{\alpha}_0^\vee \rangle > 0, \langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$, lemma 3.3
$S \setminus \{\alpha_d\}, 1 \leq d \leq n-1$	$\bar{\alpha}_{0,d-1}$	$S \setminus \{\alpha_{d-1}\}$		$-\omega = I_d \Rightarrow$ A9
	$2\bar{\alpha}_{0,d-1}$	$S \setminus \{\alpha_{d-1}\}$	no: $\omega^\sharp = 2\bar{\alpha}_{d,n-1} + \bar{\alpha}_n$, list 2.5	no: $\langle -\omega, \bar{\alpha}_d^\vee \rangle = 2$, lemma 3.3

$S(X_1)$	choice of ω	$S(X_2)$	$\omega^\#$ hom.?	$-\omega$ inhom.?
	$\bar{\alpha}_{d+1} + 2\bar{\alpha}_{d+2, n-1} + \bar{\alpha}_n$	$S \setminus \{\alpha_{d+2}\}$	no: $\omega^\# = 2\bar{\alpha}_{0,d} + \bar{\alpha}_{d+1}$, list 2.5	no: $\langle -\omega, \bar{\alpha}_d^\vee \rangle > 0$, lemma 3.3
$d = 3$	$3\bar{\alpha}_0 + 2\bar{\alpha}_2 + \bar{\alpha}_3$	$S \setminus \{\alpha_0\}$	no: $\omega^\# = \bar{\alpha}_1 + 2\bar{\alpha}_2 + 3\bar{\alpha}_{3, n-1} + \bar{\alpha}_n$, list 2.5	no: $\langle -\omega, \bar{\alpha}_3^\vee \rangle > 0$, lemma 3.3
	I_{d+1}	$S \setminus \{\alpha_{d+1}\}$		no: $\langle -\omega, \bar{\alpha}_{d+1}^\vee \rangle > 0$, lemma 3.3
$n = 2, d = 1$	$[\frac{1}{2}]\bar{\alpha}_0 + \bar{\alpha}_2$	$\{\alpha_1\}$	no: $\omega^\# = [\frac{1}{2}]\bar{\alpha}_0 + 2\bar{\alpha}_1$	no: $\langle -\omega, \bar{\alpha}_1^\vee \rangle = [\frac{1}{2}]3$, lemma 3.3
$S \setminus \{\alpha_1, \alpha_{n-1}\}$	$[\frac{1}{2}]\bar{\alpha}_0 + \bar{\alpha}_n$	$S \setminus \{\alpha_0, \alpha_3\}$	no: $\omega^\# = \frac{1}{2}\bar{\alpha}_0 + 2\bar{\alpha}_{1, n-1}$	no: corollary 3.24
	$[2]\bar{\alpha}_0$	$\alpha_2 \notin S(X_2)$ lemma 3.12		
	$[2]\bar{\alpha}_n$	$\alpha_1 \notin S(X_2)$ lemma 3.12		
$S \setminus \{\alpha_d, \alpha_{d+1}\}$ $0 \leq d \leq n-1$	$\omega \in C(0)$	$\alpha_{d+1} \notin S(X_2)$ lemma 3.12		
	$\omega \in C(n)$	$\alpha_d \notin S(X_2)$, lemma 3.12		
$S \setminus \{\alpha_d, \alpha_e\}$ with $1 \leq d < e \leq n-1$ and $n \geq 5$	$\omega \in C(0)$	$\alpha_e \notin S(X_2)$ via lemma 3.12		
	$\omega \in C(d+1)$		no: lemma 3.13	
$S \setminus \{\alpha_d, \alpha_n\}$ with $d \leq n-2$	$\omega \in C(0)$	$\alpha_n \notin S(X_2)$, lemma 3.12		
	$\omega \in C(n-1)$		no: lemma 3.13	no: corollary 3.24
$S \setminus \{\alpha_n\}$	$\omega = 2\bar{\alpha}_{0, n-1}$	$S \setminus \{\alpha_{n-1}\}$	$\omega^\# = \bar{\alpha}_n \Rightarrow A8$	
$S \setminus \{\alpha_n\}$	$\omega = \bar{\alpha}_{0, n-1}$	$S \setminus \{\alpha_{n-1}\}$		$-\omega = I_n \Rightarrow A9$
$n = 4$	$[\frac{1}{2}]3\bar{\alpha}_0 + 2\bar{\alpha}_1 + \bar{\alpha}_2$	$S \setminus \{\alpha_0\}$	no: $\omega^\#$ supported all simple roots, list 2.5	no: $\langle -\omega, \bar{\alpha}_n^\vee \rangle > 0$ lemma 3.3

7.10. $A_{2n-1}^{(2)}, n \geq 3$.

$$\begin{array}{ccccccc} \frac{1}{\alpha_1} & \frac{2}{\alpha_2} & \frac{2}{\alpha_3} & \dots & \frac{2}{\alpha_{n-2}} & \frac{2}{\alpha_{n-1}} & \frac{1}{\alpha_n} \\ & | & & & & & \\ & 1 & & & & & \\ & \alpha_0 & & & & & \end{array}$$

$S(X_1)$	choice of ω	$S(X_2)$	$\omega^\#$ hom.?	$-\omega$ inhom.?
$S \setminus \{\alpha_0\}$	$\bar{\alpha}_1 + 2\bar{\alpha}_{2, n-1} + \bar{\alpha}_n$	$S \setminus \{\alpha_2\}$	$\omega^\# = \bar{\alpha}_0 \Rightarrow A10$	
	I_1	$S \setminus \{\alpha_1\}$		$-\omega = I_0 \Rightarrow A13$
$S \setminus \{\alpha_0, \alpha_1\}$	$\bar{\alpha}_2 + 2\bar{\alpha}_{3, n-1} + \bar{\alpha}_n$	$S \setminus \{\alpha_3\}$	$\omega^\# = \bar{\alpha}_{0,2} \Rightarrow A12$	
$S \setminus \{\alpha_0, \alpha_2\}$	$[2]\bar{\alpha}_1$ or $\omega = \bar{\alpha}_3 + 2\bar{\alpha}_{4, n-1} + \bar{\alpha}_n$	$\alpha_0 \notin S(X_2)$, lemma 3.12		
$S \setminus \{\alpha_0, \alpha_d\}$, $3 \leq d \leq n-1$	$\omega \in C(n)$	$\alpha_0 \notin S(X_2)$, lemma 3.12		
	$\omega \in C(1)$		no: list 2.5	no: lemma 3.3
$S \setminus \{\alpha_0, \alpha_n\}$	$\omega \in C(1)$	$S \setminus \{\alpha_1, \alpha_{n-1}\}$	no: $\omega^\# = \bar{\alpha}_0 + \bar{\alpha}_{2, n}$, list 2.5	no: $\langle -\omega, \bar{\alpha}_0^\vee \rangle > 0, \langle -\omega, \bar{\alpha}_n^\vee \rangle > 0$, lemma 3.3
$n = 4$	$[\frac{1}{2}]\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3$	$S \setminus \{\alpha_2\}$	no: $\omega^\# = [\frac{1}{2}]\bar{\alpha}_0 + \bar{\alpha}_3 + \bar{\alpha}_4$, list 2.5	no: $\langle -\omega, \bar{\alpha}_0^\vee \rangle > 0, \langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$, lemma 3.3

$S(X_1)$	choice of ω	$S(X_2)$	$\omega^\#$ hom.?	$-w$ inhom.?
$S \setminus \{\alpha_2\}$	$\bar{\alpha}_0$	$S \setminus \{\alpha_0\}$	$\omega^\# = \bar{\alpha}_1 + 2\bar{\alpha}_{2,n-1} + \bar{\alpha}_n \Rightarrow \text{A10}$	
	$2\bar{\alpha}_0$	$S \setminus \{\alpha_0\}$	no: $\omega^\# = 2(\bar{\alpha}_1 + 2\bar{\alpha}_{2,n-1} + \bar{\alpha}_n)$, list 2.5	no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle > 0$, lemma 3.3
	I_0	$S \setminus \{\alpha_0\}$		no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle > 0$, lemma 3.3
	$\frac{1}{2}(\bar{\alpha}_0 + \bar{\alpha}_1)$	$S \setminus \{\alpha_0, \alpha_1\}$		$-\omega = I_2 \Rightarrow \text{A14}$
	$\bar{\alpha}_0 + \bar{\alpha}_1$	$S \setminus \{\alpha_0, \alpha_1\}$	no: $\omega^\# = 2\bar{\alpha}_{2,n-1} + \bar{\alpha}_n$, list 2.5	no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle = 2$, lemma 3.3
	$\bar{\alpha}_3 + 2\bar{\alpha}_{4,n-1} + \bar{\alpha}_n$	$S \setminus \{\alpha_4\}$	$\omega^\# = \bar{\alpha}_0 + \bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3$, list 2.5	no: type D , lemma 3.3
$S \setminus \{\alpha_d, \alpha_{d+1}\}$ $2 \leq d$	$\omega \in C(0, 1)$	$\alpha_{d+1} \notin S(X_2)$, lemma 3.12		
	$\omega \in C(n)$	$\alpha_d \notin S(X_2)$, lemma 3.12		
$S \setminus \{\alpha_3\}$	$\bar{\alpha}_{0,2}$	$S \setminus \{\alpha_0, \alpha_1\}$	$\omega^\# = \bar{\alpha}_2 + 2\bar{\alpha}_{3,n-1} + \bar{\alpha}_n \Rightarrow \text{A12}$	
$S \setminus \{\alpha_d\}$	$\bar{\alpha}_0 + \bar{\alpha}_1 + 2\bar{\alpha}_{2,d-1}$	$S \setminus \{\alpha_{d-1}\}$	no: $\omega^\# = 2\bar{\alpha}_{d,n-1} + \bar{\alpha}_n$, list 2.5	no: $\langle -\omega, \bar{\alpha}_d^\vee \rangle = 2$
	$\frac{1}{2}(\bar{\alpha}_0 + \bar{\alpha}_1 + 2\bar{\alpha}_{2,d-1})$	$S \setminus \{\alpha_{d-1}\}$		$-\omega = I_d \Rightarrow \text{A14}$
	I_{d+1}	$S \setminus \{\alpha_{d+1}\}$		no: lemma 3.3
	$\bar{\alpha}_{d+1} + 2\bar{\alpha}_{d+2,n-1} + \bar{\alpha}_n$	$S \setminus \{\alpha_{d+2}\}$	no: $\omega^\# = \bar{\alpha}_0 + \bar{\alpha}_1 + 2\bar{\alpha}_{2,d} + \bar{\alpha}_{d+1}$, list 2.5	no: lemma 3.3
$S \setminus \{\alpha_d, \alpha_e\}$, $3 \leq d, d+1 < e \leq n-1$	$\omega \in C(d+1)$		no: lemma 3.13	
	$w \in C(0, 1)$	$\alpha_e \notin S(X_2)$ lemma 3.12		
	$\omega \in C(n)$	$\alpha_d \notin S(X_2)$ lemma 3.12		
$S \setminus \{\alpha_d, \alpha_n\}$ $2 \leq d$	$\omega \in C(0)$	$\alpha_n \notin S(X_2)$, lemma 3.12		
	$\omega \in C(d+1)$		no: lemma 3.13	
$S \setminus \{\alpha_n\}$	$\bar{\alpha}_0 + \bar{\alpha}_1 + 2\bar{\alpha}_{2,n-1}$	$S \setminus \{\alpha_{n-1}\}$	$\omega^\# = \bar{\alpha}_n \Rightarrow \text{A11}$	
$S \setminus \{\alpha_n\}$	$\frac{1}{2}(\bar{\alpha}_0 + \bar{\alpha}_1 + 2\bar{\alpha}_{2,n-1})$	$S \setminus \{\alpha_{n-1}\}$		$-\omega = I_n \Rightarrow \text{A14}$

7.11. $D_{n+1}^{(2)}$, $n \geq 2$.

$$\begin{array}{c} \xleftarrow{1} \xrightarrow{1} \dots \xleftarrow{1} \xrightarrow{1} \\ \alpha_0 \quad \alpha_1 \quad \quad \quad \alpha_{n-1} \quad \alpha_n \end{array}$$

$S(X_1)$	choice of ω	$S(X_2)$	$\omega^\#$ hom.?	$-w$ inhom.?
$S \setminus \{\alpha_0\}$	$\bar{\alpha}_{1,n}$	$S \setminus \{\alpha_1\}$	$\omega^\# = \bar{\alpha}_0 \Rightarrow \text{D8}$	
$n = 2, S \setminus \{\alpha_0\}$	I_2	$S \setminus \{\alpha_2\}$		$-\omega = I_0 \Rightarrow \text{D5}$
$S \setminus \{\alpha_0\}, n = 3$	$[\frac{1}{2}]3\bar{\alpha}_3 + 2\bar{\alpha}_2 + \bar{\alpha}_1$	$S \setminus \{\alpha_3\}$	$\omega^\# = [\frac{1}{2}]3\bar{\alpha}_0 + 2\bar{\alpha}_1 + \bar{\alpha}_2 \Rightarrow \text{D7}$	
$S \setminus \{\alpha_0, \alpha_1\}$	$\omega \in C(2)$	$\alpha_0 \notin S(X_2)$		

$S(X_1)$	choice of ω	$S(X_2)$	$\omega^\#$ hom.?	$-\omega$ inhom.?
$S \setminus \{\alpha_0, \alpha_2\}$ $n = 3$	$\omega = [\frac{1}{2}]\bar{\alpha}_1 + \bar{\alpha}_3$	$S \setminus \{\alpha_1, \alpha_3\}$	$\omega^\# = [\frac{1}{2}]\bar{\alpha}_0 + \bar{\alpha}_2 \Rightarrow \text{D6}$	
$S \setminus \{\alpha_0, \alpha_d\}$ $2 \leq d \leq n-1$	$\omega \in C(1)$		no: lemma 3.13	
	$\omega \in C(n)$	$\alpha_0 \notin S(X_2)$, lemma 3.12		
$S \setminus \{\alpha_0, \alpha_n\}$	$\bar{\alpha}_{1,n-1}$	$S \setminus \{\alpha_1, \alpha_{n-1}\}$	$\omega^\# = \bar{\alpha}_0 + \bar{\alpha}_n \Rightarrow \text{D9}$	
$n = 4$	$[\frac{1}{2}]\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3$	$S \setminus \{\alpha_2\}$	no: lemma 3.13	
$n = 3$	$2\bar{\alpha}_1$	$S \setminus \{\alpha_1\}$	no: $\omega^\# = 2\bar{\alpha}_0 + 2\bar{\alpha}_2$, list 2.5	
$n = 2, S \setminus \{\alpha_1\}$	$\frac{1}{2}(\bar{\alpha}_0 + \bar{\alpha}_2)$	$S \setminus \{\alpha_0, \alpha_2\}$		$-\omega = I_1 \Rightarrow \text{D4}$
$S \setminus \{\alpha_d\}, 2 \leq d$	$[2]\bar{\alpha}_{0,d-1}$	$S \setminus \{\alpha_{d-1}\}$	$\omega^\# = [2]\bar{\alpha}_{d,n} \Rightarrow \text{D8}$	
$S \setminus \{\alpha_d, \alpha_{d+1}\}$	$\omega \in C(0)$	$\alpha_{d+1} \notin S(X_2)$		
	$\omega \in C(n)$	$\alpha_d \notin S(X_2)$		
$S \setminus \{\alpha_d, \alpha_e\}, 2 \leq d, d+1 < e < n$	$\omega \in C(0)$	$\alpha_e \notin S(X_2)$, lemma 3.12		
	$\omega \in C(d+1)$		no: lemma 3.13	

7.12. $E_6^{(2)}$.

$$\begin{array}{cccccc} 1 & 2 & 3 & 2 & 1 & \\ \hline \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \end{array}$$

We argue the same way we did for $F_4^{(1)}$. We have $\delta = \bar{\alpha}_0 + 2\bar{\alpha}_1 + 3\bar{\alpha}_3 + 2\bar{\alpha}_3 + \bar{\alpha}_4$. Again, $t \in \{\frac{1}{2}, 1\}$.

Let $t = 1$.

- (1) $k_\omega(\bar{\alpha}_2) = 3, \alpha_2 \in S^p$
 - (a) $\omega = \bar{\alpha}_4 + 2\bar{\alpha}_3 + 3\bar{\alpha}_2$. Then $\omega^\# = \bar{\alpha}_0 + 2\bar{\alpha}_1$ which is not a spherical root by list 2.5
 - (b) $\omega = \bar{\alpha}_4 + 2\bar{\alpha}_3 + 3\bar{\alpha}_2 + 2\bar{\alpha}_1$. Then $\omega^\# = \bar{\alpha}_0$ which is admissible. This leads to local root systems $S(X_1) = S \setminus \{\alpha_0\}, S(X_2) = S \setminus \{\alpha_1\}$. $\Rightarrow \text{E1}$
- (2) $k_\omega(\bar{\alpha}_2) = 2, \alpha_2 \in S^p$
 - (a) $\omega = \bar{\alpha}_{0,3}$. Then $\omega^\# = \bar{\alpha}_{2,4}$ which is admissible. The corresponding local root systems are $S(X_1) = S \setminus \{\alpha_4\}, S(X_2) = S \setminus \{\alpha_1\}$. $\Rightarrow \text{E2}$
- (3) $k_\omega(\bar{\alpha}_2) = \frac{3}{2}, \alpha_2 \in S^p$ Then $\omega = \frac{1}{2}(\bar{\alpha}_4 + 2\bar{\alpha}_3 + 3\bar{\alpha}_2)$. Then $\omega^\#$ supported all simple roots, which is impossible.

Now let $t = \frac{1}{2}$.

- (1) $k_\omega(\bar{\alpha}_2) = \frac{3}{2}, \alpha_2 \notin S^p$. Then $\omega = \frac{1}{2}\bar{\alpha}_4 + \bar{\alpha}_3 + \frac{3}{2}\bar{\alpha}_2$. This would lead to $\omega^\# = \frac{1}{2}\bar{\alpha}_0 + \bar{\alpha}_2$ which is not a spherical root by list 2.5.
- (2) $k_\omega(\bar{\alpha}_2) = 1, \alpha_2 \in S^p$. Then $\omega = \bar{\alpha}_2 + \bar{\alpha}_3$ which would lead to $\omega^\# = \frac{1}{2}\bar{\alpha}_0 + \bar{\alpha}_1 + \frac{1}{2}\bar{\alpha}_2 + \frac{1}{2}\bar{\alpha}_4$ which is impossible by list 2.5.

$S(X_1)$	choice of ω	$S(X_2)$	$\omega^\#$ hom.?	$-\omega$ inhom.?
$S \setminus \{\alpha_0\}$	$\omega = 2\bar{\alpha}_1 + 3\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4$	$S \setminus \{\alpha_1\}$	known \Rightarrow A1	
$S \setminus \{\alpha_1\}$	$\omega = [2]\bar{\alpha}_0$	$S \setminus \{\alpha_0\}$	done	nothing possible, lemma 3.3
	$\omega = I_0$	$S \setminus \{\alpha_0\}$		nothing possible, lemma 3.3
	$\omega = \bar{\alpha}_2 + \bar{\alpha}_3 + \bar{\alpha}_4$	$S \setminus \{\alpha_4\}$	known \Rightarrow E2	
	$\omega = 2\bar{\alpha}_{2,4}$	$S \setminus \{\alpha_4\}$	done	no: $\langle -\omega, \bar{\alpha}_1^\vee \rangle > 0$, lemma 3.3
$S \setminus \{\alpha_2\}$	$\omega = \bar{\alpha}_0 + \bar{\alpha}_1$	$S \setminus \{\alpha_0, \alpha_1\}$	done	no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle > 0$, lemma 3.3
	$\omega = I_0$	$S \setminus \{\alpha_0\}$		$\langle -\omega, \bar{\alpha}_2^\vee \rangle > 0$, lemma 3.3
	$\omega = I_1$	$S \setminus \{\alpha_1\}$		$\langle -\omega, \bar{\alpha}_2^\vee \rangle > 0$, lemma 3.3
	$\omega = \bar{\alpha}_3 + \bar{\alpha}_4$	$S \setminus \{\alpha_3, \alpha_4\}$		$\langle -\omega, \bar{\alpha}_2^\vee \rangle = 2$, lemma 3.3
	$\omega = I_3$	$S \setminus \{\alpha_3\}$		$\langle -\omega, \bar{\alpha}_2^\vee \rangle = \frac{4}{3}$, lemma 3.3
	$\omega = I_4$	$S \setminus \{\alpha_4\}$		$\langle -\omega, \bar{\alpha}_2^\vee \rangle > 0$, lemma 3.3
$S \setminus \{\alpha_3\}$	$\omega = \bar{\alpha}_{0,3}$	$S \setminus \{\alpha_0, \alpha_2\}$		$-\omega \in I_3 \Rightarrow$ E3
	$\omega = [\frac{1}{2}]\bar{\alpha}_0 + 2\bar{\alpha}_1 + \bar{\alpha}_3$	$S \setminus \{\alpha_1\}$	done	$\langle -\omega, \bar{\alpha}_3^\vee \rangle > 0$, lemma 3.3
	$\omega = I_0$	$S \setminus \{\alpha_0\}$		$\langle -\omega, \bar{\alpha}_3^\vee \rangle > 0$, lemma 3.3
	$\omega = I_2$	$S \setminus \{\alpha_2\}$		$\langle -\omega, \bar{\alpha}_3^\vee \rangle = \frac{3}{4}$, lemma 3.3
	$\omega = [2]\bar{\alpha}_4$	$S \setminus \{\alpha_4\}$		$\langle -\omega, \bar{\alpha}_3^\vee \rangle > 0$, lemma 3.3
	$\omega = I_4$	$S \setminus \{\alpha_4\}$		$\langle -\omega, \bar{\alpha}_3^\vee \rangle > 0$, lemma 3.3
$S \setminus \{\alpha_4\}$	$\omega = \bar{\alpha}_0 + 2\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3$	$S \setminus \{\alpha_1\}$	done	$\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$, lemma 3.3
	$\omega = I_0$	$S \setminus \{\alpha_0\}$		nothing possible, lemma 3.3

7.13. $D_4^{(3)}$.

$$\frac{1}{\alpha_0} \xrightarrow{\quad} \frac{2}{\alpha_1} \xleftarrow{\quad} \frac{1}{\alpha_2}$$

$S(X_1)$	choice of ω	$S(X_2)$	$\omega^\#$ hom.?	$-\omega$ inhom.?
$S \setminus \{\alpha_0\}$	$\omega = [2]2\bar{\alpha}_1 + \bar{\alpha}_2$	$S \setminus \{\alpha_1\}$	$\omega^\# = [2]\bar{\alpha}_0 \Rightarrow$ D10	
$S \setminus \{\alpha_1\}$	$\omega = [2]\bar{\alpha}_0$	$S \setminus \{\alpha_0\}$	$\omega^\# = [2]2\bar{\alpha}_1 + \bar{\alpha}_2 \Rightarrow$ D10	
	$\omega = [2]\bar{\alpha}_2$	$S \setminus \{\alpha_2\}$	no: $\omega^\# = [2]\bar{\alpha}_0 + 2\bar{\alpha}_1$, list 2.5	no: $\langle -\omega, \bar{\alpha}_1^\vee \rangle = [2]3$
	$\omega = [\frac{1}{2}]\bar{\alpha}_0 + \bar{\alpha}_2$	$\{\alpha_1\}$	$\omega^\# = [2]\bar{\alpha}_1 \Rightarrow$ D11	
	$\omega = I_0$	$S \setminus \{\alpha_0\}$		nothing possible
	$\omega = I_2$	$S \setminus \{\alpha_2\}$		no: $\langle -\omega, \bar{\alpha}_1^\vee \rangle = \frac{3}{2}$, lemma 3.3
$S \setminus \{\alpha_2\}$	$\omega = \bar{\alpha}_0 + \bar{\alpha}_1$	$\{\alpha_2\}$	no: $\omega^\# = \bar{\alpha}_1 + \bar{\alpha}_2$, list 2.5	$-\omega \in I_2 \Rightarrow$ D12
	$\omega = I_0$	$S \setminus \{\alpha_0\}$		nothing possible lemma 3.3
	$\omega = I_1$	$S \setminus \{\alpha_1\}$		no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle = \frac{2}{3}$, lemma 3.3

8. LIST OF MOMENT POLYTOPES OF RANK ONE

Theorem 3.28 (Momentum Polytopes of Rank one). *A compact multiplicity free genuine quasi-Hamiltonian K -manifold of rank one for K simple, simply connected corresponds to a spherical pair (\mathcal{P}, Λ_S) with the following properties:*

- \mathcal{P} is a line segment $[X_1 X_2]$ for X_1, X_2 found in [list 3.29](#)
- The lattice Λ_S is generated by the corresponding ω in [list 3.29](#)

We remember our notation $H(i)$ and $I_k(i)$ from [definition 3.5](#). All parameters are assumed to be non-negative integers. We mark inhomogeneous local models by printing them in **bold font**. A [2] in front of a local model says that not only ω and ω^\sharp , but also 2ω and $2\omega^\sharp$ is possible.

We also list the points X_1, X_2 where the moment polytope hits the walls of the alcove to convince the reader that these moment polytopes actually exist. Remember [section 1](#) for our choice of coordinates. We add the value of the simple roots on X_i for those simple roots that are not zero on X_i to convince the reader that the points X_1 and X_2 really lie *inside* the fundamental alcove.

We list, as discussed, the graph of the corresponding primitive spherical triples in the "local model"-columns. It contains information about the semisimple type of the local model.

We shall also add a diagram notation for each case where we draw the diagrams for ω and ω^\sharp in one diagram of the affine root system. For homogeneous models, we use Luna's diagrams, for inhomogeneous, we use the notation introduced in [definition 3.4](#). Note that we will only note the components of the local model with non-trivial operation in order to not overload the tables.

We also investigated the operation of the connected center that could appear in the cases where the diagram of the primitive triples shows an asterisk. If these \mathfrak{t}^1 acted via a scalar N , we put an \mathbb{C}_N^k in the V -layer of our Diagram. We omit this if $N = 1$ which stands for the "standard" operation. To find this scalar N , we determine a linearly independent set of primitive $\zeta \in \mathbb{Z}\Sigma$ corresponding to the center of the Levi. It is defined by the fact that its elements are perpendicular to all simple roots in the local root system. Then it is well known that $N = \langle \omega, \zeta^\vee \rangle$.

Remarkably, it turned out that $N = 1$ **for all genuine quasi-Hamiltonian manifolds of rank one!**

We shall see that this is not true for those quasi-Hamiltonians of rank one that are exponentials of Hamiltonian manifolds in the next part.

List 3.29. (1) $A_n^{(1)}$
 For $A_1^{(1)}$ only:

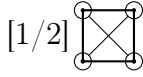
	ω	local model	ω^\sharp	local model	remarks
A1)	$2\bar{\alpha}_0 \in H(1)$ $X_1 = (\frac{1}{2}, -\frac{1}{2})$ $\alpha_1(X_1) = 1$	$\bullet \mathfrak{sl}(2)$	$2\bar{\alpha}_1 \in H(0)$ $X_1 = (0, 0)$ $\alpha_0(X_2) = 1$	$\bullet \mathfrak{sl}(2)$	



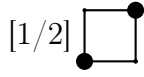
A1)

For $A_3^{(1)}$ only:

	ω	local model	ω^\sharp	local model	remarks
A2)	$[\frac{1}{2}](\bar{\alpha}_1 + \bar{\alpha}_3) \in H(0, 2)$ $X_1 = (\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})$ $\alpha_0(X_1) = \frac{1}{2} = \alpha_2(X_1)$		$[\frac{1}{2}](\bar{\alpha}_0 + \bar{\alpha}_2) \in H(1, 3)$ $X_2 = (1/2, 0, 0 - 1/2)$ $\alpha_1(X_2) = \frac{1}{2} = \alpha_3(X_2)$		
A3)	$[2]\frac{1}{2}(\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3) \in H(0)$ $X_1 = (0, 0, 0, 0)$ $\alpha_0(X_1) = 1$	$\bullet \mathfrak{sl}(4)$ $\bullet \mathfrak{sp}(4)$	$[2]\frac{1}{2}(\bar{\alpha}_3 + 2\bar{\alpha}_0 + \bar{\alpha}_1) \in H(2)$ $X_2 = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ $\alpha_2(X_2) = 1$	$\bullet \mathfrak{sl}(4)$ $\bullet \mathfrak{sp}(4)$	up to diagram automorphisms



A2)

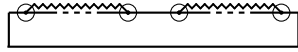


A3)

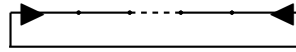
Remark 3.30. Kno showed in [Kno16] that the case A2 (without the factor 1/2) corresponds to the disymmetric manifold $SU(4)/SO(4) \times SU(4)/Sp(4)$.

For $A_n^{(1)}, l \geq 1$

	ω	local model	ω^\sharp	local model	remarks
A4)	$\bar{\alpha}_1 + \dots + \bar{\alpha}_{d-1} \in H(0, d)$ $X_1 = (a, \dots, \frac{a}{d}, b, \dots, b)$ $a = \frac{n+1-d}{2n+2}, b = \frac{-d}{2n+2}$ $\alpha_0(X_1) = \frac{1}{2} = \alpha_d(X_1)$	$\bullet \mathfrak{sl}(d)$ $\bullet \mathfrak{t}^1 + \mathfrak{sl}(d-1)$	$\bar{\alpha}_d + \dots + \bar{\alpha}_n + \bar{\alpha}_0 \in H(d-1, 1)$ $X_2 = (a, b, \dots, \frac{b}{d-1}, c, \dots, c)$ $a = \frac{2n+2-d}{2n+2}, b = \frac{n+1-d}{2n+2}$ $c = \frac{-d}{2n+2}$ $\alpha_{d-1}(X_2) = \frac{1}{2} = \alpha_1(X_2)$	$\bullet \mathfrak{sl}(n+1-d)$ $\bullet \mathfrak{t}^1 + \mathfrak{sl}(n-d)$	up to diagram automorphisms
A5)	$\omega = I_0(1)$ $X_1 = (\frac{n}{n+1}, -\frac{1}{n+1}, \dots, -\frac{1}{n+1})$ $\alpha_1(X_1) = 1$	$\bullet \mathfrak{sl}(n+1)$ $\bullet \mathfrak{sl}(n+1)$ $\bullet \mathbb{C}^{n+1}$	$-\omega = I_1(0)$ $X_2 = (0, \dots, 0)$ $\alpha_0(X_2) = 1$	$\bullet \mathfrak{sl}(n+1)$ $\bullet \mathfrak{sl}(n+1)$ $\bullet \mathbb{C}^{n+1}$	up to diagram automorphisms



A4)



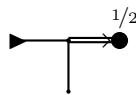
A5)

Remark 3.31. The three possible cases $\omega(A5), 2\omega(A4), 4\omega(A1)$ for $n = 2$ correspond, in that order, to the manifolds S^4 (the so-called "spinning 4-sphere", [AMW02]), $S^2 \times S^2$ and $\mathbb{P}^2(\mathbb{C})$, [Kno16, pp.36].

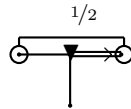
(2) $B_n^{(1)}, n \geq 3$

For $n = 3$:

	ω	local model	$\omega^\#$	local model	remarks
B1)	$\frac{1}{2}(\bar{\alpha}_0 + 2\bar{\alpha}_2 + 3\bar{\alpha}_3) \in H(1)$ $X_1 = (1, 0, 0)$ $\alpha_1(X_1) = 1$	\bullet $\mathfrak{so}(7)$ \bullet G_2	$-\omega \in I_1(\mathbf{3})$ $X_2 = (1/2, 1/2, 1/2)$ $\alpha_3(X_2) = \frac{1}{2}$	\bullet $\mathfrak{sl}(4)$ \bullet $\mathfrak{sl}(4)$ \bullet \mathbb{C}^4	or 0 and 1 switched
B2)	$\frac{1}{2}(\bar{\alpha}_1 + \bar{\alpha}_3) \in H(2)$ $X_1 = (1/2, 1/2, 0)$ $\alpha_2(X_1) = \frac{1}{2}$	$\mathfrak{sl}(2)$ \bullet $\mathfrak{sl}(2)$ \bullet $\mathfrak{sl}(2)$	$-\omega \in I_2(\mathbf{1}, \mathbf{3})$ $X_2 = (3/4, 1/4, 1/4)$ $\alpha_1(X_2) = \frac{1}{2}, \alpha_3(X_2) = \frac{1}{4}$	\bullet $\mathfrak{sl}(3)$ \bullet $\mathfrak{sl}(3)$ \bullet \mathbb{C}^3	or 0 and 1 switched



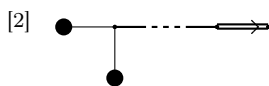
B1)



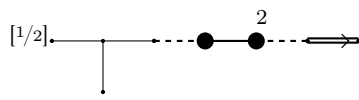
B2)

For all $n \geq 3$

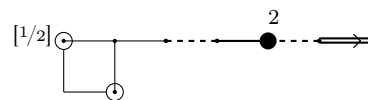
	ω	local model	$\omega^\#$	local model	remarks
B3)	$[2](\bar{\alpha}_1 + \dots + \bar{\alpha}_n) \in H(0)$ $X_1 = (0, \dots, 0)$ $\alpha_0(X_1) = 1$	\bullet $\mathfrak{so}(2n+1)$ \bullet $\mathfrak{so}(2n)$	$[2](\bar{\alpha}_0 + \bar{\alpha}_2 + \dots + \bar{\alpha}_n) \in H(1)$ $X_2 = (1, 0, \dots, 0)$ $\alpha_1(X_2) = 1$	\bullet $\mathfrak{so}(2n+1)$ \bullet $\mathfrak{so}(2n)$	
B4)	$[2](\frac{1}{2}(\bar{\alpha}_0 + \bar{\alpha}_1) + \bar{\alpha}_2 + \dots + \bar{\alpha}_d) \in H(d+1)$ $X_1 = (\frac{1}{2}, \dots, \frac{1}{2}, 0, \dots, 0)$ $\alpha_{d+1}(X_1) = \frac{1}{2}$	\bullet $\mathfrak{so}(2k+2)$ \bullet $\mathfrak{so}(2k+1)$	$[2](\bar{\alpha}_{d+1} + \dots + \bar{\alpha}_n) \in H(d)$ $X_2 = (\frac{1}{2}, \dots, \frac{1}{2}, 0, \dots, 0)$ $\alpha_d(X_2) = \frac{1}{2}$	\bullet $\mathfrak{so}(2n-2k+1)$ \bullet $\mathfrak{so}(2n-2k)$	$1 \leq d < n$



B3)



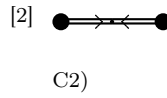
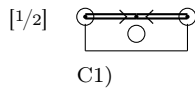
B4)



also part of B4)

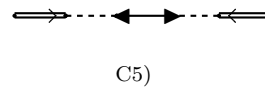
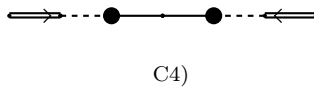
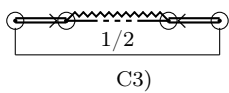
(3) $C_n^{(1)}, n \geq 2$
 For $n = 2$ only:

	ω	local model	ω^\sharp	local model	remarks
C1)	$[2] \frac{1}{2}(\bar{\alpha}_0 + \bar{\alpha}_2) \in H(1)$ $X_1 = (\frac{1}{2}, 0)$ $\alpha_1(X_1) = \frac{1}{2}$		$[2] \bar{\alpha}_1 \in H(0, 2)$ $X_2 = (\frac{1}{4}, \frac{1}{4})$ $\alpha_0(X_2) = \frac{1}{2} = \alpha_2(X_2)$	$\bullet \mathfrak{sl}(2)$	
C2)	$[2](\bar{\alpha}_0 + \bar{\alpha}_1) \in H(2)$ $X_1 = (\frac{1}{2}, \frac{1}{2})$ $\alpha_2(X_1) = 1$	$\bullet \mathfrak{so}(5)$ $\bullet \mathfrak{so}(4)$	$[2](\bar{\alpha}_1 + \bar{\alpha}_2) \in H(0)$ $X_2 = (0, 0)$ $\alpha_0(X_2) = 1$	$\bullet \mathfrak{so}(5)$ $\bullet \mathfrak{so}(4)$	



For all $n \geq 2$:

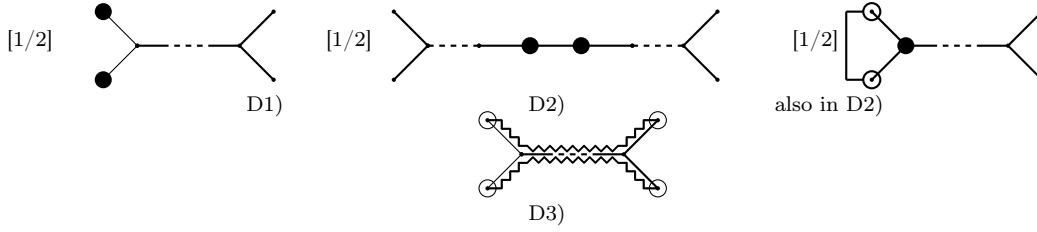
	ω	local model	ω^\sharp	local model	remarks
C3)	$\frac{1}{2}(\bar{\alpha}_0 + \bar{\alpha}_n) \in H(1, n-1)$ $X_1 = (\frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{4}, 0)$ $\alpha_1(X_1) = \frac{1}{4} = \alpha_{n-1}(X_1)$		$\bar{\alpha}_1 + \dots + \bar{\alpha}_{n-1} \in H(0, n)$ $X_2 = (\frac{1}{4}, \dots, \frac{1}{4})$ $\alpha_0(X_2) = \frac{1}{2} = \alpha_n(X_2)$	$\bullet \mathfrak{sl}(n-1)$ $\bullet \mathfrak{t}^1 + \mathfrak{sl}(n-2)$	see also $n = 2$
C4)	$\bar{\alpha}_0 + 2\bar{\alpha}_1 + \dots + 2\bar{\alpha}_{d-1} + \bar{\alpha}_d \in H(d+1)$ $X_1 = (\frac{1}{2}, \dots, \frac{1}{2}, 0, \dots, 0)$ $\alpha_{d+1}(X_1) = \frac{1}{2}$	$\mathfrak{sp}(2d+2)$ 	$\bar{\alpha}_d + 2\bar{\alpha}_{d+1} + \dots + 2\bar{\alpha}_{n-1} + \bar{\alpha}_n \in H(d-1)$ $X_2 = (\frac{1}{2}, \dots, \frac{1}{2}, 0, \dots, 0)$ $\alpha_{d-1}(X_2) = \frac{1}{2}$	$\mathfrak{sp}(2n-2d+2)$ 	$1 \leq d < n$
C5)	$\omega = I_{k-1}(k)$ $X_1 = (\frac{1}{2}, \dots, \frac{1}{2}, 0, \dots, 0)$ $\alpha_k(X_1) = \frac{1}{2}$	$\bullet \mathfrak{sp}(2k)$ $\bullet \mathfrak{sp}(2k)$ $\bullet \mathbb{C}^{2k}$	$\omega = I_k(k-1)$ $X_1 = (\frac{1}{2}, \dots, \frac{1}{2}, 0, \dots, 0)$ $\alpha_{k-1}(X_2) = \frac{1}{2}$	$\bullet \mathfrak{sp}(2n-2k+2)$ $\bullet \mathfrak{sp}(2n-2k+2)$ $\bullet \mathbb{C}^{2n-2k+2}$	$1 \leq k \leq n$



Remark 3.32. Knop determined the manifold that corresponds to case C5) in [Kno16]. More precisely, this spherical pair comes from the $\mathrm{Sp}(2n)$ -manifold structure on the quaternionic Grassmanian: $M = \mathrm{Gr}_k(\mathbb{H}^{n+1})$.

(4) $D_n^{(1)}, n \geq 4$

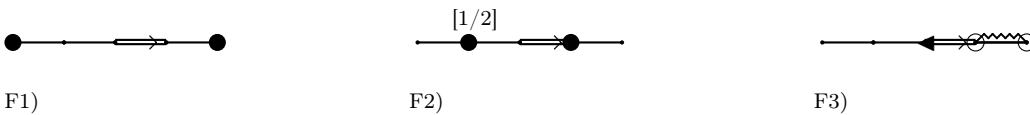
	ω	local model	ω^\sharp	local model	remarks
D1)	$[2](\bar{\alpha}_1 + \bar{\alpha}_2 + \dots + \bar{\alpha}_{n-2} + \frac{1}{2}\bar{\alpha}_{n-1} + \frac{1}{2}\bar{\alpha}_n) \in H(0)$ $X_1 = (0, \dots, 0)$ $\alpha_0(X_1) = 1$	$\bullet \mathfrak{so}(2n)$ $\bullet \mathfrak{so}(2n-1)$	$[2](\bar{\alpha}_0 + \bar{\alpha}_2 + \dots + \bar{\alpha}_{n-2} + \frac{1}{2}\bar{\alpha}_{n-1} + \frac{1}{2}\bar{\alpha}_n) \in H(1)$ $X_2 = (1, 0, \dots, 0)$ $\alpha_1(X_1) = 1$	$\bullet \mathfrak{so}(2n)$ $\bullet \mathfrak{so}(2n-1)$	
D2)	$[2](\frac{1}{2}\bar{\alpha}_0 + \frac{1}{2}\bar{\alpha}_1 + \bar{\alpha}_2 + \dots + \bar{\alpha}_d) \in H(d+1)$ $X_1 = (\frac{1}{2}, \dots, \frac{1}{2}, 0, \dots, 0)$ $\alpha_{d+1}(X_1) = \frac{1}{2}$	$\bullet \mathfrak{so}(2d+2)$ $\bullet \mathfrak{so}(2d+1)$	$[2](\bar{\alpha}_{d+1} + \dots + \bar{\alpha}_{n-2} + \frac{1}{2}\bar{\alpha}_{n-1} + \frac{1}{2}\bar{\alpha}_n) \in H(d)$ $X_2 = (\frac{1}{2}, \dots, \frac{1}{2}, 0, \dots, 0)$ $\alpha_d(X_2) = \frac{1}{2}$	$\bullet \mathfrak{so}(2n-2d)$ $\bullet \mathfrak{so}(2n-2d-1)$	$1 \leq d \leq n-2$
D3)	$\bar{\alpha}_1 + \dots + \bar{\alpha}_{n-1} \in H(0, n)$ $X_1 = (\frac{1}{4}, \dots, \frac{1}{4})$ $\alpha_0(X_1) = \frac{1}{2} = \alpha_n(X_1)$	$\bullet \mathfrak{sl}(n)$ $\bullet \mathfrak{gl}(n-1)$	$\bar{\alpha}_0 + \bar{\alpha}_2 + \dots + \bar{\alpha}_{n-2} + \bar{\alpha}_n \in H(1, n-1)$ $X_2 = (\frac{3}{4}, \frac{1}{4}, \dots, \frac{1}{4}, -\frac{1}{4})$ $\alpha_1(X_2) = \frac{1}{2} = \alpha_{n-1}(X_2)$	$\bullet \mathfrak{sl}(n)$ $\bullet \mathfrak{gl}(n-1)$	or α_n and α_{n-1} switched




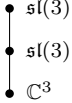

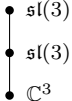
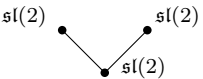
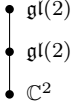
(5) $E_n^{(1)}, n \in \{6, 7, 8\}$ There are no momentum polytopes of rank 1.

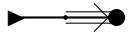
(6) $F_4^{(1)}$

	ω	local model	ω^\sharp	local model	remarks
F1)	$\bar{\alpha}_1 + 2\bar{\alpha}_2 + 3\bar{\alpha}_3 + 2\bar{\alpha}_4 \in H(0)$ $X_1 = (0, 0, 0, 0)$ $\alpha_0(X_1) = 1$	$\bullet F_4$ $\bullet \mathfrak{so}(9)$	$\bar{\alpha}_0 + \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3 \in H(4)$ $X_2 = (1, 0, 0, 0)$ $\alpha_4(X_2) = \frac{1}{2}$	$\bullet \mathfrak{so}(9)$ $\bullet \mathfrak{so}(8)$	
F2)	$\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4 \in H(1)$ $X_1 = (\frac{1}{2}, \frac{1}{2}, 0, 0)$ $\alpha_1(X_1) = \frac{1}{2}$	$\bullet \mathfrak{sl}(4)$ $\bullet \mathfrak{sp}(4)$	$\frac{1}{2}\bar{\alpha}_0 + \bar{\alpha}_1 + \frac{1}{2}\bar{\alpha}_2 \in H(3)$ $X_2 = (\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ $\alpha_3(X_2) = \frac{1}{4}$	$\bullet \mathfrak{sl}(4)$ $\bullet \mathfrak{sp}(4)$	
F3)	$\bar{\alpha}_3 + \bar{\alpha}_4 \in H(2)$ $X_1 = (\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 0)$ $\alpha_2(X_1) = \frac{1}{3}$	$\bullet \mathfrak{sl}(3)$ $\bullet \mathfrak{gl}(2)$	$-\omega \in I_2(3, 4)$ $X_2 = (\frac{5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ $\alpha_3(X_2) = \frac{1}{6}, \alpha_4(X_2) = \frac{1}{6}$	$\bullet \mathfrak{gl}(4)$ $\bullet \mathfrak{gl}(4)$ $\bullet \mathbb{C}^4$	

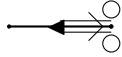


(7) $G_2^{(1)}$

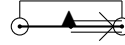
	ω	local model	ω^\sharp	local model	remarks
G1)	$\bar{\alpha}_2 + 2\bar{\alpha}_1 \in H(0)$ $X_1 = (0, 0, 0)$ $\alpha_0(X_1) = 1$		$-\omega = I_0(1)$ $X_2 = (0, -\frac{1}{3}, \frac{1}{3})$ $\alpha_1(X_2) = \frac{1}{3}$		
G2)	$\bar{\alpha}_1 \in H(2)$ $X_1 = (-\frac{1}{6}, -\frac{1}{6}, \frac{1}{3})$ $\alpha_2(X_1) = \frac{1}{2}$		$-\omega = I_2(1)$ $X_2 = (0, -\frac{1}{3}, \frac{1}{3})$ $\alpha_1(X_2) = \frac{1}{3}$		
G3)	$\frac{1}{2}(\bar{\alpha}_0 + \bar{\alpha}_1) \in H(2)$ $X_1 = (-\frac{1}{6}, -\frac{1}{6}, \frac{1}{3})$ $\alpha_2(X_1) = \frac{1}{2}$		$-\omega \in I_2(0, 1)$ $X_2 = (0, -\frac{1}{6}, \frac{1}{6})$ $\alpha_0(X_2) = \frac{1}{2}, \alpha_1(X_2) = \frac{1}{6}$		



G1)




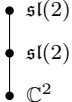


G2)



G3)

(8) $A_2^{(2)}$

	ω	local model	ω^\sharp	local model	remarks
A6)	$2\bar{\alpha}_0 \in H(1)$ $X_1 = (2)$ $\alpha_1(X_1) = 2$		$\bar{\alpha}_1 \in H(0)$ $X_2 = (0)$ $\alpha_0(X_2) = 1$		
A7)	$\bar{\alpha}_0 \in H(1)$ $X_1 = (2)$ $\alpha_1(X_1) = 2$		$-\omega = I_1(0)$ $X_2 = (0)$ $\alpha_0(X_2) = 1$		



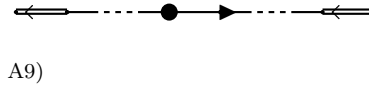
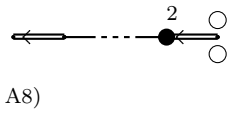
A6)



A7)

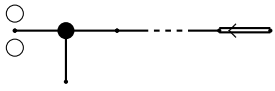
(9) $A_{2n}^{(2)}, n \geq 2$

	ω	local model	ω^\sharp	local model	remarks
A8)	$2\bar{\alpha}_0 + 2\bar{\alpha}_1 + \dots + 2\bar{\alpha}_{n-1} \in H(n)$ $X_1 = (1, \dots, 1)$ $\alpha_n(X_1) = 2$	<ul style="list-style-type: none"> $\mathfrak{so}(2n+1)$ $\mathfrak{so}(2n)$ 	$\bar{\alpha}_n \in H(n-1)$ $X_2 = (1, \dots, 1, 0)$ $\alpha_{n-1}(X_2) = 1$	<ul style="list-style-type: none"> $\mathfrak{sl}(2)$ 	
A9)	$\bar{\alpha}_0 + \bar{\alpha}_1 + \dots + \bar{\alpha}_d \in H(d+1)$ $X_1 = (1, \dots, \frac{1}{d+1}, 0, \dots, 0)$ $\alpha_{d+1}(X_1) = 1$	<ul style="list-style-type: none"> $\mathfrak{so}(2(d+1)+1)$ $\mathfrak{so}(2(d+1))$ 	$-\omega = I_{d+1}(d)$ $X_2 = (1, \dots, \frac{1}{d}, 0, \dots, 0)$ $\alpha_d(X_2) = 1$	<ul style="list-style-type: none"> $\mathfrak{sp}(2n-2d)$ $\mathfrak{sp}(2n-2d)$ \mathbb{C}^{2n-2d} 	$0 \leq d < n$

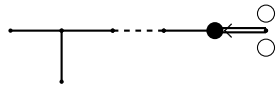


(10) $A_{2n-1}^{(2)}, n \geq 3$

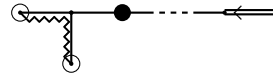
	ω	local model	ω^\sharp	local model	remarks
A10)	$\bar{\alpha}_1 + 2\bar{\alpha}_2 + \dots + 2\bar{\alpha}_{n-1} + \bar{\alpha}_n \in H(0)$ $X_1 = (0, \dots, 0)$ $\alpha_0(X_1) = 1$	<ul style="list-style-type: none"> $\mathfrak{sp}(2n)$ $\mathfrak{sl}(2)$ $\mathfrak{sp}(2n-2)$ 	$\bar{\alpha}_0 \in H(2)$ $X_2 = (\frac{1}{2}, \frac{1}{2}, 0, \dots, 0)$ $\alpha_2(X_2) = \frac{1}{2}$	<ul style="list-style-type: none"> $\mathfrak{sl}(2)$ 	or $\bar{\alpha}_0$ and $\bar{\alpha}_1$ switched
A11)	$\bar{\alpha}_0 + \bar{\alpha}_1 + 2\bar{\alpha}_2 + \dots + 2\bar{\alpha}_{n-1} \in H(n)$ $X_1 = (\frac{1}{2}, \dots, \frac{1}{2})$ $\alpha_n(X_1) = 1$	<ul style="list-style-type: none"> $\mathfrak{so}(2n)$ $\mathfrak{so}(2n-1)$ 	$\bar{\alpha}_n \in H(n-1)$ $X_2 = (\frac{1}{2}, \dots, \frac{1}{2}, 0)$ $\alpha_{n-1}(X_2) = \frac{1}{2}$	<ul style="list-style-type: none"> $\mathfrak{so}(2n-2)$ $\mathfrak{sl}(2)$ $\mathfrak{so}(2n-2)$ 	
A12)	$\bar{\alpha}_0 + \bar{\alpha}_1 + \bar{\alpha}_2 \in H(3)$ $X_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \dots, 0)$ $\alpha_3(X_1) = \frac{1}{2}$	<ul style="list-style-type: none"> $\mathfrak{sl}(4)$ $\mathfrak{gl}(3)$ 	$\bar{\alpha}_2 + 2\bar{\alpha}_3 + \dots + 2\bar{\alpha}_{n-1} + \bar{\alpha}_n \in H(0, 1)$ $X_2 = (\frac{1}{2}, 0, \dots, 0)$ $\alpha_0(X_2) = \frac{1}{2} = \alpha_1(X_2)$	<ul style="list-style-type: none"> $\mathfrak{sp}(2(n-1))$ $\mathfrak{sl}(2)$ $\mathfrak{sp}(2(n-2))$ 	
A13)	$\omega = I_1(0)$ $X_1 = (0, \dots, 0)$ $\alpha_0(X_1) = 1$	<ul style="list-style-type: none"> $\mathfrak{sp}(2n)$ $\mathfrak{sp}(2n)$ \mathbb{C}^{2n} 	$-\omega = I_0(1)$ $X_2 = (1, 0, \dots, 0)$ $\alpha_1(X_2) = 1$	<ul style="list-style-type: none"> $\mathfrak{sp}(2n)$ $\mathfrak{sp}(2n)$ \mathbb{C}^{2n} 	
A14)	$\frac{1}{2}\bar{\alpha}_0 + \frac{1}{2}\bar{\alpha}_1 + \bar{\alpha}_2 + \dots + \bar{\alpha}_d \in H(d+1)$ $X_1 = (\frac{1}{2}, \dots, \frac{1}{2}, 0, \dots, 0)$ $\alpha_{d+1}(X_1) = \frac{1}{2}$	<ul style="list-style-type: none"> $\mathfrak{so}(2(d+1))$ $\mathfrak{so}(2d+1)$ 	$-\omega = I_{d+1}(d)$ $X_2 = (\frac{1}{2}, \dots, \frac{1}{2}, 0, \dots, 0)$ $\alpha_d(X_2) = \frac{1}{2}$	<ul style="list-style-type: none"> $\mathfrak{sp}(2n-2d)$ $\mathfrak{sp}(2n-2d)$ \mathbb{C}^{2n-2d} 	$1 \leq d \leq n$



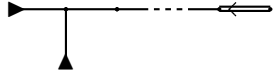
A10)



A11)



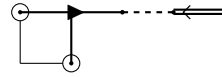
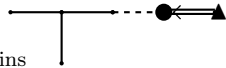
A12)



A13)



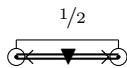
A14)



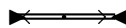
Again, note that A14) also contains these two cases:

(11) $D_{n+1}^{(2)}, n \geq 2$
 For $D_3^{(2)}$ only:

	ω	local model	$\omega^\#$	local model	remarks
D4)	$\frac{1}{2}(\bar{\alpha}_0 + \bar{\alpha}_2) \in H(1)$ $X_1 = (1, 0)$ $\alpha_1(X_1) = 1$		$-\omega \in I_1(0, 2)$ $X_2 = (\frac{1}{2}, \frac{1}{2})$ $\alpha_0(X_2) = \frac{1}{2} = \alpha_2(X_2)$	<ul style="list-style-type: none"> • $\mathfrak{gl}(2)$ • $\mathfrak{gl}(2)$ • \mathbb{C}^2 	
D5)	$\omega \in I_0(2)$ $X_1 = (1, 1)$ $\alpha_2(X_1) = 1$	<ul style="list-style-type: none"> • $\mathfrak{sl}(3)$ • $\mathfrak{sl}(3)$ • \mathbb{C}^3 	$-\omega \in I_2(0)$ $X_2 = (0, 0)$ $\alpha_0(X_2) = 1$	<ul style="list-style-type: none"> • $\mathfrak{sl}(3)$ • $\mathfrak{sl}(3)$ • \mathbb{C}^3 	



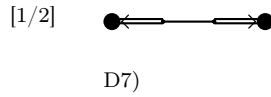
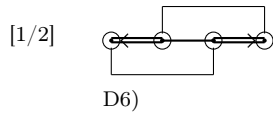
D4)



D5)

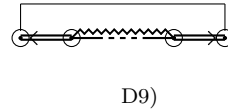
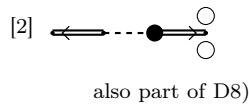
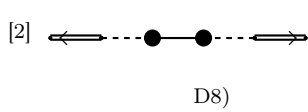
For $D_4^{(2)}$ only:

	ω	local model	ω^\sharp	local model	remarks
D6)	$[2] \frac{1}{2}(\bar{\alpha}_0 + \bar{\alpha}_2) \in H(1, 3)$ $X_1 = (1, \frac{1}{3}, \frac{1}{3})$ $\alpha_1(X_1) = \frac{2}{3}, \alpha_3(X_1) = \frac{1}{3}$		$[2] \frac{1}{2}(\bar{\alpha}_1 + \bar{\alpha}_3) \in H(0, 2)$ $X_2 = (\frac{2}{3}, \frac{2}{3}, 0)$ $\alpha_0(X_2) = \frac{1}{3}, \alpha_2(X_2) = \frac{2}{3}$		
D7)	$[2] \frac{1}{2}(3\bar{\alpha}_0 + 2\bar{\alpha}_1 + \bar{\alpha}_2) \in H(3)$ $X_1 = (1, 1, 1, 0)$ $\alpha_3(X_1) = 1$		$[2] \frac{1}{2}(\bar{\alpha}_1 + 2\bar{\alpha}_2 + 3\bar{\alpha}_3) \in H(0)$ $X_2 = (0, 0, 0, 0)$ $\alpha_0(X_2) = 1$		



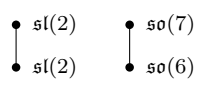
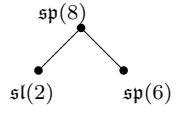

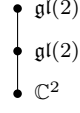
For every $n \geq 2$:

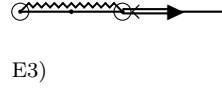
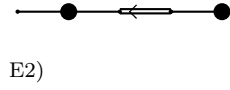
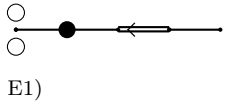
	ω	local model	ω^\sharp	local model	remarks
D8)	$[2](\bar{\alpha}_0 + \dots + \bar{\alpha}_d) \in H(d+1)$ $X_1 = (1, \dots, \frac{1}{d+1}, 0, \dots, 0)$ $\alpha_{d+1}(X_1) = 1$		$[2](\bar{\alpha}_{d+1} + \dots + \bar{\alpha}_n) \in H(d)$ $X_2 = (1, \dots, \frac{1}{d}, 0, \dots, 0)$ $\alpha_d(X_2) = 1$		$0 \leq d < n$
D9)	$\bar{\alpha}_1 + \dots + \bar{\alpha}_{n-1} \in H(0, n)$ $X_1 = (\frac{1}{2}, \dots, \frac{1}{2})$ $\alpha_0(X_1) = \frac{1}{2} = \alpha_n(X_1)$		$\bar{\alpha}_0 + \bar{\alpha}_n \in H(1, n-1)$ $X_2 = (1, \frac{1}{2}, \dots, \frac{1}{2}, 0)$ $\alpha_1(X_2) = \frac{1}{2} = \alpha_{n-1}(X_2)$		








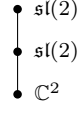
(12) $E_6^{(2)}$

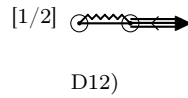
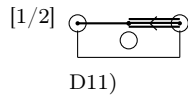
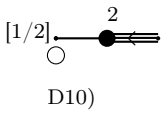
	ω	local model	ω^\sharp	local model	remarks
E1)	$2\bar{\alpha}_1 + 3\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4 \in H(0)$ $X_1 = (0, 0, 0, 0)$ $\alpha_0(X_1) = 1$		$\bar{\alpha}_0 \in H(1)$ $X_2 = (1, 0, 0, 0)$ $\alpha_1(X_2) = \frac{1}{2}$		

E2)	$\bar{\alpha}_2 + \bar{\alpha}_3 + \bar{\alpha}_4 \in H(1)$ $X_1 = (1, 0, 0, 0)$ $\alpha_1(X_1) = \frac{1}{2}$		$\bar{\alpha}_0 + 2\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3 \in H(4)$ $X_2 = (1, 1, 0, 0)$ $\alpha_4(X_2) = 1$		
E3)	$\bar{\alpha}_0 + \bar{\alpha}_1 + \bar{\alpha}_2 \in H(3)$ $X_1 = (1, \frac{1}{2}, \frac{1}{2}, 0)$ $\alpha_3(X_1) = \frac{1}{2}$		$-\omega \in I_3(0, 2)$ $X_2 = (\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ $\alpha_0(X_2) = \frac{1}{4} = \alpha_2(X_2)$		



(13) $D_4^{(3)}$

	ω	local model	$\omega^\#$	local model	remarks
D10)	$[2](2\bar{\alpha}_1 + \bar{\alpha}_2) \in H(0)$ $X_1 = (0, 0, 0)$ $\alpha_0(X_1) = 1$		$[2]\bar{\alpha}_0 \in H(1)$ $X_2 = (0, -\frac{1}{2}, \frac{1}{2})$ $\alpha_1(X_2) = \frac{1}{2}$		
D11)	$[2]\frac{1}{2}(\bar{\alpha}_0 + \bar{\alpha}_2) \in H(1)$ $X_1 = (0, -\frac{1}{2}, \frac{1}{2})$ $\alpha_1(X_1) = \frac{1}{2}$		$[2]\bar{\alpha}_1 \in H(0, 2)$ $X_2 = (\frac{1}{4}, \frac{1}{4}, -\frac{1}{2})$ $\alpha_0(X_2) = \frac{3}{4}, \alpha_2(X_2) = \frac{1}{4}$		
D12)	$\bar{\alpha}_0 + \bar{\alpha}_1 \in H(2)$ $X_1 = (-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3})$ $\alpha_2(X_1) = 1$		$-\omega \in I_2(0, 1)$ $X_2 = (0, -\frac{1}{3}, \frac{1}{3})$ $\alpha_0(X_2) = \frac{1}{3}, \alpha_1(X_2) = \frac{1}{3}$		



Part 4. Hamiltonian Manifolds of Rank one

Let us recall that every q -Hamiltonian manifold locally looks like a Hamiltonian manifold, cf. [Kno16]. Conversely, every Hamiltonian manifold locally carries the structure of a quasi-Hamiltonian manifold. We use this to apply our theory to classify momentum polytopes of compact Hamiltonian manifolds of rank one.

Hence, speaking in the quasi-Hamiltonian setting, we want to look at momentum polytopes of rank one that do not hit the wall of the alcove corresponding to α_0 , but every other wall. That means we can "forget about" the existence of this one wall corresponding to α_0 and see the Weyl chamber of a classical root system that is in bijection with the orbit space \mathfrak{k}^*/K of a Hamiltonian manifold, more details in [Kno11]. We use the classification theorem [Kno11, thm. 11.2] that this manifolds are again characterized by spherical pairs.

We shall add one more assumption on the moment polytope: we do not want our polytope to contain the origin of the Weyl chamber. The reason is the following: if our polytope did, then the local model in the origin were just a smooth affine spherical variety of rank one. As the corresponding weight points inside the Weyl chamber and does not hit any other wall, this is an unbounded polytope. Hence, the only manifolds that we would get by these polytopes are the Akhiezer-list of smooth affine spherical roots of rank one from [Akh83] and the spherical modules of rank one for type A and C .

We classify all compact convex Hamiltonian manifolds such that the corresponding spherical pair (\mathcal{P}, Λ_S) has the following properties:

- The moment polytope $\mathcal{P} = [X_1 X_2]$ is a line segment touching every wall of the dominant chamber, meaning $\alpha(X_1) = 0$ or $\alpha(X_2) = 0$ for every simple root $\alpha \in \{\alpha_1, \dots, \alpha_n\}$, but not the origin.
- The lattice Λ_S has rank one and its generator ω generates \mathcal{P} as a line segment.

So let us now look at the Weyl chamber of a classical root system and assume for the rest of this part that all manifolds are compact multiplicity free Hamiltonian manifolds whose moment polytopes have these properties. Most of our theory now works similar. We have the following structure theorems:

Theorem 4.1. *There are no bi-homogeneous polytopes of rank one for compact Hamiltonian K -manifolds.*

Proof. Suppose there was a bi-homogeneous polytope. Hence the weight ω_1 in X_1 and the weight ω_2 in X_2 are \mathbb{Z} -linear combinations of simple roots. As we suppose our polytope to hit every wall of the alcove, $\omega_2 = -\omega_1$ were possible only if there was a linear dependence between the simple roots (Which is given via the root δ in the affine case!). But as the simple roots of a classical root system are well known to be linearly independent, this is not the case. \square

We can also find a similar statement to [theorem 3.19](#):

Theorem 4.2. *Let ω be the weight of an inhomogeneous smooth affine spherical variety of rank one that is a local model in X_1 with $\langle \omega, \bar{\alpha}_k^\vee \rangle = 1$. Let \mathcal{P} be a line segment that hits every wall of the dominant chamber, but not in the origin. Then, for $X_2 = X_1 + c\omega$, we have*

$$S(X_2) = S \setminus \{\alpha_k\}$$

Proof. We argue as in the proof of [theorem 3.19](#) that $\alpha_1, \dots, \alpha_{k-1} \in S(X_2), \alpha_k \notin S(X_2)$. □

We deduce the same way as for genuine q-Hamiltonians:

Corollary 4.3.

- (1) *Bi-inhomogeneous polytopes are only possible if $|S(X_1)| = |S(X_2)| = n - 1$ and, if $S(X_1) = S \setminus \{\alpha_k\}, \langle \omega, \bar{\alpha}_j^\vee \rangle = 1$, we need $S(X_2) = S \setminus \{\alpha_j\}$ and $\langle -\omega, \bar{\alpha}_k^\vee \rangle = 1$.*
- (2) *Let ω be a homogeneous polytope for X_1 . Mixed polytopes are only possible if $|S(X_1)| = n - 1$.*

Proof.

- (1) Reason the same way as in the proof of [corollary 3.20](#)
- (2) Same reasons as for [corollary 3.24](#)

□

It follows:

Corollary 4.4. *Let M be a compact Hamiltonian K -manifold of rank one for K simple such that the corresponding moment polytope $[X_1 X_2]$ touches every wall of the chamber, but not in the origin. Then: The local root system in w.l.o.g X_1 contains exactly $n - 1$ simple roots.*

We go on and explicitly determine these polytopes, using the same short notation we used for the q -Hamiltonians. If something does not work, the reason is always [lemma 3.3](#).

For A_n , we have:

$S(X_1)$	choice of ω	$S(X_2)$	$-w$ inhom.?
$S \setminus \{\alpha_1\}$	$\bar{\alpha}_{1,n}$	$S \setminus \{\alpha_1, \alpha_n\}$	$-\omega \in I_1 \Rightarrow \mathbf{A3}$
$n = 2$	$2\bar{\alpha}_2$	$S \setminus \{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_1^\vee \rangle = 2$
$n = 4$	$\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4$	$S \setminus \{\alpha_3\}$	$-\omega \in I_4 \Rightarrow \mathbf{A2}$
$n = 4$	$\frac{1}{2}(\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4)$	$S \setminus \{\alpha_3\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle = \frac{1}{2}$,
$S \setminus \{\alpha_d\}, 0 < d < n$	$\bar{\alpha}_{1,d}$	$S \setminus \{\alpha_{1,d}\}$	$-\omega \in I_{d-1} \Rightarrow \mathbf{A3}$
$2 < d < n - 1$	I_1	$S \setminus \{\alpha_1\}$	no: $\langle -\omega, \bar{\alpha}_d^\vee \rangle > 0$
	I_{d-1}	$S \setminus \{\alpha_{d-1}\}$	$-\omega \in I_d \Rightarrow \mathbf{A4}$
$d = 2$	I_1	$S \setminus \{\alpha_1\}$	$-\omega \in I_2 \Rightarrow \mathbf{A4}$
$d = 4$	$[\frac{1}{2}]\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3$	$S \setminus \{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$
$d = 2$	$2\bar{\alpha}_1$	$S \setminus \{\alpha_1\}$	no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle = 2$
$n = 3, d = 2$	$\bar{\alpha}_1 + \bar{\alpha}_3$	$\{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle = 2$
	$\frac{1}{2}(\bar{\alpha}_1 + \bar{\alpha}_3)$	$\{\alpha_2\}$	$-\omega \in I_2 \Rightarrow \mathbf{A1}$

For $B_n, n \geq 2$, we have:

$S(X_1)$	choice of ω	$S(X_2)$	$-w$ inhom.?
$S \setminus \{\alpha_1\}$	$\bar{\alpha}_{2,n}$	$S \setminus \{\alpha_2\}$	$-\omega \in I_1 \Rightarrow \mathbf{B5}$
$S \setminus \{\alpha_1\}$	$2\bar{\alpha}_{2,n}$	$S \setminus \{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_1^\vee \rangle = 2$
$n = 4$	$\bar{\alpha}_2 + 2\bar{\alpha}_3 + 3\bar{\alpha}_4$	$S \setminus \{\alpha_4\}$	$-\omega \in I_1 \Rightarrow \mathbf{B3}$
$n = 4$	$\frac{1}{2}(\bar{\alpha}_2 + 2\bar{\alpha}_3 + 3\bar{\alpha}_4)$	$S \setminus \{\alpha_4\}$	no: $\langle -\omega, \bar{\alpha}_1^\vee \rangle = \frac{1}{2}$
$n = 3$	I_3	$S \setminus \{\alpha_3\}$	$-\omega \in I_1 \Rightarrow \mathbf{B2}$
$S \setminus \{\alpha_n\}$	$\bar{\alpha}_{1,n-1}$	$S \setminus \{\alpha_1, \alpha_{n-1}\}$	no: $\langle -\omega, \bar{\alpha}_n^\vee \rangle = 2$
$n = 4$	$\frac{1}{2}(\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3)$	$S \setminus \{\alpha_2\}$	$-\omega \in I_4 \Rightarrow \mathbf{B4}$
$n = 4$	$\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3$	$S \setminus \{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle = 2$
$n = 2$	$2\bar{\alpha}_1$	$S \setminus \{\alpha_1\}$	no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle = 4$
$S \setminus \{\alpha_n\}$	I_1	$S \setminus \{\alpha_1\}$	no: $\langle -\omega, \bar{\alpha}_n^\vee \rangle > 0$
	I_{n-1}	$S \setminus \{\alpha_{n-1}\}$	$-\omega \in I_n \Rightarrow \mathbf{B6}$
$S \setminus \{\alpha_d\}, 1 < d < n - 1$	$\omega \in C(1)$		no: Type B

$S(X_1)$	choice of ω	$S(X_2)$	$-w$ inhom.?
$d = n - 1$	$\omega \in C(1)$		no: $\langle -\omega, \bar{\alpha}_{n-1}^\vee \rangle > 0$
	I_1	$S \setminus \{\alpha_1\}$	no: $\langle -\omega, \bar{\alpha}_d^\vee \rangle > 0$
	I_{d-1}	$S \setminus \{\alpha_{d-1}\}$	no: $\langle -\omega, \bar{\alpha}_d^\vee \rangle > 0$
$S \setminus \{\alpha_d\}, 1 < d < n$	$\bar{\alpha}_{d+1,n}$	$S \setminus \{\alpha_{d+1}\}$	$-\omega \in I_d \Rightarrow \mathbf{B5}$
	$2\bar{\alpha}_{d+1,n}$	$S \setminus \{\alpha_{d+1}\}$	no: $\langle -\omega, \bar{\alpha}_d^\vee \rangle = 2$
$d = n - 3, n > 4$	$[\frac{1}{2}]\bar{\alpha}_{n-2} + 2\bar{\alpha}_{n-1} + 3\bar{\alpha}_n$	$S \setminus \{\alpha_n\}$	no: $\langle -\omega, \bar{\alpha}_{n-3}^\vee \rangle > 0$
$d = 2, n = 3$	$\frac{1}{2}(\bar{\alpha}_1 + \bar{\alpha}_3)$	$S \setminus \{\alpha_2\}$	$-\omega \in I_2 \Rightarrow \mathbf{B1}$
	$\bar{\alpha}_1 + \bar{\alpha}_3$	$S \setminus \{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle = 2$

$C_n, n \geq 3$:

$S(X_1)$	choice of ω	$S(X_2)$	$-w$ inhom.?
$S \setminus \{\alpha_1\}$	$\bar{\alpha}_2 + 2\bar{\alpha}_{3,n-1} + \bar{\alpha}_n$	$S \setminus \{\alpha_3\}$	$-\omega \in I_1 \Rightarrow \mathbf{C2}$
$n = 3$	$2\bar{\alpha}_2 + 2\bar{\alpha}_3$	$S \setminus \{\alpha_3\}$	no: $\langle -\omega, \bar{\alpha}_1^\vee \rangle = 2$
$n = 3$	$\bar{\alpha}_2 + \bar{\alpha}_3$	$S \setminus \{\alpha_3\}$	$-\omega \in I_1 \Rightarrow \mathbf{C2}$
	I_2	$S \setminus \{\alpha_2\}$	$-\omega \in I_1 \Rightarrow \mathbf{C3}$
$n = 2$	$2\bar{\alpha}_2$	$S \setminus \{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_1^\vee \rangle = 4$
$S \setminus \{\alpha_n\}$	$\bar{\alpha}_{1,n-1}$	$S \setminus \{\alpha_1, \alpha_{n-1}\}$	$-\omega \in I_n \Rightarrow \mathbf{C1}$
	I_1	$S \setminus \{\alpha_1\}$	no: $\langle -\omega, \bar{\alpha}_n^\vee \rangle > 0$
	I_{n-1}	$S \setminus \{\alpha_{n-1}\}$	$-\omega \in I_n \Rightarrow \mathbf{C3}$
$n = 4$	$[\frac{1}{2}]\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3$	$S \setminus \{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_n^\vee \rangle > 0$
$n = 2$	$2\bar{\alpha}_1$	$S \setminus \{\alpha_1\}$	no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle = 2$
$S \setminus \{\alpha_d\}, 1 < d < n$	$\bar{\alpha}_{1,d-1}$	$S \setminus \{\alpha_1, \alpha_{d-1}\}$	$-\omega \in I_d \Rightarrow \mathbf{C1}$
$2 < d$	I_1	$S \setminus \{\alpha_1\}$	no: $\langle -\omega, \bar{\alpha}_d^\vee \rangle > 0$
	I_{d-1}	$S \setminus \{\alpha_{d-1}\}$	$-\omega \in I_d \Rightarrow \mathbf{C3}$
$d = 2$	$2\bar{\alpha}_1$	$S \setminus \{\alpha_1\}$	no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle = 2$
$d = 4, n > 4$	$[\frac{1}{2}]\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3$	$S \setminus \{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$
$S \setminus \{\alpha_d\}, 1 < d < n$	$\bar{\alpha}_{d+1} + 2\bar{\alpha}_{d+2,n-1} + \bar{\alpha}_n$	$S \setminus \{\alpha_{d+2}\}$	no: $\langle -\omega, \bar{\alpha}_d^\vee \rangle = 1$
	I_{d+1}	$S \setminus \{\alpha_{d+1}\}$	$-\omega \in I_d \Rightarrow \mathbf{C3}$
$d = n - 2$	$2\bar{\alpha}_{n-1} + 2\bar{\alpha}_n$	$S \setminus \{\alpha_n\}$	no: $\langle -\omega, \bar{\alpha}_{n-2}^\vee \rangle = 2$

$S(X_1)$	choice of ω	$S(X_2)$	$-w$ inhom.?
$d = n - 1$	$2\bar{\alpha}_n$	$S \setminus \{\alpha_n\}$	no: $\langle -\omega, \bar{\alpha}_{n-1}^\vee \rangle = 4$

$D_n, n \geq 4$:

$S(X_1)$	choice of ω	$S(X_2)$	$-w$ inhom.?
$S \setminus \{\alpha_1\}$	$\bar{\alpha}_{2,n-2} + \frac{1}{2}\bar{\alpha}_{n-1} + \frac{1}{2}\bar{\alpha}_n$	$S \setminus \{\alpha_2\}$	$-\omega \in I_1 \Rightarrow \mathbf{D1}$
	$2\bar{\alpha}_{2,n-2} + \bar{\alpha}_{n-1} + \bar{\alpha}_n$	$S \setminus \{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_1^\vee \rangle = 2$
$S \setminus \{\alpha_n\}$	$\bar{\alpha}_{1,n-1}$	$S \setminus \{\alpha_1, \alpha_{n-1}\}$	$-\omega \in I_n \Rightarrow \mathbf{D2}$
	I_1	$S \setminus \{\alpha_1\}$	no: Type D
	I_{n-1}	$S \setminus \{\alpha_{n-1}\}$	$-\omega \in I_n \Rightarrow \mathbf{D3}$
$S \setminus \{\alpha_{n-2}\}$	$[2]\bar{\alpha}_n$	$S \setminus \{\alpha_n\}$	no: $\langle -\omega, \bar{\alpha}_{n-2}^\vee \rangle > 0$
	$\bar{\alpha}_{n-1} + \bar{\alpha}_n$	$S \setminus \{\alpha_{n-1}, \alpha_n\}$	no: $\langle -\omega, \bar{\alpha}_{n-2}^\vee \rangle = 2$
	$\frac{1}{2}(\bar{\alpha}_{n-1} + \bar{\alpha}_n)$	$S \setminus \{\alpha_{n-1}, \alpha_n\}$	$-\omega \in I_{n-2} \Rightarrow \mathbf{D1}$
	I_1	$S \setminus \{\alpha_1\}$	no: Type D
	I_{n-3}	$S \setminus \{\alpha_{n-3}\}$	no: $\langle -\omega, \bar{\alpha}_{n-2}^\vee \rangle > 0$
	$\omega \in C(1)$		no: $\langle -\omega, \bar{\alpha}_{n-2}^\vee \rangle > 0$
$S \setminus \{\alpha_d\}, 1 < d < n-2$	$\omega \in C(1)$		no: Type D
	$\omega = I_1$	$S \setminus \{\alpha_1\}$	no: Type D
	$\omega = I_{d-1}$	$S \setminus \{\alpha_{d-1}\}$	no: Type D
	$\bar{\alpha}_{d+1,n-2} + \frac{1}{2}\bar{\alpha}_{n-1} + \frac{1}{2}\bar{\alpha}_n$	$S \setminus \{\alpha_{d+1}\}$	$-\omega \in I_d \Rightarrow \mathbf{D1}$
	$2\bar{\alpha}_{d+1,n-2} + \bar{\alpha}_{n-1} + \bar{\alpha}_n$	$S \setminus \{\alpha_{d+1}\}$	no: $\langle -\omega, \bar{\alpha}_d^\vee \rangle = 2$

E_6 :

$S(X_1)$	choice of ω	$S(X_2)$	$-w$ inhom.?
$S \setminus \{\alpha_1\}$	$\omega \in C(6)$	$S \setminus \{\alpha_6\}$	no: Type D
$S \setminus \{\alpha_2\}$	$\bar{\alpha}_1 + \bar{\alpha}_{3,6}$	$S \setminus \{\alpha_1, \alpha_6\}$	no: Type D
	I_1	$S \setminus \{\alpha_1\}$	no: Type D
$S \setminus \{\alpha_3\}$	$[2]\bar{\alpha}_1$ or I_1	$S \setminus \{\alpha_1\}$	no: Type D
	$\bar{\alpha}_2 + \bar{\alpha}_{4,6}$	$S \setminus \{\alpha_2, \alpha_6\}$	no: $\langle -\omega, \bar{\alpha}_3^\vee \rangle > 0$
	I_2	$S \setminus \{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_3^\vee \rangle > 0$

$S(X_1)$	choice of ω	$S(X_2)$	$-w$ inhom.?
	I_6	$S \setminus \{\alpha_6\}$	no: Type D
$S \setminus \{\alpha_4\}$	$\bar{\alpha}_1 + \bar{\alpha}_3$	$S \setminus \{\alpha_1, \alpha_3\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$
	I_1	$S \setminus \{\alpha_1\}$	no: Type D
	I_3	$S \setminus \{\alpha_3\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$
	$[2]\bar{\alpha}_2$ or I_2	$S \setminus \{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$

 E_7 :

$S(X_1)$	choice of ω	$S(X_2)$	$-w$ inhom.?
$S \setminus \{\alpha_1\}$	$\omega \in C(7)$	$S \setminus \{\alpha_7\}$	no: Type D
$S \setminus \{\alpha_2\}$	$\bar{\alpha}_1 + \bar{\alpha}_{3,7}$	$S \setminus \{\alpha_1, \alpha_7\}$	no: Type D
	I_1	$S \setminus \{\alpha_1\}$	no: Type D
	I_7	$S \setminus \{\alpha_7\}$	no: Type E_6
$S \setminus \{\alpha_3\}$	$[2]\bar{\alpha}_1$ or I_1	$S \setminus \{\alpha_1\}$	no: Type D
	$\bar{\alpha}_2 + \bar{\alpha}_{4,7}$	$S \setminus \{\alpha_2, \alpha_7\}$	no: $\langle -\omega, \bar{\alpha}_3^\vee \rangle > 0$
	I_2	$S \setminus \{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_3^\vee \rangle > 0$
	I_7	$S \setminus \{\alpha_7\}$	no: Type E_6
$S \setminus \{\alpha_4\}$	$\bar{\alpha}_1 + \bar{\alpha}_3$	$S \setminus \{\alpha_1, \alpha_3\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$
	I_1	$S \setminus \{\alpha_1\}$	no: Type D
	I_3	$S \setminus \{\alpha_3\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$
	$[2]\bar{\alpha}_2$ or I_2	$S \setminus \{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$
	$\bar{\alpha}_{5,7}$	$S \setminus \{\alpha_5, \alpha_7\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$
	$[\frac{1}{2}]\bar{\alpha}_5 + 2\bar{\alpha}_6 + \bar{\alpha}_7$	$S \setminus \{\alpha_6\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$
	I_5	$S \setminus \{\alpha_5\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$
	I_7	$S \setminus \{\alpha_7\}$	no: Type E_6
$S \setminus \{\alpha_5\}$	$\bar{\alpha}_1 + \bar{\alpha}_3 + \bar{\alpha}_4 + \bar{\alpha}_2$	$S \setminus \{\alpha_1, \alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_5^\vee \rangle > 0$
	I_1	$S \setminus \{\alpha_1\}$	no: Type D
	I_2	$S \setminus \{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_5^\vee \rangle > 0$
	$\bar{\alpha}_{6,7}$	$S \setminus \{\alpha_6, \alpha_7\}$	no: Type D
	I_6	$S \setminus \{\alpha_6\}$	no: $\langle -\omega, \bar{\alpha}_5^\vee \rangle > 0$

$S(X_1)$	choice of ω	$S(X_2)$	$-w$ inhom.?
	I_7	$S \setminus \{\alpha_7\}$	no: Type E
$S \setminus \{\alpha_6\}$	$\omega \in C(1)$	$S \setminus \{\alpha_1\}$	no: Type D
	$[2]\bar{\alpha}_7$ or I_7	$S \setminus \{\alpha_7\}$	no: Type E_6
$S \setminus \{\alpha_7\}$	nothing possible: Type E_6		

 E_8 :

$S(X_1)$	choice of ω	$S(X_2)$	$-w$ inhom.?
$S \setminus \{\alpha_1\}$	$\omega \in C(8)$	$S \setminus \{\alpha_8\}$	no: Type D
$S \setminus \{\alpha_2\}$	$\bar{\alpha}_1 + \bar{\alpha}_{3,8}$	$S \setminus \{\alpha_1, \alpha_8\}$	no: Type D
	I_1	$S \setminus \{\alpha_1\}$	no: Type D
	I_8	$S \setminus \{\alpha_8\}$	no: Type E_7
$S \setminus \{\alpha_3\}$	$[2]\bar{\alpha}_1$ or I_1	$S \setminus \{\alpha_1\}$	no: Type D
	$\bar{\alpha}_2 + \bar{\alpha}_{4,8}$	$S \setminus \{\alpha_2, \alpha_8\}$	no: $\langle -\omega, \bar{\alpha}_3^\vee \rangle > 0$
	I_2	$S \setminus \{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_3^\vee \rangle > 0$
	I_8	$S \setminus \{\alpha_8\}$	no: Type E_7
$S \setminus \{\alpha_4\}$	$\bar{\alpha}_1 + \bar{\alpha}_3$	$S \setminus \{\alpha_1, \alpha_3\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$
	I_1	$S \setminus \{\alpha_1\}$	no: Type D
	I_3	$S \setminus \{\alpha_3\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$
	$[2]\bar{\alpha}_2$ or I_2	$S \setminus \{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$
	$\bar{\alpha}_{5,8}$	$S \setminus \{\alpha_5, \alpha_8\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$
	I_5	$S \setminus \{\alpha_5\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle > 0$
	I_8	$S \setminus \{\alpha_8\}$	no: Type E_7
$S \setminus \{\alpha_5\}$	$\bar{\alpha}_1 + \bar{\alpha}_3 + \bar{\alpha}_4 + \bar{\alpha}_2$	$S \setminus \{\alpha_1, \alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_5^\vee \rangle > 0$
	I_1	$S \setminus \{\alpha_1\}$	no: Type D
	I_2	$S \setminus \{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_5^\vee \rangle > 0$
	$\bar{\alpha}_{6,8}$	$S \setminus \{\alpha_6, \alpha_7\}$	no: Type D
	$[\frac{1}{2}]\bar{\alpha}_6 + 2\bar{\alpha}_7 + \bar{\alpha}_8$	$S \setminus \{\alpha_7\}$	no: $\langle -\omega, \bar{\alpha}_5^\vee \rangle > 0$
	I_6	$S \setminus \{\alpha_6\}$	no: $\langle -\omega, \bar{\alpha}_5^\vee \rangle > 0$
	I_8	$S \setminus \{\alpha_8\}$	no: Type E_7

$S(X_1)$	choice of ω	$S(X_2)$	$-w$ inhom.?
$S \setminus \{\alpha_6\}$	$\omega \in C(1)$	$S \setminus \{\alpha_1\}$	no: Type D
	$\bar{\alpha}_{7,8}$	$S \setminus \{\alpha_7, \alpha_8\}$	no: Type E_6
	I_7	$S \setminus \{\alpha_7\}$	no: $\langle -\omega, \bar{\alpha}_6^\vee \rangle > 0$
	I_8	$S \setminus \{\alpha_8\}$	no: Type E_7
$S \setminus \{\alpha_7\}$	$[2]\bar{\alpha}_8$ or I_8	$S \setminus \{\alpha_8\}$	no: Type E_7
$S \setminus \{\alpha_8\}$	nothing possible: Type E_7		

 F_4 :

$S(X_1)$	choice of ω	$S(X_2)$	$-w$ inhom.?
$S \setminus \{\alpha_1\}$	$\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4$	$S \setminus \{\alpha_3\}$	$-\omega \in I_1 \Rightarrow \mathbf{F1}$
	I_4	$S \setminus \{\alpha_4\}$	no: Type B
$S \setminus \{\alpha_2\}$	$[2]\bar{\alpha}_1$ or I_1	$S \setminus \{\alpha_1\}$	no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle > 0$
	$\bar{\alpha}_3 + \bar{\alpha}_4$	$S \setminus \{\alpha_3, \alpha_4\}$	$-\omega \in I_2 \Rightarrow \mathbf{F2}$
	I_3	$S \setminus \{\alpha_3\}$	$-\omega \in I_2 \Rightarrow \mathbf{F4}$
	I_4	$S \setminus \{\alpha_4\}$	no: Type B
$S \setminus \{\alpha_3\}$	$\bar{\alpha}_1 + \bar{\alpha}_2$	$S \setminus \{\alpha_1, \alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_3^\vee \rangle = 2$
	I_1	$S \setminus \{\alpha_1\}$	no: $\langle -\omega, \bar{\alpha}_3^\vee \rangle > 0$
	I_2	$S \setminus \{\alpha_2\}$	$-\omega \in I_3$, as already seen $\Rightarrow \mathbf{F2}$
	$[2]\bar{\alpha}_4$ or I_4	$S \setminus \{\alpha_4\}$	no: Type B
$S \setminus \{\alpha_4\}$	$\bar{\alpha}_{1,3}$	$S \setminus \{\alpha_1\}$	$-\omega \in I_4 \Rightarrow \mathbf{F3}$
	$2\bar{\alpha}_{1,3}$	$S \setminus \{\alpha_1\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle = 2$
	$\bar{\alpha}_1 + 2\bar{\alpha}_2 + 3\bar{\alpha}_3$	$S \setminus \{\alpha_3\}$	no: $\langle -\omega, \bar{\alpha}_4^\vee \rangle = 3$

 G_2 :

$S(X_1)$	choice of ω	$S(X_2)$	$-w$ inhom.?
$S \setminus \{\alpha_1\}$	$[2]\bar{\alpha}_2$	$S \setminus \{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_1^\vee \rangle = [2] \cdot 3$
	I_2	$S \setminus \{\alpha_2\}$	no: $\langle -\omega, \bar{\alpha}_1^\vee \rangle = \frac{3}{2}$

$S(X_1)$	choice of ω	$S(X_2)$	$-w$ inhom.?
$S \setminus \{\alpha_2\}$	$2\bar{\alpha}_1$	$S \setminus \{\alpha_1\}$	no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle = 2$
	$\bar{\alpha}_1$	$S \setminus \{\alpha_1\}$	$-\omega \in I_2 \Rightarrow \mathbf{G1}$
	I_1	$S \setminus \{\alpha_1\}$	no: $\langle -\omega, \bar{\alpha}_2^\vee \rangle = \frac{1}{2}$

9. LIST OF HAMILTONIAN MANIFOLDS OF RANK ONE

Theorem 4.5 (Momentum Polytopes of Rank one). *A compact multiplicity free Hamiltonian K -manifold of rank one for K simple with moment polytope $\mathcal{P} = [X_1 X_2]$ such that $\alpha(X_1) = 0$ or $\alpha(X_2) = 0$ for every simple root $\alpha_1, \dots, \alpha_n$ that does not touch the origin corresponds to a spherical pair (\mathcal{P}, Λ_S) with the following properties:*

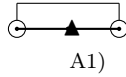
- \mathcal{P} is a line segment $[X_1 X_2]$ that is spanned as a line by $X_1 + c\omega$ for some ω found in the list below and X_1 such that $S(X_1)$ is the local root system indicated.
- The lattice Λ_S is generated by this ω .

Note that for Hamiltonian manifolds, the points X_1, X_2 are only given up to uniform scaling with the origin of the Weyl chamber as center. Hence we only list a representative with "nice" coordinates and do not mention the scaling factor to avoid overloading the tables.

List 4.6.

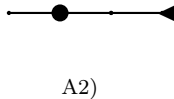
A_n For $n = 3$ only:

	ω	local model	$-\omega$	local model	remarks
A1)	$\frac{1}{2}(\bar{\alpha}_1 + \bar{\alpha}_3) \in H(2)$ $X_1 = (1, 1, -1, -1)$ $\alpha_2(X_1) = 2$		$-\omega \in I_2(\mathbf{1}, \mathbf{3})$ $X_2 = (2, 0, 0, -2)$ $\alpha_1(X_2) = 2 = \alpha_3(X_2)$		



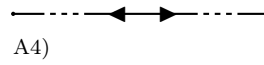
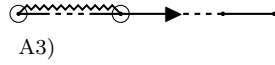
For $n = 4$ only:

	ω	local model	$-\omega$	local model	remarks
A2)	$\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3 \in H(4)$ $X_1 = (1, 1, 1, 1 - 4)$ $\alpha_4(X_1) = 5$		$-\omega \in I_4(\mathbf{2})$ $X_2 = (6, 6, -4, -4)$ $\alpha_2(X_2) = 10$		



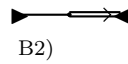
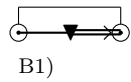
For arbitrary $n \geq 2$:

	ω	local model	$-\omega$	local model	remarks
A3)	$\bar{\alpha}_{1,k} \in H(k+1)$ $X_1 = (n-k, \dots, n-k, \dots, -k-1, \dots, -k-1)$ $\alpha_{k+1}(X_1) = n+1$	<ul style="list-style-type: none"> • $\mathfrak{sl}(k)$ • $\mathfrak{gl}(k-1)$ 	$-\omega \in I_{k+1}(\mathbf{1}, k)$ $X_2 = (2n-k+1, n-k, \dots, n-k, -k-1, \dots, -k-1)$ $\alpha_1(X_2) = n+1 = \alpha_k(X_2)$	<ul style="list-style-type: none"> • $\mathfrak{t}^1 + \mathfrak{sl}(n-k) + \mathfrak{t}^1$ • $\mathfrak{t}^1 + \mathfrak{sl}(n-k) + \mathfrak{t}^1$ • $\mathbb{C}_{1,1}^{n-k}$ 	
A4)	$\omega \in I_k(k+1) \in H(k+1)$ $X_1 = (n-k, \dots, n-k, \dots, -k-1, \dots, -k-1)$ $\alpha_{k+1}(X_1) = n+1$	<ul style="list-style-type: none"> • $\mathfrak{t}^1 + \mathfrak{sl}(k)$ • $\mathfrak{t}^1 + \mathfrak{sl}(k)$ • $\mathbb{C}_{-k/n+1}^k$ 	$-\omega \in I_{k+1}(k)$ $X_2 = (n-k+1, \dots, n-k+1, -k, \dots, -k)$ $\alpha_k(X_2) = n+1$	<ul style="list-style-type: none"> • $\mathfrak{t}^1 + \mathfrak{sl}(n-k)$ • $\mathfrak{t}^1 + \mathfrak{sl}(n-k)$ • $\mathbb{C}_{n-k/n+1}^{n-k}$ 	



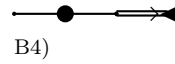
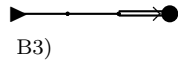
B_n For $n = 3$ only:

	ω	local model	$-\omega$	local model	remarks
B1)	$\frac{1}{2}(\bar{\alpha}_1 + \bar{\alpha}_3) \in H(2)$ $X_1 = (3, 3, 0)$ $\alpha_2(X_1) = 3$	<ul style="list-style-type: none"> • $\mathfrak{sl}(2)$ • $\mathfrak{sl}(2)$ • $\mathfrak{sl}(2)$ 	$-\omega \in I_2(\mathbf{1}, 3)$ $X_2 = (4, 2, 2)$ $\alpha_1(X_2) = 2 = \alpha_3(X_2)$	<ul style="list-style-type: none"> • $\mathfrak{t}^1 + \mathfrak{sl}(2) + \mathfrak{t}^1$ • $\mathfrak{t}^1 + \mathfrak{sl}(2) + \mathfrak{t}^1$ • $\mathbb{C}_{1/2, -1/2}^2$ 	
B2)	$\omega \in I_3(\mathbf{1})$ $X_1 = (2, 0, 0)$ $\alpha_1(X_1) = 2$	<ul style="list-style-type: none"> • $\mathfrak{t}^1 + \mathfrak{sp}(4)$ • $\mathfrak{t}^1 + \mathfrak{sp}(4)$ • $\mathbb{C}_{1/2}^4$ 	$-\omega \in I_1(\mathbf{3})$ $X_2 = (1, 1, 1)$ $\alpha_3(X_2) = 1$	<ul style="list-style-type: none"> • $\mathfrak{t}^1 + \mathfrak{sp}(4)$ • $\mathfrak{t}^1 + \mathfrak{sp}(4)$ • $\mathbb{C}_{-1/2}^4$ 	



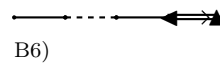
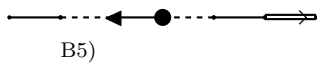
For $n = 4$ only:

	ω	local model	$-\omega$	local model	remarks
B3)	$\bar{\alpha}_2 + 2\bar{\alpha}_3 + 3\bar{\alpha}_4 \in H(1)$ $X_1 = (1, 0, 0, 0)$ $\alpha_1(X_1) = 1$	<ul style="list-style-type: none"> • $\mathfrak{so}(7)$ • G_2 	$-\omega \in I_1(4)$ $X_2 = (1, 1, 1, 1)$ $\alpha_4(X_2) = 1$	<ul style="list-style-type: none"> • $\mathfrak{t}^1 + \mathfrak{sl}(4)$ • $\mathfrak{t}^1 + \mathfrak{sl}(4)$ • \mathbb{C}_{-3}^4 	
B4)	$\frac{1}{2}\bar{\alpha}_1 + \bar{\alpha}_2 + \frac{1}{2}\bar{\alpha}_3 \in H(4)$ $X_1 = (1, 1, 1, 1)$ $\alpha_4(X_1) = 1$	<ul style="list-style-type: none"> • $\mathfrak{sl}(4)$ • $\mathfrak{sp}(4)$ 	$-\omega \in I_4(2)$ $X_2 = (2, 2, 0, 0)$ $\alpha_2(X_2) = 2$	<ul style="list-style-type: none"> • $\mathfrak{t}^1 + \mathfrak{sp}(4)$ • $\mathfrak{t}^1 + \mathfrak{sp}(4)$ • \mathbb{C}_{-1}^4 	

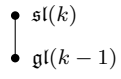
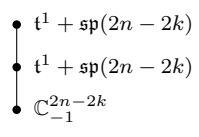
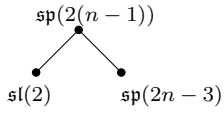
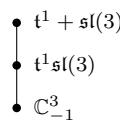
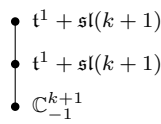
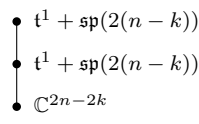


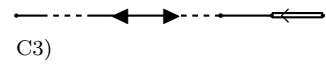
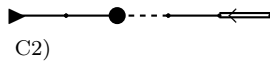
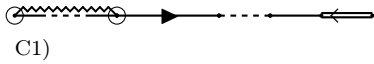
For arbitrary $n \geq 2$:

	ω	local model	$-\omega$	local model	remarks
B5)	$\bar{\alpha}_{k+1} + \dots + \bar{\alpha}_n \in H(k)$ $X_1 = (1, \dots, \underset{k}{1}, 0, \dots, 0)$ $\alpha_k(X_1) = 1$	<ul style="list-style-type: none"> • $\mathfrak{so}(2(n-k)+1)$ • $\mathfrak{so}(2(n-k))$ 	$-\omega \in I_k(k+1)$ $X_2 = (1, \dots, \underset{k+1}{1}, 0, \dots, 0)$ $\alpha_{k+1}(X_2) = 1$	<ul style="list-style-type: none"> • $\mathfrak{t}^1 + \mathfrak{sl}(k)$ • $\mathfrak{t}^1 + \mathfrak{sl}(k)$ • \mathbb{C}_{-1}^k 	
B6)	$\omega \in I_{n-1}(n)$ $X_1 = (1, \dots, 1)$ $\alpha_n(X_1) = 1$	<ul style="list-style-type: none"> • $\mathfrak{t}^1 + \mathfrak{sl}(n)$ • $\mathfrak{t}^1 \mathfrak{sl}(n)$ • \mathbb{C}^n 	$-\omega \in I_n(n-1)$ $X_2 = (2, \dots, 2, 0)$ $\alpha_{n-1}(X_2) = 2$	<ul style="list-style-type: none"> • $\mathfrak{t}^1 + \mathfrak{sl}(2)$ • $\mathfrak{t}^1 + \mathfrak{sl}(2)$ • $\mathbb{C}_{-3/2}^2$ 	



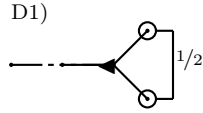
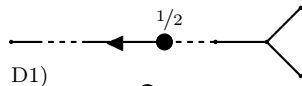
C_n For arbitrary $n \geq 2$:

	ω	local model	$-\omega$	local model	remarks
C1)	$\bar{\alpha}_1 + \dots + \bar{\alpha}_k \in H(k+1)$ $X_1 = (1, \dots, \underset{k+1}{1}, 0, \dots, 0)$ $\alpha_{k+1}(X_1) = 1$		$-\omega \in I_{k+1}(1, k)$ $X_2 = (2, 1, \dots, \underset{k}{1}, 0, \dots, 0)$ $\alpha_1(X_2) = 1 = \alpha_k(X_2)$		$1 \leq k < n$
C2)	$\bar{\alpha}_2 + 2\bar{\alpha}_3 + \dots + 2\bar{\alpha}_{n-1} + \bar{\alpha}_n \in H(1)$ $X_1 = (1, 0, \dots, 0)$ $\alpha_1(X_1) = 1$		$-\omega \in I_1(3)$ $X_2 = (1, 1, 1, 0, \dots, 0)$ $\alpha_3(X_2) = 1$		
C3)	$\omega \in I_k(k+1)$ $X_1 = (1, \dots, \underset{k+1}{1}, 0, \dots, 0)$ $\alpha_{k+1}(X_1) = 1$		$-\omega \in I_{k+1}(k)$ $X_2 = (1, \dots, \underset{k}{1}, 0, \dots, 0)$ $\alpha_k(X_2) = 1$		

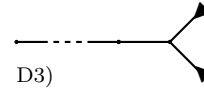
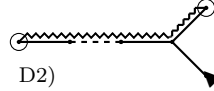


D_n For $n \geq 4$:

	ω	local model	$-\omega$	local model	remarks
D1)	$\bar{\alpha}_{k+1} + \dots + \bar{\alpha}_{n-2} + \frac{1}{2}\bar{\alpha}_{n-1} + \frac{1}{2}\bar{\alpha}_n \in H(k)$ $X_1 = (1, \dots, \frac{1}{k}, 0, \dots, 0)$ $\alpha_k(X_1) = 1$	\bullet $\mathfrak{so}(2(n-k))$ \bullet $\mathfrak{so}(2(n-k)-1)$	$-\omega \in I_k(k+1)$ $X_2 = (1, \dots, \frac{1}{k+1}, 0, \dots, 0)$ $\alpha_{k+1}(X_2) = 1$	\bullet $\mathfrak{t}^1 + \mathfrak{sl}(k)$ \bullet $\mathfrak{t}^1 + \mathfrak{sl}(k)$ \bullet \mathbb{C}_{-1}^k	
D2)	$\bar{\alpha}_1 + \dots + \bar{\alpha}_{n-1} \in H(n)$ $X_1 = (2, \dots, 2)$ $\alpha_n(X_1) = 4$	\bullet $\mathfrak{sl}(n)$ \bullet $\mathfrak{gl}(n-1)$	$-\omega \in I_n(1, n-1)$ $X_2 = (3, 2, \dots, 2, 1)$ $\alpha_1(X_2) = 1 = \alpha_{n-1}(X_2)$	\bullet $\mathfrak{t}^1 + \mathfrak{sl}(n-1) + \mathfrak{t}^1$ \bullet $\mathfrak{t}^1 + \mathfrak{sl}(n-1) + \mathfrak{t}^1$ \bullet $\mathbb{C}_{1,-1}^{n-1}$	or n and $n-1$ switched
D3)	$\omega \in I_{n-1}(n)$ $X_1 = (1, \dots, 1)$ $\alpha_n(X_1) = 2$	\bullet $\mathfrak{sl}(n)$ \bullet $\mathfrak{sl}(n)$ \bullet $\mathbb{C}_{-1/2}^n$	$-\omega \in I_n(n-1)$ $X_2 = (1, \dots, 1, -1)$ $\alpha_{n-1}(X_2) = 2$	\bullet $\mathfrak{sl}(n)$ \bullet $\mathfrak{sl}(n)$ \bullet $\mathbb{C}_{-1/2}^n$	or n and $n-1$ switched



is part of case D1)

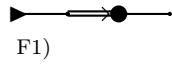


$E_n, n \in \{6, 7, 8\}$ nothing

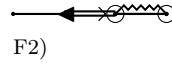
F_4 :

	ω	local model	$-\omega$	local model	remarks
F1)	$\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4 \in H(1)$ $X_1 = (2, 2, 0, 0)$ $\alpha_1(X_1) = 2$	\bullet $\mathfrak{sp}(6)$ \bullet $\mathfrak{sl}(2)$ \bullet $\mathfrak{sp}(4)$	$-\omega \in I_1(3)$ $X_2 = (3, 1, 1, 1)$ $\alpha_3(X_2) = 1$	\bullet $\mathfrak{t}^1 + \mathfrak{sl}(3)$ \bullet $\mathfrak{t}^1 + \mathfrak{sl}(3)$ \bullet \mathbb{C}_{-2}^3	
F2)	$\bar{\alpha}_3 + \bar{\alpha}_4 \in H(2)$ $X_1 = (4, 2, 2, 0)$ $\alpha_2(X_1) = 2$	\bullet $\mathfrak{sl}(3)$ \bullet $\mathfrak{gl}(2)$	$-\omega \in I_2(3, 4)$ $X_2 = (5, 1, 1, 1)$ $\alpha_3(X_2) = 1 = \alpha_4(X_2)$	\bullet $\mathfrak{t}^1 + \mathfrak{sl}(3) + \mathfrak{t}^1$ \bullet $\mathfrak{t}^1 + \mathfrak{sl}(3) + \mathfrak{t}^1$ \bullet $\mathbb{C}_{-1,-1}^3$	

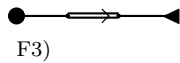
F3)	$\bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3 \in H(4)$ $X_1 = (1, 0, 0, 0)$ $\alpha_4(X_1) = \frac{1}{2}$	<ul style="list-style-type: none"> • $\mathfrak{so}(7)$ • $\mathfrak{so}(6)$ 	$-\omega \in I_4(1)$ $X_2 = (1, 1, 0, 0)$ $\alpha_1(X_2) = 1$	<ul style="list-style-type: none"> • $t^1 + \mathfrak{sp}(6)$ • $t^1 + \mathfrak{sp}(6)$ • \mathbb{C}_{-1}^6 	
F4)	$\omega \in I_3(2)$ $X_1 = (4, 2, 2, 0)$ $\alpha_2(X_1) = 2$	<ul style="list-style-type: none"> • $\mathfrak{sl}(3)$ • $\mathfrak{sl}(3)$ • \mathbb{C}_2^3 	$-\omega \in I_2(3)$ $X_2 = (3, 1, 1, 1)$ $\alpha_3(X_2) = 1$	<ul style="list-style-type: none"> • $t^1 + \mathfrak{sl}(3)$ • $t^1 + \mathfrak{sl}(3)$ • \mathbb{C}_{-2}^3 	



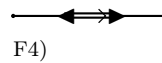
F1)



F2)



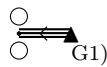
F3)



F4)

$G_2 :$

	ω	local model	$-\omega$	local model	remarks
G1)	$\bar{\alpha}_1 \in H(2)$ $X_1 = (-1, -1, 2)$ $\alpha_2(X_1) = 3$	<ul style="list-style-type: none"> • $\mathfrak{sl}(2)$ 	$-\omega \in I_2(1)$ $X_2 = (0, -2, 2)$ $\alpha_1(X_2) = 2$	<ul style="list-style-type: none"> • $t^1 + \mathfrak{sl}(2)$ • $t^1 + \mathfrak{sl}(2)$ • \mathbb{C}_{-1}^2 	



G1)

Part 5. Surjective Moment Maps

This part of the thesis is devoted to the classification of all convex quasi-Hamiltonian manifolds with surjective moment maps which is equivalent to $\mathcal{P} = \mathcal{A}$. We shall also call these polytopes surjective by abuse of notation.

We recall that it suffices to check the spherical-pair-condition for minimal faces of the moment polytope which are in this case the vertices of the fundamental alcove. We use Λ^+ for the set of dominant weights of a group and Λ for its weight lattice. We use the shortcut $S_i, i \in \{0, 1, \dots, n\}$ for $S \setminus \{\alpha_i\}$, the local root system of a vertex of the alcove where all simple roots except α_i are zero. We use X_i for the corresponding vertex $\{X \in \mathcal{A} : \alpha(X) = 0 \forall \alpha \in S \setminus \{\alpha_i\}\}$. We shall also denote Λ_i^+ for the set of fundamental dominant weights $\omega_0, \omega_1, \dots, \omega_{i-1}, \omega_{i+1}, \dots, \omega_n$ (note that these weights depend on i). We shall use Λ_i for the lattice generated by the set Λ_i^+ .

For a set of dominant weights W , we use $\mathbb{N}W$ for the monoid generated by these weights and $\mathbb{Z}W$ for the corresponding lattice.

We will go by the following three steps:

- (1) We choose an irreducible affine root system and fix one vertex of the alcove, without loss of generality X_0 .
- (2) We look for all weight monoids $\mathbb{N}\Gamma_0$ of L -saturated smooth affine spherical varieties of full rank for the Levi subgroup L corresponding to X_0 . The cone spanned by its weight monoid is exactly the tangent cone of \mathcal{P} in this vertex. This means we need a classification of all saturated weight monoids of full rank of smooth affine spherical varieties for simple groups. This was done in [PPVS17].
- (3) We check for every other vertex of the alcove if the pair $(\mathcal{P}, \mathbb{Z}\Gamma_0)$ is spherical, which means to check whether $\mathbb{Z}\Gamma_0 \cap C_{X_i}(\mathcal{P})$ is the weight monoid of a smooth affine spherical variety for every vertex $X_i, i = 1, \dots, n$ of the alcove.

We remark that the second step gives a classification of (non-compact!) Hamiltonian manifolds with surjective moment map as a side result.

10. LOCAL MODELS FOR MANIFOLDS WITH SURJECTIVE MOMENT MAP

In this section, we recall our classification of all G -saturated weight monoids of full rank for G a simple group, meaning they span the whole dominant chamber, from [PPVS17]. So this section closely follows part 3 of loc.cit. We always think of these monoids as local models for surjective moment polytopes in X_0 , where the local root system is the root system of a simple group.

Let us first recall that some of the weight monoids of spherical varieties have been known for long time, e.g. from [Kra79] in case of homogenous spherical varieties. But, as already mentioned in the introduction, all of these classifications only list so-called primitive varieties. Each is in some sense the most generic representative of a whole family a varieties. The result from [PPVS17] is the first complete classification of all G -saturated weight monoids of full rank for simple groups.

We first give this classification from [PPVS17] and then discuss major steps of the proof, in particular the smoothness criterion from [PVS15].

List 5.1. G -saturated smooth affine spherical varieties of full rank

	G	Z	$\mathbb{Z}\Gamma(Z)$	$\Sigma^N(Z)$
A1	$\mathrm{SL}(2)$	\mathbb{C}^2	$\langle \omega_1 \rangle_{\mathbb{Z}}$	\emptyset
A2	$\mathrm{SL}(2)$	$\mathrm{SL}(2)/\mathrm{T}$	$\langle 2\omega_1 \rangle_{\mathbb{Z}}$	$\{2\bar{\alpha}_1\}$
A3	$\mathrm{SL}(2)$	$\mathrm{SL}(2)/\mathrm{N}(\mathrm{T})$	$\langle 4\omega_1 \rangle_{\mathbb{Z}}$	$\{2\bar{\alpha}_1\}$
A4	$\mathrm{SL}(n+1), n \geq 2$	$\mathrm{SL}(n+1)/Z_d \mathrm{SO}(n+1)$ with $d (n+1)$ $Z_d := \{\zeta \mathrm{Id} : \zeta \in \mathbb{C}, \zeta^d = 1\}$ if n is even, $Z_d := \langle \mathrm{diag}(\xi^{\frac{n+1}{d}}, \dots, \xi^{\frac{n+1}{d}}, (-\xi)^{\frac{n+1}{d}}) \rangle$ $\xi = \text{primitive } 2(n+1)\text{-th root of } 1$ if n is odd	$2\langle \bar{\alpha}_2, \bar{\alpha}_3, \dots, \bar{\alpha}_n, d\omega_n \rangle_{\mathbb{Z}}$	$2S$
A5	$n \geq 2,$ even	$\mathrm{SL}(n+1)/H_k \mathrm{Sp}(n)$ $k \in \mathbb{N} \setminus \{0\}$ $H_k := \{\mathrm{diag}(\zeta, \dots, \zeta, \zeta^{-n}) : \zeta \in \mathbb{C}, \zeta^{2k} = 1\}$	$\mathbb{Z}S^+ \oplus \mathbb{Z}(k\omega_{n-1})$	S^+
A6	$n \geq 2,$ odd	$\mathrm{SL}(n+1) \times_{Z_{2e} \mathrm{Sp}(n+1)} \mathbb{C}^{n+1}$ $e \frac{n+1}{2};$ $0 \leq r \leq e-1$ $\zeta \mathrm{Id} \in Z_{2e}$ acts on \mathbb{C}^{n+1} as the multiplication by ζ^r	$\langle \bar{\alpha}_2 + \bar{\alpha}_3, \bar{\alpha}_3 + \bar{\alpha}_4, \dots, \bar{\alpha}_{n-1} + \bar{\alpha}_n, e\omega_{n-1}, r\omega_{n-1} + \omega_n \rangle_{\mathbb{Z}}$	S^+
B1	$\mathrm{Spin}(2n+1), n \geq 2$	$\mathrm{SO}(2n+1)/\mathrm{S}(\mathrm{O}(n+1) \times \mathrm{O}(n))$	$2\mathbb{Z}S$	$2S$
B2	$\mathrm{Spin}(2n+1), n \geq 2$	$\mathrm{SO}(2n+1)/(\mathrm{SO}(n+1) \times \mathrm{SO}(n))$	2Λ	$2S$
B3	$\mathrm{Spin}(2n+1), n \geq 2$	$\mathrm{SO}(2n+1)/\mathrm{N}(\mathrm{GL}(n))$	$\mathbb{Z}(S^+ \cup \{2\bar{\alpha}_n\})$	$S^+ \cup \{2\bar{\alpha}_n\}$
B4	$\mathrm{Spin}(2n+1), n \geq 2$	$\mathrm{SO}(2n+1)/\mathrm{GL}(n)$	$\langle \omega_1, \omega_2, \dots, \omega_{n-1}, 2\omega_n \rangle_{\mathbb{Z}}$	$S^+ \cup \{2\bar{\alpha}_n\}$
C1	$\mathrm{Sp}(2n), n \geq 3$	$\mathrm{Sp}(2n)/\mathrm{N}(\mathrm{GL}(n))$	$2\mathbb{Z}S$	$2S$
C2	$\mathrm{Sp}(2n), n \geq 3$	$\mathrm{Sp}(2n)/\mathrm{GL}(n)$	2Λ	$2S$

	G	Z	$\mathbb{Z}\Gamma(Z)$	$\Sigma^N(Z)$
C3	$\mathrm{Sp}(2n), n \geq 2$	$\mathrm{Sp}(2n) \times^{\mathrm{Sp}(2a) \times \mathrm{Sp}(2b)} \mathbb{C}^{2b}$ $a = b = n/2$ if n is even, $a = b - 1 = (n - 1)/2$ if n is odd	Λ	S^+
D1	$\mathrm{Spin}(2n), n \geq 4$	$\mathrm{SO}(2n)/\mathrm{N}(\mathrm{SO}(n) \times \mathrm{SO}(n))$	$2\mathbb{Z}S$	$2S$
D2	$\mathrm{Spin}(2n), n \geq 4$	$\mathrm{SO}(2n)/(\mathrm{SO}(n) \times \mathrm{SO}(n))$	2Λ	$2S$
D3	$\mathrm{Spin}(2n), n \geq 4$	$\mathrm{SO}(2n)/\mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(n))$	$2\langle \bar{\alpha}_1, \dots, \bar{\alpha}_{n-2}, \bar{\alpha}_n, \omega_1 \rangle_{\mathbb{Z}}$	$2S$
D4	$\mathrm{Spin}(2n), n \geq 4$ even	$\mathrm{SO}(2n)/\langle A \cdot (\mathrm{SO}(n) \times \mathrm{SO}(n)) \rangle$, where $Ae_i = -e_{n-i+1}$, $Ae_{\frac{n}{2}+i} = e_{\frac{n}{2}-i+1}$, $Ae_{n+i} = e_{2n-i+1}$, $Ae_{\frac{3n}{2}+i} = -e_{\frac{3n}{2}-i+1}$ for all $i \in \{1, \dots, n/2\}$ and $(e_i)_{i \in \{1, \dots, 2n\}}$ is the standard basis of \mathbb{C}^{2n}	$2\langle \bar{\alpha}_1, \dots, \bar{\alpha}_{n-2}, \bar{\alpha}_n, \omega_n \rangle_{\mathbb{Z}}$	$2S$
D5	$\mathrm{Spin}(2n), n \geq 4$ even	$\mathrm{SO}(2n)/\langle A \cdot \mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(n)) \rangle$	$2\langle \bar{\alpha}_1, \dots, \bar{\alpha}_{n-2}, \bar{\alpha}_n, \omega_1 + \omega_n \rangle_{\mathbb{Z}}$	$2S$
E61	E_6	$E_6/\mathbb{Z}(E_6)C_4$	$2\mathbb{Z}S$	$2S$
E62	E_6	E_6/C_4	2Λ	$2S$
E71	E_7	$E_7/\mathrm{N}(A_7)$	$2\mathbb{Z}S$	$2S$
E72	E_7	E_7/A_7	2Λ	$2S$
E8	E_8	E_8/D_8	$2\Lambda = 2\mathbb{Z}S$	$2S$
F4	F_4	$F_4/(C_3 \times A_1)$	$2\Lambda = 2\mathbb{Z}S$	$2S$
G2	G_2	$G_2/(A_1 \times A_1)$	$2\Lambda = 2\mathbb{Z}S$	$2S$

For the homogeneous case, there is a nice criterion by Luna for finding all lattices in these families of smooth affine spherical varieties: Not only G/H for the wonderful subgroup H is a smooth affine spherical variety, but for any subgroup of G between H and $N_G(H)$ (the G -normalizer of H) via Luna's augmentation, cref. [Lun01], prop. 6.4.

We translate this to the language of weight lattices and get that $\mathbb{Z}\Gamma$ is the lattice of a smooth affine spherical variety if and only if

$$(3) \quad \mathbb{Z}\Sigma \subseteq \mathbb{Z}\Gamma \subseteq \mathbb{Z}\Gamma_G,$$

where Γ_G is the weight monoid of G/H and $\Sigma = \Sigma^N(\Gamma_G)$ is the corresponding set of N -spherical roots, as defined below section 10. The N -spherical roots, weights of the variety $G/N_G(H)$, were already worked out by P. Bravi and G. Pezzini in [BP15]. In the case $\mathbb{Z}\Gamma = 2\Lambda$, we have $\Sigma = \{2\bar{\alpha}_1, \dots, 2\bar{\alpha}_n\}$, and in the case $\mathbb{Z}\Gamma = \langle \omega_1, \omega_2, \dots, \omega_{n-1}, 2\omega_n \rangle$ for support of type B , we have $\Sigma = \{\bar{\alpha}_1 + \bar{\alpha}_2, \bar{\alpha}_2 + \bar{\alpha}_3, \dots, \bar{\alpha}_{n-1} + \bar{\alpha}_n, 2\bar{\alpha}_n\}$, and for $\mathbb{Z}\Gamma = \Lambda$ in type A and C , we have $\Sigma = \{\bar{\alpha}_1 + \bar{\alpha}_2, \bar{\alpha}_2 + \bar{\alpha}_3, \dots, \bar{\alpha}_{n-1} + \bar{\alpha}_n\}$.

In particular, lattices $\mathbb{Z}\Gamma$ between these two extremes are in 1:1-correspondence to subgroups of $\mathbb{Z}\Gamma_G/\mathbb{Z}\Sigma$ by a standard argument from the theory of finitely generated abelian groups. We call the order of this quotient the **index** of $\mathbb{Z}\Sigma$ in $\mathbb{Z}\Gamma_G$. For

simple groups, the index of the root lattice in the weight lattice can be found in [Bou81].

From this it is obvious that there are no lattices strictly in between two given lattices if and only if the index is prime.

Let us now introduce the smoothness criterion from [PVS15]. Note that this smoothness criterion not only works for simple G , but every reductive group.

We assume for this part that $\mathbb{N}\Gamma$ is a normal G -saturated monoid generated by $\lambda_1, \dots, \lambda_r$ of rank r with $n \leq r < \infty$, the vector space spanned by these λ_i has dimension n and $\mathbb{N}\Gamma$ is generated by dominant weights $\lambda_i \in \mathbb{Z}\Lambda^+$.

We denote by $V(\lambda)$ the irreducible G -modules of highest weight $\lambda \in \Lambda$. Remember that we identify the weight monoid $\Gamma(Z)$ for an affine variety Z with the following submonoid of Λ :

$$\Gamma(Z) = \{\lambda \in \Lambda : \text{Hom}_G(V(\lambda), \mathbb{C}[Z]) \neq 0\}$$

We say a submonoid Γ of $\mathbb{N}\Lambda^+$ is G -saturated if:

$$\mathbb{Z}\Gamma \cap \Lambda^+ = \Gamma$$

In particular, all monoids we will talk about for the rest of this thesis are G -saturated, so we will assume all our weight monoids to be G -saturated from this point.

We call a monoid $\Gamma \subseteq \mathbb{N}\Lambda^+$ **smooth** if and only if it is the weight monoid of a smooth affine spherical variety, that is there exists a (unique by [Los09]) smooth affine G -variety Z such that $\mathbb{C}[Z] \cong \bigoplus_{\lambda \in \Gamma} V(\lambda)$.

Definition 5.2. ([PPVS17], Definition 3.1) We will say that a sublattice \mathcal{X} of Λ has **full rank** if $\text{rk } \mathcal{X} = \text{rk } \Lambda$. Furthermore, we say that a submonoid of $\mathbb{N}\Lambda^+$ has *full rank* if the lattice it generates in Λ has full rank. Finally, we say that an affine spherical G -variety Z has *full rank* if its weight monoid has full rank.

Definition 5.3. Let Z be spherical with open orbit G/H and weight monoid Γ . We define the set $\Sigma^{sc}(G)$ of spherically closed roots of G as the subset of $\mathbb{N}S$ with the following elements: an element σ of $\mathbb{N}S$ is an element of $\Sigma^{sc}(G)$ if, after renumbering the simple roots in its support like [Bou81], it is in the following table:

List 5.4.

Type of $\text{supp}(\sigma)$	σ
A_1	α_1
A_1	$2\alpha_1$
$A_1 \times A_1$	$\alpha_1 + \alpha'_1$
$A_n, n \geq 2$	$\alpha_1 + \dots + \alpha_n$
A_3	$\alpha_1 + 2\alpha_2 + \alpha_3$
$B_n, n \geq 2$	$\alpha_1 + \dots + \alpha_n$ $2(\alpha_1 + \dots + \alpha_n)$
B_3	$\alpha_1 + 2\alpha_2 + 3\alpha_3$
$C_n, n \geq 3$	$\alpha_1 + 2(\alpha_2 + \dots + \alpha_{n-1}) + \alpha_n$
$D_n, n \geq 4$	$2(\alpha_1 + \dots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$
F_4	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$
G_2	$4\alpha_1 + 2\alpha_2$ $\alpha_1 + \alpha_2$

Next, we define

$$\Sigma^N(\Gamma) := \Sigma^N(G/H) := \Sigma(G/N_G(H))$$

the set of N -**spherical roots** of $\mathbb{Z}\Gamma$. For more details on the geometric meaning of these sets of spherical roots and their construction, we refer to [VSt13]. Note that these two sets are basically the same roots, only with different normalizations. The spherically closed spherical roots correspond to the so-called spherical closure of H , and the N -spherical roots correspond to the G -normalizer of H .

Proposition 5.5. [cf. [PVS15], Prop. 1.6.]

Suppose Γ is a G -saturated submonoid of Λ^+ . If $\bar{\sigma} \in \Sigma^{sc}(G)$, then $\bar{\sigma}$ is an element of $\Sigma^N(\Gamma)$ if and only if the following conditions are all satisfied:

- (1) $\bar{\sigma}$ is not a simple root
- (2) $\bar{\sigma} \in \mathbb{Z}\Gamma$
- (3) $\bar{\sigma}$ is compatible with $S^p(\Gamma)$, that is:
 - (a) if $\bar{\sigma} = \bar{\alpha}_1 + \dots + \bar{\alpha}_n$ with support of type B_n then $\{\alpha_2, \alpha_3, \dots, \alpha_{n-1}\} \subset S^p(\Gamma)$ and $\alpha_n \notin S^p(\Gamma)$;
 - (b) if $\bar{\sigma} = \bar{\alpha}_1 + 2\bar{\alpha}_2 + \dots + 2\bar{\alpha}_{n-1} + \bar{\alpha}_n$ with support of type C_n then $\{\alpha_3, \alpha_4, \dots, \alpha_n\} \subset S^p(\Gamma)$
 - (c) if $\bar{\sigma}$ is any other element of $\Sigma^{sc}(G)$ then $\{\bar{\alpha} \in \text{supp}(\bar{\sigma}) : \langle \bar{\alpha}^\vee, \bar{\sigma} \rangle = 0\} \subset S^p(\Gamma)$;
- (4) if $\bar{\sigma} = 2\bar{\alpha}$ then $\langle \bar{\alpha}^\vee, \bar{\gamma} \rangle \in 2\mathbb{Z}$ for all $\bar{\gamma} \in \mathbb{Z}\Gamma$;
- (5) if $\bar{\sigma} = \bar{\alpha} + \bar{\beta}$ with $\alpha, \beta \in S$ and $\bar{\alpha} \perp \bar{\beta}$, then $\langle \bar{\alpha}^\vee, \bar{\gamma} \rangle = \langle \bar{\beta}^\vee, \bar{\gamma} \rangle$ for all $\bar{\gamma} \in \mathbb{Z}\Gamma$.

We particularly recall [section 2](#) for the definitions of many objects that are needed here, for example S^p .

Definition 5.6. ([PVS15], 1.9) Let S be the set of simple roots of a root system. Let S^p be a subset of S . Let Σ^N be a subset of $\mathbb{N}S$. We say that the triple (S, S^p, Σ^N) is admissible if there exists a finite set I and for every $i \in I$ a triple (S_i, S_i^p, Σ_i^N) from the [list 5.9](#) and an embedding $f_i : S_i \hookrightarrow S$ of the Dynkin diagram of S_i such that the Dynkin diagram of S is the union over $i \in I$ of the Dynkin diagrams of the S_i , that $S^p = \cup_i f_i(S_i^p)$ and that $\Sigma^N = \cup_i f_i(\Sigma_i^N)$.

One final invariant is needed to state the criterion:

Definition 5.7. (see [PVS15], Prop. 1.7) Let Γ be a G -saturated submonoid spanned by dominant weights. Among all the subsets F of the set of simple roots S such that the relative interior of the cone spanned by $\{\bar{\alpha}^\vee|_{\mathbb{Z}\Gamma} : \bar{\alpha} \in F\}$ in $\text{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Q})$ intersects the valuation cone $\mathcal{V}(\Gamma)$, there is a unique one, denoted S_Γ , that contains all the others.

The main theorem of [PVS15] (specialized to the case of G -saturated weight monoids, but still no assumptions on G but being reductive!) is the main tool we used for our classification:

Theorem 5.8 ([PVS15], 1.11-algorithm). *Let Γ be a G -saturated monoid of dominant weights Λ^+ of G , $S^p(\Gamma)$, $\Sigma^N(\Gamma)$, S_Γ as above. Then Γ is the weight monoid of a smooth affine spherical G -variety if and only if:*

- (1) $\{\bar{\alpha}^\vee|_{\mathbb{Z}\Gamma} : \bar{\alpha} \in S_\Gamma \setminus S^p(\Gamma)\}$ is a subset of a basis of $\mathbb{Z}\Gamma^*$.
- (2) for all $\bar{\alpha}, \bar{\beta} \in S_\Gamma \setminus S^p(\Gamma)$ we have $\bar{\alpha}^\vee|_{\mathbb{Z}\Gamma} = \bar{\beta}^\vee|_{\mathbb{Z}\Gamma}$ if and only if $\bar{\alpha} + \bar{\beta} \in \Sigma^N(\Gamma)$
- (3) the triple $(S_\Gamma, S^p(\Gamma) \cap S_\Gamma, \Sigma^N(\Gamma) \cap \langle S_\Gamma \rangle_{\mathbb{Z}})$ is admissible, meaning it is contained in [list 5.9](#)

We now state the list of **admissible triples**:

List 5.9. ([PVS15], 1.10)

- (1) (S, S, \emptyset) where S is the set of simple roots of an irreducible root system
- (2) $(A_n, \{\bar{\alpha}_2, \bar{\alpha}_3, \dots, \bar{\alpha}_n\}, \emptyset)$ for $n \geq 1$
- (3) $(A_n, \{\bar{\alpha}_1, \bar{\alpha}_3, \bar{\alpha}_5, \dots, \bar{\alpha}_{n-1}\}, \{\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3, \bar{\alpha}_3 + 2\bar{\alpha}_4 + \bar{\alpha}_5, \dots, \bar{\alpha}_{n-3} + 2\bar{\alpha}_{n-2} + \bar{\alpha}_{n-1}\})$ for $n \geq 4$, n even.
- (4) $(A_n \times A_k, \{\bar{\alpha}_{k+2}, \bar{\alpha}_{k+3}, \dots, \bar{\alpha}_n\}, \{\bar{\alpha}_1 + \bar{\alpha}'_1, \bar{\alpha}_2 + \bar{\alpha}'_2, \dots, \bar{\alpha}_k + \bar{\alpha}'_k\})$ for $n > k \geq 2$.
- (5) $(C_n, \{\bar{\alpha}_2, \bar{\alpha}_3, \dots, \bar{\alpha}_n\}, \emptyset)$ for $n \geq 2$
- (6) $(D_5, \{\bar{\alpha}_2, \bar{\alpha}_3, \bar{\alpha}_4\}, \{\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4 + 2\bar{\alpha}_5\})$

The following lemma is an immediate consequence of the **elementary divisors theorem** (cf. [Lan84], p. 234, theorem 5.2) and is very helpful for verifying condition (1) of [theorem 5.8](#).

Lemma 5.10. *Let $k, l \in \mathbb{N}$ with $k \geq l \geq 1$ and $A \in \text{Mat}(k \times l, \mathbb{Z})$. The following are equivalent:*

- (1) There exists a matrix $C \in \text{Mat}(k \times (k-l), \mathbb{Z})$ such that $(A|C) \in \text{GL}(k, \mathbb{Z})$.
- (2) The greatest common divisor of all $l \times l$ -minors in A is equal to 1.

In particular, we would like to recall that Luna's criterion for homogeneous smooth affine spherical varieties [eq. \(3\)](#) can be directly recovered from this algorithm:

Lemma 5.11. *If $\Sigma^N(\Gamma)$ is the set of N -spherical roots of an affine homogeneous variety, we have $S_\Gamma = \emptyset$.*

Remark 5.12. This lemma was proven together with Guido Pezzini and Bart Van Steirteghem for an early version of [\[PPVS17\]](#). However, this Lemma is not included in the final version. Nevertheless, this proof is joint work of the three authors of [loc.cit.](#).

Proof. We assume $\Sigma^N(\Gamma)$ are the spherical roots of a homogeneous affine variety. Then $\Sigma^N(\Gamma)$ is the set of spherical roots of an affine space G/H . This implies the existence of a linear combination γ of elements of $\Sigma^N(\Gamma)$ with non-negative coefficients such that

$$\langle D, \gamma \rangle > 0 \text{ for all colors } D$$

by [\[Kno91\]](#), theorem 6.7.

Suppose $S_\Gamma \neq \emptyset$, then there exists (by definition of S_Γ) a linear combination of colors $\delta = \sum_D a_D D$ such that $\langle \delta, \sigma_i \rangle \leq 0 \forall i$ where σ_i are the spherical roots. But then we find the contradiction

$$0 < \langle \sum a_D D, \gamma \rangle = \langle \delta, \gamma \rangle = \langle \delta, \sum_{\geq 0} b_i \sigma_i \rangle \leq 0.$$

Therefore, $S_\Gamma = \emptyset$. □

Now [theorem 5.8](#) yields Luna's criterion ([eq. \(3\)](#)): part (1) and (2) is trivially okay and (3) gives the empty triple which is admissible.

Two sets of spherical roots will often appear in the following discussion, so we will use the following names for them:

$$\begin{aligned} 2S &:= \{2\bar{\alpha} : \alpha \in S\} \\ S^+ &:= \{\bar{\alpha} + \bar{\beta} : \alpha, \beta \in S, \bar{\alpha} \neq \bar{\beta}, \bar{\alpha} \not\perp \bar{\beta}\} \end{aligned}$$

Lemma 5.13. *(this is [\[PPVS17\]](#), lemma 3.2) Let Γ be a submonoid of Λ^+ .*

- (1) *If $\text{rk } \mathbb{Z}\Gamma = \text{rk } \Lambda$, then $S^p(\Gamma) = \emptyset$.*
- (2) *If $\text{rk } \mathbb{Z}\Gamma = \text{rk } \Lambda$ and Γ is G -saturated, then $\Sigma^N(\Gamma) \subset 2S \cup S^+$*

Proof. Assertion (1) follows from [definition 2.4](#). Indeed, if $\alpha \in S^p(\Gamma)$, then $\mathbb{Z}\Gamma$ is a subset of the lattice $\{\lambda \in \Lambda : \langle \bar{\alpha}^\vee, \lambda \rangle = 0\}$, so that $\text{rk } \mathbb{Z}\Gamma < \text{rk } \Lambda$.

Part (2) is a consequence of [proposition 5.5](#). Indeed, it follows from (1) and condition (3) in [proposition 5.5](#) that if $\sigma \in \Sigma^N(\Gamma)$, then $\sigma \in 2S \cup S^+$ or $\sigma = \bar{\alpha} + \bar{\alpha}'$ with $\alpha, \alpha' \in S$, $\bar{\alpha} \perp \bar{\alpha}'$. The second possibility cannot occur by condition (5) of [proposition 5.5](#) since $\text{rk } \mathbb{Z}\Gamma = \text{rk } \Lambda$. □

Now let us collect some information from the lemmas 3.3-3.6 from [\[PPVS17\]](#). We do not repeat their proofs here.

Lemma 5.14. *Let Γ be a G -saturated submonoid of Λ^+ .*

- (1) *If $\alpha \in S \setminus \text{supp}(\Sigma^N(\Gamma))$, then $\bar{\alpha} \in S_\Gamma$*
- (2) *If $\bar{\alpha}, \bar{\beta} \in S_\Gamma$ with $\bar{\alpha} \neq \bar{\beta}$, then $\bar{\alpha} \perp \bar{\beta}$.*

- (3) Let $\alpha, \beta \in S$ with $\bar{\alpha} \neq \bar{\beta}$ and $\bar{\alpha} \not\perp \bar{\beta}$. Then:
- (a) $\bar{\alpha} \in \text{supp}(\Sigma^N(\Gamma))$ and $\bar{\beta} \in \text{supp}(\Sigma^N(\Gamma))$.
 - (b) suppose $\langle \bar{\alpha}^\vee, \bar{\beta} \rangle \in \{-1, -3\}$. If $2\bar{\alpha} \in \Sigma^N(\Gamma)$, then $\bar{\alpha} + \bar{\beta} \notin \Sigma^N(\Gamma)$.
- (4) Let $\alpha, \beta \in S$ with $\bar{\alpha} \perp \bar{\gamma}$, $\langle \bar{\alpha}^\vee, \bar{\beta} \rangle = -1$ and $\bar{\gamma} \not\perp \bar{\beta}$. Then the following hold:
- (a) If $2\bar{\alpha} \in \Sigma^N(\Gamma)$, then $\bar{\beta} + \bar{\gamma} \notin \Sigma^N(\Gamma)$
 - (b) Suppose $\langle \bar{\beta}^\vee, \bar{\alpha} \rangle = -1$. If $\bar{\alpha} + \bar{\beta} \in \Sigma^N(\Gamma)$, then $\bar{\beta} + \bar{\gamma} \in \Sigma^N(\Gamma)$

From this lemma we can deduce that the set of N -spherical roots for a smooth affine spherical $\text{SL}(n+1)$ -variety is either $2S$ or S^+ (and no mixtures).

This leads to the following characterization of these varieties:

Theorem 5.15. ([PPVS17], Corollary 3.15) *Let $G = \text{SL}(n+1)$ with $n \geq 1$. A G -saturated affine spherical G -variety Z of full rank is smooth if and only if it is one of the following:*

- (1) for $n = 2$: $\langle \omega \rangle_{\mathbb{Z}}, \langle 2\omega \rangle_{\mathbb{Z}}, \langle 4\omega \rangle_{\mathbb{Z}}$, these are the cases [A1](#), [A2](#), [A3](#) of list 5.1 or, if $n > 2$:
- (2) $\mathbb{Z}\Gamma(Z) = 2\langle \bar{\alpha}_2, \bar{\alpha}_3, \dots, \bar{\alpha}_n, d\omega_n \rangle_{\mathbb{Z}}$ for some $d \in \mathbb{N}$ with $d|(n+1)$, this is [A4](#)
- (3) n is even and $\mathbb{Z}\Gamma(Z) = \langle \bar{\alpha}_1 + \bar{\alpha}_2, \bar{\alpha}_2 + \bar{\alpha}_3, \dots, \bar{\alpha}_{n-1} + \bar{\alpha}_n, k\omega_{n-1} \rangle_{\mathbb{Z}}$ for some $k \in \mathbb{N} \setminus \{0\}$; this is [A5](#)
- (4) n is odd and $\mathbb{Z}\Gamma(Z) = \langle \bar{\alpha}_2 + \bar{\alpha}_3, \bar{\alpha}_3 + \bar{\alpha}_4, \dots, \bar{\alpha}_{n-1} + \bar{\alpha}_n, e\omega_{n-1}, r\omega_{n-1} + \omega_n \rangle_{\mathbb{Z}}$ for some $e, r \in \mathbb{N}$ with $e|\frac{n+1}{2}$ and $0 \leq r \leq e-1$, this is [A6](#)

Of course, the next step is to determine the smooth affine spherical varieties with G -saturated weight monoid of full rank for G simple and not of type A :

Lemma 5.16. ([PPVS17], Proposition 3.14) *Suppose G is a simply connected simple group of some type different from A . Then: A G -saturated submonoid Γ of Λ^+ of full rank is smooth if and only if one of the following holds:*

- (1) $2\mathbb{Z}\Sigma \subset \mathbb{Z}\Gamma \subset 2\Lambda$;
- (2) G is of type B_n with $n \geq 2$ and $\langle \bar{\alpha}_1 + \bar{\alpha}_2, \bar{\alpha}_2 + \bar{\alpha}_3, \dots, \bar{\alpha}_{n-1} + \bar{\alpha}_n, 2\bar{\alpha}_n \rangle_{\mathbb{Z}} \subset \mathbb{Z}\Gamma \subset \langle \omega_1, \omega_2, \dots, \omega_{n-1}, 2\omega_n \rangle_{\mathbb{Z}} =: \Lambda^B$; or
- (3) G is of type C_n with $n \geq 2$ and $\mathbb{Z}\Gamma = \Lambda$.

Recall that the lattices between 2Λ and $2\mathbb{Z}S$ are in natural bijection with the subgroups of the quotient $\Lambda/\mathbb{Z}S$ (cf. [PPVS17], Remark 3.15). The isomorphism type of the group $\Lambda/\mathbb{Z}S$ can be found in [Bou81], Planches I-IX:

Type of G	$\Lambda/\mathbb{Z}S$
A_n	$\mathbb{Z}/(n+1)\mathbb{Z}$
$B_n, n \geq 2$	$\mathbb{Z}/2\mathbb{Z}$
$C_n, n \geq 3$	$\mathbb{Z}/2\mathbb{Z}$
$D_n, n \geq 4$ even	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$D_n, n \geq 5$ odd	$\mathbb{Z}/4\mathbb{Z}$
E_6	$\mathbb{Z}/3\mathbb{Z}$
E_7	$\mathbb{Z}/2\mathbb{Z}$
E_8	$\{e\}$
F_4	$\{e\}$
G_2	$\{e\}$

Apart from A_n for n even and $\Sigma^N(G) = S^+$, there is only one more family of inhomogeneous varieties of full rank for type C and $\Sigma^N(\Gamma) = S^+$.

Theorem 5.17. *(This is a subcase of [PPVS17], Lemma 3.17. We show a more detailed proof here.)*

Let Γ be a G -saturated weight monoid of full rank for type C with $\Sigma^N(\Gamma) = S^+$. Then:

Γ is smooth if and only if $\Gamma = \mathbb{N}\Lambda^+$

Proof. We have $\Sigma^N(\Gamma) = S^+ = \{\bar{\alpha}_1 + \bar{\alpha}_2, \bar{\alpha}_2 + \bar{\alpha}_3, \dots, \bar{\alpha}_{n-1} + \bar{\alpha}_n\}$ by assumption and number the elements $\sigma_1, \dots, \sigma_{n-1}$. The next step of the "inverse PVS-algorithm" is to determine the set S_Γ . To do that, we determine the matrix giving the Cartan pairing. For our case, this is an $n \times (n-1)$ -matrix. For n even, this matrix is:

$$\begin{array}{cccccccc}
& \sigma_1 & \sigma_2 & \sigma_3 & \dots & \sigma_{n-3} & \sigma_{n-2} & \sigma_{n-1} \\
\bar{\alpha}_1^\vee & 1 & -1 & 0 & \dots & 0 & 0 & 0 \\
\bar{\alpha}_2^\vee & 1 & 1 & -1 & \dots & 0 & 0 & 0 \\
\bar{\alpha}_3^\vee & -1 & 1 & 1 & \dots & 0 & 0 & 0 \\
\bar{\alpha}_4^\vee & 0 & -1 & 1 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\bar{\alpha}_{n-3}^\vee & 0 & 0 & 0 & \ddots & 1 & -1 & 0 \\
\bar{\alpha}_{n-2}^\vee & 0 & 0 & 0 & \ddots & 1 & 1 & -1 \\
\bar{\alpha}_{n-1}^\vee & 0 & 0 & 0 & \ddots & -1 & 1 & 0 \\
\bar{\alpha}_n^\vee & 0 & 0 & 0 & \dots & 0 & -1 & 1
\end{array}$$

Obviously, $\bar{\alpha}_1, \bar{\alpha}_3, \dots, \bar{\alpha}_{n-1} \in S_\Gamma$, as $\langle \sum_{i \text{ odd}} \bar{\alpha}_i^\vee, \sigma \rangle \leq 0$ for all $\sigma \in \Sigma^N(\Gamma)$. The equation $\langle \sum \sigma_i, \bar{\alpha}_2^\vee \rangle = 1$ gives $\bar{\alpha}_2 \notin S_\Gamma$.

Now let k be an even number $4 \leq k \leq n$: We assume $\bar{\alpha}_k$ was in S_Γ . Then the equations $\langle \sigma_i, c_{k-2} \cdot \bar{\alpha}_{k-2}^\vee \rangle$ forces the coefficient $c_{k-2} > 0$ and hence $\bar{\alpha}_{k-2} \in S_\Gamma$. But then, by the same argument, also $\bar{\alpha}_{k-4}, \dots, \bar{\alpha}_4$ and finally $\bar{\alpha}_2 \in S_\Gamma$. But we already saw that $\bar{\alpha}_2 \notin S_\Gamma$. Contradiction. So $S_\Gamma = \{\bar{\alpha}_i : i \text{ odd}\}$.

We add up all spherical roots: $\sum \sigma_i = \omega_2$, so $\omega_2 \in \mathbb{Z}\Gamma$. We add $\sum_{i=3}^n \sigma_i = -\omega_2 + \omega_4$, so $\omega_4 \in \mathbb{Z}\Gamma$. We advance like this and finish with $\omega_n \in \mathbb{Z}\Gamma$. Hence, S_Γ are the odd co-roots, and hence $\{\omega_i : i \text{ even}\} \in \mathbb{Z}\Gamma$.

We can assume without loss of generality that the ω_i with i even are generators of the lattice lattice. That means the other generators are of the form $a_j^1 \omega_1 + a_j^3 \omega_3 + a_j^5 \omega_5 + \dots + a_j^l \omega_l$ with l the biggest odd number $< n$. But the condition that the odd co-roots are part of a basis of $\mathbb{Z}\Gamma^*$ gives $a_j^i = \delta_{i,j}$. We get:

$$\mathbb{Z}\Gamma = \langle \omega_1, \omega_2, \dots, \omega_n \rangle = \Lambda.$$

The proof for n odd works similar. \square

Remark 5.18. Note that [list 5.1](#) is also a classification of (non-compact) multiplicity free Hamiltonian manifolds of full rank with surjective moment map: We think of a moment polytope \mathcal{P} whose $\mathbb{R}_{\geq 0}$ -span intersected with the Weyl chamber of a classical root system is the whole chamber. Hence the only minimal face of \mathcal{P} is $X = \underline{0} = (0, \dots, 0)$ and the local models there are those smooth affine spherical varieties Z such that the cone spanned by $\Gamma(Z)$ is the tangent cone of \mathcal{P} in $\underline{0}$. But these are precisely the varieties from [list 5.1](#). So the lattices from [list 5.1](#) and \mathcal{P} the whole chamber are all spherical pairs possible in this setting.

We also note that the lattice 2Λ with N -spherical roots $2S$ corresponds to a smooth affine spherical variety for every simple group. By applying [\[Kra79\]](#) and [\[Los09\]](#), we see that these varieties are homogeneous and hence, all lattices in between 2Λ and $2S$ are also lattices of homogeneous varieties. It is also known from Krämer that the lattices Λ with N -spherical roots S^+ correspond to a homogeneous variety for $\mathrm{SL}(n+1)$, n even. The same is true for $\mathrm{Spin}(2n+1)$ and the lattice Λ^B and N -spherical roots $S^+ \cup \{2\bar{\alpha}_n\}$.

11. QUASI-HAMILTONIAN MANIFOLDS WITH SURJECTIVE MOMENT MAP

As we now know all local models for multiplicity free q-Hamiltonian manifolds with surjective moment maps, this section is devoted to the gluing.

Note that in every vertex of the alcove, we have a different set of fundamental weights for the corresponding subgroup $K(a)$ as a basis of our lattice. It is useful to have the following formulas for that change of generators.

Lemma 5.19. *Let*

$$k_0\bar{\alpha}_0 + k_1\bar{\alpha}_1 + \cdots + k_n\bar{\alpha}_n = 0$$

be the relation between the simple roots of an affine root system (given by the root δ). Then the following is a relation between co-roots:

$$k_0\|\bar{\alpha}_0\|^2\bar{\alpha}_0^\vee + k_1\|\bar{\alpha}_1\|^2\bar{\alpha}_1^\vee + \cdots + k_n\|\bar{\alpha}_n\|^2\bar{\alpha}_n^\vee = 0$$

*The coefficients in this relation are sometimes called **colabels**.*

Proof. We use the usual $\bar{\alpha}^\vee = \frac{2\bar{\alpha}}{\|\bar{\alpha}\|^2} \Rightarrow \bar{\alpha} = \frac{\|\bar{\alpha}\|^2}{2}\bar{\alpha}^\vee$ and eliminate the denominator. \square

We remember the notation X_i for the vertex of the alcove where all simple roots except $\bar{\alpha}_i$ are zero.

Theorem 5.20. *Let Φ be an affine root system. Let $\mathbb{Z}\Gamma_0$ be the weight lattice corresponding to the local root system in X_0 with fundamental weights Λ_0^+ and simple roots S_0 . Let $\tilde{\Phi}$ be a local root system for $X_e, e \in \{1, \dots, n\}$ with corresponding fundamental weights $\tilde{\omega}_0, \tilde{\omega}_1, \dots, \tilde{\omega}_{e-1}, \tilde{\omega}_{e+1}, \dots, \tilde{\omega}_n$. Then these two bases transform as follows:*

$$\begin{aligned} \tilde{\omega}_0 &= -\frac{k_0\|\bar{\alpha}_0\|^2}{k_e\|\bar{\alpha}_e\|^2}\omega_e \\ \tilde{\omega}_f &= \omega_f - \frac{k_f\|\bar{\alpha}_f\|^2}{k_e\|\bar{\alpha}_e\|^2}\omega_e \quad \forall f \in \{0, 1, \dots, n\} \setminus \{0, e\} \end{aligned}$$

Conversely:

$$\begin{aligned} \omega_e &= -\frac{k_e\|\bar{\alpha}_e\|^2}{k_0\|\bar{\alpha}_0\|^2}\tilde{\omega}_0 \\ \omega_f &= \tilde{\omega}_f - \frac{k_f\|\bar{\alpha}_f\|^2}{k_0\|\bar{\alpha}_0\|^2}\tilde{\omega}_0 \quad \forall f \in \{0, 1, \dots, n\} \setminus \{0, e\} \end{aligned}$$

Proof. We recall that the fundamental weights are defined by $\langle \omega_i, \alpha_k^\vee \rangle = \delta_{ik}$, $i, k \in \{1, \dots, n\}$. It follows that $\tilde{\omega}_0$ in $\tilde{\Phi}$ is defined by the relations

$$\begin{aligned} \langle \tilde{\omega}_0, \bar{\alpha}_0^\vee \rangle &= 1 \\ \langle \tilde{\omega}_0, \bar{\alpha}_i^\vee \rangle &= 0 \quad \forall i \in \{0, \dots, n\} \setminus \{0, e\} \end{aligned}$$

which gives us an inhomogeneous system of n linearly independent equations. We use [lemma 5.19](#) to get:

$$\begin{aligned} \langle \tilde{\omega}_0, -\frac{k_e\|\bar{\alpha}_e\|^2}{k_0\|\bar{\alpha}_0\|^2}\alpha_e^\vee \rangle &= 1 \\ \langle \tilde{\omega}_0, \bar{\alpha}_i^\vee \rangle &= 0 \quad \forall i \in \{0, \dots, n\} \setminus \{0, e\} \end{aligned}$$

So $\tilde{\omega}_0$ needs to be a multiple of ω_e . The first equation gives us

$$-\frac{k_e \|\bar{\alpha}_e\|^2}{k_0 \|\bar{\alpha}_0\|^2} \langle \tilde{\omega}_0, \bar{\alpha}_e^\vee \rangle = 1 \Rightarrow \tilde{\omega}_0 = -\frac{k_0 \|\bar{\alpha}_0\|^2}{k_e \|\bar{\alpha}_e\|^2} \omega_e$$

Now we look at another weight $\tilde{\omega}_f, f \in \{0, 1, \dots, n\} \setminus \{0, e\}$. It is defined by the n linearly independent equations

$$\begin{aligned} \langle \tilde{\omega}_f, \bar{\alpha}_0^\vee \rangle &= 0 \\ \langle \tilde{\omega}_f, \bar{\alpha}_1^\vee \rangle &= 0 \\ &\vdots \\ \langle \tilde{\omega}_f, \bar{\alpha}_{f-1}^\vee \rangle &= 0 \\ \langle \tilde{\omega}_f, \bar{\alpha}_f^\vee \rangle &= 1 \\ \langle \tilde{\omega}_f, \bar{\alpha}_{f+1}^\vee \rangle &= 0 \\ &\vdots \\ \langle \tilde{\omega}_f, \bar{\alpha}_{e-1}^\vee \rangle &= 0 \\ \langle \tilde{\omega}_f, \bar{\alpha}_{e+1}^\vee \rangle &= 0 \\ &\vdots \\ \langle \tilde{\omega}_f, \bar{\alpha}_n^\vee \rangle &= 0 \end{aligned}$$

We replace $\bar{\alpha}_0^\vee$ and $\bar{\alpha}_f^\vee$ with the relations from [lemma 5.19](#) and add the right multiples of the other equations to get:

$$\begin{aligned} \langle \tilde{\omega}_f, -\frac{k_f \|\bar{\alpha}_f\|^2}{k_0 \|\bar{\alpha}_0\|^2} \bar{\alpha}_f^\vee - \frac{k_e \|\bar{\alpha}_e\|^2}{k_0 \|\bar{\alpha}_0\|^2} \bar{\alpha}_e^\vee \rangle &= 0 & (I) \\ \langle \tilde{\omega}_f, \bar{\alpha}_1^\vee \rangle &= 0 \end{aligned}$$

$$\begin{aligned} &\vdots \\ \langle \tilde{\omega}_f, -\frac{k_0 \|\bar{\alpha}_0\|^2}{k_f \|\bar{\alpha}_f\|^2} \bar{\alpha}_0^\vee - \frac{k_e \|\bar{\alpha}_e\|^2}{k_f \|\bar{\alpha}_f\|^2} \bar{\alpha}_e^\vee \rangle &= 1 & (II) \\ &\vdots \\ \langle \tilde{\omega}_f, \bar{\alpha}_{e-1}^\vee \rangle &= 0 \\ \langle \tilde{\omega}_f, \bar{\alpha}_{e+1}^\vee \rangle &= 0 \\ &\vdots \\ \langle \tilde{\omega}_f, \bar{\alpha}_n^\vee \rangle &= 0 \end{aligned}$$

This justifies the ansatz $\tilde{\omega}_f = c_e \omega_e + c_f \omega_f$ for some real numbers c_e, c_f . We put this in equation *I* and get

$$\begin{aligned} \langle c_f \omega_f + c_e \omega_e, -\frac{k_f \|\bar{\alpha}_f\|^2}{k_0 \|\bar{\alpha}_0\|^2} \bar{\alpha}_f^\vee - \frac{k_e \|\bar{\alpha}_e\|^2}{k_0 \|\bar{\alpha}_0\|^2} \bar{\alpha}_e^\vee \rangle &= 0 \\ -c_f k_f \|\alpha_f\|^2 &= c_e k_e \|\alpha_e\|^2 \\ \frac{c_f}{c_e} &= -\frac{k_e \|\bar{\alpha}_e\|^2}{k_f \|\bar{\alpha}_f\|^2} \end{aligned}$$

We also put the ansatz in equation *II*:

$$\begin{aligned} \langle c_f \omega_f + c_e \omega_e, -\frac{k_0 \|\bar{\alpha}_0\|^2}{k_f \|\bar{\alpha}_f\|^2} \bar{\alpha}_0^\vee - \frac{k_e \|\bar{\alpha}_e\|^2}{k_f \|\bar{\alpha}_f\|^2} \bar{\alpha}_e^\vee \rangle &= 1 \\ -c_e \frac{k_e \|\bar{\alpha}_e\|^2}{k_f \|\bar{\alpha}_f\|^2} &= 1 \\ c_e \cdot \frac{c_f}{c_e} &= 1 \\ c_f &= 1 \end{aligned}$$

Hence, $c_e = -\frac{k_f \|\bar{\alpha}_f\|^2}{k_e \|\bar{\alpha}_e\|^2}$, and $\tilde{\omega}_f = \omega_f - \frac{k_f \|\bar{\alpha}_f\|^2}{k_e \|\bar{\alpha}_e\|^2} \omega_e$.

The second part of the theorem is a trivial consequence of the first part, we just have to solve the equations from part one for the desired weight. \square

We can immediately deduce:

Corollary 5.21. *Situation as in theorem 5.20. Then:*

(1) *The coefficients $\frac{k_i \|\bar{\alpha}_i\|^2}{k_0 \|\bar{\alpha}_0\|^2}, i = 1, \dots, n$ in*

$$\begin{aligned} \omega_e &= -\frac{k_e \|\bar{\alpha}_e\|^2}{k_0 \|\bar{\alpha}_0\|^2} \tilde{\omega}_0 \\ \omega_f &= \tilde{\omega}_f - \frac{k_f \|\bar{\alpha}_f\|^2}{k_0 \|\bar{\alpha}_0\|^2} \tilde{\omega}_0 \quad \forall f \in \{0, 1, \dots, n\} \setminus \{0, e\} \end{aligned}$$

are always integral.

(2) *We have $\Lambda_0 \subseteq \Lambda_k \forall k = 1, \dots, n$ (and hence, also $2\Lambda_0 \subseteq 2\Lambda_k$) for every affine root system. In other words, the lattice spanned by fundamental weights in X_0 is a sublattice of the lattice spanned by fundamental weights in every other vertex.*

(3) *If Φ is of type $A_n^{(1)}$ and $C_n^{(1)}$, all these coefficients are 1 and hence, we have $\Lambda_0 = \Lambda_k \forall k = 1, \dots, n$.*

Proof.

We remember [table 1](#) of the affine root systems, the labels and the choice of α_0 .

(1) This follows from closely looking at the affine root systems and their Dynkin labels. Case-by-case-checking gives the desired result.

- (2) Simple consequence of part 1, which says in other words that the fundamental weights for X_0 are integral linear combinations of the fundamental weights in every other vertex of the fundamental alcove.
- (3) We see that all coefficients are 1 in these two cases by closely studying the affine Dynkin diagrams.

□

11.1. The Homogeneous Case.

11.1.1. The lattices $2S \subseteq \mathbb{Z}\Gamma \subseteq 2\Lambda$.

Now let us apply these results to the homogeneous local models Γ with $2S \subseteq \mathbb{Z}\Gamma \subseteq 2\Lambda$. We note that Part 1 of [corollary 5.21](#) says: For $\mathbb{Z}\Gamma = 2\Lambda_i$, all coefficients of the $\tilde{\omega}$ in this representation of the fundamental weights ω are even. Hence, if $2\bar{\alpha}_j, j \in \{0, 1, \dots, n\} \setminus \{i\}$ is in $\mathbb{Z}\Gamma_i$, then $\bar{\alpha}_j$ is even on $\mathbb{Z}\Gamma$. The only thing left to do is to see if this $2\Lambda_i$ is the weight lattice of a smooth affine spherical variety in Φ_i is to check if all these $2\bar{\alpha}_j^\vee$ are in the lattice $2\Lambda_i$ via Luna's augmentations, as it is a straightforward argument using [\[PVS15\]](#) to see that the lattice $2S$ is also the lattice of a saturated smooth affine spherical variety for every semisimple group.

Example 5.22. Let us consider the weight monoid $\Gamma_0 = 2\Lambda_0 = \langle 2\omega_1, 2\omega_2, 2\omega_3 \rangle$ for the local root system $\Phi_0 = \{\alpha_1, \alpha_2, \alpha_3\}$ of type A_3 in $A_3^{(1)}$. We have the following expressions for the ω_i in the base $\tilde{\omega}_0, \tilde{\omega}_2, \tilde{\omega}_3$ of fundamental weights of Φ_1 :

$$\begin{aligned}\omega_1 &= -\tilde{\omega}_0 \\ \omega_2 &= \tilde{\omega}_2 - \tilde{\omega}_0 \\ \omega_3 &= \tilde{\omega}_3 - \tilde{\omega}_0\end{aligned}$$

So, the lattice $\mathbb{Z}\Gamma_0 = \langle 2\omega_1, 2\omega_2, 2\omega_3 \rangle$ can be represented as a linear combination of fundamental weights $\tilde{\omega}_i$ of Φ_1 where all ω_i are linear combinations of $\tilde{\omega}_i$ with even coefficients:

$$\begin{aligned}2\omega_1 &= -2\tilde{\omega}_0 \\ 2\omega_2 &= 2\tilde{\omega}_2 - 2\tilde{\omega}_0 \\ 2\omega_3 &= 2\tilde{\omega}_3 - 2\tilde{\omega}_0\end{aligned}$$

Hence, the spherical roots $2\bar{\alpha}_0, 2\bar{\alpha}_2, 2\bar{\alpha}_3$ fulfill Luna's axioms as $\bar{\alpha}_0^\vee, \bar{\alpha}_2^\vee, \bar{\alpha}_3^\vee$ are even on $\mathbb{Z}\Gamma_0$. We check elementarily that they are in $2\Lambda_1$. Hence $\mathbb{Z}\Gamma_0 \cap C_{X_1}(\mathcal{P})$ is also the weight monoid of a smooth affine spherical variety (e.g. by [\[PPVS17\]](#)) in the vertex X_1 .

Seeing which lattice is contained in which other lattice is not always easy, but for the case of $2\mathbb{Z}S \subseteq \mathbb{Z}\Gamma \subseteq 2\Lambda$, we found a nice rule:

Lemma 5.23.

Let Φ be an affine root system.

- (1) *The lattices $\mathbb{Z}S_i$ are the same for all vertices where the root α_i has Dynkin label 1.*
- (2) *The lattices $\mathbb{Z}S_i$ for all vertices where the root α_i has a Dynkin label > 1 are proper sublattices of the lattice from part 1.*

Proof. Both are simple consequences of the linear dependence between the simple roots indicated by the Dynkin labels. We compare two vertices X_k, X_l of an affine root system where α_k, α_l have Dynkin label 1. Then we can express the root $\bar{\alpha}_k$ as an integral linear combination of the simple roots of the local root system Φ_l , and vice versa, using $\delta = 0$.

For the second part we compare two vertices X_k, X_l where α_k has Dynkin label 1, α_l has Dynkin label > 1 . Then we can express the root $\bar{\alpha}_k$ as an integral linear combination of the roots in Φ_l , but the other way round, the linear combination will not be integral, as it is never the case that the label of α_l is a divisor of all the coefficients in this linear combination. \square

Corollary 5.24. *In every affine root system but $A_{2l}^{(2)}$ we have*

$$2\mathbb{Z}S_k \subseteq 2\mathbb{Z}S_0 \subseteq 2\Lambda_0 \subseteq 2\Lambda_k$$

for all $k \in 1, \dots, n$.

Proof. As α_0 has Dynkin label 1 in every affine root system but $A_{2l}^{(2)}$, this is a direct consequence from [corollary 5.21](#) and [lemma 5.23](#). \square

We must be a bit more careful when we go to $A_{2n}^{(2)}$, because in this case the relations between the simple roots (at least for the convention we use) is different, the simple root α_0 has Dynkin label two.

Lemma 5.25. *Considering $A_{2l}^{(2)}$ and the family $2S_0 \subseteq \mathbb{Z}\Gamma \subseteq 2\Lambda_0$, only the lattice $2\Lambda_0 = 2S_n$ gives rise to a q -Hamiltonian manifold with surjective moment map.*

Proof. We can see that $2\mathbb{Z}S_k \subseteq 2\mathbb{Z}S_0 \subseteq 2\Lambda_0 \subseteq 2\Lambda_K$ for all $k \in 1, \dots, n-1$ by using the arguments from the proof of [lemma 5.23](#). It remains to check the vertex X_n . Elementary calculations show that $2\Lambda_0 = 2\mathbb{Z}S_n$. Hence, $2\Lambda_0$ is the biggest possible lattice in X_0 and the smallest possible lattice in X_n . So it is the only lattice that comes from the weight monoid of a smooth affine spherical variety in both vertices. \square

11.1.2. The lattice Λ^B .

If Φ_0 is of type B_n , we have other homogenous models corresponding to sublattices of Λ^B . Let us consider $B_n^{(1)}$. We figure out by elementary calculations that this lattice $\mathbb{Z}\Gamma = \Lambda^B = \{(a_1, \dots, a_n) : a_i \in \mathbb{Z}\}$ does not correspond to a smooth monoid in Φ_n : To do so, $\mathbb{Z}\Gamma$ would have to lie between the lattices $2\Lambda_n = \{(a_1, \dots, a_n) : a_1 \equiv \dots \equiv a_n \pmod{2}\}$ and $2\mathbb{Z}S_n = \{(a_1, \dots, a_n) : a_i \text{ even}\}$ by our classification in [\[PPVS17\]](#), which is not the case. It follows by the same argument that also $\mathbb{Z}(S^+ \cup \{2\bar{\alpha}_n\})$ is not smooth in X_n .

But Λ^B is also a possible local model in X_0 for Φ of type $D_{n+1}^{(2)}$. Let us check all possible lattices in this case:

- (1) Let us assume $\mathbb{Z}\Gamma = \Lambda_0^B$. The corresponding set of spherical roots is $\Sigma_0 = \{\bar{\alpha}_1 + \bar{\alpha}_2, \dots, \bar{\alpha}_{n-1} + \bar{\alpha}_n, 2\bar{\alpha}_n\}$. We find (by using [theorem 5.20](#) and looking at the expressions)

$$\Lambda_0^B \subseteq \langle 2\tilde{\omega}_0, \tilde{\omega}_1, \dots, \tilde{\omega}_{k-1}, \tilde{\omega}_{k+1}, \dots, \tilde{\omega}_{n-1}, 2\tilde{\omega}_n \rangle_{\mathbb{Z}} \quad \forall k = 1, \dots, n-1,$$

so surely Λ^B gives rise to a quasi-Hamiltonian manifold: It is equal to $\mathbb{Z}\Lambda_n^B$ via [theorem 5.20](#) and the reasoning of the proof of [corollary 5.21](#), part 3).

- (2) We next consider $\mathbb{Z}\Gamma = \Sigma_0 = S_0^+ \cup \{2\bar{\alpha}_n\}$ for n odd. This is obviously the same lattice as $\Sigma_n = \langle 2\bar{\alpha}_0, \bar{\alpha}_0 + \bar{\alpha}_1, \dots, \bar{\alpha}_{n-2} + \bar{\alpha}_{n-1} \rangle$. To see this, it suffices to see that $2\bar{\alpha}_0 = -2(\bar{\alpha}_1 + \bar{\alpha}_2) - 2(\bar{\alpha}_3 + \bar{\alpha}_4) - \dots - 2(\bar{\alpha}_{n-2} + \bar{\alpha}_{n-1}) - 2\bar{\alpha}_n$.

We can see elementary that $\mathbb{Z}\Sigma_i = \langle 2\bar{\alpha}_0, \bar{\alpha}_0 + \bar{\alpha}_1, \dots, \bar{\alpha}_{i-2} + \bar{\alpha}_{i-1}, \bar{\alpha}_{i+1} + \bar{\alpha}_{i+2}, \dots, \bar{\alpha}_{n-1} + \bar{\alpha}_n, 2\bar{\alpha}_n \rangle$ for all $i = 1, \dots, n-1$ is contained in this lattice, but they are not the same: $2\bar{\alpha}_0 = -2(\bar{\alpha}_1 + \bar{\alpha}_2) - 2(\bar{\alpha}_3 + \bar{\alpha}_4) - \dots - 2(\bar{\alpha}_{n-2} + \bar{\alpha}_{n-1}) - 1 \cdot (2\bar{\alpha}_n)$. In other words, we have $\mathbb{Z}\Sigma_i \subseteq \mathbb{Z}\Sigma_0 = \mathbb{Z}\Gamma \subseteq \Lambda_0^B \subseteq \Lambda_i$, so this gives rise to a q-Hamiltonian manifold.

- (3) Let us now look at $\mathbb{Z}\Gamma = \mathbb{Z}\Sigma_0$ for n even. We claim that this lattice does not correspond to the weight monoid of a smooth affine spherical variety in X_n . To see this, we apply [\[PVS15\]](#). The lattice has the following description in the basis of fundamental weights in X_0 :

$$\begin{aligned} \mathbb{Z}\Gamma &= \langle \omega_1 + \omega_2 - \omega_3, -\omega_1 + \omega_2 + \omega_3 - \omega_4, \dots, \\ &\quad -\omega_{n-3} + \omega_{n-2} + \omega_{n-1} - 2\omega_n, -\omega_{n-2} + \omega_{n-1}, -2\omega_{n-1} + 4\omega_n \rangle \end{aligned}$$

We see (via using [theorem 5.20](#)) that the root $\bar{\alpha}_0 + \bar{\alpha}_1 = \tilde{\omega}_1 - \tilde{\omega}_2 = \omega_1 - \omega_2$ is NOT contained in that lattice: We calculate $-(\bar{\alpha}_2 + \bar{\alpha}_3) - (\bar{\alpha}_4 + \bar{\alpha}_5) + \dots - (\bar{\alpha}_{n-2} + \bar{\alpha}_{n-1}) = \omega_1 - \omega_2 + \omega_{n-1} + 2\omega_n$. So $\bar{\alpha}_n = \omega_{n-1} + 2\omega_n$ had to be in the lattice, but, as $2\alpha_n$ is a spherical root, it is not via [definition 2.7](#).

But the root $2\bar{\alpha}_0 = 2\tilde{\omega}_0 - 2\tilde{\omega}_1 = -2\omega_1 = -2(\bar{\alpha}_1 + \bar{\alpha}_2) - 2(\bar{\alpha}_3 + \bar{\alpha}_4) - 2(\bar{\alpha}_5 + \bar{\alpha}_6 - \dots - 2(\bar{\alpha}_{n-1} + \bar{\alpha}_n))$ is in the lattice and $\bar{\alpha}_0^\vee$ is even on all other spherical roots. So we have $\Sigma_N(\Gamma) = \{2\bar{\alpha}_0, \bar{\alpha}_1 + \bar{\alpha}_2, \dots, \bar{\alpha}_{n-2} + \bar{\alpha}_{n-1}\}$ for Γ in the vertex Φ_n . As

$$\langle \bar{\alpha}_{n-1}^\vee + \bar{\alpha}_{n-2} + 2\bar{\alpha}_{n-3}^\vee + 2\bar{\alpha}_{n-4}^\vee + \dots + (n/2)\bar{\alpha}_1^\vee + (n/2) \cdot 1/2\bar{\alpha}_0^\vee, \sigma_i \rangle \leq 0 \forall \sigma_i \in \Sigma_N(\Gamma),$$

we get $S_\Gamma = \{\bar{\alpha}_0, \dots, \bar{\alpha}_{n-1}\}$, so the triple is $(B_n, \emptyset, \Sigma^N(\Gamma))$, and this triple is not admissible, see [list 5.9](#). So the lattice does not correspond to a smooth affine spherical variety in X_n and hence gives not rise to a q-Hamiltonian manifold.

The lattice Λ^B could also correspond to a surjective moment map in $C_2^{(1)}$. In this case, neither Λ_0^B nor $\mathbb{Z}\Sigma_0 := \langle \{2\bar{\alpha}_1, \bar{\alpha}_1 + \bar{\alpha}_2\} \rangle$ give rise to q-Hamiltonian manifolds: We start in X_0 where we can choose $\mathbb{Z}\Gamma = \Lambda_0^B = \langle 2\omega_1, \omega_2 \rangle$ or $\mathbb{Z}\Gamma = \Sigma_0 = \langle 2\bar{\alpha}_1, \bar{\alpha}_1 + \bar{\alpha}_2 \rangle = \langle 4\omega_1 - 2\omega_2, \omega_2 \rangle$.

Let us first discuss the lattice $\mathbb{Z}\Gamma = \langle 2\omega_1, \omega_2 \rangle$. In the vertex X_2 , this is $\langle 2\tilde{\omega}_1, \tilde{\omega}_0 \rangle_{\mathbb{Z}}$ which is obviously the lattice of the same smooth affine spherical variety. We advance to X_1 . We see that $\mathbb{Z}\Gamma$ is spanned by $\{2\tilde{\omega}_0, \tilde{\omega}_0 - \tilde{\omega}_2\}$ and the corresponding monoid is $\Gamma = \langle 2\tilde{\omega}_0, \tilde{\omega}_0 - \tilde{\omega}_2, 2\tilde{\omega}_2 \rangle$ by Gordan's Lemma (cf. [\[BG09\]](#), Lemma 2.9, p.52). We have $\Sigma^N(\Gamma) = \emptyset$ and hence $S_\Gamma = \{\bar{\alpha}_0, \bar{\alpha}_2\}$. The triple $(A_1 \times A_1, \emptyset, \emptyset)$ is admissible, but $\{\bar{\alpha}^\vee |_{\mathbb{Z}\Gamma} : \bar{\alpha} \in S_\Gamma\}$ is not a subset of a basis of $(\mathbb{Z}\Gamma)^*$, so this is not smooth.

Let us now discuss the lattice $\mathbb{Z}\Sigma_0 = \langle 4\omega_1, \omega_2 \rangle_{\mathbb{Z}}$. This gives rise to the weight monoid of a smooth affine homogeneous spherical variety in X_0 . In X_1 , the lattice is described as $\langle \tilde{\omega}_2 - \tilde{\omega}_0, 4\tilde{\omega}_0 \rangle_{\mathbb{Z}}$ and corresponds (via Gordan's Lemma) to the monoid $\Gamma = \langle \tilde{\omega}_2 - \tilde{\omega}_0, 4\tilde{\omega}_0, 4\tilde{\omega}_2 \rangle_{\mathbb{N}}$. Obviously, we have $\Sigma^N(\Gamma) = \emptyset$ and hence $S_\Gamma = \{\bar{\alpha}_0, \bar{\alpha}_2\}$. Again, this is not part of a basis of $(\mathbb{Z}\Gamma)^*$.

Note that Λ_n^B is a local model in $A_{2n}^{(2)}$ as part of the series seen in [lemma 5.25](#).

But for $A_4^{(2)}$, we could also have Λ_0^B as a local model in X_0 that we have not seen until now. We have to discuss that.

Let us start with $\Gamma = \langle 2\omega_1, \omega_2 \rangle_{\mathbb{N}}$ for X_0 in the fundamental alcove of $A_4^{(2)}$ which is the weight monoid of a smooth affine homogenous spherical variety and known to be smooth by [Kra79]. We look at X_1 where this lattice corresponds to the monoid $\mathbb{Z}\Gamma_1 = \langle -4\tilde{\omega}_0, \tilde{\omega}_2 - 2\tilde{\omega}_0 \rangle$ which leads to $\Sigma^N(\Gamma) = \{2\bar{\alpha}_0\}$. Hence $S_{\Gamma} = \{\bar{\alpha}_2\}$, the triple $(\bar{\alpha}_2, \emptyset, \emptyset)$ is admissible, and $\bar{\alpha}_2^{\vee}$ is part of a basis of $(\mathbb{Z}\Gamma)^*$. So this is smooth by theorem 5.8. In X_2 , we have $\mathbb{Z}\Gamma_2 = \langle 2\tilde{\omega}_1 - 4\tilde{\omega}_0, -2\tilde{\omega}_0 \rangle$. It follows $\Sigma^N(\Gamma) = \{2\bar{\alpha}_0, 2\bar{\alpha}_1\}$ and $S_{\Gamma} = \emptyset$, so this is smooth by theorem 5.8. Hence this lattice gives rise to a surjective moment map.

We also have to discuss the monoid $\Gamma = \langle 2\bar{\alpha}_1, \bar{\alpha}_1 + \bar{\alpha}_2 \rangle = \langle 4\omega_1 - 2\omega_2, \omega_2 \rangle$ for X_0 . This means we have $\Sigma^N(\Gamma) = \{2\bar{\alpha}_1, \bar{\alpha}_1 + \bar{\alpha}_2\}$ again and this is known to be smooth. But in X_2 , things are different: We have $\mathbb{Z}\Gamma = \langle 4\tilde{\omega}_1, 2\tilde{\omega}_0 \rangle$ which leads to $\Sigma^N(\Gamma_2) = \{2\bar{\alpha}_1\}$. This gives the triple $(\{\bar{\alpha}_0, \bar{\alpha}_1\}, \emptyset, 2\bar{\alpha}_1)$ which is not admissible in the sense of theorem 5.8. So the root lattice does not give rise to a q-Hamiltonian manifold.

11.1.3. Lattices $\mathbb{Z}\Gamma$ with $\Lambda \supseteq \mathbb{Z}\Gamma \supseteq S^+$ in type A , n even.

In case of $A_n, n \equiv 0 \pmod{2}$, the lattice $\mathbb{Z}\Gamma = \Lambda_0$ is also a homogenous local model where the spherical roots are the sums of neighboring roots S^+ . The manifold corresponding to the whole weight lattice $\Lambda_S = \Lambda$ was discussed in [Kno16], we shall also classify those for all sublattices:

Lemma 5.26. *Let Φ be of type $A_n^{(1)}$. Then it is a necessary condition for a local model in a X_0 with lattice $\mathbb{Z}\Gamma$ such that $\Lambda \supseteq \mathbb{Z}\Gamma \supseteq \Sigma^N(G) = \{\bar{\alpha}_1 + \bar{\alpha}_2, \dots, \bar{\alpha}_{n-1} + \bar{\alpha}_n\}$ to form a spherical pair $(\mathbb{Z}\Gamma, \mathcal{A})$ that $\bar{\alpha}_0 + \bar{\alpha}_1$ and $\bar{\alpha}_0 + \bar{\alpha}_n$ are in the lattice.*

Proof. The lattice Λ intersected with the tangent cone $C_{X_0}(\mathcal{P})$ gives a well known smooth monoid with spherical roots S_0^+ . We know from [PPVS17] that all smooth G -saturated weight monoids for A_n have either all 2α or all sums of consecutive simple roots as spherical roots. For $n \geq 3$, $\bar{\alpha}_1, \bar{\alpha}_2$ or $\bar{\alpha}_{n-1}, \bar{\alpha}_n$ is found in every local root system. As all sums of consecutive roots not containing $\bar{\alpha}_0$ are in the lattice by construction and are compatible by construction, at least one sum of consecutive roots appears as a spherical roots in every local root system. Then there are smooth monoids for these local root systems if and only if $\Sigma^N(\Gamma) = S^+$. This is the case if and only if $\bar{\alpha}_0 + \bar{\alpha}_1$ and $\bar{\alpha}_0 + \bar{\alpha}_n$ are in the lattice and we have S^+ as N -spherical roots in every vertex.

For $n = 2$, there are no obvious spherical roots in the lattice $\mathbb{Z}\Gamma$ in X_1, X_2 . But, as $\bar{\alpha}_1 + \bar{\alpha}_2$ is in the lattice by construction, also $\bar{\alpha}_0 = -(\bar{\alpha}_1 + \bar{\alpha}_2)$ is. So $2\bar{\alpha}_0$ can not be a spherical root by Luna's axioms. But a smooth monoid with the given properties for A_2 supports every simple root in at least one spherical root. So this could only be smooth if $\bar{\alpha}_0 + \bar{\alpha}_1$ and $\bar{\alpha}_2 + \bar{\alpha}_0$ are in the lattice and hence spherical roots. \square

Lemma 5.27. *The necessary condition from lemma 5.26 for the local model in X_0 is also sufficient if n is even.*

Proof. If n is even, a variety fulfilling the conditions from [lemma 5.26](#) has weight lattice $S^+ \oplus k\omega_{n-1}$ (cf. [list 5.1](#)) and is known to be homogeneous. Then, as discussed in [section 10](#), $\mathbb{Z}\Gamma$ is smooth if and only if it contains all roots S_i^+ in every vertex X_i . \square

Theorem 5.28. *Let Φ be of type $A_n^{(1)}$, n even. Then the lattice $\mathbb{Z}\Gamma = S_0^+ \oplus k\omega_{n-1}$ corresponds to a smooth local model in every vertex if and only if $k|n+1$.*

Proof. We start with the lattice $S_0^+ \oplus k\omega_{n-1}$ and number the generators $\lambda_1, \dots, \lambda_n$. By the two lemmas above it is necessary and sufficient to show that this lattice contains $\bar{\alpha}_0 + \bar{\alpha}_1$ and $\bar{\alpha}_n + \bar{\alpha}_0$. We first check if $\bar{\alpha}_0 + \bar{\alpha}_1$ is in the lattice. We have the relation $\bar{\alpha}_0 + \bar{\alpha}_1 + \dots + \bar{\alpha}_n = 0$, and hence, we have $-(\bar{\alpha}_0 + \bar{\alpha}_1) = \bar{\alpha}_2 + \dots + \bar{\alpha}_n = -\omega_1 + \omega_2 + \omega_n$. We seek a linear combination $\sum_{i=1}^n a_i \lambda_i$ for integral a_i that gives $-(\bar{\alpha}_0 + \bar{\alpha}_1)$. This gives n equations in n unknowns a_i . They are:

$$\begin{aligned} -1 &= a_1 - a_2 \\ 1 &= a_1 + a_2 - a_3 \\ 0 &= -a_1 + a_2 + a_3 - a_4 \\ &\vdots \\ 0 &= -a_{n-4} + a_{n-3} + a_{n-2} - a_{n-1} \\ 0 &= -a_{n-3} + a_{n-2} + a_{n-1} + k \cdot a_n \\ 1 &= -a_{n-1} + a_n \end{aligned}$$

Number these equations $1, \dots, n$. We can easily verify that adding all even numbered equations gives $a_1 = 2$. Then the first equation says $a_2 = 3$, the third $a_3 = 4$, advance like this until equation $n-2$, skip equation $n-1$ and use the last one to get $a_{n-1} = n$. Then putting all this information into equation $n-1$ gives

$$\begin{aligned} (-n+2) + (n-1) + n + k \cdot a_n &= 0 \\ k \cdot a_n &= -(n+1) \end{aligned}$$

So this system has an integral solution if and only if k divides $n+1$.

Now let us check $\bar{\alpha}_0 + \bar{\alpha}_n = \omega_1 + \omega_{n-1} - \omega_n$. This yields slightly different equations:

$$\begin{aligned} 1 &= a_1 - a_2 \\ 0 &= a_1 + a_2 - a_3 \\ 0 &= -a_1 + a_2 + a_3 - a_4 \\ &\vdots \\ 0 &= -a_{n-4} + a_{n-3} + a_{n-2} - a_{n-1} \\ 1 &= -a_{n-3} + a_{n-2} + a_{n-1} + k \cdot a_n \\ -1 &= -a_{n-1} + a_n \end{aligned}$$

We again add up the even numbered equations, but now get $a_1 = -1, a_2 = -2, \dots, a_{n-1} = -(n-1)$. We put this in equation $n-1$:

$$\begin{aligned} 1 &= -(-(n-3)) + (-(n-2)) + (-(n-1)) + k \cdot a_n \\ 1 &= n-3 - n+2 - n+1 + k \cdot a_n \\ 1 &= -n + k \cdot a_n \\ 1+n &= k \cdot a_n \end{aligned}$$

and this is the same condition $k|n+1$. It follows from [list 5.1](#) that these were the only candidates. \square

Putting all results in this section together, we get:

Theorem 5.29. *Let M be a multiplicity free q -Hamiltonian manifold of rank n . Then the following weight lattices $\mathbb{Z}\Gamma$ correspond to spherical pairs $(\mathcal{A}, \mathbb{Z}\Gamma)$ and are all that correspond to homogeneous spherical varieties in X_0 :*

- (1) $\mathbb{Z}\Gamma$ with $2\mathbb{Z}S_0 \subseteq \mathbb{Z}\Gamma \subseteq 2\Lambda_0$ for arbitrary G but $A_{2n}^{(2)}$.
- (2) $\mathbb{Z}\Gamma = S^+ \oplus k\omega_{n-1}$ with $k|n+1$ for $A_n^{(1)}, n \equiv 0(2)$
- (3) Λ_0^B for $D_{n+1}^{(2)}$
- (4) $\mathbb{Z}(S^+ \cup \{2\bar{\alpha}_n\})$ for $D_{n+1}^{(2)}, n$ odd
- (5) Λ_0^B for $A_4^{(2)}$
- (6) $2\Lambda_0$ for $A_{2n}^{(2)}$

11.2. The inhomogeneous case. Now we take a look at the gluing of the inhomogeneous lattices. We have determined the lattices $\mathbb{Z}\Gamma_0$ that are smooth in X_0 . The ones that also fulfill the same criterion in every other vertex are the ones corresponding to smooth affine spherical varieties that give rise to a q -Hamiltonian manifold. In case of $A_n^{(1)}, n$ odd, this gives one more condition:

Theorem 5.30. *Let Φ be of type $A_n^{(1)}, n$ odd. Then*

$$\mathbb{Z}\Gamma = \langle \bar{\alpha}_2 + \bar{\alpha}_3, \dots, \bar{\alpha}_{n-1} + \bar{\alpha}_n, e\omega_{n-1}, r\omega_{n-1} + \omega_n \rangle_{\mathbb{Z}}, \quad e \mid \frac{n+1}{2}, \quad 0 \leq r \leq e-1$$

is the lattice of a q -Hamiltonian manifold with surjective moment map if and only if:

- (1) *The lattice $\mathbb{Z}\Gamma$ contains $\bar{\alpha}_0 + \bar{\alpha}_1$ and $\bar{\alpha}_n + \bar{\alpha}_0$.*
- (2) *Both $\bar{\alpha}_1^\vee, \bar{\alpha}_3^\vee, \dots, \bar{\alpha}_n^\vee$ and $\bar{\alpha}_0^\vee, \bar{\alpha}_2^\vee, \bar{\alpha}_4^\vee, \dots, \bar{\alpha}_{n-1}^\vee$ are part of a basis of $\mathbb{Z}\Gamma^*$.*

Proof. The first part the necessary condition from lemma [lemma 5.26](#).

We know from [[PPVS17](#)] that all these lattices are constructed such that the odd coroots are part of a basis. For every Φ_k with k even, the roots that have to be part of a basis are exactly these. When we start with this lattice and go to a vertex X_k where an α_k with odd k is missing, the ‘‘odd coroots’’ of this local root systems are

the original roots $\bar{\alpha}_0, \bar{\alpha}_2, \dots, \bar{\alpha}_{n-1}$. As the corresponding local model also has S^+ as N -spherical roots and contains an odd number of roots, the monoid in this vertex is smooth if and only if the coroots $\bar{\alpha}_0^\vee, \bar{\alpha}_2^\vee, \bar{\alpha}_4^\vee, \dots, \bar{\alpha}_{n-1}^\vee$ are part of a basis by [list 5.1](#). It follows that conditions 1 and 2 are necessary and sufficient. \square

Lemma 5.31. *The lattice $\mathbb{Z}\Gamma = \langle \bar{\alpha}_2 + \bar{\alpha}_3, \dots, \bar{\alpha}_{n-1} + \bar{\alpha}_n, e\omega_{n-1}, r\omega_{n-1} + \omega_n \rangle_{\mathbb{Z}}$, $e \mid \frac{n+1}{2}$, $0 \leq r \leq e-1$, n odd contains $\bar{\alpha}_0 + \bar{\alpha}_1$ and $\bar{\alpha}_n + \bar{\alpha}_0$ for every r, e, n .*

Proof. We have $-(\bar{\alpha}_0 + \bar{\alpha}_1) = (\bar{\alpha}_2 + \bar{\alpha}_3) + (\bar{\alpha}_4 + \bar{\alpha}_5) + \dots + (\bar{\alpha}_{n-1} + \bar{\alpha}_n)$ and $-(\bar{\alpha}_n + \bar{\alpha}_0) = (\bar{\alpha}_1 + \bar{\alpha}_2) + (\bar{\alpha}_3 + \bar{\alpha}_4) + \dots + (\bar{\alpha}_{n-2} + \bar{\alpha}_{n-1})$. As the lattice contains $\bar{\alpha}_1 + \bar{\alpha}_2, \bar{\alpha}_2 + \bar{\alpha}_3, \dots, \bar{\alpha}_{n-1} + \bar{\alpha}_n$ by construction, the sums on the right side are obviously in the lattice. \square

Lemma 5.32. *The second condition of [theorem 5.30](#) is also true for $\mathbb{Z}\Gamma = \langle \bar{\alpha}_2 + \bar{\alpha}_3, \dots, \bar{\alpha}_{n-1} + \bar{\alpha}_n, e\omega_{n-1}, r\omega_{n-1} + \omega_n \rangle_{\mathbb{Z}}$, $e \mid \frac{n+1}{2}$, $0 \leq r \leq e-1$, n for every e, r, n .*

Proof. All lattices from this family fulfill that $\bar{\alpha}_1^\vee, \bar{\alpha}_3^\vee, \dots, \bar{\alpha}_n^\vee$ is part of a basis by construction. It remains to show the same for the even coroots. We recall from [lemma 5.19](#) that $\bar{\alpha}_0^\vee = -(\bar{\alpha}_1^\vee + \bar{\alpha}_2^\vee + \dots + \bar{\alpha}_n^\vee)$. To see if the even coroots are part of a basis we look at the following matrix with n columns and $(n+1)/2 =: d$ rows, giving the pairing of the even coroots with the generators of the lattice:

	$\bar{\alpha}_2 + \bar{\alpha}_3$	$\bar{\alpha}_3 + \bar{\alpha}_4$	$\bar{\alpha}_4 + \bar{\alpha}_5$	$\bar{\alpha}_5 + \bar{\alpha}_6$	\dots	$\bar{\alpha}_{n-2} + \bar{\alpha}_{n-1}$	$\bar{\alpha}_{n-1} + \bar{\alpha}_n$	$e\omega_{n-1}$	$r\omega_{n-1} + \omega_n$
$\bar{\alpha}_0^\vee$	0	0	0	0	\dots	0	-1	-e	-r-1
$\bar{\alpha}_2^\vee$	1	-1	0	0	\ddots	0	0	0	0
$\bar{\alpha}_4^\vee$	-1	1	1	-1	\ddots	0	0	0	0
$\bar{\alpha}_6^\vee$	0	0	-1	1	\ddots	0	0	0	0
\vdots	\dots	\dots	\ddots	\ddots	-1	\ddots	\vdots	\vdots	\vdots
$\bar{\alpha}_{n-3}^\vee$	0	0	0	0	\dots	-1	0	0	0
$\bar{\alpha}_{n-1}^\vee$	0	0	0	0	\dots	1	1	e	r

We see that that columns 1 and 2, 3 and 4, \dots , $n-2$ and $n-1$ are multiples of each other. So all $d \times d$ sub-determinants that contain one of these pairs is zero. We choose the submatrix built from columns 1, 3, \dots , $n-4, n-1$ and n . (In fact, taking column $n-2$ instead of $n-1$ is just a special case of this). If we chose e.g. column 2 instead of 1, this would just change the sign of the determinant which is

not interesting when looking at being coprime. We get the $d \times d$ -submatrix:

$$M = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -1 & -e & -r-1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & e & r \end{pmatrix}$$

We see that the second row contains only one entry $\neq 0$, so we develop the determinant for that row.

$$\det M = (-1) \cdot M = \begin{pmatrix} 0 & 0 & \dots & 0 & -1 & -e & -r-1 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 1 & \dots & 0 & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & -1 & e & r \end{pmatrix}$$

Again, we develop for the second row of this submatrix and advance like that until we reach at

$$\det M = \pm \det \begin{pmatrix} -e & -r-1 \\ e & r \end{pmatrix} = \pm(-er + er + 1) = \pm 1.$$

So all $d \times d$ subdeterminants are 0 or ± 1 and hence coprime. It follows from the elementary divisors theorem that the even coroots are part of a basis, independent of e, r, n . \square

Now let us advance to the inhomogeneous lattice of type C . We have already seen that the lattice Λ glues in $C_n^{(1)}$, for example by applying [corollary 5.21](#), (3). As there are no other possibilities in the inhomogeneous case in C_n , we are done: It is the only inhomogeneous lattice for type $C_n^{(1)}$ that gives rise to a spherical pair (\mathcal{P}, Λ_M) .

The corresponding variety is also a local model in $A_{2n}^{(2)}$ and glues for the same reason: it is equal to the lattice

$$\langle 2\tilde{\omega}_0, \tilde{\omega}_1, \dots, \tilde{\omega}_{k-1}, \tilde{\omega}_{k+1}, \dots, \tilde{\omega}_n \rangle_{\mathbb{Z}}$$

for every X_k , $k = 1 \dots n$ (use again the reasoning of [corollary 5.21](#)), and these are known to be lattices of smooth affine spherical varieties.

The monoid Λ_0^+ could also be a local model in $A_{2n-1}^{(2)}$ in the vertex X_0 . We claim that the lattice $\Lambda = \langle \omega_1, \dots, \omega_n \rangle_{\mathbb{Z}}$ does not correspond to the weight monoid of a smooth affine spherical variety in X_n . We can see by applying [theorem 5.20](#) that

the lattice has generators $\langle \tilde{\omega}_1 - \tilde{\omega}_0, \tilde{\omega}_2, \dots, \tilde{\omega}_{n-1}, -2\tilde{\omega}_0 \rangle$ in this vertex. So clearly, no $2\bar{\alpha}$ can be a N -spherical root, as the corresponding $\bar{\alpha}^\vee$ would not be even on the lattice, contradicting [definition 2.7](#), (4).

So we clearly have $\Sigma^N(\Gamma) = S_n^+$, the set of sums of neighboring roots. For n odd, we have $\langle \bar{\alpha}_0^\vee + \bar{\alpha}_1^\vee + \dots + \bar{\alpha}_{n-1}^\vee, \sigma_i \rangle = 0$ for all spherical roots, for n even, we have $\langle \bar{\alpha}_0^\vee + \dots + \bar{\alpha}_n^\vee, \sigma_i \rangle = 0$ for all spherical roots. So clearly for $n > 3$, the triple $(S_\Gamma, S^p(\Gamma) \cap S_\Gamma, \Sigma^N(\Gamma) \cap \langle S_\Gamma \rangle_{\mathbb{Z}})$ is never in [list 5.9](#).

For $n = 3$, we have a product of the admissible triples $(\alpha_0, \emptyset, \emptyset) \times (\alpha_1, \emptyset, \emptyset)$. But then, $\{\bar{\alpha}_0^\vee, \bar{\alpha}_1^\vee\}$ is not part of a basis of $(\mathbb{Z}\Gamma)^*$: The Matrix

$$\begin{array}{cccc} & \tilde{\omega}_1 - \tilde{\omega}_0 & \tilde{\omega}_2 & -2\tilde{\omega}_0 \\ \bar{\alpha}_0^\vee & -1 & 0 & 2 \\ \bar{\alpha}_1^\vee & 1 & 0 & 0 \end{array}$$

has the 2×2 -subdeterminants $0, 0, 2$ which are not coprime.

11.3. List Of surjective Quasi-Hamiltonian Manifolds.

We have shown:

Theorem 5.33. *A compact multiplicity free quasi-Hamiltonian $K\tau$ -manifold with K simple and spherical pair $(\mathcal{A}, \mathbb{Z}\Gamma)$ exists if and only if $\mathbb{Z}\Gamma$ is found in [list 5.35](#).*

Remark 5.34. The moment polytopes for surjective moment maps for groups of rank one are already part of [list 3.29](#).

List 5.35. Surjective moment polytopes

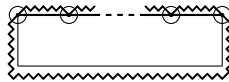
	affine root system	Lattice
1A	$A_n^{(1)}$	$2\langle \bar{\alpha}_2, \bar{\alpha}_3, \dots, \bar{\alpha}_n, d\omega_n \rangle_{\mathbb{Z}}$ with $d n+1$
1B	$A_n^{(1)}$, n even	$\langle S^+ \oplus k\omega_{n-1} \rangle_{\mathbb{Z}}$ with $k n+1$.
1C	n odd	$\langle \bar{\alpha}_2 + \bar{\alpha}_3, \dots, \bar{\alpha}_{n-1} + \bar{\alpha}_n, e\omega_{n-1}, r\omega_{n-1} + \omega_n \rangle_{\mathbb{Z}}, e \frac{n+1}{2}, 0 \leq r \leq e-1$.
2A	$B_n^{(1)}$	$2\mathbb{Z}S_0$
2B		$2\Lambda_0$
3A	$C_n^{(1)}$	$2\mathbb{Z}S_0$
3B		$2\Lambda_0$
3C		Λ_0
4A	$D_n^{(1)}$	$2\mathbb{Z}S_0$
4B		$2\Lambda_0$
4C		$2\langle \bar{\alpha}_1, \dots, \bar{\alpha}_{n-2}, \bar{\alpha}_n, \omega_1 \rangle_{\mathbb{Z}}$
4D	n even	$2\langle \bar{\alpha}_1, \dots, \bar{\alpha}_{n-2}, \bar{\alpha}_n, \omega_n \rangle_{\mathbb{Z}}$
4E	n even	$2\langle \bar{\alpha}_1, \dots, \bar{\alpha}_{n-2}, \bar{\alpha}_n, \omega_1 + \omega_n \rangle_{\mathbb{Z}}$
5A	$E_6^{(1)}$	$2\mathbb{Z}S_0$
5B		$2\Lambda_0$
6A	$E_7^{(1)}$	$2\mathbb{Z}S_0$
6B		$2\Lambda_0$
7	$E_8^{(1)}$	$2\Lambda_0$
8	$F_4^{(1)}$	$2\Lambda_0$
9	$G_2^{(1)}$	$2\Lambda_0$

	affine root system	Lattice
10A	$A_4^{(2)}$	$\langle 2\omega_1, \omega_2 \rangle_{\mathbb{Z}}$
10B	$A_{2n}^{(2)}$	$2\Lambda_0$
10C		Λ_0
11A	$A_{2n-1}^{(2)}$	$2\mathbb{Z}S_0$
11B	$A_{2n-1}^{(2)}$	$2\Lambda_0$
12A	$D_{n+1}^{(2)}$	$2\mathbb{Z}S_0$
12B		$2\Lambda_0$
12C		Λ_0^B
12D	n odd	$\langle \bar{\alpha}_1 + \bar{\alpha}_2, \bar{\alpha}_2 + \bar{\alpha}_3, \dots, \bar{\alpha}_{n-1} + \bar{\alpha}_n, 2\bar{\alpha}_n \rangle_{\mathbb{Z}}$
13	$E_6^{(2)}$	$2\Lambda_0$
14	$D_4^{(3)}$	$2\Lambda_0$

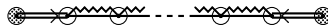
The corresponding diagrams are:

List 5.36. (1) All 2α for 1A, 2A, 2B, 3A, 3B, 4A, 4B, 4C, 4D, 4E, 5A, 5B, 6A, 6B, 7, 8, 9, 10B, 11A, 11B, 12A, 12B, 13, 14.

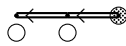
(2) 1B, 1C



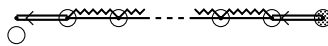
(3) 3C



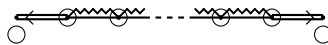
(4) 10A



(5) 10C



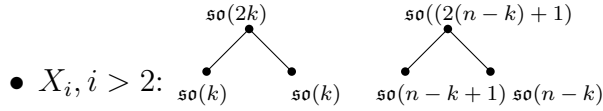
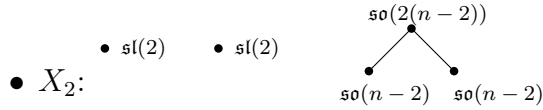
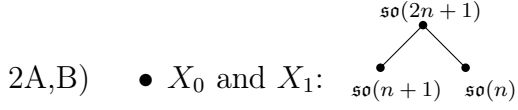
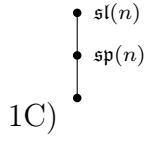
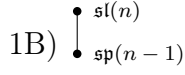
(6) 12C, 12D



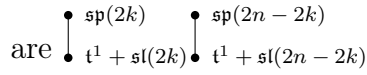
Next, let us discuss the local models.

1A) The local model is obviously the same for every vertex of the alcove. It is of

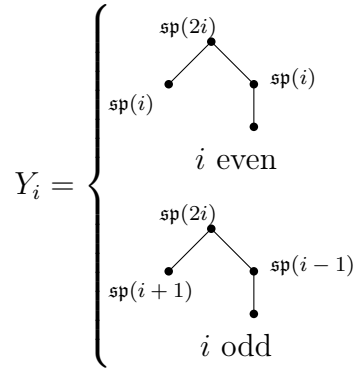
$$\text{type } \begin{array}{c} \bullet \\ \text{sl}(n) \\ \vdots \\ \bullet \\ \text{so}(n) \end{array} .$$



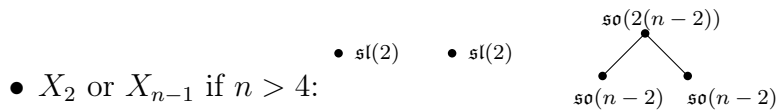
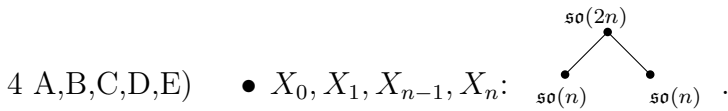
3A,B) The centralizers $K_{X_i}^{\mathbb{C}}$ are $\mathfrak{sp}(2k)\mathfrak{sp}(2n-2k)$ for every X_k . The local models



3C) We define Y_i as:



Then $Y_i Y_{n-i}$ are the triples of the local models in X_i (see also [Kno16, p.39]).



$$\bullet \mathfrak{sl}(2) \quad \bullet \mathfrak{sl}(2) \quad \bullet \mathfrak{sl}(2) \quad \bullet \mathfrak{sl}(2)$$

- X_2 and $n = 4$:

- $X_i, 2 < i < n - 2$: $\begin{array}{c} \bullet \mathfrak{so}(2i) \\ \swarrow \quad \searrow \\ \bullet \mathfrak{so}(i) \quad \bullet \mathfrak{so}(i) \end{array}$ $\begin{array}{c} \bullet \mathfrak{so}(2(n-i)) \\ \swarrow \quad \searrow \\ \bullet \mathfrak{so}(n-i) \quad \bullet \mathfrak{so}(n-i) \end{array}$

5A,B) • For X_0, X_1, X_6 we have $\begin{array}{c} \bullet E_6 \\ \downarrow \\ \bullet C_4 \end{array}$

- For X_3, X_2, X_5 we have $\begin{array}{c} \bullet \mathfrak{sl}(6) \\ \downarrow \\ \bullet \mathfrak{so}(6) \end{array}$ $\bullet \mathfrak{sl}(2)$

- For X_4 , we have $\begin{array}{c} \bullet \mathfrak{sl}(3) \\ \downarrow \\ \bullet \mathfrak{so}(3) \end{array}$ $\begin{array}{c} \bullet \mathfrak{sl}(3) \\ \downarrow \\ \bullet \mathfrak{so}(3) \end{array}$ $\begin{array}{c} \bullet \mathfrak{sl}(3) \\ \downarrow \\ \bullet \mathfrak{so}(3) \end{array}$

6A,B) • For X_0 and X_7 , we have $\begin{array}{c} \bullet E_7 \\ \downarrow \\ \bullet A_7 \end{array}$

- For X_1 and X_6 , we have $\bullet \mathfrak{sl}(2)$ $\begin{array}{c} \bullet \mathfrak{so}(12) \\ \swarrow \quad \searrow \\ \bullet \mathfrak{so}(6) \quad \bullet \mathfrak{so}(6) \end{array}$

- For X_2 , we have $\begin{array}{c} \bullet \mathfrak{sl}(8) \\ \downarrow \\ \bullet \mathfrak{so}(8) \end{array}$

- For X_3 and X_5 , the triple is $\begin{array}{c} \bullet \mathfrak{sl}(3) \\ \downarrow \\ \bullet \mathfrak{so}(3) \end{array}$ $\begin{array}{c} \bullet \mathfrak{sl}(6) \\ \downarrow \\ \bullet \mathfrak{so}(6) \end{array}$

- For X_4 , the triple is $\begin{array}{c} \bullet \mathfrak{sl}(4) \\ \downarrow \\ \bullet \mathfrak{so}(4) \end{array}$ $\begin{array}{c} \bullet \mathfrak{sl}(4) \\ \downarrow \\ \bullet \mathfrak{so}(4) \end{array}$ $\bullet \mathfrak{sl}(2)$

7) • X_0 : $\begin{array}{c} \bullet E_8 \\ \downarrow \\ \bullet D_8 \end{array}$

- X_1 : $\begin{array}{c} \bullet \mathfrak{so}(16) \\ \swarrow \quad \searrow \\ \bullet \mathfrak{so}(8) \quad \bullet \mathfrak{so}(8) \end{array}$

- X_2 : $\begin{array}{c} \bullet \mathfrak{sl}(9) \\ \downarrow \\ \bullet \mathfrak{so}(9) \end{array}$

- X_3 : $\bullet \mathfrak{sl}(2)$ $\begin{array}{c} \bullet \mathfrak{sl}(8) \\ \downarrow \\ \bullet \mathfrak{so}(8) \end{array}$

- X_4 : $\begin{array}{c} \bullet \mathfrak{sl}(2) \\ \downarrow \\ \bullet \mathfrak{so}(2) \end{array}$ $\bullet \mathfrak{sl}(2)$ $\begin{array}{c} \bullet \mathfrak{sl}(6) \\ \downarrow \\ \bullet \mathfrak{so}(6) \end{array}$

• X_5 : $\begin{array}{c} \bullet \text{ } \mathfrak{sl}(5) \\ | \\ \bullet \text{ } \mathfrak{so}(5) \end{array}$ $\begin{array}{c} \bullet \text{ } \mathfrak{sl}(5) \\ | \\ \bullet \text{ } \mathfrak{so}(5) \end{array}$

• X_6 : $\begin{array}{c} \bullet \text{ } \mathfrak{so}(10) \\ / \quad \backslash \\ \bullet \text{ } \mathfrak{so}(5) \quad \bullet \text{ } \mathfrak{so}(5) \end{array}$ $\begin{array}{c} \bullet \text{ } \mathfrak{sl}(4) \\ | \\ \bullet \text{ } \mathfrak{so}(4) \end{array}$

• X_7 : $\begin{array}{c} \bullet \text{ } E_6 \\ | \\ \bullet \text{ } C_4 \end{array}$ $\begin{array}{c} \bullet \text{ } \mathfrak{sl}(3) \\ | \\ \bullet \text{ } \mathfrak{so}(3) \end{array}$

• X_8 : $\begin{array}{c} \bullet \text{ } E_7 \\ | \\ \bullet \text{ } A_7 \end{array}$ $\bullet \text{ } \mathfrak{sl}(2)$

8) • X_0 : $\begin{array}{c} \bullet \text{ } F_4 \\ / \quad \backslash \\ \bullet \text{ } C_3 \quad \bullet \text{ } A_1 \end{array}$

• X_1 : $\bullet \text{ } \mathfrak{sl}(2)$ $\begin{array}{c} \bullet \text{ } \mathfrak{sp}(6) \\ | \\ \bullet \text{ } \mathfrak{gl}(3) \end{array}$

• X_2 : $\begin{array}{c} \bullet \text{ } \mathfrak{sl}(3) \\ | \\ \bullet \text{ } \mathfrak{so}(3) \end{array}$ $\begin{array}{c} \bullet \text{ } \mathfrak{sl}(3) \\ | \\ \bullet \text{ } \mathfrak{so}(3) \end{array}$

• X_3 : $\begin{array}{c} \bullet \text{ } \mathfrak{sl}(4) \\ | \\ \bullet \text{ } \mathfrak{so}(4) \end{array}$ $\bullet \text{ } \mathfrak{sl}(2)$

• X_4 : $\begin{array}{c} \bullet \text{ } \mathfrak{so}(9) \\ / \quad \backslash \\ \bullet \text{ } \mathfrak{so}(5) \quad \bullet \text{ } \mathfrak{so}(4) \end{array}$

9) • X_0 : $\begin{array}{c} \bullet \text{ } G_2 \\ / \quad \backslash \\ \bullet \text{ } A_1 \quad \bullet \text{ } A_1 \end{array}$

• X_1 : $\begin{array}{c} \bullet \text{ } \mathfrak{sl}(3) \\ | \\ \bullet \text{ } \mathfrak{so}(3) \end{array}$

• X_2 : $\bullet \text{ } \mathfrak{sl}(2)$ $\bullet \text{ } \mathfrak{sl}(2)$

10A) • X_0 and X_2 : $\begin{array}{c} \bullet \text{ } \mathfrak{so}(5) \\ | \\ \bullet \text{ } \mathfrak{t}^1 + \mathfrak{sl}(2) \end{array}$

• X_1 : $\bullet \text{ } \mathfrak{sl}(2)$ $\bullet \text{ } \mathfrak{sl}(2)$

10B) For every X_i , we have $\begin{array}{c} \bullet \text{ } \mathfrak{so}(2i+1) \\ / \quad \backslash \\ \bullet \text{ } \mathfrak{so}(i+1) \quad \bullet \text{ } \mathfrak{so}(i) \end{array}$ $\begin{array}{c} \bullet \text{ } \mathfrak{sp}(2(n-i)) \\ | \\ \bullet \text{ } \mathfrak{t}^1 + \mathfrak{sl}(2(n-i)) \end{array}$

10C) For every X_i , we have Y_{n-i} $\begin{array}{c} \bullet \text{ } \mathfrak{so}(2i+1) \\ | \\ \bullet \text{ } \mathfrak{t}^1 + \mathfrak{sl}(n) \end{array}$ with Y_i defined as for 3B).

11A,B) • X_0 and X_1 : $\begin{array}{c} \bullet \text{ sp}(2n) \\ | \\ \bullet \text{ t}^1 + \text{sl}(2n) \end{array}$

• X_2 : $\begin{array}{ccc} \bullet \text{ sl}(2) & \bullet \text{ sl}(2) & \bullet \text{ sp}(2(n-2)) \\ & & | \\ & & \bullet \text{ t}^1 + \text{sl}(2(n-2)) \end{array}$

• $X_i, i > 2$: $\begin{array}{ccc} & \bullet \text{ so}(2i) & \\ / & & \backslash \\ \bullet \text{ so}(i) & & \bullet \text{ so}(i) \end{array}$ $\begin{array}{c} \bullet \text{ sp}(2(n-i)) \\ | \\ \bullet \text{ t}^1 + \text{sl}(2(n-i)) \end{array}$

12A,B) For every X_i , we have $\begin{array}{ccc} & \bullet \text{ so}(2i+1) & \\ / & & \backslash \\ \bullet \text{ so}(i+1) & & \bullet \text{ so}(i) \end{array}$ $\begin{array}{ccc} & \bullet \text{ so}(2(n-i)+1) & \\ / & & \backslash \\ \bullet \text{ so}((n-i)+1) & & \bullet \text{ so}(n-i) \end{array}$

12C,D) For every X_i , we have $\begin{array}{c} \bullet \text{ so}(2i+1) \\ | \\ \bullet \text{ t}^1 + \text{sl}(i) \end{array}$ $\begin{array}{c} \bullet \text{ so}(2(n-i)+1) \\ | \\ \bullet \text{ t}^1 + \text{sl}(n-i) \end{array}$

13) • X_0 : $\begin{array}{ccc} & \bullet F_4 & \\ / & & \backslash \\ \bullet C_3 & & \bullet A_1 \end{array}$

• X_1 : $\begin{array}{ccc} & \bullet \text{ so}(7) & \\ / & & \backslash \\ \bullet \text{ sl}(2) & & \bullet \text{ so}(3) \end{array}$

• X_2 : $\begin{array}{ccc} \bullet \text{ sl}(3) & & \bullet \text{ sl}(3) \\ | & & | \\ \bullet \text{ so}(3) & & \bullet \text{ so}(3) \end{array}$

• X_3 : $\begin{array}{ccc} \bullet \text{ sl}(4) & & \bullet \text{ sl}(2) \\ | & & \\ \bullet \text{ so}(4) & & \end{array}$

• X_4 : $\begin{array}{c} \bullet \text{ sp}(8) \\ | \\ \bullet \text{ t}^1 + \text{sl}(4) \end{array}$

14A,B) • X_0 : $\begin{array}{ccc} & \bullet G_2 & \\ / & & \backslash \\ \bullet A_1 & & \bullet A_1 \end{array}$

• X_1 : $\begin{array}{ccc} \bullet \text{ sl}(2) & & \bullet \text{ sl}(2) \end{array}$

• X_2 : $\begin{array}{c} \bullet \text{ sl}(3) \\ | \\ \bullet \text{ so}(3) \end{array}$

Remark 5.37. Knop determined in [Kno16] the local models for those varieties where the lattice is the whole weight lattice (cases 1B (for $k=1$), 1C(for $e=1$), 3C and 10C) on the group level. However, we would like to emphasize that the explicit determination of these local models on the group level is not that easy. We checked some examples and found out that in the general case, where the Dynkin diagrams of the local root systems are not connected, the local models almost never correspond

to a direct product of monoids from [list 5.1](#), but contain some finite abelian group acting "diagonally" on the direct product of two varieties from [list 5.1](#).

Part 6. Conclusion

Other important questions might be if it is possible to generalize the techniques developed in this thesis to momentum polytopes of ranks different from one or n . This seems rather difficult, the local models get much more complicated and we don't have any control of what the momentum polytopes do inside the alcove.

It might be a very important question what the global geometry of these manifolds is.

As there is a strong connection between q -Hamiltonian manifolds and physics, it might also be important to discuss possible connections with theoretical physicists.

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