Finite Element Simulation of Flow-Induced Noise using Lighthill’s Acoustic Analogy

Der Technischen Fakultät der Universität Erlangen-Nürnberg

zur Erlangung des Grades

DOKTOR - INGENIEUR

vorgelegt von

M.Sc. Max Escobar

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Als Dissertation genehmigt von
der Technischen Fakultät der
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Tag der Einreichung: 23. April 2007
Dekan: Prof. Dr.-Ing. A. Leipertz
Berichterstatter: PD Dr.-techn. M. Kaltenbacher
Prof. Dr. C.-D. Munz
Prof. Dr. Dr. h.c. F. Durst
Finite Elemente Simulation
von strömungsinduziertem Lärm nach Lighthills akustischer Analogie

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Zusammenfassung
Abstract

The main focus of the present work is the numerical simulation of flow-induced noise using the Finite Element method (FEM). In recent years, Computational Aeroacoustics (CAA) has expanded from its original aeronautical application field and has become popular among research groups, due to the growing concern shown by diverse industrial sectors for understanding and reducing the noise generated by turbulent flows in applications involving low Mach numbers.

The CAA methodology introduced in this thesis follows a hybrid approach using the weak formulation of Lighthill’s acoustic analogy. Therefore, the flow-induced noise is computed in a coupled two-step procedure. First of all, results from an incompressible unsteady flow simulation capable to resolve the turbulent scales responsible for the noise generation are used to compute acoustic source terms directly on the fine fluid grid. As a second step, after conservative interpolation of the acoustic nodal sources to the acoustic grid, the acoustic propagation is computed using the Finite Element method.

The Finite Element derivation of the aeroacoustic problem is obtained by performing a volume discretization of the weak formulation of Lighthill’s inhomogeneous wave equation. Therewith, reflections and diffraction of the generated acoustic waves due to the presence of solid/elastic bodies are implicitly taken into account, and there is no need to use explicit terms to include these effects, as occurs when using CAA methodologies based on integral formulations. The discretization of the weak formulation represents a key advantage of the FE method over other numerical methods, e.g. the Finite Difference (FD) method, in which discretization of the original differential operator (strong formulation) is required.

Unsteady flow computations following LES (Large Eddy Simulation) and SAS (Scale Adaptive Simulation) turbulence modelling approaches are used for the evaluation of the acoustic sources.

For the validation of the implementation, numerical results of the acoustic pressure for the test case of a co-rotating vortex pair are compared against the corresponding analytical far-field solution. The applicability of the coupled numerical scheme is demonstrated by means of 2D and 3D examples, in both the time and frequency domain. Results from a 2D example of the flow-induced noise from a square cylinder are used to perform qualitative analyses and to estimate appropriate parameters to be able to perform 3D numerical simulations in an efficient way while preserving a good accuracy of the acoustic results. Finally, results are presented from 3D simulations of the noise generated by wall-mounted cylinders with two different geometry profiles. The obtained numerical results compare well to measured data for similar configurations.

Keywords: computational aeroacoustics, finite element method, computational fluid dynamics, Lighthill’s acoustic analogy, absorbing boundary conditions, Perfectly Matched Layer
# List of symbols

## Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1D, 2D, 3D</td>
<td>one-, two-, three-dimensional</td>
</tr>
<tr>
<td>FE</td>
<td>Finite Element</td>
</tr>
<tr>
<td>FD</td>
<td>Finite Difference</td>
</tr>
<tr>
<td>FV</td>
<td>Finite Volume</td>
</tr>
<tr>
<td>DG</td>
<td>Discontinuous Galerkin</td>
</tr>
<tr>
<td>CAA</td>
<td>Computational Aeroacoustics</td>
</tr>
<tr>
<td>APE</td>
<td>Acoustic Perturbation Equations</td>
</tr>
<tr>
<td>LPCE</td>
<td>Linearized Perturbed Compressible Equations</td>
</tr>
<tr>
<td>LEE</td>
<td>Linearized Euler Equations</td>
</tr>
<tr>
<td>MAE</td>
<td>Matched Asymptotic Expansion</td>
</tr>
<tr>
<td>CFD</td>
<td>Computational Fluid Dynamics</td>
</tr>
<tr>
<td>LES</td>
<td>Large Eddy Simulation</td>
</tr>
<tr>
<td>SAS</td>
<td>Scale Adaptive Simulation</td>
</tr>
<tr>
<td>PML</td>
<td>Perfectly Matched Layer</td>
</tr>
<tr>
<td>PDE</td>
<td>Partial Differential Equation</td>
</tr>
</tbody>
</table>

## Mathematical conventions

- $\Omega$: simulation domain
- $\Gamma$: boundary of simulation domain
- $\int_{\Omega} d\Omega$: volume integral
- $\int_{\Gamma} d\Gamma$: surface integral
- $\nabla$: gradient
- $\nabla \cdot$: divergence
- $\nabla \times$: curl
- $\Delta$: laplacian
- $\partial/\partial x$: spatial partial derivative
- $\partial/\partial t$: temporal partial derivative
- $\partial/\partial n$: directional derivative with respect to $n$
### Symbols

- $x, y, z$: global coordinates
- $\xi, \eta, \zeta$: local coordinates
- $b$: complex radius
- $\beta_H$: integration parameter
- $\gamma_H$: integration parameter
- $c$: speed of sound
- $c_0$: undisturbed speed of sound
- $\delta_{ij}$: kronecker symbol
- $E$: Finite Element
- $e_{ij}$: viscous stress tensor
- $\gamma$: adiabatic index
- $h_a$: acoustic element size
- $h_f$: fluid cell size
- $H^1$: Sobolev space of weighting functions
- $i$: imaginary part
- $k$: wave number
- $\lambda$: wave length
- $\Lambda$: dissipation length of eddy
- $L$: spatial Length
- $l$: length scale of turbulence
- $M$: Mach number
- $\mu$: dynamic viscosity
- $N$: interpolation function
- $\nu$: kinematic viscosity
- $\omega$: circular frequency
- $p$: pressure
- $P$: hydrodynamic pressure
- $\Phi$: complex potential function
- $Re$: Reynolds number
- $\rho$: density
- $\rho_0$: undisturbed density
- $r$: radius
- $R$: reflection coefficient
- $\sigma_x, \sigma_y, \sigma_z$: damping functions
- $\vartheta$: collections of trial functions
- $\vartheta_h$: finite-dimensional space of trial functions
- $t$: time
- $\Delta t$: time step size
- $T$: period
- $f$: frequency
- $T_{ij}$: Lighthill’s tensor
- $u_i$: fluid velocity components
- $w$: weighting functions
z complex coordinate
Z_a acoustic impedance

Indices

sd subscript, number of space dimensions
en subscript, number of element nodes
eq subscript, number of equations
' superscript, fluctuating quantity
\cdot superscript, time derivative
\bar superscript, mean quantity
\hat superscript, complex quantity
\text{a} superscript, acoustic related quantity
\text{inc} superscript, incompressible quantity
i,j counter indices

In this work matrices are always represented in boldface letters. In a few sections boldface is also used to represent vectors in order to enhance clarity of the equations.

Vectors

n normal vector
S source vector
E flux vector
F flux vector
H vector of mean shear terms

Matrices

K stiffness matrix
C damping matrix
M mass matrix
M* effective mass matrix
K* effective stiffness matrix
1. Introduction

1.1. Problem definition

Since years noise levels due to the rapid growth of air and ground traffic densities have become an issue for urban communities. Additionally to these noise sources, many other machines producing significant noise levels surround our daily activities and contribute to deterioration of quality of life. The natural response to this problem has been, on the governmental side the definition of strict noise regulations, and on the individual side the development of a greater demand for machines with more acoustic comfort, in the common search of a quieter place to live. These demands have motivated the manufacturers to develop noise reduction strategies and to set noise reduction goals, in special in the airplane and automobile industry. Over twenty years ago researchers in the aeronautic field considered airframe noise of secondary importance in comparison to engine noise and priority was given to the reduction of the latter. Nowadays, after significant success in the engine noise reduction, airframe and engine noise have become, on the approach to landing stages, comparable. Therefore recently significant efforts have been dedicated to reduce airframe noise, being this at the moment the most active field of research in the computational aeroacoustics (CAA) community. Similarly, in the automobile industry, and in general in the ground transportation industry, at speeds over 100 km/h the aerodynamic noise becomes dominant. In this case noise is generated by the interaction of the flow with the main body of the car and with the external accessories like side mirrors or roof devices. Additionally to these noise sources, at lower speeds, internal devices as the air conditioning system also reduce the acoustic interior comfort of the vehicles. Car manufacturers are nowadays aware that acoustic comfort has become an important selection criteria for customers.

In the last years several other industries have shown concern on the aerodynamic noise level of their products and have also started to consider it as a relevant design parameter. Among these applications an important case is the aerodynamic noise from rotating systems as in wind turbines, pumps, fans, etc. Because of this growing demand to reduce noise levels and to fulfill noise regulations in more diverse industrial applications, the motivation exists to investigate the basic aeroacoustic phenomena and mechanisms of sound generation and propagation.

Besides experimental methods, several theories describing the noise generation exist and can now be put into practice to simulate aeroacoustic problems using numerical methods. However, the simulation of aeroacoustic problems poses a broad range of challenges involving physical and numerical aspects, e.g. energy disparity between fluid and acoustic fields, length
1. INTRODUCTION

scale disparity, simulation of unbounded domains, reflection effects, dissipative errors, non-mean background flows, etc. Each of the available aeroacoustic methodologies considers only some of these issues, which restricts their applicability, making them, in many cases, problem dependent methodologies. So far, because of its field of origin, most of the methodologies in CAA have been developed for solving problems in the aeronautical industry in which open domain problems in large acoustic propagation regions are of interest. In contrast, in the industrial sectors more recently interested in aeroacoustic research other requirements arise which have to be fulfilled by new CAA methodologies. In these cases, numerical approaches are required for solving low Mach number problems involving more general geometries and non-compact acoustic sources. Additionally, for confined aeroacoustic problems, or if structural/acoustic effects are considered, numerical approaches different from the traditional aeroacoustic methodologies are required to account for the interactions between the solid surfaces and the flow-induced noise directly in the acoustic simulation.

Regarding the aforementioned aspects, computational aeroacoustic approaches, with emphasis on Lighthill’s acoustic analogy and its use for the numerical simulation of flow-induced noise, are the main topic of this work.

1.2. State of the art and current trends

1.2.1. Background

The beginning of modern aeroacoustics is considered to have been established in 1952 by Sir James Lighthill with his two-part paper published in 1952, 1954 [63, 64] on aerodynamically generated sound. In his work, he developed the jet noise theory by deriving an acoustic analogy which states that the sound field generated by a turbulent flow is that which would be produced by a distribution of acoustic quadrupoles computed from the instantaneous fluctuations of the velocity field in the flow. After that, in 1968, J. Ffowcs and D. Hawkings [37] extended his theory to include the radiated noise from surfaces in arbitrary motion. This extension to the acoustic analogy includes dipole and monopole distributions. The dipole distribution (loading noise) is determined by the force that acts on the fluid as a result of the presence of the surfaces of the body. The monopole source (thickness noise) is determined by the geometry and kinematics of the body. Although the theoretical formulations have been available for a relative long time, only since recently their practical usage became possible mainly with the arrival of high performance computers (HPC) and in great part driven by the development of computational fluid dynamics. These formulations can all be employed in a hybrid two-step approach. The first step consists in computing the turbulent flow field from which the acoustic source terms are evaluated. Subsequently the far field acoustic radiation is computed, using as source for the noise generation the acoustic source terms evaluated from the flow computation. Another important assumption in this kind of hybrid methodology is that in general no feedback from the acoustic field to the turbulent flow is considered.

By utilizing computational aeroacoustic tools based on acoustic analogy formulations considerable noise level reductions have already been achieved mainly in the aeronautical industry, where large funding has been dedicated to CAA research [70]. However, for this
industry it still remains a big challenge to meet the always stricter noise regulations imposed at regions near airports. As powerful computational resources have become more accessible, many other research groups, besides the aeronautical sector, have also started to solve aeroacoustic problems following these theoretical formulations. An important industrial sector which is doing significant CAA research is the automobile industry. In this field the typical applications include the noise reduction from side mirrors, air conditioning systems, sunroofs, exhaust sound design, etc. Some of these problems are interior or confined problems in which the reflection of the propagated waves with the solid surfaces is of importance. Therefore, in the last few years aeroacoustic formulations have been developed for this type of problems or the existing ones have been modified to describe these effects more precisely.

1.2.2. Current CAA approaches

As mentioned above, when following acoustic analogy formulations the flow simulation and the acoustic propagation are performed in separate computations. This fact allows to divide the description of the current developments and trends in CAA numerical methods in two main categories, the numerical methods used for resolving the turbulent flow fields from which the acoustic sources are obtained, and the numerical methods used to compute the subsequent acoustic propagation.

**Acoustic propagation schemes**

The main emphasis of this work is the numerical computation of the acoustic radiation induced by the turbulent flow. Therefore, first of all current trends concerning the numerical methods used to compute the acoustic propagation are introduced. Currently, the methods which are widely spread for solving open simulation problems in large acoustic domains like airframe noise, landing gear noise simulation, fan (turbines) noise, rotor noise, etc. are the integral methods. This type of approach is derived by using free field Green’s functions and can also be subdivided in volume and surface integral methods. Among the volume integral methods we find the traditional acoustic analogy as derived by Lighthill and its extension derived by Ffowcs Williams and Hawkings (FW-H). This extension accounts for the effects of a body in the flow by means of a surface integral over the body which involves the force exerted on it. One main characteristic of these approaches is that they require the volume integration of the quadrupole term in the source region. In the group of surface integral methods there are the Kirchhoff method and the Porous FW-H method. In the Kirchhoff approach the far field acoustic solution is found from a linear Kirchhoff formulation evaluated on a control surface surrounding the nonlinear region which is assumed to enclose all noise sources. As for the Porous FW-H approach, it is similar to the usual FW-H method, but in this case the surface is assumed to be permeable instead of solid and impenetrable, and in this way a general equation can be derived which allows to place the integration surface away from the solid surface. The main advantage of this latter method is that the integration surface can even be located in the nonlinear region, where the use of the Kirchhoff formula would give significant errors [14]. Thus, the porous FW-H method is at the moment the most common surface integral method used in CAA. However discussions exist concerning the effect on the acoustic results of the passage of non-acoustic disturbances through the
permeable surface. An investigation regarding this aspect suggests the inclusion of corrections depending on the location of the permeable surface [69]. An analytical comparison of the Kirchhoff and the porous FW-H methods has been presented by Brentner and Farassat [14] and a comprehensive review of surface methods can be found in [72].

Another approach for computing the radiated sound following the acoustic analogy formulation is based on the variational (also called weak) formulation of Lighthill’s acoustic analogy. This type of approach was first presented by Oberai [80, 81] in 2000 for determining structural-acoustic responses due to hydrodynamic sources. Although not so widely used as the integral methods, the key advantage of this formulation consists in the fact that it can be solved using volume discretization methods like the Finite-Element-Method (FEM). This allows to consider flexible structures and non-compact acoustic sources in the acoustic computation with relative ease. The starting point for the derivation of the formulation is to apply the weighted residuals method to Lighthill’s inhomogeneous wave equation which is obtained by multiplying it with a test function and integrating over the whole acoustic domain. Since in this type of approach a discretization of the whole computational domain is required, in general the computational costs are higher than for the integral methods previously mentioned. On the other hand, by using the variational formulation of Lighthill’s analogy, the acoustic solver implicitly takes into account the interactions between the solid/flexible surfaces and the induced noise. If these effects are significant (e.g. for non-compact solid boundaries, and for interior/ducted aeroacoustic problems) this can be a crucial advantage of the variational formulation over the integral formulations. A discussion comparing these two methodologies can be found in [16].

Active research is also being done in hybrid approaches using other volume discretization methods, such as finite differences (FD), discontinuous galerkin (DG) or finite volume (FV) schemes, for simulating the acoustic propagation. Among the approaches using these full discretization methods the most common is the one based on linearized Euler equations (LEE) with source terms [12, 97, 100]. Additionally, we also find an approach which simulates the acoustic field by solving acoustic perturbation equations (APE) [34] which in contrast to LEE are stable for arbitrary mean flow fields containing hydrodynamic instabilities. Another alternative for the simulation of the acoustic field are the linearized perturbed compressible equations (LPCE) [93] which similarly to the APE method also suppress the errors coming from hydrodynamic instabilities.

As a main issue for the simulation of unbounded domains using interior methods remains the boundary treatment which needs to be applied to avoid the reflection of the outgoing waves on the boundary of the computational domain. Besides acoustics, this is a numerical problem common to several fields of application, such as electromagnetics, aerodynamics, geophysics, meteorology and oceanography. Hence, this issue continues to be a topic of active research and among the numerical techniques that have been developed since the 1970s we find: infinite element methods [10, 5], non-reflecting boundary conditions (NRBC) [54, 45, 49], absorbing boundary conditions (ABCs) [28, 7], Dirichlet-to-Neumann (DtN) [42] and more recently, since the mid-1990s, the perfectly matched layer (PML) [9, 55]. This latter method being considered by many researchers as a breakthrough, due to its relative simple applicability and excellent absorption for any frequency and any angle of incidence.
1.2. STATE OF THE ART AND CURRENT TRENDS

CFD schemes

Concerning the computation of the turbulent flows, parallel to the developments for solving the acoustic radiation, significant advancements have been made in the field of computational fluid dynamics (CFD). Nowadays, improvements in computational performance and a better understanding of the characteristics of turbulent flows allow the simulation of physically accurate turbulent fluid fields for many real applications. Although CFD computations and models are not the primary topic of this work, it is important to describe the current trends since the CFD results are the source for conducting the acoustic computations and their inaccuracies can significantly influence the final acoustic radiation. Therewith, an overview of the current developments and application of the CFD methods commonly used for CAA simulations is given. First of all, as the most accurate CFD method appears the Direct Numerical Simulation (DNS). In this approach all the relevant scales of turbulence are directly resolved and no modeling to approximate the turbulence is employed. It is still unfeasible for practical applications due to its extremely high computational cost. However, because of its high accuracy, it remains very attractive and hope exists that with the further development of computer power higher Reynolds number can be computed. So far, DNS can be applied to problems having low Reynolds numbers and simple geometries. Another method which has been employed by CAA researchers is the Reynolds Averaged Navier Stokes (RANS) \[33, 79, 71\]. Although this method has also been a topic of research in the last years and computations can be performed for some practical turbulent flows, its use remains limited. Its main limitation is due to the fact that it computes only mean flow properties and that it relies heavily on the turbulence models to model all relevant scales of turbulence. Unfortunately, for many flows the turbulent part can be very large and of the same order as the mean \[98\]. CAA methodologies combining RANS for the fluid computation and LEE for the acoustic propagation have been presented in \[33, 71\]. Since noise generation is a multi-scale problem that involves a wide range of length and time scales, unsteady Reynolds Averaged Navier Stokes (uRANS) appears as a better option to RANS being its usage more widely spread \[98\]. On the other hand, Large Eddy Simulation (LES) is currently widely used for solving the turbulent fluids required for acoustic analogy computations. In the LES method the large scales are directly resolved and the effect of the small scales on the large scales are modeled. Although still with a relatively high computational cost, with the LES method it is already possible to simulate turbulent flows with high Reynolds numbers and complex geometries. Therefore, this method has been adopted by researchers as one of the standard methods for computing the turbulent near field in CAA problems. Additionally we find the group of hybrid RANS/LES methods \[47\]. This type of methods blend statistical approaches with LES, in order to yield enhanced predictions of both turbulence statistics and unsteady flow dynamics at a fraction of the cost of traditional LES. Under the hybrid RANS/LES methods we find Detached Eddy Simulation (DES) \[82\] which allows the turbulence model to transition from an uRANS method for attached boundary layers to a LES in separated regions. Another very novel hybrid approach is the Scale-Adaptive Simulation method (SAS) \[74\] which instead of producing the large-scale unsteadiness, typically observed in URANS simulations, it adjusts to the already resolved scales in a dynamic way and allows the development of a turbulent spectrum in the detached regions. It therefore behaves in a way much similar to a DES model, but without the explicit grid dependence in the RANS regime.
1. INTRODUCTION

[74]. A thorough review of the current ongoing research in CFD methods used for CAA simulations can be found in [99].

1.3. Solution approach

The main focus of the present work is the numerical simulation of flow induced noise using the Finite Element method (FEM). The CAA approach followed corresponds to a hybrid methodology using the weak formulation of Lighthill’s acoustic analogy. As usual for a hybrid approach, the flow induced noise is computed in a coupled two-step procedure. First of all, a turbulent flow field is computed in an incompressible flow simulation, using a CFD method capable to resolve the turbulent scales responsible for the noise generation. From the flow quantities the corresponding acoustic source terms are evaluated. As a second step, the acoustic propagation is computed, using as input for the acoustic solver the acoustic sources computed from the hydrodynamic quantities. The computational domain where the flow simulation is resolved is chosen to be just large enough to enclose all turbulent scales producing noise. On the other hand, the numerical domain for the acoustic computation encompasses the region corresponding to the CFD region together with the propagation region.

The FE-derivation of the problem is obtained by performing a volume discretization of the weak formulation of Lighthill’s inhomogeneous wave equation. Therewith, the effects on the acoustic radiation due to the presence of any solid/elastic body are implicitly taken into account, and there is no need to use explicit source terms to include these effects, as occurs when using CAA methodologies based on integral formulations. The discretization of the weak formulation represents a key advantage of the FE method over other numerical methods, e.g. the Finite Difference (FD) method, in which discretization of the original differential operator (strong formulation) is required.

Concerning the turbulent flow fields used for computing the acoustic sources, in this work CFD results have been obtained from LES and SAS computations carried out with the codes FASTEST-3D [26] and ANSYS-CFX [18] respectively.

From the fluid velocity field we evaluate, directly on the fine fluid resolution, the divergence of Lighthill’s tensor as required by the weak formulation of Lighthill’s acoustic analogy and from this quantity we compute the acoustic loads using the FE method. The resulting sources are interpolated as acoustic nodal loads in a conservative way from the fine CFD grid to the coarser acoustic grid. The conservative interpolation scheme preserves the acoustic energy responsible for the sound generation. This novel procedure for the computation and transfer of the source quantities is an important aspect for the flow-induced noise simulation, providing robustness to the acoustic computation by allowing the use of very coarse acoustic grids without affecting the solution of the radiated sound field.

For time domain computations of the radiated sound field, flow and acoustic field simulations are performed simultaneously, transferring the acoustic nodal sources from the fluid to the acoustic computation at each coupling step. This approach implies that the noise induced by the flow does not affect the characteristics of the flow field itself. Due to the scale disparities between fluid and acoustic computations we allow for the involved physical fields different spatial and temporal discretizations. The nodal acoustic loads are interpolated,
from the fluid mesh to the acoustic mesh at every acoustic time step using the Mesh-based parallel Code Coupling Interface MpcCCI [39].

In order to carry out harmonic acoustic simulations of the flow-induced noise, first of all the interpolated (coarse) transient acoustic sources are stored. A Fourier transformation is then performed to the transient acoustic sources, producing the corresponding acoustic nodal sources in frequency domain which are the input for the harmonic acoustic computation.

The flow-induced noise formulation presented in this work, allowing a direct coupling between flow and acoustic field computations in time domain as well as a sequential coupling in frequency domain has been implemented in the in-house simulation package CFS++ [17].

Some main remarks about the solution approach proposed in this work in comparison to traditional CAA methodologies based on integral formulations are

- The only acoustic source term is given by the distribution of volume quadrupoles evaluated from the fluid velocity field. Therefore, no pressure values from the fluid computation are required for the acoustic computation.
- The solution approach takes into account reflection effects of the aerodynamic sound by the solid surfaces implicitly, in contrast to integral CAA formulations based on Green’s functions, in which these effects must be taken into account by explicit surface integrals.
- Full discretization methodologies, are more appropriate for extension to include convection effects of wave propagation in non-uniform background flows.

In problems involving non-compact solid boundaries or confined aeroacoustic problems, the fact that the acoustic source term is exclusively based on the fluid velocity field can be a crucial advantage over the integral formulations, where source terms are usually obtained from CFD simulations which do not represent the acoustic pressure fluctuations accurately enough to handle the various reflections [16]. Additionally, if structural/acoustic effects are considered, it is also more appropriate to use a full discretization method, as the one presented in this work, to account for the interactions between the surfaces and the flow-induced noise directly in the acoustic simulation. In such cases, integral formulations would require a priori knowledge of a hard-wall Green’s function that is not known for complex geometries [80].

1.4. Outline of the work

In the following lines an overview of the present thesis is introduced by giving a brief description of its chapters. Some of the material presented in the chapters has been published in [29, 30, 32, 60, 3, 31, 2].

After the introduction to CAA given in the present chapter, in Chap. 2, current CAA methodologies, with emphasis on hybrid formulations, will be presented in more detail. The derivations of the fundamental equations for the different acoustic analogy formulations are described. Additionally, the appropriate interpretation of the explicit source terms introduced in the FW-H analogy which are required if an integral method is chosen for computing the radiated sound is given. Restrictions concerning their applicability to practical
problems are presented. Furthermore, main differences and key features of methods based on integral solutions of the acoustic analogy and of those based on the weak formulation are also discussed in detail.

In Chap. 3 we perform the volume discretization of Lighthill’s acoustic analogy by applying the FE method. After deriving the weak formulation of Lighthill’s acoustic analogy, the semidiscrete Galerkin formulation for the inhomogeneous wave equation is obtained. Since in the numerical scheme presented in this work we can compute the radiated sound field in either time or frequency domain, in this chapter we introduce the corresponding transient and harmonic formulations used in the numerical implementation. For the transient formulation, the discretization in time using the Newmark method is presented and for the harmonic formulation we obtain the corresponding complex algebraic system of equations from the semidiscrete Galerkin formulation by applying a Fourier transformation. Besides these aspects, the procedure employed for the numerical evaluation of the acoustic source term is also presented.

Chapter 4 is dedicated to the numerical techniques used as boundary treatment for the simulation of unbounded domains. The theory describing absorbing methods is introduced starting from the general concept of absorbing boundary conditions (ABCs) and finalizing with the principle of the perfectly matched layer (PML) method. Additionally, implementation aspects are given for the absorbing boundary methods and PML employed in the numerical scheme developed in this work.

In Chap. 5, the aspects around the coupling approach chosen to transfer the acoustic sources from the flow computation to the acoustic computation are described. First of all, the flowchart for the coupling computation is presented. The differences in the procedures used for performing transient or harmonic acoustic analyses are presented. Secondly, the theory behind the coupling interface used to perform the interpolation, is introduced, describing the non-conservative and conservative interpolation schemes. Additionally, a comparison of acoustic results obtained using these two schemes is presented to emphasize the influence that the interpolation of the quantities can have on the final acoustic results.

Numerical examples in Chap. 6 demonstrate the performance of the methodology proposed in this work for solving flow-induced noise problems using FEM. For the validation of the numerical implementation we compute the noise generated by the hydrodynamic field induced by a co-rotating vortex pair. This validation example has also been employed by other authors in the past, e.g. in [38, 62, 34, 27], for the validation of various CAA approaches. Thereby, two hybrid approaches are investigated, the first approach being based on an acoustic analogy in terms of pressure perturbations and the second following the numerical implementation of Lighthill’s acoustic analogy as implemented in this work. The numerical acoustic results for the co-rotating vortex pair example are validated with the far-field analytical solution, obtained by the matched asymptotic expansion method (MAE) [77].

In Chap. 7, the applicability of the coupled numerical scheme is demonstrated by means of 2D and 3D examples. Initially a 2D coupled simulation is carried out to compute the flow-induced noise due to a square cylinder immersed in a turbulent flow with a Reynolds number around 13000. In the 2D coupled computations, the fluid simulations are carried out in three-dimensions and then a two-dimensional slice is extracted from the results. The subsequent 2D acoustic radiation, either in time or in frequency domain, is computed with the
acoustic sources evaluated on the extracted cut-plane. Concerning 3D coupled computations, aeroacoustic simulations are performed for the flow around wall-mounted cylinders, with a similar Reynolds number as in the 2D computation. For this case the influence of the cylinder profile is investigated using two different geometry variations, a square cylinder and a cylinder with half-ellipse profile in the downstream direction. Finally, results are evaluated and compared with measurements for similar configurations carried out in the anechoic wind tunnel at our department.

Finally, in Chap. 8, conclusions from the results obtained from this thesis are drawn and the main features of our numerical approach as well as an outlook on future work to extend the implementation are given.
1. INTRODUCTION
2. CAA Methodologies

Since the beginning of computational aeroacoustics several numerical methodologies have been proposed, each of these trying to overcome the challenges that the specific problems under investigation pose for an effective and accurate computation of the radiated sound. The difficulties which have to be considered for the simulation of flow noise problems include [51, 52]:

- **Energy disparity and acoustic inefficiency:** There is a large disparity between the energy of the flow in the non-linear field and the acoustic energy in the far field. Therefore, the radiated acoustic energy of an unsteady flow is a very small fraction of the total energy in the flow. In general, the total radiated power of a turbulent jet scales with $O(U^8/c^5)$, and for a dipole source arising from pressure fluctuations on surfaces inside the flow scales with $O(U^6/c^3)$. This shows that an aeroacoustics process at low Mach number is rather a poor sound emitter.

- **Length scale disparity:** Large disparity also occurs between the size of an eddy in the turbulent flow and the wavelength of the generated acoustic noise. Low Mach number eddies have a characteristic length scale $l$, velocity $U$, a life time $l/U$ and a frequency $\omega$. This eddy will then radiate acoustic waves of the same characteristic frequency, but with a much larger length scale

$$\lambda \propto c \frac{l}{U} = \frac{l}{M}.$$  

Furthermore, energy coming from an eddy will be dissipated in $\Lambda = l/Re^{3/4}$. Thus, the ratio of both lengths (acoustic wavelength/dissipation length of eddy) is given by $\lambda/\Lambda = Re^{3/4}/M$, which means that for a high Reynolds number and a Mach number lower than 1, the ratio $\lambda/\Lambda$ will be large.

- **Preservation of multipole character:** The numerical analysis must preserve the multipole structure of the acoustic source, which applies to quadrupoles representing turbulent eddies or forces acting on a surface inside the flow. In order to estimate the source strength, it is necessary to resolve the whole structure of the source at retarded time.

- **Dispersion and dissipation:** The discrete form of the acoustic wave equation can not precisely represent the dispersion relation of the acoustic sound. Numerical discretisation in space and time converts the original non-dispersive system into a dispersive discretised one, which exhibits wave phenomena of two kinds:
1. Long wavelength components approaching the solution of the original differential equation as the grid is refined.

2. Short wavelength components (spurious waves) without counterpart in the original equation evolving in the numerical scheme disturbing the solution.

The wave equation shows a non-dissipative behaviour; as such, dissipative errors must be avoided by a numerical implementation, in which both the amplitude and phase of the wave are of crucial importance.

- **Flows with high Mach and Reynolds number:** Aeroacoustic problems often involve both high Mach and Reynolds numbers. Flows at a high Mach number may induce new non-linear sources and convective effects while flows at a high Reynolds number introduce multiple scale difficulties due to the disparity between the acoustic wavelength $\lambda$ and the size of the energy dissipating eddies.

- **Simulation of unbounded domains:** As a main issue for the simulation of unbounded domains using volume discretization methods remains the boundary treatment which needs to be applied to avoid the reflection of the outgoing waves on the truncating boundary of the computational domain. This remains to be a very active field of research and several numerical formulations have been developed for both transient and harmonic analyses.

Currently available aeroacoustic methodologies overcome only some of these broad range of physical and numerical issues, which restricts their applicability, making them, in many cases, problem dependent methodologies. The application of Direct Numerical Simulation (DNS) is becoming more feasible with the permanent advancement in computational resources. However, due to the large disparities of length and energy scales between fluid and acoustic fields, DNS remains restricted to low Reynolds number flows. In a DNS, all relevant scales of turbulence are resolved and no turbulence modelling is employed. Some relevant work has been done in this direction by Freund [40], however, the simulation of practical problems involving high Reynolds numbers requires very high resolutions which are still far beyond the capabilities of current supercomputers [98]. In this respect, more recently a promising heterogeneous domain decomposition technique [97] has been proposed, as a way to overcome the scales disparities and still allow a direct simulation. In this manner, the equations, numerical methods, grids, and time steps within each subdomain could be adapted to meet the local physical requirements. Meanwhile, hybrid methodologies still remain as the approach most commonly used for aeroacoustic computations, due to the practical advantages provided by the separate treatment of fluid and acoustic computations. In these schemes, the computational domain is split into a non-linear source region and a wave propagation region, and different physical models are used for the flow and acoustic computations. Herewith, first a turbulence model is used to compute the unsteady flow in the source region. Secondly, from the fluid field, acoustic sources are evaluated, which are used as input for the computation of the acoustic propagation. Therewith, a common aspect to hybrid methodologies lies in the assumption that no significant physical influence occurs from the acoustic propagation to the flow field.
Figure 2.1 shows the typical numerical methods which are employed when using any of these hybrid methodologies. Among the group of hybrid approaches, integral methods remain widely used in CAA for solving open domain problems in large acoustic domains like airframe noise, landing gear noise simulation, fan (turbines) noise, rotor noise, etc. One reason which motivates the use of integral formulations in such applications is that, in general, their acoustic sources can be considered to be compact and only an extension of the acoustic solution at a few points in the far field is desired. Therefore, in such cases, integral methods as Lighthill’s acoustic analogy, Curle’s formulation, Ffowcs Williams and Hawkings formulation, Kirchhoff Method or extension thereof are computationally less expensive than volume discretization methods where a whole discretization of the acoustic domain is required. Integral approaches are derived by using free field Green’s functions, being the Ffowcs Williams and Hawkings formulation the most widely used among these. The reason is that it is generalized to account for noise generation effects due to moving surfaces immersed in a turbulent flow.

On the other hand, for confined aeroacoustic problems, where non-compact solid boundaries are present, or if structural/acoustic effects are considered, it is more appropriate to use a volume discretization method to account for the interactions between the solid surfaces and the flow-induced noise directly in the acoustic simulation. In such cases, integral formulations would require a priori knowledge of a hard-wall Green’s function that is not known for complex geometries [80]. Furthermore, integral methods do not allow for a straightforward inclusion of the elastic effects of structures in the flow. An additional advantage in inner methods is that they can also be used to include the effects of wave propagation in non-uniform background flows.

Among the volume discretization methods used in CAA we find finite differences (FD), discontinuous galerkin (DG) or finite volume (FV) schemes, generally employed to solve aeroacoustic formulations based on Linearized Euler Equations (LEE) or systems of equations alike (APE, LPCE, etc.). Additionally, we find the Finite Element method (FEM) used to solve the variational formulation of Lighthill’s acoustic analogy as treated in this work. Fig. 2.1 also depicts the general configuration when using these methods. Herewith, \( \Omega_F \) denotes the area where the flow field is firstly computed and where the acoustic sources are interpolated from the fluid simulation to the acoustic computation. The acoustic propagation region is given by \( \Omega_F \cup \Omega_A \) where the acoustic field is computed in a second step, by solving the inhomogeneous wave equation or a corresponding set of equations depending on the CAA methodology followed. Since interior methods require the whole discretization of the propagation domain, usually they are used to compute the radiated sound until an intermediate region in the far field (i.e. until \( \Gamma_A \) in Fig. 2.1), before moving to an integral formulation in which the acoustic solution from the volume discretization method at the interface is used as input for computing pressure levels at further points. Such a combined scheme has been presented by Manoha et al. [73] solving the Euler equations for the intermediate inhomogenous flow region and a 3D Kirchhoff method for the far field noise.
2. Hybrid approaches based on acoustic analogies

2.1.1. Lighthill’s acoustic analogy

Most of the hybrid approaches employed in CAA, including the one presented in this work, are based on the inhomogeneous wave equation as derived by Lighthill in [63]. This formulation allows the calculation of the acoustic radiation from relatively small regions of turbulent flow embedded in a homogeneous fluid. For the derivation of the inhomogenous wave equation Lighthill starts from the continuity and momentum equations which, by using the summation convention, can be written as [63, 46]

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0 \]  
(2.1)

\[ \rho \frac{\partial u_i}{\partial t} + u_j \frac{\partial \rho u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial e_{ij}}{\partial x_j} \]  
(2.2)

where \( e_{ij} \) is the \((i,j)\)th component of the viscous stress tensor. For a Stokesian gas it can be expressed in terms of the velocity gradients by
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\[ e_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right), \quad (2.3) \]

where \( \mu \) is the viscosity of the fluid and \( \delta_{ij} \) the Kronecker delta.

Multiplying the continuity equation (2.1) by \( u_i \), adding the result to the momentum equation, and combining terms yields

\[ \frac{\partial \rho u_i}{\partial t} = -\frac{\partial}{\partial x_j} (\rho u_i u_j + \delta_{ij} \rho - e_{ij}) \quad (2.4) \]

which after adding and subtracting the term \( c_0^2 \partial \rho/\partial x_i \), becomes

\[ \frac{\partial \rho u_i}{\partial t} + c_0^2 \frac{\partial \rho}{\partial x_i} = -\frac{\partial T_{ij}}{\partial x_i}, \quad (2.5) \]

where \( T_{ij} \) is the Lighthill’s turbulence stress tensor given as

\[ T_{ij} = \rho u_i u_j + \delta_{ij}[(p - p_0) - c_0^2(\rho - \rho_0)] - e_{ij}, \quad (2.6) \]

being \( \rho_0 \) and \( p_0 \) the atmospheric values of density and pressure respectively. Now it is possible to differentiate (2.1) with respect to \( t \), take the divergence of (2.5), and subtract the results to obtain Lighthill’s inhomogeneous wave equation solving for the acoustic density fluctuation \( \rho' \) \([63, 46]\)

\[ \frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}. \quad (2.7) \]

Approximation of Lighthill’s Stress Tensor

An important aspect in Lighthill’s acoustic analogy, in order to be able to compute the noise radiation, is the assumption that Lighthill’s stress tensor is a known source term or at least can be evaluated to a certain degree of approximation. Additionally, this source term is assumed to vanish outside the turbulent region. Indeed, for a turbulent flow embedded in a uniform atmosphere at rest, Lighthill’s stress tensor \( T_{ij} = \rho u_i u_j + \delta_{ij}[(p - p_0) - c_0^2(\rho - \rho_0)] - e_{ij} \) can be neglected outside the turbulent region itself. In this outer region the velocity \( u_i \) consists only of the small motions characteristics of sound. Furthermore, this velocity appears quadratically in the calculation of the tensor as \( \rho u_i u_j \). Moreover, the effects of viscosity and heat conduction are expected to cause only a slow damping due to the conversion of acoustic energy into heat and to have a significant effect only over very large distances. Therefore, it should be possible to neglect \( e_{ij} \) entirely \([46]\).

The term \( (p - p_0) - c_0^2(\rho - \rho_0) \), becomes of importance only for anisotropic media, when the Mach number in the acoustic domain is significantly different from the one in the fluid domain. Otherwise, for isentropic flows in which \( (p - p_0)/p_0 \) and \( (\rho - \rho_0)/\rho_0 \) are very small, the isentropic relation

\[ (p - p_0) = c_0^2(\rho - \rho_0) \quad (2.8) \]

can be assumed. Therefore, the resulting approximate of Lighthill’s tensor is given by
Lighthill pointed out in [63], that the resulting approximate tensor could also be found from an approach which makes approximations in the equations of motion right from the beginning of the derivation of Lighthill’s inhomogeneous equation, but that approximations at that early stage could introduce dipole or monopole sources whose contribution to the radiated sound might be relatively large.

The explicit separation of propagation and generation as derived by Lighthill has arised many discussions ever since, which have motivated the derivation of improved acoustic analogy formulations. A difficulty of Lighthill’s equation is the interpretation of the source term where mean flow effects on the wave propagation are included [86]. In such cases, a compressible velocity field is required for the evaluation of the source term. In order to obtain a formulation to describe the noise propagation in a transversally sheared mean flow, Lilley [65, 66] proposed a third-order wave operator. The mean flow and any refraction it causes are explicit in this wave operator. However, this formulation does not provide any obvious simplification of the sources, still leaving complex combinations of terms which must be modeled in making predictions. Since there is no clear simplification that comes about by including more propagation physics into the wave operator, it can therefore be argued that the Lighthill approach is no worse off than such a more sophisticated approach. Since acoustic analogies are exact, an accurate representation of the source in the original Lighthill’s analogy provides the correct sound despite the multiple physical effects grouped into it. Since different acoustic analogies require different source models, it is difficult to make clear direct comparisons between them and thus, difficult to judge if one is preferable to others [41].

Acoustic analogy formulation in terms of pressure perturbations

If linear acoustic propagation can be assumed, then the isentropic relation $p' = c_0^2 \rho'$ is fulfilled. Using this relation and due to the hybrid nature of both sides of Lighthill’s inhomogeneous wave equation, a change of the acoustic variables can be applied on the left-hand side of (2.7) to solve for the pressure instead of the density as

$$\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}. \quad (2.10)$$

Now, concerning fluid dynamics, the approximation of a compressible fluid field through an incompressible one demands the splitting of the pressure fluctuations into two components, $p' = p^{inc} + p^a$, where $p^{inc}$ represents the incompressible pressure fluctuations (or so called pseudo-sound pressure) and $p^a$ the actual far-field acoustic pressure [85]. Additionally, if in the flow region the flow can be described as nearly incompressible, then the pressure fluctuations are dominated by inertial effects instead of by compressibility and they can be directly determined from the velocity field by solving Poisson’s equation [46]

$$\Delta p^{inc} = -\rho_0 \frac{\partial^2 (u_i u_j)}{\partial x_i \partial x_j}. \quad (2.11)$$

Based on (2.11) and by making use of the splitting of the pressure variables, derives Roger [87] the following inhomogenous wave equation solving for the acoustic pressure fluctuations

$$T_{ij} \approx \rho u_i u_j. \quad (2.9)$$
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Equation (2.12) is equivalent to Lighthill’s acoustic analogy from (2.7), but in contrast, its right-hand side is defined in terms of the second time derivative of the incompressible hydrodynamic pressure fluctuations \( p^{inc} \). This equation is valid as long as compressibility effects are considered to occur only at the second order where the acoustic motion takes place \[87\]. In general this occurs in nearly incompressible low Mach number problems in which the part of the density fluctuations related to the hydrodynamic or pseudo-sound pressure can be neglected.

2.1.2. Volume integral formulations

Far-field solution when no solid boundaries are present

The original methodology proposed by Lighthill for solving the inhomogenous wave equation from (2.7) corresponds to an integral formulation. The far-field solution for this equation is expressed in terms of the free-space Green’s function and corresponds to the volume integral \[63, 46\]

\[
\rho' = \frac{1}{4\pi c_0^2} \frac{\partial^2}{\partial x_i \partial x_j} \int_{\Omega} \frac{T_{ij}(y, t - \frac{|x-y|}{c_0})}{|x-y|} dy, \tag{2.13}
\]

in which the integration is performed over the turbulent domain where the sound sources are evaluated. At points far enough from the flow to be in the radiation field of each quadrupole, that is, at a distance large compared with \((2\pi)^{-1}\) times a typical wave-length, the spatial differentiation in (2.13) can be replaced by a temporal differentiation \[63\]. The derivation of this transformation is also presented by Goldstein \[46\], using multipole expansion, which for any function \( f \) of \((t - \frac{|x|}{c_0})\) results in the form

\[
\frac{\partial}{\partial x_i} f \left( t - \frac{|x|}{c_0} \right) = - \frac{x_i}{c_0 |x|} \frac{\partial}{\partial t} f \left( t - \frac{|x|}{c_0} \right). \tag{2.14}
\]

Employing this latter equivalence in the spatial differentiation from (2.13) results in

\[
\frac{\partial^2}{\partial x_i \partial x_j} \frac{T_{ij}(y, t - \frac{|x-y|}{c_0})}{|x-y|} = \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^3} \frac{1}{c_0^2} \frac{\partial^2 T_{ij}(y, t - \frac{|x-y|}{c_0})}{\partial t^2}. \tag{2.15}
\]

Using (2.15), it is then possible to rewrite the far-field solution of Lighthill’s acoustic analogy as

\[
\rho' = \frac{1}{4\pi c_0^2} \int_{\Omega} \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^3} \frac{1}{c_0^2} \frac{\partial^2 T_{ij}}{\partial t^2} \left( y, t - \frac{|x-y|}{c_0} \right) dy. \tag{2.16}
\]
In addition, if the observation distances are large compared with the dimensions of the flow region one may approximate $x_i - y_i$ by $x_i$ in (2.16), assuming that the origin of the coordinate system is in the source region. This yields the simpler form for the radiated far-field [63]

$$\rho' = \frac{1}{4\pi c_0^2 |\mathbf{x}|^3} \int_{\Omega} \frac{1}{c_0^2} \frac{\partial^2 T_{ij}}{\partial t^2} \left( \mathbf{y}, t - \frac{|\mathbf{x} - \mathbf{y}|}{c_0} \right) d\mathbf{y}. \quad (2.17)$$

On the other hand, as pointed out by Lighthill, for extensive flows the replacement of $x_i - y_i$ by $x_i$ may be inadequate at distances from the flow at which one can conveniently make measurements.

**Extension to account for the effect of solid surfaces**

The integral formulation proposed by Lighthill for evaluating the far-field radiation described by the inhomogenous wave equation from (2.7) assumes that there are no solid boundaries immersed in the fluid (or if there are, that their effect can be neglected). His formulation was originally developed for the prediction of jet noise where in general no solid surfaces are considered. On the other hand, in many other real applications these effects play a significant, and sometimes dominant role for the generation of noise. Therefore, Ffowcs Williams and Hawkings [37] extended Lighthill’s integral formulation to explicitly take into account the *surface loading* (dipole) and *thickness loading* (monopole) contributions due to the presence of solid boundaries.

The starting point used for the derivation of Ffowcs Williams and Hawkings formulation (FW-H), in order to include the effects of surfaces, is to make Lighthill’s analogy valid for an unbounded fluid which is partitioned into regions by mathematical surfaces *which exactly correspond to the real surfaces*. In the derivation of the model, a volume of fluid $\Omega$ enclosed by a surface $\Gamma$ is considered. $\Omega$ is divided into regions 1 and 2 by a surface of discontinuity $\Gamma_{12}$ encroaching on region 2 with velocity $\mathbf{v}$, as shown in Fig. 2.2. The motion of the new fluid on and outside the mathematical surfaces is defined to be completely identical with the real motion, whereas the interior flow can be specified arbitrarily. The interior motion is assumed to be very simple, and consequently does not match the exterior flow at the boundaries. Thus, mass and momentum forces are introduced to maintain these discontinuities, and these ultimately act as sound generators. After the reformulation of mass and momentum equations they become [37]

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i} (\bar{\rho} \bar{u}_i) = \rho_0 v_i \delta(f) \frac{\partial f}{\partial x_i} \quad (2.18)$$

$$\frac{\partial \bar{\rho} \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j} (\bar{\rho} \bar{u}_i \bar{u}_j + \bar{P}_{ij}) = P_{ij} \delta(f) \frac{\partial f}{\partial x_j} \quad (2.19)$$

where $\rho_0$ is the density of the fluid at rest and $v_i$ is the surface velocity. The presence and orientation of the surface is accounted by the function $f$. The equation $f = 0$ defines the
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Figure 2.2. Mathematical partitions assumed for the derivation of Ffowcs Williams-Hawkings acoustic analogy.

surface, \( f < 0 \) is true for the region considered to be inside the surface and \( f > 0 \) is valid in the region exterior to the surface. Herewith, \( \delta(f) \) is the one-dimensional delta function, which is zero everywhere except where \( f = 0 \), which means, on the surface. Equations (2.18) and (2.19) are the equations governing the unbounded fluid, and are valid throughout space, being their variables defined in the following way

\[
\tilde{\rho} = \begin{cases} \rho & f > 0 \\ \rho_0 & f < 0 \end{cases} \tag{2.20}
\]

\[
\tilde{\rho}_i u_i = \begin{cases} \rho u_i & f > 0 \\ 0 & f < 0 \end{cases} \tag{2.21}
\]

and

\[
\tilde{P}_{ij} = \begin{cases} (p - p_0)\delta_{ij} & f > 0 \\ 0 & f < 0 \end{cases} \tag{2.22}
\]

If there are no discontinuities present in the fluid, the mass and momentum sources in (2.18) and (2.19) vanish, leaving the usual conservation equations.

In order to obtain the inhomogeneous wave equation governing the generation and propagation of sound, mass and momentum equations are differentiated with respect to \( t \) and \( x_i \) respectively and subsequently \( \tilde{\rho}_i u_i \) is eliminated from the set of equations. Finally, by changing the dependent variable to be the density perturbation, Ffowcs Williams and Hawkings (FW-H) formulation reads

\[
\frac{\partial^2 \rho'}{\partial t^2} - c^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \underbrace{\frac{\partial^2 \tilde{T}_{ij}}{\partial x_i \partial x_j}}_{\text{Quadrupole}} - \underbrace{\frac{\partial}{\partial x_i} \left( P_{ij}(f) \frac{\partial f}{\partial x_j} \right)}_{\text{Dipole}} + \underbrace{\frac{\partial}{\partial t} \left( \rho_0 v_i \delta(f) \frac{\partial f}{\partial x_i} \right)}_{\text{Monopole}} \tag{2.23}
\]
In (2.23), $P_{ij}$ represents the difference of the stress tensor from its mean value $p_0\delta_{ij}$, and the complete term containing this tensor yields an integral over the surface of the obstacle.

The generalized tensor $\bar{T}_{ij}$ is equal to the Lighthill’s stress tensor from (2.6) outside any surfaces, and is zero within them. In most practical applications, the last two terms will have a negligible effect in the evaluation of the tensor (see Section 2.1.1).

The integral representation from FW-H formulation, solving for the density perturbation $\rho'$, is expressed as \[37\]

$$
\rho' = \frac{1}{4\pi c_0^2} \frac{\partial^2}{\partial x_i \partial x_j} \int_{\Omega} \frac{T_{ij}(y, t - \frac{|x-y|}{c_0})}{|x-y|} dy - \frac{1}{4\pi c_0^2} \frac{\partial}{\partial x_i} \int_{\Gamma} \frac{P_{ij} n_j(y, t - \frac{|x-y|}{c_0})}{|x-y|} dy
$$

$$
+ \frac{1}{4\pi c_0^2} \frac{\partial}{\partial t} \int_{\Gamma} \frac{\rho_0 v_n(y, t - \frac{|x-y|}{c_0})}{|x-y|} dy,
$$

(2.24)

where the volume integral is taken over the region exterior to the surfaces and the integral arguments are evaluated at retarded time \[37\]. In the case of fixed surfaces the third term on the right hand side of the equation, containing the velocity normal to the solid surfaces $v_n$ vanishes reducing the equation to Curle’s equation [21].

The right hand side of (2.24) represents that the sound can be regarded as generated by three source distributions. The first of these is a distribution of acoustic quadrupoles of strength density $T_{ij}$ distributed throughout the region exterior to the solid surfaces (corresponding to Lighthill’s right-hand term) \[63\]. This is supplemented by surface distributions of acoustic dipoles of strength density given by $P_{ij} n_j$ \[21, 14\], and if the surfaces are moving, by further surface distributions of sources essentially monopole in character representing a volume displacement effect.

Similarly as in Lighthill’s integral formulation, for the evaluation of the radiated sound, at points far enough from the flow to be in the radiation field of each quadrupole, it is possible to replace spatial derivatives in (2.24) by time derivatives to write the integral representation of the solution as

$$
\rho' = \frac{1}{4\pi c_0^2} \frac{\partial^2}{\partial x_i \partial x_j} \int_{\Omega} \frac{(x_i - y_i)(x_j - y_j) T_{ij}(y, t - \frac{|x-y|}{c_0})}{|x-y|^3} dy
$$

$$
+ \frac{1}{4\pi c_0^2} \frac{\partial}{\partial t} \int_{\Gamma} \frac{(x_i - y_i) P_{ij} n_j(y, t - \frac{|x-y|}{c_0})}{|x-y|^2} d\Gamma
$$

$$
+ \frac{1}{4\pi c_0^2} \frac{\partial}{\partial t} \int_{\Gamma} \frac{\rho_0 v_n(y, t - \frac{|x-y|}{c_0})}{|x-y|} dy.
$$

(2.25)

As emphasized by Ffowcs Williams and Hawkings in their original derivation, although the quadrupole, dipole and monopole source terms in (2.24) may not be the physical origin of
the sound, they do completely specify the field. Since these terms are explicit, it is possible to compute them separately and to neglect those which do not give a significant contribution to the source field. In general, when following integral approaches in cases involving low Mach numbers, the quadrupole term may be neglected. This is due to the fact that if the quadrupole and dipole distributions have similar spatial and temporal scales, the quadrupole distribution is smaller than that of the dipole distribution by the factor $L/(1 - M_r T c_0)$, where $M_r$ represents the Mach number at which the source approaches the field point, $L$ corresponds to a characteristic source dimension and $T$ to the characteristic time for the source fluctuations \[46\].

### 2.1.3. Surface integral formulations

In surface integral formulations it is assumed that a control surface conveniently placed in the non-linear flow field can enclose all significant acoustic sources responsible for the noise generation. Based on this, it is then possible to obtain a far field solution of the acoustic pressure in terms of an integral over this control surface. Two surface integral formulations which are commonly found in CAA literature and which have been investigated in detail \[14, 22, 35, 72\] are the Porous FW-H method and the Kirchhoff method.

Usually, in the FW-H formulation described in Sec. 2.1.2, the control surface $f = 0$ is chosen to coincide with the solid body and is impenetrable. Since the original formulation is not restricted by this latter condition \[37\], the control surface can be assumed to be porous. In this way, the integration surface does not need to coincide with the solid surface and the pressure field can be computed based on source terms evaluated from the fluid field data on this control surface. In this general case, the formulation is termed as Porous or Permeable FW-H and can be written in its differential form, assuming $p' = c_0^2 \rho'$ outside the flow region, as \[14\]

$$
\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = \frac{\partial}{\partial x_i} \{ T_{ij} H(f) \} - \frac{\partial}{\partial x_i} \{ [P_{ij} n_j + \rho u_i (u_n - v_n)] \delta(f) \} + \frac{\partial}{\partial t} \{ [\rho v_n + \rho (u_n - v_n)] \delta(f) \} \equiv Q_{FWH\text{ permeable}}, \tag{2.26}
$$

where the Heaviside function in the quadrupole term delimits the evaluation of this volume term to the region outside the permeable surface and $v_n = -\partial f/\partial t$ is the normal velocity of the moving body. This general representation of FW-H equation, in contrast to (2.23) which is limited to $u_n = v_n$, allows a more direct comparison with the governing equation of Kirchhoff formula for moving surfaces to be presented below. In the FW-H formulation, it is known that the quadrupole term is responsible for noise generation as well as distortion of the acoustic waveform \[69, 14\]. On the other hand, it has been shown by Di Francescantonio \[22\] that, if the integration surface is placed away from the solid surface, the turbulent structures inside this surface which contribute to the far field noise are accounted for by their mathematical projection as a distribution of source terms on the permeable surface. Based
on this, if the integration surface encompasses most of the strong quadrupole sources, the weaker quadrupoles sources can be neglected, which would simply provide a small distortion of the acoustic waveform. It is under this assumption, which eliminates the volume term in (2.26), that the porous FW-H formulation can be categorised as a surface integral method.

The FW-H equation is based on the conservation laws of fluid mechanics rather than on the wave equation. On the other hand, the classical Kirchhoff formula describes the solution of the wave equation (commonly used for both electrodynamics as well as acoustics) in the exterior of a surface \( S \) as a surface integral over \( S \). Farassat and Myers [36, 78] extended Kirchhoff’s original equation solving for the acoustic pressure \( p' \) to account for moving surfaces as\(^1\)

\[
\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = - \left( \frac{\partial p'}{\partial t} \frac{M_n}{c_0} + \frac{\partial p'}{\partial n} \right) \delta(f) \\
- \frac{\partial}{\partial t} \left[ p' \frac{M_n}{c_0} \delta(f) \right] - \frac{\partial}{\partial x_i} \left[ p' n_i \delta(f) \right] \equiv Q_{\text{KIRCH}} , \tag{2.27}
\]

where \( M_n = v_n/c_0 \) and the surface \( f = 0 \) is defined such that the acoustic sources are contained inside the surface. The Kirchhoff formulation does not require volume integration because it contain only surface terms. Unlike the FW-H source terms, the Kirchhoff source terms are not explicitly related to thickness, loading, nonlinear effects, nor any physical mechanisms [14]. The main disadvantage of the Kirchhoff method is that the Kirchhoff surface must be chosen to be in the linear flow region (the surface \( f = 0 \) must contain all acoustic sources), such that the input acoustic pressure \( p' \) and its derivatives \( \partial p'/\partial t \) and \( \partial p'/\partial n \) are compatible with the wave equation (i.e., solution of the wave equation on \( f = 0 \)). After rigorous mathematical manipulation, Brentner and Farassat [14] presented a detailed comparison of the source terms in both FW-H and Kirchhoff formulations yielding

\[
Q_{\text{FWH permeable}} = Q_{\text{KIRCH}} + \frac{\partial^2 \rho u_i u_j}{\partial x_i \partial x_j} H(f) . \tag{2.28}
\]

Therefore, the only contribution in FW-H sources, not contained in the Kirchhoff formulation, is a second order term involving the velocity components \( u_i \). This demonstrates that FW-H and Kirchhoff formulations are equivalent if the integration surface is placed in the linear region, where the changes of \( u_i \) are small and this term can be neglected. Additionally, this is the reason why the Kirchhoff formula is more sensitive to the choice of the control surface, which can lead to large errors if the surface is positioned in the non-linear region [35].

### 2.1.4. Variational formulation of Lighthill’s acoustic analogy

As presented in previous sections, several aeroacoustic methodologies follow numerical approaches which use wave equation solutions formulated as volume or surface integrals. Such

\(^1(2.27)\) was also derived and compared with the FW-H formulation by Ffowcs Williams and Hawkings in their original work [37].
approaches are efficient for computing the far field acoustic solution at specific observer locations, requiring explicit surface source terms to take into account surface distributions of dipole or monopole sources, which arise from the forces and motions imparted to the surface by the unsteady flow. Another alternative to compute the aerodynamically generated sound following Lighthill’s acoustic analogy, consists in solving the variational formulation of the inhomogeneous wave equation using an interior method as the Finite Element method. In such a way, the acoustic solution is obtained at all discrete points inside the numerical domain. Furthermore, interior numerical methods are implicit in the sense that the interactions between the solid surfaces and the aerodynamic noise are taken into account by the acoustic solver without requiring explicit surface source terms as in integral formulations.

The starting point for the derivation of the variational formulation consists in applying the weighted residuals method to Lighthill’s inhomogeneous wave equation by multiplying with a test function \( w \) and integrating over the whole acoustic domain which yields

\[
\int_{\Omega} \frac{\partial^2 \rho'}{\partial t^2} w \, d\Omega - c_0^2 \int_{\Omega} \frac{\partial^2 \rho'}{\partial x_i^2} w \, d\Omega = \int_{\Omega} \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} w \, d\Omega.
\]  

(2.29)

After applying Green’s integration by parts theorem to (2.29) to the second spatial derivative of \( \rho' \) as well as to the acoustic source term containing \( T_{ij} \), the equation can be rearranged in the variational or weak formulation of Lighthill’s acoustic analogy as

\[
\int_{\Omega} w \frac{\partial^2 \rho'}{\partial t^2} d\Omega + \int_{\Omega} c_0^2 \frac{\partial w}{\partial x_i} \frac{\partial \rho'}{\partial x_i} d\Omega - \int_{\Gamma} c_0^2 w \frac{\partial \rho'}{\partial n} d\Gamma = - \int_{\Omega} \frac{\partial w}{\partial x_i} \frac{\partial T_{ij}}{\partial x_j} d\Omega.
\]  

(2.30)

Equation (2.30) represents the natural basis for the Finite Element implementation of Lighthill’s acoustic analogy as treated in this work and its derivation will be presented in detail in Chap. 3. This equation can be used to derive a semidiscrete Galerkin formulation from which corresponding transient and harmonic formulations are obtained.

2.1.5. Comparison of integral and variational formulations of the acoustic analogy

Integral formulations, such as those based on Curle’s or FW-H equations employ a specific free-space Green’s function for Lighthill’s equation to develop semi-analytical formulae for taking into account the effects of solid boundaries [80]. This results in additional source terms, explicitly describing each type of source distribution (dipole and monopole terms), besides the original volume (quadrupole) term. Reflection effects of the aerodynamic sound by the solid surfaces must be taken into account by these explicit surface integrals. On the other hand, the weak formulation of Lighthill’s acoustic analogy does not separate the different contributions. In contrast, it is implicit and its practical use requires a discretization of the whole acoustic domain (e.g. Finite Element method). Additionally, since the only source term is the distribution of volume quadrupoles, it does not utilize the pressure obtained from the fluid computation. Therefore, the interactions between the solid surfaces and the induced noise are taken into account by the acoustic solver rather than by surface terms.
involving the hydrodynamic pressure. For non-compact solid boundaries or for inner/ducted aeroacoustic problems, this can be a crucial advantage over the integral formulations where source terms are usually obtained from CFD simulations which do not represent the acoustic pressure fluctuations accurately enough to handle the various reflections [16]. Hence, integral formulations are preferable as long as the sources can be considered to be compact and only an extension of the acoustic solution at a few points in the far field is desired. In such cases, these methods are computationally less expensive than volume discretization methods, where a whole discretization of the acoustic domain is required. In contrast, for confined aeroacoustic problems, or if structural/acoustic effects are considered, it is more appropriate to use an acoustic interior method to account for the interactions between the solid surfaces and the flow-induced noise directly in the acoustic simulation. In such cases, integral formulations would require a priori knowledge of a hard-wall Green’s function that is not known for complex geometries [80]. Furthermore, integral methods do not allow for a straightforward inclusion of the elastic effects of structures in the flow. Additionally, a significant advantage in volume discretization methods is that they can also be extended to include convection effects of wave propagation in non-uniform background flows.

2.2. Approaches based on perturbation quantities

In the last years several approaches have been proposed which use perturbation quantities as source quantities for linearized Euler equations [73, 53, 12, 97] or set of equations alike [34, 93]. The main advantage which makes it appealing to compute the noise propagation using these approaches, is the possibility to consider noise propagation effects in non-uniform flows outside the turbulent region. Additionally, since refraction effects are considered by these numerical methods, the effects due to the presence of complex geometries in the flow are implicitly taken into account in the acoustic solution. On the other hand, a major disadvantage in methods based on perturbation quantities, is that in general they need to solve for five perturbation variables (velocity, pressure and density) which can result in higher computational costs.

2.2.1. LEE based methods

The computation of flow noise using source terms in linearized Euler equations has been initially treated comprehensively by Bogey et al [12]. They use linearized Euler equations (LEE) forced with aerodynamic sources to compute the acoustic far field. These Euler equations linearized around a stationary mean flow are written as [6, 94]

\[
\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} + H = S
\]  

(2.31)

being \( U = [\rho', \rho_0u'_1, \rho_0u'_2, p']^T \) the unknown vector, where the prime superscript denotes the perturbation quantities, and \( E \) and \( F \) the two-dimensional flux vectors. The vector \( H \) contains terms related to the gradients of the mean flow, which are equal to zero when the mean flow is uniform [94]. Vectors \( E, F \) and \( H \) are given respectively, in the two-dimensional form, by
2.2. APPROACHES BASED ON PERTURBATION QUANTITIES

\[ E = \begin{pmatrix} \rho'\bar{u}_1 + \bar{p}' u_1' \\ \bar{u}_1 \bar{p}' u_1' + p' \\ \bar{u}_1 \bar{p}' u_2' \\ \bar{u}_1 p' + \gamma \bar{p}' u_1' \end{pmatrix}, \quad F = \begin{pmatrix} \rho'\bar{u}_2 + \bar{p}' u_2' \\ \bar{u}_2 \bar{p}' u_1' \\ \bar{u}_2 \bar{p}' u_2' + p' \\ \bar{u}_2 p' + \gamma \bar{p}' u_2' \end{pmatrix}, \]

\[ H = \begin{pmatrix} 0 \\ \langle \bar{p} u_1' + \rho' \bar{u}_1 \rangle \frac{\partial}{\partial x_1} + \langle \bar{p} u_2' + \rho' \bar{u}_2 \rangle \frac{\partial}{\partial x_2} \\ \langle \bar{p} u_1' + \rho' \bar{u}_1 \rangle \frac{\partial}{\partial x_1} + \langle \bar{p} u_2' + \rho' \bar{u}_2 \rangle \frac{\partial}{\partial x_2} \\ (\gamma - 1) p' \nabla \cdot \bar{u} - (\gamma - 1) u' \cdot \nabla \bar{p} \end{pmatrix}, \]

where \( \gamma \) represents the adiabatic index, and \( \bar{u}, \bar{p}, \bar{\rho} \) are mean quantities.

The vector \( S \) represents possible unsteady sources in the flow, which is found by analogy with Lilley’s equation [65, 66] and is given in terms of the divergence of the velocity fluctuations, reading

\[ S = \begin{pmatrix} 0 \\ S_1 - \bar{S}_1 \\ S_2 - \bar{S}_2 \\ 0 \end{pmatrix} \]

where

\[ S_i = -\frac{\partial \rho u_i' u_j'}{\partial x_j}, \quad \bar{S}_i = -\frac{\partial \rho \bar{u}_i' u_j'}{\partial x_j}, \quad (2.32) \]

being \( u_i' = u_i - \bar{u}_i \) the velocity fluctuations. Linearized Euler equations support both acoustic disturbances and instability waves, which are not decoupled in a sheared mean flow. As a result, physical growing instability waves can be excited by the source terms from (2.32) when using this hybrid approach. Therefore, a simplified formulation of LEE is used for computing the noise propagation. This simplification consists in cancelling in the LEE system the vector \( H \) containing mean shear terms. In [12], an investigation is presented to demonstrate that no significant effects are induced in the actual noise propagation by cancelling this term.

2.2.2. Acoustic Perturbation Equations

The family of Acoustic Perturbation Equations (APE) proposed by Ewert et al. [34] is also a hybrid approach for the simulation of the acoustic fields in space and time from unsteady flow simulations. In the APE systems perturbation quantities are defined to be deviations from time-averaged mean quantities as
Here \( \bar{u} \) denotes the time-averaged mean flow, \( u^v \) is a solenoidal vortical perturbation, and \( u^a \) is an irrotational acoustic perturbation. Using this definition of perturbation quantities, four different APE formulations are presented in which the source terms are filtered to suppress the vortical modes, leaving source terms which excite purely acoustic modes.

In the APE-1 formulation, which is the basis for the other APE variants, the derivation of the system equations starts by rewriting the continuity and Navier-Stokes equations with the enthalpy \( h \) and the velocity \( \mathbf{u} \) as variables. These equations contain both an acoustic and a vorticity eigenmode on the left-hand side such that also vorticity waves are excited by the right-hand side. From the combined acoustic and vorticity sources in the equations, a filtered source vector is obtained which will excite either acoustical or vortical modes. By means of this filtering procedure, it is then possible to write a system of equations which generate only responses of the acoustic eigenmodes. After rewriting the system of equations using the perturbation pressure \( p' \) as variable, the complete system of acoustic perturbation equations (APE-1) for the perturbation pressure and acoustic perturbation velocity \( (p', u^a) \) reads [34]

\[
\frac{\partial p'}{\partial t} + \bar{c}^2 \nabla \cdot \left( \bar{\rho} u^a + \bar{\mathbf{u}} \frac{p'}{\bar{c}^2} \right) = \bar{c}^2 q_c \tag{2.34}
\]

\[
\frac{\partial u^a}{\partial t} + \nabla (\bar{\mathbf{u}} \cdot u^a) + \nabla \left( \frac{p'}{\bar{\rho}} \right) = q_m \tag{2.35}
\]

with sources

\[
q_c = - \nabla \rho \cdot u^v + \frac{\bar{\rho}}{c_p} \frac{\bar{D} s'}{\bar{D} t} \tag{2.36}
\]

\[
q_m = \nabla \Phi_p + \nabla q_\rho + \frac{T'}{\nabla T} \nabla s - s' \nabla T \tag{2.37}
\]

where \( \bar{D}/\bar{D} t = \partial/\partial t + \bar{\mathbf{u}} \cdot \nabla \) represents the substantial time derivative and a bar over a quantity denotes its mean value. Additionally, \( \gamma \) represents the adiabatic index, \( c_p \) the specific heat capacity, \( s' \) the entropy fluctuations and \( T \) the temperature. This fundamental system describes wave propagation in a non-uniform mean flow field \( \bar{\mathbf{u}} \). Since in the derivation the non-linear terms containing acoustic perturbations are dropped, non-linear acoustic propagation and sound generation effects due to non-linear mode interaction are not considered by the system. In (2.36) and (2.37), terms I and III are the major source terms for turbulent induced or vortex sound. If combustion noise is considered, terms II and V are important since they involve entropy and temperature fluctuations, however, they might be negligible for vortex sound. The remaining source IV describes sound generation due to acoustic/mean-vorticity interaction, and is of relevance in wave propagation in mean
flows with significant vorticity. From the remaining APE formulations, APE-2 and APE-4 are of interest for practical CAA computations using incompressible and compressible CFD computations respectively.

For the derivation of the APE-2 formulation, the perturbation pressure is decomposed as $p' = \bar{\rho} \Phi + p^a$, under the assumption of an incompressible flow. This decomposition means that the hydrodynamic pressure $P' \approx \bar{\rho} \Phi$ can be excluded from the pressure fluctuations $p'$ and that $p^a$ can be regarded as an acoustic perturbation pressure with pseudosound excluded. Using this decomposition, the system (2.34) and (2.35) with sources (2.36) and (2.37) is rearranged to allow the computation of the acoustic sources from an incompressible flow simulation as

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho' \bar{u} + \bar{\rho} u^a) = -\nabla \bar{\rho} \cdot \bar{u}$$

$$\frac{\partial u^a}{\partial t} + \nabla (\bar{u} \cdot u^a) + \nabla \left( \frac{p^a}{\rho} \right) = \nabla \bar{q}_\omega + T' \nabla \bar{s} - s' \nabla \bar{T}$$

$$\frac{\partial p^a}{\partial t} - \bar{c}_2 \frac{\partial \rho'}{\partial t} = -\bar{\rho} \frac{\partial \Phi}{\partial t} \underbrace{\text{IIb}}_{\text{IIIb}} + \frac{\gamma}{} \bar{c}_p \frac{\partial s'}{\partial t} \underbrace{\text{IIb}}_{\text{IIb}}.$$ (2.40)

Considering only vortex sound at small Mach numbers, in this system of equations all source terms but IIIb disappear and this remaining term can be approximated as the time derivative of the perturbation pressure of the incompressible flow simulation, $\partial P'/\partial t$.

For computing the acoustic field from compressible fluid simulations, the fourth formulation (APE-4) is more appropriate since in this case there is no need to solve a Poisson equation. The source term for this formulation is based on the Lamb vector $L = \omega \times u$. Furthermore, this latter formulation is not restricted to low Mach number problems.

2.2.3. Perturbed Compressible Equations

The perturbed compressible equations (PCE) proposed by Moon [93] are based on investigation of the perturbed vorticity transport equations. In PCE, the perturbed Euler equations are modified by introducing the perturbed viscous stresses in the momentum equations and the perturbed energy equation is formally derived from the compressible thermal energy equation. By handling the generation and diffusion of perturbed vorticity properly, they attempt to correct the false generation of perturbed vorticity common in perturbed Euler equations. For the PCE, if an inappropriate acoustic grid resolution is used, the perturbed vorticity can become unstable. In such cases the length scale of the perturbed vorticity is not comparable to that of acoustic waves but closer to the hydrodynamic vortical scale. Therefore, to eliminate the dependency of the solution on the acoustic grids, the authors propose the linearized perturbed compressible equations (LPCE). In LPCE, which are a modified version of the original PCE, the generation of perturbed vorticity is completely suppressed by dropping the non-linear coupling terms that contribute to generate vortical components in the perturbed system. The resulting LPCE system is given as [93]
\[
\frac{\partial \rho'}{\partial t} + (\bar{u} \cdot \nabla) \rho' + \bar{\rho} (\nabla \cdot \bar{u}') = 0 \quad (2.41)
\]
\[
\frac{\partial \bar{u}'}{\partial t} + \nabla (u' \cdot \bar{u}) + \frac{1}{\bar{\rho}} \nabla p' = 0 \quad (2.42)
\]
\[
\frac{\partial p'}{\partial t} + (\bar{u} \cdot \nabla) p' + \gamma P (\nabla \cdot \bar{u}') + (u' \cdot \nabla) P = -\frac{DP}{Dt}. \quad (2.43)
\]

In this set of linearized perturbed compressible equations, similarly to the APE-2 formulation, the only explicit noise source term is given in terms of the hydrodynamic pressure, as \(DP/Dt\), although, in this case the substantial or total derivative of \(P\) is required.

### 2.3. Heterogeneous domain decomposition for aeroacoustics

A methodology, in which fluid and acoustic phenomena can be solved in a direct simulation has been proposed in [97]. The proposed technique makes use of a non-overlapping domain decomposition method, where equations, numerical methods, grids and time steps are then adapted to meet the local requirements in each of the resulting subdomains. In this adaptive way it is possible to overcome the energy and scale disparities between fluid and acoustic fields. In contrast to the standard hybrid approaches, here the influence of the acoustic waves on the flow field is taken into account.

By using different equations in each subdomain, it is then possible to include all nonlinear as well as viscous effects just in the noise generation region. Farther away, the viscous parts may already be neglected and a simpler system of nonlinear Euler equations can be used. In the far field, in general nonlinear effects become less important, therefore the linearized Euler equations can be used. In this methodology, the subdomains are coupled at their common boundary only by interpolating or averaging the data in the coupling ghostcells. These ghosts cells are then used by the corresponding numerical method to update the inner cells in each subdomain. Additionally, the data between the subdomains is exchanged at the common timelevel. The local CFL number in each subdomain is adjusted in such a way, that it is assured, that neighboring domains have the same time level if the domain with the larger timestep is updated once. The CFL number is adapted for each subdomain automatically at each subcycle [97].

Another important characteristic of this approach consists in using high order schemes for all subdomains. The ADER (Arbitrary Derivatives) schemes developed by Toro et al. [96] are adopted and optimized for linear and nonlinear equations. Such approach has been presented in [91] for solving linear two-dimensional hyperbolic problems using the finite volume (FV) method on cartesian grids. The same concept was applied also in two dimensions in the framework of a Discontinuous Galerkin (DG) method implementation, to allow the use of unstructured meshes. More recently, extensions of both ADER-FV and ADER-DG implementations for solving linear and non-linear hyperbolic problems in three dimensions
using unstructured meshes have been presented by Käser [61] and Dumbser [24]. The combination of the extended ADER-FV and ADER-DG schemes can provide more flexibility for an efficient simulation of complex three-dimensional problems arising in CAA.
2. CAA METHODOLOGIES
3. Finite Element formulation of Lighthill’s acoustic analogy

In this chapter the finite element analysis for solving the partial differential equation derived by Lighthill as introduced in Sec. 2.1.1 will be presented. Firstly, the semidiscrete Galerkin formulation is derived and the corresponding transient and harmonic formulations of the numerical implementation are obtained thereof. For the transient formulation the discretization in time using the Newmark method is presented. On the other hand, concerning the harmonic formulation we obtain the corresponding complex algebraic system of equations after Fourier transformation of the semidiscrete Galerkin formulation. Besides these aspects, in the end of the chapter, the procedure for the numerical evaluation of the acoustic source term following the Finite Element method is introduced.

3.1. Strong Formulation of the Inhomogeneous Wave Equation

As presented in Chap. 2, for the simulation of the wave propagation in a hybrid approach using an interior method we consider an acoustic computational domain as displayed in Fig. 3.1.
Therein, \( \Omega_1 \) denotes the computational domain, where we have to solve the inhomogeneous wave equation using the acoustic sources obtained from the hydrodynamic velocity field, and \( \Omega_2 \) the domain outside the turbulent region where the acoustic sources become zero, and where we are interested in the radiated sound. The surface \( \Gamma_S \) defines the interaction between a solid/elastic body and the flow and \( \Gamma_I \) the boundary of the total computational domain.

We will perform a volume discretization of Lighthill’s equation \([63]\) by applying the finite element method (FEM). Therewith, the interactions of any solid/elastic body with the turbulent flow field will be implicitly taken into account by the acoustic/mechanical formulation. The governing PDE for our aeroacoustic problem is given by the original Lighthill’s inhomogeneous wave equation introduced in Chapter 2, as

\[
\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}, \tag{2.7}
\]

with \( \rho' \) representing the acoustic density fluctuation and \( T_{ij} \) the components of the Lighthill’s tensor \([T]\) approximated as \( T_{ij} \approx \rho u_i v_j \).

The initial/boundary-value problem for this equation may be stated as

Given:

\[
\begin{align*}
\rho_0' & : \Omega \to \mathbb{R} \quad (3.1) \\
\dot{\rho}_0' & : \Omega \to \mathbb{R} \quad (3.2) \\
c_0 & : \Omega \to \mathbb{R} \quad (3.3) \\
h & : \Gamma \times (0, T) \to \mathbb{R} \quad (3.4)
\end{align*}
\]

we need to find:

\[
\rho' : \overline{\Omega} \times (0, T) \to \mathbb{R}, \quad (3.5)
\]

such that:

\[
\begin{align*}
\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \Delta \rho' &= \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \quad \text{on } \Omega \times (0, T) \quad (3.6) \\
c_0^2 \frac{\partial \rho'}{\partial n_i} &= h \quad \text{on } \Gamma \times (0, T) \quad (3.7) \\
\rho'(\vec{x}, 0) &= \rho_0' \quad \vec{x} \in \Omega \quad (3.8) \\
\frac{\partial \rho'}{\partial t} (\vec{x}, 0) &= \dot{\rho}_0' \quad \vec{x} \in \Omega, \quad (3.9)
\end{align*}
\]

### 3.2. Weak Formulation

The weak formulation of the initial/boundary-value problem given by equations (3.6)-(3.9) can be derived applying the method of weighted residuals and by making use of a usual space of functions, \( \vartheta \), defined as
\[ \vartheta = \{ \rho'(\cdot, t) \mid \rho'(\vec{x}, t) \in H^1, \vec{x} \in \Omega \}, \quad (3.10) \]

where \(H^1\) denotes the Sobolev space defined as (see e.g. [1])
\[ H^1 = \{ u \in L^2 \mid \partial u/\partial x_i \in L^2 \} \quad (3.11) \]
and \(L^2\) the space of square integrable functions.

After multiplication by an appropriate test function \(w\) from the space of functions given in (3.10) and integration over the whole domain \(\Omega\), the problem can be formulated in the integral form as
\[
\int_{\Omega} \frac{\partial^2 \rho'}{\partial t^2} w \, d\Omega - c_0^2 \int_{\Omega} \frac{\partial^2 \rho'}{\partial x_i^2} w \, d\Omega = \int_{\Omega} \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} w \, d\Omega
\]
\[
\int_{\Omega} \rho'(\vec{x}, 0) w \, d\Omega = \int_{\Omega} \rho_0 w \, d\Omega
\]
\[
\int_{\Omega} \frac{\partial \rho'}{\partial t}(\vec{x}, 0) w \, d\Omega = \int_{\Omega} \rho'_0 w \, d\Omega \quad \forall \ w \in H^1. \quad (3.12)
\]

Now, we apply Green's integral theorem in the first equation in (3.12) to the second spatial derivative of \(\rho'\) as well as to the acoustic source term containing \(T_{ij}\). This operation results in the following relations
\[
\int_{\Omega} c_0^2 w \frac{\partial^2 \rho'}{\partial x_i^2} \, d\Omega = \int_{\Gamma_S \cup \Gamma_I} c_0^2 \frac{\partial \rho'}{\partial n} \, d\Gamma - \int_{\Omega} c_0^2 \frac{\partial w}{\partial x_i} \frac{\partial \rho'}{\partial x_i} \, d\Omega \quad (3.13)
\]
\[
\int_{\Omega} w \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \, d\Omega = \int_{\Gamma_S} w \frac{\partial T_{ij}}{\partial x_j} n_i \, d\Gamma - \int_{\Omega} \frac{\partial w}{\partial x_i} \frac{\partial T_{ij}}{\partial x_j} \, d\Omega. \quad (3.14)
\]

Herewith, it is important to emphasise, that the boundary integral in (3.14) is just over the surface \(\Gamma_S\) of any solid/elastic body since we assume \(T_{ij} = 0\) on the limits of the computational domain \(\Gamma_I\), whereas in (3.13) we have to integrate over \(\Gamma_S\) as well as over \(\Gamma_I\). By using the momentum conservation law and assuming a solid body, e.g. \(v_i n_i = 0\) where \(v\) denotes the velocity of the body, we can express the surface integral in (3.14) by
\[
\int_{\Gamma_S} w \frac{\partial T_{ij}}{\partial x_j} n_i \, d\Gamma = - \int_{\Gamma_S} c_0^2 w \frac{\partial \rho'}{\partial n} \, d\Gamma. \quad (3.15)
\]

Therewith, we can combine the two surface integrals to a single one just performed over the outer boundary \(\Gamma_I\). This remaining surface integral corresponds to the natural boundary condition associated with the weak formulation and will be employed later for applying absorbing boundary conditions. Finally, we arrive at the weak form of (2.7): Find \(\rho' \in H^1\) such that
\[
\int_{\Omega} w \frac{\partial^2 \rho'}{\partial t^2} \, d\Omega + \int_{\Omega} c_0^2 \frac{\partial w}{\partial x_i} \frac{\partial \rho'}{\partial x_i} \, d\Omega - \int_{\Gamma_I} c_0^2 w \frac{\partial \rho'}{\partial n} \, d\Gamma = - \int_{\Omega} \frac{\partial w}{\partial x_i} \frac{\partial T_{ij}}{\partial x_j} \, d\Omega \quad (3.16)
\]
is fulfilled for all \(w \in H^1\).
3.3. Spatial Discretization

The semidiscrete Galerkin formulation is obtained from the weak formulation of the equation after discretization of the domain and introduction of finite element spaces. The entire domain, \( \Omega = \Omega_1 \cup \Omega_2 \), is discretized by an approximate decomposition \( \sum_h \) of \( \Omega \) into \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) finite elements. From our previously defined space \( \vartheta \), we choose a finite dimensional space \( \vartheta_h \subset \vartheta \). We can now write a variational equation of the form of (3.16) in terms of \( \rho'_h \in \vartheta_h \) (using the nabla differential operator for better readability) as

\[
\int_{\Omega_h} \dot{\rho}'_h w_h \, d\Omega + c_0^2 \int_{\Omega_h} \nabla \rho'_h \cdot \nabla w_h \, d\Omega = - \int_{\Omega_h} (\nabla \cdot T_{ij}) \cdot \nabla w_h \, d\Omega
\]
\[
+ c_0^2 \int_{\Gamma_h} w_h \frac{\partial \rho'}{\partial n_i} \, d\Gamma \quad \forall \ w_h \in \vartheta_h.
\]

(3.17)

Using standard nodal finite elements, we can approximate the continuous density perturbation \( \rho' \) as well as the test function \( w \) in terms of the interpolation functions as \(^1\)

\[
\rho'(t) \approx \rho'_h(t) = \sum_{i=1}^{n_{eq}} N_i(\vec{x}) \rho'_i(t)
\]

\[
w \approx w_h = \sum_{i=1}^{n_{eq}} N_i(\vec{x}) c_i
\]

(3.18)

(3.19)

where \( N_i(\vec{x}) \) denote appropriate interpolation functions and \( c_i \) corresponding weights.

From (3.17), we can now write the semidiscrete Galerkin formulation in matrix form as

\[
M \ddot{\rho}'(t) + K \rho'_h(t) = f(t)
\]

(3.20)

where the matrices \( M \) and \( K \) are computed as follows:

\(^1\)The evaluation of the source terms using the same kind of interpolation functions is considered in section 3.6, while at this point they are treated as given continuous values sampled at the finite element nodes.
3.4. Time discretization

In order to discretize our problem in the time domain, a predictor-corrector method is typically implemented (see e.g. [56]). Let us start from the semidiscrete Galerkin formulation in matrix form

$$M \ddot{\rho}'_{n+1} + K \rho'_{n+1} = f_{n+1}$$

(3.24)

with $M$ the mass matrix, $K$ the stiffness matrix, $f$ the right hand side, $\rho'$ the vector of unknowns at the finite element nodes and $\rho''$ its second derivative with respect to time. For hyperbolic partial differential equations the Newmark scheme is generally used. Therewith, we have (see e.g. [56])

$$\rho'_{n+1} = \rho'_{n} + \Delta t \dot{\rho}'_{n} + \frac{\Delta t^2}{2} \left( (1 - 2\beta_H) \rho''_{n} + 2\beta_H \rho''_{n+1} \right)$$

(3.25)

$$\rho''_{n+1} = \dot{\rho}'_{n} + \Delta t \left( (1 - \gamma_m) \rho''_{n} + \gamma_m \rho''_{n+1} \right).$$

(3.26)
In (3.25)-(3.26) \( n \) denotes the time step counter, \( \Delta t \) the time step value and \( \beta, \gamma \) the integration parameters. Substituting \( \rho'_{n+1} \) and \( \dot{\rho}'_{n+1} \) according to (3.25) and (3.26) in (3.24) leads to the following algebraic system of equations

\[
M^*\ddot{\rho}'_{n+1} = f_{n+1} - K\left(\rho'_n + \gamma \Delta t \dot{\rho}'_n + \frac{\Delta t^2}{2} (1 - 2\beta)\dot{\rho}'_n\right) \tag{3.27}
\]

\[
M^* = M + \beta \Delta t^2 K. \tag{3.28}
\]

Writing the solution process for one time step as a predictor-corrector algorithm we arrive at the effective mass as well as effective stiffness formulations.

1. Effective Mass Formulation

   - Perform predictor step:
     
     \[
     \tilde{\rho}' = \rho'_n + \Delta t \dot{\rho}'_n + (1 - 2\beta)\Delta t^2 \dot{\rho}'_n \tag{3.29}
     \]
     
     \[
     \tilde{\dot{\rho}} = \dot{\rho}'_n + \Delta t (1 - \gamma)\ddot{\rho}'_n \tag{3.30}
     \]

   - Solve algebraic system of equations:
     
     \[
     M^*\ddot{\rho}'_{n+1} = f_{n+1} - K\tilde{\rho}' \tag{3.31}
     \]
     
     \[
     M^* = M + \beta \Delta t^2 K \tag{3.32}
     \]

   - Perform corrector step:
     
     \[
     \rho'_{n+1} = \tilde{\rho}' + \beta \Delta t^2 \tilde{\rho}'_{n+1} \tag{3.33}
     \]
     
     \[
     \dot{\rho}'_{n+1} = \tilde{\dot{\rho}} + \gamma \Delta t \ddot{\rho}'_{n+1} \tag{3.34}
     \]

2. Effective Stiffness Formulation

   According to (3.25) and (3.26) we can express \( \ddot{\rho}'_{n+1} \) and \( \dot{\rho}'_{n+1} \) as follows

   \[
   \ddot{\rho}'_{n+1} = \frac{\rho'_{n+1} - \tilde{\rho}'}{\beta \Delta t^2} \tag{3.35}
   \]
   \[
   \dot{\rho}'_{n+1} = \tilde{\dot{\rho}} + \gamma \Delta t \ddot{\rho}'_{n+1} = \dot{\rho}'_n + \frac{\gamma}{\beta \Delta t} (\ddot{\rho}'_{n+1} - \tilde{\rho}'). \tag{3.36}
   \]

   Thereafter, we obtain

   - Perform predictor step:
     
     \[
     \dot{\rho}' = \rho'_n + \Delta t \ddot{\rho}'_n + (1 - 2\beta)\frac{\Delta t^2}{2} \dot{\rho}'_n \tag{3.37}
     \]
     
     \[
     \dot{\dot{\rho}} = \dot{\rho}'_n + \Delta t (1 - \gamma)\dot{\rho}'_n \tag{3.38}
     \]
3.5. Harmonic formulation

In the following, the harmonic representation of the variational formulation of Lighthill’s acoustic analogy is introduced. By performing a harmonic analysis, it is then possible to compute the sound radiation for specific frequency components present in the acoustic sources. In this way, we obtain the complex acoustic pressure at each node in the numerical domain. For deriving the harmonic counterpart of the implementation presented in Sec. 3.4 we apply a Fourier-transformation to the semidiscrete Galerkin formulation from Eq. (3.24), obtaining the following complex algebraic system of equations

\[
(-\omega^2 \mathbf{M} + \mathbf{K}) \mathbf{\hat{\rho}} = \mathbf{\hat{f}},
\]

where the source term \( \mathbf{\hat{f}} \) represents the complex nodal acoustic sources, which are obtained by applying a Fourier transformation to the dataset of transient nodal sources interpolated from the fluid computation. In general, this overhead of transforming the acoustic nodal sources on the acoustic side is small in comparison to the numerical advantages that computing the acoustic field in frequency domain can provide. First of all, in a harmonic computation only the relevant frequency components are computed, avoiding in this way the high frequency numerical noise commonly present in transient results. This improves the quality of the solution of the acoustic far field. Furthermore, as will be presented later in this work, the perfectly matched layer method (PML) can be conveniently included in the harmonic FE implementation as boundary treatment, which significantly reduces reflections at the acoustic boundary. This allows the usage of smaller acoustic domains without the solution being contaminated, representing in the end a significant reduction in computational times.
3. FE FORMULATION OF LIGHTHILL’S ACOUSTIC ANALOGY

3.6. FE evaluation of the acoustic source term

In this section, we present the procedure followed for the numerical evaluation of the acoustic nodal sources using the velocity field values from the fluid computation. This aspect plays an important role in the FE implementation of the hybrid approach following Lighthill’s acoustic analogy. The acoustic sources need to be integrated over each element and the resulting nodal contributions are added to the complete algebraic system of equations as usual in the Finite Element method.

From Eq. (3.17), the term responsible for the noise generation is written as

$$ f(\vec{x}, t) = -\int_{\Omega_h} (\nabla \cdot T_{ij}) \cdot \nabla w_h \, d\Omega. $$

(3.44)

In expression (3.44), the numerical evaluation of the vector resulting from $\nabla \cdot T_{ij}$ has to be performed at each time step. The subindex $ij$ gives the component of the tensor, where $1 \leq i, j \leq n_sd$. Assuming a constant density $\rho$ as it is the case for incompressible Navier-Stokes fluid computations, this vector quantity reads explicitly

$$ \nabla \cdot T_{ij} \approx \nabla \cdot \rho \dot{u}_i \dot{u}_j = \rho \begin{pmatrix} \frac{\partial u_1^2}{\partial x_1} + \frac{\partial u_1 u_2}{\partial x_2} + \frac{\partial u_1 u_3}{\partial x_3} \\ \frac{\partial u_2 u_1}{\partial x_1} + \frac{\partial u_2^2}{\partial x_2} + \frac{\partial u_2 u_3}{\partial x_3} \\ \frac{\partial u_3 u_1}{\partial x_1} + \frac{\partial u_3 u_2}{\partial x_2} + \frac{\partial u_3^2}{\partial x_3} \end{pmatrix} $$

(3.45)

where $u_1$, $u_2$ and $u_3$ are the velocity components of the flow in the $x_1$, $x_2$ and $x_3$ directions respectively. Then, we can evaluate the vector components by applying the chain rule

$$ \frac{\partial u_i u_j}{\partial x_i} = \frac{\partial u_i}{\partial x_i} u_j + \frac{\partial u_j}{\partial x_i} u_i, $$

(3.46)

which for the evaluation of the first vector component results in

$$ \frac{\partial u_1^2}{\partial x_1} = \frac{\partial u_1}{\partial x_1} u_1 + \frac{\partial u_1 u_1}{\partial x_1} = 2u_1 \frac{\partial u_1}{\partial x_1}, $$

$$ \frac{\partial u_1 u_2}{\partial x_2} = u_2 \frac{\partial u_1}{\partial x_2} + u_1 \frac{\partial u_2}{\partial x_2}, $$

$$ \frac{\partial u_1 u_3}{\partial x_3} = u_3 \frac{\partial u_1}{\partial x_3} + u_1 \frac{\partial u_3}{\partial x_3}. $$

After analogous evaluation of the other vector components, the divergence of Lighthill’s tensor results in
\[ \nabla \cdot T_{ij} = \rho \begin{pmatrix} \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_1} + u_1 \frac{\partial u_1}{\partial x_2} + u_3 \frac{\partial u_3}{\partial x_1} + u_1 \frac{\partial u_3}{\partial x_3} \\ \frac{\partial u_1}{\partial x_1} + \frac{2u_2}{\partial x_2} + u_3 \frac{\partial u_2}{\partial x_2} + u_3 \frac{\partial u_3}{\partial x_2} + 2u_3 \frac{\partial u_3}{\partial x_3} \\ \frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_1} + \frac{\partial u_3}{\partial x_2} + \frac{\partial u_3}{\partial x_2} + 2u_3 \frac{\partial u_3}{\partial x_3} \end{pmatrix} \] (3.47)

Now, by using the discrete nodal values \( u_1^{(n)} \) of \( \vec{u} \), the finite element evaluation for the terms of the first vector component is given by

\[
2u_1 \frac{\partial u_1}{\partial x_1} = 2 \left( \sum_{n=1}^{n_{en}} N_n(\vec{x})u_1^{(n)} \right) \left( \sum_{n=1}^{n_{en}} \frac{\partial N_n(\vec{x})}{\partial x_1} u_1^{(n)} \right) \] (3.48)

\[
u_2 \frac{\partial u_1}{\partial x_2} = \left( \sum_{n=1}^{n_{en}} N_n(\vec{x})u_2^{(n)} \right) \left( \sum_{n=1}^{n_{en}} \frac{\partial N_n(\vec{x})}{\partial x_2} u_2^{(n)} \right) \] (3.49)

\[
u_1 \frac{\partial u_2}{\partial x_2} = \left( \sum_{n=1}^{n_{en}} N_n(\vec{x})u_1^{(n)} \right) \left( \sum_{n=1}^{n_{en}} \frac{\partial N_n(\vec{x})}{\partial x_2} u_2^{(n)} \right) \] (3.50)

\[
u_3 \frac{\partial u_1}{\partial x_3} = \left( \sum_{n=1}^{n_{en}} N_n(\vec{x})u_3^{(n)} \right) \left( \sum_{n=1}^{n_{en}} \frac{\partial N_n(\vec{x})}{\partial x_3} u_1^{(n)} \right) \] (3.51)

\[
u_1 \frac{\partial u_3}{\partial x_3} = \left( \sum_{n=1}^{n_{en}} N_n(\vec{x})u_1^{(n)} \right) \left( \sum_{n=1}^{n_{en}} \frac{\partial N_n(\vec{x})}{\partial x_3} u_3^{(n)} \right) \] (3.52)

where \( n_{en} \) corresponds to the number of element nodes. Similar as with expressions (3.48)-(3.52), we can evaluate the other components of the divergence of Lighthill’s tensor at each time step for the three-dimensional case. The resulting vector can now be multiplied by the gradient of the test function following (3.23) and by numerical integration it is then possible to obtain the acoustic source value for the right hand side of equation (3.17) at each finite element node.
3. FE FORMULATION OF Lighthill’s Acoustic Analogy
4. Simulation of Unbounded Domains

4.1. Introduction

As a main issue for the simulation of unbounded domains using volume discretization methods like FEM, remains the boundary treatment which needs to be applied to avoid the reflection of the outgoing waves on the boundary of the computational domain. Besides acoustics, this is a numerical problem common to several fields of application, such as electromagnetics, aerodynamics, geophysics, meteorology and oceanography.

Among the main challenges that are found in the development of absorbing boundary methods are, besides appropriate absorption of the waves, that they have to be stable, accurate, efficient and easy to implement [45]. Hence, since 30 years or so, this issue has been and continues to be a topic of active research. The 1970s and early 1980s produced some low-order local absorbing boundary conditions (ABCs) that became well-known, e.g. the Engquist-Majda ABCs [28] and the Bayliss-Turkel ABCs [7]. In addition, the infinite element method was invented [10], and boundary element methods for the solution of infinite-domain problems became popular. The period between the late 1980s and mid 1990s has been characterized by the emergence of exact nonlocal non-reflecting boundary conditions (NRBCs) [54] and those methods based on the Dirichlet-to-Neumann (DtN) map [42], by the development of new infinite elements especially designed for wave problems [5], and by the invention of the perfectly matched layer (PML) [9]. This latter method being considered by many researchers as a breakthrough, due to its relative simple applicability and excellent absorption for any frequency and any angle of incidence. Reviews on the development of boundary treatment methods for the simulation of unbounded domains are given, for example, in [43, 5].

Currently, parallel to the further development of the PML method, development of ABCs has continued with the purpose of deriving practical local high-order boundary conditions for both transient and harmonic approaches. Despite the many years of research, until today, absorbing boundary conditions beyond second- or third-order have remained impractical from an implementation point of view [43]. Very recently, Hagstrom and Warburton [49] introduced a promising high-order absorbing boundary conditions based on auxiliary variables allowing arbitrary-order in their formulation. In 2006, Givoli et al. [44] presented a Finite Element implementation for this high-order formulation including numerical tests solving simple problems governed by the scalar wave equation. Further generalization, such as extension to polygonal and curved boundaries and automatic choice of absorbing parameters, is still required to make this implementation amenable for its practical use.
In the following, an introduction to the absorbing boundary conditions (ABCs) and to the perfectly matched layer (PML) methods is given, presenting their corresponding formulations used for the Finite Element implementation in the acoustic solver. Additionally, both boundary treatment methods are evaluated by means of numerical examples.

4. SIMULATION OF UNBOUNDED DOMAINS

4.2. Absorbing Boundary Conditions

4.2.1. Derivation of local ABCs

The fundamentals for the derivation of local low-order absorbing boundary conditions were first introduced by Engquist and Majda [19, 28]. They developed theoretical perfectly absorbing boundary conditions for general classes of wave equations. Subsequently, by making use of Padé approximations of the solutions to the wave equation representing plane waves travelling to the left, they derived a hierarchy of highly absorbing local boundary conditions which approximate the theoretical nonlocal initially obtained. The local boundary conditions are more suitable from the computational point of view, since their locality preserves the sparseness of the matrix, whereas nonlocal ABCs require information from nodes over the entire boundary, making their application impractical for the FE method. The hierarchy of highly absorbing local boundary conditions has then to be built satisfying the following two criteria:

1. These boundary conditions are local.
2. The boundary conditions lead to a well-posed boundary value problem for the wave equation.

- **Theoretical Nonlocal Boundary Condition**

Let us consider the second order wave equation (for simplicity the speed of sound $c_0$ is set to 1)

\[
\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x_1^2} - \frac{\partial^2 p}{\partial x_2^2} = 0
\]

in the half-space $x_1 \geq 0$. We denote the dual variables to $(x_2, t)$ by $(\omega, \xi)$. Now, we can give the special families of solutions to the wave equation representing waves travelling to the left by the plane waves [28]

\[
p = e^{i(\sqrt{\xi^2 - \omega^2}x_1 + \xi t + \omega x_2)}
\]

with $\xi^2 - \omega^2 > 0$, $\xi > 0$. If $(\omega, \xi)$ is held fixed, one first order boundary condition which annihilates $p$ has the form

\[
\left(\frac{d}{dx_1} - i\sqrt{\xi^2 - \omega^2}\right) p|_{x_1=0} = 0.
\]
The expression above means that for all waves as in (4.2) with \((\omega, \xi)\) held fixed, the boundary condition in (4.3) produces no reflections. This is the basic starting point for the derivation of nonreflecting boundary conditions. A complete procedure for the obtention of the \textit{perfectly absorbing boundary conditions} for general wave packets travelling to the left, can be found in [28]. Results of this procedure gives a boundary condition of the form

\[
\left( \frac{d}{dx_1} - \eta \left( \frac{\partial}{\partial x_2}, \frac{\partial}{\partial t} \right) \sqrt{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_2^2}} \right) p|_{x_1=0} = 0, \tag{4.4}
\]

where \(\eta\) is a factor which can be adjusted to \textit{exactly} annihilate the reflected waves. However, as previously mentioned, it is not possible in general to construct boundary conditions in such a way that all impinging waves are absorbed, instead it is possible to obtain conditions for which only smooth low amplitude reflections still remain.

- \textit{Highly Absorbing Local Boundary Conditions}

Using the same initial expression contained in (4.3) from which (4.4) was derived, given as

\[
\frac{d}{dx_1} - i\xi \sqrt{1 - \omega^2/\xi^2}, \tag{4.5}
\]

we can obtain a hierarchy of boundary conditions, depending on the order of approximation used for the square root term at normal incidence \((\omega = 0)\). Using the approximation \(\sqrt{1 - \omega^2/\xi^2} = 1 + O(\omega^2/\xi^2)\) and recalling that \(i\xi\) corresponds to \(\partial/\partial t\), we obtain

\textbf{1st order approximation:}

\[
\left( \frac{\partial}{\partial x_1} - \frac{\partial}{\partial t} \right) p|_{x_1=0} = 0. \tag{4.6}
\]

By using the next approximation (the first Taylor or Padé approximation) to the square root, \(\sqrt{1 - \omega^2/\xi^2} = 1 - \frac{1}{2}\omega^2/\xi^2 + O(\omega^4/\xi^4)\) in (4.5) and multiplying by \(i\xi\), we arrive at

\[
i\xi \frac{\partial}{\partial x_1} + \xi^2 - \frac{1}{2}\omega^2. \tag{4.7}
\]

This yields the boundary condition of

\textbf{2nd order approximation:}

\[
\left( \frac{\partial^2}{\partial x_1 \partial t} - \frac{\partial^2}{\partial t^2} + \frac{1}{2} \frac{\partial^2}{\partial x_2^2} \right) p|_{x_1=0} = 0. \tag{4.8}
\]

For the two approximations given above, when \(\xi = 1\), \(\omega\) has the interpretation of \(\omega = \sin \theta\) where \(\theta\) is the angle of incidence. Thus, if a wave of the form
\[ a(\xi, \omega)e^{i\sqrt{\xi^2 - \omega^2 x_1 + i\omega x_2 + i\xi t}} \] (4.9)

strikes the boundary at the angle \( \theta \), the first approximation produces a reflected wave of

\[ be^{-i\sqrt{\xi^2 - \omega^2 x_1 + i\omega x_2 + i\xi t}} \] (4.10)

with amplitude \( b \) given by

\[ b = a \left( \frac{\cos \theta - 1}{\cos \theta + 1} \right). \] (4.11)

This means, that only for waves travelling orthogonal to the boundary, no reflection will be produced, while the second approximation produces a \textit{weaker} reflected wave

\[ be^{-i\sqrt{\xi^2 - \omega^2 x_1 + i\omega x_2 + i\xi t}} \] (4.12)

with amplitude

\[ b = -a \left( \frac{\cos \theta - 1}{\cos \theta + 1} \right)^2. \] (4.13)

These results are valid at both \textit{low} and \textit{high frequencies}. Standard energy estimates can be used to prove that the 1st order approximation is a maximal dissipative boundary condition and therefore is trivially well posed, while the verification of the well-posedness of the 2nd order approximation can be done through more elaborated methods. Both approximations satisfy the two desired criteria for absorbing boundary conditions [28].

As can be expected, by trying higher order approximations to \( \sqrt{1 - \omega^2 / \xi^2} \), it is possible to achieve boundary conditions with even better absorption. For boundary conditions reflecting waves with amplitudes diminished by

\[ \left| \frac{\cos \theta - 1}{\cos \theta + 1} \right|^3, \] (4.14)

the second Padé approximation can be used. This is given by

\[ \sqrt{1 + x} = 1 + \frac{x}{2 + x/2} + O(|x|^3) \] (4.15)

and similarly to the lower order approximations, the third order boundary condition is obtained

\textit{3rd order approximation:}
4.2. ABSORBING BOUNDARY CONDITIONS

\[
\left( \frac{\partial^3}{\partial t^2 \partial x_1} - \frac{1}{4} \frac{\partial^3}{\partial x_1 \partial x_2^2} - \frac{\partial^3}{\partial t^3} + \frac{3}{4} \frac{\partial^3}{\partial t \partial x_2^2} \right) p|_{x_1=0} = 0. \tag{4.16}
\]

In [28] it is also proved that this high order boundary condition is well posed, whereas by using the second Taylor approximation the resulting fourth approximation is strongly ill posed. In general, when Padé approximations are used instead of a Taylor series, the resulting boundary conditions are stable. In [48] and [7], remarks about well-posedness and stability concerning absorbing boundary conditions can be found.

Even though the third order approximation provides excellent absorption, in general high order absorbing boundary conditions are not only more difficult to implement numerically but also increase the computational cost. These aspects have to be considered, when choosing the absorbing boundary conditions to be implemented.

4.2.2. Finite Element Formulation

In the implementation of the absorbing boundary conditions for our case of study, choosing a first order approximation as the one given in (4.6), results in a straightforward procedure given by the introduction of a damping. The 1st order boundary condition for the acoustic equation in the \( x_1 \) direction is given as

\[
\left( c_0 \frac{\partial}{\partial x_1} - \frac{\partial}{\partial t} \right) \rho'|_{x_1=0} = 0. \tag{4.17}
\]

We may start the procedure for the implementation of this absorbing boundary condition by recalling equation (3.16)

\[
\int_{\Omega} \dot{\rho}' w d\Omega + c^2 \int_{\Omega} \nabla \rho' \cdot \nabla w d\Omega = - \int_{\Omega} \left( \nabla \cdot T_{ij} \right) \cdot \nabla w d\Omega \\
+ c_0^2 \int_{\Gamma} \left. \frac{w}{\partial \vec{n}} \right|_{\Gamma} d\Gamma \tag{3.16}
\]

from the derivation of the weak formulation of the problem. At that point we assumed homogeneous Neumann boundary conditions which eliminated the surface integral containing the normal derivative of \( \rho' \). Equation (4.17) represents waves travelling to the left along \( x_1 \), with normal incidence on the surface (\( \omega = 0 \)). Therewith, in all directions we can express the one-dimensional spatial derivative in (4.17) as the normal derivative in (3.16). By doing so, special attention has to be given to the direction of the normal vector relative to the positive direction of the \( x_1 \) axis. In our case, the normal points out of the boundary towards the infinite domain, that is, in the same direction of the outgoing wave, whereas \( x_1 \) is positive in the direction opposite to the propagation of the normal wave. Finally, changing the direction of the normal vector and substituting (4.17) in (3.16), yields
\[
\int_\Omega \dot{\rho}' w \, d\Omega + c_0^2 \int_\Omega \nabla \rho' \cdot \nabla w \, d\Omega = - \int_\Omega (\nabla \cdot T_{ij}) \cdot \nabla w \, d\Omega - c_0 \int_\Gamma w \frac{\partial \rho'}{\partial t} \, d\Gamma.
\]

This new surface integral including a first time derivative of the acoustic quantity, may be seen as a damping matrix \( C \) acting only on the surface of the computational domain. Therewith, the semidiscrete Galerkin formulation changes to

\[
M \ddot{\rho}'(t) + C \dot{\rho}'(t) + K \rho'(t) = f(t),
\]

where the new matrix \( C \) computes as follows

\[
C = [C_{ij}],
\]

\[
C_{ij} = c_0 \int_\Gamma N_i N_j \, d\Gamma \quad 1 \leq i, j \leq n_{eq}.
\]

This damping matrix is almost empty, since only terms along the boundary \( \Gamma \) of the domain contribute to its nonzero entries.

Regarding the time discretization of the problem, initially we had implemented the Newmark method without considering the presence of a damping matrix. Therewith, the previously obtained algebraic system of equations and the \textit{predictor-corrector} algorithm have to be modified. The counterpart for the algebraic system represented by equations (3.27) and (3.28) is now given as

\[
M^* \ddot{\rho}'_{n+1} = f_{n+1} - C \left( \dot{\rho}'_n + (1 - \gamma_H) \Delta t \ddot{\rho}'_n \right) - K \left( \dot{\rho}'_n + \gamma_H \Delta t \dot{\rho}'_n + \frac{\Delta t^2}{2} (1 - 2\beta_H) \ddot{\rho}'_n \right)
\]

\[
M^* = M + \gamma_H \Delta t C + \beta_H \Delta t^2 K.
\]

In this case, we use an implicit algorithm where \( \beta_H \neq 0 \) and \( \gamma_H \neq 0 \). Here, the matrix \( M^* \) computes as the sum of the mass matrix \( M \), the damping matrix \( C \) and the stiffness matrix \( K \) and values for the integration factors are typically chosen to be \( \beta_H = 0.25 \) and \( \gamma_H = 0.5 \).

We can now write again the solution process for one time step of the \textit{predictor-corrector} algorithm as an \textit{Effective Stiffness Formulation} according to (3.25) and (3.26), obtaining this time [58]

\textit{Effective Stiffness Formulation}
4.2. ABSORBING BOUNDARY CONDITIONS

- Perform predictor step:

\[
\tilde{\rho}' = \rho'_n + \Delta t \tilde{\rho}'_n + (1 - 2\beta_H) \frac{\Delta t^2}{2} \tilde{\rho}'_n \\
\tilde{\rho}' = \rho'_n + \Delta t (1 - \gamma_H) \rho''_n
\]  

(4.23) \hspace{1cm} (4.24)

- Solve algebraic system of equations:

\[
K^* \rho'_{n+1} = f_{n+1} - C \tilde{\rho}' + \left( \frac{1}{\beta_H \Delta t^2} M + \frac{\gamma_H}{\beta_H \Delta t} C \right) \tilde{\rho}'
\] 

(4.25)

\[
K^* = K + \frac{\gamma_H}{\beta_H \Delta t} C + \frac{1}{\beta \Delta t^2} M
\]  

(4.26)

- Perform corrector step:

\[
\tilde{\rho}'_{n+1} = \frac{\rho'_n + \Delta t^2}{\beta \Delta t^2} \tilde{\rho}'_{n+1} - \tilde{\rho}'
\] 

(4.27)

\[
\rho'_{n+1} = \tilde{\rho}' + \gamma_H \Delta t \rho''_{n+1}
\]  

(4.28)

4.2.3. Evaluation of the ABC Implementation

In this section, the absorbing boundary conditions with the first order approximation are tested. Comparisons for a simple pulse in a rectangular domain are carried out, using the implicit homogeneous Neumann boundary condition and the first order absorbing boundary condition due to Madja and Engquist presented in Sec. 4.2.1.

Pressure Field in an Unbounded Domain

Figure 4.1 shows the acoustic pressure distribution due to a sinusoidal signal with frequency \( f = 10 \) kHz applied at the center of a rectangular domain for different times. The dimensions of the numerical region are \( L \times L = 100 \text{ mm} \times 100 \text{ mm} \) where its center is coinciding with \( (x_1, x_2) = (0, 0) \). Using a speed of sound of \( c_0 = 343 \text{ m/s} \) for air, the wave-length for this problem results in \( \lambda = 34.3 \text{ mm} \). In the computation time is dicretized using a time step size \( \Delta t = 2.5 \mu \text{s} \) corresponding to a discretization ratio of 40 time steps per period. Concerning the spatial discretization, an element size of \( h = 1 \text{ mm} \) is chosen and the simulation is solved using linear quadrilateral elements. For this discretization, the number of elements per wave-length corresponds to \( \lambda/h = 34.3 \). In Fig. 4.1 (a), results using the implicit homogeneous Neumann boundary condition are presented. This condition gives a total reflection disturbing the pressure fluctuation in the interior of the domain. At time \( t = 195 \mu \text{s} \), the reflected wave has already dominated the solution, losing the computation its physical meaning. As expected for this boundary condition, the effective reflection rate \( |R| \), defined as the ratio between the amplitude of the acoustic wave when impinging the boundary and the amplitude of the reflected wave is

\[
|R| = 1,
\]  

(4.29)
representing the effect of a rigid wall.

Results using first order boundary conditions are shown in Fig. 4.1 (b). At time $t = 165 \mu$s, the impinging wave-front is mostly normal to the surfaces, still giving a very good absorption. At later times, the angle of incidence becomes less orthogonal, starting a distortion of the solution. However results remain still meaningful, as can be observed at time $t = 195 \mu$s. Finally, at time $t = 230 \mu$s, the pressure distribution deviates significantly from the one that would be obtained in a real infinite domain.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.1.png}
\caption{Acoustic pulse in a uniform flow. Two-dimensional pressure field $p'$, obtained using: (a) homogeneous Neumann boundary conditions (b) first order absorbing boundary conditions. Logarithmic representation of 10 iso-contours.}
\end{figure}

In Fig. 4.2, the pressure distribution in time, for a fixed point in the rectangular do-
main located at \((x_1, x_2) = (25 \text{ mm}, 0 \text{ mm})\), is considered. Both homogeneous Neumann, and first order absorbing boundary conditions are plotted. For the hard wall conditions, the reflected wave is superposed to the impinging wave producing a total wave of about double the amplitude of the outgoing wave, which corresponds to the effective reflection rate for these boundary conditions. As for the absorbing conditions, no significant reflection can be observed for the whole time of the computation, \(t = 300 \mu s\).

![Graph showing acoustic pressure vs. time for different boundary conditions](image)

**Figure 4.2.** Evaluation of the acoustic pressure using homogeneous Neumann and first order absorbing conditions, for a point located at \((x_1, x_2) = (25 \text{ mm}, 0 \text{ mm})\).

In the three dimensions, tests of the absorbing boundary conditions were performed using two simple cases, a cube with side length \(L = 50 \text{ mm}\) and a sphere with radius \(r = 25 \text{ mm}\), discretized with hexahedrals and tetrahedrals elements respectively. Similar simulation parameters were used in both computations. A sinusoidal excitation signal with frequency \(f = 10 \text{ kHz}\) has been applied at the centroids of the geometries. In the computations time is discretized using a time step size \(\Delta t = 5 \mu s\) corresponding to a discretization ratio of 20 time steps per period. Sizes of hexahedrals and tetrahedrals elements are chosen to be \(h = 1.25 \text{ mm}\). In this discretization, the ratio of elements per wave-length corresponds to \(\lambda/h \approx 27\) where, for propagation in air, the wave-length is \(\lambda = 34.3 \text{ mm}\).

In the case of the cube, results for a fixed point in space located at 12.5mm from the acoustic source are presented in Fig. 4.3. Similarly, results for a fixed point located at 12.5 mm from the center of the sphere are given in Fig. 4.4. When using the absorbing boundary conditions acting on quadrilaterals boundary elements belonging to hexahedrals, although eventually a total absorption is observed and only minor reflections are still present until approximately \(t = 900 \mu s\), superposition of the reflected waves and the impinging ones distorts the solution significantly having these an amplitude about 40 percent higher than the one that can be seen before the reflected waves have influenced them. Results obtained for the implementation acting on the triangular surface elements corresponding to tetrahedrals show better absorption due mainly to the normal direction of the impinging waves with
respect to the boundary of the sphere. From Fig. 4.4 a reflection rate could be estimated to be around 30 percent. In both cases, the results after a time $t = 150 \mu s$ show already the cancellation effect of reflected waves from opposite directions in the center of the domain.

Figure 4.3. Pressure distribution using homogeneous Neumann and first order absorbing conditions for the cube case, of a point located at $(x_1, x_2, x_3) = (12.5 \text{ mm}, 0 \text{ mm}, 0 \text{ mm})$ relative to the center of the cube.
4.3. Perfectly Matched Layer - PML

The perfectly matched layer method was first developed by Berenger [9] for solving unbounded electromagnetic problems with the finite-difference method in time domain. PML is based on the use of an absorbing layer surrounding the physical domain to absorb outgoing waves. This method is known as perfectly matched because the theoretical reflection factor of the waves impinging on the interface between the physical domain and the absorbing layer is zero at any frequency and at any incidence angle [9]. Apart from problems involving Maxwell’s equations solved with finite differences, the PML has been used successfully in other wave propagation fields, e.g. acoustics and mechanics. The use of the PML technique in the field of aeroacoustics has been investigated by Hu in [55] for the linearized Euler Equations using finite differences. The split formulation of Berenger suits the FEM well and has mainly been used for computations in the frequency domain [88]. More recently, derivations of PML formulations for the time domain finite element method for Maxwell’s equations have been presented in [57, 88].

In the following, basics ideas concerning impedance matching, derivation of an enhanced PML technique for the acoustic time-harmonic case and its FE formulation [59] are presented. The implementation of this numerical technique in the in-house FE solver allows to truncate the computational domain in the acoustic near field, without compromising the acoustic solution.
4.3.1. Basic Ideas of Matching

For linear acoustics in the one-dimensional case the relation between the acoustic particle velocity \( v' = v_\xi e_\xi \) and the acoustic pressure \( p' \) is given by (conservation of mass and momentum)

\[
\frac{\partial p'}{\partial t} = -\rho_0 c_0^2 \frac{\partial v'_\xi}{\partial \xi} \quad (4.30)
\]

\[
\frac{\partial v'_\xi}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial \xi}. \quad (4.31)
\]

In (4.30) and (4.31) \( \rho_0 \) denotes the mean density of the fluid, \( c_0 \) the speed of sound and \( e_\xi \) the direction of propagation. Applying a time derivative to (4.30), a divergence operation to (4.31) and then subtracting the so obtained equations leads to the well known linear wave equation for \( p' \)

\[
\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \frac{\partial}{\partial \xi} \frac{\partial p'}{\partial \xi} = 0. \quad (4.32)
\]

We know, that for a plane wave the acoustic impedance \( Z_a \), also called the characteristic impedance, is

\[
Z_a = \rho_0 c_0. \quad (4.33)
\]

In addition, the reflection coefficient \( R \) at an interface for a plane wave impinging perpendicular to the interface (see Fig. 4.5) computes as

\[
R = \frac{Z_{a2} - Z_{a1}}{Z_{a2} + Z_{a1}} \quad \text{with} \quad Z_{a1} = \rho_0 c_0, \quad Z_{a2} = \tilde{\rho} \tilde{c}. \quad (4.34)
\]

Clearly, to make \( R \) equal to zero, \( Z_{a2} \) has to match \( Z_{a1} \). However, we are free in choosing the two quantities \( \tilde{\rho} \) and \( \tilde{c} \) in the damping region in such a way, that just their product equals \( \rho_0 c_0 \). Therefore, by setting

\[
\tilde{\rho} = \rho_0 (1 - j \sigma_\xi) \quad \tilde{c} = \frac{c_0}{1 - j \sigma_\xi} \quad (4.35)
\]
in the damping region, we obtain for the acoustic impedance $Z_{a2} = Z_{a1}$. Furthermore, the wave equation (4.32) reads in the time-harmonic case as

$$\frac{\omega^2}{c_0^2} \hat{\dot{p}} + \frac{\partial}{\partial \xi} \frac{\partial \hat{p}}{\partial \xi} = 0$$  \hspace{1cm} (4.36)$$

with $\omega$ the pulsation of the wave and $\hat{p}$ the acoustic pressure. Without any restrictions, we choose $\sigma = \sigma_0$ (constant within the damping region) and obtain

$$\frac{\omega^2}{c_0^2} (1 - j\sigma_0)^2 \hat{p} + \frac{\partial^2 \hat{p}}{\partial \xi^2} = 0.$$  \hspace{1cm} (4.37)$$

Therewith, we arrive at a wave equation with a complex wave number $\tilde{k}$

$$\tilde{k} = \frac{\omega}{c_0} (1 - j\sigma_0) = k(1 - j\sigma_0).$$  \hspace{1cm} (4.38)$$

Since the general solution of (4.37) is

$$\hat{p} = \hat{p}_0 e^{j(\omega t - \tilde{k} \xi)} = \hat{p}_0 e^{j(\omega t - k \xi)} e^{-\sigma_0 \xi},$$

we achieve our goal, that we have impedance matching and damping of the wave at the same time.

However, if a plane wave impinging at an angle $\varphi$ to the normal vector of the interface is considered (see Fig. 4.5), then the acoustic impedances computed by (see e.g., [84])

$$Z_{a1} = \frac{\rho_0 c_0}{\cos \varphi_1} \quad Z_{a2} = \frac{\hat{\rho} c}{\cos \varphi_2}$$

and our method will be not working. Therefore, in order to obtain perfectly matching, we need a further degree of freedom, which will be the discussion of the subsequent section.

### 4.3.2. Construction of Perfectly Matched Layers

We start at the mass as well as momentum conservation equations for linear acoustics, which read as follows [58]

$$\frac{\partial p'}{\partial t} = -\rho_0 c_0^2 \nabla v'$$  \hspace{1cm} (4.39)$$

$$\frac{\partial v'}{\partial t} = -\frac{1}{\rho_0} \nabla p'.$$  \hspace{1cm} (4.40)$$

According to [9] we apply a splitting of the acoustic pressure $p'$ into $p'_x$, $p'_y$ and $p'_z$ and introduce artificial damping. Therewith, the mass as well as momentum conservation equation for linear acoustics change to

$$\frac{\partial p'_x}{\partial t} + \sigma_x p'_x = -\rho_0 c_0^2 \frac{\partial v'_x}{\partial x}$$

$$\frac{\partial p'_y}{\partial t} + \sigma_y p'_y = -\rho_0 c_0^2 \frac{\partial v'_y}{\partial y}$$

$$\frac{\partial p'_z}{\partial t} + \sigma_z p'_z = -\rho_0 c_0^2 \frac{\partial v'_z}{\partial z}$$

$$\frac{\partial v'_x}{\partial t} + \sigma_x v'_x = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x}$$

$$\frac{\partial v'_y}{\partial t} + \sigma_y v'_y = -\frac{1}{\rho_0} \frac{\partial p'}{\partial y}$$

$$\frac{\partial v'_z}{\partial t} + \sigma_z v'_z = -\frac{1}{\rho_0} \frac{\partial p'}{\partial z}$$
In the above equations $\sigma_x$, $\sigma_y$ and $\sigma_z$ are damping functions, which are zero within the acoustic propagation domain and which are different from zero within the PML-layer enclosing the acoustic propagation domain (see Fig. 4.6). Applying a Fourier-transformation to (4.41)

- (4.43) and rearranging the involved terms, we arrive at the following equations

\[
\hat{p}_x = -\rho \frac{c_0^2}{j\omega + \sigma_x} \frac{1}{\eta_x} \frac{\partial \hat{v}_x}{\partial x}, \quad \hat{v}_x = \frac{1}{\rho_0} \frac{1}{j\omega + \sigma_x} \frac{\partial \hat{p}_x}{\partial x} \tag{4.44}
\]

\[
\hat{p}_y = -\rho \frac{c_0^2}{j\omega + \sigma_y} \frac{1}{\eta_y} \frac{\partial \hat{v}_y}{\partial y}, \quad \hat{v}_y = \frac{1}{\rho_0} \frac{1}{j\omega + \sigma_y} \frac{\partial \hat{p}_y}{\partial y} \tag{4.45}
\]

\[
\hat{p}_z = -\rho \frac{c_0^2}{j\omega + \sigma_z} \frac{1}{\eta_z} \frac{\partial \hat{v}_z}{\partial z}, \quad \hat{v}_z = \frac{1}{\rho_0} \frac{1}{j\omega + \sigma_z} \frac{\partial \hat{p}_z}{\partial z} \tag{4.46}
\]

with $\omega = 2\pi f$ the pulsation of the wave and $f$ the frequency. As can be seen from (4.44) - (4.46), we can directly compute $\hat{v}_x$, $\hat{v}_y$ and $\hat{v}_z$ as a function of $\hat{p}_x$, $\hat{p}_y$ and $\hat{p}_z$. Therewith, we obtain for the total acoustic pressure $\hat{p} = \hat{p}_x + \hat{p}_y + \hat{p}_z$ the following modified Helmholtz equation

\[
\eta_y \eta_z \frac{\partial}{\partial x} \left( \frac{1}{\eta_x} \frac{\partial \hat{p}}{\partial x} \right) + \eta_x \eta_z \frac{\partial}{\partial y} \left( \frac{1}{\eta_y} \frac{\partial \hat{p}}{\partial y} \right) + \eta_x \eta_y \frac{\partial}{\partial z} \left( \frac{1}{\eta_z} \frac{\partial \hat{p}}{\partial z} \right) + \eta_x \eta_y \eta_z k^2 \hat{p} = 0. \tag{4.47}
\]

In (4.47) $k = \omega/c_0$ denotes the acoustic wave number and the functions $\eta_1$, $\eta_2$ and $\eta_3$ compute as follows

\[
\eta_x = 1 - j \frac{\sigma_x}{\omega}, \quad \eta_y = 1 - j \frac{\sigma_y}{\omega}, \quad \eta_z = 1 - j \frac{\sigma_z}{\omega}. \tag{4.48}
\]

This decomposition of the acoustic pressure $p'$ into its $x$-component, $y$-component and $z$-component is the key point of the PML-technique. For a physical interpretation we will consider an interface between a propagation- and a PML-region, which is parallel to the $y$-axis and has a normal vector pointing in $x$-direction (see Fig. 4.7). Now, the $x$-component $p'_x$ of the total acoustic pressure $p'$ can be regarded as a plane wave propagating just in $x$-direction and which will be damped in the PML-region with the damping coefficient $\sigma_x$. Therewith, we have the one-dimensional case and achieve our goal of a reflexion-less interface, if simply the impedances matches (see previous section). This simple consideration already provides us with the information, where to choose the individual damping coefficients $\sigma_x$, $\sigma_y$ and $\sigma_z$ different from zero. Figure 4.8 displays the PML-technique for the 2D- and 3D-case.
At this point, it is important to note, that there is a second quite general method to derive (4.47). This is known as a mapping of the solution of Helmholtz equation in the real coordinate space to a complex coordinate space, e.g. an analytic continuation of the solution (see e.g., [20, 95]).

4.3.3. Choice of Damping Functions

Let us consider the case, in which a wave is propagating within the PML-layer in $y$-direction and having an angle of $\varphi$ with respect to the $y$-axis. In this case the total pressure $p'$ computes as [59]

$$p' = p'_x + p'_y = p_{x0} e^{j(\omega t - \tilde{k}_x x - \tilde{k}_y y)} + p_{y0} e^{j(\omega t - \tilde{k}_x x - \tilde{k}_y y)}.$$  (4.49)

Substituting the values for $\tilde{k}_x$, $\tilde{k}_y$ according to [59]

$$\tilde{k}_x = \frac{\omega}{c_0} \eta_x \sin \varphi = k \sin \varphi \left( 1 - j \frac{\sigma_x}{\omega} \right),$$  (4.50)

and

$$\tilde{k}_y = \frac{\omega}{c_0} \eta_y \cos \varphi = k \cos \varphi \left( 1 - j \frac{\sigma_y}{\omega} \right),$$  (4.51)
and considering that for the chosen case $\sigma_x = 0$, we obtain

$$p' = (p_{x0} + p_{y0}) e^{i(\omega t - kx \sin \varphi - ky \cos \varphi)} e^{-(\sigma_y / c_0) \cos \varphi} = p_0 e^{-(\sigma_y / c_0) \cos \varphi}.$$  

(4.52)

Assuming a layer thickness of $L$, the damped wave will be totally reflected at the outer boundary of the PML-region and this reflected wave at the interface between propagation- and PML-region takes on the following value

$$p'_r = p_0 e^{-(2/c_0) \cos \varphi \int_0^L \sigma_y(y) dy} = p_0 R_0.$$  

(4.53)

A reasonable choice of the reflection factor $R$ is $10^{-3}$, since we have to take care, that a too strong damping in a too small PML-region can strongly disturb the numerical solution. In addition, in order to get rid of the dependence of the overall damping on the speed of sound $c_0$, we will choose all damping functions $\sigma_y$ direct proportional to $c_0$ (see (4.53)).

In a first case, we will assume a constant damping $\sigma_y = \sigma_0$. Therewith, we obtain from (4.53) the following relation for $\sigma_0$

$$\sigma_0 = -\frac{c_0 \ln R}{2L \cos \varphi}.$$  

(4.54)

In a second case, we consider a quadratically increasing damping function, hence we set

$$\sigma_y = \sigma_0^q \frac{y^2}{L^2},$$

and assume, that $y$ is equal to zero at the interface and is increasing within the PML-region. Again, exploiting (4.53) we arrive at a relation for the constant factor $\sigma_0^q$

$$\sigma_0^q = \frac{3c_0 \ln R}{2L \cos \varphi}.$$  

(4.55)

In a last step, we will introduce a singular function, given by

$$\sigma_y = \frac{c_0}{L - y},$$  

(4.56)

which means, that we increase the damping inverse with the distance.

We want to emphasise, that the damping functions according to (4.54) as well as (4.56) are different from zero at the interface, and therefore will introduce a discontinuity at the interface. However, due to the properties of the PML-technique, no spurious reflections will occur [59].

4.3.4. Finite Element Formulation

A volume discretization of the acoustic wave equation is performed by applying the finite element method (FEM). In a strong setting, we have to solve the following PDE [59]:

Given:

$$\hat{f} : \Omega \rightarrow \mathbb{C}$$

$$c_0, \rho_0 : \Omega \rightarrow \mathbb{C}.$$  

Find:  $$\hat{p} : \Omega \rightarrow \mathbb{C}.$$
\[ \eta_x \eta_y \frac{\partial}{\partial x} \left( \frac{1}{\eta_x} \frac{\partial \hat{p}}{\partial x} \right) + \eta_x \eta_z \frac{\partial}{\partial y} \left( \frac{1}{\eta_y} \frac{\partial \hat{p}}{\partial y} \right) + \eta_x \eta_y \frac{\partial}{\partial z} \left( \frac{1}{\eta_z} \frac{\partial \hat{p}}{\partial z} \right) \]

\[ + \omega^2 \frac{\eta_x \eta_y \eta_z}{c_0^2} \hat{p} - \hat{f} = 0. \quad (4.57) \]

Boundary conditions
\[ \frac{\partial \hat{p}}{\partial n} = 0 \text{ on } \Gamma \quad (4.58) \]

In (4.57) and (4.58) \( \hat{f} \) denotes any acoustic source term, \( \mathcal{C} \) the set of complex numbers, \( \Gamma \) the outer boundary of the computational domain and \( \eta_x, \eta_y, \eta_z \) the damping functions, which are equal to 1 in the propagation domain and computed according to (4.48) within the PML-region. In a first step, we multiply (4.57) by an appropriate test function \( v \) and integrate over the whole domain \( \Omega \)
\[ \int_\Omega v \left( \eta_y \eta_z \frac{\partial}{\partial x} \left( \frac{1}{\eta_x} \frac{\partial \hat{p}}{\partial x} \right) + \eta_x \eta_z \frac{\partial}{\partial y} \left( \frac{1}{\eta_y} \frac{\partial \hat{p}}{\partial y} \right) + \eta_x \eta_y \frac{\partial}{\partial z} \left( \frac{1}{\eta_z} \frac{\partial \hat{p}}{\partial z} \right) \right. \]

\[ \left. + \omega^2 \frac{\eta_x \eta_y \eta_z}{c_0^2} \hat{p} \right) \, d\Omega = \int_\Omega \hat{f} \, d\Omega. \quad (4.59) \]

Now, we apply Green’s integral theorem to the second order spatial derivatives and incorporate the homogeneous Neumann boundary condition (4.58). These steps lead to the following weak formulation of (4.58): Find \( \hat{p}' \in H^1 \) such that
\[ \int_\Omega \eta_x \eta_z \frac{\partial v}{\partial x} \frac{\partial \hat{p}}{\partial x} + \eta_x \eta_z \frac{\partial v}{\partial y} \frac{\partial \hat{p}}{\partial y} + \frac{\partial v}{\partial z} \frac{\partial \hat{p}}{\partial z} \, d\Omega - \omega^2 \int_\Omega \eta_x \eta_y \eta_z \frac{v \hat{p}}{c_0^2} \, d\Omega = \int_\Omega v \hat{f} \, d\Omega \quad (4.60) \]
for any \( v \in H^1 \). With \( H^1 \) we denote the Sobolev space defined as (see e.g. [1])
\[ H^1 = \{ v \in L^2 | \partial v/\partial x_i \in L^2 \} \quad (4.61) \]
and \( L^2 \) the space of square integrable functions. Using standard nodal finite elements, we arrive at the following discrete complex algebraic system of equations [58]
\[ (\mathbf{K} - \omega^2 \mathbf{M}) \hat{\mathbf{p}} = \hat{\mathbf{f}} \quad (4.62) \]
with \( \mathbf{K} \) the stiffness matrix, \( \mathbf{M} \) the mass matrix, \( \hat{\mathbf{p}} \) the nodal vector of complex acoustic pressure and \( \hat{\mathbf{f}} \) the complex nodal vector of the right hand side. For a stability investigation we refer, e.g., to [8, 13]. In (4.62), it is important to notice that, in contrast to (4.19), no further absorption by means of a damping matrix \( \mathbf{C} \) acting on the outer boundary elements is considered. Although the PML domain is of finite length, numerical investigations demonstrate that even for PML layers being just a fraction of the wave-length, reflections that occur at the PML outer boundaries are already negligible. When the outgoing waves reach the outer boundaries their magnitudes have already been reduced significantly. Furthermore, even when some reflection occurs, these waves will travel through the PML domain again and be damped further before they re-enter the physical domain [55]. Therefore, implicit homogeneous Neumann boundary conditions are assumed in the derivation of (4.62) to avoid additional computational cost.
4.3.5. Evaluation of the PML Implementation

A 2D example, similar to the one which was used in Sec. 4.2.3 to compare the ABC boundary treatment with the Neumann boundary conditions, is used in this section for the evaluation of the PML implementation. In this case, results from a harmonic computation using PML are compared with results from a transient computation using ABC, as well as with transient results obtained in a sufficiently large domain which avoids any influence of reflections. For the computations using PML and ABC, the numerical domain has dimensions $L \times L = 150 \, \text{mm} \times 150 \, \text{mm}$ being its center located at $(x_1, x_2) = (0, 0)$. A PML with thickness $5 \, \text{mm}$ (corresponding to about $\lambda/7$) has been used for the harmonic computation. For the transient computation using a large domain, the dimensions of the domain were chosen to be $L \times L = 300 \, \text{mm} \times 300 \, \text{mm}$. Concerning spatial discretizations, in all three computations the size of the linear quadrilateral elements is $h = 1 \, \text{mm}$. In the transient computations, time is discretized using a time step size $\Delta t = 2.5 \, \mu\text{s}$ corresponding to a discretization ratio of 40 time steps per period. As in Sec. 4.2.3, the excitation frequency is $f = 10 \, \text{kHz}$ which for acoustic propagation in air results in a wave-length of $\lambda = 34.3 \, \text{mm}$.

Figure 4.9 presents the contour plots of the acoustic pressure obtained from the transient computation using a large domain and the corresponding harmonic computation using PML. Transient results correspond to time $t = 370 \, \mu\text{s}$. At this simulation time, the wave front has traveled 127 mm and has not yet reached the nearest boundaries located at $(x_1, x_2) = (\pm 150 \, \text{mm}, \pm 150 \, \text{mm})$, which guarantees that no influence of reflections is present in the solution. A good qualitative agreement is observed between both solutions even at the corner regions although slight lower amplitudes are noticeable in the transient results.

![Figure 4.9](image_url)

(a) Acoustic pressure field obtained from transient computation using large domain
(b) Acoustic pressure field at frequency $f = 10 \, \text{kHz}$ using PML with thickness $5 \, \text{mm}$ ($\approx \lambda/7$)

Additionally, Fig. 4.10 compares the decay of the acoustic pressure values along the diagonal line $x_1 = x_2$ between the computation using PML and the transient results at time $t = 370 \, \mu\text{s}$, in which the decay of the acoustic pressure obtained using Neumann BC is shown just to emphasize the relevance of the absorbing boundary treatment. This latter curve results in a non-physical decay at distances over 0.04 m from the center of the domain.
In contrast, the other curves show a good agreement, although again results in time domain show some numerical dissipation of the amplitudes which is not present in the harmonic results using PML. In the results where the domain has been truncated using ABC, as expected, a significant loss in amplitudes is noticed specially at distances above 0.08 m, which corresponds to the corner region. This is mainly due to the fact that at this simulation time, waves with a 45° incidence angle, for which the highest reflection occurs ($|R| \approx 0.17$), have already reached the corners and have partially reflected about 21mm into the numerical domain.

![Figure 4.10. Decay of acoustic pressure values along the diagonal line $x_1 = x_2$](image-url)
4. SIMULATION OF UNBOUNDED DOMAINS
5. Coupling of fluid and acoustic computations

5.1. Simulation schemes

For the computation of flow-induced noise problems using a hybrid methodology as introduced in this work, two simulation schemes are followed. The choice of any of these depends on whether the acoustic propagation is solved in time or frequency domain. Fig. 5.1 presents a schematic of these two configurations. For a directly coupled transient computation as presented in Fig. 5.1(a), the flowchart for solving the flow-induced noise problem is presented in Fig. 5.2. Acoustic nodal sources are evaluated directly on the fine discretization of the fluid computation at each coupled time step. These sources are then interpolated to the acoustic solver, where they are used as input for solving the acoustic propagation. On the other hand, Fig. 5.1(b) depicts the procedure to be followed, if a harmonic acoustic computation is performed. In this case, first of all a transient coupling is performed between the fluid computation and the acoustic grid, in which a dataset of the interpolated nodal sources is stored. Secondly, a Fourier transformation is performed to the transient dataset and the resulting sources in frequency domain are then used as input for the acoustic computation in harmonic mode.

5.2. Transfer of the coupling quantities

Coupled field computations like CAA simulations following hybrid approaches, where quantities from a fine resolution fluid computation are used to compute the acoustic radiation, need special algorithms to transfer the coupling quantities between the different numerical grids. Specially in three-dimensional computations, interpolation of the data in problems involving solid or flexible geometries having complicated surfaces is not a trivial task. To perform this transfer correctly it is necessary to know exactly how the coupled regions fit together based on the grid information given on the fluid and acoustic sides of the computation. In this work we make use of the coupling interface software MpCCI [39]. Some of the basic principles used by this interface for the interpolation of the data are presented in the following sections.
5. COUPLING OF FLUID AND ACOUSTIC COMPUTATIONS

(a) Directly coupled approach for transient acoustic simulation
(b) Sequentially coupled approach for harmonic acoustic simulation

Figure 5.1. Coupling approaches in time and frequency domains for the flow-induced noise coupled computation.

5.2.1. Neighborhood Search

The main purpose of the neighborhood search is to determine how the coupling regions fit together based on the mesh definition provided on each physical field. This neighborhood interpolation is important because it determines in which way communication between processes of different codes will take place. The results from the neighborhood computation must be such that they can be used for interpolating the coupling quantities in an efficient way. The main difficulties of the neighborhood search are that coupling regions could be distributed over several partitions of several processes and that initial distances are allowed between the coupling surfaces of the different codes.

The result of the neighborhood search is a relation between the geometric components of different meshes. In MpCCI there are three types of relationships: point-element relations for standard interpolation, element-element relations for mappings based on intersection algorithms and point-point relations for matching grids and nearest neighbor interpolation. Geometric relationships are connected with one pair of meshes. For different mesh pairs different geometric relationships can be computed [39]. In the following we only describe the point-element relation since this is the one which is used to perform standard interpolation of the fluid quantities to the acoustic grid.
5.2. TRANSFER OF THE COUPLING QUANTITIES

Point-Element relationship

To apply the standard interpolation for each node of one grid, one suited element of the other grid must be computed. In addition, the local coordinates of the point with respect to the chosen element are gained while the neighborhood computation takes place. For this purpose, a criterion to reject or accept pairs of elements and points must be used. For surface coupling, a preceding projection must be carried out because in general, the target point $q$ will not lie exactly in an element of the source surface.

When using MpCCI, the neighborhood search is done in two steps: first, a number of elements is computed most likely to contain the point or the projection of the point. This will be done in the pre-contact search. In the second step, the local coordinates of the point with respect to those elements will be computed and according to a matching criterion, the best match will be chosen among the result of the pre-contact search. In this context, a point may be a node of an element or its midpoint [39].

Figure 5.2. Flowchart of the directly coupled computation
5. COUPLING OF FLUID AND ACOUSTIC COMPUTATIONS

5.2.2. Search Algorithm

The algorithms needed to find the pairs of elements and points are search algorithms. The most straightforward of such an algorithm is to compare all pairs of points and elements (linear search). However, this may be inefficient in comparison with the time needed to solve the actual problem on a subregion. This can be demonstrated taking for example a cubic subregion with a uniform grid of \( n_V = l \times l \times l \) nodes. Then for explicit solvers the computing time on that subregion will be [39]

\[
 t_{\text{subregion}} = \mathcal{O}(n_V). \tag{5.1}
\]

A single face of this cubic region has size \( n_A = l \times l \) nodes. If we now assume both the number of nodes and elements to be \( \mathcal{O}(n_A) \), the computational cost of the linear search algorithm becomes

\[
 t_{\text{search}} = \mathcal{O}(n_A) = \mathcal{O}(l^4) = \mathcal{O}(n_V^2) \tag{5.2}
\]

Comparing Equations (5.1) and (5.2), it is possible to notice that the amount of time needed for the linear search algorithm may increase faster with problem size than that for the solution of a problem on a subregion. In large three-dimensional computations this can be a critical issue. This aspect can become even more relevant when we have to consider moving meshes or adaptive grids, where neighborhood relations are not fixed during the simulation time. Therefore, better algorithms are needed. The search algorithms behind MpCCI are mostly based on a bucket search algorithm. Practical experience with this type of algorithm has shown that, although possibly theoretically inferior, they can compete with more theoretically optimal algorithms in many application areas. The reason is that optimality deals with theoretical limit cases, which are rarely reached in practice. A detailed explanation of bucketing techniques applied to computational geometry can be found in [39, 4].

5.2.3. Interpolation

From the results of the neighborhood search, interpolation of the coupling quantities can be performed. The main interpolation schemes employed by MpCCI can be subdivided in two categories:

- Non-conservative interpolated quantities: are functions of the spatial coordinates and time. Examples are temperature, pressure, and velocity. Fig. 5.3 depicts the concept of the standard non-conservative procedure used by MpCCI for interpolating the quantities. In the case of surface interpolation the receiver mesh has been projected on the sender mesh to gain the result of the neighborhood search. The interpolation from nodes on nodes supported by MpCCI is based directly on the element definition. The source values \( s_i \) to be interpolated are located at the nodes and the target quantity \( t \) is required at points inside the elements by the local coordinates \( \xi \) and \( \eta \). A straightforward interpolation is done using the base functions as follows:

\[
 t = \sum_{i=1}^{n_{\text{en}}} N_i(\xi, \eta) s_i \tag{5.3}
\]
5.2. TRANSFER OF THE COUPLING QUANTITIES

Vectors (e.g. velocities) are interpolated component wise. This interpolation is constant, linear or quadratic depending on the source element. With this kind of interpolation, the conservation of physical fluxes is not guaranteed because no balances are made.

![Figure 5.3. Standard non-conservative interpolation](image)

- Conservative interpolated quantities: are amounts of conserved quantities crossing the contact surface through a surface element. Examples of these quantities are the heat flux through an element area and the force exerted on an element area. Non-conservative interpolation (e.g. bilinear) is meaningless for such quantities. A common property of these quantities is that their sum must be preserved during interpolation. In general conservativity in MpCCI is defined as the preserving of the sum of the values.

In contrast to the non-conservative case, in this case the sender mesh has been projected on the receiver mesh to gain the result of the neighborhood search. Conservativity is meant here as preserving the sum of the values, also called redistribution. The standard conservative interpolation supported by MpCCI is also based on the element definition. However, the situation is opposed to that for the fields, because here the source value is located at a single point, and the target locations are the nodes as shown in Fig. 5.4.

By evaluating the basis functions $N_i$ that define the element in the point where the coupling quantity is given, we obtain four weights

$$w_i = N_i(\xi, \eta),$$  

one for each corner, that satisfy

$$\sum_{i=1}^{\text{non}} w_i = 1$$
because of
\[
\sum_{i=1}^{n_{\text{ele}}} N_i(\xi, \eta) \equiv 1 \quad \forall (\xi, \eta) \in E,
\] (5.6)
each node \(i\) then gets a portion \(w_i \cdot s\) of the given coupling value \(s\). Therefore, by its definition, this interpolation preserves the sum of the quantities. Vectors (e.g. forces) are interpolated componentwise.

In the present work, acoustic results from simulations using both the non-conservative and the conservative interpolation types have been evaluated. In both cases the computation of the acoustic sources is performed using the FE formulation introduced in Sec. 3.6. In the non-conservative case, initially the velocity field from the fluid computation has been interpolated from the fine fluid mesh to the coarse acoustic mesh as a vector field disregarding the preserving of the quantities. From the interpolated velocity field, we computed the acoustic loads on the coarse acoustic resolution. As for the conservative interpolation, we compute the acoustic loads directly on the fine fluid resolution. The resulting fine sources are then interpolated in a conservative way as nodal acoustic loads to the coarser acoustic grid.

An example of resulting acoustic pressure values using the two interpolation schemes is presented in Fig. 5.5. Herewith we compare results using the non-conservative interpolation scheme (interpolation of the velocity vector) with the conservative approach (interpolation of the scalar nodal sources). Results where the acoustic sources were computed on the acoustic grid using the interpolated velocity values show very high amplitudes and a non-physical offset of 0.12 Pa. Whereas, when evaluating the acoustic sources directly on the fluid side, results at the same point show amplitudes which are about 10 times smaller in which almost no offset is present.

The conservative interpolation procedure is prefered for the transfer of the acoustic sources in the hybrid approach introduced in this chapter, since by preserving the actual acoustic energy responsible for the noise generation it provides flexibility for the definition of the acoustic grid. As will be demonstrated in Chap. 7, by using conservative interpolation, mesh independent results for the acoustic computation can be obtained for acoustic grids containing large acoustic-element-size to fluid-cell-size ratios.
5.2. TRANSFER OF THE COUPLING QUANTITIES

Figure 5.5. Influence of the procedure for the evaluation and interpolation of the acoustic sources on the acoustic pressure at a point \((x, y) = (0\,\text{m}, 5\,\text{m})\).
5. COUPLING OF FLUID AND ACOUSTIC COMPUTATIONS
6. Validation of the implementation

6.1. Introduction

In the present chapter the acoustic far field caused by an incompressible, purely unsteady vortical flow is investigated following two of the hybrid approaches presented in Chapter 3, the first approach being based on the acoustic analogy in terms of pressure perturbations and the second following the numerical implementation of the original Lighthill’s acoustic analogy. The inhomogeneous wave equation is forced with the acoustic sources obtained from the analytical hydrodynamic field induced by a co-rotating vortex pair. CAA investigations using these types of flow fields are a practical way to validate the simulation of flow-induced noise problems and have been used by other authors in the past ([23, 62, 34, 27, 67]) as a benchmark for the validation of their numerical methods. The resulting acoustic field represents the basic acoustic field generated by turbulent shear flows, jet flows, edge tones, etc.[62]. Therefore, the main objective of this numerical investigation is to verify the applicability of Lighthill’s acoustic analogy in its FEM implementation, together with the associated coupling schemes and boundary conditions methods introduced in previous chapters. Parts of the results of this chapter have been published in [30].

An important motivation for investigating this type of vortex sound problem is the existence of the analytical solution for the acoustic far field, which allows a direct validation of the acoustic results. This solution is obtained employing the method of matched asymptotic expansion (MAE) for spinning vortices first presented by Müller and Obermeier[77], and it is used in the following sections to validate the numerical results from the hybrid computations.

For the conservative interpolation of the acoustic sources from the fine fluid grid to the acoustic domain, the coupling approach as presented in Chap. 5 has been employed, and in the case of harmonic analyses a Fourier transformation has been applied to the interpolated nodal sources.

6.2. Theoretical Approach

In the numerical aeroacoustic investigations presented in this chapter, the acoustic sources in the flow region are computed from the hydrodynamic quantities of the flow field induced by a spinning vortex pair. This corotating vortex pair consists of two point vortices which are separated by a fixed distance of $2r_0$ with circulation intensity $\Gamma$. The schematic of the corotating vortices is presented in Fig. 6.1. These vortices rotate around each other with a period $T = \frac{8\pi^2 r_0^2}{\Gamma}$. Each vortex induces on the other a velocity $v_\theta = \frac{\Gamma}{(4\pi r_0)}$. 

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The configuration results in a rotating speed \( \omega = \Gamma / (4\pi r_0^2) \), and rotating Mach number \( M_r = v_0/c_0 = \Gamma / (4\pi r_0 c_0) = 2\pi r_0/T c_0 \). The rotating noncircular streamlines are directly associated with the hydrodynamic field of the rotating quadrupole [62]. The incompressible, inviscid flow can be determined numerically by the evaluation of a complex potential function \( \Phi(z,t) \) [38, 34, 27]

\[
\Phi(z,t) = \frac{\Gamma}{2\pi i} \ln(z-b) + \frac{\Gamma}{2\pi i} \ln(z+b) = \frac{\Gamma}{2\pi i} \ln z^2 \left( 1 - \frac{b^2}{z^2} \right), \tag{6.1}
\]

where \( z = re^{i\theta} \) and \( b = r_0 e^{i\omega t} \). For \(|z/b| \gg 1\), (6.1) can be approximated by

\[
\Phi(z,t) \approx \frac{\Gamma}{\pi i} \ln z - \frac{\Gamma}{2\pi i} \left( \frac{b}{z} \right)^2 = \Phi_0 + \Phi_1. \tag{6.2}
\]

The first term on the right hand side of (6.2) represents a steady vortical flow, whereas the second term represents the fluctuation with the fundamental frequency due to the vortex motion [62].

From (6.1) it is then possible to derive the expressions representing the hydrodynamic quantities required to compute the acoustic sources when following the acoustic analogy approach. The hydrodynamic velocity is obtained by differentiating (6.1) with respect to \( z \), and the hydrodynamic pressure \( p^{inc} \) is obtained by the unsteady Bernoulli’s equation as

\[
u_x -iu_y = \frac{\partial \Phi(z,t)}{\partial z} = \frac{\Gamma}{i\pi} \frac{z}{z^2 - b^2}. \tag{6.3}
\]

\[
p^{inc} = p_0 - \rho_0 \frac{\partial}{\partial t} \{ \text{Re}(\Phi(z,t)) \} - \frac{1}{2} \rho_0 (u_x^2 + u_y^2). \tag{6.4}
\]

In the acoustic computation a linear propagation is assumed outside the fluid region, governed by the homogeneous acoustic wave equation.
6.3. NUMERICAL INVESTIGATION

For comparison with the numerical results, the analytical solution of the acoustic pressure fluctuations from the corotating vortex pair, obtained with the matched asymptotic expansion (MAE) method, is given by [77]

\[ p' = \frac{\rho_0 \Gamma^4}{64 \pi^3 r_0^4 c_0^2} [J_2(kr) \cos(\Psi) - Y_2(kr) \sin(\Psi)], \quad (6.5) \]

where \( k = 2\omega/c_0 \), \( J_2(kr), Y_2(kr) \) are the second-order Bessel functions of the first and second kind and \( \Psi = 2(\omega t - \theta) \). An equivalent expression for this quadrupole-like solution has also been derived by Mitchell et al. [75] starting from the vortex sound theory proposed by Möhring [76].

6.3. Numerical Investigation

In this section two hybrid approaches are evaluated and validated using the flow field induced by the corotating vortex pair. The first approach is based on the acoustic analogy in terms of pressure perturbations, in which acoustic sources are computed as a function of the second temporal derivative of the hydrodynamic pressure fluctuation given by 6.4. Secondly, the original acoustic analogy derived by Lighthill [64] as implemented in this work, is validated. In this case the acoustic sources are obtained from the hydrodynamic velocity field. This vector field is obtained by spatial differentiation of (6.1) and is used to evaluate the weak formulation of Lighthill’s acoustic source term.

In all computations presented in this section we have evaluated the flow field induced by the spinning vortex pair in a numerical region with dimensions 200 m \( \times \) 200 m. This region corresponds to the region where the acoustic nodal sources for the inhomogeneous wave equation are computed. The acoustic propagation is computed in a larger numerical domain with dimensions 400 m \( \times \) 400 m, as the one depicted in Fig. 6.2 where the inner square domain corresponds to the source region. For evaluating the complex potential function the spinning radius is chosen to be \( r_0 = 1 \text{ m} \), the circulation intensity \( \Gamma = 1.00531 \text{ m}^2/\text{s} \) and the speed of sound \( c_0 = 1 \text{ m/s} \). This results in a wave length \( \lambda \approx 39 \text{ m} \) and a rotating Mach number \( M_r = 0.0796 \). Similar parameters for the flow field induced by the corotating vortex pair have been used by other authors[38, 90] for the validation of aeroacoustic simulations using different numerical methods.

6.3.1. Validation of vortex sound propagation using the perturbation formulation

For the first validation of the acoustic propagation using the FEM, the acoustic analogy based on pressure perturbations as presented in Sec. 2.1.1 is followed. Therefore the inhomogeneous wave equation in this case is given as

\[ \frac{1}{c_0^2} \frac{\partial^2 p^a}{\partial t^2} - \frac{\partial^2 p^a}{\partial x_i^2} = - \frac{1}{c_0^2} \frac{\partial^2 p^{inc}}{\partial t^2}, \quad (2.12) \]

under the assumption that in an incompressible flow the pressure fluctuations can be explicitly split into two components, \( p' = p^{inc} + p^a \) where \( p^{inc} \) represents the incompressible
pressure fluctuations and $p^a$ the acoustic pressure fluctuations with pseudo-sound excluded. Similar to the derivation of the FE formulation of the original Lighthill’s acoustic analogy presented in Chap. 3, by applying a volume discretization to the weak formulation of (2.12) a corresponding FE formulation is obtained.

The second temporal derivative of the incompressible pressure required for the computation of the acoustic sources is obtained after differentiating $p^{inc}$ from (6.4) as

$$\frac{\partial^2 p^{inc}}{\partial t^2} = -\rho_0 (\dddot{\Phi}_{Re} + \dot{u}_x^2 + u_x \ddot{u}_x + \dot{u}_y^2 + u_y \ddot{u}_y),$$

(6.6)

where $\dddot{\Phi}_{Re}$ represents the third temporal derivative of the real part of the complex potential function and $u_x$ and $u_y$ are the velocity components obtained from (6.3). For the evaluation of the acoustic sources from (6.6) care must be taken since the hydrodynamic velocities and pressure have very steep and large gradients near the point vortices. This singularity issue has also been treated by other authors [62, 34, 27, 89]. In the works from Lee [62] and Schram [89] a vortex core model was used to replace the point vortices, whereas Ekaterinaris [27] overcomes this problem by placing the mesh points far enough from the vortex centers, as depicted in the grid configuration from Fig. 6.1 which corresponds to a cut-off of the values at this region. For this formulation based on the hydrodynamic pressure, good acoustic results have been obtained applying a cut-off of the source values for nodes located at distances $r/r_0 \geq 1.5$. Large amplitudes and wrong frequency values were obtained if no cut-off of the sources was applied.

For the computations, a structured spatial discretization has been chosen in the acoustic source region using two different element sizes. In Fig. 6.2 the numerical grid with coarse element size is shown. In this coarse discretization with element size $h = 3.33$ m, the number of elements per wave length corresponds to 11, whereas in the finer grid with an element size $h = 1.66$ m, the discretization results in about 23 elements per wave length.

Figure 6.3 (a) shows the acoustic sources obtained from the computation at time $t = 120$ s, using a time step size $\Delta t = 0.5$ s. As can be observed, the source region is large enough to avoid significant truncation of the source terms. The amplitudes of the sources at the
boundary of the source region, are about 1% of their maximum amplitude reached in the source domain. Figure 6.3 (b) corresponds to the resulting acoustic pressure field for the same time.

Figure 6.3. Instantaneous visualization of acoustic results from the simulation following the perturbation formulation using fine grid, at time $t = 120\, s$.

Figure 6.4 compares the analytical solution obtained using MAE with the numerical solution using the finer grid at time $t = 280\, s$. This simulation time corresponds to the time needed by the propagated wave front to reach the corners of the numerical domain. Although very good qualitative agreement, in the numerical solution the spurious effect of reflections at the corners can already be observed since first order absorbing boundary conditions are used.

In Fig. 6.5 a comparison of the acoustic pressure decay along the x-axis between the analytical solution and the numerical results using the two different spatial discretizations is presented. The numerical result obtained with the coarser spatial discretization shows a significant numerical dispersion in the waves reaching the acoustic boundary. In contrast, when using a twice finer discretization this numerical dispersion does not appear and the

Figure 6.4. Comparison of contour plots of the acoustic pressure distribution at time $t=280\, s$. Left: analytical solution. Right: numerical solution using fine grid.
solution matches very well the acoustic pressure from the analytical solution. The differences near the origin (center of the domain) can be due to the fact that, to avoid the singularity at this point, acoustic sources are only computed for grid points at distances $r/r_0 \geq 1.5$ as suggested by Ekaterinaris [27]. This means that sources near the origin are neglected.

Figure 6.6 presents a comparison in time at a point $(x, y) = (80 \text{ m}, 0 \text{ m})$ of the analytical solution with the numerical results using the fine spatial discretizations. A good agreement between analytical and numerical results is obtained after the wave front has reached the monitoring point. Amplitudes from the numerical results are slightly lower (less than 5%) in comparison to the numerical ones.

![Figure 6.5. Decay of the acoustic pressure values along the x-axis](image)

### 6.3.2. Validation of vortex sound propagation following Lighthill’s acoustic analogy

The purpose of this section is to validate the numerical implementation of the weak formulation of Lighthill’s acoustic analogy as derived in Chapter 3. In contrast to the computation following the perturbation formulation, where scalar sources corresponding to the second time derivative of the hydrodynamic pressure were directly obtained from the complex potential function, in Lighthill’s acoustic analogy we require the velocity vector field to compute the velocity gradients contained in the acoustic source term from (3.45). This aspect makes it necessary to evaluate the fluid field using a much finer resolution than that of the acoustic field, so that the velocity gradients can be computed accurately. Therefore, for the validation of Lighthill’s analogy we employ the two-step coupling approach presented in Chapter 4.18. The first step consists in evaluating the analytical velocity field and the acoustic loads on a fine fluid grid. Secondly, after interpolation of the acoustic sources from the fine fluid grid to the coarser acoustic grid, we solve the inhomogeneous wave equation using these sources to obtain the acoustic solution.
An additional issue for the computation is the singularity of the velocity field at the point vortices. Since the acoustic contribution of the region near the point vortices is important for the final acoustic field, a cut-off of the sources at these region, as proposed in [27], does not produce good results. Therefore, the application of a vortex core model, as proposed in [62, 34, 89, 83], is required to obtain the desingularized tangential velocity field around these points. A desingularized kernel following the Scully model [92, 11] and presented in [89, 83] is applied at each point vortex for radii \( r_{\text{vortex}} < 0.15 \text{ m} \). Within these regions the expressions for the velocity components are given by

\[
\begin{align*}
  u_x &= -\frac{\Gamma}{2\pi r_{\text{core}}} \frac{y}{r_{\text{core}} + x + y^2}; \\
  u_y &= \frac{\Gamma}{2\pi r_{\text{core}}} \frac{x}{r_{\text{core}} + x^2 + y^2},
\end{align*}
\]  

(6.7)

where the coordinates \((x, y)\) are taken with respect to each vortex and \(r_{\text{core}}\) corresponds to the vortex core radius, which is the distance from the point vortex where the maximum tangential velocity values occur. For the computations, a value \(r_{\text{core}} = 0.10 \text{ m}\) has been chosen. The use of the vortex core model allows a very fine discretization of the source region where the flow field is resolved, which is of importance for the accurate evaluation of the acoustic sources. Figure 6.7 presents the velocity magnitude and vector field around the point vortices, where Fig. 6.7 (b) shows a close up of the tangential velocity when no vortex core model is used and Fig. 6.7 (d) after applying the desingularized kernel.

In the discretization of the fluid region, several element sizes, \(h_f\), have been tested for the evaluation of the velocity field, until no dependence on the fluid mesh was observed in the resulting acoustic field. For the computations presented here we have used the finer grid size which corresponds to \(r_{\text{core}}/h_f = 5\) in a region around the corotating vortices with dimensions \(6 \text{ m} \times 6 \text{ m}\). Outside this area the grid is coarsened in the radial directions. For the transient acoustic computation, the grid is correspondingly coarsened in the coupled region and it extends 500 m in all directions to avoid the effects of boundary reflections on the solution.
6. VALIDATION OF THE IMPLEMENTATION

Figure 6.7. Velocity magnitude and vector field around the point vortices

(a) Velocity field around the point vortices (no vortex core model used)

(b) Velocity magnitude around single vortex, depicting vector field of nodes along x-axis (no vortex core model used)

(c) Velocity field around the point vortices when using vortex core model

(d) Velocity magnitude around single vortex, depicting vector field of nodes along x-axis using vortex core model

In this case the element size for the acoustic grid in the region next to the spinning vortices is chosen to be $h_a = 0.1$ m which corresponds to $h_a/h_f = 5$. Since the applied vortex core model is an approximation of the velocity field near the point vortices, the evaluation of the velocity gradients at these regions still remains very sensible and numerical oscillations are still present even with a very fine discretization. Therefore, transient results of the acoustic pressure field show strong numerical noise as can be observed from the contour plot of the acoustic field at time $t = 413$ s presented in Fig. 6.8, where a time step size $\Delta t = 0.5$ s was used. Another common reason for spurious high frequency noise in transient simulations results from the sudden application of the initial acoustic sources [68]. Regarding these issues, to improve the quality of the solution an additional transient simulation was performed using a finer grid and a temporal ramping with cosine shape to gradually ramp up the acoustic sources during the initial time $t \leq 50$ s. As can be observed in the contour plot from Fig. 6.9, a better acoustic solution containing almost no spurious noise is obtained from this simulation.

To completely suppress transient numerical noise in the acoustic pressure field, the interpolated transient acoustic sources are transformed to the frequency domain and a harmonic
6.3. NUMERICAL INVESTIGATION

Figure 6.8. Contour plot of acoustic pressure field at time $t = 413$ s showing strong high frequency numerical noise. Distance scale in meters.

Figure 6.9. Contour plot of acoustic pressure field at time $t = 413$ s computed using a temporal ramping of the initial acoustic sources. Distance scale in meters.

Analysis is carried out for the main frequency component of the problem, $f = 1/T \approx 0.026$ Hz, following the coupling approach presented in Chap. 5. A comparison of the acoustic field from the numerical results with the analytical solution is presented in Fig. 6.10. Good agreement in both the spiral pattern as well as in amplitudes is found in the far-field acoustic pressure except at the center of the computational domain, where the evaluation of the analytical solution resulted in a non-physical behaviour.

Figure 6.11 compares the decay of the acoustic pressure along the positive x-axis between the transient results using the ramping function, the harmonic results and the analytical solution. Apart from the spurious noise in the transient simulation at distances $x < 40$ m within the source region, numerical decays present good correspondence with the far-field analytical values. The discrete evaluation of the acoustic sources can be one reason for the slightly lower amplitudes in the numerical results, corresponding in general to around 95% of the analytical values.
6. VALIDATION OF THE IMPLEMENTATION

(a) Acoustic pressure field at frequency $f = 0.026$ Hz, from hybrid computation following Lighthill’s acoustic analogy

(b) Analytical acoustic pressure field obtained using MAE

Figure 6.10. Comparison of sound pressure field obtained numerically using Lighthill’s acoustic analogy with analytical solution obtained using MAE method. Distance scale in meters.

Figure 6.11. Decay of the acoustic pressure values along the x-axis
7. Application

In the following sections the applicability of the coupled numerical scheme is evaluated by means of 2D and 3D examples of the flow-induced noise due to cylinder geometries. Some of the material presented in this chapter has been published in [29, 32, 60]

7.1. Flow-Induced Noise from a 2D square cylinder

As a first test case for the FE implementation of Lighthill’s acoustic analogy, we compute the flow-induced noise from the flow around a square cylinder with a Reynolds number around 13000. This problem corresponds to a typical tonal aeroacoustic phenomena. Herewith, the acoustic propagation is considered in two dimensions, which eases the investigation of the influence of several time and spatial discretizations. Therefore, for the computation of the acoustic sources, a two-dimensional velocity field is extracted on a slice from a 3D fluid simulation. The hybrid domain used in the computation is defined in Fig. 7.1, in which the square cylinder is located in the centre of the circular acoustical domain. The domain \( \Omega_1 \) matches the fluid region and corresponds to the coupled area, where the scalar acoustic nodal sources are interpolated from the fluid grid to the acoustic grid at \( z = 2D \) with \( D = 20 \text{ mm} \) (see Fig. 7.2). The acoustic simulation is performed in the complete domain, \( \Omega_1 \cup \Omega_2 \).

![Figure 7.1. Schematic of two-dimensional hybrid domain.](image-url)
7.1.1. Fluid Simulation

The 3D fluid simulation, from which the acoustic sources are computed, has been carried out at the Institute of Fluid Mechanics (LSTM) using the CFD code FASTEST-3D [26]. A Large-Eddy-Simulation (LES) based on Smagorinsky model implemented in this code, was used to resolve the flow field. A description of LES can be found in (A.1). The numerical domain for the problem is displayed in Fig. 7.2 and the boundary conditions are listed in Tab. 7.1. The square cylinder is located at \((x, y) = (10D, 5.5D)\). Dimensions of the numerical grid are \(L_x \times L_y \times L_z = 192 \times 96 \times 128\), with a stretching factor of 1.05 downstream from cylinder wall, 1.35 upstream from cylinder wall and 1.18 in spanwise direction from cylinder. A total number of 2.3 million control volumes is used in the computation. For the case, where inlet velocity \(u_0 = 10 \text{ m/s}\), \(D = 20 \text{ mm}\) and fluid air at 25 C we achieve a Reynolds number around 13000. The fluid computation was performed using second order spatial discretisation with Crank-Nicolson time stepping scheme, and the value used for the Smagorinsky constant was \(C_s = 0.065\).

<table>
<thead>
<tr>
<th>Position</th>
<th>Boundary Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(X = 0)</td>
<td>inlet, (u_0 = 10 \text{ m/s})</td>
</tr>
<tr>
<td>(X = 40D)</td>
<td>convective exit</td>
</tr>
<tr>
<td>(Y = 0, Y = 11D)</td>
<td>symmetry</td>
</tr>
<tr>
<td>(Z = 0, Z = 4D)</td>
<td>periodic</td>
</tr>
<tr>
<td>cylinder</td>
<td>no slip</td>
</tr>
</tbody>
</table>

Table 7.1. Boundary conditions for square cylinder simulation

The isosurface plot from Fig. 7.3 highlights the three-dimensional structures in the flow around the square cylinder and in the downstream region. Numerical fluid simulations were validated by comparing with published experimental work from Durão et al. [25]. The comparisons were performed at the centreline distribution for the axial velocity components (see Fig. 7.4) and for the normal Reynolds stress components (Figures 7.5 and 7.6). The turbulent kinetic energy (TKE) characteristics are displayed in Fig. 7.7.
The fact that a wall function was used to resolve the boundary layer in the LES computation can be a reason for discrepancies observed in Figures 7.4-7.6. Such an approach was used in order to avoid doing a DNS at the wall and to keep the computational effort not too high. Furthermore, the fact that in the experimental investigation from Durão the cylinder was mounted inside a channel, can be an additional cause for differences between measurements and simulation results.

7.1.2. Investigation of the interpolated acoustic sources

Initially, before computing the radiated sound field, several characteristics of the interpolated nodal sources are investigated. As presented in Chap. 3, for the computation of the acoustic source term, the numerical integration of the right hand side of (3.16) is performed using the FE method directly on the fine resolution of the fluid domain. Next, the resulting scalar
values, which represent acoustic loads in each finite element node, are interpolated to the acoustic region using the conservative scheme. This approach guarantees a higher accuracy in the computation of the acoustic sources, as demonstrated in Sec. 5.2.3.

Several characteristic points within the turbulent region have been chosen to investigate the properties of the interpolated acoustic sources. Herewith, we have analysed nodal sources interpolated on the numerical grid depicted in Fig. 7.11 (a), having an element size, $h = 10 \text{ mm}$. The exact locations of the points under investigation are depicted in Fig. 7.8. Initially, analyses are carried out at two different characteristic points, $p_5 = (x, y) = (0.1 \text{ m}, 0.01 \text{ m})$ and $p_6 = (x, y) = (0.1 \text{ m}, 0.0 \text{ m})$, located at $5 \times D$ in the downstream direction along the fringe of the cylinder and along the x-axis, respectively. Figure 7.9 presents the results for these two points. Their characteristics confirm the measurements from a turbulent flow around a square cylinder [2]. For the point located along the fringe, among other frequency modes, we find the 65 Hz component, which is the actual main component for this tonal
problem. On the other hand, for the point located exactly along the x-axis we find, as main frequency component, a value which is twice higher than the main frequency of 65 Hz. This fact can be associated with the combination at the center line in the downstream flow of the upper and lower main vortices, each having a frequency of 65 Hz. Significant differences in the source values are found at a point \( p_7 = (x, y) = (0.2 \text{ m}, 0.06 \text{ m}) \) located outside the central region of the wake, as presented in Fig. 7.10. In this case several other frequency components are found, but still the first two dominant components are present.

Figure 7.8. Schematic representation of source region depicting the square cylinder with selected points (distance in mm).

Figure 7.9. Acoustic source values in time and frequency spectrum at points \( p_5 \) and \( p_6 \) located at \( 5 \times D \) in the downstream direction along the fringe of the cylinder and along the x-axis respectively.

### 7.1.3. Acoustic Propagation

In the following, transient and harmonic investigations concerning the flow-induced noise propagation for the 2D square cylinder test case are presented. The acoustic field is computed in the complete domain \( \Omega_1 \cup \Omega_2 \), as depicted in Fig. 7.1. Dimensions of the coupled region are given in Fig 7.8. These dimensions correspond to the size of the numerical grid in the fluid computation, so they are the same in transient and harmonic analyses.
Figure 7.10. Acoustic source values in time and frequency domain at point \( p_7 = (x, y) = (0.2 \text{ m}, 0.06 \text{ m}) \) located outside the central region of the wake.

**Transient Analysis**

Several spatial discretizations have been used for the computation of the acoustic propagation in time domain. Fig. 7.11 depicts two different discretizations of the coupled region of the acoustic domain from which results are evaluated in this section. The mapped mesh has an element size \( h = 10 \text{ mm} \). In the mesh from Fig. 7.11 (b), the element size near the cylinder is just as fine as the cell size used in the fluid computation \( (h = 1 \text{ mm}) \) and it coarsens until reaching an element size \( h = 10 \text{ mm} \).

Figure 7.11. Two different acoustic coupled regions used for the evaluation in transient computations

Fig. 7.12 corresponds to a closeup of a characteristic grid as used in transient computations, depicting the progressive coarsening of the elements from the coupled region towards the propagation region. In the transient case, the radius of the complete acoustic domain has been chosen to be \( r = 40 \text{ m} \). Hereby, it is possible to investigate the acoustic solution during several periods, without any influence from spurious reflections of waves impinging not or-
7.1. FLOW-INDUCED NOISE FROM A 2D SQUARE CYLINDER

Through on the acoustic absorbing boundary. In this way, only discretization parameters should affect the acoustic propagation and results can be more directly compared with those obtained later from harmonic computations. The discretization of the farthermost elements in the propagation region, \( \Omega_2 \), located near the absorbing boundary, corresponds to about 7 elements per acoustic wave length. The total number of elements in the mesh from Fig. 7.12 is \( N_{\text{elems}} = 41584 \). Results presented in this section have been obtained using quadrilateral finite elements of second order.

Figure 7.12. Section of acoustic grid depicting the near-field region where sources are interpolated, \( \Omega_1 \), and the far-field region, \( \Omega_2 \).

Figure 7.13 presents the results at a point located 5 m away from the cylinder using the two different discretizations in the coupled region from Fig. 7.11. Simulations performed with the mapped mesh in the coupled region, produce a slight better quality in the solution when compared with the results obtained using the progressively coarsened mesh from 7.11(b). More importantly, it can be noticed that amplitudes are not affected by the different mesh discretizations, which can be ascribed to the use of a conservative interpolation scheme.

Additional to the mesh sensitivity aspect, simulations using different acoustic time step sizes have been performed in order to estimate an appropriate time discretization for which accurate results can be expected. In the following, results from two different time discretizations are presented.

In the three-dimensional computation of the turbulent flow field, the time step size was chosen to be \( \Delta t_f = 10 \mu s \). As for the acoustic computations, the purpose is to use a time step which correctly discretizes the periods of the main frequency components expected in the wave propagation. At the same time, it is important to try to keep it coarse enough so that the usage of computational resources and the data exchange between fluid and acoustics are kept minimal.

In FE computations using a Newmark method as time discretization scheme for hyperbolic problems, a general choice consists in employing 20 time steps per period. For the specific problem of the square cylinder the expected main frequency component is 65 Hz. Hence, choosing an acoustic time step size of \( \Delta t_a = 0.5 \text{ ms} \) would correspond to about 30 time steps per period at this main frequency. Theoretically such time step size, where \( \Delta t_a/\Delta t_f = 50 \),
would be expected to provide good results for the acoustic computation in the far field. However, as demonstrated with the following results, the quality of the solution decreases significantly even for finer time step sizes, which can be due to the presence of significantly higher frequency components up to around 400 Hz in the acoustic nodal loads, as presented in 7.10.

In Fig. 7.14 acoustic pressure values at a point \((x, y) = (0 \text{ m}, 5.0 \text{ m})\) are presented using two different time step sizes, \(\Delta t_a = 0.2 \text{ ms}\) and \(\Delta t_a = 0.1 \text{ ms}\). In the solution using a time step size 20 times coarser than the one used on the fluid computation \(\Delta t_a = 0.2 \text{ ms}\), significant numerical noise is present. In this case, the time step size does not suffice for an accurate discretization of signals due to the higher frequency components present in the acoustic nodal sources. For example, for 400 Hz, such discretization results in just 12.5 time steps per period. On the other hand, the computation using a twice finer time step size \((\Delta t_a = 0.1 \text{ ms})\) shows a much smoother behaviour. Nevertheless, it is important to notice that no significant differences are found in the amplitudes of the acoustic solution from both computations.

Contour plots of the acoustic pressure in the near and the far field using the spatial and time discretization parameters, which produced results showing the best quality, are presented in Fig. 7.15. In the near field a strong radiation in the upstream and in the downstream is observed. Here it is important to mention that although this does not seem to correspond to the expected dipole radiation in the cross-flow direction, it is from the reciprocal oscillation of these two radiation patterns that the far field dipole characteristic originates. The contour far field pressure from Fig. 7.15 (b) at time \(t = 140 \text{ ms}\) helps to understand this sound generation mechanism. Near the square cylinder the acoustic field shows the strong radiation in the upstream and downstream direction whereas in the far field the expected dipole radiation for this problem dominates. To corroborate this fact, in the following section we present results from corresponding harmonic computations for the same problem, performed following the simulation scheme in the frequency domain as introduced.

Figure 7.13. Acoustic pressure values in time for a point \((x, y) = (0 \text{ m}, 5 \text{ m})\) located 5 m away along the y-axis.
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Figure 7.14. Acoustic pressure results in point \((x, y) = (0 \text{ m}, 5 \text{ m})\) using two different time step sizes.

Harmonic Analysis

In order to carry out a harmonic acoustic simulation, first of all we store the interpolated (coarse) acoustic nodal sources in the time domain. From the resulting dataset a Fourier transformation is performed producing the corresponding acoustic nodal sources in frequency domain. There are several advantages for computing the acoustic field in frequency domain. First, only the relevant frequency components are computed, avoiding in this way the high frequency numerical noise in transient results. This improves the quality of the solution of the acoustic far field. Furthermore, in harmonic analysis our FE implementation offers the
perfectly matched layer method (PML) as boundary treatment, which practically avoids all reflections at the acoustic boundary. This allows the usage of a smaller acoustic domain without the solution being contaminated, which can represent in the end a significant reduction in computational times.

The numerical domain used for the harmonic computations is depicted in Fig. 7.16. The dimensions of the complete domain without considering the PML region are \( L_x \times L_y = 4.4\,\text{m} \times 3.3\,\text{m} \) with the cylinder located at \((x, y) = (2\,\text{m}, 0\,\text{m})\). The PML region and its discretization is also shown in Fig. 7.16, to emphasize that a small and coarse mesh with 612 elements suffices for this region to obtain accurate results. For direct comparison with the transient results, an initial harmonic computation has been performed using a numerical domain with the same discretization in the coupled region as in the transient mesh from Fig. 7.11 (a). The total number of elements in this case is \( N_{\text{elems}} = 8488 \), including the PML region. Figure 7.17 presents the amplitude and phase values of the acoustic pressure field in the whole acoustic domain, computed for the main frequency component, \( f = 65\,\text{Hz} \). Similar to the transient results, we find the expected dipole sound radiation in the acoustic field. Directivity patterns obtained from the transient and harmonic analyses at a radius \( r = 1\,\text{m} \) away from the cylinder, normalized around the harmonic values are compared in Fig. 7.18. In this comparison, only small numerical differences are noticeable, which can be due to the evaluation of the amplitudes from the transient pressure signals. Moreover, these results have demonstrated that the excellent performance of the PML allows to use a small numerical domain, which in all directions encompasses just a fraction of the acoustic wave length for the main frequency component (about \( \lambda/3 \) where for \( f = 65\,\text{Hz} \), \( \lambda = 5.27\,\text{m} \)).

![Figure 7.16. Schematic of numerical domain used in harmonic computations. Distance scale in meters.](image)

In the following, additional harmonic acoustic computations are performed using several spatial discretizations to evaluate the robustness of the conservative interpolation scheme and its influence on the results. Hereby, we interpolate the computed acoustic nodal sources from the fine flow grid containing 91492 cell points, to four different acoustic meshes with a total number of nodes in the coupled region ranging from 10307 to 590. As previously men-
7.1. FLOW-INDUCED NOISE FROM A 2D SQUARE CYLINDER

Figure 7.17. Contour visualization of harmonic results for frequency $f = 65$ Hz

Figure 7.18. Comparison of directivity patterns at radius $r = 1.0$ m between harmonic and transient computations. Amplitudes normalized around harmonic results

mentioned, all computations are performed using second order quadrilateral elements (8 finite element nodes per element). Fig. 7.19 presents the four different discretization of the acoustic coupled region used for the investigation, where Fig. 7.19 (b) corresponds to the mesh
previously used for comparison with the transient results. The finest numerical grid corresponds to a ratio between the acoustic and fluid discretization sizes, $h_a/h_f \approx 5$. Fig. 7.19d) is an extremely coarse mesh containing 590 nodes, which corresponds to a ratio $h_a/h_f \approx 20$ in the region directly around the cylinder and $h_a/h_f \approx 60$ at the outermost regions. The aim for using this latter numerical grid is mainly to estimate the limits of the interpolation for achieving acceptable results. Table 7.2, presents the discretization ratios for the different meshes with respect to the wavelength $\lambda_{min}$ obtained for the 400 Hz component found in the acoustic nodal sources (see Fig. 7.10).

![Figure 7.19](image)

(a) 10307 nodes  
(b) 5480 nodes  
(c) 1616 nodes  
(d) 590 nodes

Figure 7.19. Different spatial discretizations of acoustic coupled region as used for sensitivity analysis.

<table>
<thead>
<tr>
<th>Total number of nodes</th>
<th>$\lambda_{min}/h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10307</td>
<td>187</td>
</tr>
<tr>
<td>5480</td>
<td>84</td>
</tr>
<tr>
<td>1616</td>
<td>84-22</td>
</tr>
<tr>
<td>590</td>
<td>40-20</td>
</tr>
</tbody>
</table>

Table 7.2. Discretization ratios for different acoustic meshes with respect to $\lambda_{min} = 0.85$ m.

First of all, we evaluate the performance of the different numerical grids by applying two 65 Hz sinusoidal sources with opposite phases, to simulate a dipole radiation. These artificial sources were applied on the faces of the cylinder parallel to the x-axis (upper and lower faces). The directivity plot in Fig. 7.20 compares the amplitudes, 1 m away from cylinder, from the results using the four numerical grids presented in Fig. 7.19. Although small, some discrepancies are noticed in the results obtained using the coarsest mesh. Besides numerical errors, differences in this latter case are mainly due to the slightly different locations of the monitoring nodes used to evaluate the amplitudes. All in all, amplitudes from the four different computations show good agreement. The practically mesh independent results allow to employ the numerical grids to exclusively evaluate the influence of the conservative interpolation scheme in the flow-induced noise computation for the 2D square cylinder case.
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Fig. 7.21 presents the results from the flow-induced noise coupled computations for all numerical grids. Results are normalized around the values obtained using the grid with 1616 nodes. Except for the coarsest computation, where amplitudes are significantly smaller, all other results are in very good agreement with each other. Despite the coarse discretization from the grid from Fig. 7.19 (\( h_a/h_f \) being around 10 near the cylinder and 50 at outer regions), results still show very similar amplitudes in comparison to the results from the finest numerical grid. For this flow-induced noise computation, this fact demonstrates that the conservative interpolation scheme makes the implementation a robust and efficient approach, producing good results even for very coarse acoustic resolutions in comparison to the CFD discretization.

**7.2. Flow-Induced Noise from 3D wall-mounted cylinders**

In the present section the implementation is used to perform 3D computations of the flow-induced noise from wall-mounted cylinders using two different geometry profiles, a standard square cylinder geometry and a cylinder with elliptic shape in the downstream direction. The general set-up for the coupled simulation is depicted in Fig. 7.22. \( \Omega_1 \) denotes the area...
Figure 7.21. Directivity patterns from flow-induced noise harmonic computation using different grids. Normalized amplitudes 1 m away from cylinder.

where the flow field is computed and where the acoustic sources are interpolated from the fluid grid to the acoustic grid. The region encompassing $\Omega_1$ and $\Omega_2$ corresponds to the region where the acoustic propagation is computed.

The two different geometry profiles evaluated in the numerical investigations are depicted in Fig. 7.23. Based on the crossflow side length of $D = 20$ mm, the Reynolds number for both fluid calculations results in $Re \approx 13000$.

7.2.1. Fluid computation of wall-mounted square cylinder

In the flow-induced noise investigations for the square cylinder profile depicted in Fig. 7.23(a) fluid results are used which have been computed with the in-house CFD code FASTEST-3D[26], developed at LSTM and with the commercial CFD code ANSYS-CFX[18]. The numerical domain is described in Fig. 7.24, where $D = 20$ mm. The Reynolds number $Re$ based on the side length $D$ for an inflow velocity $u_x = 10$ m/s is about 13000.

The boundary conditions used in the fluid computation with respect to the configuration from Fig. 7.24 are described in Table 7.3.

In the simulation performed with the code FASTEST-3D, LES (Large Eddy Simulation) using Smagorinsky model was applied to simulate the transient flow field with complete resolution of the boundary layer. A description of LES methodology can be found in (A.1).
Simulations were performed on a SGI-ALTIX system with 16 processors using a numerical domain containing approximately 3.1 million volume cells and a time step size of $\Delta t_f = 10 \, \mu s$.

For the simulation of the flow using the code ANSYS-CFX a turbulence modeling approach
Figure 7.24. Numerical domain used for fluid computations depicting dimensions. D = 20 mm.

Based on SAS (Scale Adaptive Simulation) was employed. A description of this method is introduced in (A.2). The SAS approach allows to use coarser grids than those used in LES computations. Therefore approximately 1.1 million cells were used in this case, which resulted in a shorter computational time and less memory usage. Regarding time discretization, a time step size of $\Delta t_f = 20 \mu s$ was used in this simulation.

Figure 7.25 shows the transient flow field obtained from both computations. In the FASTEST-3D results the coherent structures are shown as iso-surface, for the values of $\lambda_{sym} = -1000$, where $\lambda_{sym}$ is the eigenvalue of the symmetric tensor $S^2 + \Omega^2$, $S$ and $\Omega$ being the symmetric and antisymmetric parts of $\nabla \vec{u}$. In the results from the SAS computation isosurfaces of $\Omega^2 - S^2 = 100000 \text{s}^{-2}$ colored with the eddy viscosity are used for the visualization of the turbulent structures, with $\Omega$ representing here the vorticity and $S$ the strain rate.

7.2.2. Fluid computation of wall-mounted cylinder with elliptic profile

As second geometry profile for the 3D simulations, a wall-mounted square cylinder with elliptic shape in the downstream direction as depicted in Fig. 7.23(b) has been used. For this case, despite the same inflow velocity $u_x = 10 \text{ m/s}$, measurements [50] predict significantly higher SPL values than those obtained using the square cylinder profile.

The configuration of the numerical domain used for the flow computation as well as boundary conditions are analogous to the ones used for the square cylinder case, presented in Fig. 7.24 and in Table 7.3, respectively. The dimensions of the cylinder are identical to the square cylinder, except for the extension with the half ellipse form in the downstream direction, as depicted in Fig. 7.23. Similarly as in the results for the square cylinder, Fig. 7.26 visualizes the turbulent structures from the SAS computation for this configuration by means of the isosurfaces of $\Omega^2 - S^2 = 100000 \text{s}^{-2}$ colored with the eddy viscosity, with $\Omega$ denoting the vorticity and $S$ the strain rate.
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7.2.3. Acoustic Computations

Square wall-mounted cylinder

For the square cylinder profile, acoustic simulations have been computed in time and frequency domain, using the results from the fluid computations presented in Sec. 7.2.1 performed with the CFD codes FASTEST-3D and ANSYS-CFX.

Directly coupled transient computations following the coupling approach from Sec. 5.1 were performed, using an acoustic grid with radius $r = 3$ m, consisting of 3683670 tetrahedral elements with quadratic basis functions which resulted in 4986115 finite element nodes. The
coupled region has been discretized with 135173 nodes. Since the conservative interpolation scheme from MpCCI is restricted to volume interpolation of linear 3D elements, the acoustic sources have been interpolated on the 17925 corner (linear) nodes of the quadratic tetrahedral elements.

Figure 7.27 presents the numerical grid in which first order absorbing boundary conditions, as introduced in Chap. 4, have been applied on the half sphere boundary using 51896 triangular surface elements. A closer view of the coupled region depicting the edges of the tetrahedral elements on the bottom plane and on the cylinder surface is shown in Fig. 7.28.

Fig. 7.29 represents isosurfaces of the near-field acoustic pressure obtained from coupled computations using both CFD results for the evaluation of the acoustic sources. Some differences are noticed mainly in the negative isosurfaces above the square cylinder and in the turbulent structures down the wake. The computation performed using the acoustic sources obtained from the ANSYS-CFX simulation produces longer turbulent structures in the wake behind the cylinder and negative isosurface values located closer to the top of the cylinder in comparison to the one using the FASTEST-3D flow field. On the other hand, since the LES simulation was performed using a twice shorter time step (\(\Delta t_{LES} = 100 \mu s\)) than that used for the SAS (\(\Delta t_{SAS} = 200 \mu s\)), more numerical noise is present in the acoustic field using sources from this latter computation. Based on the evaluation of the time discretization performed for the 2D flow-induced noise simulation from Sec. 7.1.3 the time step size for the 3D acoustic computation was chosen to be \(\Delta t_a = 200 \mu s\). For these low frequency tonal noise problems, further quantitative analysis of 3D transient results is restricted due to the large wavelength expected for these specific problems (\(\lambda \approx 6.2 m\) for \(f = 55 Hz\)) and the high computational cost due to the very large number of elements required to cover higher frequency signals present in the acoustic sources. Therefore, for more practical analysis of this type of aeolian tone problems, in the following sections harmonic analyses for the main frequencies components are performed employing the coupling approach for the simulation of the acoustic propagation in frequency domain as introduced in Sec. 5.1.
7.2. FLOW-INDUCED NOISE FROM 3D WALL-MOUNTED CYLINDERS

Similarly as in the 2D test case, in order to perform an acoustic computation in frequency domain, first of all we store the interpolated (coarse) acoustic nodal sources in the time domain. From the resulting dataset a Fourier transformation is performed producing the corresponding acoustic nodal sources in frequency domain. This latter set of nodal files is then used as input for the harmonic computation. As previously mentioned, the main advantage of computing in frequency domain is that the acoustic field is solved only for relevant frequency components, avoiding in this way high frequency transient numerical noise. In 3D computations we can also use the PML method as introduced in Chap. 4 which allows smaller acoustic domains producing almost no spurious reflections. This aspect
7. APPLICATION

(a) View point used for isosurface visualization of acoustic pressure

(b) using acoustic sources from FASTEST-3D LES simulation

(c) using acoustic sources from CFX-ANSYS SAS simulation

Figure 7.29. Isosurfaces of acoustic field for values $\pm 2\text{ Pa}$ at time $t=10\text{ ms}$ obtained using fluid results from FASTEST-3D and CFX-ANSYS simulations.

significantly reduces the computational time.

In Fig. 7.30 the configuration of the simulation domain for the harmonic computation showing the monitoring points used for directivity analysis is presented. Due to the rectangular configuration of the acoustic domain, the discretization could be performed using hexahedral elements with quadratic basis functions which increases the accuracy of the computation. A spatial discretization has been chosen based on the observations made from the analysis for the 2D square cylinder case previously presented in this chapter, in which a ratio of $h_a/h_f = 10$ in the region directly around the cylinder sufficed to produce mesh independent results. Following this discretization criteria three different mesh resolutions have been tested with the number of linear nodes in the coupled region ranging from 8092 to 24177. Figure 7.31 presents results for the amplitudes at a radius $r = 1\text{ m}$ using the three different spatial discretizations. While the finest mesh produces a smoother directivity pattern, for the other grids only minimal differences are observed in the amplitudes which results in mesh independent results even for the case using 8092 acoustic nodal sources.

In Fig. 7.32 isosurface of the acoustic pressure for $p' = 5\text{ mPa}$ is shown for the main frequency component present in the computation, $f = 55\text{ Hz}$. The isosurface is colored with its corresponding phase values. Opposite phases can be observed around the cylinder from the general $180^\circ$ phase shift present on the isosurface. These results represent the
7.2. FLOW-INDUCED NOISE FROM 3D WALL-MOUNTED CYLINDERS

Figure 7.30. Schematic drawing of the acoustic domain used for the harmonic computation showing points used for directivity analysis. Distance scale in m.

Figure 7.31. Comparison of amplitudes at radius $r = 1$ m on the crossflow $yz$-plane using three different spatial discretizations in the acoustic coupled region.

characteristic dipole-like radiation expected for this tonal noise problem. Additionally, Fig. 7.33 presents several isosurfaces of the acoustic pressure clipped at $yz$-plane. Values of the isosurfaces range from 5 mPa to 54 mPa, the outermost corresponding to about 48 dB.
Figure 7.32. Isosurface of the acoustic pressure, $p' = 5\, \text{mPa}$, colored with phase values, obtained for the main frequency component, $f = 55\, \text{Hz}$. Dotted region represents PML.

Figure 7.33. Isosurface of acoustic pressure for wall-mounted square cylinder at $f = 55\, \text{Hz}$, clipped at $yz$-plane. Dotted region represents PML.
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Wall-mounted cylinder with elliptic profile

Similarly as for the square cylinder profile, harmonic acoustic computations have been performed for the case using the profile from Fig. 7.23(b) and its SPL values and directivity pattern have been analysed. The acoustic sources have been evaluated using the velocity components from the CFD results computed with CFX-ANSYS using SAS turbulence modeling presented in Fig. 7.26. The acoustic computation has been performed using a similar configuration for the computational domain as the one depicted in Fig. 7.30, except for the cylinder geometry. Figure 7.34 presents a close-up of the acoustic grid showing the corresponding elliptic profile employed in this case. Regarding the acoustic coupled region, the fine acoustic sources from the fluid resolution have been interpolated on 21160 hexahedral corner nodes.

![Figure 7.34. Close up of acoustic grid for the case of the elliptic profile used for harmonic computations](image)

In the acoustic computations, the main frequency component found for this problem was $f = 39\, \text{Hz}$. For this frequency value, isosurfaces of the acoustic pressure clipped at the yz-plane are presented in Fig. 7.35. Values of the isosurfaces range from 22 mPa to 100 mPa and the outermost isosurface corresponds in this case to 61 dB and, similarly as in the case using the square cylinder profile, the directivity pattern for this main frequency value results in a dipole-like acoustic field.

Evaluation of 3D results

In the following, acoustic results from the harmonic computations for the two wall-mounted cylinder profiles are evaluated and compared with experimental results from [50].

Table 7.4 presents the main frequency components found in the computations and the frequency values found in the measurements carried out in the anechoic wind tunnel at our department [50]. The higher values obtained in the simulations for both cylinder profiles first appear in the CFD results, and are then carried over to the acoustic simulations. At the moment this is a common issue to the CFD computations of wall-mounted cylinders using symmetry boundary conditions, as employed in this work.

In Table 7.5 a comparison of the SPL values between the numerical and the experimental results is presented for both cylinder profiles. For both geometries higher SPL values are
obtained in the simulation. At this point it is important to note that, due to practical reasons, in the experimental case the cylinders were mounted at the center of a wall with a crossflow length of $W_{\text{exp}} = 0.66 \text{ m}$, whereas on the simulation the wall covers the complete domain width $W_{\text{sim}} = 2.22 \text{ m}$. Reflection of the acoustic waves on the larger wall from the simulation domain is one of the reasons for the higher SPL values at the monitoring points compared. Therefore, the relative difference of the SPL values from both profiles can give a better estimate of the agreement between measurements and simulations. In Table 7.5 this difference is observed to be 16 dB in the experiments and 14 dB in the numerical computations.

<table>
<thead>
<tr>
<th>SPL value</th>
<th>Square Profile</th>
<th>Elliptic Profile</th>
<th>Relative SPL difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulation</td>
<td>47 dB</td>
<td>61 dB</td>
<td>14 dB</td>
</tr>
<tr>
<td>Measurements</td>
<td>44 dB</td>
<td>60 dB</td>
<td>16 dB</td>
</tr>
</tbody>
</table>

Table 7.5. Numerical and experimental SPL values of the flow-induced noise for the two cylinder profiles evaluated at $0^\circ$ on the yz-plane

Finally, directivity plots of the SPL levels at a radius $r = 1 \text{ m}$ away of the flow-induced
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noise for the two wall-mounted cylinders investigated in this section are presented in Fig. 7.36. In this plot it can be observed the significantly higher amplitudes at all monitoring points obtained in the flow-induced noise computation for the cylinder with elliptic profile.

Figure 7.36. Directivity patterns for two cylinder profiles at radius $r = 1.0$ m on the crossflow yz-plane
8. Conclusions and outlook

In this thesis a FE formulation for the hybrid computation of flow-induced noise using Lighthill’s acoustic analogy has been presented, in which acoustic sources are computed by means of the FE method from unsteady flow fields. The capabilities of the implementation were investigated by means of a validation example and two- and three-dimensional test cases of the flow-induced noise generated by cylinder geometries.

For the validation of the numerical implementation, simulation results have been compared against the analytical solution of the flow-induced noise generated by two co-rotating vortices. A two dimensional test case computed in time and frequency domain is used to evaluate the performance of the absorbing boundary and the robustness of the acoustic results regarding the conservative interpolation of the quantities. Additionally, three-dimensional simulations of the noise generated by wall-mounted cylinders with two different geometry profiles have shown a good agreement in comparison with experiments using a similar configuration and the same inflow velocity.

Concerning the validation of the numerical implementation by means of the co-rotating vortices test case, the coupled numerical simulations following the original Lighthill’s acoustic analogy formulation reproduced correctly the frequency value and the spiral pattern observed in the analytical far-field. Additionally, a good quantitative agreement was obtained in the amplitudes for the far-field acoustic pressure between the numerical results and the values from the analytical solution.

The two-dimensional example was used to estimate appropriate parameters in order to perform numerical simulations in an efficient way while preserving a good accuracy of the acoustic results. The influence of temporal and spatial discretizations was evaluated. Additionally the good performance of the Perfectly Matched Layer technique (PML) when truncating the acoustic domain even at fractions of the main wave length was demonstrated. Based on the experience obtained from these 2D investigations, computational parameters were estimated and considered in the preparation of the numerical set-up for the full 3D test cases, which allowed a more efficient usage of the computational resources. In the following, some of the general aspects observed from 2D and 3D numerical investigations are summarized:

- The conservative interpolation scheme used to transfer the acoustic sources as nodal loads to the acoustic solver provides a robust way to reduce the computational cost by allowing the usage of acoustic meshes which are several times coarser than the grids from the fluid computation. In the 2D and 3D test cases evaluated in this work it was observed that with elements using quadratic basis functions, a discretization ratio
between acoustic and fluid cell sizes of as much as \( h_a / h_f = 10 \) on the cylinder surfaces and up to \( h_a / h_f \approx 50 \) at the outermost locations of the coupled region can suffice to produce good acoustic results.

- In the 2D transient investigations, the choice of an acoustic time step size 10 times coarser than the one used on the fluid computation still provided a good quality of the acoustic pressure values. For a twice coarser acoustic time step size \( (\Delta t_a / \Delta t_f = 20) \) it was observed that although amplitudes of acoustic pressure for the main frequency value remained unchanged, high frequency numerical noise appeared which significantly polluted the time evolution of the acoustic pressure values. On the other hand, this is not so relevant for harmonic analyses, since the acoustic field is solved only at specific frequencies. Therefore, in this latter case, transient datasets of the acoustic nodal sources having a coarser temporal resolution can be used for the Fourier transformation, provided the periods for the corresponding frequencies of interest are appropriately discretized (e.g. 25 sampling points per period). Depending on the specific problems and with preliminary knowledge of the frequency spectra deduced from the CFD results and/or from measurements, the use of coarser datasets can result in shorter times for the interpolation of the transient nodal sources and significant saving of storage space.

- Through the 2D harmonic test case it was verified that in the simulation for the main frequency component, the PML allowed the truncation of the acoustic domain at a distance around \( \lambda / 3 \) (\( \lambda \) denoting the acoustic wave-length) from the cylinder and still produced accurate results in comparison to those from the transient simulation. Furthermore, in this case the PML width was chosen to be just around \( \lambda / 20 \) and discretized with only 3 quadratic elements in the radial direction.

- Additionally, 3D harmonic simulations of the noise generated by wall-mounted cylinders demonstrated that the numerical scheme in the frequency domain can effectively determine the acoustic radiation patterns and their amplitudes for different geometries quite well, with relative ease and without any additional treatment as would be required in numerical schemes based on integral formulations.

Besides the aforementioned aspects, results from 3D transient computations for the wall-mounted cylinder with square profile confirmed the need for higher order absorbing boundary conditions in the time domain. This topic still remains challenging and can be the focus of future research. Currently, with the implemented first order absorbing boundary conditions, the domain sizes for transient computations have to be chosen large enough to reduce the influence on the acoustic solution of the partial reflection of waves impinging non-orthogonally on the domain boundary. In this respect, besides better absorbing boundary conditions, it may also be advantageous to extend the current FE implementation with a non-matching grid functionality to make 3D transient acoustic computations more computationally efficient. The greater flexibility for the mesh generation provided by such a feature would allow to use a fine spatial discretization near the geometries in the coupled region and a faster transition to much coarser elements towards the propagation region in which only the main frequency spectra for the specific problem needs to be covered.
Additionally, since the nature of the finite element method allows solving problems involving arbitrary geometries, the implementation presented in this work could also be used in conjunction with a fluid-structure scheme to provide a tool for solving transient fluid-structure-acoustics interaction problems. Once again, in such a case a non-matching grid capability would play an important role in making the simulation of 3D practical applications feasible.

Finally, for further development of the flow-induced noise methodology based on a volume discretization method, as implemented in this work, a viable and promising alternative to the standard FEM would be a high-order finite element technique based on higher degree piecewise polynomial basis functions like the spectral element method (SEM). While keeping the geometric flexibility of finite elements this type of method also exhibits several favourable computational properties, like naturally diagonal mass matrices which increase the computational efficiency, permitting to solve problems involving higher frequencies with less computational resources.
A. Turbulence modelling

In the following sections, a brief description of the LES and SAS approaches for the simulation of unsteady incompressible flows is presented for completion. Turbulence modelling is an extensive field on its own. Therefore, for a detailed description of these methodologies, more comprehensive bibliographical references are recommended, like [99, 15, 74] on which the contents of this chapter are based.

A.1. LES approach

Large Eddy Simulation is classified under the group of averaged or filtered simulations. The main concept behind this type of turbulence modelling approach consists in applying an averaging or filtering procedure to the Navier-Stokes equations, yielding new equations for a variable that is smoother than the original from the Navier-Stokes equations because the filtering procedure removes the small scales or high frequencies of the solution.

The smallest scales are parametrized via the use of a statistical model referred as a subgrid scale model. The removal of the highest frequencies is carried out considering the following assumptions [99]

- Large scales of the flow
  - characterize the flow
  - are sensitive to boundary conditions and so are anisotropic
  - are assumed to contain around 80% to 90% of the total kinetic energy of the flow

- Small (subgrid) scales of the flow
  - have a universal character and are isotropic.
  - are only responsible for the viscous generation
  - are weak, so they only contain a few percent of the total kinetic energy

Applying a convolution filter to the Navier-Stokes equations for incompressible flows, and taking into account all the properties of the filter, one obtains the governing equations for LES

\[
\frac{\partial \tilde{u}_i}{\partial t} + \frac{\partial \tilde{u}_i u_j}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \tilde{p}}{\partial x_i} + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} \quad (A.1)
\]

\[
\frac{\partial \tilde{u}_j}{\partial x_j} = 0 \ , \quad (A.2)
\]
where the overline represents filtered terms. This set of equations can not be directly used because the nonlinear term on the left-hand-side of (A.1) must first be decomposed as a function of the only acceptable variables, which are $\overline{u_i}$ and $u'_i$. For the decomposition of the nonlinear term, Leonard’s or Germano’s decomposition approaches can be used [99].

A.2. SAS approach

The recent SAS approach can be classified under the group of hybrid RANS-LES methods. This type of methods blends statistical approaches with LES, in order to yield enhanced predictions of both turbulence statistics and unsteady flow dynamics at a fraction of the cost of traditional LES. Such methods can be considered as a form of very large eddy simulation with subgrid stresses that are designed to reach RANS levels in certain limits of coarse or highly stretched meshes. In general in hybrid RANS-LES approaches a potential problem can arise when flow is transported across regions of varying mesh resolution. For separated flows around simple geometries, suitable mesh generation can eliminate this problem by maintaining uniformly high-aspect-ratio cells in the near wall layer. However, for realistic geometries involving abrupt curvatures, it can become difficult to satisfy such stringent meshing constraints [99]. The SAS approach is a recent attempt to overcome this problem proposed by Menter et al. [74] who have investigated the development of improved uRANS methods which can provide a LES-like behaviour in detached flow regions. Its concept is based on the introduction of the von Karman length-scale into the turbulence scale equation. This introduces a dynamical behaviour into the turbulence model. Instead of producing the large-scale unsteadiness, typically observed in URANS simulations, the model adjusts to the already resolved scales in a dynamic way and allows the development of a turbulent spectrum in the detached regions. It therefore behaves in a way much similar to a DES model, but without the explicit grid dependence in the RANS regime [74].
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Kurzfassung


Für die Auswertung der akustischen Quellterme werden instationäre Strömungsfelder verwendet, die aus LES (Large Eddy Simulation) und SAS (Scale Adaptive Simulation) Berechnungen erhalten werden.

Für die Validierung der numerischen Implementierung, wird der strömungsinduzierte Lärm, der durch ein rotierendes Wirbelpaar verursacht wird, berechnet. Die numerischen akustischen Ergebnisse werden mit der entsprechenden analytischen Lösung verglichen.


Schlagwörter: Computational Aeroacoustics, Finite-Elemente-Methode, Computational Fluid Dynamics, Akustische Analogie, absorbierende Randbedingungen, Perfectly Matched Layer
1. Einleitung

1.1. Problemstellung


Außer den experimentellen Methoden gibt es mehrere Ansätze, die die physikalischen Zusammenhänge der Lärmerzeugung beschreiben und mit Hilfe von numerischen Metho-
1. EINLEITUNG


Ursprünglich wurden die meisten Methoden der CAA für Probleme in der Luftfahrtindustrie entwickelt, wobei die Freifeldabstrahlung in große akustische Ausbreitungsgebiete immer noch von großen Interesse ist. Im Gegensatz dazu entstehen in Industriebranchen, die sich erst vor kurzem für die aeroakustische Forschung interessiert haben, andere Bedürfnisse, die durch die neuen Methoden der CAA erfüllt werden müssen. In diesen Fällen sind numerische Methoden für die Lösung von Problemen mit niedriger Mach-Zahl nötig, die allgemeinere Geometrien und nicht kompakte akustische Quellen mit einbeziehen. Wenn darüber hinaus abgeschlossene aeroakustische Probleme bzw. Effekte mit Struktur-Akustik-Interaktion betrachtet werden müssen, sind andere numerische Methoden als die traditionellen aeroakustischen Verfahren erforderlich, um die Interaktion zwischen der festen Oberfläche und dem strömungsinduzierten Lärm direkt in der akustischen Simulation zu berücksichtigen.

Insofern sind aeroakustische Verfahren Thema dieser Arbeit, wobei die numerische Simulation von strömungsinduziertem Lärm mit Lighthills akustischer Analogie im Mittelpunkt steht.

1.2. Stand der Technik

1.2.1. Hintergrund

breitung mit den Quelltermen als Eingangsgröße durchgeführt. Eine wichtige Annahme in dieser Art der hybrid Methode ist, dass im Allgemeinen keine Rückwirkung aus dem akustischen Feld zu den turbulenten Strömungen berücksichtigt wird.


1.2.2. Aktuelle CAA Verfahren

Wie oben erwähnt, werden bei der gewählten Formulierung der akustischen Analogie die Strömungsrechnung und die akustische Ausbreitung in unterschiedlichen Berechnungen durchgeführt. Aus diesem Grund können die Beschreibungen der aktuellen Trends sowie der Fortschritte bei den numerischen Methoden der CAA in zwei Hauptkategorien unterteilt werden. Zum Einen in die Beschreibung der numerischen Methoden, die für die Lösung der Strömungsfelder benutzt werden und aus denen man die akustischen Quellen erhält. Zum Anderen erfolgt eine Übersicht über die numerischen Verfahren, die für die akustische Ausbreitung verwendet werden.

Numerische Verfahren für die akustische Ausbreitung


Neben der Anwendung von FE-Methoden findet derzeit auch eine Weiterentwicklung im Bereich der hybriden Verfahren statt, welche andere Volumendiskretisierungsmethoden, wie Finite Differenzen (FD), “discontinuous” Galerkin (DG) oder Finite Volumen (FV) verwenden, um die akustische Ausbreitung zu berechnen. Die numerischen Ansätze, die Vo-
lumendiskretisierungsmethoden benutzen, berechnen die Schallentstehung und -ausbreitung üblicherweise mittels linearisierter Euler Gleichungen (LEE) mit Quelltermen [12, 97, 100]. Zusätzlich findet man auch einen numerischen Ansatz, der das akustische Feld durch die Lösung akustischer Störungsgleichungen (APE) [34] simuliert. Diese Gleichungen sind, im Gegensatz zu LEE, für beliebige Hintergrundströmungen, die hydrodynamische Instabilitäten enthalten, stabil. Andere Alternativen für die Simulation des akustischen Feldes sind die linearisiert-gestör-kompresiblen-Gleichungen (LPCE) [93], die, in ähnlicher Weise wie die APE, die aufgrund von hydrodynamischen Instabilitäten verursachten Fehlern ausblenden.


CFD Verfahren


1.3. Lösungsansatz


Die FE-Formulierung des Problems wird durch die Volumendiskretisierung von Lighthills akustischer Analogie in der schwachen Form erhalten. Hiermit können die Effekte von festen bzw. flexiblen Körpern auf die akustische Abstrahlung implizit miteinbezogen werden. Im Gegensatz zu Integralformulierungen werden hier keine zusätzlichen Terme benötigt um solche Effekte berücksichtigen zu können. Die Diskretisierung der schwachen Formulierung stellt
1.3. LÖSUNGSANSATZ

einen Hauptvorteil der FE-Methode gegenüber anderer numerischer Methoden, wie z. B. der Finite-Differenzen Methode (FD), dar. Bei derartigen Methoden ist die Diskretisierung des ursprünglichen Differentialoperators (starke Formulierung) erforderlich.


Für die Durchführung von harmonischen Simulationen werden die auf das Akustikgitter interpolierten akustischen Quellen im Zeitbereich zuerst in einem Datensatz gespeichert. Auf diesen wird eine Fourier-Transformation angewendet, um die entsprechenden Quellen im Frequenzbereich zu erhalten. Der resultierende Datensatz wird dann wiederum als Einganggröße für die harmonische akustische Simulation verwendet.


Als einige Anmerkungen zu dem in dieser Arbeit verfolgten Lösungsansatz, im Vergleich zu den auf Integralformulierungen basierenden CAA Methoden sind zu nennen:

- Die akustischen Quellterme werden allein aus der Volumenverteilung der Quadrupole des Strömungsgeschwindigkeitsfeldes evaluierter. Deswegen werden keine Druckwerte aus der Strömungsberechnung benötigt.

- Der Lösungsansatz berücksichtigt implizit die Effekte von Reflexionen des strömungsinduzierten Lärms durch feste Oberflächen. Im Gegensatz dazu müssen diese Effekte bei CAA Formulierungen, die auf Greenschen Funktionen basieren, durch explizite Oberflächenintegrale berücksichtigt werden.
• Volumendiskretisierungsmethoden sind im Allgemeinen besser für die Betrachtung von konvektiven Effekten bei der Wellenausbreitung in nichtuniformen Hintergrundströmungen geeignet.

Bei Problemen, die nichtkompakte Ränder der Quellgebiete beinhalten bzw. bei aeroakustischen Problemen in geschlossenen Räumen, bietet die Tatsache, dass sich die akustischen Quelltermen allein auf das Strömungsgeschwindigkeitsfeld beziehen, einen entscheidenden Vorteil gegenüber Integralformulierungen. In letzterem Fall können Quellterme aus CFD Simulationen, die die Druckfluktuationen nicht korrekt auflösen, nicht genau genug sein, um die verschiedenen Reflexionen nachzubilden zu können [16]. Falls Struktur-Akustik Effekte mitberücksichtigt werden müssen oder bei Innenraumproblemen, sind Volumendiskretisierungsmethoden, wie die in dieser Arbeit vorgestellte, auch besser geeignet, um die Interaktionen zwischen den Oberflächen und dem strömungsinduzierten Lärm direkt in der akustischen Simulation miteinbeziehen zu können. In solchen Fällen würden Integralformulierungen eine a priori Kenntnis der Greenschen Funktion der festen Flächen erfordern, welche im Allgemeinen für komplexe Geometrien nicht bekannt ist [80]. Des Weiteren erlauben Integralmethoden keine einfache Einbeziehung von Effekten aufgrund elastischer Strukturen im Strömungsfeld in die Akustiksimulation.

1.4. Überblick der Arbeit

Im folgenden Abschnitt werden die Inhalte der einzelnen Kapitel der vorliegenden Doktorarbeit kurz zusammengefasst. Teile dieser Kapiteln wurden in [29, 30, 32, 60, 3, 31, 2] veröffentlicht.

Nachdem in diesem Kapitel ein Überblick über die CAA-Methode gegeben wurde, werden in Kapitel 2 hybride CAA Ansätze im Detail vorgestellt, die zurzeit zur Lösung der praktischen Probleme verwendet werden. Die Ableitungen der grundlegenden Gleichungen für die verschiedenen akustischen Analogie-Formulierungen werden hier ebenfalls beschrieben. Die Beschränkungen hinsichtlich der Anwendbarkeit dieser Formulierungen auf praktische Probleme werden vorgestellt. Zusätzlich werden die Hauptunterschiede und die entscheidenden Eigenschaften der Methoden, die sowohl auf der integralen Lösung der akustischen Analogie und als auch auf der schwachen Formulierung basieren, ausführlich diskutiert.

Im Kapitel 3 wird die FE Methode angewendet, um die Volumendiskretisierung der Lighthillschen akustischen Analogie durchzuführen. Nach Ableitung der schwachen Formulierung von der Lighthillschen akustischen Analogie, erhält man die semidiskrete Galerkin Formulierung für die inhomogene Wellengleichung. In diesem Kapitel werden die entsprechenden transienten und harmonischen Formulierungen vorgestellt, welche zur Berechnung des abgestrahlten Schallfeldes sowohl im Zeit- als auch im Frequenzbereich genutzt werden. Einerseits wird die Newmark Methode verwendet, um die Zeitdiskretisierung der transienten Formulierung herzuleiten. Um die harmonische Formulierung andererseits zu erhalten, wird die Fourier Transformation angewandt, welche das entsprechende komplexe algebraische Gleichungssystem aus der semidiskreten Galerkin Formulierung ergibt. Des Weiteren wird das Verfahren vorgestellt, das für die numerische Auswertung der akustischen Quelltermen verwendet wurde.
1.4. ÜBERBLICK DER ARBEIT

Kapitel 4 konzentriert sich auf numerische Verfahren, die am Rand des Rechengebietes benutzt werden, um Freifeldsimulationen zu ermöglichen. Die grundlegende Theorie dieser Verfahren wird vorgestellt, wobei als Ausgangspunkt das allgemeine Konzept der absorbierenden Randbedingungen (ABC) beschrieben und anschließend das Prinzip der Perfectly Matched Layer eingeführt wird. Des Weiteren werden die Aspekte zur Implementierung der absorbierenden Methoden, die in dieser Arbeit angewandt werden, vorgestellt.

In Kapitel 5 werden die Aspekte des Kopplungsverfahrens, das für den Austausch der akustischen Quellen verwendet wird, beschrieben. Zuerst wird der Ablauf der Kopplung der Strömungs- und Akustikfelder dargestellt. Das unterschiedliche Vorgehen, um transiente oder harmonische Simulationen der akustischen Ausbreitung durchzuführen, werden ebenfalls vorgestellt. Im Anschluss werden die Eigenschaften der Kopplungsschnittstelle MpCCI eingeführt, indem auf ihre nicht-konservativen und konservativen Interpolationsansätze eingegangen wird. Letztlich werden akustische Ergebnisse gezeigt, die den Einfluss dieser zwei Interpolationsverfahren auf die akustischen Amplituden verdeutlichen.


In Kapitel 8 wird der Inhalt der Arbeit kurz zusammengefasst und ein Ausblick auf Forschungsarbeiten, um die Anwendbarkeit der Implementierung zu erweitern, gegeben.
2. Zusammenfassung und Ausblick


- Das konservative Interpolationsschema, welches für die Übergabe der akustischen Quellen von dem CFD-Gitter auf die Knoten des akustischen Gitters verwendet wird, bietet eine wesentliche Reduktion der Rechenkosten durch die Nutzung akustischer Gitter die mehrfach gröber sind als die Strömungsgitter. In den 2D- und 3D-Testfällen die hierbei
evaluier wurden, wurde festgestellt, dass bei der Nutzung von Elementen mit quadratischen Basisfunktionen ein Verhältnis zwischen Akustik- und Strömungs-Diskretisierung von \( h_a/h_f \leq 10 \) auf der Zylinderoberfläche und \( h_a/h_f \leq 50 \) bei den äußersten Regionen ausreicht, um genaue akustische Ergebnisse zu erzielen.

- Bei transienten 2D-Untersuchungen, bei denen die akustische Zeitschrittweite 10 fach gröber gewählt wurde als bei der Strömungsberechnung, ergibt sich eine gute Qualität der akustischen Druckwerte. Bei einer zweifach gröberen akustischen Zeitschrittweite (\( \Delta t_a/\Delta t_f = 20 \)) wurde hochfrequentes numerisches Rauschen verursacht, das den Zeitverlauf der Druckwerte verschlechtert, obwohl die Amplituden des akustischen Druckes unverändert bleiben. Auf der anderen Seite ist dies bei harmonischen Analysen nicht sehr relevant, da hierbei das akustische Feld ausschließlich für bestimmte Frequenzen berechnet wird. Deswegen können in diesem Fall gröbere transiente Datensätze der akustischen Quellterme für die Fourier-Transformation verwendet werden. Vorraussetzung ist, dass das Zeitsignal ausreichend diskretisiert ist, um die Transformation der betrachteten Frequenzen korrekt durchführen zu können (z.B. eine Abtastung von 25 Werten pro Periode). Je nach Problemstellung und unter Berücksichtigung der Frequenzspektra die aus den CFD-Ergebnissen und/oder Messungen entnommen werden, kann die Nutzung gröberer transienter Datensätze sowohl die Interpolationszeit der transienten Knotenquellgrössen als auch den Speicherplatzverbrauch deutlich reduzieren.

- Im harmonischen 2D-Testbeispiel wurde nachgewiesen, dass in der Simulation der Hauptfrequenzkomponente mit der PML das Rechengebiet auf eine Entfernung zum Zylinder von ungefähr \( \lambda/3 \) (wobei \( \lambda \) die akustische Wellenlänge ist) begrenzen lässt. Um eine ausreichende Genauigkeit zu erzielen, genügte es in diesem Fall, die PML mit einer Breite von ungefähr \( \lambda/20 \) entsprechend 3 Elementen mit quadratischen Basisfunktionen in der radialen Richtung zu diskretisieren. Ergebnisse aus dieser harmonischen Simulation haben eine gute Übereinstimmung mit den entsprechenden transienten Rechnungen gezeigt.

- Des Weiteren haben harmonische 3D-Simulationen des Schallfeldes an einer Wand befestigter, umströmter Zylindergeometrien demonstriert, dass mit dem Berechnungsschema im Frequenzbereich die Richtcharakteristiken und die Schalldruckamplituden für unterschiedliche Zylindergeometrien relativ einfach und ohne Behandlung von Wandeffekten, die bei Integralmethoden erforderlich wären, bestimmt werden können.

durchgeführt werden können. Die grössere Flexibilität bei der Gittergenerierung durch eine solche Erweiterung würde eine feinere Ortsdiskretisierung im Quellgebiet und einen schnelleren Übergang zu gröberen und nicht verzerrten Elementen im Ausbreitungsgebiet, wo nur die Hauptfrequenzspektra aufgelöst werden müssen, erlauben.


Um ein auf Volumendiskretisierung basiertes Verfahren wie das, das auch in dieser Arbeit eingesetzt wurde, weiter zu entwickeln, wäre die Spektrale-Elemente-Methode (SEM) eine versprechende Möglichkeit zu der Standard-FEM. Die SEM entspricht einer Finite-Elemente-Methode höherer Ordnung, in derstückweise, polynomiale Basisfunktionen höherer Ordnung verwendet werden. Diese Methode weist ebenfalls die geometrische Flexibilität der FEM auf und bietet darüber hinaus mehrere günstige numerische Eigenschaften, wie diagonale Massenmatrizen, welche die Recheneffizienz erhöhen und es ermöglichen, Probleme mit höheren Frequenzkomponenten mit weniger Computerressourcen zu berechnen.
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