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Corrector estimates for the homogenization of a locally periodic medium with areas of low and high diffusivity

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We prove an upper bound for the convergence rate of the homogenization limit $\epsilon \to 0$ for a linear transmission problem for an advection–diffusion(–reaction) system posed in areas with low and high diffusivity, where $\epsilon$ is a suitable scale parameter. In this way we rigorously justify the formal homogenization asymptotics obtained in [37] (van Noorden, T. and Muntean, A. (2011) Homogenization of a locally-periodic medium with areas of low and high diffusivity. Eur. J. Appl. Math. 22, 493–516). We do this by providing a corrector estimate. The main ingredients for the proof of the correctors include integral estimates for rapidly oscillating functions with prescribed average, properties of the macroscopic reconstruction operators, energy bounds, and extra two-scale regularity estimates. The whole procedure essentially relies on a good understanding of the analysis of the limit two-scale problem.

\textbf{Key words:} Corrector estimates; Transmission condition; Homogenization; Micro–macro transport; Reaction–diffusion system in heterogeneous materials

1 Introduction

We study the averaging of a system of reaction–diffusion equations with linear transmission condition posed in a class of highly heterogeneous media, including areas of low and high diffusivity. Our aim is twofold: On the one hand, we wish to justify rigorously the formal asymptotics expansions performed in [37], while on the other we wish to understand the error caused by replacing a heterogeneous solution by an approximation (averaged) estimate. To this end, we prove an upper bound for the convergence rate of the limit procedure $\epsilon \to 0$, where $\epsilon$ is a suitable scale parameter (see Section 2.1 for the definition of $\epsilon$). The materials science scenario we have in view is motivated by a very practical problem: the sulphate corrosion of concrete. We refer the readers to [19] for a brief presentation of the physico-chemical problem and for the formal (two-scale) averaging of locally periodic distributions of unsaturated pores attacked by sulphuric acid.
Fatima and Muntean’s paper [20] contains the rigorous proof of the limiting procedure $\epsilon \to 0$ treating the uniformly periodic case of the same corrosion scenario. In [20], the main working tools involve the two-scale convergence concept in the sense ofNguetseng and Allaire\(^1\) combined with a periodic unfolding of the oscillatory boundary. For that reaction–diffusion scenario, we have used the periodic unfolding technique to get correctors very much in the spirit of Griso [21]. Here the situation we are looking at is conceptually different – the microstructures are now allowed to be distributed in a locally (non-uniform!) periodic way and unfolding-like techniques are not yet developed to treat this case (nor for the quite related case of weakly stochastic distributions of microstructures [26]). In this paper, we use an energy-type method for a simpler model problem, still related to the corrosion issue.

The framework that we tackle in this context is essentially a deterministic one. We assume that a distribution of the Locally periodic array of microstructures\(^2\) is known a priori. We refer the reader to [7] for related discussion of continuum mechanical descriptions of balance laws in continua with microstructure and to [28] for a few worked-out averaging examples away from the periodic setting. At the mathematical level, we succeed in combining successfully the philosophy of getting correctors as explained in the analysis by Chechkin and Piatnitski [10] with the intimate two-scale structure of our system; see also \([8,9]\) for related settings where similar averaging strategies are used. For methodological hints on how to get convergence rates for homogenization-like limits, we refer the reader to the analysis shown in [16] for the case of a reaction–diffusion phase field-like system posed in fixed domains. For a higher level of discussion, see \([5,11]\), both of which are nicely written introductory texts on periodic homogenization methods and applications. It is worth noting that the periodic homogenization of linear transmission problems is a well-understood subject (cf. \([2,17,22,33]\), for example); however, much less is known if one steps away from the periodic setting even if one stays within the deterministic case. If the microstructures are distributed in a suitable random fashion, then concepts such as random fields (see \([4]\) or \([6]\) and references cited therein; \([25]\) and intimate connection with the stochastic geometry of the perforations) can turn out to be helpful in getting averaged equations (eventually also capturing new memory terms), but corrector estimates seem to be very hard to get; (cf., e.g. \([3]\) ). Disordered media for which the stationarity and/or ergodicity assumptions on the random measure do not hold or situations where at a given time scale length scales are non-separated are typical situations that cannot be averaged with existing techniques (so it makes no sense to search here for correctors). This is the place where renormalization group methods could potentially contribute \([1]\).

\(^1\) For an introduction to two-scale convergence, see \([27]\).

\(^2\) Here we deviate from a purely periodic setting. It is also worth noting that the assumption of statistically homogeneity of distribution of microstructures, which is a crucial restriction of stochastic homogenization, does not cover all possible configurations of locally periodic geometries. Also, as we formulate the working framework, we cannot treat neither randomly placed microstructures nor stochastic distributions of micro-geometries. Consequently, the exact connection between these two averaging techniques is hard to make precise.
Equally important is the computability issue: One also needs to design a smart (i.e. computationally cheap) way to use the correctors in designing a posteriori estimates for multi-scale numerical approximations; see the context of [34] and the general philosophy of [14]. Our research is ongoing in this direction.

This paper is organized as follows: In Section 2 we introduce the necessary notation and basic geometry as well as the microscale and the two-scale limit models. We comment only briefly on what sort of physics these models describe. Section 3 contains the main result of our paper – Theorem 3.1.

Essentially, we state in Theorem 3.1 that we can find a constant \( c > 0 \) independent of the choice of the scale parameter \( \epsilon \) such that suitable error/corrector norms in terms of the involved concentrations decrease with decreasing the parameter \( \epsilon \), viz.

\[
\begin{align*}
||u_\epsilon - u_0||_{L^\infty(I, L^2(\Omega_\epsilon^h))} + ||v_\epsilon - v_0||_{L^\infty(I, L^2(\Omega_\epsilon^l))} \\
+ ||u_\epsilon - u_1||_{L^\infty(I, H^1(\Omega_\epsilon^h))} + \epsilon||v_\epsilon - v_0||_{L^\infty(I, H^1(\Omega_\epsilon^l))} \leq c \sqrt{\epsilon}.
\end{align*}
\]

The reader is kindly referred to Section 3 for the clarification of the above estimate.

The motivation triggering this research is twofold:

(i) Mathematically, we expand on the line of Meier’s recent PhD thesis [30] by providing a methodology that rigorously connects rather general distributed microstructure reaction–diffusion-flow systems with the homogenization of the corresponding microscopic systems posed in locally periodic distributed domains. In a broad sense, the convergence rates stated in (1.1) are the best upper bounds that one can get for this setting.

(ii) From the application point of view, we wish to get a better fundamental understanding of cement-based materials in general, and of acid attack on concrete (e.g. the sulphatation reaction) in particular; see [19] for the description of the real-world problem we have in mind.

Both items (i) and (ii) are parts of a larger research initiative targeting the modelling, analysis, and simulation of reaction–diffusion-flow problems in deformable materials, where the mechanics of the medium is altered by multi-scale phase transitions (in this context phase transitions mostly refer to those fast first-order chemical reactions arising in concrete that can induce the mechanical deterioration of the material).

In Section 4 we introduce the technical assumptions needed to obtain the correctors. At this point, we also collect the known well-posedness and regularity results for both the microscopic model and the two-scale limit model. The rest of the paper consists of the proof of the convergence rate (1.1), which is the subject of Section 5.

2 Statement of the problem

In this section we introduce our notation, the microscale model and the two-scale limit problem. We start with the definition of the locally periodic heterogeneous medium.
2.1 Definition of the locally periodic micro-medium

We consider a heterogenous medium consisting of areas of high and low diffusivity. The medium in the present paper is represented by a two-dimensional domain. To define the locally periodic medium with inclusions, we follow the main lines of [10]. We denote the two-dimensional bounded domain by $\Omega \subset \mathbb{R}^2$, with boundary $\Gamma$. Denote

$$J^\epsilon := \{ j \in \mathbb{Z}^2 \mid \text{dist}(e_j, \Gamma) \geq \epsilon \sqrt{2} \},$$
$$\mathcal{U} := \{ y \in \mathbb{R}^2 \mid -1/2 \leq y_i \leq 1/2 \text{ for } i = 1, 2 \}.$$

A convenient way to parameterize the interface $\Gamma^\epsilon$ between the high and low diffusivity areas is to use a level set function, which we denote by $S^\epsilon(x)$:

$$x \in \Gamma^\epsilon \iff S^\epsilon(x) = 0. \quad (2.1)$$

Since we allow the size and shape of the perforations to vary with the macroscopic variable $x$, we use the following characterization of $S^\epsilon$:

$$S^\epsilon(x) := S(x, x/\epsilon), \quad (2.2)$$

where $S : \Omega \times \mathcal{U} \to \mathbb{R}$ is 1-periodic in its second variable, $S \in C^2(\Omega \times \mathcal{U})$ and $S$ is independent of $\epsilon$. We assume that

$$S(x, 0) < \text{const.} < 0 \text{ and } S(x, y)|_{y \in \partial \mathcal{U}} > \text{const.} > 0 \text{ for all } x \in \Omega \quad (2.3)$$

so that the substructures in each unit cell do not touch each other. We set

$$Q^\epsilon_j := \{ x \in \mathcal{U} + j \mid S(x, x/\epsilon) < 0 \},$$

and introduce the area of low diffusivity $\Omega^\epsilon_l$ as follows:

$$\Omega^\epsilon_l := \bigcup_{j \in J^\epsilon} Q^\epsilon_j,$$

and the area of high diffusivity $\Omega^\epsilon_h$ as:

$$\Omega^\epsilon_h := \Omega \setminus \Omega^\epsilon_l.$$

We call a medium of which the geometry is specified with a level set function of the type given in (2.2) a locally periodic medium [10]. In Figure 1 a schematic picture is given of how such a medium might look like for a given $\epsilon > 0$. Note that by construction, the area of high diffusivity $\Omega^\epsilon_h$ is connected and the area of low diffusivity $\Omega^\epsilon_l$ is disconnected. The interface between high and low diffusivity areas $\Gamma^\epsilon$ is now given by $\Gamma^\epsilon := \partial \Omega^\epsilon_l$ and the boundary of $\Omega^\epsilon_h$ is given by $\partial \Omega^\epsilon_h := \Gamma^\epsilon \cup \Gamma$.

3 For the ease of presentation, we choose to work in a two-dimensional setting. Both formal and rigorous analysis work for the three-dimensional case as well. Obviously, to obtain the same corrector estimates, a higher regularity price needs to be payed.
 Furthermore, we use the notation $\Omega_1^\varepsilon := \Omega \setminus \bigcup_{j \in J_\varepsilon} (\varepsilon(\mathbb{W} + j))$, and introduce for later use the smooth cut-off function $\chi_\varepsilon(x)$ that satisfies $0 \leqslant \chi_\varepsilon(x) \leqslant 1$, $\chi_\varepsilon(x) = 0$ if $x \in \Omega_1^\varepsilon$ and $\chi_\varepsilon(x) = 1$ if $\text{dist}(x, \Omega_1^\varepsilon) > \text{dist}(\Gamma^\varepsilon, \Omega_1^\varepsilon)$. Moreover, $\varepsilon |\nabla \chi_\varepsilon| \leqslant C$ and $\varepsilon^2 |\Delta \chi_\varepsilon| \leqslant C$, with $C$ independent of $\varepsilon$, and also

$$
\|1 - \chi_\varepsilon\|_{L^2(\Omega)} \leqslant \varepsilon^{1/2} C,
\|\nabla \chi_\varepsilon\|_{L^2(\Omega)} \leqslant \varepsilon^{-1/2} C,
\|\Delta \chi_\varepsilon\|_{L^2(\Omega)} \leqslant \varepsilon^{-3/2} C,
$$

with $C$ again independent of $\varepsilon$ (see e.g. [10,16] for more insight on the role of such cut-off functions).

In addition, we need to expand the normal $v^\varepsilon$ to $\Gamma^\varepsilon$ in a power series in $\varepsilon$. This can be done in terms of the level set function $S^\varepsilon$, which we assume to be sufficiently regular so that all the following computations make sense (see assumption (B1) in Section 4). We have

$$
v^\varepsilon = \frac{\nabla S^\varepsilon(x)}{||\nabla S^\varepsilon(x)||} = \frac{\nabla S(x, x/\varepsilon)}{||\nabla S(x, x/\varepsilon)||} = \frac{\nabla_x S + \frac{1}{\varepsilon} \nabla_y S}{||\nabla_x S + \frac{1}{\varepsilon} \nabla_y S||} \quad \text{at} \quad x \in \Gamma^\varepsilon.
$$
First we expand $|\nabla S^\epsilon|$. Using the Taylor series of the square-root function, we obtain

$$|\nabla S^\epsilon| = \frac{1}{\epsilon} |\nabla_y S| + O(\epsilon^0). \quad (2.6)$$

In the same fashion, we get

$$\nu^\epsilon = \nu_0 + \epsilon \nu_1 + O(\epsilon^2),$$

where

$$\nu_0 := \frac{\nabla_y S}{|\nabla_y S|} \quad (2.7)$$

and

$$\nu_1 := \frac{\nabla_x S}{|\nabla_y S|} - \frac{(\nabla_x S \cdot \nabla_y S)}{|\nabla_y S|^2} \nabla_y S. \quad (2.8)$$

If we introduce the normalized tangential vector $\tau_0$, with $\tau_0 \perp \nu_0$, we can rewrite $\nu_1$ as

$$\nu_1 = \tau_0 \frac{\nabla_x S}{|\nabla_y S|}. \quad (2.9)$$

Later on we will also use the notation

$$\nu_1^\epsilon = \epsilon^{-1} (\nu^\epsilon - \nu_0) = \nu_1 + O(\epsilon).$$

### 2.2 Microscopic equations and their upscaled form

We focus on the following microscopic model,

$$\begin{cases}
\partial_t u^\epsilon = \nabla \cdot (D_h \nabla u^\epsilon - q^\epsilon u^\epsilon) & \text{in } \Omega_h^\epsilon, \quad (2.10) \\
\partial_t v^\epsilon = \epsilon^2 \nabla \cdot (D_l \nabla v^\epsilon) & \text{in } \Omega_l^\epsilon, \quad (2.11) \\
v^\epsilon \cdot (D_h \nabla u^\epsilon) = \epsilon^2 v^\epsilon \cdot (D_l \nabla v^\epsilon) & \text{on } \Gamma^\epsilon \quad (2.12) \\
u^\epsilon = v^\epsilon & \text{on } \Gamma^\epsilon \quad (2.13) \\
u^\epsilon(x,t) = u_0(x,t) & \text{on } \Gamma, \quad (2.14) \\
u^\epsilon(x,0) = u^I_h(x) & \text{in } \Omega_h^\epsilon, \quad (2.15) \\
v^\epsilon(x,0) = v^I_l(x) & \text{in } \Omega_l^\epsilon. \quad (2.16)
\end{cases}$$

$(P^\epsilon)$ describes in a minimal fashion the following prototypical scenario: A chemical species (tracer) moves by diffusion and convection through a highly porous region $\Omega_h^\epsilon$, penetrates via perfect transmission conditions, i.e. (see (2.12) and (2.13)) low porosity zones surrounded by $\Gamma^\epsilon$, where they diffusive slowly. Here we denote the tracer concentration in the high diffusivity area by $u^\epsilon$, the concentration in the low diffusivity area by $v^\epsilon$ and the velocity of the fluid phase by $q^\epsilon$. Furthermore, $D_h$ denotes the diffusion coefficient in the high diffusivity region, $D_l$ denotes the diffusion coefficient in the low diffusivity regions, $v^\epsilon$ denotes the unit normal to the boundary $\Gamma^\epsilon(t)$, $u_0$ denotes the Dirichlet boundary data.
for the concentration \( u_\epsilon \) and \( u'_\epsilon \) and \( v'_\epsilon \) are the initial values for the active concentrations \( u_\epsilon \) and \( v_\epsilon \), respectively.

In order to formulate the upscaled equations and obtain closed formulae for the effective transport coefficients, we use the notation

\[
B(x) := \{ y \in \mathcal{U} : S(x,y) < 0 \},
\]

(2.17)

and

\[
Y(x) := \mathcal{U} - B(x),
\]

(2.18)

and we define the following \( x \)-dependent cell problem\(^4\)

\[
\begin{aligned}
\Delta_y M_j(x,y) &= 0 &\text{for all } x \in \Omega, y \in Y(x), \\
v_0 \cdot \nabla_y M_j(x,y) &= -v_0 \cdot e_j &\text{for all } x \in \Omega, y \in \partial B(x), \\
M_j(x,y) &= \text{\( y \)-periodic,}
\end{aligned}
\]

(2.19)

for \( j \in \{1,2\} \). To ensure the uniqueness of weak/strong solutions to this cell problem, suitable conditions on the spatial averages of cell functions need to be added. Note that the solution \( M \) to the cell problem (2.19) has the standard (expected) elliptic regularity [23].

The solution to this cell problem allows us to write the results of the formal homogenization procedure in the form of the following distributed microstructure (two-scale) model:

\[
\begin{aligned}
\partial_t v_0(x,y,t) &= D_l \Delta_y v_0(x,y,t) &\text{for } y \in B(x), x \in \Omega, \\
\partial_t \left( \theta(x) u_0 + \int_{B(x)} v_0 dy \right) &= \nabla_x \cdot (\mathcal{D}(x) \nabla_x u_0 - \bar{q} u_0) &\text{for } x \in \Omega, \\
v_0(x,y,t) &= u_0(x,t) &\text{for } y \in \partial B(x), \\
u_0(x,t) &= u_b(x,t) &\text{for } x \in \Gamma, \\
u_0(x,0) &= \bar{u}_b(x) &\text{for } x \in \Omega, \\
v_0(x,y,0) &= \bar{v}_t(x,y) &\text{for } y \in B(x), x \in \Omega,
\end{aligned}
\]

(2.20)

(2.21)

(2.22)

where the porosity \( \theta(x) \) of the medium is given by

\[
\theta(x) := |Y(x)| = 1 - |B(x)|,
\]

(2.23)

the effective diffusivity tensor \( \mathcal{D}(x) \) is defined by

\[
\mathcal{D}(x) := D_h \int_{Y(x)} (I + \nabla_y M(x,y)) dy,
\]

(2.24)

with \( M := (M_1,M_2) \), while the averaged convection vector \( \bar{q} \) is given (and has to be understood in the spirit of (4.1); see Remark 4.1).

Obviously, \( u_0 = u_0(x,t) \) is now a macroscopic object (the concentration of the macroscopic species \( u_0 \)), while \( u_b \) and \( u_t \) are the corresponding macroscopic boundary and initial

\(^4\) See, e.g. [12] for the role of cell problems in formulating averaged equations.
concentration. On the other hand, \( v_0 = v_0(x, y, t) \) is a distributed microscopic concentration and \( v_I \) represents its distributed initial data. Furthermore, \( \bar{q} \) is a given macroscopic convective flux.

3 Main result

Recall that \((u_\epsilon, v_\epsilon)\) is the solution vector for the micro problem and \((u_0, v_0)\) is the solution vector for the two-scale problem. We now introduce the macroscopic reconstructions\(^5\) \( u_\epsilon^0, v_\epsilon^0, u_\epsilon^1 \), which are defined as follows:

\[
\begin{align*}
    u_\epsilon^0(x, t) &:= u_0(x, t) \quad \text{for all } x \in \Omega^\epsilon_h, \\
v_\epsilon^0(x, t) &:= v_0(x, x/\epsilon, t) \quad \text{for all } x \in \Omega^\epsilon_l, \\
u_\epsilon^1(x, t) &:= u_\epsilon^0(x, t) + \epsilon M(x, x/\epsilon) \nabla u_\epsilon^0(x, t) \quad \text{for all } x \in \Omega^\epsilon_h.
\end{align*}
\]

In the same spirit, we introduce the corresponding reconstructions \( u_\epsilon^0_I \) and \( v_\epsilon^0_I \) for the macroscopic initial data \( u_I \) and \( v_I \), respectively.

The main result of our paper is stated in the next theorem. The applicability of this result confines to our working assumptions (A1), (A2), (B1) and (B2) that we introduce in Section 4.

**Theorem 3.1** Assume (A1), (A2), (B1) and (B2). Then the following convergence rate holds

\[
\|u_\epsilon - u_\epsilon^0\|_{L^\infty(I,L^2(\Omega^\epsilon_h))} + \|v_\epsilon - v_\epsilon^0\|_{L^\infty(I,L^2(\Omega^\epsilon_l))} + \|u_\epsilon - u_\epsilon^1\|_{L^\infty(I,H^1(\Omega^\epsilon_h))} + \epsilon \|v_\epsilon - v_\epsilon^0\|_{L^\infty(I,H^1(\Omega^\epsilon_l))} \leq c \sqrt{\epsilon},
\]

where \( I = (0, T] \), and where the constant \( c \) is independent of \( \epsilon \).

The remainder of the paper is concerned with the proof of Theorem 3.1.

4 Technical preliminaries

4.1 Technical assumptions. Known results

4.1.1 Assumptions for the microscopic model

**Assumption (A1).** We assume the following restrictions on data and parameters: Take \( D_h, D_l \in ]0, \infty[, u_\epsilon^0 \in H^1(\Omega^\epsilon_h), v_\epsilon^0 \in H^1(\Omega^\epsilon_l), \) and \( u_b \in H^1(I; H^2(\Gamma)) \). Furthermore, we assume

\[
\begin{align*}
    \|u_\epsilon^I - u_\epsilon^0\|_{L^\infty(\Omega^\epsilon_I)} &= \mathcal{O}(\epsilon^{\theta_1}) \quad \text{with } \theta_1 \geq \frac{1}{2}, \\
    \|u_\epsilon^I - u_\epsilon^0\|_{L^\infty(\Omega^\epsilon_I)} &= \mathcal{O}(\epsilon^{\theta_2}) \quad \text{with } \theta_2 \geq \frac{1}{2}.
\end{align*}
\]

\(^5\) We borrowed this terminology from [16]. Note, however, that the concept of reconstruction operators appears in various other frameworks like for the heterogeneous multi-scale method [15,34].
Assumption (A2). \( q_e \in H^2(\Omega^\epsilon;\mathbb{R}^d) \) with \( \nabla \cdot q_e = 0 \) a.e. in \( \Omega^\epsilon \). In addition, we assume that

\[
\left\| q_e - \frac{1}{\theta} \bar{q} \right\|_{L^2(\Omega^\epsilon;\mathbb{R}^d)} = O(e^{\theta_3}) \text{ with } \theta_3 \geq \frac{1}{2}.
\]

Remark 4.1 If one wishes to replace \( q_e \) with the solution of the stationary Stokes or Navier–Stokes equations, then a few additional things have to be taken into account. One of the most striking facts is that the exponent \( \theta_3 \) arising in (4.1) seems to be restricted to \( \theta_3 = \frac{1}{6} \). Consequently, this worsens essentially the convergence rate; see, for instance, [29] (Theorem 1) for a discussion of the periodic case.

It is important to note that in this paper \( q_e \) depends on \( \epsilon \) only because of the \( \epsilon \)-dependence of its domain of definition. Special scalings in \( \epsilon \) could be studied, but then they would need to be checked carefully to see which of them are reasonable in the frame of the Stokes/Navier–Stokes equations. Herein we rely on Assumption (A2) to avoid such discussions on flow effects.

4.1.2 Assumptions for the two-scale model

Assumption (B1). The level set function \( S : \Omega \times \mathcal{U} \to \mathbb{R} \) is 1-periodic in its second variable and is in \( C^2(\Omega \times \mathcal{U}) \) (cf. the structure indicated in (2.2)). In addition, we also need that the porosity \( \theta \) (defined cf. (2.23)) of the medium is non-degenerate and stays bounded, i.e. there exists \( \delta > 0 \) such that \( \delta \leq \theta(x) < 1 \) for all \( x \in \Omega \).

Assumption (B2). We assume the following restrictions on data and parameters:

\[
\begin{align*}
\theta, D &\in H^2(\Omega), \theta \geq \delta, D \in [0, \infty[, \\
\bar{q} &\in H^2(\Omega;\mathbb{R}^d) \text{ with } \nabla \cdot \bar{q} = 0 \text{ a.e. in } \Omega, \\
u_\text{b} &\in H^1(I;H^2(\Gamma)) \text{ with positive and bounded extension on } \Omega \times I, \\
\partial_t u_\text{b} &\leq 0 \text{ a.e. } (x,t) \in \Omega \times I, \\
u_\text{l} &\in L^\infty(\Omega) \cap H^1(\Omega), \\
v_\text{l}(x,\cdot) &\in L^\infty(\Omega) \cap L^2(\Omega;H^1(B(x))) \text{ for a.e. } x \in \Omega.
\end{align*}
\]

Remark 4.2 Following the lines of [31] and [35], Assumption (B1) implies in particular that the measures \( |\partial B(x)| \) and \( |B(x)| \) are bounded away from zero (uniformly in \( x \)). Consequently, the following direct Hilbert integrals (cf. for instance Part II, Chapter 2 in [13]),

\[
L^2(\Omega;H^1(B(x))) := \{ u \in L^2(\Omega;L^2(B(x))) : \nabla \cdot u \in L^2(\Omega;L^2(B(x))) \}
\]

\[
L^2(\Omega;H^1(\partial B(x))) := \{ u : \Omega \times \partial B(x) \to \mathbb{R} \text{ measurable such that } \int_\Omega \| u(x) \|_{L^2(\partial B(x))}^2 < \infty \},
\]

are well-defined separable Hilbert spaces, and, in addition, the distributed trace

\[
\gamma : L^2(\Omega;H^1(B(x))) \to L^2(\Omega,L^2(\partial B(x)))
\]
given by
\[ \gamma u(x, s) := (\gamma_x U(x))(s), \quad x \in \Omega, s \in \partial B(x), u \in L^2(\Omega; H^1(B(x))) \] (4.2)
is a bounded linear operator. For each fixed \( x \in \Omega \), the map \( \gamma_x \), which is arising in (4.2), is the standard trace operator from \( H^1(B(x)) \) to \( L^2(\partial B(x)) \). We refer the reader to [30] for more details on the construction of these spaces and to [32] for the definitions of their duals as well as for a less regular condition (compared to (B1)) allowing to define these spaces in the context of a certain class of anisotropic Sobolev spaces.

Note that the restriction in the second part of (B1) is not fulfilled automatically as the following example shows: Take \( \Omega = ]0, 1[^2 \) and define \( \forall (x, y) \in \Omega \times U \) such that
\[ S(x, y) := |y_1|^\psi(x) + |y_2|^\psi(x) - \frac{1}{\pi}, \]
where \( \psi(x) := \frac{1}{x_1 x_2} \) with \( x = (x_1, x_2) \in \Omega \) and \( y = (y_1, y_2) \in U \). A simple calculation shows that for \( \hat{x} = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \) we get \( \theta(\hat{x}) = 0 \), and hence \( \delta = 0 \). This is not desired here; the two-scale system then changes its type from parabolic to elliptic at all the degeneration points. Physically, the pointwise vanishing of \( \theta \) means that the grey discs in Figure 1 are allowed to touch each other blocking the transport locally. In terms of a double porosity situation, we can then speak about clogging of the pores; see (3.10) in [36].

**Remark 4.3** Assumption (B1), together with (2.1) and (2.2), requires that the regularity of \( \Gamma_\epsilon \) is compatible with the one of \( \Gamma \) (e.g. \( C^2 \) for both would be sufficient).

### 4.1.3 Analysis of microscopic equations

**Definition 4.4** Assume (A1), (A2) and (B1) and let \( u_\epsilon = U_\epsilon + u_b \). The pair
\[ (U_\epsilon, v_\epsilon) \in L^2(0, T; H^2(\Omega^\epsilon_h)) \cap H^1((0, T) \times \Omega^\epsilon_h) \times L^2(0, T; H^2(\Omega^\epsilon_l)) \cap H^1((0, T) \times \Omega^\epsilon_l) \]
is a solution of problem \((P_\epsilon)\) if all equations of problem \((P_\epsilon)\) hold in the spirit of
\[
\int_{\Omega^\epsilon_h} \partial_t (U_\epsilon + u_b) \phi \, dx + \int_{\Omega^\epsilon_h} (D_h \nabla (U_\epsilon + u_b) - q_\epsilon (U_\epsilon + u_b)) \cdot \nabla \phi \, dx \\
= - \int_{\Gamma^\epsilon} v_\epsilon \cdot (e^2 D_l \nabla v_\epsilon) \phi \, ds, \tag{4.3}
\]
\[
\int_{\Omega^\epsilon_l} \partial_t v_\epsilon \psi \, dx + \int_{\Omega^\epsilon_l} e^2 D_l \nabla v_\epsilon \cdot \nabla \psi \, dx = \int_{\partial \Omega^\epsilon} v_\epsilon \cdot (e^2 D_l \nabla v_\epsilon) \psi \, ds, \tag{4.4}
\]
for all test functions \((\phi, \psi) \in H^1(\Omega^\epsilon_h; \partial \Omega) \times H^1(\Omega^\epsilon_l) \) and \( t \in I \), where \( H^1(\Omega^\epsilon_h; \partial \Omega) := \{ \phi \in H^1(\Omega^\epsilon_h) \mid \phi = 0 \text{ a.e. on } \partial \Omega \} \). Note that the normal \( v_\epsilon \) points into \( \Omega^\epsilon_h \).

In particular, it is worth observing that the transmission conditions (2.12) and (2.13) are fulfilled a.e. on \( \Omega \times \partial B(x) \).

**Theorem 4.5** Assume (A1), (A2) and (B1). Problem \((P_\epsilon)\) admits a unique global-in-time solution in the sense of Definition 4.4.
Proof Since we are dealing here with a linear transmission problem, the proof of the Theorem can be done with standard techniques (see e.g. [18]).

4.1.4 Analysis of two-scale equations

This section contains basic results concerning the well-posedness of the two-scale problem, which we reformulate here as follows:

\[
\begin{align*}
\theta(x)\partial_t u_0 - \nabla_x \cdot (\mathcal{Q}(x)\nabla_x u_0 - \bar{q}u_0) &= -\int_{\partial B(x)} v_0 \cdot (D_l \nabla_y v_0) d\sigma & \text{in } \Omega, \\
\partial_t v_0 - D_l \partial_y v_0 &= 0 & \text{in } B(x), \\
u_0(x, t) &= v_0(x, y, t) & (x, y) \in \Omega \times \partial B(x), \\
u_0(x, t) &= u_b(x, t) & x \in \partial \Omega, \\
u_0(x, y, 0) &= v_l(x, y) & (x, y) \in \partial \Omega \times \overline{B(x)},
\end{align*}
\]

where $v_0$ is the same as in (2.7). Before starting to discuss the existence and uniqueness of solutions to problem $(P)$, we write $U := u - u_b$ and note that $U = 0$ at $\partial \Omega$. Furthermore, we introduce for convenience the evolution triple $(\mathcal{V}, \mathcal{H}, \mathcal{V}^*)$, with

\[
\mathcal{V} := \{ (\phi, \psi) \in H^2(\Omega) \times L^2(\Omega; H^2(\mathcal{B}(x))) \mid \phi(x) = \psi(x, y) \text{ for } x \in \Omega, y \in \partial B(x) \},
\]

\[
\mathcal{H} := H^1_0(\Omega) \times L^2(\Omega; H^1(\mathcal{B}(x))).
\]

Definition 4.6 Assume (B1) and (B2). The pair $(u, v)$, with $u = U + u_b$ where $(U, v) \in L^2(0, T; \mathcal{V}) \cap H^1((0, T) \times \mathcal{H})$, is a solution of problem $(P)$ if all equations of problem $(P)$ hold in the spirit of

\[
\int_{\Omega} \theta \partial_t (U + u_b) \phi dx + \int_{\Omega} (D \nabla_x (U + u_b) - \bar{q}(U + u_b)) \cdot \nabla_x \phi dx \\
= -\int_{\Omega} \int_{\partial B(x)} v_0 \cdot (D_l \nabla_y v_0) \phi d\sigma dx,
\]

\[
\int_{\Omega} \int_{\partial B(x)} \partial_t v \psi dy dx + \int_{\Omega} \int_{\partial B(x)} D_l \nabla_y \psi dy dx = \int_{\Omega} \int_{\partial \Omega} v_0 \cdot (D_l \nabla_y v) \phi d\sigma dx,
\]

for all $(\phi, \psi) \in \mathcal{V}$ and $t \in I$.

Proposition 4.7 (Uniqueness) Assume (B1) and (B2). Problem $(P)$ admits at most one solution in the sense of Definition 4.6.

Proof Since Problem $(P)$ is linear, uniqueness follows in the standard way and can be done directly in the $x$-dependent function spaces: One takes two different solutions to $(P)$ satisfying the same initial data. Testing with their difference and using Gronwall’s inequality conclude the proof. □
Theorem 4.8  Assume (B1) and (B2). Problem (P) admits at least a global-in-time solution in the sense of Definition 4.6.

Proof  See the proof of Theorem 5.11 in [37].

To get the corrector estimates stated in Theorem 3.1, we need more two-scale regularity for the macroscopic reconstruction of the concentration field $v_0$. We state this fact in the following result.

Lemma 4.9 (Additional two-scale regularity) Assume (B1) and (B2). Then

$$v_0 \in L^2(I; H^2(\Omega; H^2(B(x)))).$$  \hspace{1cm} (4.9)

Proof  The proof of this result is similar to the regularity lift proven in Claim 5.10 in [37]. The main ingredients are fixing of the boundary and testing with difference quotients in the weak formulation posed in fixed domains. We omit the details.

Theorem 4.10  Assume (B1) and (B2). Problem (P) admits one global-in-time solution such that for a.e. $t \in I$ we have $(u_0(t), v_0(t)) \in H^2(\Omega) \times H^2(\Omega; H^2(B(x)))$.

Since we rather wish that the reader focusses on our strategy of getting the correctors, we omit the detailed proof of this regularity result. We only point out that under the assumptions (B1) and (B2), Theorem 4.8 guarantees the existence of global in time sufficiently regular solutions. A similar calculation is done in [18] (Theorem 5, pp. 360–364). For our purpose, we have, for instance,

$$||\nabla \partial_t u_0||_{L^2(I \times \Omega^h)} \leq c,$$  \hspace{1cm} (4.10)

$$||u_0||_{L^2(I; H^2(\Omega^h; H^2(B(x))}}} \leq c,$$  \hspace{1cm} (4.11)

where $u_0^e$ is the macroscopic reconstruction of $u_0$.

5 Proof of Theorem 3.1

In this section, we give the proof of the main result of our paper, i.e. of Theorem 3.1. The proof uses the auxiliary results stated in the lemmas below. They mainly concern integral estimates for rapidly oscillating functions with prescribed average. Related integral estimates can be found, for instance, in [10] and Section 1.5 in [11].

Lemma 5.1  Assume the hypothesis of Theorem 3.1 to hold. Then

$$\int_{Y(x)} \left( \nabla_x \cdot ((I + \nabla_y M) \nabla_x u_0) - \frac{1}{\theta(x)} \nabla_x \cdot \int_{Y(x)} (I + \nabla_y M) \nabla_x u_0 dy \right) dy$$

$$= - \int_{\partial B(x)} \nu_1 \cdot (I + \nabla_y M) \nabla_x u_0 d\sigma.$$
Proof We compute
\[
\int_{Y(x)} \left( \nabla_x \cdot ((I + \nabla_y M) \nabla_x u_0) - \frac{1}{\theta(x)} \nabla_x \cdot \int_{Y(x)} (I + \nabla_y M) \nabla_x u_0 \, dy \right) \, dy = \int_{Y(x)} \nabla_x \cdot ((I + \nabla_y M) \nabla_x u_0) \, dy - \nabla_x \cdot \int_{Y(x)} (I + \nabla_y M) \nabla_x u_0 \, dy.
\]

Reynolds’s transport theorem (see, for instance, Chapter 33 in [24] for various generalization of Gauß differentiation formulas applied for transport in \( x \)- and \( t \)-dependent domains) gives
\[
\nabla_x \cdot \int_{Y(x)} (x) \left( I + \nabla_y M \right) \nabla_x u_0 \, dy = \int_{Y(x)} \nabla_x \cdot \left( \left( I + \nabla_y M \right) \nabla_x u_0 \right) \, dy + \int_{\partial B(x)} \frac{\nabla_x S}{|\nabla_y S|} (I + \nabla_y M) \nabla_x u_0 \, d\sigma.
\]

By the boundary condition in (2.19), we have
\[
v_0(I + \nabla_y M) = \frac{\nabla_x S}{|\nabla_y S|} (I + \nabla_y M) = 0 \text{ on } \partial B(x)
\]
so that we can write, using (2.8),
\[
\frac{\nabla_x S}{|\nabla_y S|} (I + \nabla_y M) \nabla_x u_0 = \left( \frac{\nabla_x S}{|\nabla_y S|} - \frac{\nabla_x S \cdot \nabla_y S}{|\nabla_y S|^2} \frac{\nabla_y S}{|\nabla_y S|} \right) (I + \nabla_y M) \nabla_x u_0
\]
\[
= v_1 \cdot (I + \nabla_y M) \nabla_x u_0
\]
on \( \partial B(x) \). Combining the above expressions proves the conclusion of this lemma. \( \square \)

Lemma 5.2 Assume the hypotheses of Theorem 3.1 hold. Let \( Q(x, y) \in L^2(\Omega; L^2(B(x))) \) and \( p \in L^2(\Omega; L^2(\partial B(x))) \). Furthermore, suppose that \( \int_{Y(x)} Q(x, y) \, dy = \int_{\partial Y(x)} p(x, y) \, d\sigma \). Then the inequality
\[
\left| \int_{\Omega^c_\epsilon} Q(x, x/\epsilon) \phi(x) \, dx - \epsilon \int_{\Gamma^c} p(x, x/\epsilon) \phi(x) \, ds \right| \leq C \epsilon \| \phi \|_{H^1(\Omega^c_\epsilon)}
\]
holds for every \( \phi \in H^1(\Omega^c_\epsilon; \partial \Omega) \). The constant \( C \) does not depend on the choice of \( \epsilon \).

Proof The problem
\[
\Delta_y \Psi(x, y) = Q(x, y) \text{ in } Y(x)
\]
\[
v_0 \cdot \nabla_y \Psi = p(x, y) \text{ on } \partial B(x)
\]
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has a 1-periodic in $y$ solution that is unique up to an additive constant. We multiply the first equation above with $\phi$ and integrate over $\Omega_0^\epsilon$ and use the obtained equality to get

$$
\left| \int_{\Omega_0^\epsilon} Q(x, x/\epsilon) \phi(x) \, dx - \epsilon \int_{\Gamma_0^\epsilon} p(x, x/\epsilon) \phi(x) \, ds \right|
= \left| \int_{\Omega_0^\epsilon} \Delta_y \Psi(x, y) \big|_{y = \frac{x}{\epsilon}} \phi(x) \, dx - \epsilon \int_{\Gamma_0^\epsilon} p(x, x/\epsilon) \, ds \right|
= \left| \epsilon \int_{\Gamma_0^\epsilon} (\nabla_y \Psi(x, y) \big|_{y = \frac{x}{\epsilon}} - \nabla_x \nabla_y \Psi(x, y) \big|_{y = \frac{x}{\epsilon}}) \phi(x) \, dx - \epsilon \int_{\Gamma_0^\epsilon} p(x, x/\epsilon) \, ds \right|
= \left| \epsilon \int_{\Gamma_0^\epsilon} (v_0 + \epsilon v_1^\epsilon) \cdot \nabla_y \Psi(a, y) \big|_{y = \frac{x}{\epsilon}} \phi(x) \, ds - \epsilon \int_{\Gamma_0^\epsilon} \nabla_y \Psi \big|_{y = \frac{x}{\epsilon}} \nabla_x \phi(x) \, dx \right|
- \epsilon \int_{\Omega_0^\epsilon} \nabla_x \nabla_y \Psi(x, y) \big|_{y = \frac{x}{\epsilon}} \phi(x) \, dx - \epsilon \int_{\Gamma_0^\epsilon} p(x, x/\epsilon) \, ds \right|
\leq \epsilon^2 \left| \int_{\Gamma_0^\epsilon} v_1^\epsilon \cdot \nabla_y \Psi(a, y) \big|_{y = \frac{x}{\epsilon}} \phi(x) \, ds \right| + \epsilon \left| \int_{\Omega_0^\epsilon} \nabla_y \Psi \big|_{y = \frac{x}{\epsilon}} \nabla_x \phi(x) \, dx \right|
+ \epsilon \left| \int_{\Omega_0^\epsilon} \nabla_x \nabla_y \Psi(x, y) \big|_{y = \frac{x}{\epsilon}} \phi(x) \, dx \right| \leq \epsilon C \| \phi \|_{H^1(\Omega_0^\epsilon)}.
$$

The lemma is now proved. \hfill \Box

The last auxiliary lemma is a special case of Lemma 4 in [10], and therefore we state it here without proof.

**Lemma 5.3** Let $\Pi_\epsilon$ be a subset of \{ $x \in \Omega \mid \text{dist}(x, \partial\Omega) \leq \frac{\sqrt{2}}{2} \epsilon$ \}. Then the following inequality

$$
\left| \int_{\Pi_\epsilon} \nabla_x u_0 \phi \, dx \right| \leq C \epsilon^{3/2} \| \phi \|_{H^1(\Pi_\epsilon)}
$$

holds for all $\phi \in H^1(\Omega_0^\epsilon; \partial\Omega)$. The constant $C$ does not depend on $\epsilon$.

**Proof** (of Theorem 3.1) We define

$$
z_\epsilon(x, t) := u_0(x, t) + \epsilon \xi_\epsilon(x) M(x, x/\epsilon) \nabla u_0(x, t) - u_\epsilon(x, t),
$$
$$
w_\epsilon(x, t) := v_0(x, x/\epsilon, t) - v_\epsilon(x, t).
$$

By, for example, Theorem 4 in [27], the functions $M(x, x/\epsilon)$ and $v_0(x, x/\epsilon, t)$ are well-defined functions in $H^1(\Omega_0^\epsilon)$ and $L^2(I; H^1(\Omega_0^\epsilon))$, respectively, and furthermore, we know by construction that there exists a $C > 0$ such that

$$
\| z_\epsilon \|_{L^2(I; H^1(\Omega_0^\epsilon))} \leq C, \quad \text{(5.1)}
$$
$$
\| w_\epsilon \|_{L^2(I; H^1(\Omega_0^\epsilon))} \leq \epsilon C, \quad \text{(5.2)}
$$

where the constant $C$ is independent of the choice of $\epsilon$. In the following we will use the notation $u_1(x, y, t) = M(x, y)\nabla u_0(x, t)$. We compute:

$$
\Delta z_\epsilon(x, t) = \Delta u_0(x, t) + \epsilon \chi_\epsilon \Delta_x u_1(x, y, t)|_{y=\frac{z}{\epsilon}} + 2 \epsilon \chi_\epsilon \nabla_x \cdot \nabla_j u_1(x, y, t)|_{y=\frac{z}{\epsilon}} \\
+ \frac{1}{\epsilon} \chi_\epsilon \Delta_y u_1(x, y, t)|_{y=\frac{z}{\epsilon}} + \epsilon \Delta \chi_\epsilon u_1(x, y, t)|_{y=\frac{z}{\epsilon}} - \Delta u_\epsilon(x, t) \\
+ 2 \epsilon \chi_\epsilon \nabla_x \cdot \nabla_j u_1(x, y, t)|_{y=\frac{z}{\epsilon}} + 2 \epsilon \chi_\epsilon \cdot \nabla_j u_1(x, y, t)|_{y=\frac{z}{\epsilon}},
$$

$$
\Delta_w(x, \frac{X}{\epsilon}, t) = \Delta_w v_0(x, t) + \epsilon^{-1} \nabla_x \cdot \nabla_j v_0(x, y, t)|_{y=\frac{z}{\epsilon}} \\
+ \epsilon^{-1} \nabla_y \cdot \nabla_j v_0(x, y, t)|_{y=\frac{z}{\epsilon}} + \frac{1}{\epsilon} \chi_\epsilon v_0(x, y, t)|_{y=\frac{z}{\epsilon}} - \Delta v_\epsilon(x, t).
$$

We use

$$
\theta(x)\partial_t u_0 - \nabla_x \cdot (\nabla(x)\nabla_x u_0 - \bar{q} u_0) = - \int_{\partial B(x)} v_0 \cdot (D_j \nabla_j v_0) d\sigma \text{ in } \Omega
$$

and $\Delta_j u_1(x, y) = 0$ and $\partial_t u_\epsilon = \nabla \cdot (D_h\nabla u_\epsilon - q_\epsilon u_\epsilon)$ to obtain

$$
D_h \Delta_x z_\epsilon - \partial_t z_\epsilon = D_h \Delta_x u_0 + \epsilon \chi_\epsilon D_h \Delta_x u_1 + 2 \epsilon \chi_\epsilon D_h \nabla_x \cdot \nabla_j u_1 + \epsilon D_h \Delta \chi_\epsilon u_1 \\
+ 2 \epsilon D_h \nabla \chi_\epsilon \cdot \nabla_j u_1 + 2 \epsilon D_h \chi_\epsilon \nabla_j u_1 \\
- \frac{1}{\theta} \left( \nabla_x \cdot (\nabla(x)\nabla_x u_0 - \bar{q} u_0) - \int_{\partial B(x)} v_0 \cdot (D_j \nabla_j v_0) d\sigma \right) - \epsilon \chi_\epsilon \partial_t u_1 \\
- \nabla \cdot (q_\epsilon u_\epsilon),
$$

$$
e^2 D_l \Delta_x w_\epsilon - \partial_t w_\epsilon = e^2 D_l \Delta_x v_0 + e D_l (\nabla_x \cdot \nabla_j v_0 + \nabla_j \cdot \nabla_x v_0).
$$

On $\Gamma^\epsilon$ we have

$$
v^\epsilon \cdot \nabla z_\epsilon = -v^\epsilon \cdot \nabla u_\epsilon + v^\epsilon \cdot \nabla_x u_0 + \epsilon v^\epsilon \cdot \nabla_x u_1 + v^\epsilon \cdot \nabla_j u_1 \\
= -e^2 \frac{D_l}{D_h} v^\epsilon \cdot \nabla v_\epsilon + v^\epsilon \cdot \nabla x u_0 + \epsilon v^\epsilon \cdot \nabla x u_1 + v^\epsilon \cdot \nabla_j u_1,
$$

$$
v^\epsilon \cdot \nabla w_\epsilon = -v^\epsilon \cdot \nabla v_\epsilon + v^\epsilon \cdot \nabla x v_0 + e^{-1} v^\epsilon \cdot \nabla_j v_0.
$$

Now we multiply with $\phi$ and integrate by parts to get

$$
\int_{\Omega_h} \partial_t z_\epsilon \phi dx + \int_{\Omega_h^*} (D_h \nabla z_\epsilon \cdot \nabla \phi + q^\epsilon \cdot \nabla z_\epsilon \phi) dx \\
= \epsilon \int_{\Omega_h} \chi_\epsilon \partial_t u_1 \phi dx - \int_{\Omega_h^*} D_h \Delta x u_0 \phi dx \\
- \epsilon \int_{\Omega_h^*} D_h \chi_\epsilon \Delta x u_1 \phi dx - 2 \int_{\Omega_h^*} D_h \chi_\epsilon \nabla x \cdot \nabla_j u_1 \phi dx \\
+ \int_{\Omega_h} \frac{1}{\theta} \left( \nabla_x \cdot (\nabla(x)\nabla_x u_0) - \int_{\partial B(x)} v_0 \cdot (D_j \nabla_j v_0) d\sigma \right) \phi dx
$$

(5.3)
\[ + \epsilon^2 \int_{\Gamma^e} D_l v^e \cdot \nabla v^e \phi ds - D_h \int_{\Gamma^e} v^e \cdot \nabla_x u_0 \phi ds \]

\[ - \epsilon D_h \int_{\Gamma^e} v^e \cdot \nabla_x u_1 \phi ds - D_h \int_{\Gamma^e} v^e \cdot \nabla_y u_1 \phi ds \]

\[- \int_{\Omega^e_h} (\epsilon D_h \Delta \chi u_1 + 2 \epsilon D_h \nabla \chi \cdot \nabla_x u_1 + 2 \epsilon D_h \nabla \chi \cdot \nabla_y u_1) \phi dx \]

\[- \int_{\Omega^e_h} \left( \frac{1}{\theta} \partial_t q_e - q_e \right) \cdot \nabla u_0 \phi dx - \epsilon \int_{\Omega^e_h} \chi \epsilon u_1 q_e \cdot \nabla \phi dx, \quad (5.5) \]

and

\[
\int_{\Omega^l} \partial_t w \cdot \psi dx + \epsilon^2 \int_{\Omega^l} D_l \nabla w \cdot \nabla \psi dx
\]

\[= - \epsilon^2 \int_{\Omega^l} D_l \Delta_v v_0 \psi dx - \epsilon \int_{\Omega^l} D_l (\nabla_x \cdot \nabla_y v_0 + \nabla_y \cdot \nabla_x v_0) \psi dx - \epsilon^2 \int_{\Gamma^e} D_l v^e \cdot \nabla v \psi ds
\]

\[+ \epsilon^2 \int_{\Gamma^e} D_l v^e \cdot \nabla_x v_0 \psi ds + \epsilon \int_{\Gamma^e} D_l v^e \cdot \nabla_y v_0 \psi ds. \quad (5.6) \]

We take into account the identity

\[(\nabla_y \cdot \nabla_x u_1(x,y)) \mid_{y=x/\epsilon} = \epsilon \nabla_x \cdot (\nabla_x u_1(x,x/\epsilon)) - \epsilon (\Delta \chi u_1(x,y)) \mid_{y=x/\epsilon}, \]

which gives

\[\epsilon D_h \int_{\Gamma^e} v^e \cdot \nabla_x u_1 \mid_{y=x/\epsilon} z_e ds = \epsilon D_h \int_{\Omega^e_h} \nabla_x u_1 \mid_{y=x/\epsilon} \cdot \nabla (\chi \epsilon z_e) + \chi \epsilon z_e \nabla_x \cdot (\nabla_x u_1 \mid_{y=x/\epsilon}) \mid_{y=x/\epsilon} dx
\]

\[= \epsilon D_h \int_{\Omega^e_h} \nabla_x u_1 \cdot \nabla (\chi \epsilon z_e) + \chi \epsilon z_e \Delta u_1 dx
\]

\[+ D_h \int_{\Omega^e_h} \chi \epsilon z_e \nabla_x u_1 dx,
\]

and also the boundary condition \(v_0 \cdot \nabla_y u_1 = -v_0 \cdot \nabla_x u_0\) for \(y \in \partial B(x)\), which holds also on \(\Gamma^e\), we add the two equations (5.3) and (5.6), substitute \(\phi = z_e\) and \(\psi = w_e\) and obtain

\[
\frac{1}{2} \partial_t \|z_e\|_{L^2(\Omega^e)}^2 + \frac{1}{2} \partial_t \|w_e\|_{L^2(\Omega^l)}^2 + \int_{\Omega^e_h} (D_h \nabla z_e \cdot \nabla z_e + q^e \cdot \nabla z_e) dx + \epsilon^2 \|\nabla w_e\|_{L^2(\Omega^l)}^2
\]

\[= - \int_{\Omega^e_h} D_h \Delta u_0 z_e dx - \int_{\Omega^e_h} \chi \epsilon D_h (\nabla_x \cdot \nabla_y u_1) z_e dx + \int_{\Omega^e_h} \frac{1}{\theta} \nabla_x \cdot (\zeta(x) \nabla_x u_0) z_e dx
\]
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\[-\epsilon D_h \int_{\Gamma^\epsilon} v^\epsilon_1 \cdot (\nabla_x u_0 + \nabla_y u_1) z_\epsilon \, ds\]

\[-\int_{\Omega_h^\epsilon} \left( \int_{\partial B(x)} v_0 \cdot (D_l \nabla_y v_0) \, d\sigma \right) z_\epsilon \, dx + \epsilon \int_{\Gamma^\epsilon} D_l v^\epsilon_1 \cdot \nabla_y v_0 z_\epsilon \, ds\]

\[+ \epsilon \int_{\Omega_h^\epsilon} \chi_\epsilon \partial_t u_1 z_\epsilon \, dx - \epsilon \int_{\Omega_h^\epsilon} D_h \nabla_x u_1 \cdot \nabla (\chi_\epsilon z_\epsilon) \, dx\]

\[-e^2 \int_{\Omega_h^\epsilon} D_l \Delta_x v_0 w_e \, dx - e \int_{\Omega_h^\epsilon} D_l (\nabla_x \cdot (\nabla_x v_0 + \nabla_y \cdot \nabla_x v_0)) w_e \, dx + e^2 \int_{\Gamma^\epsilon} D_l v^\epsilon_1 \cdot \nabla_y v_0 w_e \, ds\]

\[+ e^2 \int_{\Gamma^\epsilon} D_l v^\epsilon_1 \cdot (\epsilon \nabla v_e - \nabla_y v_0) u_1 \, ds - \int_{\Omega_h^\epsilon} (\epsilon D_h \Delta x u_1 + 2\epsilon D_h \nabla \chi_\epsilon \cdot \nabla_y u_1 + 2D_h \nabla \chi_\epsilon \cdot \nabla_y u_1) z_\epsilon \, dx\]

\[-\int_{\Omega_h^\epsilon} \left( \frac{1}{\theta} \bar{q} - q_\epsilon \right) \cdot \nabla u_0 z_\epsilon \, dx - e \int_{\Omega_h^\epsilon} \chi_\epsilon u_1 q_\epsilon \cdot \nabla z_\epsilon \, dx,\]

where we have also used \(\phi - \psi = z_\epsilon - w_e = \epsilon u_1\) on \(\Gamma^\epsilon\) and \(\phi = \epsilon u_1\) on \(\partial \Omega\).

We know that there exist \(\beta > 0\) and \(\gamma > 0\) such that

\[\beta \|z_\epsilon\|_{H^1(\Omega^\epsilon)} \leq \int_{\Omega^\epsilon} (D_h \nabla z_\epsilon \cdot \nabla z_\epsilon + q^\epsilon \cdot \nabla z_\epsilon) \, dx + \gamma \|z_\epsilon\|_{L^2(\Omega^\epsilon)},\]

and we use this to estimate

\[\frac{1}{2} \partial_t \|z_\epsilon\|_{L^2(\Omega^\epsilon)}^2 + \frac{1}{2} \partial_t \|w_e\|_{L^2(\Omega^\epsilon)}^2 + \beta \|z_\epsilon\|_{H^1(\Omega^\epsilon)}^2 + e^2 \|\nabla w_e\|_{L^2(\Omega^\epsilon)}^2 \leq \gamma \|z_\epsilon\|_{L^2(\Omega^\epsilon)} + I_1 + I_2 + \cdots + I_{11},\]

where

\[I_1 = \int_{\Omega^\epsilon} D_h \nabla_x \cdot (\nabla_x u_0 + \nabla_y u_1) z_\epsilon \, dx - \int_{\Omega^\epsilon} \frac{1}{\theta} \nabla_x \cdot (\chi(x) \nabla_x u_0) z_\epsilon \, dx\]

\[+ \epsilon D_h \int_{\Gamma^\epsilon} v_1 \cdot (\nabla_x u_0 + \nabla_y u_1) z_\epsilon \, ds,\]

\[I_2 = \int_{\Omega^\epsilon} \left( \int_{\partial B(x)} v_0 \cdot (D_l \nabla_y v_0) \, d\sigma \right) z_\epsilon \, dx - e \int_{\Gamma^\epsilon} D_l v_0 \cdot \nabla_y v_0 z_\epsilon \, ds,\]

\[I_3 = \int_{\Omega^\epsilon} (1 - \chi_\epsilon) D_h (\nabla_x \cdot \nabla_y u_1) z_\epsilon \, dx,\]

\[I_4 = \epsilon^2 \int_{\Gamma^\epsilon} (D_h (v_2 + O(\epsilon)) \cdot (\nabla_x u_0 - \nabla_y u_1) + D_l v^\epsilon_1 \cdot \nabla_y v_0) z_\epsilon \, ds,\]

\[I_5 = \int_{\Omega^\epsilon} (\epsilon D_h \Delta x u_1 + 2\epsilon D_h \nabla \chi_\epsilon \cdot \nabla_x u_1 + 2D_h \nabla \chi_\epsilon \cdot \nabla_y u_1) z_\epsilon \, dx,\]
\[ I_6 = \epsilon \left| \int_{\Omega_\epsilon^I} \chi_\epsilon \partial_t u_1 z_\epsilon \, dx \right|, \]
\[ I_7 = \epsilon \left| \int_{\Omega_\epsilon^I} D_h \nabla_x u_1 \cdot \nabla (\chi_\epsilon z_\epsilon) \, dx \right|, \]
\[ I_8 = \epsilon^2 \int_{\Omega_\epsilon^I} D_1 \Delta x z_\epsilon w_\epsilon \, dx + \epsilon \left| \int_{\Omega_\epsilon^I} D_1 (\nabla_x \cdot \nabla_y v_0 + \nabla_x v_0 \cdot \nabla_y v_0) w_\epsilon \, dx - \epsilon^2 \int_{\Gamma_\epsilon^I} D_1 v_\epsilon^I \cdot \nabla_x v_0 w_\epsilon \, ds \right|, \]
\[ I_9 = \epsilon^2 \int_{\Gamma_\epsilon^I} D_1 v_\epsilon^I \cdot (\epsilon \nabla v_\epsilon - \nabla_y v_0) u_1 \, ds, \]
\[ I_{10} = \int_{\Omega_\epsilon^I} \left( \frac{1}{\partial q} - q_0 \right) \cdot \nabla u_0 z_\epsilon \, dx, \]
\[ I_{11} = \epsilon \int_{\Omega_\epsilon^I} \chi_\epsilon u_1 q_0 \cdot \nabla z_\epsilon \, dx. \]

For \( I_1 \) we use \( u_1 := MV u_0 \) and \( \partial (x) := D_h \int_{Y(x)} (I + \nabla y M) \, dy. \) We set
\[ Q(x, y) = \nabla_x \cdot ((I + \nabla y M) \nabla_x u_0) - \frac{1}{\partial (x)} \nabla_x \cdot \int_{Y(x)} (I + \nabla y M) \nabla_x u_0 \, dy, \]
\[ p(x, y) = -v_0 \cdot (I + \nabla y M), \]
and use Lemma 5.2 to obtain \( I_1 \leq \epsilon C \| z_\epsilon \|_{H^1(\Omega^I_\epsilon)} \). Lemma 5.1 asserts that the conditions of Lemma 5.2 are satisfied for these choices of \( Q \) and \( p \).

For \( I_2 \) we also apply Lemma 5.2, this time for the choice
\[ Q(x, y) = \frac{1}{\partial (x)} \int_{\partial B(x)} v_0 \cdot \nabla y v_0 \, d\sigma, \]
\[ p(x, y) = v_0 \cdot \nabla y v_0, \]
and we again get \( I_2 \leq \epsilon C \| z_\epsilon \|_{H^1(\Omega^I_\epsilon)} \) with \( C \) independent of \( \epsilon \). For \( I_3 \) we have \( I_3 \leq \sqrt{\epsilon} C \| z_\epsilon \|_{H^1(\Omega^I_\epsilon)} \). Application of the regularity results in Lemma 4.9 and Theorem 4.10 results in \( I_4 + I_6 \leq C \epsilon \| z_\epsilon \|_{H^1(\Omega^I_\epsilon)} \). With the use of Lemma 5.3 and the properties of \( \chi_\epsilon \), we estimate \( I_5 + I_7 \leq \sqrt{\epsilon} C \| z_\epsilon \|_{H^1(\Omega^I_\epsilon)} \). Another application of the regularity results gives \( I_8 \leq C \epsilon \| w_\epsilon \|_{H^1(\Omega^I_\epsilon)} \) and \( I_9 \leq C \epsilon \| u_0 \|_{H^1(\Omega^I_\epsilon)} (\| v_\epsilon \|_{H^1(\Omega^I_\epsilon)} + \| v_0 \|_{H^1(\Omega^I; H^2(B(x)))}) \). For \( I_{10} \) and \( I_{11} \) we use the assumptions on \( q \) and \( q^I \) (as stated in (A2) and (B2)) to get \( I_{10} + I_{11} \leq \sqrt{\epsilon} C \| z_\epsilon \|_{H^1(\Omega^I_\epsilon)} \).

Now, we obtain
\[
\frac{1}{2} \partial_t \| z_\epsilon \|_{L^2(\Omega^I_\epsilon)}^2 + \frac{1}{2} \partial_t \| w_\epsilon \|_{L^2(\Omega^I_\epsilon)}^2 + D_h \| \nabla z_\epsilon \|_{L^2(\Omega^I_\epsilon)}^2 \leq \epsilon C_1 \| u_0 \|_{H^1(\Omega^I_\epsilon)} (\| v_\epsilon \|_{H^1(\Omega^I_\epsilon)} + \| v_0 \|_{H^1(\Omega^I; H^2(B(x)))}) + \sqrt{\epsilon} C_2 \| z_\epsilon \|_{H^1(\Omega^I_\epsilon)} + \epsilon C_3 \| w_\epsilon \|_{H^1(\Omega^I_\epsilon)},
\]
Using the energy bounds (5.1) and (5.2) together with the Gronwall-type argument lead to

\[ \|z_\epsilon\|_{L^2(I,L^2(\Omega_\epsilon^h))}^2 + \|w_\epsilon\|_{L^2(I,L^2(\Omega_\epsilon^l))}^2 + \|z_\epsilon\|_{L^2(I,H^1(\Omega_\epsilon^h))}^2 \]
\[ + \epsilon \|w_\epsilon\|_{L^2(I,H^1(\Omega_\epsilon^l))}^2 \leq \epsilon \tilde{C}_1 + \sqrt{\epsilon} \tilde{C}_2 \|z_\epsilon\|_{L^2(I,H^1(\Omega_\epsilon^h))}, \]  

(5.7)

which, in particular, implies

\[ \|z_\epsilon\|_{L^2(I,H^1(\Omega_\epsilon^h))}^2 \leq \epsilon \tilde{C}_1 + \sqrt{\epsilon} \tilde{C}_2 \|z_\epsilon\|_{L^2(I,H^1(\Omega_\epsilon^h))} \]

and thus

\[ \|z_\epsilon\|_{L^2(I,H^1(\Omega_\epsilon^h))} \leq \sqrt{\epsilon} \left( \frac{1}{2} (\tilde{C}_2^2 + \sqrt{\tilde{C}_2 + 4 \tilde{C}_1}) \right). \]

Combining this with (5.7) gives the result

\[ \|z_\epsilon\|_{L^2(I,L^2(\Omega_\epsilon^h))} + \|w_\epsilon\|_{L^2(I,L^2(\Omega_\epsilon^l))} + \|z_\epsilon\|_{L^2(I,H^1(\Omega_\epsilon^h))} + \epsilon \|w_\epsilon\|_{L^2(I,H^1(\Omega_\epsilon^l))} \leq c \sqrt{\epsilon}, \]

where the constant \( c \) is independent of \( \epsilon \). The last step uses the evident estimate \( \|\epsilon u_1(1 - \chi_\epsilon)\|_{H^1(\Omega_\epsilon^h)} \leq C \sqrt{\epsilon} \), and the theorem is proven.

6 Conclusions

We proved a uniform upper bound for the convergence rate of the homogenization limit \( \epsilon \to 0 \) for a linear advection–diffusion–reaction system with transmission conditions posed in areas with low and high diffusivity. By means of this estimate, we rigorously justified the formal homogenization asymptotics obtained in [37]. It is worth mentioning that the techniques developed for this context are applicable to a larger system of coupled partial differential equations posed in media with locally periodic heterogeneities.

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References


