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A regular version of Smilansky model

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We discuss a modification of Smilansky model in which a singular potential “channel” is replaced by a regular, below unbounded potential which shrinks as it becomes deeper. We demonstrate that, similarly to the original model, such a system exhibits a spectral transition with respect to the coupling constant, and determine the critical value above which a new spectral branch opens. The result is generalized to situations with multiple potential “channels.” © 2014 AIP Publishing LLC.

I. INTRODUCTION

In the inspirative paper Smilansky discussed a simple example of quantum dynamics which could exhibit a behavior of two substantial different types controlled by the value of the coupling constant. The model in which it can be demonstrated allows for interpretation in different ways, as a one-dimensional system coupled to a heat bath, as a simple quantum graph to the vertices of which one or more one-dimensional harmonic oscillators are attached with a position-dependent coupling strength, or as a two-dimensional quantum system described by the Hamiltonian

\[ H_{Sm} = -\frac{\partial^2}{\partial x^2} + \frac{1}{2} \left( -\frac{\partial^2}{\partial y^2} + y^2 \right) + \lambda y \delta(x). \] (1.1)

The harmonic oscillator frequency can be, of course, changed by scaling to any positive \( \omega^2 \), what is important is the last term. It was argued in Ref. 1 that the behavior of the system depends crucially on the coupling parameter: if \( |\lambda| > \sqrt{2} \) the particle can escape to infinity along the singular “channel” in the \( y \) direction.

The claim can be made mathematically rigorous in terms of the spectral properties of such an operator: one can prove that that for \( |\lambda| > \sqrt{2} \) the system has an additional branch of absolutely continuous spectrum which is not bounded from below.\(^2\) The model was subsequently generalized to the situation when one has more than one singular “channel” – cf. Refs. 3 and 4 – and its further properties were studied, in particular, the discrete spectrum in the subcritical case.

It has appeared that there is also another motivation to investigate such systems. Recently Guarneri has used the model – or rather its modification in which the motion in the \( x \) direction is restricted to a finite interval with periodic boundary conditions – to describe quantum measurements;\(^5\) he studied the time evolution in such a situation identifying the escape along a particular “channel” with reduction of the wave packet.

The paper\(^5\) concludes with expressing the hope that “similar behavior may be reproducible with smoother interaction potentials and also in purely classical models.” The aim of the present paper

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is to demonstrate that this is indeed the case. We are going to investigate a model in which the δ
interaction with y-dependent strength is replaced by a smooth potential channel of increasing depth,
and to show that it exhibits the analogous spectral transition as the coupling parameter exceeds a
critical value.

Replacing the δ interaction by a regular potential, however, requires modifications, in particular,
the coupling cannot be linear in y and the profile of the channel has to change with y; in this respect
our present problem is similar to another model we have investigated recently.6 To understand the
reason one should realize that the essence of the effect lays in the fact that far from the x-axis the
variables in the solution to the Schrödinger equation effectively decouple – one can regard it as a
sort of adiabatic approximation – and the oscillator potential competes with the principal eigenvalue
of the “transverse” part of the operator, which in the singular case equals to \( \frac{1}{4}\lambda^2 y^2 \). If we want to
approximate the δ interaction by a family of shrinking potential in the usual way7 (Sec. I.3.2) we
have to match the integral of the potential with the δ coupling parameter, \( \int U(x, y) \, dx \sim y \), which can
be achieved, e.g., by choosing \( U(x, y) = \lambda y^2 V(xy) \) for a fixed function \( V \).

Inspired by these considerations we are going to investigate the model described by the partial
differential operator on \( L^2(\mathbb{R}^2) \) acting as

\[
H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \omega^2 y^2 - \lambda y^2 V(xy) \chi_{(|x| \leq a)}(x),
\]

(1.2)

where \( \omega, a \) are positive constants, \( \chi_{(|y| \leq a)} \) is the indicator function of the interval \((-a, a)\), and
the potential \( V \) with \( \text{supp} \, V \subset [-a, a] \) is a nonnegative function with bounded first derivative. By
Faris-Lavine theorem8 (Theorems X.28 and X.38) the above operator is essentially self-adjoint on
\( C_0^\infty(\mathbb{R}^2) \); the same is true for its generalization,

\[
H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \omega^2 y^2 - \sum_{j=1}^{N} \lambda_j y^2 V_j(xy) \chi_{(|x|+|y| \leq a_j)}(x)
\]

(1.3)

with a finite number of potential channels, where the functions \( V_j \) are positive with bounded
first derivative, with the supports contained in the intervals \((b_j - a_j, b_j + a_j)\) and such that
\( \text{supp} \, V_j \cap \text{supp} \, V_k = \emptyset \) holds for \( j \neq k \).

Our aim in the present paper is to demonstrate existence of a critical coupling separating two
different situations: below it the spectrum is bounded from below while above it covers the whole
real line. Discussion of further properties such as the discrete spectrum in the subcritical case or
time evolution of wave packets is postponed to a later paper. To finish the introduction, let us add a
few comments which may sound simple but hopefully they can prevent some misunderstandings:

- The results discussed here depend substantially on the asymptotic behavior of the potential
channels and would not change if the cut-off is absent or the potential is modified in the vicinity
of the x-axis in a different way, for instance, by replacing the cut-off functions in (1.2) and
(1.3) with \( \chi_{(|x| \geq a)}(y) \) and \( \chi_{([y] \geq a)}(y) \), respectively. The reason why we introduce the cut-offs is
that they make it easier to treat generalization of the model to the multiple-channel situation.
Needless to say, their presence and concrete form influences other spectral properties such as
the discrete spectrum in the subcritical case.
- It is also not important that in contrast to the original model with the Hamiltonian (1.1) we
assume, mostly for simplicity, that the potential channels depth increases in both directions
parallel to the y-axis; an exact analogue would have \(-\lambda |y| \delta(x) \) as the interaction term in (1.1).
- Let us notice that if we replace the potential \( V \) by the family \( V_{\epsilon} : V_{\epsilon}(u) = \frac{\epsilon}{2} V(\frac{u}{\epsilon}) \)
the quadratic forms of the corresponding operators \( H_{\epsilon} \) converge to the form of operator \( H_{Sm} \) as \( \epsilon \to 0 \),
and because of this one can understand the model described by Hamiltonian \( H \) as the regular
modification of \( H_{Sm} \).
II. SUBCRITICAL CASE

To state the result we will employ a one-dimensional comparison operator

\[ L = -\frac{d^2}{dx^2} + \omega^2 - \lambda V(x) \]  

(2.1)
on \mathcal{L}^2(\mathbb{R}) with the domain \mathcal{H}^2(\mathbb{R}); as long as there is no danger of misunderstanding we refrain from labeling the symbol by \omega, \lambda, and V. The important property will be the sign of its spectral threshold; since V is supposed to be non-negative, the latter is a monotonous function of \lambda and there is a \lambda_{\text{crit}} > 0 at which the sign changes.

We shall first focus on the subcritical coupling case.

**Theorem 2.1.** Under the stated assumptions, the spectrum of operator H given by (1.2) is bounded from below provided the operator L is non-negative.

**Proof.** It is obviously sufficient to prove the claim for \lambda = 1. We employ Neumann bracketing. Let \( h_n \) and \( \tilde{h}_n \) be, respectively, the restrictions of operator H to the strips \( G_n = \mathbb{R} \times \{ y : \ln n < y \leq \ln(n+1) \} \) and \( \tilde{G}_n = \mathbb{R} \times \{ y : -\ln(n+1) < y \leq -\ln n \}, n = 1, 2, \ldots, \) with Neumann boundary conditions. Then we have the inequality

\[ H \geq \bigoplus_{n=1}^{\infty} (h_n \oplus \tilde{h}_n), \]  

(2.2)
and to prove the claim we have to demonstrate that the sets \( \sigma(h_n) \) and \( \sigma(\tilde{h}_n) \) have a uniform lower bound as \( n \to \infty. \) Using the fact that the function V has a bounded derivative and it is compactly supported we find

\[ V(xy) - V(x \ln n) = O\left(\frac{1}{n \ln n}\right), \quad y^2 - \ln^2 n = O\left(\frac{\ln n}{n}\right), \]

for any \( (x, y) \in G_n, \) and consequently

\[ y^2 V(xy) - \ln^2 n V(x \ln n) = O\left(\frac{\ln n}{n}\right). \]

Similarly, we have

\[ y^2 V(xy) - \ln^2 n V(-x \ln n) = O\left(\frac{\ln n}{n}\right). \]

for for any \( (x, y) \in \tilde{G}_n. \) These relations yield asymptotic inequalities

\[ \inf \sigma(h_n) \geq \inf \sigma(l_n) + O\left(\frac{\ln n}{n}\right), \]

\[ \inf \sigma(\tilde{h}_n) \geq \inf \sigma(\tilde{l}_n) + O\left(\frac{\ln n}{n}\right), \]  

(2.3)
in which the Neumann operators \( l_n := -\frac{d^2}{dx^2} - \frac{\partial^2}{\partial y^2} + \omega^2 \ln^2 n - \ln^2 n V(x \ln n) \) on \( G_n \) and \( \tilde{l}_n := -\frac{d^2}{dx^2} - \frac{\partial^2}{\partial y^2} + \omega^2 \ln^2 n - \ln^2 n V(-x \ln n) \) on \( \tilde{G}_n \) have separated variables. Since the minimal eigenvalue of \( -\frac{d^2}{dx^2} \) on the interval with Neumann boundary conditions defined on intervals \( \ln n < y \leq \ln(n+1), n = 1, 2, \ldots, \) is zero, we have \( \inf \sigma(l_n) = \inf \sigma(l_n^{(1)}), \) where

\[ l_n^{(1)} = -\frac{d^2}{dx^2} + \omega^2 \ln^2 n - \ln^2 n V(x \ln n) \]
acts on \( \mathcal{L}^2(\mathbb{R}). \) Note that the cut-off function \( \chi(\{ |x| \leq n \}) \) in (1.2) plays no role in the asymptotic estimate as it affects a finite number of terms only. By the change of variable \( x = \frac{t}{\ln n} \) the last operator is unitarily equivalent to \( \ln^2 n L \) which is non-negative as long as \( L \) is non-negative. In the same way
one proves that $\tilde{T}_{\mu}$ is non-negative under the assumption of the theorem; this in combination with (2.3) concludes the proof. \hfill \Box

By a straightforward modification of the proof we get the following claim.

**Corollary 2.2.** Let $H$ be the operator on $(-c, c) \times \mathbb{R}$ for some $c \geq 0$ given the differential expression (1.2) with Dirichlet (Neumann, periodic) boundary conditions in the variable $x$. The spectrum of $H$ is bounded from below if $L \geq 0$ holds, where $L$ is the operator (2.1) on $L^2(-c, c)$ with Dirichlet (respectively, Neumann or periodic) boundary conditions.

### III. SUPERCRITICAL CASE

Let us turn to the case where the “escape to infinity” is possible.

**Theorem 3.1.** Under our hypotheses, $\sigma(H) = \mathbb{R}$ holds if $\inf \sigma(L) < 0$.

**Proof.** To prove that any real number $\mu$ belongs to essential spectrum of operator $H$ we are going to use Weyl’s criterion (Theorem VII.12): we have to find a sequence $\{\psi_k\}_{k=1}^{\infty} \subset D(H)$ such that $\|\psi_k\| = 1$ which contains no convergent subsequence and

$$
\|H\psi_k - \mu\psi_k\| \to 0, \quad \text{as} \quad k \to \infty
$$

holds. Since the claim is invariant under scaling transformations we can suppose without loss of generality that $\inf \sigma(L) = -1$. The spectral threshold is easily seen to be a simple isolated eigenvalue; we denote the corresponding normalized eigenfunction of $L$ by $h$. Our aim is to show first that $0 \in \sigma_{\text{ess}}(H)$. In fact, it would be enough to show that zero belongs to $\sigma(H)$ but we get the stronger claim at no extra expense.

We fix a positive $\varepsilon$ and choose a natural number $k = k(\varepsilon)$ with which we associate a function $\chi_k \subset C^2_0(1, k)$ satisfying the following conditions:

$$
\int_1^k \frac{1}{z} \chi_k^2(z) \, dz = 1 \quad \text{and} \quad \int_1^k z(\chi_k'(z))^2 \, dz < \varepsilon. \tag{3.1}
$$

To give an example, consider the function

$$
\tilde{\chi}_k(z) = \frac{8 \ln^3 z}{\ln k} \chi_{[1 \leq z \leq \sqrt{k}]}(z) + \frac{2 \ln k - 2 \ln z}{\ln k} \chi_{[\sqrt{k} \leq z \leq k^{1/2}]}(z)
$$

$$
+ g_k(z) \chi_{[\sqrt{k} < z < k^{1/2}]}(z) + q_k(z) \chi_{[k^{1/2} < z \leq k]}(z),
$$

where $g_k$ and $q_k$ are interpolating functions chosen in such a way that $\tilde{\chi}_k \in C^2_0(1, k)$. The first integral in (3.1) is positive for $\tilde{\chi}_k$; in fact we have $\int_1^{\sqrt{k}} \frac{1}{z} \chi_k^2(z) \, dz \geq \frac{1}{4}$, hence we can define $\chi_k(z) = \left( \int_1^k \frac{1}{z} \chi_k^2(z) \, dz \right)^{-1/2} \tilde{\chi}_k(z)$. This function satisfies by definition the first condition of (3.1) and one can check that it also satisfies the second one provided $k$ is sufficiently large; this follows from the fact that $\int_1^k z(\chi_k'(z))^2 \, dz = O\left( \frac{1}{\ln k} \right)$ as $k \to \infty$.

Such functions allow us to construct the Weyl sequence we seek. Given a function $\chi_k$ with the described properties, we define

$$
\psi_k(x, y) := h(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) + \frac{f(xy)}{y^2} e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right), \tag{3.2}
$$

where $f(t) := -\frac{1}{2} t^2 h(t)$, $t \in \mathbb{R}$, and $n_k \in \mathbb{N}$ is a positive integer to be chosen later, using the following auxiliary result.

**Lemma 3.2.** Let $\psi_k, k = 1, 2, \ldots$, be defined by (3.2). Then for any given $k$ one can achieve that

$$
\|\psi_k\|_{L^2(\mathbb{R}^2)} \geq \frac{1}{2}
$$

holds by choosing $n_k$ large enough.
Proof. The following estimates show
\[
\int_{\mathbb{R}} \left| h(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy = \int_{n_k}^{k_{\text{max}}} \int_{\mathbb{R}} \left| h(xy) \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy
\]
\[
= \int_{n_k}^{k_{\text{max}}} \int_{\mathbb{R}} \left| \frac{1}{y} h(t) \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dy \, dt = \int_{\mathbb{R}} \left| h(t) \right|^2 \, dt \int_{n_k}^{k_{\text{max}}} \frac{1}{y} \left| \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dy
\]
\[
= \int_{n_k}^{k_{\text{max}}} \frac{1}{y} \left| \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dy = \int_{1}^{k_{\text{max}}} \frac{1}{y} \left| \chi_k(z) \right|^2 \, dz = 1
\] (3.3)
and
\[
\int_{\mathbb{R}} \left| \frac{1}{y^2} f(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy = \int_{n_k}^{k_{\text{max}}} \int_{\mathbb{R}} \left| \frac{1}{y^2} f(xy) \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy
\]
\[
= \int_{n_k}^{k_{\text{max}}} \int_{\mathbb{R}} \left| \frac{1}{y^2} f(t) \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dy \, dt \leq \frac{1}{n_k^2} \int_{n_k}^{k_{\text{max}}} \int_{\mathbb{R}} \left| f(t) \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dt \, dy
\]
\[
= \frac{1}{n_k^2} \int_{\mathbb{R}} \left| f(t) \right|^2 \, dt \int_{1}^{k_{\text{max}}} \left| \chi_k(z) \right|^2 \, dz < \frac{1}{16} ;
\] (3.4)

note that since the potential \( V \) has a compact support by assumption, the ground state eigenfunction \( h \) decays exponentially as \( |x| \to \infty \), hence the first integral in the last expression converges. This concludes the proof. \( \square \)

We need one more auxiliary result:

Lemma 3.3. Let \( \psi_k, k = 1, 2, \ldots , \) be again functions defined by (3.2). Then the inequality \( \| H \psi_k \|_{L^2(\mathbb{R}^2)} < c \varepsilon \) with a fixed constant \( c \) holds for \( k = k(\varepsilon) \).

Proof. By a straightforward calculation one gets

\[
\frac{\partial^2 \psi_k}{\partial x^2} = y^2 h''(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) + f''(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right)
\]
and

\[
\frac{\partial^2 \psi_k}{\partial y^2} = x^2 h''(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) + 2 i xy h'(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right)
\]
\[
+ \frac{2x}{n_k} h''(xy) e^{iy^2/2} \chi_k' \left( \frac{y}{n_k} \right) - y^2 h(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right)
\]
\[
+ i h(xy) e^{iy^2/2} \chi_k' \left( \frac{y}{n_k} \right) + 2 \frac{2x}{n_k} h(xy) e^{iy^2/2} \chi_k' \left( \frac{y}{n_k} \right)
\]
\[
+ \frac{1}{n_k^2} h(xy) e^{iy^2/2} \chi_k'' \left( \frac{y}{n_k} \right) + \frac{x^2}{y^2} f''(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right)
\]
\[
+ 2 \frac{i x}{y} f'(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) + \frac{2x}{n_k y^2} f'(xy) e^{iy^2/2} \chi_k' \left( \frac{y}{n_k} \right)
\]
\[
- f(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) + \frac{i}{y^2} f(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right)
\]
\[
+ \frac{4x}{y^3} f'(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) - \frac{4i}{y^2} f(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right)
\]
\[
- \frac{4}{n_k y^3} f(xy) e^{iy^2/2} \chi_k' \left( \frac{y}{n_k} \right) + \frac{6}{y^4} f(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) ;
\] (3.5)
We want to show that choosing \( n_k \) sufficiently large one can make the terms at the right-hand side of (3.5) as small as we wish. Changing the integration variables, we get the following estimate,

\[
\int_{\mathbb{R}^2} \left| x^2 h''(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy = \int_{n_k}^{k n} \int_{\mathbb{R}} \left| x^2 h'(xy) \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy
\]

\[
= \int_{n_k}^{k n} \frac{1}{y^2} \left| \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dy \int_{\mathbb{R}} t^4 |h''(t)|^2 \, dt \leq \frac{1}{n_k^2} \int_{n_k}^{k n} |\chi_k(z)|^2 \, dz \int_{\mathbb{R}} t^4 |h''(t)|^2 \, dt, \]

where the last integral again converges from the reason described above. In the same way we establish the remaining inequalities which we need to demonstrate our claim:

\[
\int_{\mathbb{R}^2} \left| \frac{x}{n_k} h''(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy = \int_{n_k}^{k n} \int_{\mathbb{R}} \left| \frac{x}{n_k} h'(xy) \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy
\]

\[
\leq \frac{1}{n_k^2} \int_{n_k}^{k n} |\chi_k(z)|^2 \, dz \int_{\mathbb{R}} t^4 |h''(t)|^2 \, dt, \]

\[
\int_{\mathbb{R}^2} \left| \frac{x^2}{2} f''(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy = \int_{n_k}^{k n} \int_{\mathbb{R}} \left| \frac{x^2}{2} f'(xy) \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy
\]

\[
\leq \frac{1}{n_k^2} \int_{n_k}^{k n} |\chi_k(z)|^2 \, dz \int_{\mathbb{R}} t^2 |f''(t)|^2 \, dt, \]

\[
\int_{\mathbb{R}^2} \left| \frac{x}{n_k y} f'(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy = \int_{n_k}^{k n} \int_{\mathbb{R}} \left| \frac{x}{n_k y} f'(xy) \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy
\]

\[
\leq \frac{1}{n_k^3} \int_{n_k}^{k n} |\chi_k(z)|^2 \, dz \int_{\mathbb{R}} t^2 |f'(t)|^2 \, dt, \]

\[
\int_{\mathbb{R}^2} \left| \frac{1}{y^2} f(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy = \int_{n_k}^{k n} \int_{\mathbb{R}} \left| \frac{1}{y^2} f(xy) \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy
\]

\[
\leq \frac{1}{n_k^2} \int_{n_k}^{k n} |\chi_k(z)|^2 \, dz \int_{\mathbb{R}} |f(t)|^2 \, dt, \]

\[
\int_{\mathbb{R}^2} \left| \frac{1}{n_k^2 y^2} f(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy = \int_{n_k}^{k n} \int_{\mathbb{R}} \left| \frac{1}{n_k^2 y^2} f(xy) \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy
\]

\[
\leq \frac{1}{n_k^2} \int_{n_k}^{k n} |\chi_k''(z)|^2 \, dz \int_{\mathbb{R}} |f(t)|^2 \, dt, \]
\[
\int_{\mathbb{R}^2} \left| \frac{1}{n_k} f(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy = \int_{n_k}^{kn_k} \int_{\mathbb{R}} \frac{1}{n_k} f(t) \chi_k \left( \frac{y}{n_k} \right) \, dt \, dy
\]

\[
\leq \frac{1}{n_k^2} \int_{1}^{k} |\chi_k(z)|^2 \, dz \int_{\mathbb{R}} |f(t)|^2 \, dt.
\]

\[
\int_{\mathbb{R}^2} \left| \frac{x}{y^3} f(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy = \int_{n_k}^{kn_k} \int_{\mathbb{R}} \frac{1}{y} f'(t) \chi_k \left( \frac{y}{n_k} \right) \, dt \, dy
\]

\[
\leq \frac{1}{n_k} \int_{1}^{k} |\chi_k(z)|^2 \, dz \int_{\mathbb{R}} |t^2 f'(t)|^2 \, dt.
\]

\[
\int_{\mathbb{R}^2} \left| \frac{1}{y^2} f(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy = \int_{n_k}^{kn_k} \int_{\mathbb{R}} \frac{1}{y^2} f(t) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \, dt \, dy
\]

\[
\leq \frac{1}{n_k} \int_{1}^{k} |\chi_k(z)|^2 \, dz \int_{\mathbb{R}} |f(t)|^2 \, dt.
\]

Consequently, choosing \( n_k \) large enough we can achieve that the sum of all the integrals at the left-hand sides of the above inequalities is less than \( \varepsilon \). Using then the inequality

\[
\left| \sum_{k=1}^{M} a_k \right|^2 \leq M \sum_{k=1}^{M} |a_k|^2
\]

valid for any \( \{a_k\}_{k=1}^{M} \subset \mathbb{C}^M, \ M < \infty \), in combination with properties of supports of the functions \( \psi_k \) we obtain the estimate

\[
\int_{\mathbb{R}^2} |H \psi_k|^2(x, y) \, dx \, dy = \int_{\mathbb{R}^2} \left[ -\frac{\partial^2 \psi_k}{\partial x^2} - \frac{\partial^2 \psi_k}{\partial y^2} + \omega^2 y^2 \psi_k - y^2 V(xy) \psi_k \right]^2 \, dx \, dy
\]

\[
\leq 14 \int_{n_k}^{kn_k} \int_{\mathbb{R}} \left[ y^2 h''(xy) \chi_k \left( \frac{y}{n_k} \right) + f''(xy) \chi_k \left( \frac{y}{n_k} \right) + 2i xy h'(xy) \chi_k \left( \frac{y}{n_k} \right) \right.
\]

\[
+ih(xy) \chi_k \left( \frac{y}{n_k} \right) - y^2 h(xy) \chi_k \left( \frac{y}{n_k} \right) + \frac{2y}{n_k} h(xy) \chi_k \left( \frac{y}{n_k} \right)
\]

\[
-f(xy) \chi_k \left( \frac{y}{n_k} \right) - \omega^2 y^2 h(xy) \chi_k \left( \frac{y}{n_k} \right) - \omega^2 f(xy) \chi_k \left( \frac{y}{n_k} \right)
\]

\[
+\left. y^2 V(xy) h(xy) \chi_k \left( \frac{y}{n_k} \right) + V(xy) f(xy) \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy + 14\varepsilon
\]

\[
= 14 \int_{n_k}^{kn_k} \int_{\mathbb{R}} \left[ y^2 (h''(xy) - \omega^2 h(xy) + V(xy) h(xy) - h(xy)) \chi_k \left( \frac{y}{n_k} \right) \right.
\]

\[
+ih(xy) \chi_k \left( \frac{y}{n_k} \right) + f''(xy) \chi_k \left( \frac{y}{n_k} \right) + 2i xy h'(xy) \chi_k \left( \frac{y}{n_k} \right) \right.
\]

\[
+\left. + \frac{2y}{n_k} h(xy) \chi_k \left( \frac{y}{n_k} \right) - f(xy) \chi_k \left( \frac{y}{n_k} \right) - \omega^2 f(xy) \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dx \, dy + 14\varepsilon.
\]
where the coefficient in front of the integrals comes from the number of the summands. Using the fact that $Lh = -h$ and applying the Cauchy inequality, the above result implies

$$\frac{1}{14} \int_{\mathbb{R}^2} |H\psi_k|^2(x, y) \, dx \, dy < \int_{n_k}^{\infty} \left( f''(xy) + 2i xy h'(xy) + ih(xy) - f(xy) \right) \left( \frac{y}{n_k} \right) + 2i y h(xy) \frac{y}{n_k} \frac{y}{n_k} \, dx \, dy + \varepsilon$$

$$\leq 2 \int_1^k \left| \chi_k(\varepsilon) \right|^2 \frac{d\varepsilon}{\varepsilon} \int_\mathbb{R} \left| f''(t) + f(t) (1 + \omega^2 - V(t)) - 2iht'(t) - ih(t) \right|^2 \, dt$$

$$+ 8 \int_1^k z \left| \chi_k'(\varepsilon) \right|^2 \, dz + \varepsilon$$

$$\leq 2 \int_\mathbb{R} \left| f''(t) + f(t) (1 + \omega^2 - V(t)) - 2iht'(t) - ih(t) \right|^2 \, dt + 9\varepsilon.$$

It is easy to check that $-(r^2 h'(t))'' + r^2 h(t)(1 + \omega^2 - V(t)) = -4\mu h'(t) - 2h(t)$, hence $-f''(t) + f(t)(1 + \omega^2 - V(t)) - 2iht'(t) - ih(t) = 0$ and the last integral in the above estimate vanishes, which gives

$$\int_{\mathbb{R}^2} |H\psi_k|^2(x, y) \, dx \, dy \leq 126\varepsilon; \quad (3.6)$$

this concludes the proof. \hfill \Box

Using the above two lemmata, we are able to complete the proof of the theorem. We fix a sequence \( \{\varepsilon_j\}_{j=1}^\infty \) such that \( \varepsilon_j \searrow 0 \) holds as \( j \to \infty \) and to any \( j \) we construct a function \( \psi_{k(\varepsilon_j)} \), with the corresponding numbers chosen in such a way that \( n_{k(\varepsilon_j)} > k(\varepsilon_{j-1}) n_{k(\varepsilon_{j-1})} \). The norms of \( H\psi_{k(\varepsilon_j)} \) satisfy inequality which \( (3.6) \) with \( 9\varepsilon_j \) on the right-hand side, and since the supports of \( \psi_{k(\varepsilon_j)}, \ j = 1, 2, \ldots, \) do not intersect each other by construction, their sequence converges weakly to zero.

This yields the sought Weyl sequence for zero energy; for any nonzero real number \( \mu \) we use the same procedure replacing the above \( \psi_k \) with

$$\psi_k(x, y) = h(xy) e^{i\epsilon_{\mu}(y)} \chi_k \left( \frac{y}{n_k} \right) + \frac{f(xy)}{y^2} e^{i\epsilon_{\mu}(y)} \chi_k \left( \frac{y}{n_k} \right),$$

where \( \epsilon_{\mu}(y) := \int_{\sqrt{|y|}}^y \sqrt{t^2 + \mu} \, dt \), and furthermore, the functions \( f, \chi_k \) defined in the same way as above. The second derivatives of those functions are

$$\frac{\partial^2 \psi_k}{\partial x^2} = y^2 h''(xy) e^{i\epsilon_{\mu}(y)} \chi_k \left( \frac{y}{n_k} \right) + f''(xy) e^{i\epsilon_{\mu}(y)} \chi_k \left( \frac{y}{n_k} \right).$$
and
\[
\frac{\partial^2 \psi_k}{\partial y^2} = x^2 h(x y) e^{i \epsilon y} \chi_k \left( \frac{y}{n_k} \right) + 2 i x \sqrt{y^2 + \mu} h(x y) e^{i \epsilon y} \chi_k \left( \frac{y}{n_k} \right) \\
+ \frac{2 x}{n_k} h(x y) e^{i \epsilon y} \chi_k' \left( \frac{y}{n_k} \right) - (y^2 + \mu) h(x y) e^{i \epsilon y} \chi_k \left( \frac{y}{n_k} \right) \\
+ i \frac{y^2}{\sqrt{y^2 + \mu}} h(x y) e^{i \epsilon y} \chi_k \left( \frac{y}{n_k} \right) + 2 i \frac{\sqrt{y^2 + \mu}}{n_k} h(x y) e^{i \epsilon y} \chi_k' \left( \frac{y}{n_k} \right) \\
+ \frac{1}{n_k^2} h(x y) e^{i \epsilon y} \chi_k'' \left( \frac{y}{n_k} \right) + \frac{2 x}{n_k} f(x y) e^{i \epsilon y} \chi_k \left( \frac{y}{n_k} \right) \\
- \frac{y^2 + \mu}{y^2} f(x y) e^{i \epsilon y} \chi_k \left( \frac{y}{n_k} \right) + \frac{i}{\sqrt{y^2 + \mu}} f(x y) e^{i \epsilon y} \chi_k' \left( \frac{y}{n_k} \right) \\
+ \frac{4 x}{y^2} f(x y) e^{i \epsilon y} \chi_k \left( \frac{y}{n_k} \right) - \frac{4 i \sqrt{y^2 + \mu}}{y^3} f(x y) e^{i \epsilon y} \chi_k' \left( \frac{y}{n_k} \right) \\
- \frac{4}{n_k} f(x y) e^{i \epsilon y} \chi_k' \left( \frac{y}{n_k} \right) + \frac{6}{y^2} f(x y) e^{i \epsilon y} \chi_k \left( \frac{y}{n_k} \right).
\]

It is not difficult to check that for any positive \( \epsilon \) there is a number \( n_k \) large enough such that
\[
\left\| \frac{\partial^2 \psi_k}{\partial y^2} e^{-i \epsilon y} \to 0 \right\|_{L^2(\mathbb{R}^2)} < \epsilon.
\]

Using further the identity \( \frac{\partial^2 \psi_k}{\partial x^2} e^{-i \epsilon x} = e^{-i \epsilon^2 / 2} \frac{\partial^2}{\partial y^2} \left( \psi_k e^{-i \epsilon y} + i y^2 / 2 \right) \) we obtain the estimate
\[
\left\| H \psi_k - \mu \psi_k \right\|_{L^2(\mathbb{R}^2)} = \left\| (H \psi_k) e^{-i \epsilon x} - \mu \psi_k e^{-i \epsilon x} \right\|_{L^2(\mathbb{R}^2)} < \epsilon
\]

Summarizing this discussion and that of Sec. II, we have obtained the following result:

**Theorem 3.4.** The spectrum of operator \( H \) given by (1.2) is bounded from below if and only if the operator \( L \) is non-negative, otherwise the spectrum of \( H \) covers the whole real axis.

**IV. INTERVALS AND MULTIPLE CHANNELS**

Let us look next how the above result changes if the motion in the \( x \) direction is restricted. We have the following result:

**Theorem 4.1** Let \( H \) be the operator on \( L^2(-c, c) \otimes L^2(\mathbb{R}) \) for some \( c > 0 \) given by the differential expression (1.2) with Dirichlet condition at \( x = \pm c \) and denote by \( L \) the corresponding Dirichlet operator (2.1) on \( L^2(-c, c) \). If the spectral threshold of \( L \) is negative, the spectrum of \( H \) covers the whole real axis.
Proof. Without loss of generality we may suppose that $c = 1$. We shall apply again Weyl's criterion modifying the argument of Sec. III. By Dirichlet bracketing, one has that $L \leq \Theta_{k=1}^{n_k} L_k$, where $\tilde{L}$ is the original comparison operator $(2.1)$, $\tilde{L} = -\frac{d^2}{dx^2} + \omega^2 - V$ on $L^2(\mathbb{R})$, while $L_1$ and $L_{j,j}$ are given by the same differential expression on $L^2(-1,1)$ and $L^2(-\infty, -1), L^2(1, \infty)$, respectively. Thus under the assumption the spectral threshold of $\tilde{L}$ is negative, and without loss of generality we may suppose that its ground state satisfies $\tilde{L} \psi = -\psi$ with $\|\psi\| = 1$ and show that $0 \in \sigma_{ess}(\tilde{L})$. The functions $(3.2)$ are now changed as follows,

$$
\psi_k(x, y) = h(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x) + \frac{f(xy)}{y^2} e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x)
$$

with $\phi \in C^2_0(-1, 1)$ such that $\phi(x) = 1$ holds for $|x| \leq \frac{1}{2}$, while the numbers $k = k(\varepsilon)$, $n_k \in \mathbb{N}$ and functions $\chi_k, f$ are the same as before. Instead of the estimates $(3.3)(3.4)$ we now have for large enough $n_k$ the inequalities

$$
\int_{-1}^{1} \int_{n_k}^{n_k} \left| h(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x) \right|^2 \, dx \, dy 
$$

and

$$
\int_{-1}^{1} \int_{n_k}^{n_k} \left| f(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x) \right|^2 \, dx \, dy 
$$

which means that $\|\psi_k\|_{L^2(\mathbb{R})} \geq \frac{1}{2} - 2\sqrt{\varepsilon}$ holds for $n_k$ large enough; our aim is to show that $\|H \psi_k\|_{L^2(\mathbb{R})}^2 < d \varepsilon$ with a fixed $d > 0$. Let us first compute the partial derivatives

$$
\frac{\partial^2 \psi_k}{\partial x^2} = y^2 h''(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x) + 2y h'(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x) 
$$

$$
+ h(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi''(x) + f''(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x) 
$$

$$
+ \frac{2}{y} f'(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi'(x) + \frac{1}{y^2} f(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi''(x)
$$
and

\begin{align*}
\frac{\partial^2 \psi_k}{\partial y^2} &= x^2 h''(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x) + 2ixy h'(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x) \\
+ &\frac{2x}{n_k} h'(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x) - y^2 h(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x) \\
+ &ih(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x) + 2i\frac{y}{n_k} h(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x) \\
+ &\frac{1}{n_k} h(xy) e^{iy^2/2} \chi_k'' \left( \frac{y}{n_k} \right) \phi(x) + x^2 \frac{y^2}{n_k} f''(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x) \\
+ &2i \frac{x}{y} f'(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x) + \frac{2x}{n_k y^2} f'(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x) \\
- &\frac{f(xy)}{y^2} e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x) + i \frac{1}{y^2} f(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x) \\
+ &\frac{1}{y^3 n_k} f(xy) e^{iy^2/2} \chi_k'' \left( \frac{y}{n_k} \right) \phi(x) + \frac{2i}{y^2 n_k} f(xy) e^{iy^2/2} \chi_k' \left( \frac{y}{n_k} \right) \phi(x) \\
- &4 \frac{x}{y^3} f'(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x) + 4i \frac{1}{y^2} f(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x) \\
- &\frac{4}{n_k y^3} f(xy) e^{iy^2/2} \chi_k' \left( \frac{y}{n_k} \right) \phi(x) + \frac{6}{y^4} f(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi(x).
\end{align*}

Using the exponential decay of $h$ and the fact that $\phi$ is constant on $[-1/2, 1/2]$ we find that for all sufficiently large $n_k$ we have

\begin{align*}
\int_{-1}^{1} \int_{n_k}^{kn_k} \left| y h'(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi'(x) \right|^2 \, dx \, dy \\
&\leq \left\| \phi' \right\|_{L^2(\mathbb{R})}^2 \int_{|y| > \frac{1}{2}} \int_{n_k}^{kn_k} \left| y h'(xy) \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dy \, dx \\
&\leq \left\| \phi' \right\|_{L^2(\mathbb{R})}^2 \int_{n_k}^{kn_k} \int_{n_k/2}^{\infty} \frac{1}{y} \left| y h'(t) \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dt \, dy \\
&\leq \int_{n_k}^{kn_k} \int_{n_k/2}^{\infty} \frac{1}{y} \left| y h'(t) \chi_k \left( \frac{y}{n_k} \right) \right|^2 \, dt \, dy \\
&= n_k^2 \left\| \phi' \right\|_{L^2(\mathbb{R})}^2 \int_{1}^{k} \int_{n_k/2}^{\infty} z \chi_k^2(z) \, dz \int_{n_k/2}^{\infty} |h'(t)|^2 \, dt \\
&\leq n_k^2 \left\| \phi' \right\|_{L^2(\mathbb{R})}^2 \int_{1}^{k} \int_{n_k/2}^{\infty} z \chi_k^2(z) \, dz \int_{n_k/2}^{\infty} |h'(t)|^2 \, dt < \varepsilon.
\end{align*}

in a similar way,

\begin{align*}
\int_{-1}^{1} \int_{n_k}^{kn_k} \left| h(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi''(x) \right|^2 \, dx \, dy < \varepsilon, \\
\int_{-1}^{1} \int_{n_k}^{kn_k} \left| \frac{1}{y^2} f(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi''(x) \right|^2 \, dx \, dy < \varepsilon, \\
\int_{-1}^{1} \int_{n_k}^{kn_k} \left| \frac{1}{y} f'(xy) e^{iy^2/2} \chi_k \left( \frac{y}{n_k} \right) \phi'(x) \right|^2 \, dx \, dy < \varepsilon.
\end{align*}
As for the remaining term in the partial derivative expressions, we simply repeat our calculations from Sec. III. In this way we are able to conclude that for large enough \( k \), and, respectively, \( n_k \) we have

\[
\int_{\mathbb{R}} |H\psi_k|^2 \, dx \, dy < \|\phi\|^2_{L^2(\mathbb{R})} \int_{n_k}^{k n_k} \int_{-1}^{1} |y^2 h''(xy)\chi_k \left( \frac{y}{n_k} \right) + f''(xy)\chi_k \left( \frac{y}{n_k} \right) + f(x)\chi_k \left( \frac{y}{n_k} \right) + V(xy)f(xy)\chi_k \left( \frac{y}{n_k} \right)|^2 \, dx \, dy + \varepsilon.
\]

Using the assumption about the ground state of \( \tilde{L} \), the last equation implies

\[
\int_{-1}^{1} \int_{\mathbb{R}} |H\psi_k|^2 \, dx \, dy < \|\phi\|^2_{L^2(\mathbb{R})} \int_{n_k}^{k n_k} \int_{-1}^{1} \left| f''(xy) + 2i xy h'(xy) + i h(xy) \right|^2 \chi_k \left( \frac{y}{n_k} \right) + 2i y h(xy)\chi_k \left( \frac{y}{n_k} \right) - f(xy)\chi_k \left( \frac{y}{n_k} \right) + V(xy)f(xy)\chi_k \left( \frac{y}{n_k} \right)|^2 \, dx \, dy + \varepsilon.
\]

Using the fact that \( f(t) = -i\frac{d^2}{dt^2} h(t) \) we conclude in the same way as in Sec. III that the right-hand side of the last inequality can be estimated by \( 9\|\phi\|^2_{L^2(\mathbb{R})} \varepsilon \).

The rest of the proof follows the same routine. We pick a sequence \( \{\varepsilon_j\}_{j=1}^{\infty} \) such that \( \varepsilon_j \downarrow 0 \) holds as \( j \to \infty \) and to any \( j \) we construct a function \( \psi_{k(e_j)} \) with the corresponding numbers chosen in such a way that \( n_k(e_j) > k(e_{j-1})n_k(e_{j-1}) \). The norms of \( H\psi_{k(e_j)} \) satisfy inequality which (3.6) with \( 9\|\phi\|^2_{L^2(\mathbb{R})} \varepsilon \) on the right-hand side, and the sequence \( \{\psi_{k(e_j)}\}_{j=1}^{\infty} \) converges weakly to zero by construction, their sequence converges weakly to zero. This proves that \( 0 \in \sigma_{ess}(H) \); for any nonzero real number \( \mu \) we proceed in the same way replacing the above \( \psi_k \) with

\[
\psi_k(x, y) = h(xy) e^{i\phi(y)} \chi_k \left( \frac{y}{n_k} \right) \phi(x) + \frac{f(xy)}{y^2} e^{i\phi(y)} \chi_k \left( \frac{y}{n_k} \right) \phi(x),
\]

where \( \mu = \int_{\mathbb{R}} \sqrt{r^2 + \mu} \, dr \), and furthermore, the functions \( f, \chi_k, \phi \) defined in the same way as above.

Observing the domains of the quadratic form associated with such operators we can extend the result in the following way:

**Corollary 4.2.** The claim of Theorem IV.1 remains valid if the Dirichlet boundary conditions at \( x = \pm c \) are replaced by any other self-adjoint boundary conditions.
The result also allows us to answer the question about spectral transition for the model with multiple singular channels.

**Theorem 4.3.** Let $H$ be the operator (1.3) with the potentials satisfying the stated assumptions, namely the functions $V_j$ are positive with bounded first derivative and $\text{supp } V_j \cap \text{supp } V_k = \emptyset$ holds for $j \neq k$. Denote by $L_j$ the operator (2.1) on $L^2(\mathbb{R})$ with the potential $V_j$ and $t_V := \min_j \inf \sigma(L_j)$. Then $H$ is bounded from below if and only if $t_V \geq 0$ and in the opposite case its spectrum covers the whole real axis.

**Proof.** The claim follows by bracketing. By assumption we can choose points $x_j$ such that

$$x_0 < v_j^- < v_j^+ < x_1 < v_2^- < \cdots < x_{n-1} < v_n^- < v_n^+ < x_n,$$

where $v_j^- := \inf \text{supp } V_j$ and $v_j^+ := \sup \text{supp } V_j$ and impose additional Neumann and Dirichlet boundary conditions at them. The spectrum in the intervals $(-\infty, x_0)$ and $(x_n, \infty)$ is found trivially, to the other components of the direct sum obtained in this way we apply Corollary 2.2 and Theorem 4.1, respectively.

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