C-arm Computed Tomography with Extended Axial Field-of-View

C-Bogen-Computertomographie mit erweitertem Sichtfeld in axialer Richtung

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To my Dad, Mom and Wife.
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Abstract

C-arm computed tomography (CT) is an innovative imaging technique in the interventional room that enables a C-arm system to generate 3D images like a CT system. Clinical reports demonstrate that this technique can help reduce treatment-related complications and may improve interventional efficacy and safety. However, currently, C-arm CT is only capable of imaging axially-short object, because it employs a single circular data acquisition geometry. This shortcoming can be a problem in some intraoperative cases when imaging a long object, e.g., the entire spine, is crucial. A new technique, C-arm CT for axially-long objects, namely extended-volume C-arm CT, has to be developed. This thesis aims at achieving this development. In particular, this thesis designs and analyzes data acquisition geometries as well as develops and implements reconstruction algorithms for extended-volume C-arm CT.

The thesis consists of three parts. In the first part, we studied three data acquisition geometries and invented two thereof. For these geometries, we investigated their feasibility on a C-arm system and analyzed their possibility for efficient, theoretically-exact and -stable (TES) reconstruction algorithms. We observed that the reverse helical trajectory is a good start for real data test and the novel ellipse-line-ellipse trajectory is a good candidate for efficient TES image reconstruction. In the second part, we developed and implemented geometry-specific reconstruction algorithms. For the reverse helix, we designed three Feldkamp-Davis-Kress (FDK)-type reconstruction methods. Among the three methods, the Fusion-RFDK and Fusion-HFDK methods are preferred as they are more practical and produce acceptable images for extended-volume C-arm CT. For the ellipse-line-ellipse trajectory, we established an efficient TES reconstruction scheme, which makes proficient use of the geometry of this trajectory. In the third part, we conducted the first experiment for extended-volume C-arm CT on a laboratorial Artis zeego system. In this experiment, cone-beam data were reliably acquired using the reverse helical trajectory and 3D images were successfully reconstructed by the Fusion-RFDK method. The consistency among theoretical understanding, simulation results and achieved image quality from a real system strongly demonstrate feasibility of extended-volume C-arm CT in the interventional room.
Zusammenfassung

Die C-Bogen Computertomographie (CT) ist eine bahnbrechende Bildgebungstechnologie, die es im Interventionsraum erlaubt, mit einem angiographischen C-Bogen System tomographische 3D Bilder wie mit einem CT System zu erzeugen. Klinische Studien haben gezeigt, dass diese Technologie in der interventionellen Behandlung helfen kann, Komplikationen zu reduzieren und die Effizienz und Sicherheit des Eingriffs zu verbessern. Jedoch kann heute die C-Bogen CT nur einen axial kurzen Abschnitt des Objektes darstellen, da nur eine einzige, kreisförmige Rotationsaufnahme durchgeführt wird. Diese Limitation kann in einigen intraoperativen Situationen zu Beeinträchtigungen führen, wenn die Darstellung eines längeren Abschnitts wichtig wäre wie z.B. bei Wirbelsäuleneingriffen. Ein neues Verfahren, die C-Bogen CT langer Objekte, soll entwickelt werden. Diese Arbeit möchte zu dieser Entwicklung beitragen. Für die C-Bogen CT langer Objekte werden Aufnahmegeometrien und zugehörige Rekonstruktionsalgorithmen entwickelt, analysiert und implementiert.

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Chapter 1

Introduction

Good seasons start with good beginnings.

SPARKY ANDERSON

1.1 C-arm Computed Tomography

C-arm computed tomography (CT) is an innovative imaging technique that is more and more useful in today’s interventional radiology. This technique enables a C-arm system to generate CT-like three-dimensional (3D) images using a set of two-dimensional (2D) X-ray projections, which are typically acquired from a flat panel detector.

C-arm CT was first proposed in the middle of 1990’s in order to perform endovascular imaging on a C-arm system [Sain 94, Fahr 97, Bani 98]. At that time, most C-arm systems acquired projection data with an X-ray image intensifier, which is large in size and may produce distorted images with a veiling glare [Rowl 00]. Consequently, C-arm CT was primarily used for high-contrast 3D imaging.

In the late 1990’s, the flat panel detector was introduced on a C-arm system. Compared to an X-ray image intensifier, the flat panel detector has a significantly smaller volume and saves space for patient positioning. In terms of quantization, the pixel depth of a state-of-the-art flat panel detector is 16 bits, whereas the pixel depth of an X-ray image intensifier is typically 12 bits. In addition, the flat panel detector offers distortion free images with high “low-contrast resolution”. With these marked improvements in detector technology, a C-arm system is able to assist doctors with better angiography imaging as well as minimally-invasive surgery guidance. In particular, low-contrast C-arm CT is now possible on a C-arm system that is equipped with a flat panel detector.

C-arm systems have been continuously refined, and the capability of the state-of-the-art C-arm systems is far beyond the initial objective. The advancement of the C-arm system brings a lot of potential for new clinical applications of C-arm CT. Recently, encouraging clinical applications of C-arm CT were reported in neurovascular imaging, interventional procedure guidance, and post-surgery assessment [Wall 08]. For example, C-arm CT
was used for placing stent grafts for aortic aneurysm [Bink 06], localizing and assessing chemoembolization [Hiro 06, Meye 07], guiding hepatic arterial interventions [Wall 07], as well as assessing the distribution of cement after vertebroplasty [Hode 06]. These promising cases demonstrate that C-arm CT can reduce treatment-related complications and may improve interventional efficacy and safety [Wall 08].

1.2 Motivation: a New Clinical Demand

While there are many reports of the successful applications of C-arm CT, new clinical demands also arise, one of which is axially-extended volume imaging. Because current C-arm systems only employ a circular trajectory for data acquisition, they can only offer a CT image of short coverage along the direction of the patient table. This limitation was encountered as a problem in hepatic vascular interventions where imaging the entire liver is required [Wall 08]. Therefore, axially-extended volume imaging is demanded for C-arm CT. For convenience, we will call this demand extended-volume C-arm CT. That is, by extended-volume we always refer to volume extension in the axial direction, and this abbreviation will be applied throughout the rest of this thesis.

Besides hepatic vascular interventions, extended-volume C-arm CT has potentials in other clinical settings. For some intraoperative or emergency situations, such as localizing embolization within the aorta, or finding structural damage in the vertebral body, extended-volume imaging during the intervention could be crucial, particularly because it would prevent transferring the patient to the CT room, which is time consuming and increases risk for the patient.

1.3 Technical Challenges

Extended-volume C-arm CT is a long-object problem, which aims to reconstruct a portion of an object of unbounded axial support. The long-object problem has been challenging for helical CT, and may not be expected to be different for C-arm CT. Two technical challenges should be considered: i) a proper cone-beam (CB) reconstruction algorithm to address axial data truncation, and ii) a proper source trajectory for data acquisition. Data truncation means part of the projection data is missing. Data truncation typically occurs when the detector is relatively too small for the object. The above two aspects are interdependent, because CB reconstruction algorithms are usually source trajectory specific and source trajectories are usually designed with knowledge of the reconstruction algorithms. Therefore, these two aspects should be conjointly addressed.

In the following, we will first provide a brief review of CB reconstruction algorithms, then explain the constraints on the source trajectory design for extended-volume C-arm CT.
1.3. Technical Challenges

1.3.1 A Brief Review of CB Reconstruction Algorithms

This review aims to provide critical background knowledge about analytical CB reconstruction algorithms that is useful for the problem at hand, i.e., extended-volume C-arm CT. In particular, we review four categories of CB reconstruction algorithms: (i) approximate algorithms, (ii) theoretically-exact and -stable (TES) but impractical algorithms, (iii) the cone-beam-filtered-backprojection (CB-FBP) algorithms as well as (iv) R-line based algorithms. An R-line is a segment of line that connects any two points on a source trajectory. By TES, we mean that the true value of an image can be accurately recovered if noise and discretization errors are ignored, and that these effects produce limited errors. In contrast, by approximate we mean only an approximation of the true image can be reconstructed.

The key features that distinguish algorithms of the above four categories are: exactness, practicability and the ability to handle axial data truncation. In the following, we will first explain the necessary and sufficient data condition for TES image reconstruction, then describe the four categories of reconstruction algorithms.

- **Data sufficiency**
  One natural question is: what type of source trajectories are needed to allow TES image reconstruction? The answer to this question is known as Tuy’s condition [Tuy 83, Finc 85], which is as follows: to reconstruct one point-of-interest (POI), all planes passing through this point must intersect the source trajectory. From a practical viewpoint, it is important to notice that any point within the convex hull [Boyd 04] of a connected source trajectory always satisfies Tuy’s condition. However, C-arm CT usually employs a single circular trajectory, and Tuy’s condition is only satisfied in the plane where the trajectory lies. In other words, TES image reconstruction is not possible for the region-of-interest (ROI) that is outside the trajectory plane. Tuy’s condition is necessary and sufficient, and thus is an important guide for source trajectory design.

- **Approximate algorithms**
  As indicated by the name, approximate reconstruction algorithms generate non-exact results. Inexactness in the results produced by these approximate algorithms is caused by CB artifacts, which is false information added to the true image value due to incorrect handling of projection data or lack of Tuy’s condition. In practical viewpoint, approximate reconstruction algorithms, such as the Feldkamp-Davis-Kress (FDK) algorithm [Feld 84] and the Single-Slice-Rebinning algorithm [Noo 98], are efficient, stable and easy to implement. In addition, most approximate reconstruction algorithms allow data truncation. The above mentioned features are all favorable for medical imaging and make approximate algorithms competitive [Wang 07]. Among all approximate reconstruction algorithms, two kinds are of particular interest. Algorithms of the first kind consist of two steps: a rebinning process and 2D image reconstruction [Tagu 98, Noo 99, Hu 99]. The rebinning process rearranges CB projection data to a stack of fan-beam (FB) or parallel-beam (PB) projection data.
These FB or PB projections are then used for 2D image reconstruction. The final reconstruction results are obtained by combining all these 2D reconstruction results. Algorithms of the second kind are the FDK algorithm \cite{Feld84} and its variants \cite{Yan92, Wang93, Sour03, Kudo04}. These algorithms borrow concepts from 2D reconstruction algorithms and directly extend these concepts for 3D reconstruction. The FDK-type algorithms are of FBP style, which is efficient, stable and easy to implement. Although the FDK algorithm \cite{Feld84} was introduced empirically, it was subsequently demonstrated that this algorithm is a good approximation to some TES CB reconstruction algorithms. Specifically, as will be discussed later, for the circular trajectory, which does not produce sufficient projection data for TES image reconstruction outside the orbit plane, the TES CB-FBP method of Defrise \cite{Defr94} degenerates to the FDK algorithm.

- **TES but impractical algorithms**
  Algorithms in this category are TES and are good references for developing new reconstruction algorithms. These algorithms are capable of producing exact image reconstructions without data truncation. With truncated projection data, these algorithms become approximate.

  One fundamental formula for exact 3D CB reconstruction is the inverse formula for the 3D Radon transform \cite{Rado05, Natt01}. A 3D Radon transform is an integral taken over a plane that is referred to as a Radon plane. Using this formula for TES reconstruction of a given POI, the value of the 3D Radon transforms is required to be known for all the Radon planes that go through this POI and its neighbourhood. This requirement indicates that data truncation is not allowed. Also, this formula utilizes the 3D Radon transform, which is usually not readily available in practice. Without a link between the 3D Radon transform and CB projection data, this formula is impractical for TES CB image reconstruction.

  A well-known link and a TES inversion formula was proposed by Tuy in 1983 \cite{Tuy83}. Although Tuy’s formula directly applies to CB projection data, no implementation scheme was suggested by Tuy. Also, Tuy’s formula does not allow data truncation.

  In 1985, Smith proposed another TES inversion method \cite{Smit85, Defr93} for CB reconstruction. The main result of this method is an intermediate function that links the 3D Radon transform to CB projection data. Once the results from the intermediate function are obtained, TES image reconstruction can be obtained by a convolution-backprojection scheme. The Smith method does not allow data truncation.

- **CB-FBP algorithms**
  The CB-FBP algorithms \cite{Defr94, Kudo94b} are TES, but not straightforward for truncated projection data. These algorithms use Grangeat’s formula \cite{Gran91}, which is a significant result that effectively links the 3D Radon transform to CB projection data. The name of the CB-FBP algorithms comes from the fact that they follow the filtered-backprojection scheme, which is a marked improvement from the Smith
1.3. Technical Challenges

method in terms of implementation cost. Note that for reconstruction of a single
POI, filtering lines of all radial directions are usually involved in the detector plane
for each projection view, which is computationally intensive. Because of this rea-
son, TES image reconstruction from truncated CB projection data is in general very
challenging for the CB-FBP algorithms. However, in some circumstances, a cer-
tain degree of data truncation is allowed if the redundancy in the projection data is
addressed smartly \cite{Kudo94a,Noo98}.

- **R-line based algorithms**

Algorithms in this category are capable of efficiently producing TES reconstruction
results for any POI that lies on an R-line. More importantly, these algorithms are
able to produce TES reconstructions with truncated projection data. Three kinds of
algorithms belong to this category: (i) the Katsevich’s algorithm, (ii) the R-line based
FBP algorithms and (iii) the DBP algorithms. Here, DBP stands for differentiated-
backprojection. The three kinds of algorithms come with the same limitation: TES
image reconstruction can be achieved only at positions that lie on R-lines.

The Katsevich’s algorithm \cite{Kats02} was first proposed for helical CT in 2002. This
algorithm allows TES image reconstruction in a simple convolution-backprojection
manner. A significant advantage of the Katsevich’s algorithm compared to the CB-
FBP algorithms is: for reconstruction of a POI, only a few filtering lines in the de-
tector plane are involved for each projection view. As a result, if the filtering lines
are not parallel to the axial direction, data truncation in the axial direction can be
allowed. However, the filtering directions are not trivial. This is because Katse-
vich’s algorithm requires a sophisticated piece-wise weighting function to address
data redundancy and the directions of the filtering lines are partially determined by
this weighting function. The Katsevich’s algorithm is strongly related to Grangeat’s
formula and the CB-FBP algorithms; a detailed description of this link can be found
in \cite{Noo12}.

The R-line based FBP algorithms \cite{Zou04a,Pack05c,Ye05a} are important variants
of the Katsevich’s algorithm. Different from Katsevich’s algorithm, these formulae
select the filtering lines at the very beginning, and do not require any specific form
for the weighting function. Hence, data truncation handling in these algorithms with
given filtering lines is more transparent than in Katsevich’s algorithm. However,
noise propagation from the CB data to the image is less obvious, because the function
weighting redundancies in CB data is never built.

The DBP algorithms \cite{Pack05b,Zou04b,Ye05b} are based on the finite inverse
Hilbert transform \cite{Mikl57}. One important characteristic of the DBP algorithms is
that: they perform the filtering operation in the spacial domain after backprojection.
This characteristic brings a lot of flexibility in terms of data truncation, especially
because they are capable of addressing data truncation not only in the axial direc-
tion but also in the transverse direction, which is usually neither possible for the
Katsevich’s algorithm nor for the R-line based FBP algorithms.


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\subsection*{1.3.2 Constraints on the Source Trajectory}

When a source trajectory for extended-volume C-arm CT is considered, three aspects require special attention: (i) mechanical limitations, (ii) reconstruction algorithms and (iii) practical concerns, as discussed below.

- **Mechanical limitations:** because of the open design and the lack of slip-ring technology, a C-arm system cannot rotate non-stop in a single direction. In some clinical situations, a C-arm system may rotate 400° in one direction. However, in most cases, this angular length is limited to a short scan (e.g., 240°). These motion constraints imply that a source trajectory for extended-volume C-arm CT must present a reverse pattern in its rotational direction, preferably with a short scan per rotation.

- **Reconstruction algorithms:** TES reconstruction algorithms are preferred, hence source trajectories should satisfy Tuy’s condition. Among these TES reconstruction algorithms, R-line based TES reconstruction algorithms \cite{Kats02, Zou04a, Pack05c, Ye05a, Pack05b, Zou04b, Ye05b} are attractive, because they are capable of handling data truncation in the axial direction. These algorithms require the ROI to be fully covered by R-lines. Therefore, in terms of algorithms, a source trajectory for extended-volume C-arm CT is preferred if it satisfies Tuy’s condition and offers full R-line coverage in the ROI. R-line coverage refers to the space that is covered by R-lines.

- **Practical concerns:** source trajectories that correspond to smooth and continuous scans are preferred, because long pauses in X-ray exposure can be avoided and scan time can be reduced. For vascular interventions, reduction of scan time is crucial to motion suppression and injection of contrast medium.

\subsection*{1.3.3 Demands on R-lines}

We would like to emphasize how important the R-line coverage of a data acquisition geometry is for extended-volume C-arm CT. As mentioned in Section 1.1, the state-of-the-art C-arm system is equipped with a flat panel detector, which is fairly wide compared to detectors on a CT system. For example, the flat panel detector on the Artis zeego system (Siemens AG, Healthcare Sector, Erlangen, Germany) is about the size 30 cm × 40 cm and is capable of producing projection images of maximum size 2480 × 1920. This big detector yields a large cone angle for data acquisition.

Larger cone angles can introduce more CB artifacts for approximate reconstruction algorithms, especially for trajectories that do not satisfy Tuy’s condition. In some cases, CB artifacts can impair image quality so much that they are intolerable for radiologists \cite{Sieb09}. In these circumstances, TES reconstruction algorithms are strongly desired.

Extended-volume C-arm CT is a long-object problem, and axial data truncation cannot be avoided. In order to perform TES image reconstruction with axially-truncated data, the most attractive candidates are the R-line based TES reconstruction algorithms, which
strongly rely on the R-line coverage of a source trajectory. Therefore, the R-line coverage becomes a key performance indicator of a source trajectory that is used for extended-volume C-arm CT. Moreover, according to the R-line based TES reconstruction algorithms, understanding the spatial distribution of the R-lines of a source trajectory is crucial to develop an efficient image reconstruction method. In this thesis, a good amount of effort will be devoted to analysis of R-line coverage.

1.4 State-of-the-Art

In the literature, to our knowledge, three geometries are possibly feasible for extended-volume C-arm CT data acquisition: (i) the sequential circular trajectory [Kohl 01, Hsie 06, Zhen 12], (ii) the two-circles-plus-one-line trajectory [Lu 12] and (iii) the reverse helical trajectory [Cho 08].

- **The sequential circular trajectory**
  This trajectory consists of multiple parallel concentric circles that are of equal distance. To implement this trajectory on a C-arm system, the adjacent circles should be of opposing rotational directions. Despite the existence of spiral CT, this trajectory is still attractive for practical long-object CB CT [Kohl 01, Hsie 06], and may be beneficial for linear-accelerator-mounted CB CT in image-guided radiation therapy [Zhen 12]. However, this trajectory does not satisfy Tuy’s condition, and thus TES image reconstruction is not possible; reconstruction results from this trajectory suffer from CB artifacts.

- **The two-circles-plus-one-line trajectory**
  This trajectory consists of two parallel concentric circles connected by a line that is parallel to the central axis. It was proposed for CB CT on a Kodak Image Station [Lu 12]. This trajectory not only satisfies Tuy’s condition, but also has full R-line coverage in its convex hull, and thus can be used for efficient TES image reconstruction. Although we can introduce a reverse pattern by making the rotational directions of the two circles opposite, it is difficult to implement this trajectory on a C-arm system, because it requires the two circles to be complete (360° long). To employ this trajectory for TES extended-volume C-arm CT, some modifications are needed [Yu 11b, Yu 12b].

- **The reverse helical trajectory**
  This trajectory was originally proposed for image-guided radiation therapy using a linear-accelerator-mounted CT system. It consists of two consecutive helical turns with opposing rotational directions. This trajectory is the most attractive among the three for extended-volume C-arm CT, because it i) possesses a reverse pattern in its rotational direction, ii) allows short scans for each helical turn, and iii) satisfies Tuy’s condition. One challenge with this trajectory is that there is a region right in the center of its convex hull that is not covered by R-lines, and thus efficient
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TES image reconstruction becomes challenging, particularly in the presence of data truncation. At this stage, two TES solutions have been developed for the reverse helical trajectory, and we explain them as follows.

One of the solutions was proposed by the group of F. Noo at the University of Utah [Noo 09]. Their algorithm consists of two steps. First, the region of the object that is covered by R-lines is reconstructed by the DBP method. The forward projection of the intermediate reconstruction result is then subtracted from the original projection data to obtain non-truncated projection data of the region of the object that is not covered by R-lines. In the second step, the FBP formulation of the method of Grangeat is employed to obtain the region of the object that does not have R-line coverage. The other solution was proposed by the group of X. Pan at the University of Chicago [Cho 10]. The idea from Pan’s group is the same as that of Noo’s group, except that they use the Pack-Noo formula [Pack 05a] in the second step for reconstruction of the region that is not covered by R-lines.

Both solutions produce TES reconstruction results while allowing axial data truncation. However, both of them need one forward projection and two backprojections, and thus are not computationally efficient. Moreover, they require combining volumes with different shift-variant resolutions [Noo 09, Cho 10], which tends to yield undesirable artifacts.

1.5 Original Contributions

In this thesis, we present two novel source trajectories, four new reconstruction methods, and the first experimental demonstration of extended-volume C-arm CT using a state-of-the-art C-arm system. These original contributions are briefly described as follows.

- **Source trajectories for extended-volume C-arm CT**
  We have developed two novel source trajectories for extended-volume C-arm CT, i.e., the arc-extended-line-arc (AELA) trajectory and the ellipse-line-ellipse (ELE) trajectory. The AELA trajectory can be implemented on a C-arm system, because it not only possesses a reverse pattern in rotational direction, but also provides sufficient R-line coverage in the ROI while permitting short scans. However, this trajectory is discontinuous and requires relatively long pauses in X-ray exposure. To overcome this problem, we then proposed the ELE trajectory, which is smooth and continuous while retaining all the advantages of the AELA trajectory. We have performed thorough analyses of the R-line coverage of these two source trajectories. The AELA trajectory was presented at an international conference [Yu 10] and was published in a peer-reviewed journal article [Yu 11b]. The ELE trajectory was presented at an international conference [Yu 12b] and is ready to be submitted as a journal article [Yu 13c].
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- **Three approximate reconstruction methods for the reverse helix**
  We have developed three practical FDK-type reconstruction methods for the reverse helical trajectory, namely, the Fusion-RFDK, Fusion-HFDK and Voxel-Dependent-HFDK. These methods are computationally efficient and do not have the problem of resolution matching as presented in [Noo 09, Cho 10]. Although CB artifacts and discretization errors are present, all three FDK-type reconstruction methods are capable of producing results of acceptable image quality. These methods were presented at an international conference [Yu 11a], and are ready to be submitted as a journal article [Yu 13a].

- **A TES reconstruction method for the ELE trajectory**
  We have established a novel TES reconstruction scheme for the ELE trajectory. First, based on the understanding of the spatial distribution of the R-lines, we create a set of R-line surfaces that fully covers the ROI. Then, we perform image reconstruction on these R-line surfaces using the DBP method. Finally, image reconstructions are obtained by a rebinning process from the R-line surfaces to the Cartesian image space. This reconstruction scheme is efficient and practical. We demonstrate these characteristics using computer-simulated data. This result was presented at an international conference [Yu 13b].

- **Real C-arm data**
  We have conducted the first experiment for extended-volume C-arm CT using a reverse helical trajectory that was implemented on a laboratorial floor-mounted Artis zeego system (Siemens AG, Healthcare Sector, Erlangen, Germany). First, we designed a practical method for trajectory calibration. This method can be considered as an extension of the method presented in [Stro 03], which was proposed for a circular trajectory. In particular, a new axially-long calibration phantom was designed to serve for a long-object imaging purpose. On the other side, for noise and artifact reduction, the projection data were preprocessed using classical data corrections. Next, an ideal reverse helix that best fitted the calibrated trajectory was found through an optimization procedure, and the projection data of this ideal reverse helix were created through a rebinning process. Finally, we performed image reconstruction using the Fusion-RFDK method [Yu 11a, Yu 13a]. These first experimental results demonstrate that a reverse helical trajectory can be practically implemented on a state-of-the-art multi-axis C-arm system and 3D imaging using projection data from such a source trajectory is achievable. We believe this work is a step forward towards feasible extended-volume C-arm CT in the interventional suit. This work was presented at an international conference [Yu 12a].

In addition, although not presented in this thesis, the following publications represent two important by-products of our investigation.

- **Simulation tools for 2D X-ray CT experiments**
  We have developed tools for computer simulation using a 2D phantom, namely the
Chapter 1. Introduction

2D FORBILD head phantom, which models the central axial slice through the FORBILD head phantom. In detail, we first offer the definitions of the 2D FORBILD head phantom for monochromatic and polychromatic X-ray beams. Using these phantom definitions, we then provide two functions, one to generate the digital 2D FORBILD head phantom and the other to generate its line integrals. These two functions are provided at [Yu 12d, Appendix B] and are ready-to-use by copy and paste. Also note that these two functions can be used for other 2D phantoms as well. This tool is important for development of new reconstruction algorithms. This work was published in a peer-reviewed journal [Yu 12d].

- A novel formula for efficient TES CB reconstruction outside R-lines

We have developed a new inversion scheme for TES CB image reconstruction. This scheme can be considered as an extension to the scheme that was established by Pack and Noo [Pack 05c, Pack 05a]. Based upon this new scheme, we proposed a novel inversion formula that is capable of performing efficient TES reconstructions outside R-lines. The new formula is comparable to Pack-Noo formula D [Pack 05a] but with different data acquisition flexibility and filtering choices. This original contribution was presented at an international conference [Yu 12c].

Altogether, this thesis is supported by six conference presentations [Yu 10, Yu 11a, Yu 12b, Yu 12a, Yu 12c, Yu 13b] and four journal articles [Yu 11b, Yu 12d, Yu 13a, Yu 13c].

1.6 Thesis Outline

This thesis is composed of three parts: (I) source trajectory design, (II) development of trajectory-specific reconstruction algorithms, and (III) real data experiments. The structure of the thesis is as follows.

Before giving these three parts, we describe some basics of a C-arm system in Chapter 2 and introduce our mathematical notations and geometry conventions in Chapter 3.

Part I consists of Chapters 4, 5 and 6. Each chapter introduces one source trajectory for extended-volume C-arm CT. Other than the reverse helix, the AELA and ELE trajectories are novel. We provide a brief description of these three chapters as below.

- Chapter 4: The geometry of the reverse helical trajectory is described, and the \( \pi \)-line and R-line coverages are investigated.

- Chapter 5: The arc-line-arc (ALA) and the AELA trajectories are introduced, and their R-line coverages are studied. Each trajectory can be decomposed into three fundamental trajectories, i.e., the arc-line, arc-arc, and line-arc trajectories. The R-line coverage of each fundamental trajectory is then carefully studied. The R-line coverage of the ALA or AELA can be considered as the union of the R-line coverages

--1http://www.imp.uni-erlangen.de/phantoms/head/head.html
of these fundamental trajectories. In particular, we provide an analytical method to identify all the R-lines that go through any given POI.

- Chapter 6: The ELE trajectory is introduced, and its R-line coverage is thoroughly analyzed. This trajectory can be decomposed into three fundamental trajectories, and each fundamental trajectory is carefully studied in terms of R-line coverage. The R-line coverage of the ELE trajectory is considered as the union of the R-line coverages of these fundamental trajectories. Similar to the ALA (or AELA) trajectory, we also provide an analytical method to identify all the R-lines that go through each POI.

Part II consists of Chapters 7 and 8. Chapter 7 provides three approximate reconstruction algorithms for the reverse helix, whereas Chapter 8 provides a TES reconstruction algorithm for the ELE trajectory. Details of these two chapters are as follows.

- Chapter 7: Three FDK-type reconstruction algorithms are introduced for the reverse helix. These algorithms are thoroughly evaluated using computer simulations. The FORBILD head and the XCAT phantoms are used for data simulation, and the resolution matched reconstruction results are evaluated with respect to image bias and image noise. Image bias is a combination of CB artifacts and discretization errors, whereas image noise is caused by adding Poisson noise to the projection data.

- Chapter 8: A reconstruction scheme based on the DBP algorithm is proposed for the ELE trajectory. We first introduce the DBP algorithm with mathematical details. Then we offer a method to select R-lines of the ELE trajectory for the DBP algorithm. We assess this reconstruction scheme by computer simulations using a modified FORBILD head phantom.

Part III is about real C-arm data, and is presented in Chapter 9. In this chapter, we demonstrate the first experiment for extended-volume C-arm CT. This experiment was achieved by implementing a five-turn reverse helical trajectory on a laboratorial Artis zee-goo multi-axis C-arm system (Siemens AG, Healthcare Sector, Erlangen, Germany). We extended the method presented in [Stro 03] and performed a dedicated trajectory calibration for the reverse helix. We obtained 3D image reconstruction by using the Fusion-RFDK method [Yu11a, Yu13a]. The results of trajectory calibration and image reconstruction show that the reverse helical trajectory can be reliably implemented on a state-of-the-art multi-axis C-arm system and that 3D imaging for axially extended field-of-view (FOV) is possible using projection data acquired from such a trajectory. This experiment successfully demonstrates the first example of extended-volume C-arm CT in the interventional suit.

We provide the conclusion and outlook of this thesis in Chapter 10.
Chapter 1. Introduction
Chapter 2

Basics of a C-arm System

Since its introduction in 1960, the C-arm system has evolved into a powerful tool for interventional radiologists. Today, some low-end C-arm systems still employ an X-ray image intensifier, whereas high-end C-arm systems are typically equipped with a flat panel detector. To perform CB CT, a C-arm system equipped with a flat panel detector is usually needed. Such a system is currently provided by various vendors, such as syngo DynaCT (Siemens AG, Healthcare Sector, Erlangen, Germany), Xper CT (Philips Healthcare, Andover, USA), Innova CT (GE Healthcare, Chalfont St. Giles, UK), Infinix X-ray (Toshiba, America Medical Systems, Tustin, USA), O-ARM CT (Medtronic Navigation, Louisville, USA) and Vision CT (Ziehm Imaging GmbH, Nuremberg, Germany).

2.1 Major Components

The major components of a typical C-arm system in the intervention room consist of a C-arm gantry, a supporting stand and the patient table. On one end of the C-arm gantry, an X-ray tube and a collimator are attached. Typically, an X-ray beam is generated with automatic exposure control [Stro 09 Schw 10], which is crucial for image quality. On the other end of the C-arm gantry, a flat panel detector is attached. This detector, e.g., the one on the Artis zeego system, is usually 30 cm × 40 cm with square pixels of side 154 µm and bit depth 16 bits. The X-ray tube, the collimator and the detector are exactly aligned on the C-arm gantry. Theoretically, these three components should be stationary to each other during scanning.

The C-arm gantry can be supported by various types of stands, such as a mobile stand, a floor- or ceiling-mounted stand, or a multi-axis floor mounted robot, as shown in Figure 2.1 These configurations offer different motion flexibility to a C-arm system, and hence provide radiologists more planning choices in the interventional room.
Chapter 2. Basics of a C-arm System

Figure 2.1: Four examples of Siemens C-arm systems with CB CT capabilities (Siemens AG, Healthcare Sector, Erlangen, Germany). Top left: the Cios Alpha mobile C-arm system. Top right: the Artis zee floor-mounted system. Bottom left: the Artis zeeego multi-axis system. Bottom right: the Artis zee ceiling-mounted system.

A typical C-arm system in the interventional room is equipped with a special patient table. Unlike the patient table of a CT system, the patient table of a C-arm system has a floating feature so that physicians can easily adjust the patient position for interventional operations. The cost of this floating feature is that, the patient table of a C-arm system can not move as precisely as a CT system.

2.2 C-arm CT Data Acquisition

Because of the open design, a C-arm system has tremendous flexibility in mechanical motions, e.g., the Artis zeeego system can produce different motions using its multi-axis stand. On the other side, the open design reduces motion stability, hence a C-arm system is not able to produce an ideal trajectory for data acquisition. Consequently, a calibration process is required to address deviations from the ideal trajectory [Roug93, Kopp95, Silv00, Wies00, Stro03, Hopp07]. The specific trajectory has to be calibrated before the patient is
2.3. **C-arm CT Data Correction**

scanned. Fortunately, a state-of-the-art C-arm system only needs to be calibrated less than once a year.

Regarding data acquisition from a C-arm system, different vendors may offer different procedures. In this context, we take the Siemens syngo DynaCT as an example, and describe its procedure for data acquisition as follows.

First, the patient (or a phantom) is carefully positioned such that the ROI is right in the center of the FOV. A safety run is then performed using a pre-calibrated source trajectory, i.e., the C-arm gantry is moved to the end position of the trajectory, then to the start position along the trajectory to exclude potential collision. Once the safety run is completed, a fluoroscopic X-ray pulse is activated for automatic exposure control. The C-arm is then ready for CB data acquisition.

The time needed for data acquisition is dependent on C-arm models and operating modes. With an open design, the scan speed of a state-of-the-art C-arm system is very impressive. For example, the rotational speed of the Artis zeego system (Siemens AG, Healthcare Sector, Erlangen, Germany) can be up to 60°/sec in the 3D-acquisition mode.\(^1\)

2.3 **C-arm CT Data Correction**

Before raw data can be used for CB image reconstruction, a logarithm should be first applied to obtain an approximation of the line integral of the linear attenuation coefficients. Then, a series of preprocessing procedures should be employed so as to combat noise and various artifacts. These procedures include overexposure correction, scatter correction, beam hardening correction and truncation correction. For technical details, we refer to [Zell 05, Stro 09].

Chapter 2. Basics of a C-arm System
Chapter 3

Mathematical Notations, Cone-Beam Geometry and Line Concepts

We shape our tools and afterwards our tools shape us.

MARSHALL McLuhan

This chapter provides some conventions that are used throughout this thesis.

3.1 Mathematical Notations

We denote a vector or a position vector by a lowercase letter with an underline. We use this notation for spaces of different dimensions. For example, in 2D, we denote $\mathbf{x}$ as a point in Cartesian coordinates $(x, y)$, whereas in 3D, we refer to $\mathbf{x}$ as a point in coordinates $(x, y, z)$. As a special case, we denote a 2D unit vector with polar angle $\theta$ as $\mathbf{\theta} = (\cos \theta, \sin \theta)$.

Three vector operators will be intensively used: the Euclidean vector norm $|| \cdot ||$, the inner vector product $\cdot$ and the cross vector product $\times$. For example, let $\mathbf{x}$ and $\mathbf{y}$ be two vectors. Then, the Euclidean norms of $\mathbf{x}$ and $\mathbf{y}$ are indicated by $||\mathbf{x}||$ and $||\mathbf{y}||$, respectively, whereas the inner and cross products of $\mathbf{x}$ and $\mathbf{y}$ are indicated by $\mathbf{x} \cdot \mathbf{y}$ and $\mathbf{x} \times \mathbf{y}$, respectively.

In some circumstances, for geometrical description, we use uppercase letter to symbolize a point and an underline below two uppercase letters to indicate a vector. The distance between two points is denoted by two vertical bars. For example, let $A$ and $B$ be two points, we refer to $\overrightarrow{AB}$ as the vector that points from $A$ to $B$, and denote $|\overrightarrow{AB}|$ as the Euclidean distance between $A$ and $B$.

3.2 CB Geometry Conventions

We denote the linear attenuation coefficient of X-rays as $f$, and its value at a specific location $\mathbf{x}$ as $f(\mathbf{x})$. The function $f(\mathbf{x})$ is assumed to be smooth and compactly supported.
within a cylinder $\Omega$ of radius $r$ that is centered on the $z$-axis. The parameter $r$ is typically interpreted as the radius of the FOV.

We call the source trajectory the vertex path, and the points on it vertex points. The vertex path can be a single curve or a union of several curves. Throughout this thesis, we assume all vertex paths are located on a cylindrical surface $S_c$ that is centered on the $z$-axis. We call $S_c$ the trajectory cylindrical surface, and denote the radius of $S_c$ as $R$, which can be interpreted as the scan radius. Moreover, any curve forming the vertex path can be described by a single parameter called $\lambda$, and a vertex point on this curve is denoted as $a(\lambda)$. Practically, we assume the vertex path to be outside the FOV.

In terms of data acquisition of a C-arm system, we consider a flat panel detector that moves assembled with the source position at a constant distance $D$ from $a(\lambda)$. The detector is positioned such that it is parallel to the $z$-axis and orthogonal to the plane that contains $a(\lambda)$ and the $z$-axis (Figure 3.1).

![Figure 3.1: CB scan geometry with a flat panel detector. The detector is oriented such that it is parallel to the $z$-axis and orthogonal to the plane that contains the vertex point $a(\lambda)$ and the $z$-axis. The distance between $a(\lambda)$ and the detector is set to be constant $D$. The detector rows and columns are identified by Cartesian coordinates $u$ and $v$, respectively. The origin of the detector, $O_d$, is defined to be the orthogonal projection of $a(\lambda)$ onto the detector.

The flat panel detector is spanned by two orthonormal vectors $e_u$ and $e_v$, and a point in this detector can be specified by two Cartesian coordinates, $u$ and $v$, which are measured along $e_u$ and $e_v$, respectively. The origin of the $(u,v)$-coordinate system is defined as the orthogonal projection of the vertex point $a(\lambda)$ onto the detector, and this origin is denoted as $O_d$. To complete the 3D Cartesian coordinate system, we introduce the unit vector $e_z$.}
3.3. Line Concepts

\( \mathbf{e}_w = \mathbf{e}_u \times \mathbf{e}_v \). Theoretically, \( \mathbf{e}_v \) is parallel to the z-axis, whereas \( \mathbf{e}_u \) and \( \mathbf{e}_w \) are parallel to the \((x,y)\)-plane.

Let \( \mathbf{a} \) be a unit vector of an X-ray beam pointing from \( \mathbf{a}(\lambda) \) to the detector, and let \( g(\lambda, \mathbf{a}) \) be the line integral of \( f(\mathbf{x}) \) along this X-ray beam. Then the line integral can be expressed as below

\[
g(\lambda, \mathbf{a}) = \int_{-\infty}^{\infty} f(\mathbf{a}(\lambda) + t\mathbf{a}) \, dt. \tag{3.1}
\]

Depending on the context, \( \mathbf{a} \) can be identified in different ways. If the X-ray beam is identified by the \((u,v)\)-coordinates of the position where it hits the detector, then \( \mathbf{a} \) can be formulated as below

\[
\mathbf{a}(\lambda, u, v) = \frac{u \mathbf{e}_u + v \mathbf{e}_v - D \mathbf{e}_w}{\sqrt{u^2 + v^2 + D^2}}.
\]

Otherwise, if the X-ray beam is identified by a point of the object, \( \mathbf{x} \), then \( \mathbf{a} \) can be expressed as

\[
\mathbf{a}(\lambda, \mathbf{x}) = \frac{\mathbf{x} - \mathbf{a}(\lambda)}{||\mathbf{x} - \mathbf{a}(\lambda)||}.
\]

Consequently, with a slight abuse of notation, we sometimes also denote the line integral by \( g(\lambda, u, v) \) or \( g(\lambda, \mathbf{x}) \), depending on the context.

### 3.3 Line Concepts

In this section, we introduce three types of lines: (i) \( \pi \)-line, (ii) C-line and (iii) R-line. These line concepts are important for both trajectory analysis and image reconstruction.

- **\( \pi \)-line**: A \( \pi \)-line is defined for the helix. It refers to a segment of line that connects two source positions on a helix with an angular distance less than 360°; see Figure 3.2 (top). The portion of the helix that is bounded by a \( \pi \)-line is called a \( \pi \)-line segment, whereas the region covered by \( \pi \)-lines is called \( \pi \)-line coverage.

- **C-line**: A C-line is defined for the reverse helix. A reverse helix is composed of two helical turns called the upper helix and the lower helix. These two helical turns have opposite rotational directions and are connected to each other via a so-called kink point; see Figure 3.2 (middle). More details can be found in Chapter 4. A C-line is defined as a segment of line that connects a source point on the upper helix to a source point on the lower helix. The portion of the reverse helix bounded by a C-line is called a C-line segment, whereas the region covered by C-lines is called C-line coverage. Here, \( C \) stands for cross.

- **R-line**: An R-line is a segment of line that connects any two source points on any arbitrary trajectory; see Figure 3.2 (bottom). The portion of the source trajectory that is bounded by an R-line is called an R-line segment, whereas the region covered by R-lines is called R-line coverage. By definition, both \( \pi \)-lines and C-lines are R-lines. Here, \( R \) stands for redundant.
Figure 3.2: Illustration of a $\pi$-line (top), a C-line (middle) and an R-line (bottom).
Chapter 4

Reverse Helical Trajectory

There is no royal road to geometry.

EUCLID OF ALEXANDRIA

The reverse helical trajectory was proposed in 2008 [Cho08] for linear-accelerator-mounted CT systems. This trajectory is feasible on a C-arm system and satisfies Tuy’s condition, and thus has considerable potential for exact CB image reconstruction. In this chapter, we will first describe the geometry of the reverse helix, then we will study the $\pi$-line and R-line coverage of this trajectory. A $\pi$-line is a special kind of R-line, which can be defined as any line segment that connects two source points on helix with an angular distance less than 360°.

4.1 Geometry

The reverse helix is very similar to the classical helix that is used in CT. In fact, both source trajectories involve a rotation around the $z$-axis as well as a translation in $z$ that depends linearly on the rotation angle. The only difference between the reverse helix and the classical helix is the direction of rotation. For the classical helix, the direction of rotation is always the same, whereas for the reverse helix, the direction of rotation alternates between clockwise and counter-clockwise. The alternation occurs once the rotation angle has reached a fixed maximum value that is denoted as $\lambda_m$ hereafter. Figure 4.1 illustrates a reverse helix that consists of 5 turns around the patient table with $\lambda_m = 300^\circ$.

As discussed in Section 1.3.2, the reverse helix defines a practical method to perform continuous long-object imaging with a C-arm system. In this context, the rotation angle $\lambda_m$ that triggers alternations in the rotational direction is the result of real physical constraints: the X-ray tube on a C-arm system can only be rotated by a fixed amount in a single direction.

For the purposes of image reconstruction, it is sufficient to focus on two consecutive turns of the reverse helix. Let $H$ be the distance covered in $z$ per helical turn, then two
Chapter 4. Reverse Helical Trajectory

Figure 4.1: Illustration of a reverse helix with 5 turns. In this example, the direction of rotation is alternated every 300 degrees. The turns are numbered from 1 to 5 as shown.

consecutive turns cover a distance of $2H$ and these two turns are used for reconstruction of a central cylindrical region of length $H$. Thus, in Figure 4.1, the 5 turns yields 4 pairs of consecutive turns, namely turns 1-2, 2-3, 3-4 and 4-5, each of which allows a reconstruction over a cylindrical region of length $H$. The 4 regions that result from these 4 pairs are adjacent and cover together a region of length $4H$.

Given the above observations, the reverse helix will be pictured, from now on, as the simple union of two consecutive turns. Without loss of generality, the first turn is chosen to be in the clockwise direction and the second turn in the counterclockwise direction, as shown in Figure 4.2. The mathematical description of this specific reverse helix is as follows.

Figure 4.2: Reverse helix defined with $\lambda_m = 300^\circ$ and $z_K = 0$. The cylinder $\Omega$ is along the $z$-axis and encompasses the scanned object. Left: 3D visualization. Middle: projection onto the $(x,z)$-plane. Right: projection onto the $(x,y)$-plane.

First, let $K$ be the point where the direction of rotation is reversed, and let $z_K$ be the $z$-coordinate of $K$. This point is called the kink point, and the plane orthogonal to the $z$-axis through $K$ is called the kink plane. The reverse helix is oriented such that the kink point is in the $(x,z)$-plane. The helical turn below the kink plane (in $z$) is referred to as the
lower helix and the one above as the upper helix. The endpoints of the lower helix are \( B \) and \( K \), whereas the endpoints of the upper helix are \( K \) and \( T \). Note that, at the kink point, the \( x \) and \( z \) components of the motion vector of the lower helix are the same as that of the upper helix. Hence, the orthogonal projection of the reverse helix onto the \((x,z)\)-plane has a smooth transition at \( K \); see the middle of Figure 4.2.

Now let \( R \) be the rotational radius and let \( H \) be the distance covered in \( z \) over a rotation of \( \lambda_m \). The X-ray source position along the reverse helix is described using a parameter \( \lambda \) that varies from \(-\lambda_m\) to 0 for the lower helix, and from 0 to \( \lambda_m \) for the upper helix. The expression for the source position at \( \lambda \) is

\[
a(\lambda) = \begin{cases} 
(R \cos \lambda, -R \sin \lambda, H \lambda / \lambda_m + z_h), & \lambda \in [-\lambda_m, 0] \\
(R \cos \lambda, R \sin \lambda, H \lambda / \lambda_m + z_h), & \lambda \in [0, \lambda_m]
\end{cases}
\]  

(4.1)

Geometrically, \(|\lambda|\) is equivalent to the polar angle \( \theta \) that characterizes the \( x \) and \( y \) coordinates of the source position. For the lower helix, the first source position is at \( \theta = \lambda_m \) and \( \theta \) decreases towards 0 as the X-ray source is rotated clockwise. For the upper helix, the first source position is at \( \theta = 0 \) and \( \theta \) increases towards \( \lambda_m \) as the X-ray source is rotated counter-clockwise. Accordingly, the kink point is at \( \theta = 0 \), which corresponds to \( \lambda = 0 \). To guarantee sufficient data for reconstruction, \( \Omega \) should be completely within the convex hull of the reverse helix. Therefore, we always select \( \lambda_m \geq \pi + 2\arcsin(r/R) \).

### 4.2 \( \pi \)-line Coverage

Recall that a \( \pi \)-line is any line segment that connects two source points on a regular helix that are separated by less than a \( 360^\circ \) rotation; see [Defr 00] for a detailed discussion on \( \pi \)-lines and their properties. Given that the reverse helix consists of the union of two helices, a number of \( \pi \)-lines can be associated with the reverse helix. In this subsection, we identify the set of points that belong to \( \pi \)-lines within \( \Omega \). Note that this discussion is particularly important for the Voxel-Dependent-HFDK algorithm that will be explained in Chapter 7.

First, we introduce four surfaces of \( \pi \)-lines, called \( S_{1}^{+}, S_{1}^{-}, S_{0}^{+} \) and \( S_{0}^{-} \). These surfaces are depicted in Figure 4.3 for the case where \( \lambda_m \leq 2\pi \). Note that, although the notations of these four surfaces also contain the letter \( S \), they are not related to the cylindrical surface \( S_{c} \). Surface \( S_{1}^{+} \) consists of \( \pi \)-lines that connect endpoint \( T \) of the upper helix to the points that are below and within \( 360^\circ \) of \( T \) on the upper helix, whereas \( S_{1}^{-} \) is obtained by connecting endpoint \( B \) of the lower helix to the points above and within \( 360^\circ \) of \( B \) on the lower helix. On the other hand, \( S_{0}^{+} \) and \( S_{0}^{-} \) are created by connecting \( K \) to the points that are within \( 360^\circ \) of \( K \) on the upper and the lower helices, respectively. Note that the kink plane separates \( S_{0}^{+} \) and \( S_{0}^{-} \) at the kink point \( K \). Note also that \( K \) only belongs to \( S_{1}^{+} \) and \( S_{1}^{-} \) when \( \lambda_m \leq 2\pi \). Moreover, due to properties of \( \pi \)-lines, \( S_{1}^{+} \) is above \( S_{0}^{+} \), whereas \( S_{1}^{-} \) is below \( S_{0}^{-} \).

Next, it is important to note that the reverse helix is a connected trajectory, and Tuy’s condition is satisfied within its convex hull. The convex hull, denoted as \( \Lambda \), is bounded by
Figure 4.3: Spatial distribution of the $\pi$-lines of the reverse helix with $\lambda_m = 300^\circ$. Left: $\pi$-line surfaces corresponding to endpoints $T$ and $B$. Middle: $\pi$-line surfaces corresponding to the kink point. Right: spatial distribution of $\pi$-lines in $\Omega_{\Lambda}$; the regions called $\Omega_{\pi}^u$ and $\Omega_{\pi}^c$ are covered by $\pi$-lines, whereas $\Omega_{\pi}^l$ is not. Here, $\Omega_{\Lambda}$ is the portion of $\Omega$ that is bounded by $S_1^+$ and $S_1^-$. 

three surfaces when $\lambda_m \geq 2\pi$. These surfaces are $S_1^+$, $S_1^-$ and the cylindrical surface of the trajectory $S_c$. When $\lambda_m < 2\pi$, there is one more surface bounding $\Lambda$: the plane that is parallel to the $z$-axis through $T$, $K$ and $B$.

There are two sets of points covered by $\pi$-lines within $\Lambda$: i) the region that is bounded by $S_0^+$ and $S_1^+$, and ii) the region that is bounded by $S_0^-$ and $S_1^-$. We call these the upper and the lower $\pi$-line regions. Within each of these two regions, there is one and only one $\pi$-line passing through each point. In the upper (resp. lower) region, a $\pi$-line connects points from the upper (resp. lower) helix. From these observations, $\Lambda$ can be seen as the union of three non-overlapping subsets: the upper and the lower $\pi$-line regions, and the set of points that is between $S_0^+$ and $S_1^-$. 

Now, the $\pi$-line coverage within $\Omega$ can be identified. Recall that $\lambda_m > \pi + 2\arcsin(r/R)$ is assumed. Under this condition, let $\Omega_{\Lambda}$ be the portion of $\Omega$ between $S_1^+$ and $S_1^-$. By construction, $\Omega_{\Lambda}$ is entirely within $\Lambda$. Therefore, based on the discussion above, $\Omega_{\Lambda}$ can be divided into three non-overlapping regions, called $\Omega_{\pi}^u$, $\Omega_{\pi}^c$ and $\Omega_{\pi}^l$. Here, $\pi$ stands for $\pi$-line, whereas $u$, $c$, $l$ stand for upper, central and lower, respectively. Region $\Omega_{\pi}^u$ identifies the points that belong to $\pi$-lines from the upper helix, whereas $\Omega_{\pi}^l$ identifies the points that belong to $\pi$-lines from the lower helix. Region $\Omega_{\pi}^c$ is between $\Omega_{\pi}^u$ and $\Omega_{\pi}^l$ and consists of points that do not belong to any $\pi$-line. We call this region the core of missing $\pi$-lines. As illustrated in the right of Figure 4.3, $\Omega_{\pi}^c$ is fairly large compared to $\Omega_{\pi}^u$ and $\Omega_{\pi}^l$.

### 4.3 R-line Coverage

Let a C-line be a line segment that connects one point on the upper helix to another point on the lower helix, and let the set of points that belong to the C-lines be the C-line coverage. By definition, the R-line coverage is the union of the $\pi$-line coverage and the C-line
4.3. R-line Coverage

coverage. Note that $\pi$-line surfaces $S_0^+$ and $S_0^-$ belong to both the $\pi$-line coverage and the $R$-line coverage of the reverse helix.

Recall from the previous section that $\Omega^\pi_c$ is not covered by $\pi$-lines. Unfortunately, the C-lines can only cover part of this missing $\pi$-line region, leaving a region that is not covered by any type of R-lines. Because this is the only region that is not covered by any type of R-lines inside $\Lambda$, we call this region of missing R-line coverage the core, and denote it as $\Omega^R_c$. We denote $S_R$ as the portion of the bounding surface of $\Omega^R_c$ that is inside $\Lambda$. Note that for $\lambda_m \geq 2\pi$, $\Omega^R_c$ is solely bounded by $S_R$, whereas for $\lambda_m < 2\pi$, $\Omega^R_c$ is bounded by one more surface, i.e., the planar surface of $\Lambda$.

Figure 4.4: The R-line surface of the reverse helix above the $(x, y)$-plane for $\lambda_m = 300^\circ$. We denote $S_R^+$ as the half portion of $S_R$ that is above the $(x, y)$-plane. We illustrate the R-line surface together with the $(x, y)$-plane and the $\pi$-line surface $S_1^+$. (a) two angles that define the view point: the azimuth angle (AZ) and the elevation angle (EL). (b) Visualization of the R-line surfaces at AZ $= 30^\circ$ and EL $= 40^\circ$.

Therefore, the R-line coverage of the reverse helix is the region between $S_R$ and the bounding surface of $\Lambda$. To our knowledge, unlike the $\pi$-line surfaces, an analytical expression or a clear geometrical interpretation is extremely difficult for the R-line surface $S_R$, if not impossible. We have performed numerical simulations to obtain the R-line surface. The results for the case of $\lambda_m = 300^\circ$, $R = 10$ cm and $H = 8.33$ cm are illustrated in Figure 4.4. Note that, for a better visualization, here we only show the portion of the R-line surface that is above the $(x, y)$-plane, which is denoted as $S_R^+$. This visualization is specified by the azimuth and elevation angles. Based on the $\pi$-line and R-line surfaces, we then obtain the R-line coverage, whose cross sections are shown in Figure 4.5. Note that a significant amount of $\Omega^R_c$ is located in the $(x, y)$-plane around the $z$-axis. Therefore, in this case, $\Omega^R_c$ intersects with the ROI $\Omega$. 
Chapter 4. Reverse Helical Trajectory

Figure 4.5: Cross sections of the R-line coverage of the reverse helix with $\lambda_m = 300^\circ$, $R = 10\text{ cm}$ and $H = 8.33\text{ cm}$. Color white indicates regions covered by R-lines. Left: the R-line coverage in the sagittal section at $x = 0\text{ cm}$. Middle: the R-line coverage in the transverse section at $z = 0.01\text{ cm}$. Right: the R-line coverage in the coronal section at $y = 0\text{ cm}$.

4.4 Conclusion and Discussion

The reverse helix is attractive, because it fulfills the motion constraint of a C-arm system and satisfies Tuy’s condition. However, the reverse helix does not provide sufficient R-line coverage in the ROI. This lack of R-line coverage makes it challenging to employ efficient TES reconstruction algorithms for long-object imaging on the reverse helix. Possible solutions may involve one efficient TES reconstruction in the region covered by R-lines and another TES reconstruction in $\Omega^R_c$, e.g., the DBP-FBP methods in [Noo09, Cho10]. These solutions are exact and allow axial data truncation, but they are computationally expensive. More importantly, they require combining different volumes of incompatible resolutions that tends to yield undesirable artifacts.

An alternative would be approximate reconstruction algorithms, e.g., the FDK-type image reconstruction algorithms. These algorithms are efficient, stable and easy to implement, but they generate reconstructions entangled with CB artifacts. To make these results clinically acceptable, it is necessary to suppress CB artifacts. To this end, special modifications to the existing approximate reconstruction algorithms are needed to accommodate the reverse helix. In addition, we should also borrow knowledge from exact reconstruction algorithms for improvement of approximate reconstruction algorithms. For example, knowledge of the spatial distribution of the $\pi$-lines may be incorporated in FDK-type algorithms. We will present three practical image reconstruction solutions for the reverse helix in Chapter 7.
Chapter 5

Circular-Arcs-Plus-Line Trajectories

To improve is to change; to be perfect is to change often.

Winston Churchill

In this chapter, we introduce two source trajectories that consist of two parallel concentric circular arcs connected by a line segment. These trajectories possess a reverse pattern in their direction of rotation and thus are suitable for long-object imaging on a C-arm system. Moreover, these trajectories satisfy Tuy’s condition so that exact image reconstruction is possible.

Additionally, we perform an analysis of the R-line coverage for such trajectories. More specifically, we examine to what extent R-lines for such trajectories cover a central ROI. We organize this chapter as follows. First, we describe the source trajectories of interest. Next, we discuss the R-line coverage resulting from two parallel circular arcs, and also the R-line coverage resulting from connecting a line orthogonally to the endpoint of a circular arc. Subsequently, we are able to present the R-line coverage for the entire source trajectory. Our analysis shows that the R-line coverage is inadequate when the line scans have their endpoints on the arcs and these arcs have an angular length less than 360°. On the other hand, when the line scans are allowed to extend beyond the arcs, we find that full coverage of a central cylindrical ROI becomes possible. The required line extension is analyzed as a function of the ROI radius and the length of the circular arcs.

5.1 Geometry and Preliminaries

5.1.1 Source Trajectories

We consider extended-volume imaging using periodic duplicates of a source trajectory consisting of two circular arcs connected by a line segment. The patient is assumed to lie along the z-axis, the arcs are in parallel planes that are orthogonal to this axis, and the line is orthogonal to each arc through one of its endpoints. Figure 5.1 depicts this trajectory.
Two options are considered: (a) the line is spatially limited by the arcs, (b) the line extends beyond the arcs. In the first option, the trajectory is called the arc-line-arc (ALA) trajectory; in the second option, it is called the arc-extended-line-arc (AELA) trajectory. Figure 5.1 also shows how each circular arc is oriented relative to $x$- and $y$-axes that form together with the $z$-axis a Cartesian system of coordinates. The distance in $z$ between the arcs is $2H$, the radius of the arcs is $R$, and the line extension in option (b) is $\Delta H$ on each end. Also, the $(x,y)$-plane is chosen to be at mid-distance between the two arcs.

![Figure 5.1: Geometries of the line plus circular arc trajectories. Extended-volume imaging is performed using duplicates of a path consisting of two circular arcs plus a line. Two options are considered for this path: (left) the line is tightly fit between the arcs, so that each endpoint of the line corresponds to one endpoint of an arc, and (middle) the line extends beyond the arcs by a distance $\Delta H$ on each side. The right figure shows the orthogonal projection of the source trajectory onto the $(x,y)$-plane. Points $O_- = (0,0,-H)$ and $O_+ = (0,0,H)$ are the centers of the lower and upper arcs, respectively. Points $E_- = (R\cos \lambda_s, R\sin \lambda_s, -H - \Delta H)$ and $E_+ = (R\cos \lambda_e, R\sin \lambda_e, H + \Delta H)$ are the extremities of the extended line. The polar angles $\lambda_s$ and $\lambda_e$ define the start and end points of each arc.

Throughout the text, the circular arcs and the line segment that together form the source trajectory are referred to as the T-arcs and the T-line, where $T$ stands for trajectory. The T-arcs at $z = H$ and $z = -H$ are distinguished from each other using the terms upper and lower T-arcs, respectively. By extension, the term T-arc is also used to denote the orthogonal projection of either the upper or the lower T-arc onto any plane that is parallel to the $(x,y)$-plane. The T-arc terminology is especially useful as it avoids confusion with other arcs that will appear later.

We specify a point on the upper or lower T-arc by a polar angle $\lambda$, which ranges from $\lambda_s$ to $\lambda_e$. By convention, the superscript $s$ refers to the start point on the T-arc and the superscript $e$ refers to the end point. We denote a point on the upper (resp. lower) T-arc at position $\lambda$ as $A^\lambda_+$ (resp. $A^\lambda_-$). For example, the endpoints of the upper T-arc are denoted as $A^\lambda_+$ and $A^\lambda_-$, whereas the endpoints of the lower T-arc are denoted as $A^\lambda_-$ and $A^\lambda_+$. The line segment is defined to pass through $A^\lambda_+$ and $A^\lambda_-$. 
5.1. Geometry and Preliminaries

By Tuy’s condition \cite{Tuy83,Finspang85}, exact reconstruction of the X-ray linear attenuation coefficient from CB projections is only possible within the convex hull of the source trajectory. Hence, we do not investigate the R-line coverage outside this convex hull. To be more precise, we only investigate R-line coverage within a specific cylinder centered on the z-axis where Tuy’s condition is fulfilled. This cylinder extends from \( z = -H \) to \( z = H \) and has radius \( R_m \) determined by the angular length, \( \lambda_m = \lambda_e - \lambda_s \), of the T-arcs according to the equation

\[
R_m = R \sin \left( \frac{(\lambda_m - \pi)}{2} \right). \tag{5.1}
\]

Basically, \( R_m \) is the radius of the largest central cylinder within the convex hull of the source trajectory.

5.1.2 Preliminaries

Below, we introduce some terminology used in this chapter. First, note that we will make extensive use of the conical surface that is defined by a point and a circular arc, called the base arc, that are not coplanar. To simplify the text, we loosely refer to such a surface as being a partial cone surface. In order to aid with understanding of our terminology and for later developments, we give the following theorem on partial cone surfaces that is proven in Section A.1.

**Theorem 1.** As illustrated in the left of Figure 5.2, the intersection between a partial cone surface and a plane that is parallel to the base of the cone is an arc that is a translation of a scaled copy of the base arc, with the scaling being isotropically applied relative to the center of the base arc.

![Figure 5.2: Left: illustration of Theorem 1. Plane \( \Pi_z \) is parallel to the base plane and intersects the partial cone surface from point \( A \) along a circular arc that is centered at \( O_b' \); this arc can be obtained by isotropically-scaling and translating the base arc. Right: illustration of \( S^A_+ \): the positive partial cone surface from \( A^A_+ \).](image)

Figure 5.2: Left: illustration of Theorem 1. Plane \( \Pi_z \) is parallel to the base plane and intersects the partial cone surface from point \( A \) along a circular arc that is centered at \( O_b' \); this arc can be obtained by isotropically-scaling and translating the base arc. Right: illustration of \( S^A_+ \): the positive partial cone surface from \( A^A_+ \).
As shown in the right of Figure 5.2, a partial cone surface may be obtained by selecting one point $A^+_\lambda$ on the upper T-arc and connecting it to all of the points on the lower T-arc. Such a partial cone surface is called a positive partial cone surface, and is denoted as $S^+_{\lambda}$. Similarly, a partial cone surface may be obtained by connecting $A^-_{\lambda}$ on the lower T-arc to all of the points on the upper T-arc; such a partial cone surface is called a negative cone surface, and is denoted as $S^-_{\lambda}$.

Let $\Pi_z$ be the horizontal plane that intersects the $z$-axis at $(0,0,z)$, as illustrated in the right of Figure 5.2. Theorem 1 shows that the intersection between $\Pi_z$ and any positive or negative cone surface is a circular arc. Throughout the paper, such an arc is called an R-arc, because it consists of points covered by R-lines. Furthermore, Theorem 1 implies that the angular length of any R-arc is the same as the angular length of the T-arcs, and it also implies that the line connecting the endpoints of any R-arc is parallel to the line connecting the endpoints of either T-arc.

Two more interesting properties of R-arcs need to be noted. First, by construction, an R-arc is within the disk delimited by the T-arc inside $\Pi_z$. As defined in the second paragraph of Section 5.1.1, “T-arc” in the previous sentence means the orthogonal projection of the upper or lower T-arc onto $\Pi_z$. Second, an R-arc always shares one (and only one) point with the T-arc; this point is the intersection of $\Pi_z$ with the line parallel to the $z$-axis through the vertex of the partial cone surface defining the R-arc. Consequently, every R-arc is tangent to the T-arc. See Figures 5.3 and 5.4.

We denote the center of the R-arc resulting from the intersection of $\Pi_z$ and $S^+_{\lambda}$ (resp. $S^-_{\lambda}$) as $O^+_\lambda$ (resp. $O^-_{\lambda}$). If an arc can be unambiguously identified by its center, we will denote it with the label of its center point; for example, we denote the lower T-arc in the right of
Figure 5.2 as Arc\((O_{-})\) and the R-arc in the left of Figure 5.4 as Arc\((O_{+}^{\lambda_{z}})\). Similarly, a circle that is unambiguously identifiable by its center, \(O\), is referred to as Circ\((O)\).

Figure 5.4: Arc-arc R-line coverage when the length of the T-arcs is 234°. Left: coverage in Π\(_z\) due to the start point of the upper T-arc; this coverage is an R-arc centered on a point denoted as \(O_{+}^{\lambda_{z}}\) in the figure. Middle: coverage in the same plane as in the left figure, but due to several points on the upper T-arc. Right: full AA R-line coverage in Π\(_z\), as obtained using a numerical simulation. The white region is covered by R-lines, whereas the black region is not covered by R-lines.

For convenience, we introduce two more notations. We denote a point of the T-arc in Π\(_z\) with polar angle \(\lambda\) as \(A_{\lambda_{z}}^{\lambda}\), and we denote the intersection between Π\(_z\) and the R-line connecting \(A_{\lambda_{z}}^{\lambda_{1}}\) and \(A_{\lambda_{z}}^{\lambda_{2}}\) as \(A_{\lambda_{z}}^{\lambda_{1},\lambda_{2}}\).

### 5.2 ALA Trajectory

In this section, we discuss the R-line coverage for the ALA trajectory. First, we examine the R-line coverage that is generated by connecting points from one T-arc to the other. Next, we consider the additional R-line coverage that results from connecting points on the T-line to points on the T-arcs. Note that the geometry of the problem at hand is mirror symmetric relative to the \(z = 0\) plane. Therefore, the R-line coverage in the plane \(z = z_{0}\) is the same as that in the plane \(z = -z_{0}\) for any \(0 \leq z_{0} \leq H\). Hence, we only discuss R-line coverage at positions \(z \geq 0\).

#### 5.2.1 Arc-Arc R-line Coverage

To understand the R-line coverage for the elemental arc-arc (AA) trajectory, we start by considering the simpler case where each T-arc has a length of 360°. We choose a value for \(z \in [0, H]\), draw Π\(_z\), and then find the R-line coverage within this plane. The situation is depicted in Figure 5.3. Basically, we take a point on the upper T-arc, \(A_{\lambda_{z}}^{\lambda_{1}}\), and connect it to all points on the lower T-arc as shown in the left of Figure 5.3. Doing so, we create
a cone surface that intersects $\Pi_z$ along a circle, called an R-circle (in analogy with the earlier-defined notion of an R-arc). This circle defines the R-line coverage coming from $A^\lambda_+$ in $\Pi_z$. By continuously moving $A^\lambda_+$ along the upper T-arc, we obtain an infinite number of additional R-circles, a coarse sampling of which is shown in the right of Figure 5.3. The union of all the R-circles is the full R-line coverage in $\Pi_z$. This union is an annular region with external boundary $D_0$ and internal boundary $D_1$ as shown in the right of Figure 5.3. The boundary $D_0$ is the T-arc in $\Pi_z$.

As illustrated in the right of Figure 5.3, the R-line coverage within $\Pi_z$ is conveniently described by two parameters $r_1$ and $r_2$: (i) $r_1$ is the radius of the R-circle; (ii) $r_2$ is the radius of the circle formed by the union of the centers of the R-circles. By construction, both $r_1$ and $r_2$ are functions of $z$, and are easily found to be

$$r_1 = \frac{H - z}{2H} R \quad \text{and} \quad r_2 = R - r_1 = \frac{H + z}{2H} R.$$  \hspace{1cm} (5.2)

Using the above formulae for $r_1$ and $r_2$, it follows that the radius of $D_1$ is identical to $r_2 - r_1 = Rz/H$.

Now, we turn to the general situation where the T-arcs have an angular range of less than 360°. We again consider the R-line coverage in plane $\Pi_z$ with $z \geq 0$, as illustrated in Figure 5.4. Comparing the left of Figure 5.3 with the left of Figure 5.4, we see that the R-line coverage resulting from $A^\lambda_+$ is not an R-circle, it is an R-arc, albeit of the same radius, $r_1$. By moving $A^\lambda_+$ along the upper T-arc, we obtain a set of R-arcs whose union defines the AA R-line coverage within $\Pi_z$, as shown in the right of Figure 5.4. Contrasting the right of Figure 5.3 with the middle of Figure 5.4, we see that the R-line coverage for two T-arcs of angular length shorter than 360° is a subset of that for two 360° T-arcs. The R-line coverage resulting from a numerical simulation is shown in the right of Figure 5.4.

Note the following important property from the middle of Figure 5.4: points $O_z$, $O^\lambda_+z$ and $A^\lambda_+$ are aligned, where $O_z$ is the center of the T-arc, $O^\lambda_+z$ is the center of the R-arc, and $A^\lambda_+$ is the orthogonal projection of $A^\lambda_+$ onto $\Pi_z$. This property is due to $A^\lambda_+$ being the point on the R-arc defined from $A^\lambda_+$ that is also on the T-arc. Indeed, under such circumstances, the line connecting $O_z$ to $A^\lambda_+$, and the line connecting $O^\lambda_+z$ to $A^\lambda_+$ must be both orthogonal to the tangent to the T-arc at $A^\lambda_+$.

We now describe an efficient numerical procedure to identify the R-line coverage within any plane $\Pi_z$. Afterwards, we will give geometrical insight into the origin of the R-line coverage shown in the right of Figure 5.4. Efficient computation of the R-line coverage in $\Pi_z$ is made possible by the following corollary to Theorem 1.

**Corollary 1.** For the AA trajectory, all of the R-arcs in plane $\Pi_z$ are translated versions of each other (with their radius being equal to $r_1$, as given by Equation 5.2).

**Proof.** The corollary is a direct consequence of the following observations: (i) as discussed at the beginning of this section and illustrated with Figure 5.3, all R-arcs in $\Pi_z$ have the same radius, namely $r_1$, (ii) by Theorem 1 all R-arcs in $\Pi_z$ have the same angular length,
and (iii) by Theorem 1 all R-arcs in $\Pi_z$ are oriented in the same way as the T-arc, with the line connecting their endpoints being parallel to the line connecting the endpoints of the T-arc.

Analytically, the corollary can also be explained as follows. Consider two R-arcs in $\Pi_z$. By definition, each of these two R-arcs comes from the intersection of a positive partial cone surface with $\Pi_z$, so that, by Theorem 1, any point $x$ on the lower T-arc can be associated with a point $x_1 = s_1x + l_1$ on the first R-arc and a point $x_2 = s_2x + l_2$ on the second R-arc, with $s_1$ and $s_2$ being scaling constants and $l_1$ and $l_2$ being translation vectors, all independent of $x$. Since all R-arcs in $\Pi_z$ have the same radius, we have $s_1 = s_2$ and thus $x_2 = x_1 + \xi$ where $\xi = l_2 - l_1$ is independent of $x$. Hence, the two R-arcs are translations of each other.

Figure 5.5: The R-line coverage in $\Pi_z$ can be fully parameterized with two angles, $\lambda_u$ and $\lambda_l$, varying both over $[0, \lambda_m]$. Angle $\lambda_u$ is used to specify the center of the R-arc, whereas angle $\lambda_l$ is used to specify a location on the R-arc, which is identified as $P$ in the figure. The meaning of the other symbols is as follows: $R$ is the radius of the T-arc; $r_1$ is the radius of the R-arc; $r_2$ is the distance between the z-axis and the center of the R-arc; $n_s$ is the unit vector pointing from the center $O_z$ to the start point $A^\lambda_u$, whereas $\lambda_l$ is the location on the R-arc; $e_z$ be the unit vector pointing in the z-direction. Thanks to the corollary, any point $\bar{x} \in \Pi_z$ that belongs to an R-line can be parameterized using two angles, $\lambda_u$ and $\lambda_l$, according to

$$\bar{x} = r_2 (\cos \lambda_u n_s + \sin \lambda_u n_s^\perp) + r_1 (\cos \lambda_l n_s + \sin \lambda_l n_s^\perp) + z e_z,$$

where $n_s$ is the unit vector in $\Pi_z$ that goes from the center of the T-arc towards the start point on the T-arc, and $n_s^\perp$ is obtained by rotating $n_s$ about the z-axis by 90° in the counterclockwise direction. This equation may be best understood by looking at Figure 5.5 where
two R-arcs are shown in \( \Pi_z \): Arc\((O_+^{\lambda_{u}z})\) which starts at the start point on the T-arc, \( A^\lambda_+\), and Arc\((O_-^{\lambda_{u}z})\) which is arbitrary, with its start point denoted as \( A^\lambda_-\). By Corollary 1 the line that connects \( O_-^{\lambda_{u}z} \) to \( A^\lambda_- \) is parallel to the line that passes through \( O_z \), \( O_+^{\lambda_{u}z} \) and \( A^\lambda_+ \), where \( O_z \) is the center of the T-arc in \( \Pi_z \). Vector \( n_z \) is the common direction of these two lines. The first term in Equation 5.3 selects the position of the center of the arbitrary R-arc relative to the line of direction \( n_z \) through \( A^\lambda_+ \), whereas the second term selects a point on the arbitrary R-arc, relative to the line of direction \( n_z \) through \( A^\lambda_- \). Naturally, angles \( \lambda_u \) and \( \lambda_l \) in Equation 5.3 are not allowed to take arbitrary values; they are constrained to lie between 0 and \( \lambda_m \).

An efficient algorithm for computation of the R-line coverage in \( \Pi_z \) is obtained by inverting Equation 5.3 to obtain \( \lambda_u \) and \( \lambda_l \) as functions of the first two coordinates of \( x \), denoted as \( x \) and \( y \). First, note from Figure 5.3 that for a given \((x, y)\), there are at most two solutions, denoted \((\lambda^1_u, \lambda^1_l)\) and \((\lambda^2_u, \lambda^2_l)\), and these solutions only exist if \((r_1 - r_2)^2 \leq x^2 + y^2 \leq (r_1 + r_2)^2\). Since \( n_z = [\cos \lambda_z, \sin \lambda_z, 0] \), we have

\[
\begin{align*}
   x &= r_2 \cos(\lambda_u + \lambda_s) + r_1 \cos(\lambda_l + \lambda_s) \\
   y &= r_2 \sin(\lambda_u + \lambda_s) + r_1 \sin(\lambda_l + \lambda_s)
\end{align*}
\]  

(5.4)

and thus

\[
(x - r_2 \cos(\lambda_u + \lambda_s))^2 + (y - r_2 \sin(\lambda_u + \lambda_s))^2 = r_1^2.
\]  

(5.5)

Using \( \rho \) and \( \theta \) for the polar coordinates of \((x, y)\), Equation 5.5 yields

\[
\cos(\lambda_u + \lambda_s - \theta) = \frac{\rho^2 + r_2^2 - r_1^2}{2\rho r_2},
\]

which gives the following expressions for \( \lambda^1_u \) and \( \lambda^2_u \):

\[
\begin{align*}
   \lambda^1_u &= \mod(\theta - \lambda_s + \eta, 2\pi) \\
   \lambda^2_u &= \mod(\theta - \lambda_s - \eta, 2\pi)
\end{align*}
\]

with \( \mod(u, 2\pi) \) being equal to \( u \) modulo \( 2\pi \), and with

\[\eta = \arccos\left(\frac{\rho^2 + r_2^2 - r_1^2}{2\rho r_2}\right),\]

which is a real number as long as \( r_2 - r_1 \leq \rho \leq r_2 + r_1 \). Rewriting Equation 5.4 in the form

\[
\begin{align*}
   r_1 \cos(\lambda_l + \lambda_s) &= x - r_2 \cos(\lambda_u + \lambda_s) \\
   r_1 \sin(\lambda_l + \lambda_s) &= y - r_2 \sin(\lambda_u + \lambda_s)
\end{align*}
\]

we then see that \( \lambda^1_l \) and \( \lambda^2_l \) can be expressed as

\[
\lambda^{1,2}_l = \mod(-\lambda_s + \arctan2(y - r_2 \sin(\lambda^{1,2}_u + \lambda_s), x - r_2 \cos(\lambda^{1,2}_u + \lambda_s)), 2\pi)
\]
where \( \text{atan2}(v, u) \) is the four-quadrant inverse tangent function, which gives the polar angle of point \((u, v)\) in the \((x, y)\)-plane in the range \([-\pi, \pi]\). The solution \((\lambda_1 u, \lambda_1 l)\) is admissible only if \(\lambda_1 u \in [0, \lambda_m]\) and \(\lambda_1 l \in [0, \lambda_m]\). Similarly, the solution \((\lambda_2 u, \lambda_2 l)\) is admissible only if \(\lambda_2 u \in [0, \lambda_m]\) and \(\lambda_2 l \in [0, \lambda_m]\). If both solutions are admissible, then \((x, y)\) belongs to two R-lines, otherwise \((x, y)\) belongs to either one R-line or no R-line depending on whether one of the two solutions is admissible or not.

The above procedure to evaluate if a point \((x, y, z) \in \Pi_z\) lies on an R-line is straightforward to implement on a computer. The result for \(z = 0.2H\) was shown in the right of Figure 5.4 for a 234° T-arc. Figure 5.6 shows the AA R-line coverage for \(z = 0\). In the rest of this section, we provide some geometrical insight into why these R-line coverage diagrams appear as they do.

Figure 5.6: Arc-arc R-line coverage in the \((x, y)\)-plane and the length of the T-arcs is 234°. Left: R-line coverage of the AA trajectory due to several points on the upper T-arc. Right: full R-line coverage in \(\pi_z\), as obtained using numerical simulation.

Recall our construction of the R-line coverage in \(\Pi_z\) as the union of translated R-arcs defined by positive partial cone surfaces with vertices on the upper T-arc; see Figure 5.4. The region of R-line coverage in \(\Pi_z\) may be geometrically understood by identifying the path traced by each extremity of the R-arcs as \(\lambda_u\) is increased from 0 to \(\lambda_m\). These two paths are the dashed arcs shown in the top left of Figure 5.7, the black-colored dashed arc corresponds to \(A_z^{\lambda_u, \lambda_e}\), whereas the gray-colored dashed arc corresponds to the other extremity, \(A_z^{\lambda_u, \lambda_e}\). Observe that any point on the path of \(A_z^{\lambda_u, \lambda_e}\) belongs to an R-line connecting the upper T-arc with \(A_z^{\lambda_e}\). Similarly, any point on the path of \(A_z^{\lambda_u, \lambda_e}\) belongs to an R-line connecting the upper T-arc with \(A_z^{\lambda_e}\). Therefore, the paths of \(A_z^{\lambda_u, \lambda_e}\) and \(A_z^{\lambda_u, \lambda_e}\) can be alternatively interpreted as the intersections of \(\Pi_z\) with negative partial cone surfaces \(S_z^{\lambda_u}\) and \(S_z^{\lambda_e}\); see the bottom row of Figures 5.7. Analytically, the paths of \(A_z^{\lambda_u, \lambda_e}\) and \(A_z^{\lambda_u, \lambda_e}\) are solutions of Equation 5.4 with \(\lambda_l = 0\) and \(\lambda_l = \lambda_m\), respectively, where \(\lambda_u \in [0, \lambda_m]\).
Chapter 5. Circular-Arcs-Plus-Line Trajectories

Figure 5.7: Arc-arc R-line coverage in $\Pi_z$ when the length of the T-arcs is $234^\circ$. Top left: the R-arcs corresponding to $\lambda_u = 0$ and $\lambda_u = \lambda_m$ and the path traced by each extremity of these R-arcs as $\lambda_u$ increases from 0 to $\lambda_m$. Top right: the R-line coverage in $\Pi_z$ is characterized by five curves: the paths traced by $A_{z,\lambda_s}^\lambda$ and $A_{z,\lambda_e}^\lambda$ when $\lambda$ increases from $\lambda_s$ to $\lambda_e$, the R-arcs at $\lambda_u = 0$ and $\lambda_u = \lambda_m$ and the circle $D_i$. Bottom left: the path traced by $A_{z,\lambda_s}^\lambda$ as $\lambda$ increases from $\lambda_s$ to $\lambda_e$, seen as the intersection of $\Pi_z$ with the negative cone surface, $S_{z}^{\lambda_s}$. Bottom right: the path traced by $A_{z,\lambda_e}^\lambda$ as $\lambda$ increases from $\lambda_s$ to $\lambda_e$, seen as the intersection of $\Pi_z$ with the negative cone surface $S_{z}^{\lambda_e}$. 
5.2. ALA Trajectory

As illustrated in the top right of Figure 5.7, the R-line coverage in $\Pi_z$ is fully characterized by five curves: the paths followed by $A^\lambda_{\lambda_1,\lambda_2}$ and $A^\lambda_{\lambda_1,\lambda_2}$ as $\lambda$ increases from $\lambda_1$ to $\lambda_2$, the R-arcs corresponding to $\lambda_u = 0$ and $\lambda_u = \lambda_m$, and the circle $D_i$ that was introduced in the right of Figure 5.3. In particular, note that the R-line coverage includes two small regions denoted as $\Omega_1$ and $\Omega_2$. Region $\Omega_1$ is the area bounded by the path traced by $A^\lambda_{\lambda_1,\lambda_2}$, the R-arc at $\lambda_u = 0$ and the circle $D_i$. This region and the set of points that belong to both the convex hull of the R-arc at $\lambda_u = 0$ and the convex hull of the path of $A^\lambda_{\lambda_1,\lambda_2}$ are covered by the R-arcs corresponding to $\lambda_u \in [0, \lambda_1]$. Region $\Omega_2$ is the mirror image of $\Omega_1$ relative to the line that connects the midpoint of the T-arc to the center of $D_i$; this region is created with $\lambda_u \in [\pi, \lambda_m]$. The above description for the R-line coverage in $\Pi_z$ simplifies considerably when $z = 0$. First, the regions $\Omega_1$ and $\Omega_2$ disappear because $r_1 = r_2$ and thus $D_i$ is reduced to a point. Second, the paths followed by $A^\lambda_{\lambda_1,\lambda_2}$ and $A^\lambda_{\lambda_1,\lambda_2}$ become equivalent to the R-arcs defined by $S^\lambda_{\lambda_1}$ and $S^\lambda_{\lambda_1}$, respectively; this is due to the mirror symmetry relative to the $(x, y)$-plane. Hence, the R-line coverage in $\Pi_0$ is determined entirely by the paths followed by $A^\lambda_{\lambda_1,\lambda_2}$ and $A^\lambda_{\lambda_1,\lambda_2}$, as shown in Figure 5.6.

5.2.2 Arc-Line R-line Coverage

Here, we examine the coverage associated with the R-lines that connect a point on the T-line to a point on the T-arcs. This arc-line (AL) coverage is most easily analyzed in two steps. First, we find the coverage from the T-line and the lower T-arc. Next, we find the coverage due to the T-line and the upper T-arc. The union of these two regions is the complete AL coverage.

To analyze the AL R-line coverage, we will need Lemma 1 which is proven in Section A.2. According to Lemma 1, the R-line coverage for the AL trajectory composed of the lower T-arc and the T-line is the convex hull of the positive partial cone surface, $S^\lambda_{\lambda_1}$. This convex hull is called $\Lambda^+$. Lemma 1. Any point within the convex hull of a partial cone surface belongs to an R-line.

The intersection of $\Lambda^+$ with $\Pi_z$ characterizes the R-line coverage in $\Pi_z$. This region has the form of a partial disk, as illustrated in the left and right of Figure 5.8 for the case $z = 0$, and can be computed with the following relations:

\[
\begin{align*}
(x - r_2 \cos \lambda_s)^2 + (y - r_2 \sin \lambda_s)^2 &\leq r_1^2 \\
x \cos \alpha + y \sin \alpha &\geq d
\end{align*}
\]

(5.6)

where $d = R \cos (\lambda_m/2)$ and $\alpha = (\lambda_s + \lambda_e)/2$.

The above two relations may be explained as follows, using Figure 5.9. First, the curved portion of the boundary for the partial disk in $\Pi_z$ is an R-arc defined by the positive partial
Chapter 5. Circular-Arcs-Plus-Line Trajectories

Figure 5.8: Arc-line R-line coverage when the length of the T-arc is $234^\circ$. The line is attached orthogonally to one of the endpoints of the T-arc. Left: partial disk of R-line coverage in $\Pi_0$. Middle: due to mirror symmetry relative to $\Pi_0$, the partial disk of R-line coverage in $\Pi_0$ is the same for both of the AL trajectories. Right: the numerical result in $\Pi_0$.

cone surface from $A^{+^s}_\lambda$, i.e., $S^{+^s}_\lambda$. Because points on this R-arc satisfy Equation 5.3 with $\lambda_u = 0$ and $\lambda_l \in [0, \lambda_m]$, it follows that the curved portion of the boundary for the partial disk belongs to the circle given by the first row in Relation 5.6. Second, the straight portion of the boundary for the partial disk is determined by the line connecting $A^{+^s}_\lambda z$ and $A^{-^s}_\lambda e z$, which is orthogonal to vector $m = (\cos \alpha, \sin \alpha, 0)$ and thus yields the second row in Relation 5.6.

Figure 5.9: R-line coverage in $\Pi_z$ for the AL trajectory composed of the lower T-arc and the T-line.

The R-line coverage for the AL trajectory composed of the upper T-arc and the T-line is found by symmetry: we just have to mirror the coverage for the previous AL trajectory with respect to $\Pi_0$. To help visualize the situation, the positive and negative partial cone surfaces of R-lines resulting from mirror symmetric AL trajectories are illustrated in the
middle of Figure 5.8. Hence, the sought union of R-lines is the convex hull of the negative partial cone surface from $A_{\lambda}^s$, i.e., $S_{\lambda}^s$. This convex hull is called $\Lambda^\leftarrow$. The partial disk characterizing the R-line coverage in $\Pi_z$ for the AL trajectory composed of the upper T-arc and the T-line is defined by the relations

\[
\begin{align*}
(x - r_1 \cos \lambda_s)^2 + (y - r_1 \sin \lambda_s)^2 &\leq r^2_2 \\
(x \cos \alpha + y \sin \alpha) &\geq d
\end{align*}
\]  
(5.7)

The first inequality follows from the fact that the intersection of $\Pi_z$ with $S_{\lambda}^s$ is the arc described by Equation 5.3 with $\lambda_l = 0$ and $\lambda_u \in [0, \lambda_m]$. In the discussion at the end of the previous section, this arc was identified as the path followed by $A_{\lambda}^s, \lambda_s$. The second inequality is of course the same as in Relation 5.6.

Note that the R-line coverage in $\Pi_0$ is the same for both AL trajectories, due to the mirror symmetry relative to $\Pi_0$. Also, for $z > 0$, the R-line coverage in $\Pi_z$ that comes from the T-line and the lower T-arc is always included within the coverage that comes from the T-line and the upper T-arc.

### 5.2.3 Combined R-line Coverage

Using the results of the previous sections, we can obtain the R-line coverage for the ALA trajectory. The ALA trajectory can be decomposed into three components: (i) the elemental AA trajectory consisting of the upper and lower T-arcs, (ii) the elemental AL trajectory consisting of the T-line and the lower T-arc, and (iii) the elemental AL trajectory consisting of the T-line and the upper T-arc. It follows that the union of the three sets of R-lines corresponding to these three elemental trajectories yields the complete set of R-lines for the AL trajectory.

The ALA R-line coverage in $\Pi_z$ can be computed numerically by combining the procedure described in Section 5.2.1 for the AA trajectory together with Relations 5.6 and 5.7 for the AL trajectories. As an example, the R-line coverage in the plane $z = 0.2H$ for a 234° ALA trajectory is illustrated in Figure 5.10. The top row of Figure 5.10 shows the sets of R-line coverage for the AA trajectory and for the two AL trajectories, respectively, and the union of these sets is depicted in the bottom row of Figure 5.10. From these figures, we observe that the sets of R-line coverage for the AA and AL trajectories compensate each other quite well in the plane $z = 0.2H$.

Practically, it is desirable to determine whether or not every point inside a specified ROI belongs to an R-line. The following theorem, which is proven in Section A.3, and its corollary enable us to answer this question for a cylindrical ROI centered on the z-axis with radius $r \leq R_m$ and delimited by the planes $z = -H$ and $z = H$. For the statement of the theorem, we define the line $L(x_0, y_0)$ to be the line parallel to the z-axis that passes through $(x_0, y_0, 0)$. 
Figure 5.10: Combined R-line coverage resulting from arc-arc and arc-line trajectories in the plane $z = 0.2H$. Top left: R-line coverage for the AA trajectory. Top right: outlines of the R-line coverage for the AL trajectories; the small partial disk is for the lower T-arc and T-line, whereas the larger partial disk is for the upper T-arc and T-line. Bottom left: result of superimposing the top right onto the top left. Bottom right: combined coverage.

**Theorem 2.** For the ALA trajectory, if $(x_0, y_0)$ satisfies $x_0^2 + y_0^2 \leq r^2$, then there exists a coordinate $\hat{z}$ with $|\hat{z}| \leq H$ such that the points on the line $\mathcal{L}(x_0, y_0)$ that are covered by R-lines have z-coordinates in the set $[-H, -\hat{z}] \cup [\hat{z}, H]$.

**Corollary 2.** Suppose that $(x_0, y_0)$ satisfies $x_0^2 + y_0^2 \leq r^2$. For the ALA trajectory, if a point $(x_0, y_0, z_0)$ lies on an R-line, then all points $(x_0, y_0, z)$ with $z \in [-H, -|z_0|] \cup [|z_0|, H]$ also lie on R-lines.

The first row of Figures 5.11 depicts the R-line coverage in the plane $\Pi_0$ for the ALA trajectories of angular ranges of $234°$ and $309.6°$, respectively. These two angles were arbitrarily chosen in the ranges $[230°, 250°]$ and $[290°, 310°]$, respectively, so as to induce some randomness. In both figures, an ROI with radius $r = R\sin(27°)$ is drawn (this is
the largest possible ROI radius, $R_m$, for the $234^\circ$ trajectory). Examining these figures, we observe that the R-lines do not fully cover the ROI for either of the ALA trajectories. In fact, it turns out that as long as the angular range of the ALA trajectory is less than $360^\circ$, there is always a moon-shaped region touching the origin that is not covered by R-lines. Conversely, if the angular range of the ALA trajectory is $360^\circ$, then any central ROI with $r \leq R$ is completely covered by R-lines. In the next section, we propose a modification to the ALA trajectory that addresses this problem.

Figure 5.11: Numerical results for the R-line coverage in $\Pi_0$ of arc-line-arc and arc-extended-line-arc trajectories. Top left: coverage of the ALA trajectory with $234^\circ$ T-arcs. Top right: coverage of the ALA trajectory with $309.6^\circ$ T-arcs. Bottom left: coverage of the AELA trajectory with $234^\circ$ T-arcs and $\Delta H/(2H) = 0.82$. Top right: coverage of the AELA trajectory with $309.6^\circ$ T-arcs and $\Delta H/(2H) = 0.48$. 
5.3 AELA Trajectory

To overcome the lack of R-line coverage mentioned in the previous section, we propose the AELA trajectory, i.e., the arc-extended-line-arc trajectory, which was described in Section 5.1.1. Below, we will see that the R-lines for the AELA trajectory can completely cover the ROI cylinder if the line extension is properly chosen.

Although Theorem 2 and Corollary 2 were stated for the ALA trajectory, it is important to realize that they also apply to the AELA trajectory. This is true because the proof of Theorem 2 is general enough to also apply to the AELA trajectory. Therefore, Corollary 2 implies that if all points inside the ROI in $\Pi_0$ belong to an R-line of the AELA trajectory, then the entire ROI cylinder extending from $z = -H$ to $z = H$ is covered by R-lines. For this reason, the following arguments only focus on the R-line coverage in $\Pi_0$.

As we did for the ALA trajectory, the R-line coverage for the AELA trajectory is numerically obtained by combining the results for the elemental AA and AL trajectories. Of course, the AA coverage remains unchanged. Only the AL coverage changes; it is now given by Relations 5.6 and 5.7 upon substituting $r'_1$ and $r'_2$ for $r_1$ and $r_2$, respectively, with

$$r'_1 = \frac{H + \Delta H - z}{2H + \Delta H} R$$

and

$$r'_2 = R - r'_1.$$

Examples of the R-line coverage for the AELA trajectory in $\Pi_0$ are given in the bottom row of Figure 5.11 for source angular ranges of $234^\circ$ and $309.6^\circ$ and T-line extensions of $\Delta H/(2H) = 0.82$ and $\Delta H/(2H) = 0.48$, respectively. In these Figures, a circular ROI of radius $r = R \sin(27^\circ)$ is shown centered on the origin. The bottom row of Figure 5.11 demonstrates that the entire ROI is covered by R-lines when the T-line extensions are 82% and 48% of the total axial length, $2H$, when $\lambda_m = 234^\circ$ and $309.6^\circ$, respectively. Compared with the corresponding examples of the ALA R-line coverage depicted in the first row of Figure 5.11, the second row shows how the T-line extension can improve the R-line coverage.

The R-line coverage for the AELA trajectory in $\Pi_0$ may be understood geometrically, as illustrated in Figure 5.12. This figure depicts both the AA and AL contributions to the R-line coverage. From the figure, it is seen that if the T-line is extended far enough beyond the T-arcs, the partial disk of R-line coverage due to the AL elemental trajectories increases in size such that the central ROI is covered by R-lines.

An important practical question for the AELA trajectory is how large the T-line extension, $\Delta H$, needs to be so that an ROI of a given radius, $r$, is covered by R-lines. The remainder of this section is devoted to answering this question. Let $\lambda_m^*$ be such that the short-scan from $\lambda_s$ to $\lambda_s + \lambda_m^*$ just encompasses the ROI, i.e., $\lambda_m^* = \pi + 2 \arcsin(r/R)$. We will determine how the required $\Delta H$ varies as $\lambda_m$ increases from $\lambda_m^*$ to $2\pi$.

Figure 5.13 contains a geometrical construction depicting the R-line coverage for the AELA trajectory in $\Pi_0$. In this figure, the contributions of both the AA and AL trajectories to the R-line coverage are shown in three cases: (a) $\lambda_m < \lambda_c$, (b) $\lambda_m = \lambda_c$, and (c) $\lambda_m > \lambda_c$, where $\lambda_c$ is a critical angle that will be defined in the next paragraph. Observe that the R-line coverage for the AA trajectory is delimited by the T-arc, $\text{Arc}(O_+^{\lambda_c}, 0)$, and $\text{Arc}(O_0^{\lambda_c}, 0)$,
5.3. AELA Trajectory

Figure 5.12: The R-line coverage in $\Pi_0$ when the T-line of the ALA trajectory is extended beyond the T-arcs by $\Delta H$. This extension helps to increase the R-line coverage in the center. The dashed circle delineates the extra coverage resulting from the T-line extension.

Figure 5.13: Geometrical constructions depicting the AELA R-line coverage in $\Pi_0$ for a given ROI and different values of $\lambda_m \geq \lambda^*_m$, where $\lambda^*_m = \pi + 2 \arcsin(r/R)$ is the minimum allowable angular length for the T-arcs, beyond which Tuy’s condition is not satisfied. Two points on the boundary of the ROI play a particular role: $K$ and $D$; $K$ is the point on $\text{Arc}(O \lambda_e^0, O)$ (i.e., the R-arc from $A \lambda_e^0$) that is not within the triangle formed by $A \lambda_s^0$, $A \lambda_e^0$ and $O$, whereas $D$ is the point on the ROI boundary that is the furthest away from $A \lambda_e^0$. As illustrated in the figures, there exists a critical angle, $\lambda_c$, for $\lambda_m$ where $D$ and $K$ become identical. The required extension for the T-line, namely $\Delta H$, takes different expressions depending on whether $\lambda_m < \lambda_c$ or not.

and that the partial disk of R-line coverage for the AL trajectory is delimited by the dashed circle and the line connecting $A \lambda_s^0$ and $A \lambda_e^0$. We need to calculate $\Delta H$ such that the contribution of the AL trajectory to the R-line coverage includes the portion of the ROI that is not covered by the AA R-lines. This calculation requires a separation between two cases,
depending on the relative position of two particular points, $D$ and $K$, on the boundary of the ROI.

Let $D$ be the point on the boundary of the ROI that is the furthest away from $A_0^\lambda$. Let $K$ be the intersection of the ROI boundary with $\text{Arc}(O_+^\lambda,0)$ that is not within the triangle formed by $A_0^\lambda$, $A_0^\lambda$ and $O$. As $\lambda_m$ increases away from $\lambda^*_m$, $\text{Arc}(O_+^\lambda,0)$ rotates counterclockwise around $O$, and $K$ moves towards $D$. We define $\lambda_c$ to be the critical value of $\lambda_m$ for which $K$ coincides with $D$; see Figure 5.13(b). Using Figures 5.13(a) and 5.13(b), it is straightforward to see that $\lambda_c = \pi + \arccos(r/R)$. For $\lambda_m < \lambda_c$, $K$ does not reach $D$, and the necessary $\Delta H$ is independent of $\lambda_m$. To reach this conclusion, we identify all points $Q$ that meet the following condition while being on the boundary of the ROI: any neighborhood of the required $\Delta H$ contains a subset of non-zero size that is not covered by AA R-lines. Of all such points $Q$ containing $\lambda_m$, $\lambda^*_m$ is always the point that is the furthest away from $O$ and $K$. As the distance from $K$ to $A_0^\lambda$ decreases with $\lambda_m$, we find that the required $\Delta H$ decreases with $\lambda_m$ once $\lambda_m > \lambda_c$; see Figure 5.13(c). Using these observations together with the geometrical construction of Figure 5.13, we show in Section A.4 that the minimum relative length of the T-line extension required to cover an ROI of radius $r$ with R-lines is

$$\frac{\Delta H}{2H} = \begin{cases} \frac{r}{(R-r)}, & \lambda^*_m \leq \lambda_m \leq \lambda_c \\ \frac{r_s}{(R-r_s)}, & \lambda_c < \lambda_m \leq 2\pi, \end{cases}$$

(5.8)

where

$$r_x = |OF| = \frac{d}{\cos \gamma - R},$$

(5.9)

with

$$d = \sqrt{R^2 + r^2 - 2Rr\cos \eta},$$

(5.10)

$$\eta = \lambda_m - \arccos\left(\frac{r}{R}\right),$$

(5.11)

$$\gamma = -\arcsin\left(\frac{(r \sin \eta) / d}{\sin \eta}\right).$$

(5.12)

Figure 5.14 plots the required $\Delta H/2H$ as a function of $\lambda_m$ for $r/R = 0.15, 0.30$, and 0.45. The ratios $r/R = 0.15$ and $r/R = 0.45$ are representative of current C-arm and CT scanners, respectively. Not surprisingly, Figure 5.14 shows that the required T-line extension increases as the relative size of the ROI grows. In addition, as $\lambda_m$ increases to $360^\circ$, the required $\Delta H$ converges to zero, which is consistent with our findings for the full scan ALA trajectory in Section 5.2.3.

Equation 5.8 can be inverted to find how $\Delta H/2H$ varies with $r/R$ at fixed $\lambda_m$. Figure 5.15 shows this behavior for $\lambda_m = 220^\circ$, $270^\circ$, $310^\circ$ and $340^\circ$; as expected, the required T-line extension increases with increasing ROI size and decreases with increasing $\lambda_m$.

The plots in Figures 5.14 and 5.15 indicate that the required T-line extension for the AELA trajectory is reasonable for $r/R$ ratios that are typical of current C-arm scanners (roughly 0.15). On the other hand, for larger ratios, such as those needed in CT, $\Delta H$ can be
Figure 5.14: Minimum $\Delta H/(2H)$ required to fully cover the ROI with R-lines, plotted versus $\lambda_m$ for $r/R = 0.15, 0.30$, and 0.45.

Figure 5.15: Minimum $\Delta H/(2H)$ required to fully cover the ROI with R-lines, plotted versus $r/R$ for $\lambda_m = 220^\circ, 270^\circ, 310^\circ$, and $340^\circ$.

fairly large when $\lambda_m = \lambda_m^*$. Fortunately, the required $\Delta H$ may be reduced to any practical length by employing a suitable, larger value for $\lambda_m$. 
5.4 Conclusion and Discussion

We have presented two arc-plus-line trajectories, i.e., the ALA and AELA trajectories, and have analyzed their R-line coverage thoroughly. Our analysis demonstrates that the R-line coverage of the ALA trajectory is insufficient for any size of cylindrical ROI, when the angular length is less than 360°. On the other hand, when the angular length of the ALA trajectory is 360°, ROIs of any size within the convex hull of the ALA trajectory are fully covered by R-lines.

To provide sufficient R-line coverage in the ROI for an angular length less than 360°, we proposed the AELA trajectory, which is a modification of the ALA trajectory with the T-line extended beyond the T-arcs. We have shown that the line extension helps to improve the R-line coverage in the ROI. Quantitatively, we also provide a function to calculate the length of the line extension to guarantee full R-line coverage in the ROI with a given radius. This function demonstrates that the length of the line extension is reasonable for a typical C-arm system.

Recall from Section 1.3.2 that, practically, we prefer the source trajectory to be continuous for extended-volume C-arm CT. This continuity is not satisfied with the AELA trajectory. To overcome this shortcoming while retaining the advantages of the AELA trajectory, we will present another trajectory called the ellipse-line-ellipse trajectory in the next chapter.
Chapter 6

Ellipse-Line-Ellipse Trajectory

In this chapter, we present a new scan geometry called the ellipse-line-ellipse (ELE) trajectory. This trajectory not only retains the advantages of the AELA trajectory, but also allows smooth and continuous scanning. We identify the R-line coverage of the ELE trajectory using the same method as that used for the ALA trajectory in Chapter 5. We first decompose the ELE trajectory into elemental trajectories called the arc-arc trajectory and the arc-line trajectory. Here for the ELE trajectory, an arc means an elliptical arc. We then study the R-line coverage of these elemental trajectories and investigate their combinations. We were able to demonstrate that full R-line coverage is obtainable within any typical ROI for a properly designed ELE trajectory. This result indicates that, using the ELE trajectory, R-line based TES image reconstructions are achievable for extended-volume C-arm CT.

6.1 Geometry and Notation

The ELE trajectory is composed of periodic duplicates of a fundamental source trajectory that consists of two elliptical arcs connected by a line segment. The elliptical arcs are centered on the center line of the patient table, whereas the line segment is parallel to the patient table and connects the elliptical arcs through one of their endpoints; see the solid portion of Figure 6.1.

The ELE trajectory is a practical geometry for extended-volume C-arm CT. It can be realized by first rotating the C-arm in one direction along one elliptical arc, then translating along the line segment, and finally rotating in the reverse direction along the other elliptical arc. Because one period is representative for the whole ELE trajectory, in the rest of this work, we will only focus on the fundamental trajectory and simply call it the ELE trajectory.
Chapter 6. Ellipse-Line-Ellipse Trajectory

We use similar notations to those used for the ALA trajectory in Chapter 5. We refer to the elliptical arcs as T-arcs, and refer to the line segment as T-line, where “T” stands for trajectory. Let \((x, y, z)\) be a Cartesian coordinate system with the origin at \(O\), and the \(z\)-axis being the center line of the patient table. The ELE trajectory lies on the cylindrical surface \(S_c\) that is centered on the \(z\)-axis with radius \(R\). The two T-arcs are mirror symmetric relative to the \((x, y)\)-plane (Figure 6.2). To distinguish these two T-arcs, we call the one above the \((x, y)\)-plane the upper T-arc and the one below the lower T-arc. We denote \(O_+\) and \(O_-\) as the centers of the upper and lower T-arcs, respectively. We define \(H\) and \(-H\) as the \(z\) coordinates of \(O_+\) and \(O_-\), respectively. By definition, the \((x, y)\) coordinates of \(O_+\) and \(O_-\) are \((0, 0, H)\) and \((0, 0, -H)\), respectively.

We now describe the ELE trajectory mathematically. We denote each point on the upper (resp. lower) T-arc as \(A^\lambda_+\) (resp. \(A^\lambda_-\)), where \(\lambda\) is the polar angle as illustrated in the right of Figure 6.2. The \((x, y, z)\) coordinates of \(A^\lambda_+\) and \(A^\lambda_-\) can be expressed as below:

\[
A^\lambda_+ : \left( R \cos \lambda , R \sin \lambda , \mathcal{H}(\lambda) \right) \\
A^\lambda_- : \left( R \cos \lambda , R \sin \lambda , -\mathcal{H}(\lambda) \right),
\]

with \(\lambda \in (\lambda_s, \lambda_e)\), and

\[
\mathcal{H}(\lambda) = H + \Delta H \cos \lambda,
\]

where \(\lambda_s\) and \(\lambda_e\) are the polar angles of the start and end points of the T-arcs, and \(\Delta H\) is a positive constant that is smaller than \(H\). We define \(\lambda_s\) and \(\lambda_e\) so that they are symmetric about \(\pi/2\), i.e., \(\lambda_s = -\gamma_m\) and \(\lambda_e = \pi + \gamma_m\), where \(\gamma_m \in (0, \pi/2)\). The parameter \(\gamma_m\) is usually interpreted as the fan-angle. According to the definition of \(\mathcal{H}(\lambda)\), the upper T-arc reaches the maximum and minimum positions in \(z\) at \(A^0_+\) and \(A^{\pi}_+\), respectively. By definition, \(2\Delta H\) is the distance in \(z\) between these two points. Geometrically, the upper T-arc can be considered as part of the intersection between the cylindrical surface \(S_c\) and the plane perpendicular to the \((x, z)\)-plane that contains \(A^0_+\) and \(A^{\pi}_+\). Similar observations

Figure 6.1: The ELE trajectory is composed of periodic duplicates of a fundamental source trajectory that consists of two elliptical arcs and a line segment. The elliptical arcs are centered on the center line of the patient table, whereas the line segment is parallel to the patient table. To achieve one period, the C-arm is first rotated around the table in one direction, then translated along the line segment, and finally rotated in the reversal direction.
Figure 6.2: The ELE trajectory is composed of two elliptical arcs and a line segment. It lies on a cylindrical surface, $S_c$, that is centered on the $z$-axis with radius $R$. Therefore, the projection of the ELE trajectory onto the $(x,y)$-plane is a circular arc. Points on the elliptical arcs can be identified by their polar angle $\lambda$, and points on the line segment can be identified by their $z$ coordinate. Left: 3D view. Right: projection onto the $(x,y)$-plane.

can be made for the lower T-arc. Regarding the T-line, by construction, it connects $A_{1+}^{\lambda_s}$ and $A_{2-}^{\lambda_s}$, and all points on it share the same $(x,y)$ coordinates. We refer to a point on the T-line as $A_{2z}^{\lambda_s}$; its $(x,y,z)$ coordinates can be expressed as below:

$$
A_{2z}^{\lambda_s} : (R \cos \lambda_s, R \sin \lambda_s, z), \quad \text{with} \quad z \in [-H(\lambda_s), H(\lambda_s)].
$$

(6.3)

### 6.2 Preliminaries

This section provides some preliminary insight about the R-lines of the ELE trajectory. Most results will be explained geometrically through lemmas, theorems and corollaries. For convenience, we introduce the following notations. We denote $\Pi_z$ as the horizontal plane that goes through $(0,0,z)$, and denote $A_{2z}^{\lambda_1,\lambda_2}$ as the intersection between $\Pi_z$ and the R-line that connects $A_{1+}^{\lambda_1}$ and $A_{2-}^{\lambda_2}$.

### 6.2.1 Introduction of an R-arc

Similar to the ALA trajectory, we now introduce the concept of an R-arc for the ELE trajectory, which will be used extensively in the next section. Let $S_{2+}^{\lambda}$ be the positive partial cone surface that is the union of all the half lines that start from $A_{1+}^{\lambda}$ and go through all the points on the lower T-arc; see Figure 6.3. Here, “positive” indicates that the vertex of the partial cone surface is located on the upper T-arc. Similarly, we call the union of all the half lines that start from $A_{2-}^{\lambda}$ and go through all the points on the upper T-arc the negative partial
cone surface, and denote it as $S^\lambda_\pm$. An R-arc is defined as the intersection between $\Pi_z$ and a partial cone surface, as illustrated in Figure 6.3. We will call an R-arc from a positive partial cone surface a positive R-arc, and an R-arc from a negative partial cone surface a negative R-arc. Whereas an R-arc of the ALA trajectory is circular (Section 5.1.2), an R-arc of the ELE trajectory is elliptical, as will be explained hereafter. In the rest of this subsection, we will focus on properties of the R-arcs of the ELE trajectory. If not specified, an R-arc in this chapter will always refer to an R-arc of the ELE trajectory.

Figure 6.3: Illustration of R-arc related terminology for the ELE trajectory. A positive partial cone surface, $S^\lambda_+$, connects $A^\lambda_+$ to all the points on the lower T-arc; a negative partial cone surface, $S^\lambda_-$, connects $A^\lambda_-$ to all the points on the upper T-arc. An R-arc can be considered as the intersection between a partial cone surface and $\Pi_z$, which is the horizontal plane that goes through $(0, 0, z)$.

R-arcs are elliptical. Consider an arbitrary $S^\lambda_+$, the line connecting $A^\lambda_+$ and $A^\lambda_-$ on this positive partial cone surface is perpendicular to horizontal planes. Also, recall from Section 6.1 that $\Delta H < H$, which indicates that the horizontal plane through $A^\lambda_+$ will never intersect the lower T-arc. According to Lemma 2 below (a proof of which can be found in Section A.5), the R-arcs from $S^\lambda_+$ are elliptical arcs. Because $S^\lambda_+$ is arbitrary chosen and it is mirror-symmetric to $S^\lambda_-$, all R-arcs are elliptical arcs. For convenience, we denote $O^\lambda_z$ (resp. $O^-\lambda_z$) as the center of the R-arc that results from intersecting $\Pi_z$ with $S^\lambda_+$ (resp. $S^\lambda_-$). If identification is needed, we will denote an R-arc using the label of its center, e.g., R-arc $(O^\lambda_z)$.

**Lemma 2.** Consider a point $A$ and an ellipse lying in a plane that does not contain $A$, which together define a cone. Let $B$ be a point on the ellipse, and let $\alpha = \frac{AB}{|AB|}$. Also, let $\Pi(A, \alpha)$ be the plane normal to $\alpha$ through $A$. If $\Pi(A, \alpha)$ has no intersection with the ellipse, then any plane that is parallel to $\Pi(A, \alpha)$ (and different from $\Pi(A, \alpha)$) is such that its intersection with the cone surface is an ellipse.
Aside from being elliptical, R-arcs are tangent to cylindrical surface $S_c$. Consider $S^\lambda_+$ and the line that goes through $A^\lambda_+$ and $A^\lambda_-$, which we denote as $L_\lambda$. Let $\zeta_\lambda$ be a plane that is tangent to $S_c$ at $L_\lambda$. By construction, $\zeta_\lambda$ shares this (and only this one) line with $S^\lambda_+$. Consequently, $S^\lambda_+$ is tangent to $S_c$ at $L_\lambda$, which indicates that all R-arcs from $S^\lambda_+$ are tangent to $S_c$. The same observations can be made for $S^\lambda_-$. Hence, all R-arcs are tangent to $S_c$. Let $D_0$ be the intersection between $\Pi_z$ and $S_c$. Consequently, R-arc $(O^\lambda_+, z)$ is tangent to $D_0$, as shown in Figure 6.3.

Besides the above two general properties, special relations can be observed for R-arcs that are from the same partial cone surface. These relations are clarified in Theorem 1, which indicates that all R-arcs from the same partial cone surface are similar to each other. Note that, according to Section A.1, Theorem 1 is also valid for an elliptical partial cone surface. As an example, given R-arc $(O^\lambda_+, z)$, R-arc $(O^\hat{\lambda}, \hat{z})$ can be obtained by translating Arc$(O^\lambda_+, z)$ by vector $O^\lambda_+O_x$, where Arc$(O^\lambda_+, z)$ is an isotropically scaled version of R-arc $(O^\lambda_+, z)$ and $O_x$ is the intersection between $A^\lambda_+O^\lambda_+z$ and $\Pi_z$. Figure 6.4: Illustration of Theorem 1 for an elliptical partial cone surface. Given R-arc $(O^\lambda_+, z)$, R-arc $(O^\hat{\lambda}, \hat{z})$ can be obtained by translating Arc$(O^\lambda_+, z)$ by vector $O^\lambda_+O_x$, where Arc$(O^\lambda_+, z)$ is an isotropically scaled version of R-arc $(O^\lambda_+, z)$ and $O_x$ is the intersection between $A^\lambda_+O^\lambda_+z$ and $\Pi_z$. Because $O^\lambda_+z$ is the center of R-arc $(O^\lambda_+, z)$ and Arc$(O^\lambda_+, z)$, $O_x$ is the center of R-arc $(O^\lambda_+, z)$, i.e., $O_x$ is identical to $O^\lambda_+z$. Consequently, we have another corollary to Theorem 1 as below.
Corollary 3. The centers of all the R-arcs from the same partial cone surface are colinear.

6.2.2 Geometrical Construction of an R-arc

Because an R-arc is elliptical, it can be identified by its center, and the directions and magnitudes of its semi-major and -minor axes. In this subsection, we provide a geometrical method to find these parameters. Though the method is described for a positive partial cone surface, it is also applicable to a negative partial cone surface due to the symmetry of the ELE trajectory.

We can identify the center, the major axis and the minor axis of R-arc \( O_{\lambda,z} \) according to Theorem 3 (top row of Figure 6.5), which is proven in Section A.6. Based on this information, we can also find the lengths of the semi-major and -minor axes of R-arc \( O_{\lambda,z} \), as described below.

Theorem 3. As shown in the top row of Figure 6.5, the center of R-arc \( O_{\lambda,z} \), which is \( O_{\lambda,z} \), is located at the middle of the line segment \( A_{\lambda,0}^{\lambda,0} A_{\lambda,\pi}^{\lambda,\pi} \). The major axis is on the line through \( O_{\lambda,z} \) that is parallel to \( A_{\lambda,0}^{\lambda,0} A_{\lambda,\pi}^{\lambda,\pi} \), whereas the minor axis is on the line through \( O_{\lambda,z} \) that is parallel to \( A_{\lambda,0}^{\lambda,0} A_{\lambda,\pi}^{\lambda,0} \). Quantitatively, the polar angle between the x-axis and the major axis is \( \lambda/2 \).

We start with two special cases, i.e., \( \lambda = 0 \) and \( \lambda = \pi \). We take the first case as an example, as shown in the middle row of Figure 6.5. According to Theorem 3, the middle point of \( A_{\lambda,0}^{\lambda,0} A_{\lambda,\pi}^{\lambda,\pi} \) is the center of the R-arc, i.e., \( O_{\lambda,z}^{0,0} \). Now, extend \( A_{\lambda,0}^{0,0} O_{\lambda,z}^{0,0} \) until it intersects \( A_{\lambda,\pi}^{0,0} \) at \( O' \). In the plane where the lower T-arc lies, we draw a line through \( O' \) that is perpendicular to \( A_{\lambda,0}^{0,0} A_{\lambda,\pi}^{0,0} \), and denote the intersection between this line and the portion of the lower T-arc of polar angle between 0 and \( \pi \) as \( Q' \). Furthermore, we denote the intersection between \( \Pi_z \) and the line that connects \( A_{\lambda,0}^{0,0} \) to \( Q' \) as \( Q \). Then \( |O_{\lambda,z}^{0,0} A_{\lambda,0}^{0,0}| \) is the length of the semi-major axis, and \( |O_{\lambda,z}^{0,0} Q| \) is the length of the semi-minor axis. A similar process can be applied to get the semi-major and minor axes for the case of \( \lambda = \pi \).

For a general case when \( \lambda \neq 0 \) or \( \pi \), the process is more complicated. To describe the procedure, we take the semi-minor axis as an example, as shown in the bottom row of Figure 6.5. First, draw the line through \( O_{\lambda,z}^{0,0} \) that is parallel to \( A_{\lambda,0}^{\lambda,0} A_{\lambda,\pi}^{\lambda,\pi} \), and denote the intersection between this line and \( A_{\lambda,0}^{\lambda,0} A_{\lambda,\pi}^{\lambda,\pi} \) as \( P \). Next, denote \( O' \) as the intersection between \( A_{\lambda,0}^{0,0} A_{\lambda,\pi}^{0,0} \) and the line that goes through \( A_{\lambda,0}^{\lambda,0} \) and \( O_{\lambda,z}^{0,0} \), and refer to the intersection between \( O_{\lambda,0}^{\lambda,0} A_{\lambda,\pi}^{\lambda,\pi} \) and the line that goes through \( A_{\lambda,0}^{\lambda,0} \) as \( P' \). Then, extend \( O'P' \) until it intersects the ellipse where the lower T-arc lies at \( Q' \). Finally, denote the intersection between the line that goes through \( A_{\lambda,0}^{\lambda,0} \) and \( Q' \), and the line that goes through \( O_{\lambda,z}^{0,0} \) and \( P \), as \( Q \). The line segment, \( O_{\lambda,z}^{0,0} Q \), is the semi-minor axis of R-arc \( O_{\lambda,z}^{0,0} \). An analogous approach can be used to obtain the semi-major axis.
Figure 6.5: Illustration for parameter identification of R-arc \((O_+^{z,\lambda})\). Top left: the line through \(A_+^{\lambda,\lambda}\) and \(A_+^{\lambda,\pi}\) intersects \(\Pi_z\) at \(A_+^{\lambda,\pi}\), whereas the line through \(A_+^{\lambda,\pi}\) and \(A_0^{\lambda,\pi}\) intersects \(\Pi_z\) at \(A_0^{\lambda,\pi}\). The middle of \(A_+^{\lambda,0} A_0^{\lambda,\pi}\) is the center of R-arc \((O_+^{z,\lambda})\). Top right: orthogonal projection of the trajectory onto \(\Pi_z\). The major axis lies on the line that is through \(O_+^{z,\lambda}\) and is parallel to \(A_+^{\lambda,\lambda} A_+^{\lambda,\pi}\), whereas the minor axis lies on the line that is through \(O_+^{z,\lambda}\) and is parallel to \(A_+^{\lambda,\lambda} A_0^{\lambda,\pi}\). Middle row: 3D (left) and 2D (right) illustration for identification of the semi-major and minor axes when \(\lambda = 0\). Bottom row: identification of the semi-minor axis for a general case. For more details, see the text.
6.3 R-line Coverage

By Tuy’s condition \cite{Tuy83, Finc85}, TES reconstruction is only possible within the convex hull of the ELE trajectory. Therefore, we only investigate the R-lines inside this convex hull. The convex hull of the ELE trajectory is bounded by four surfaces: (i) the cylindrical surface $S_c$, (ii) the plane where the upper T-arc lies, (iii) the plane where the lower T-arc lies and (iv) the plane that goes through the endpoints of the upper and lower T-arcs, which we denote as $\Pi_{\text{cut}}$.

We study two types of R-line coverage, one is that generated by connecting points from one T-arc to the other, and the other is that generated by connecting points from the T-line to either T-arc. Note that the ELE geometry is mirror-symmetric relative to the $(x, y)$-plane, and hence the R-line coverage in $\Pi_z$ is the same as in $\Pi_{-z}$ for any $0 \leq z < H + \Delta H$. In the rest of this section, we will only focus on the R-line coverage at positions $z \geq 0$.

6.3.1 Arc-Arc R-line Coverage

In this subsection, we study the R-line coverage that is generated by connecting points from one T-arc to the other. We consider $A_{\lambda u, \lambda l}^z$, which is the intersection between $\Pi_z$ and the line that connects $A_{\lambda u}^z +$ and $A_{\lambda l}^z$, with $0 \leq z \leq H(\lambda_u)$. The upper bound is needed so that $A_{\lambda u, \lambda l}^z$ does not go beyond the plane where the upper T-arc lies. From basics of Euclidean geometry, the $(x, y)$ coordinates of $A_{\lambda u, \lambda l}^z$ are

\[
\begin{align*}
\frac{x}{R} &= \frac{(1+e) \cos \lambda_u + (1-e) \cos \lambda_l + 2d \cos \lambda_u \cos \lambda_l}{2 + d (\cos \lambda_u + \cos \lambda_l)} \\
\frac{y}{R} &= \frac{(1+e) \sin \lambda_u + (1-e) \sin \lambda_l + d \sin(\lambda_u + \lambda_l)}{2 + d (\cos \lambda_u + \cos \lambda_l)},
\end{align*}
\]

where $e = z/H$ and $d = \Delta H/H$.

Note that in Equation 6.4, $x$ and $y$ are both functions of $z$, $\lambda_u$ and $\lambda_l$.

Next, we show that this arc-arc (AA) R-line coverage is a union of R-arcs in $\Pi_z$. According to Equation 6.4, the R-line coverage in $\Pi_z$ is the union of all $A_{\lambda u, \lambda l}^z$ for all possible values of $\lambda_u$ and $\lambda_l$. Therefore, the AA coverage in $\Pi_z$ can be achieved in two steps. First, with $\lambda_u$ fixed, the union of $A_{\lambda u, \lambda l}^z$ for $\lambda_l \in [\lambda_s, \lambda_e]$ can be interpreted as the intersection between $\Pi_z$ and the positive partial cone surface $S_+^{\lambda_u}$, which, by definition, is R-arc $(O_{\lambda u, z}^{\lambda u})$ (recall Section 6.2.1). Secondly, by increasing $\lambda_u$ from $\lambda_s$ to $\lambda_e$, i.e., by continuously moving $A_{\lambda u}^z$ counter-clockwise on the upper T-arc from $A_{\lambda u}^z$ to $A_{\lambda e}^z$, we obtain a set of R-arcs whose union defines the AA coverage in $\Pi_z$. Note that for $z \in [0, H - \Delta H)$, all the positive partial cone surfaces participate in the formation of the AA coverage in $\Pi_z$. However, for $z \in [H - \Delta H, H + \Delta H]$, only the positive partial cone surfaces due to the points of the upper T-arc that are above $\Pi_z$ contribute to the AA coverage in $\Pi_z$. 

6.3. R-line Coverage

According to the above two-step procedure and Equation 6.4, we can simulate the AA coverage. We start with the simplest case where the angular length of the T-arcs is $2\pi$ (i.e., $\gamma_m = \pi/2$) so that all the R-arcs become complete ellipses. The results for the case of $H = 10\text{ cm}$, $R = 10\text{ cm}$ and $\Delta H = 3\text{ cm}$ are shown in Figure 6.6. For illustration purpose, only the positive R-arcs due to several points on the upper T-arc (a coarse sampling in $\lambda_u$) are displayed. As explained in Section 6.2.1 all the positive R-arcs are tangent to $D_0$. In $\Pi_0$ (Figure 6.6(a)), these R-arcs go through a common point that is denoted as $I_1$, and the entire area inside $D_0$ is covered by R-lines. This common point is not easy to identify, and we will define it later in Section 6.4.2. For $\Pi_\zeta$ with $0 < \zeta < H - \Delta H$, the union of these R-arcs is bounded by $D_0$ and another curve, which we denote as $D_1$; see the results in $\Pi_{H/2}$ in Figure 6.6(b). The curve $D_1$ is also not trivial, and we will define it mathematically later in this subsection. When the slice position, $\zeta$, increases, $D_1$ becomes larger and is not necessarily fully within $D_0$. In $\Pi_{H-\Delta H}$, which goes through $A_0^\ell$, $D_1$ is inscribed to $D_0$ (Figure 6.6(c)), whereas in $\Pi_\zeta$ with $H - \Delta H < \zeta < H + \Delta H$, $D_1$ intersects $D_0$, and the AA coverage looks like a crescent moon (Figure 6.6(d)).

Now, we consider the general case where the angular length of the T-arcs is less than $2\pi$ (i.e., $0 < \gamma_m < \pi/2$). In this case, all the positive R-arcs become partial elliptical arcs, and the AA coverage is not simply bounded by $D_0$ and $D_1$ anymore, as shown in Figure 6.7, but these curves still play a role. Comparing Figure 6.6 with Figure 6.7, we observe, as expected, that the AA coverage in $\Pi_\zeta$ for $0 < \gamma_m < \pi/2$ is a subset of that for $\gamma_m = \pi/2$.

In contrast to the preceding two-step procedure that is driven by $\lambda_u$ and $\lambda_l$, we now provide an efficient method to identify the R-lines for given $(x, y, z)$ coordinates of a point, which is particularly important as far as TES reconstruction is considered. Afterward, we will offer some geometrical insight in the AA coverage. This method requires the inversion of Equation 6.4. To achieve this inversion, we first perform a change of variable, namely

\[
\begin{align*}
\alpha &= (\lambda_u + \lambda_l)/2, \\
\beta &= (\lambda_u - \lambda_l)/2,
\end{align*}
\]

or

\[
\begin{align*}
\lambda_u &= \alpha + \beta, \\
\lambda_l &= \alpha - \beta.
\end{align*}
\]

Using this change of variable, Equation 6.4 becomes:

\[
\begin{align*}
\frac{x}{R} &= \frac{\cos \alpha \cos \beta - e \sin \alpha \sin \beta + d (\cos^2 \alpha - \sin^2 \beta)}{1 + d \cos \alpha \cos \beta}, \\
\frac{y}{R} &= \frac{\sin \alpha \cos \beta + e \sin \beta \cos \alpha + d \sin \alpha \cos \alpha}{1 + d \cos \alpha \cos \beta}.
\end{align*}
\]

Now the problem is converted to the inversion of Equation 6.6 i.e., for given $(x, y, z)$, find all the possible pairs of $(\alpha, \beta)$. We provide the details of this inversion in Section A.7 and present the results below. Let

\[
\begin{align*}
a &= y + xyd/R, \\
b &= y^2d/R - x - Rd,
\end{align*}
\]

and

\[
\begin{align*}
c_1 &= R^2e^2 - (a^2 + b^2 + e^2x^2 + e^2y^2)/2, \\
c_2 &= (a^2 - b^2 + e^2x^2 - e^2y^2)/2, \\
c_3 &= ab + e^2xy.
\end{align*}
\]
All possible solutions of Equation 6.6 are as follows, provided that $\alpha \in [-\pi/2, 3\pi/2]$ and $\beta \in [-\pi, \pi]$.

$$
\begin{align*}
\alpha_1 &= \theta + \omega - \frac{n_1 \pi}{2} \\
\beta_1 &= \cos \frac{x \cos \alpha_1 + y \sin \alpha_1}{R}
\end{align*}
\begin{align*}
\alpha_2 &= \theta - \omega - \frac{n_2 \pi}{2} \\
\beta_2 &= -\cos \frac{x \cos \alpha_2 + y \sin \alpha_2}{R}
\end{align*}
\begin{align*}
\alpha_3 &= \theta - \omega + \frac{m_1 \pi}{2} \\
\beta_3 &= \cos \frac{x \cos \alpha_3 + y \sin \alpha_3}{R}
\end{align*}
\begin{align*}
\alpha_4 &= \theta - \omega + \frac{m_2 \pi}{2} \\
\beta_4 &= -\cos \frac{x \cos \alpha_4 + y \sin \alpha_4}{R}
\end{align*}

Figure 6.6: AA R-line coverage due to a coarse sampling in $\lambda_u$ for the case where $\gamma_m = \pi/2$. Parameters used for the experiments were as follows: i) $R = 10$ cm, ii) $H = 10$ cm, and iii) $\Delta H = 3$ cm.
where $\text{acos}$ is the inverse cos function that returns a polar angle in the range $[0, \pi]$, \( \{n_1, n_2\} = \{-1, 0, 1\} \), \( \{m_1, m_2\} = \{0, 1, 2\} \),

\[
\theta = \text{atan2}(c_3, c_2) \quad \text{and} \quad \omega = \text{acos} \left( \frac{c_1}{\sqrt{c_2^2 + c_3^2}} \right),
\]

with $\text{atan2}(v, u)$ being the four-quadrant inverse tangent function that gives the polar angle of point $(u, v)$ in the $(x, y)$-plane in the range $[-\pi, \pi]$. Note that $\omega$ should be restricted to real value.
According to the change of variable (Equation 6.5), \((\lambda_u, \lambda_l)\) can be obtained as:

\[
\begin{align*}
\lambda_{uk} &= \text{mod}(\alpha_k + \beta_k - \lambda_s, 2\pi) + \lambda_s, \\
\lambda_{lk} &= \text{mod}(\alpha_k - \beta_k - \lambda_s, 2\pi) + \lambda_s,
\end{align*}
\]

with \(k = \{1, 2, 3, 4\}\), and \(\text{mod}(u, 2\pi)\) being equal to \(u\) modulo \(2\pi\). It is important to notice that all possible pairs of \((\lambda_{uk}, \lambda_{lk})\) can be obtained from \((\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)\) and \((\alpha_4, \beta_4)\) using \(n_1 = n_2 = m_1 = m_2 = 0\). Other values of \(n_1, n_2, m_1\) and \(m_2\) will result in redundant pairs of \((\lambda_{uk}, \lambda_{lk})\). This observation can be easily verified using the property of the modulo operation and the property of an acos function \((\text{acos}(-c) = \pi - \text{acos}(c))\).

According to the above paragraph, we now have 4 solutions for \((\lambda_u, \lambda_l)\). Among the 4 candidates, the solution \((\lambda_{uk}, \lambda_{lk}) (k = \{1, 2, 3, 4\})\) is admissible only if the following requirements are satisfied,

- \(\lambda_{uk} \leq \lambda_e\) and \(\lambda_{lk} \leq \lambda_e\),
- \(\omega\) is not complex, i.e., \(|c_1/\sqrt{c_2^2 + c_3^2}| \leq 1\),
- \((\lambda_{uk}, \lambda_{lk})\) satisfies Equation 6.4,
- \(-\mathcal{H}(\lambda_{lk}) \leq z \leq \mathcal{H}(\lambda_{uk})\).

The third requirement is needed to eliminate suspicious solutions introduced during the square operation (Equation A.23) in Section A.7, and the last requirement enforces the solution to be within the convex hull of the ELE trajectory. Note that, for the central slice, i.e., \(z = e = 0\), we have \(c_1/\sqrt{c_2^2 + c_3^2} = -1\), which can yield unstable decisions for the second requirement due to computational error. To solve this problem, we can enforce \(\omega\) to be \(\pi\) when \(z = 0\).

The above procedure to identify the R-lines from the AA trajectory for a given point is straightforward to implement on a computer. With \(H = 10\) cm, \(R = 10\) cm and \(\Delta H = 3\) cm, the results in \(\Pi_0, \Pi_{H/2}\) and \(\Pi_H\) for \(\gamma_m = \pi/2\) and \(\gamma_m = \pi/6\) are shown in Figure 6.8. In this figure, the white regions are covered by 2 R-lines, whereas the gray regions are covered by 1 R-line. In the rest of this subsection, we provide some geometrical insight into why these AA coverage diagrams appear as they do.

First, we consider the coverage for the case where \(\gamma_m = \pi/2\), as shown in the first row of Figure 6.8. In this case, the ranges of \(\lambda_u\) and \(\lambda_s\) are both \([-\pi/2, 3\pi/2]\), and hence the ranges of \(\alpha\) and \(\beta\) are \([-\pi/2, 3\pi/2]\) and \([-\pi, \pi]\), respectively. Therefore, as long as \(\omega\) is real, solutions of \(\alpha\) and \(\beta\) are guaranteed to satisfy the first requirement. To make sure \(\omega\) is real, the second requirement has to be satisfied, i.e., \(|c_1/\sqrt{c_2^2 + c_3^2}| \leq 1\). Note that the third requirement is for exclusion of the fake solutions and the fourth requirement is to enforce the solutions to be within the convex hull of the ELE trajectory. Therefore, in the convex hull of the ELE trajectory, condition \(|c_1/\sqrt{c_2^2 + c_3^2}| \leq 1\) fully characterizes the AA coverage in \(\Pi_\gamma\) for \(\gamma_m = \pi/2\). This condition is equivalent to

\[c_1^2 - c_2^2 - c_3^2 \leq 0.\]
6.3. *R*-line Coverage

![Figure 6.8: Computer simulation of the AA R-line coverage. The white regions are covered by 2 R-lines, whereas the gray regions are covered by 1 R-line. Parameters used for the experiments were as follows: i) $R = 10$ cm, ii) $H = 10$ cm, and iii) $\Delta H = 3$ cm. Note that in the last column, $\Pi_H$ intersects with the upper T-arc, and hence only the portion of $\Pi_H$ that is below the upper T-arc can have R-line coverage.](image)

Replacing $c_1, c_2, c_3$ by $a$ and $b$, and replacing $a$ and $b$ by $x$ and $y$, the above relation becomes

$$e^2 \left( R^2 - x^2 - y^2 \right) \left( R^2 e^2 - (x + R d)^2 - (1 - d^2) y^2 \right) \leq 0.$$  \(6.8\)

Note that for $e = 0$, any point of interest in the convex hull of the ELE trajectory satisfies the above relation, which indicates that in $\Pi_0$, the entire area inside $D_o$ is fully covered by R-lines. This observation is consistent with the results in Figure 6.6(a) and Figure 6.8(a).

Now, we consider the case where $e > 0$. Because we only consider the points inside the convex hull of the ELE trajectory, Relation 6.8 is equivalent to the following relations

$$\begin{align*}
&\begin{cases}
x^2 + y^2 \leq R^2, \\
\frac{(x + R d)^2}{R^2 e^2} + \frac{y^2}{(Re/\sqrt{1-d^2})^2} \geq 1.
\end{cases} \quad (6.9)
\end{align*}$$

Relation 6.9 requires the point of interest to be inside $D_o$, whereas Relation 6.10 requires the point of interest to be outside an ellipse, which is centered at $(x,y) = (-R d, 0)$ with $Re$...
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as its semi-major axis and $Re/\sqrt{1-d^2}$ as its semi-minor axis. This ellipse is $\mathcal{D}_1$ as was mentioned early in this subsection. Therefore, the AA coverage in $\Pi_z$ for $\gamma_m = \pi/2$ is the set of points that are inside $\mathcal{D}_0$ and outside $\mathcal{D}_1$, as shown in Figure 6.6 and in the first row of Figure 6.8. Observe that the semi-major and -minor axes of $\mathcal{D}_1$ are proportional to $e$ (or $z$). Hence, $\mathcal{D}_1$ degenerates to a point for $e = 0$ and becomes larger when $e$ increases. The relation between $\mathcal{D}_1$ and $\mathcal{D}_0$ is as follows.

- $0 < e < 1 - d$, i.e., $0 < z < H - \Delta H$, $\mathcal{D}_1$ is completely inside $\mathcal{D}_0$;
- $e = 1 - d$, i.e., $z = H - \Delta H$, $\mathcal{D}_1$ is inscribed in $\mathcal{D}_0$;
- $1 - d < e < 1 + d$, i.e., $H - \Delta H < z < H + \Delta H$, $\mathcal{D}_1$ intersects $\mathcal{D}_0$;
- $e = 1 + d$, i.e., $z = H + \Delta H$, $\mathcal{D}_0$ is inscribed in $\mathcal{D}_1$.

Next, we study the AA coverage for the case where $0 < \gamma_m < \pi/2$. We start by considering the coverage in $\Pi_z$ with $0 < z < H - \Delta H$. Recall from the very beginning of this subsection that the region of the AA coverage in $\Pi_z$ may be geometrically understood by identifying the path traced by the endpoints of the positive R-arcs as $\lambda_u$ is increased from $\lambda_s$ to $\lambda_e$. Note that for $R-arc(\Omega^+_{s,z})$, the first point, $A^\lambda_u \lambda_{s,z}$, lies on the R-line that connects $A^\lambda_u$ to $A^\lambda_{-s}$, whereas the last point, $A^\lambda_{s,u} \lambda_{e,z}$, lies on the R-line that connects $A^\lambda_{-s}$ to $A^\lambda_{u,e}$. Therefore, the two paths traced by the endpoints of the positive R-arcs are $R-arc(\Omega^+_{s,z})$ and $R-arc(\Omega^+_{e,z})$; see the left of Figure 6.9.

As illustrated in the right of Figure 6.9, the AA R-line coverage in $\Pi_z$ for $0 < z < H - \Delta H$ is fully characterized by 6 curves: $R-arc(\Omega^+_{s,z})$, $R-arc(\Omega^+_{e,z})$, $R-arc(\Omega^+_{u,z})$, $R-arc(\Omega^+_{e,z})$, $\mathcal{D}_0$ and $\mathcal{D}_1$. This coverage includes two small regions denoted as $\Omega_1$ and $\Omega_2$, we describe Region $\Omega_1$ as an example. Region $\Omega_1$ is delimited by $R-arc(\Omega^+_{s,z})$, $R-arc(\Omega^+_{u,z})$ and $\mathcal{D}_1$. The set of points within $\Omega_1$ is covered by the positive R-arcs corresponding to $\lambda_u \in [\lambda_s, \lambda_e]$, where $\lambda_e$ is the polar angle such that $R-arc(\Omega^+_{s,z})$ is tangent to $\mathcal{D}_1$ at its last point, $A^\lambda_{s,u} \lambda_{e,z}$.

Now we analyze the AA coverage for $0 < \gamma_m < \pi/2$ in other planes. In $\Pi_0$, recall that $\mathcal{D}_1$ degenerates to point $I_1$, whereas $R-arc(\Omega^+_{s,z})$ and $R-arc(\Omega^+_{u,z})$ are identical to $R-arc(\Omega^+_{s,z})$ and $R-arc(\Omega^+_{u,z})$, respectively. Therefore, the AA coverage in $\Pi_0$ is bounded by three curves: $R-arc(\Omega^+_{s,z})$, $R-arc(\Omega^+_{u,z})$ and $\mathcal{D}_0$. In $\Pi_z$ with $H - \Delta H < z < H + \Delta H$, the region of the AA coverage can be understood in the same way as it was interpreted in $\Pi_z$ with $0 < z < H - \Delta H$, except that some portions of the six curves will disappear due to truncation from the plane where the upper T-arc lies.

6.3.2 Arc-Line R-line coverage

In this subsection, we examine the R-line coverage that results from connecting a point on the T-line to a point on the T-arcs. We study this arc-line (AL) coverage in two steps.
6.3. R-line Coverage

Figure 6.9: Analysis of the AA R-line coverage in $\Pi_z$ for $z \in (0, H - \Delta H)$. The first point of each positive R-arc lies on R-arc $O_{\lambda_s,z}^\lambda$, and the last point of each positive R-arc lies on R-arc $O_{\lambda_e,z}^\lambda$. Left: the special R-arcs in $\Pi_z$, with $z = 0.2H$, $\Delta H = 0.2H$, $[\lambda_s, \lambda_e] = [-\pi/6, 7\pi/6]$ and $\lambda_e = \pi/2$. Right: illustration for $\Omega_1$ and $\Omega_2$.

First, we analyze the coverage from the T-line and the lower T-arc. Next, we investigate the coverage from the T-line and the upper T-arc. The union of these covered regions completes the AL coverage.

We start with some notations. We denote $\Lambda_+$ as the space that is bounded by the positive partial cone surface $S_{\lambda_s}^\lambda$, the plane where the lower T-arc lies, and plane $\Pi_{\text{cut}}$ that contains the T-line and $A_{\lambda_s,0}^\lambda$. Also, let $\Lambda_-$ be the space that is bounded by $S_{\lambda_e}^\lambda$, the plane where the upper T-arc lies, and $\Pi_{\text{cut}}$. According to Lemma 1, each point of $\Lambda_+$ and $\Lambda_-$ belongs to an R-line.

The AL coverage from the T-line and the lower T-arc in $\Pi_z$ is the intersection between $\Lambda_+$ and $\Pi_z$, which is a partial elliptical disk (Figure 6.10). By the definition of $\Lambda_+$, this partial elliptical disk is bounded by R-arc $O_{\lambda_s}^\lambda$, and a cutting line, which is the intersection between $\Pi_z$ and $\Pi_{\text{cut}}$. These two geometries are analyzed as below. Note that these geometries are in horizontal planes, hence only 2D notations will be considered for the mathematical description.

First, consider R-arc $O_{\lambda_s}^\lambda$. Let $a$ and $b$ be its semi-major and -minor axes, respectively, and denote $(x_0, y_0)$ as the center of R-arc $O_{\lambda_s}^\lambda$. Also, we refer to $\phi$ as the polar angle from the $x$-axis to the major axis of R-arc $O_{\lambda_s}^\lambda$. Recall from Theorem 3 in Section 6.2.2 that $x_0$ and $y_0$ can be obtained from the $(x, y)$ coordinates of $A_{\lambda_s,0}^\lambda$ and $A_{\lambda_e,\pi}^\lambda$ (Figure 6.5), which can be calculated according to Equation 6.4 whereas $\phi = \lambda_s/2$. Regarding $a$ and $b$, they can be obtained using the method that is offered at the end of Section 6.2.2 and we provide a method for quantitative calculation in Section A.8.
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Figure 6.10: The AL R-line coverage of in Π is a union of the coverage from the T-line and the lower T-arc and that from the T-line and the upper T-arc. The former coverage is bounded by R-arc \( O^{λ_{+}; z} \) and the cutting line, whereas the latter coverage is delimited by R-arc \( O^{λ_{-}; z} \) and the cutting line. Left: 3D illustration. Right: 2D illustration.

Next, we consider the cutting line. This line is the intersection between Π and Π_{cut}, and has a normal that is parallel to the y-axis. We define this normal as \( n = (0, 1) \). Let \( r_m = R \sin \gamma_m \), then the signed distance from (0, 0) to this cutting line along \( n \) is \( -r_m \).

Let \((x, y)\) be a point within the partial elliptical disk. According to the above analysis, it satisfies the relations below

\[
\begin{align*}
\frac{(x-x_0) \cos \phi + (y-y_0) \sin \phi}{a^2} + \frac{(x-x_0) \sin \phi - (y-y_0) \cos \phi}{b^2} & \leq 1 \\
y & \geq -r_m
\end{align*}
\] (6.11)

The AL coverage from the T-line and the upper T-arc in Π can be understood as the intersection between Π and Λ_{-}. Note that Λ_{+} and Λ_{-} are mirror-symmetric relative to the \((x,y)\)-plane, hence the AL coverage from the T-line and the upper T-arc in Π is identical to that from the T-line and the lower T-arc in Π_{-}, which can be obtained according to Relation 6.11. Therefore, for \( z > 0 \), the AL coverage from the T-line and the lower T-arc is always included within that from the T-line and the upper T-arc, whereas for \( z = 0 \), they are identical. According to the properties of an R-arc, these two partial elliptical disks are not only tangent to each other at \( A^{λ_{+}; z} \), but also are transformable to each other by a linear translation and isotropical scaling. Also note that, similar to the AA coverage, when \( z \in [H - \Delta H, H + \Delta H] \), the AL coverage will be truncated by the plane where the upper T-arc lies.

According to the above analysis of the AL R-line coverage, it is now easy to perform computer simulation using Relation 6.11. The results for \( R = 10 \) cm, \( H = 10 \) cm, and \( \Delta H = 3 \) cm are shown in Figure 6.11. For illustration, we have chosen three slices, i.e., Π₀, Π_{H/2} and Π_H, as well as two angular ranges, i.e., \([-\pi/6, \pi/6]\) and \([-\pi/2, 3\pi/2]\). In Figure 6.11 the white regions are covered by 2 R-lines, whereas the gray regions are
6.3. R-line Coverage

Figure 6.11: Computer simulation of the AL R-line coverage. The white regions are covered by 2 R-lines, whereas the gray regions are covered by 1 R-line. Parameters used for the experiments were as follows: i) $R = 10\, \text{cm}$, ii) $H = 10\, \text{cm}$, and iii) $\Delta H = 3\, \text{cm}$. Note that in the last column, $\Pi_H$ intersects with the upper T-arc, and hence only the portion of $\Pi_H$ that is below the upper T-arc can have R-line coverage.

covered by 1 R-line. As observed, the AL coverage from the T-line and the lower T-arc (white regions) is always included in that from the T-line and the upper T-arc (union of gray and white regions).

6.3.3 Combined R-line Coverage

Using the results from the previous two subsections, we can obtain the ELE R-line coverage. The ELE trajectory can be decomposed into three elemental trajectories: i) the elemental AA trajectory consisting of the two T-arcs, ii) the elemental AL trajectory consisting of the T-line and the lower T-arc, iii) the elemental AL trajectory consisting of the T-line and the upper T-arc. The union of the three sets of R-lines corresponding to these three elemental trajectories is the complete set of R-lines for the ELE trajectory. Therefore, the ELE R-line coverage can be computed by combining the procedures described in Section 6.3.1 for the AA coverage as well as Relation 6.11 for the AL coverage.
Complete R-line coverage of the ELE trajectory can be now simulated and the simulation results for $R = 10$, $H = 10$, and $\Delta H = 3$ are shown in Figure 6.12. The number in each region indicates the number of R-lines. The range of the number of R-lines is 1 to 4. According to the procedures explained in the previous two sections, we can figure out the source of the R-lines that form each region. For example, in Figure 6.12(a), the gray region is covered by 2 R-lines that are from the AA elemental trajectory, whereas the white region is covered by 4 R-lines, with 2 from the AA elemental trajectory, and the other 2 from the AL elemental trajectories. Also, observe that the AA and AL R-line coverage complement each other very well due to the shared boundary of R-arc $O^{\lambda_m,z}$; see Figures 6.9 and 6.10.

Figure 6.12: Computer simulation of the ELE R-line coverage. The number in each region indicates the number of covering R-lines. Parameters used for the experiments were as follows: i) $R = 10$ cm, ii) $H = 10$ cm, and iii) $\Delta H = 3$ cm. Note that in the last column, $\Pi_H$ intersects with the upper T-arc, and hence only the portion of $\Pi_H$ that is below the upper T-arc can have R-line coverage.

### 6.4 R-line Coverage in the ROI

In this section, we study the R-line coverage within the ROI, which is important in practice. More precisely, we are interested in the portion of the ROI that is within the convex hull of
the ELE trajectory. We denote this portion of the ROI as $\Omega_\Lambda$. By definition, $\Omega_\Lambda$ is bounded by three surfaces, i.e., i) a cylindrical surface that is centered on the $z$-axis with radius $r \leq r_m (r_m = R \sin \gamma_m)$, ii) the plane where the upper T-arc lies and iii) the plane where the lower T-arc lies.

In the rest of this section, we will first prove that the R-line coverage in $\Omega_\Lambda$ is worst in $\Pi_0$. Hence, we will then perform a thorough analysis of the R-line coverage in $\Pi_0$. With this knowledge, for a given ELE trajectory, we will be able to find the maximum radius, $r$, such that, the whole $\Omega_\Lambda$ is fully covered by R-lines.

### 6.4.1 Central Plane $\Pi_0$

We found that if the ROI in the central plane $\Pi_0$ is fully covered by R-lines, then the whole ROI is fully covered by R-lines. This observation can be clarified using Theorem 4 and Corollary 4 below. For the statement of this theorem, we define $\mathcal{L}(x_0, y_0)$ to be the line parallel to the $z$-axis that goes through the point $(x_0, y_0, 0)$. We define

$$
\mathcal{G}(x_0, y_0) = \mathcal{H}\left(\cos(x_0/R)\right),
$$

which is the $z$-coordinate of the intersection between $\mathcal{L}(x_0, y_0)$ and the plane where the upper T-arc lies. Here, $\cos$ is the inverse cos function that returns a polar angle in the range $[0, \pi]$, whereas $\mathcal{H}(\lambda)$ is the function defined in Equation 6.12. Hence, a point on $\mathcal{L}(x_0, y_0)$, with $x_0^2 + y_0^2 \leq R^2$, has a $z$-coordinate with a range of $[-\mathcal{G}(x_0, y_0), \mathcal{G}(x_0, y_0)]$.

**Theorem 4.** For the ELE trajectory, if $(x_0, y_0)$ satisfies $x_0^2 + y_0^2 \leq r_m^2$, then there exists a coordinate $\hat{z}$ with $|\hat{z}| \leq \mathcal{G}(x_0, y_0)$ such that the points on the line $\mathcal{L}(x_0, y_0)$ that are covered by R-lines have $z$ coordinates in the set $[-\mathcal{G}(x_0, y_0), -\hat{z}] \cup [\hat{z}, \mathcal{G}(x_0, y_0)]$.

**Corollary 4.** Suppose that $(x_0, y_0)$ satisfies $x_0^2 + y_0^2 \leq r_m^2$. For the ELE trajectory, if a point $(x_0, y_0, z_0)$ lies on an R-line, then all points $(x_0, y_0, z)$ with $z \in [-\mathcal{G}(x_0, y_0), -|z_0|] \cup [|z_0|, \mathcal{G}(x_0, y_0)]$ also lie on R-lines.

Corollary 4 indicates that it is sufficient to examine the ELE R-line coverage in the central plane $\Pi_0$. If every point of ROI in $\Pi_0$ is covered by R-lines, then full R-line coverage inside the whole ROI is guaranteed.

Note that Theorem 4 and Corollary 4 are similar to Theorem 2 and Corollary 2 that were introduced in Chapter 5 for the ALA trajectory. The only difference is that $2\mathcal{G}(x_0, y_0)$ replaces $2H$ as the length of $\mathcal{L}(x_0, y_0)$ that is bounded by the upper and lower T-arc planes. Therefore, Theorem 4 can be proven by Section A.3 with $H$ replaced by $\mathcal{G}(x_0, y_0)$ therein.

### 6.4.2 R-line Coverage in $\Pi_0$

Recall from Section 6.3 that we have qualitatively shown the computer simulation results of the ELE coverage in $\Pi_0$; see the first columns in Figures 6.8, 6.11, and 6.12. In this subsection, we will provide a quantitative analysis of the ELE R-line coverage in $\Pi_0$. 
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We first analyze the AA R-line coverage. We start with the two intersections between R-arc\( (O^λ_s,0) \) and R-arc\( (O^λ_e,0) \) (Figure 6.13). The first intersection is \( I_1 \). We have observed from the top left in Figure 6.6 that, for \( γ_m = \pi/2 \), all the R-arcs in \( Π_0 \) become ellipses and they intersect at \( I_1 \). We obtain the \((x,y)\) coordinates of \( I_1 \) using a degenerated version of Equation 6.6. In \( Π_0 \), we have \( z = e = 0 \), and Equation 6.6 becomes

\[
\begin{align*}
\frac{x}{R} &= \frac{\cos α \cos β + d (\cos^2 α - \sin^2 β)}{1 + d \cos α \cos β} \\
\frac{y}{R} &= \frac{\sin α \cos β + d \sin α \cos α}{1 + d \cos α \cos β}.
\end{align*}
\] (6.13)

Note that when \( \cos β + d \cos α = 0 \), we have \((x,y) = (-Rd,0)\), which is constant if \( d = ΔH/H \) is fixed. These are the \((x,y)\) coordinates of \( I_1 \). For the second intersection, we consider \( A_0^λ_s,λ_e \) and \( A_0^λ_e,λ_s \), which are the last point of R-arc\( (O^λ_s,0) \) and the first point of R-arc\( (O^λ_e,0) \), respectively. Because the ELE trajectory is mirror-symmetric relative to \( Π_0 \), \( A_0^λ_s,λ_e \) and \( A_0^λ_e,λ_s \) are identical, which indicates that they are the other intersection between R-arc\( (O^λ_s,0) \) and R-arc\( (O^λ_e,0) \). We denote the second intersection as \( I_2 \). By construction, \( I_2 \) is on the cutting line.

![Figure 6.13: Quantitative analysis of the ELE R-line coverage in the central plane Π₀. Left: AA R-line coverage is composed of Region I and Region II. Middle: AL R-lines cover Region III. Right: Region IV does not have R-line coverage.](image)

Next, we quantitatively define the AA R-line coverage. We denote Ell\( (O^λ_s,0) \) and Ell\( (O^λ_e,0) \) as the ellipses on which R-arc\( (O^λ_s,0) \) and R-arc\( (O^λ_e,0) \) lie, respectively. We define Region I as the area that is within both Ell\( (O^λ_s,0) \) and Ell\( (O^λ_e,0) \), and define Region II as the area that is i) within \( D₀ \), ii) above the cutting line, and iii) outside both Ell\( (O^λ_s,0) \) and Ell\( (O^λ_e,0) \); see the left of Figure 6.13. Note that Region I is delimited by \( I_1 \) and \( I_2 \). By construction, Region I is entirely within the convex hull of the ELE trajectory. Now, the AA R-line coverage can be precisely defined by the following Theorem, which is proven in Section A.9.
6.4. R-line Coverage in the ROI

**Theorem 5.** The AA R-line coverage in $\Pi_0$ is composed of Region I and Region II.

Regarding the AL R-line coverage in $\Pi_0$ (left column in Figure 6.11), Section 6.3.2 already gives a quantitative description. We restate the AL coverage as follows. The AL coverage is Region III, which is i) above the cutting line and ii) inside $\text{Ell}(O_+^{\lambda_s,0})$; see the middle of Figure 6.13. Note that Region I is included in Region III. Therefore, the R-line coverage of the ELE trajectory in $\Pi_0$ is the union of Region II and Region III.

We now study the area in $\Pi_0$ that belongs to the convex hull but is not covered by R-lines. We call this area Region IV, which is inside $\text{Ell}(O_+^{\lambda_e,0})$, outside $\text{Ell}(O_+^{\lambda_s,0})$, and above the cutting line, as shown in the right of Figure 6.13. Note that Region IV always exists for $0 < \gamma_m < \pi/2$, and it always touches the point $I_1$. This observation indicates that as long as $r > d R$, the ROI will intersect with Region IV; see the small dashed circle in the right of Figure 6.13. Hence, for $\sin \gamma_m > d$, the maximum possible radius for the ROI of full R-line coverage is $d R$, and this condition does not change when $\gamma_m$ increases. For this reason, we set

$$\gamma_m = \text{asin}(d),$$

where asin is the inverse sin function that returns a polar angle in the range $[-\pi/2, \pi/2]$. Note that, by this configuration, we also have

$$r_m = R \sin \gamma_m = d R.$$  \hspace{1cm} (6.15)

### 6.4.3 Maximum ROI

In this subsection, for given $R$, $H$ and $\Delta H$, we define the fan-angle, $\gamma_m$, by Equation 6.14. By doing so, we have $r_m = d R$ (Equation 6.15). According to Theorem 4 and Corollary 4, it is sufficient to focus on the R-line coverage of the ROI in $\Pi_0$. As explained at the end of the last subsection, with the above configuration, the upper bound of the radius of the ROI with full R-line coverage is $r_m$. Therefore, in the rest of this subsection, we will first study the R-line coverage in the ROI that has radius of $r_m$. With this knowledge, we will be able to find the maximum ROI with full R-line coverage for a given ELE trajectory.

The ROI in $\Pi_0$ is a circular disk, and we divide this disk into 4 quadrants. We will study the R-line coverage for each quadrant, respectively. First, we study the R-line coverage in the first, second and fourth quadrants; see the left of Figure 6.14. By construction, the boundary of the ROI goes through $I_1$, whose $(x,y)$ coordinates are $(-r_m,0)$, and is tangent to the cutting line at $I_3$ with $(x,y)$ coordinates $(0,-r_m)$. Note that $I_2$ is the intersection between $\Pi_0$ and the line that connects $A_+^{\lambda_e}$ and $A_+^{\lambda_s}$ (or $A_+^{\lambda_e}$ and $A_+^{\lambda_s}$). By setting $e = 0$, $\lambda_u = \pi + \gamma_m$ and $\lambda_l = -\gamma_m$ in Equation 6.4 and using relations presented in Equations 6.14 and 6.15 we obtain the $(x,y)$ coordinates of $I_2$, i.e., $(-\cos^2 \gamma_m r_m, -r_m)$. Let $L$ be the line that connects $I_1$ and $I_2$. According to the $(x,y)$ coordinates of $I_1$ and $I_2$, $L$ has a negative slope, which indicates that the first and second quadrants of the ROI are on the right side of $L$. On the other side, $L$ is a cord line of $\text{Ell}(O_+^{\lambda_s,0})$, and the solid portion of this ellipse is entirely on the left side of $L$ (see the left of Figure 6.14). Therefore, the first and the
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Figure 6.14: Analysis of the maximum radius of the ROI for a given \( d = \Delta H / H \), with \( \gamma_m = \arcsin(d) \). Left: R-line coverage in the first, second, and fourth quadrants of the ROI in \( \Pi_0 \). Middle: R-line coverage of the 3rd quadrant of the ROI in \( \Pi_0 \). Right: analytical simulation of the R-line coverage in \( \Pi_0 \) using \( d = 0.8588 \); the gray region is covered by two R-lines, whereas the white region is covered by four R-lines; the black circle indicates the ROI, which is tangent to R-arc \((O_{+}^{L}, 0)\).

Secondly, we try to find out whether the third quadrant is entirely within Region III; see the middle of Figure 6.14. Consider the portion of R-arc \((O_{+}^{L}, 0)\) that is delimited by \( I_1 \) and \( I_2 \) (the solid curve), let \((x_3, y_3)\) be a point on this portion of R-arc \((O_{+}^{L}, 0)\). In Section A.10 we were able to prove that, for \( 0 < d < 0.8588 \), we have \( x_3^2 + y_3^2 \geq r_m^2 \). Therefore, as long as \( d = \Delta H / H \leq 0.8588 \), the third quadrant of the ROI with radius \( r_m \) is fully covered by R-lines.

According to the previous two paragraphs as well as Theorem 4 and Corollary 4, we can conclude that, for a given ELE trajectory with \( d = \Delta H / H \in (0, 0.8588] \), we should set \( \gamma_m = \arcsin(d) \), and the maximum radius of the ROI that has full R-line coverage is \( r = r_m = R \sin \gamma_m \). The analytical simulation of the ELE R-line coverage in \( \Pi_0 \) with \( d = 0.8588 \) is shown in the right of Figure 6.14. As illustrated, the ROI of radius \( r = r_m \) is tangent to R-arc \((O_{+}^{L}, 0)\) in the third quadrant. Note that, for impractical cases of \( 0.8588 < d < 1 \), a larger \( \gamma_m \) than \( \arcsin(d) \) is required to obtain full R-line coverage in the ROI of radius \( r = R d \).

On the other hand, for a given ROI with radius \( r < 0.8588 R \), full R-line coverage in the ROI can be achieved from a ELE trajectory using the following two configurations:

\[ i) \quad \gamma_m = \arcsin(r/R) \quad \text{and} \quad ii) \quad \Delta H = r H / R. \]
6.5 Conclusion and Discussion

We have presented a new data acquisition geometry called the ELE trajectory. This new source trajectory possesses a reverse pattern in the direction of rotation and can be periodically repeated along the patient table. Hence it is suitable for extended-volume C-arm CT. Moreover, the ELE trajectory allows a smooth and continuous scan so that interruption of X-ray source exposure can be avoided. Also, we were able to demonstrate that, using a short scan and a simple design of the T-arc height ($\Delta H/H$), this new source trajectory can always guarantee full R-line coverage for a typical ROI, which opens the door for applications of the R-line based reconstruction algorithms to extended-volume C-arm CT.

The ELE R-line coverage is composed of the AA and AL coverage, and these regions of coverage largely complement each other. Through simulations for both regions of coverage, we offered a method to identify the R-lines that go through a given point of interest. We observed that, when $\Pi_z$ is closer to the central plane, the AA coverage in the ROI increases, whereas the AL coverage in the ROI decreases. However, the union of the AA and AL coverage, i.e., the ELE R-line coverage, reaches a minimum in the central plane, as indicated by Corollary 4. This is because, in the ROI, the AL coverage has more impact than the AA coverage. As explained, the central slice is of particular importance, because when the ROI in this plane is covered by R-lines, the whole ROI is covered by R-lines.

A nice property is that a short scan suffices to achieve full R-line coverage for a typical ROI ($r \leq 0.8588R$). A larger angular length is not necessary, because it does not change the size of the ROI with full R-line coverage, and the only benefit will be that the number of R-lines from the AA trajectory will increase. For a large ROI ($0.8588R < r < R$), an angular length larger than the short scan is required to ensure the ROI is fully covered by R-lines. However, this big ROI is not practical for a C-arm system. In the extreme case, when $\Delta H = 0$, the ELE trajectory becomes the ALA trajectory, and a full scan is always needed, regardless of the size of the ROI.

Compared with the AELA trajectory, the ELE trajectory not only allows a smooth and continuous scan, but also employs a less complex design strategy, because the height of the T-arc ($\Delta H/H$) is solely dependent on the radius of the ROI, whereas for the AELA, the design of the T-line extension depends on both the radius of the ROI and the scan angular length.

In the ROI, different areas are covered by different numbers of R-lines, some are from the AA elementary trajectory and the others are from the AL elementary trajectories. Although we have offered a method to identify these R-lines for a given point of interest, efficient usage of the R-lines for image reconstruction is task and algorithm dependent. We will present an R-line selection scheme for the DBP method in Chapter 8.
Chapter 7

Practical Image Reconstruction Methods for the Reverse Helical Trajectory

The most practical solution is a good theory.

ALBERT EINSTEIN

The reverse helix is compatible with motion restrictions of the C-arm system and allows fast movement along the patient table. Therefore, the reverse helical trajectory is attractive for extended-volume C-arm CT. However, as shown in Chapter 4, the reverse helical trajectory does not have sufficient R-line coverage in the ROI. Hence, it is challenging to apply the efficient R-line based TES reconstruction algorithms to this trajectory.

In this chapter, we propose three approximate but practical reconstruction algorithms for the reverse helical trajectory. These algorithms are of FDK-type, and thus are efficient and stable. In the rest of this chapter, we will first briefly review the geometry of the reverse helix as a reminder, then we will provide details of the the three reconstruction methods. We evaluate these reconstruction methods by computer simulations of the FORBILD head phantom and the XCAT phantom, and assess the image quality using image bias and image noise, both by visual inspections and quantitative metrics. Here, image bias is considered as the combination of CB artifacts and discretization errors, whereas image noise is caused by addition of Poisson noise in projection data.

### 7.1 The Reverse Helix

As a reminder, we briefly review the reverse helix in this section. For a detailed description, we refer to Chapter 4. The reverse helix is composed of the upper and lower helices; see Figure 4.2. Each vertex point on this trajectory can be identified by a variable \( \lambda \), whose absolute value can be interpreted as the polar angle of the vertex point. The range of \( \lambda \) for
the upper and lower T-arcs is \([0, \lambda_m]\) and \([-\lambda_m, 0]\), respectively. The intersection between the upper and lower T-arcs is called the kink point, which is located in the \((x,z)\)-plane. We denote the \(z\)-coordinate of the kink point as \(z_k\), and call the horizontal plane that goes through the kink point the kink plane. By horizontal, we mean the plane is parallel to the \((x,y)\)-plane. The length of the reverse helix along the \(z\)-direction is \(2H\). We denote the vertex point at \(\lambda\) on the reverse helix as \(a(\lambda)\); the mathematical expression of \(a(\lambda)\) is given in Equation 4.1.

The CB scan geometry of the reverse helix is as introduced in Chapter 3; see Figure 3.1. In particular, we define the coordinate system that is attached to the detector as

\[
\begin{align*}
    \mathbf{e}_u(\lambda) &= (\cos \lambda, \sin |\lambda|, 0) \\
    \mathbf{e}_n(\lambda) &= (-\sin |\lambda|, \cos \lambda, 0) \quad \text{(7.1)} \\
    \mathbf{e}_v(\lambda) &= (0, 0, 1)
\end{align*}
\]

By definition, this coordinate system is solely reliant on the polar angle \(|\lambda|\).

Now, we restate the spatial distribution of the \(\pi\)-line coverage for the reverse helix. Recall from Chapter 3 that \(\Omega_\Lambda\) is the portion of the FOV, \(\Omega\), that is within the convex hull of the reverse helix. According to Section 4.2, \(\Omega_\Lambda\) is composed of three regions called \(\Omega_\pi^u\), \(\Omega_\pi^l\) and \(\Omega_\pi^c\). The first two regions are covered by \(\pi\)-lines, whereas the last region is not; see Figure 4.3. This knowledge of the \(\pi\)-line coverage is important for one of the three reconstruction methods.

### 7.2 Reconstruction Methods

In this section, we describe all three reconstruction methods: Fusion-RFDK (FRFDK), Fusion-HFDK (FHFDK) and Voxel-Dependent-HFDK (VDHFDK). FRFDK is an extension of the conventional Feldkamp-Davis-Kress (FDK) algorithm \cite{Feld84, Wang93} that uses the ramp filter, whereas FHFDK and VDHFDK are extensions of another type of FDK algorithm \cite{Kudo04} that employs view-dependent differentiation and the Hilbert transform. For convenience, we call the ramp-filter-based FDK algorithms RFDK, and the Hilbert transform based algorithms HFDK. For all three reconstruction methods, the ramp filtering and the Hilbert transform are both horizontal (along the \(u\)-axis).

The motivation for FHFDK over FRFDK comes from studies related to the conventional helical trajectory \cite{Turb99, Sour03}, which have shown that oblique filtering along the direction of the tangent to the helix can significantly reduce CB artifacts; the view-dependent differentiation step in FHFDK induces such an oblique component within the filtering operation. Additionally, in the backprojection step, FRFDK uses the second order of the distance weighting, whereas FHFDK uses the first order of the distance weighting. The latter is more effective in noise suppression. To understand how the third method was conceived, note that TES or almost-TES reconstruction can be easily achieved within the regions covered by \(\pi\)-lines. VDHFDK makes proficient use of this feature so that it almost produces TES reconstructions within the \(\pi\)-line regions.
In the rest of this section, we explain the technical details and the mathematical expressions for each method. Without loss of generality, we assume that the kink plane is the \((x,y)\)-plane, i.e., \(z_k = 0\). For FRFDK and FHFDK, the target volume to be reconstructed is the portion of the FOV that is bounded by the planes \(\Pi_{-H/2}\) and \(\Pi_{H/2}\), where \(\Pi_c\) is the horizontal plane that goes through \((0,0,z)\). We denote this region \(\Omega_R\). For VDHFDK, the target volume is \(\Omega_A\), which is the portion of the FOV that is bounded by the top and bottom \(\pi\)-line surfaces \(S_1^+\) and \(S_1^-\) (Figure 4.3). These targeted volumes were defined so that they can be seamlessly extended when the reverse helix is duplicated along the \(z\)-axis.

### 7.2.1 Fusion-RFDK

Let \(z' = z - z_k\), and let \(\Omega^+\) and \(\Omega^-\) be the portion of \(\Omega\) corresponding to \(z' \in [-H_F/2,H/2]\), and \(z' \in [-H/2,H/2]\), respectively, where \(0 \leq H_F \leq H\) is a design parameter. Moreover, let \(\Omega_F = \Omega^+ \cap \Omega^-\), i.e., \(\Omega_F\) is the portion of \(\Omega\) corresponding to \(z' \in [-H_F/2,H_F/2]\). Note that \(\Omega_R = \Omega^+ \cup \Omega^-\) is the target volume to be reconstructed.

![Figure 7.1: Illustration for Fusion-RFDK. Two volumes, denoted as \(\Omega^+\) and \(\Omega^-\), are reconstructed by the ramp-kernel-based FDK method using projection data from the upper and lower helices, respectively. These two volumes are then combined by a fusion process using a pair of weighting functions, \(\omega^+_F(z)\) and \(\omega^-_F(z)\). This fusion process is applied to the region common to \(\Omega^+\) and \(\Omega^-\), which corresponds to \(z \in [z_k - H_F/2,z_k + H_F/2]\).](image)
The FRFDK method consists of two stages; see Figure 7.1. First, two reconstructions, i.e., \( f^+_R(\chi) \) for \( \chi \in \Omega^+ \) and \( f^-_R(\chi) \) for \( \chi \in \Omega^- \), are obtained using the principles of RFDK independently from the upper and lower helices, respectively. Second, the final reconstruction result is obtained by a fusion process using a pair of weighting functions. This reconstruction result is denoted as \( f_{FR}(\chi) \), and is defined for \( \chi \in \Omega_R \) by the equation

\[
f_{FR}(\chi) = f^+_R(\chi) \omega^+_F(z) + f^-_R(\chi) \omega^-_F(z),
\]

where \( \omega^+_F(z) \) and \( \omega^-_F(z) \) are weighting functions defined as

\[
\omega^+_F(z) = \begin{cases} 
1 & z' \in [H_F/2, H/2] \\
\sin^2 \left( \frac{\pi(z' + H_F/2)}{2H_F} \right) & z' \in [-H_F/2, H_F/2] \\
0 & \text{otherwise}
\end{cases},
\]

\[
\omega^-_F(z) = \begin{cases} 
1 & z' \in [-H/2, -H_F/2] \\
\cos^2 \left( \frac{\pi(z' + H_F/2)}{2H_F} \right) & z' \in (-H_F/2, H_F/2] \\
0 & \text{otherwise}
\end{cases}.
\] (7.2)

Observe that the effective region for this fusion process is \( \Omega_F \). Hence, \( \Omega_F \) is called the fusion zone, and \( H_F \) is called the fusion length.

In summary, \( f_{FR}(\chi) \) for \( \chi \in \Omega_R \) can be obtained by FRFDK using the following steps:

- **Step 1:** CB length correction and Parker-like weighting:

\[
g_1(\lambda, u, v) = \frac{D}{\sqrt{u^2 + v^2 + D^2}} \omega_p(\lambda, u; \lambda_m) g(\lambda, u, v),
\]

where \( \omega_p(\lambda, u; \lambda_m) \) is the Parker-like weighting function defined as

\[
\omega_p(\lambda, u; \lambda_m) = \begin{cases} 
\omega^+_p(\lambda, u; \lambda_m) & \lambda \in [0, \lambda_m] \\
\omega^-_p(\lambda, u; \lambda_m) & \lambda \in [-\lambda_m, 0]
\end{cases},
\]

with \( \omega^+_p(\lambda, u; \lambda_m) = \omega^-_p(-\lambda, u; \lambda_m) \). Function \( \omega^+_p(\lambda, u; \lambda_m) \) is the same as that used in [Noo 02, Equation 29], and can be expressed as

\[
\omega^+_p(\lambda, u; \lambda_m) = \sum_{k=-\infty}^{\infty} \frac{c(\lambda)}{c(\lambda + k2\pi) + c(\lambda + \pi - 2 \arctan(u/D) + k2\pi)}
\]

where

\[
c(\lambda) = \begin{cases} 
\sin^2(\pi\lambda/\lambda_m) & \lambda \in [0, \lambda_m] \\
0 & \text{otherwise}
\end{cases}.
\]
7.2. Reconstruction Methods

- Step 2: ramp filtering along the $u$-axis:

\[ g_R(\lambda, u, v) = \int_{-\infty}^{\infty} du' h_R(u-u') g_1(\lambda, u', v), \]

where

\[ h_R(u) = \int_{-\infty}^{\infty} d\sigma |\sigma| \Theta(\sigma) e^{i2\pi\sigma u} \]

is the ramp filter apodized by a window, $\Theta(\sigma)$;

- Step 3: backprojection:

\[
\begin{align*}
 f_{R+}^R(x) &= \int_{\lambda_m}^{\lambda_m} d\lambda \frac{RD}{(R-x \cdot e_w(\lambda))^2} g_R(\lambda, u^*, v^*) \\
 f_{R-}^R(x) &= \int_{-\lambda_m}^{0} d\lambda \frac{RD}{(R-x \cdot e_w(\lambda))^2} g_R(\lambda, u^*, v^*),
\end{align*}
\]

where

\[
\begin{cases}
 u^* = \frac{D(x-a(\lambda)) \cdot e_u(\lambda)}{R-x \cdot e_w(\lambda)} \\
 v^* = \frac{D(x-a(\lambda)) \cdot e_v(\lambda)}{R-x \cdot e_w(\lambda)}
\end{cases}
\] \hspace{1cm} (7.3)

- Step 4: fusion:

\[ f_{FR}(x) = f_{R+}^R(x) \omega_{FR+}(z) + f_{R-}^R(x) \omega_{FR-}(z). \] \hspace{1cm} (7.4)

Note that FRFDK retains an important property from the RFDK algorithms \cite{Feld84, Wang93, Kudo04}, i.e., if $f(x)$ is independent of $z$, then FRFDK is exact. This beneficial property will be involved later, in Section 7.3.3, for resolution matching purposes.

7.2.2 Fusion-HFDK

FHF DK follows the same fusion process as FRFDK; the difference is that the reconstructions for $\Omega^+$ and $\Omega^-$, denoted as $f_{H+}^R(x)$ and $f_{H-}^R(x)$, are obtained using principles of HFDK instead of RFDK. Let the final reconstruction result for $\Omega_R$ from FHF DK be $f_{FH}(x)$. The implementation steps can be described as follows.

- Step 1: CB length correction and view-dependent differentiation

\[ g_2(\lambda, u, v) = \frac{D}{\sqrt{u^2 + v^2} + D^2} g'(\lambda, \alpha(\lambda, u, v)), \]

with $g'(\lambda, \alpha)$ as the partial derivative of $g(\lambda, \alpha)$ with respect to $\lambda$ for a fixed $\alpha$;
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- Step 2: Hilbert filtering along the u-axis
  \[ g_H(\lambda, u, v) = \int_{-\infty}^{\infty} du' h_H(u - u') g_2(\lambda, u', v), \]
  where \( h_H(u) = 1/(\pi u) \) is the kernel of the Hilbert transform;

- Step 3: Parker-like weighting
  \[ g_P(\lambda, u, v) = \omega_P(\lambda, u; \lambda_m) g_H(\lambda, u, v); \]

- Step 4: backprojection
  \[ f_{H}^+(x) = \frac{1}{2\pi} \int_0^{\lambda_m} d\lambda R x \cdot e_{\omega}(\lambda) g_P(\lambda, u^*, v^*), \]
  \[ f_{H}^-(x) = \frac{1}{2\pi} \int_{-\lambda_m}^{0} d\lambda R x \cdot e_{\omega}(\lambda) g_P(\lambda, u^*, v^*); \]

- Step 5: fusion
  \[ f_{FH}(x) = f_{H}^+(x) \omega_{F}^+(z) + f_{H}^-(x) \omega_{F}^-(z). \]  

Note that \( \omega_P(\lambda, u; \lambda_m), u^*, v^*, \omega_{F}^+(z) \) and \( \omega_{F}^-(z) \) used in FHFDK are the same as in FRFDK.

FHFDK and FRFDK share several properties. First, they require the same amount of projection data. In other words, the detector requirements for these two methods are the same; for further details, see Section 7.4. Second, FHFDK is also exact if \( f(x) \) is independent of \( z \). Finally, they have similar computational complexity; see Section 7.2.4.

7.2.3 Voxel-Dependent-HFDK

The third method is called VDHFDK, because the angular range of the backprojection and the weighting scheme are voxel dependent. Image reconstruction using this method can be achieved using the following steps.

- Step 1: CB length correction and view-dependent differentiation
  \[ g_2(\lambda, u, v) = \frac{D}{\sqrt{u^2 + v^2 + D^2}} g'_{\omega}(\lambda, \omega(\lambda, u, v)); \]

- Step 2: Hilbert transform along the u-axis
  \[ g_H(\lambda, u, v) = \int_{-\infty}^{\infty} du' h_H(u - u') g_2(\lambda, u', v); \]
7.2. Reconstruction Methods

• Step 3: voxel-dependent weighting
  \[ g_W(\lambda, u^*, v^*) = \omega_v(\lambda, x_0) g_H(\lambda, u^*, v^*) ; \]

• Step 4: voxel-dependent backprojection
  \[ f(\lambda) = \frac{1}{2\pi} \int_{\Phi(\lambda)} d\lambda \frac{1}{R - x_0 \cdot e^\omega(\lambda)} g_W(\lambda, u^*, v^*) . \]

Note that the above steps are very similar to the first four steps of FHFDK. The only difference is that VDHFDK employs a voxel-dependent weighting function called \( \omega_v(\lambda, x_0) \) in step 3 and a voxel-dependent backprojection angular range called \( \Phi(\lambda) \) in step 4. These two functions are designed according to the spatial distribution of \( \pi \)-lines in \( \Omega_\Lambda \); see Figure 4.3. In the following, we divide our description of \( \omega_v(\lambda, x_0) \) and \( \Phi(\lambda) \) into two cases:

(i) \( \pi \)-line regions: \( x_0 \in \{ \Omega_{\pi u} \cup \Omega_{\pi l} \} \)

As shown in [Defr00], there is one and only one \( \pi \)-line that goes through \( x_0 \). An example for \( x_0 \in \Omega_{\pi u} \) is provided in Figure 7.2. We call the portion of the reverse helix bounded by this \( \pi \)-line the \( \pi \)-line segment, and refer to its starting and ending angular positions as \( \lambda_{\pi s}(x_0) \) and \( \lambda_{\pi e}(x_0) \), respectively. Here, \( s \) stands for start and \( e \) stands for end.

Figure 7.2: Illustration of a \( \pi \)-line segment for the upper helix \( (x_0 \in \Omega_{\pi u}) \). There is one and only one \( \pi \)-line that goes through \( x_0 \). This \( \pi \)-line intersects the upper helix at \( a(\lambda_{\pi s}(x_0)) \) and \( a(\lambda_{\pi e}(x_0)) \). The portion of the upper helix that is delimited by this \( \pi \)-line is the \( \pi \)-line segment of \( x_0 \).

For reconstruction at \( x_0 \), we perform the backprojection within the \( \pi \)-line segment using a uniform weighting of 1. For \( x_0 \in \{ \Omega_{\pi u} \cup \Omega_{\pi l} \} \), let \( \Phi(\pi)(x_0) \) be the backprojection angular range, and let \( \omega(\pi)(x_0) \) be the corresponding weighting function. Then \( \Phi(\pi)(x_0) \) and \( \omega(\pi)(\lambda, x_0) \) can be expressed as

\[ \Phi(\pi)(x_0) = [\lambda_{\pi s}(x_0), \lambda_{\pi e}(x_0)] , \quad (7.6) \]
and

\[
\omega_\pi(\lambda, \mathbf{x}) = \begin{cases} 
1 & \lambda \in \Phi_\pi(\mathbf{x}) \\
0 & \text{otherwise.}
\end{cases} \tag{7.7}
\]

Note that the backprojection in step 4 can be efficiently implemented by using the Tam-Danielsson window [Dani 97] [Tam 98].

(ii) \(\mathbf{x} \in \Omega_\pi^c\)

Here we focus on the volume \(\{\mathbf{x} = (x, y, z) : \mathbf{x} \in \Omega_\pi^c \text{ and } z \geq z_k\}\); similar results can be obtained for the case when \(\mathbf{x}\) is below the kink plane. To start, we first introduce some notations. Recall from Section 5.2.3 that \(\mathcal{L}(x, y)\) is the vertical line through \(\mathbf{x}\); see Figure 7.3. This vertical line together with the kink point \(K\) defines a vertical plane that intersects the upper helix at rotational angle \(\lambda\pi(\mathbf{x})\). Also, we refer to the \(z\) coordinate of the intersection between \(\mathcal{L}(x, y)\) and the \(\pi\)-surface, \(S_0^+\), as \(z_0^+(x, y)\); see Figure 4.3.

Figure 7.3: \(\mathcal{L}(x, y)\) is parallel to the \(z\)-axis and goes through \((x, y, 0)\). This line together with the kink point \(K\) define a plane \(\Pi\) that intersects the upper helix at \(a(\lambda\pi(\mathbf{x}))\). The \(z\) coordinate of the intersection between \(\mathcal{L}(x, y)\) and the line connecting \(K\) and \(a(\lambda\pi(\mathbf{x}))\) is denoted as \(z_0^+(x, y)\).

For \(\mathbf{x}\) with \(z \in [z_k, z_0^+(x, y)]\), we define the backprojection range, denoted as \(\Phi_c(\mathbf{x})\), as below

\[
\Phi_c(\mathbf{x}) = [-\lambda^c(\mathbf{x}), \lambda^c(\mathbf{x})], \tag{7.8}
\]
where

\[
\begin{aligned}
\lambda^x_c(x) &= \frac{z_0^+(x,y) - z}{z_0^+(x,y) - z_k} (\lambda_m - \lambda_{\pi}(x)) + \lambda_{\pi}(x) \\
\lambda^\pi_c(x) &= \frac{z_0^+(x,y) - z}{z_0^+(x,y) - z_k} \lambda^\pi_c(x).
\end{aligned}
\]  

(7.9)

In the above notation, \(c\) implies that \(x\) belongs to \(\Omega^c\). According to Equation (7.9), both \(\lambda^x_c(x)\) and \(\lambda^\pi_c(x)\) gradually increase when \(x\) moves down along \(L(x,y)\) from \(S_0^+\) to the kink plane. For the extreme case when \(z = z_k\), \(\lambda^x_c(x) = \lambda^\pi_c(x) = \lambda_m\), which indicates that the backprojection for \(x\) in this case requires data from the entire upper and lower helices. For the other extreme case when \(z = z_0^+(x,y)\), \(\lambda^x_c(x) = \lambda_{\pi}(x)\) and \(\lambda^\pi_c(x) = 0\), which implies that the backprojection range becomes the \(\pi\)-line segment of \(x\). This feature ensures a smooth transition of the backprojection range from \(\Omega_u^c\) to \(\Omega_\pi^c\).

Figure 7.4: Backprojection and weighting schemes for \(\mathbf{x} \in \Omega^c\). Left: the backprojection angular range. Right: the weighting scheme.

Next, we define the weighting function for \(\mathbf{x}\), which is denoted as \(\omega_c(\lambda, \mathbf{x})\). The initial idea is that we first calculate \(\omega^+_p(\lambda, u^*; \lambda_m)\) for the angular range \([0, \lambda^c]\). Then, we use \(\omega^+_p(\lambda, u^*; \lambda_m)\) for \(\lambda \in [\lambda^c(x), \lambda^\pi(x)]\), and use \(\omega^+_p(|\lambda|, u^*; \lambda_m)/2\) for \(\lambda \in [-\lambda^c(x), \lambda^\pi(x)]\); see the right of Figure 7.4. This approach is admissible because the projection that goes through \(\overline{a(\lambda)}\) and \(x\) and the projection that goes through \(\overline{a(-\lambda)}\) and \(x\) share the same \(u\) coordinate. Based on the above description, we obtain the weighting function

\[
\omega_c(\lambda, \mathbf{x}) = \begin{cases} 
\omega^+_p(\lambda, u^*; \lambda_m) & \lambda \in (\lambda^c(x), \lambda^\pi(x)) \\
\omega^+_p(|\lambda|, u^*; \lambda_m)/2 & \lambda \in [-\lambda^c(x), \lambda^\pi(x)].
\end{cases}
\]  

(7.10)

Using this weighting function, VDHFDK is exact if \(f(x)\) is independent of \(z\). However, this weighting function is discontinuous at \(\overline{a(\lambda^c)}\), and does not vanish to zero at \(\overline{a(-\lambda^c)}\). This discontinuity will produce undesirable artifacts. To overcome this shortcoming, we introduce a split weighting function called \(\omega_s(\lambda, \mathbf{x})\), and define

\[
\omega_s(\lambda, \mathbf{x}) = \omega_c(\lambda, \mathbf{x}) \omega^+_p(|\lambda|, u^*; \lambda_m),
\]  

(7.11)
such that $\omega_c(\lambda, x)$ is continuous around $\varphi(\lambda^c)$, and gradually reduces to zero around $\varphi(-\lambda^c)$. To make sure VDHFDK is exact when $f(x)$ is independent of $z$, the split weighting function should be non-negative and satisfy:

$$\omega_s(\lambda, x) + \omega_s(-\lambda, x) = 1, \quad \text{for } \lambda \in (-\lambda^c, \lambda^c)$$  \hspace{1cm} (7.12)$$

so that

$$\omega_c(\lambda, x) \omega_s^+ (|\lambda|, u^*; \lambda_m) + \omega_s(-\lambda, x) \omega_s^+ (|\lambda|, u^*; \lambda_m) = \omega_p^+ (|\lambda|, u^*; \lambda_m).$$

For the case when $x$ is above the kink plane, the split weighting function is defined as below

$$\omega_s(\lambda, x) = \begin{cases} 
1 & \lambda^c(x) + \Delta \leq \lambda \leq \lambda^c(x) \\
1/2 + \sin^2 \left( \frac{\lambda + \Delta - \lambda^c(x)}{4\Delta} \pi \right)/2 & \lambda^c(x) - \Delta \leq \lambda < \lambda^c(x) + \Delta \\
1/2 & -\lambda^c(x) + \Delta \leq \lambda < \lambda^c(x) - \Delta \\
1/2 - \sin^2 \left( \frac{\lambda - \Delta + \lambda^c(x)}{4\Delta} \pi \right)/2 & -\lambda^c(x) - \Delta \leq \lambda < -\lambda^c(x) + \Delta \\
0 & -\lambda^c(x) \leq \lambda < -\lambda^c(x) - \Delta 
\end{cases}$$  \hspace{1cm} (7.13)$$

with

$$\Delta = \beta \min \left( \lambda^c(x) - \lambda^c(x), \lambda^c(x) \right),$$  \hspace{1cm} (7.14)$$

where $\beta \in (0, 1)$ is a free parameter that controls $\Delta$, which will be explained next. A graphical depiction of $\omega_s(\lambda, x)$ is given in Figure 7.5.

![Figure 7.5: Illustration of the split weighting function $\omega_s(\lambda, x)$. The definition of $\omega_s(\lambda, x)$ for $x$ above the kink plane is shown in Equation 7.13.](image)

To understand how $\Delta$ plays a role in $\omega_s(\lambda, x)$, we consider the case when $\Delta = 0$ (i.e., $\beta = 0$). In this case, $\omega_s(\lambda, x)$ reduces to Equation 7.10 and becomes discontinuous. Therefore, $\Delta$ provides a buffer zone in $\omega_s(\lambda, x)$. The larger the $\Delta$ (or $\beta$) is, the smoother $\omega_s(\lambda, x)$ will be. Note that the introduction of $\Delta$ requires a slight modification of the backprojection range given in Equation 7.8.

Mathematically, it must be enlarged to

$$\Phi_c(x) = [-\Delta - \lambda^c(x), \lambda^c(x)].$$  \hspace{1cm} (7.15)$$
Note also that the subtraction of $\Delta$ in $\Phi_c$ will not change the backprojection range for $\chi$ belonging to $S_1^+$ or the $(x,y)$-plane, because, according to Equation 7.14, $\Delta = 0$ in these two cases.

For the case when $\chi \in \{\Omega_\pi^\pi \text{ and } z \leq z_k\}$, $\Phi_c(\chi)$ and $\omega_c(\lambda, \chi)$ can be obtained similarly.

Given the analysis of the backprojection angular ranges and the weighting functions described in the above two cases, we can now summarize the functions $\omega_c(\lambda, \chi)$ and $\Phi(\chi)$ that are used in step 3 and 4 of VDHFDK. Specially, for $\chi \in \Omega_\Lambda$, we have

$$\Phi(\chi) = \begin{cases} 
\Phi_\pi(\chi) & \chi \in \{\Omega_\pi^\pi \cup \Omega_1^\pi\}, \\
\Phi_c(\chi) & \chi \in \Omega_c^\pi 
\end{cases},$$

(7.16)

and

$$\omega_c(\lambda, \chi) = \begin{cases} 
\omega_\pi(\lambda, \chi) & \chi \in \{\Omega_\pi^u \cup \Omega_1^\pi\}, \\
\omega_c(\lambda, \chi) & \chi \in \Omega_c^\pi 
\end{cases}. $$

(7.17)

### 7.2.4 Implementation and Computational Complexities

We first briefly comment on the implementation complexities of the above three methods. FRFDK uses the RFDK method, which is well-established and widely used in modern CT systems. Implementation of the ramp filter is mature in industry and has been shown to be stable and efficient. FHFDK uses the HFDK method, which decompose the ramp filter into view-dependent differentiation and Hilbert transform. View-dependent differentiation is relatively new and difficult to implement. Compared to FRFDK, FHFDK is more complex in terms of implementation. Similar to FHFDK, VDHFDK also uses the HFDK method but with a voxel-dependent weighting and backprojection scheme, which adds another layer of implementation complexity. Therefore, among all three method, FRFDK is the most practical, whereas VDHFDK is the most complex.

Although all three methods are quite different in terms of implementation complexity, they are comparable in terms of computational complexity, because they have the same bottleneck: backprojection. Let the volume-of-interest be composed of $N_x \times N_y \times N_z$ voxels, and let the reverse helix consist of $N_\Lambda$ projections. Then the total computational cost for all three methods is

$$C_{\text{FDK}} = O(N_\Lambda N_x N_y N_z),$$

where $O$ stands for in the order of. Note that, for the VDHFDK method, the voxel-dependent weighting and backprojection steps are implemented together and have the same computational cost $C_{\text{FDK}}$. 
7.3 Comparative Evaluation

In this section, we evaluate the three reconstruction methods using computer-simulated data of the FORBILD head phantom and the XCAT phantom. First, to perform a fair comparison, we made the image resolution of each method the same. Then, we performed image-quality assessment in terms of image bias and image noise, using both visual inspections and quantitative metrics.

This section is organized as follows. First, we provide the details of the data simulations in Section 7.3.1. Then, in Section 7.3.2, we clarify implementation details for image reconstruction and explain how to design the fusion length $H_F$. Next, in Section 7.3.3, we explain how to match image resolution for all three methods. Finally, we present the reconstruction results and assess the image quality in Section 7.3.4.

7.3.1 Data Simulation

For data simulation, we selected the scan radius, $R$, the source-to-detector distance, $D$, and the bin size of the detector according to Table 7.1. This table is representative for the Artis zeego C-arm system (Figure 2.1, Siemens AG, Healthcare, Erlangen, Germany). Note that a quarter pixel shift was applied in both the $u$- and $v$-axes.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius of the reverse helix: $R$</td>
<td>78.5 cm</td>
</tr>
<tr>
<td>Source-to-detector distance: $D$</td>
<td>119.9 cm</td>
</tr>
<tr>
<td>Isotropic detector bin size:</td>
<td>0.0616 cm</td>
</tr>
</tbody>
</table>

Table 7.1: Scanning parameters of the Artis zeego system.

Regarding the size of the detector, the width (in the $u$ direction) was set to be large enough such that no transverse data truncation would occur, whereas the height (in the $v$ direction) was set such that it allowed axial data truncation but provided sufficient data for all three reconstruction methods. The requirement of the detector height is dependent on $H$. In our experiments, we selected $H = 6$ cm so that sufficient projection data can be obtained by a practical C-arm panel detector.

Although a short angular range is preferred for extended-volume C-arm CT, it is interesting to see the impact of $\lambda_m$ on image reconstruction results. For this purpose, we selected three values for $\lambda_m$, i.e., $240^\circ$, $300^\circ$ and $360^\circ$. Also, we fixed the number of vertex points on each helical turn at 480, regardless of the angular range.

Each measurement was simulated as a straight line integral through the phantom. For the FORBILD head phantom, the line integral was calculated analytically. For the XCAT phantom, the line integral was computed by applying Joseph’s method to a voxelized version of the phantom; cubic voxels of side 0.5 mm were used, with each voxel value obtained as the average of 8 uniformly-distributed sub-values.
7.3. Comparative Evaluation

7.3.2 Image Reconstruction

The implementation details for all three reconstruction methods are as follows. First, the ramp filter of FRFDK was implemented as a discrete convolution of the ramp kernel and the projection data in $u$. The ramp kernel was generated using the Hamming apodization window with Nyquist’s frequency indicated by the step size of $u$. Second, the view-dependent differentiation was implemented using the scheme suggested in [Noo 07]. A detailed description of this scheme is available in Section 8.3.1. For this work, $\lambda_0$ and $H_F$ in Equation 8.14 were defined to be $(0,0,0)$ and $(0,0,1)$, respectively. Third, regarding the Hilbert transform in FHFDK and VDHFDK, we applied a half pixel shift in $u$ for ring artifact reduction, as suggested in [Noo 04]. Finally, the backprojection in all three methods was implemented using bilinear interpolation.

To perform a fair comparison between FRFDK (FHFDK) and VDHFDK, the fusion length, $H_F$, should be carefully designed. Recall from the description of VDHFDK that, for reconstruction of $x \in \Omega_\pi$, projection data from both the upper and lower helices are used with a weighting function (Figure 7.5). This process can be viewed as a fusion process, and the segment of $L(x,y)$ bounded by $S_0^+$ and $S_0^-$ can be interpreted as a fusion zone that is dependent on $(x,y)$. For this reason, the fusion zone of FRFDK and FHFDK was defined to have the $z$ range $[z_0^-(0,0),z_0^+(0,0)]$. Because we set the height of each helical turn to be constant ($H = 6$ cm) regardless of the angular length, the fusion length $H_F$ is dependent on $\lambda_m$. In particular, for the cases when $\lambda$ equals to $240^\circ$, $300^\circ$ and $360^\circ$, $H_F$ is equal to 4.5 cm, 3.6 cm and 3 cm, respectively.

We compared the reconstruction results over the volume $\Omega_R$, which is the targeted volume for FRFDK and FHFDK. The comparison between FRFDK (or FHFDK) and VDHFDK is only possible if $\Omega_R \subset \Omega_\Lambda$. Using the scan configurations presented in Section 7.3.1, this condition is satisfied if $r \leq 22$ cm. Note that $\Omega_R$ of radius 22 cm is large enough to encompass both the FORBILD head phantom and the XCAT phantom in the transverse direction.

7.3.3 Resolution Matching

In this subsection, we explain the techniques that are used for resolution matching. It is important to make the image resolution the same for each reconstruction method, because the metrics we used for quality assessment, i.e., image bias and image noise, are significantly affected by resolution.

Several implementation steps may yield a difference in image resolution. These steps include: (i) the apodization window used in the ramp filter of FRFDK, (ii) the view-dependent differentiation used in both FHFDK and VDHFDK, which involves bilinear interpolation and a resolution control parameter $\varepsilon$, and (iii) the backprojection step that involves bilinear interpolation. Since the same backprojection is applied to all three methods, only the first and second steps are expected to yield some differences in image resolution. The apodization window only affects image resolution in horizontal directions for FRFDK, because the ramp filter is defined along the $u$-axis. Here, horizontal means the directions
are parallel to the \((x, y)\)-plane. On the other hand, the view-dependent differentiation has an impact on image resolution in all directions for FHFDK and VDHFDK.

In our experiments, we performed resolution matching for all three methods in two steps. In the first step, we performed resolution matching in horizontal directions. We fixed the apodization window of the ramp filter so that the image resolution of FRFDK was kept unaltered. Then we tuned \(\varepsilon\) in the view-dependent differentiation to make the resolutions of FHFDK and VDHFDK the same as FRFDK. In the second step, we used the results of the first step, and performed resolution matching in the \(z\) direction.

The reason for us to split the process of resolution matching into two steps is that, for FDK-type reconstruction algorithms, measuring image resolution in the \(z\) direction is more difficult than in the \((x, y)\)-plane. FDK-type algorithms suffer from CB artifacts, which have an influence on image resolution. To measure image resolution accurately, it is necessary to remove the impact of CB artifacts. Recall that all three reconstruction methods are exact if the object is independent of \(z\). Using this property, image resolution in the \((x, y)\)-plane can be measured without CB artifacts. However, this method for resolution measurement in the \((x, y)\)-plane does not apply in the \(z\) direction. Therefore, we have to use another method to remove the effect of CB artifacts for resolution matching in \(z\).

For each reconstruction method, image resolution is shift variant, and thus it is impossible to match resolution everywhere. In our experiments, we matched resolution along a circle in the kink plane that was centered on the \(z\)-axis.

**Resolution Matching in Horizontal Directions**

For horizontal directions, we performed resolution matching along the edge of a disk in the kink plane. This disk is centered on the \(z\)-axis and has a radius of 6 cm. To avoid CB artifacts, the phantom used for data simulation is an infinitely long cylinder of uniform attenuation coefficient that is centered on the \(z\)-axis and has a radius of 6 cm; see Figure 7.6(a).

The metric used for resolution measurement was the modulation transfer function (MTF) corresponding to the edge profile of the disk in the kink plane. The MTF curve was obtained as follows. First, a 0.1 cm thick slice in the kink plane was reconstructed on a centered grid of 400 \(\times\) 400 isotropic voxels of side 0.1 cm. Next, all the reconstructed voxels were ordered by their distances to the center, and voxel values with distances between 3 cm and 9 cm were extracted to generate an edge profile. Then, the edge profile was differentiated and Fourier transformed. This Fourier transform was normalized to obtain a value of 1 at frequency 0, and the outcome of this normalization was defined as the MTF.

To match image resolution of all three methods, we selected the Hamming window for the ramp filter of FRFDK. Then, we tuned \(\varepsilon\) for FHFDK and VDHFDK until their MTF curves were close to that obtained from FRFDK. Using visual inspection, we selected 0.049, 0.039 and 0.033 as the optimal values of \(\varepsilon\) for the cases when \(\lambda_m\) was identical to 240°, 300° and 360°. The matched MTF curves are shown in Figure 7.6. Note that the MTF curves of FHFDK and VDHFDK almost overlap to each other. This is because both
reconstruction methods use the same view-dependent differentiation scheme and the same Hilbert transform kernel.

Resolution Matching in $z$

The method used in the previous section to exclude the impact of CB artifacts does not work in the $z$ direction, because resolution measurement cannot be achieved in $z$ using an object that is independent of $z$. Instead, we designed a disk phantom that is centered on the $z$-axis with radius 6 cm and $|z - z_K| \leq 0.1$ cm. This phantom has a uniform linear attenuation coefficient of value 0 HU, and we denote the profile of this phantom along the $z$-axis as $f(z)$. By definition, $f(z)$ is a rectangle that is centered at $z_K$ with width 0.2 cm.

For each of the reconstruction methods, the cause of resolution loss in $z$ is the finite sampling of the projection data in the detector, because it is not practical to make the
detector bin size in \( z \) infinitely small. More specifically, resolution loss in \( z \) is caused by the linear interpolation in \( v \), which is involved (a) in the backprojection step of FRFDK and (b) in both the backprojection and the view-dependent differentiation steps of FHFDK and VDHFDK. In this work, we model the linear interpolation in \( v \) as the effect of a zero-mean Gaussian filter. We denote this filter as \( \text{gauss}(z, \sigma) \), where \( \sigma \) is the standard deviation. Note that \( \text{gauss}(z_1, \sigma_a) \ast \text{gauss}(z_2, \sigma_b) = \text{gauss}(z, \sqrt{\sigma_a^2 + \sigma_b^2}), \) (7.18)

where “\( \ast \)” stands for convolution.

Based upon the above modeling, we measure image resolution in \( z \) using the following steps. Let \( \Delta u \) and \( \Delta v \) be the detector bin size in \( u \) and \( v \). Typically, the detector bin is isotropic, i.e., \( \Delta u = \Delta v \). Let \( c_v \) be a scalar that is much smaller than 1.

- **Data Simulation:** First, we simulate two sets of projection data using the disk phantom, with the first set obtained using \( \Delta u \) and \( c_v \Delta v \) as the detector bin size in \( u \) and \( v \) and the second using \( \Delta u \) and \( \Delta v \).

- **Image Reconstruction:** Next, we perform image reconstruction using these two sets of projection data and obtain two images. Note that we use the Hamming window for FRFDK and use the values of \( \varepsilon \) established in the previous section for FHFDK and VDHFDK. By averaging the values of voxels in each slice for both images, we obtain two profiles in \( z \). We denote the profile of the image obtained from the first set of projection data \( \hat{f}_1(z) \), and refer to the profile of the image obtained from the second set of projection data as \( \hat{f}_2(z) \).

- **Profile Matching:** Finally, we try to find a \( \sigma_* \) such that the profiles, \( \hat{f}_2(z) \) and \( \hat{f}_1(z) \ast \text{gauss}(z, \sigma_*) \), are matched. We achieve this goal using the equation
  \[
  \sigma_* = \arg\min_{\sigma} \left( \text{mean} \left( (\hat{f}_2(z) - \hat{f}_1(z) \ast \text{gauss}(z, \sigma))^2 \right) \right). \tag{7.19}
  \]

We use \( \sigma_* \) as the figure of merit for resolution measurement in \( z \). We illustrate the above process in Figure 7.7.

We now explain this figure of merit as follows. Ideally, if \( \Delta v \) is infinitely small, then the reconstruction result would not have resolution loss in \( z \). We denote the profile of this reconstruction results obtained by averaging the values in each slice as \( \hat{f}(z) \). Note that, \( \hat{f}(z) \) contains CB artifacts, and these CB artifacts do not change even if \( \Delta v \) is not infinitely small. In other words, CB artifacts do not depend on \( \Delta v \). Under the assumption that the linear interpolation in \( v \) can be modeled as applying a Gaussian filter, we have

\[
\hat{f}_1(z) \approx \hat{f}(z) \ast \text{gauss}(z, \sigma_1) \quad \text{and} \quad \hat{f}_2(z) \approx \hat{f}(z) \ast \text{gauss}(z, \sigma_2).
\]

According to Equation (7.18) and the definition of \( \sigma_* \), we have

\[
\sigma_* \approx \sqrt{\sigma_2^2 - \sigma_1^2}, \quad \tag{7.20}
\]
7.3. Comparative Evaluation

Figure 7.7: Scheme for resolution measurement in z. The value of $\sigma_*$ is chosen such that the difference between $\hat{f}_1(z)$ and $\hat{f}_2(z)$ is minimum. We use $\sigma_*$ as a figure of merit for resolution measurement in z.

Recall that $c_v$ is much smaller than 1, hence $\sigma_2$ is much larger than $\sigma_1$. This observation indicates that $\sigma_*$ can be used to approximate $\sigma_2$. Because $\sigma_2$ is independent on CB artifacts, $\sigma_*$ can be used as an indicator of image resolution in z.

Using the above method, we calculated $\sigma_*$ for each angular range and each reconstruction method. The results are shown in Table 7.2. These results were obtained using the following configurations. Regarding data simulation, $\Delta v = 0.0616$ cm and $c_v = 0.1$ were used. For image reconstruction, the targeted volume consisted of 300 slices occupying a z range $[-0.75 \text{ cm}, 0.75 \text{ cm}]$. Each slice was composed of $11 \times 11$ voxels of $(x,y,z)$ size $0.5 \text{ cm} \times 0.5 \text{ cm} \times 0.005 \text{ cm}$. Regarding the convolution in Equation 7.19, we adopted a centered Gaussian filter that consisted of 61 points with grid size $0.005 \text{ cm}$.

<table>
<thead>
<tr>
<th>$\sigma_*$</th>
<th>FRFDK</th>
<th>FHFDK</th>
<th>VDHFDK</th>
</tr>
</thead>
<tbody>
<tr>
<td>240°</td>
<td>0.0166</td>
<td>0.0261</td>
<td>0.0259</td>
</tr>
<tr>
<td>300°</td>
<td>0.0165</td>
<td>0.0259</td>
<td>0.0250</td>
</tr>
<tr>
<td>360°</td>
<td>0.0164</td>
<td>0.0258</td>
<td>0.0243</td>
</tr>
</tbody>
</table>

Table 7.2: Optimal $\sigma_*$ for profile matching between $\hat{f}_1(z)$ and $\hat{f}_2(z)$.

Two observations can be made regarding $\sigma_*$. First, $\sigma_*$ can be obtained stably using our numerical results. Take FRFDK with $\lambda_m = 300^\circ$ as an example. The mean square error of $(\hat{f}_2(z) - \hat{f}_1(z) \ast \text{gauss}(z, \sigma_*))$ varies in conjunction with $\sigma$; see the plot in the left of Figure 7.8. In this plot, there exists a global minimum for $\sigma \in [0, 0.1]$, i.e., $\sigma_* = 0.0259$. We plot the difference between $\hat{f}_2(z)$ and $\hat{f}_1(z) \ast \text{gauss}(z, \sigma_*)$ in Figure 7.8(right). The flat solid curve around 0 HU indicates that the convoluted results of $\hat{f}_1(z)$ with $\text{gauss}(z, \sigma_*)$ matches $\hat{f}_2(z)$ well.

We also observed that the model that views the linear interpolation in $v$ as a Gaussian filter, is practical. We take FRFDK with $\lambda_m = 240^\circ$ as an example. Note that linear interpolation in $v$ can be considered as a triangular filter in the spatial domain. Let $\text{FWHM}_T$ be
Figure 7.8: Resolution measure in $z$ using $\sigma_*$ for Fusion-HFDK with $\lambda_m = 300^\circ$. Left: the mean square error: $(\hat{f}_2(z) - \hat{f}_1(z) \ast \text{gauss}(z, \sigma))^2$ for $\sigma \in [0, 0.1]$. The unique minimum appears at $\sigma_* = 0.0259$. Right: dashed line indicates the difference between $\hat{f}_2(z)$ and $\hat{f}_1(z)$, whereas the solid line indicates the difference between $\hat{f}_2(z)$ and the convolution result of $\hat{f}_1(z)$ and $\text{gauss}(z, \sigma_*)$.

the effective full width at half maximum (FWHM) of this triangular filter corresponding to $\Delta v = 0.0616$ cm. According to Table 7.1 FWHM$_T$ can be obtained using the equation

$$\text{FWHM}_T = \frac{\Delta v R}{D} = \frac{0.0616 \times 78.5}{119.9} = 0.0403 \text{ cm}.$$  

According to Table 7.2 $\sigma_* = 0.0166$ cm for FRFDK when $\lambda_m = 240^\circ$. Let FWHM$_G$ be the FWHM of the Gaussian filter corresponding to $\sigma_* = 0.0166$ cm. Then FWHM$_G$ can be calculated as

$$\text{FWHM}_G = 2\sqrt{2\ln 2} \sigma_* = 0.0391 \text{ cm}.$$  

As we can see, FWHM$_T$ and FWHM$_G$ are quite close, which demonstrates that our model is satisfactory.

Now, the resolution matching in $z$ can be achieved by bringing all $\sigma_*$s to the same value, which we call the target standard deviation, $\sigma_T$. According to Table 7.2 $0.0259$ cm was selected as the value of $\sigma_T$.

The target standard deviation could have been reached by applying a proper Gaussian filter in $v$ to the projection data. However, the Gaussian filter is not practical in this case. Its standard deviation would have been too small in comparison with $\Delta v$ because of the marginal difference between $\sigma_*$ and $\sigma_T$. This relatively small standard deviation could result in too few effective filtering points (possibly only one point).

Therefore, instead of a Gaussian filter, we adopted a three point filter that is defined by $v \in [0, 0.5]$. Let $\{g_k : k = 1, 2, \ldots, N-1, N\}$ be a column (along the $v$-axis) of projection
data that contains $N$ detector bins, then the filtered signal, denoted as $\hat{g}_k$, can be obtained by the three point filter using the formula

$$
\hat{g}_k = \begin{cases} 
(1 - \nu) g_k + \nu g_{k+1} & k = 1 \\
\nu g_{k-1} + (1 - 2\nu) g_k + \nu g_{k+1} & k = 2 : N - 1 \\
\nu g_{k-1} + (1 - \nu) g_k & k = N
\end{cases}
$$

Using this three point filter, we found that the values of $\nu$ listed in Table 7.3 are optimal to bring $\sigma_*$ to $\sigma_T$ for each case. Hereafter, Table 7.3 will be applied to evaluate the reconstruction results for the FORBILD head phantom and the XCAT phantom.

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>FRFDK</th>
<th>FHFDK</th>
<th>VDHFDK</th>
</tr>
</thead>
<tbody>
<tr>
<td>240°</td>
<td>0.120</td>
<td>N.A.</td>
<td>N.A.</td>
</tr>
<tr>
<td>300°</td>
<td>0.123</td>
<td>N.A.</td>
<td>0.0015</td>
</tr>
<tr>
<td>360°</td>
<td>0.124</td>
<td>N.A.</td>
<td>0.0028</td>
</tr>
</tbody>
</table>

Table 7.3: Optimal $\nu$ for the three point filter to bring $\sigma_*$ to $\sigma_T$ (N.A.: not applicable).

## 7.3.4 Evaluation

In this section, we evaluate the three reconstruction methods. This evaluation was accomplished by studying image bias and image noise using the FORBILD head phantom and the XCAT phantom.

Data used for this evaluation were prepared as follows. Both phantoms were simulated using the scan configuration presented in Section 7.3.1. The kink plane was located at $z_K = 1$ cm for the FORBILD head phantom and at $z_K = 0$ cm for the XCAT phantom. Projection data of both phantoms were then filtered by the three point filter using values of $\nu$ presented in Table 7.3. Image reconstructions were obtained using implementation details presented in Section 7.3.2. In particular, the Hamming window was chosen for the ramp filter of FRFDK, and the values of $\epsilon$ presented in Figure 7.6 were selected for the view-dependent differentiation step of FHFDK and VDHFDK.

We studied image bias and image noise by both visual impression and quantitative assessment. Every result was expressed using Hounsfield units (HU). To convert the linear attenuation coefficients to HU, we assume the energy used for data simulation was 80 KeV. In this case, the linear attenuation coefficient for water is 0.183 cm$^{-1}$; see NIST.$^1$

To generate image bias and image noise, two images were obtained from one of the three reconstruction methods for each $\lambda_m$, one without noise, and the other with Poisson noise added to the projection data. The image bias was obtained by subtracting the ground truth image from the image without noise, whereas the noise was acquired by subtracting the image with noise from the image without noise.

---

$^1$http://www.nist.gov/pml/data/xray_gammaray.cfm
For quantitative assessment, the bias and noise were evaluated in a homogeneous region of each phantom. In particular, the homogeneous regions of the FORBILD head phantom and the XCAT phantom were extracted using a brain mask and a heart mask, respectively; see Figures 7.11 and 7.12. These two masks were generated as follows. First, to exclude the impact of resolution on CB artifacts and noise, the ground truth images were filtered in $(x, y, z)$ directions by a Gaussian filter that was composed of 7 points with standard deviation $\hat{\sigma} = 0.05\text{ cm}$. Note that $3\hat{\sigma}$ was much larger than the half width of the edge spread function that was used for resolution matching in the $(x, y)$-plane (Section 7.3.3). Then, the brain mask was generated by identifying the voxels of value equal to 50 HU in the filtered FORBILD head phantom, and the heart mask was generated by identifying the voxels of value equal to 64.82 HU in the filtered XCAT phantom.

![Figure 7.11: The FORBILD head brain mask. This mask was generated by identifying the homogeneous brain matter (50 HU) in the low pass filtered FORBILD head phantom. This low pass filter consisted of 7 points with standard deviation 0.05 cm. First row: coronal slice at $y = 6\text{ cm}$; second row: transverse slice at $z = 1.7\text{ cm}$; third row: sagittal slice at $x = 0\text{ cm}$.](image1)

![Figure 7.12: The XCAT heart mask. This mask was generated by identifying the homogeneous heart chambers (64.82 HU) in the low pass filtered XCAT phantom. This low pass filter consisted of 7 points with standard deviation 0.05 cm. Slice positions from left to right: $z = -3\text{ cm}$, $z = -1.5\text{ cm}$, $z = 0\text{ cm}$ and $z = 1.5\text{ cm}$.](image2)

We then performed quantitative assessments using voxels within these two masks. As figures of merit, we calculated the mean of the absolute image bias (MAIB) as well as the standard deviation of the image noise. Both quantities are presented in two forms, i.e., the slice-wise form and the volume-wise form. In the slice-wise form, the quantity is first calculated for each slice within the corresponding mask. Then, this quantity is plotted against $z$. In the volume-wise form, the quantity is calculated using every voxel that is within the corresponding 3D mask.
7.3. Comparative Evaluation

The FORBILD Head Phantom

Image reconstructions of the FORBILD head phantom were obtained over a volume that was composed of $600 \times 600 \times 120$ voxels of side 0.05 cm. The smallest $(x, y, z)$ coordinates of this volume were $(-15 \text{ cm}, -15 \text{ cm}, -2 \text{ cm})$. For addition of Poisson noise, we used 80,000 photons per X-ray beam.

The noise free image reconstructions for the case when $\lambda_m = 240^\circ$ are shown in Figure 7.13. It is observed that FRFDK and FHFDK are comparable, and they produced fewer CB artifacts than VDHFDK in the middle slice. On the other hand, VDHFDK produced fewer CB artifacts around the top of the frontal sinus (black ellipsoid), as indicated in the last row of Figure 7.13. Similar observations can also be made for the other two cases when $\lambda_m = 300^\circ$ and $\lambda_m = 360^\circ$.

For a better understanding of the CB artifacts around the top of the frontal sinus, we illustrate the reconstruction results for the slice at $z = 3 \text{ cm}$ in Figure 7.14. Images obtained from VDHFDK were significantly better than those from the other two reconstruction methods. Also, observe that images obtained from FRFDK and FHFDK using a small $\lambda_m$ had more CB artifacts than those using a larger $\lambda_m$. This figure also indicates that FHFDK produced slightly fewer CB artifacts than FRFDK in the slice at $z = 3 \text{ cm}$.

We show the MAIB slice-wise in Figure 7.15 (left) and volume-wise in Table 7.4. Both figure and table indicate that the image bias was object and location dependent, and that it increased when the angular range became larger. Note that, in the last two rows of Figure 7.15 (left), VDHFDK produced less image bias than the other two methods in the top $\pi$-line region, but not in the bottom $\pi$-line region.

Next, we present reconstruction results obtained by using projection data with Poisson noise. We illustrate these results in Figure 7.16.
Figure 7.14: Reconstruction results for the FORBILD head phantom in Π_z with \( z = 3 \) cm. From top to bottom: \( \lambda_m = 240^\circ, \lambda_m = 300^\circ \) and \( \lambda_m = 360^\circ \). From left to right: FRFDK, FHFDK and VDHFDK. Display window: \([0, 100]\)HU.

<table>
<thead>
<tr>
<th>HU</th>
<th>FRFDK</th>
<th>FHFDK</th>
<th>VDHFDK</th>
</tr>
</thead>
<tbody>
<tr>
<td>240°</td>
<td>2.58</td>
<td>2.36</td>
<td>2.52</td>
</tr>
<tr>
<td>300°</td>
<td>3.67</td>
<td>3.41</td>
<td>3.43</td>
</tr>
<tr>
<td>360°</td>
<td>5.36</td>
<td>4.77</td>
<td>4.72</td>
</tr>
</tbody>
</table>

Table 7.4: The MAIB of the FORBILD head phantom using voxels within the brain mask.

For quantitative assessment, we plot the standard deviation of the image noise slice-wise in Figure 7.15 (right). The following observations are made from this figure. First, VDHFDK produced more noise than FHFDK in general, and they produced similar amount of noise in the kink plane. Second, FRFDK produced more noise than FHFDK, and the amount of noise difference was almost constant in each slice. Finally, it is also observed that when \( \lambda_m \) increased, the noise produced by all three reconstruction methods also increased, especially that produced by VDHFDK. The standard deviation is also reported.
Figure 7.15: Image bias and image noise within the brain mask in each slice of the FOR-BILD head phantom. Left: the mean of the absolute image bias (HU). Right: the standard deviation of the image noise (HU). From top to bottom: $\lambda_m = 240^\circ$, $\lambda_m = 300^\circ$ and $\lambda_m = 360^\circ$. Horizontal axis: $z$ coordinate of each transverse slice.
Figure 7.16: Reconstruction results for the FORBILD head phantom with Poisson noise added to the projection data. The number of the input photons per X-ray beam is 80,000. Top image: $\lambda_m = 240^\circ$; middle image: $\lambda_m = 300^\circ$; bottom image: $\lambda_m = 360^\circ$. First column: Fusion-RFDK; second column: Fusion-HFDK; third column: VDHFDK. From the top row to the bottom row in each image: coronal slice at $y = 6$ cm, transverse slice at $z = 1.7$ cm, and sagittal slice at $x = 0$ cm. Display window: $[0, 100]$HU.
7.3. Comparative Evaluation

Observations from this table are consistent with those from Figure 7.15 (right).

<table>
<thead>
<tr>
<th>HU</th>
<th>FRFDK</th>
<th>FHF DK</th>
<th>VDHFDK</th>
</tr>
</thead>
<tbody>
<tr>
<td>240°</td>
<td>26.56</td>
<td>24.61</td>
<td>24.76</td>
</tr>
<tr>
<td>300°</td>
<td>28.32</td>
<td>26.53</td>
<td>28.27</td>
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<tr>
<td>360°</td>
<td>29.18</td>
<td>26.92</td>
<td>30.77</td>
</tr>
</tbody>
</table>

Table 7.5: Standard deviation of the image noise (HU) of the FORBILD head phantom using voxels within the brain mask.

The XCAT Phantom

Image reconstructions of the XCAT phantom were obtained over a volume that was composed of \(440 \times 280 \times 60\) voxels of side 0.1 cm. The smallest \((x,y,z)\) coordinates of this volume were \((-22\,\text{cm}, -14\,\text{cm}, -3\,\text{cm})\). For addition of Poisson noise, we used 160,000 photons per X-ray beam.

For visual inspection, we illustrate the image bias of the XCAT phantom in Figure 7.17 and show the image noise in Figure 7.18.

For quantitative assessment, we report the MAIB volume-wise in Table 7.6 and present the standard deviation of the image noise volume-wise in Table 7.7.

Observations from the results of the XCAT phantom are consistent with those from the results of the FORBILD head phantom, except that the impact of the angular range on image bias is different. Table 7.6 indicates that the image bias did not change monotonically with the augmentation of \(\lambda_m\).

<table>
<thead>
<tr>
<th>HU</th>
<th>FRFDK</th>
<th>FHF DK</th>
<th>VDHFDK</th>
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<td>240°</td>
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<tr>
<td>300°</td>
<td>4.99</td>
<td>4.37</td>
<td>4.86</td>
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<tr>
<td>360°</td>
<td>5.40</td>
<td>5.11</td>
<td>4.70</td>
</tr>
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</table>

Table 7.6: The MAIB for the XCAT phantom using voxels within the heart mask.

<table>
<thead>
<tr>
<th>HU</th>
<th>FRFDK</th>
<th>FHF DK</th>
<th>VDHFDK</th>
</tr>
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<tbody>
<tr>
<td>240°</td>
<td>25.51</td>
<td>24.60</td>
<td>25.23</td>
</tr>
<tr>
<td>300°</td>
<td>28.55</td>
<td>27.80</td>
<td>28.62</td>
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<tr>
<td>360°</td>
<td>30.57</td>
<td>29.24</td>
<td>31.14</td>
</tr>
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</table>

Table 7.7: Standard deviation of the image noise (HU) of the XCAT phantom using voxels within the heart mask.
Figure 7.17: Reconstruction results for the XCAT phantom. Top image: $\lambda_m = 240^\circ$; middle image: $\lambda_m = 300^\circ$; bottom image: $\lambda_m = 360^\circ$. First column: FRFDK; second column: FHFDK; third column: VDHFDK. From the top row to the bottom row in each image: coronal slice at $y = 0$ cm, transverse slice at $z = 0$ cm and sagittal slice at $x = 0$ cm. Display window: $[-50, 100]$ HU.
7.3. Comparative Evaluation

Figure 7.18: Reconstruction results for the XCAT phantom using projection data with Poisson noise. The number of input photons per X-ray beam is 160,000. Top image: \( \lambda_m = 240^\circ \); middle image: \( \lambda_m = 300^\circ \); bottom image: \( \lambda_m = 360^\circ \). First column: FRFDK; second column: FHFDK; third column: VDHFDK. From the top row to the bottom row in each image: coronal slice at \( y = 0 \) cm, transverse slice at \( z = 0 \) cm and sagittal slice at \( x = 0 \) cm. Display window: \([-50, 100]\) HU.
In this section, we study the impact of the fusion zone on image bias, image noise and detector requirement.

First, we investigate the impact of the fusion zone on image bias. Judging from visual inspection, the CB artifacts in the reconstruction results for $\Omega^+$ and $\Omega^-$, which are separately obtained from the upper and lower helices, can complement each other to a certain extent. As an example, we illustrate the reconstruction results for the FORBILD head phantom from FHFDK for $\lambda_m = 240^\circ$. The reconstruction results in the fusion zone before and after the fusion process are shown in Figure 7.19. It can be observed that the CB artifacts were markedly suppressed by the fusion process. One possible explanation for this phenomenon is that the reverse helix is symmetric relative to the kink plane. This geometrical symmetry induces projection data symmetry that helps CB artifacts suppression.

Figure 7.19: Image reconstruction results for the FORBILD head phantom within the fusion zone ($z \in [-1.25 \text{ cm}, 3.25 \text{ cm}]$). These results were obtained by FHFDK using projection data with $\lambda_m = 240^\circ$. Left: reconstruction results from the upper helix; middle: reconstruction results after the fusion process; right: reconstruction results from the lower helix. From the top row to the bottom row: coronal slice at $y = 6 \text{ cm}$, transverse slice at $z = 0.95 \text{ cm}$, and sagittal slice at $x = -3 \text{ cm}$. The horizontal and vertical lines in the second row indicate the positions of the sagittal and coronal slices, respectively. Display window: $[0, 100] \text{HU}$.

It is also observed that less image bias was produced when a larger fusion length was employed. This observation is valid for both FRFDK and FHFDK. As an example, we plot the MAIB for the FORBILD head phantom for $\lambda_m = 240^\circ$ in Figure 7.20, both volume-wise and slice-wise. In the volume-wise form (Figure 7.20(left)), the MAIB monotonically decreased when the fusion length increased. In the slice-wise form (Figure 7.20(right)), the MAIB also decreased when the fusion length increased, but the amount of the decrease in each slice was dependent on the slice position. Especially in the kink plane, this amount became insignificant.
7.4. Effect of the Fusion Zone

It is also important to notice that for any given fusion length, FHFDK produced less image bias (in the volume-wise form) than FRFDK. An example is shown in Figure 7.20 (left).

Second, the fusion zone helps suppress noise, as indicated by Figure 7.15 (right). Theoretically, the larger the fusion length is, the more projection data will be used; hence, the more noise will be suppressed. However, this noise suppression only happens in the fusion zone and causes a significant noise variation along the \( z \)-axis. Fortunately, this variation is smooth, hence will have little impact on visual inspection; see Figures 7.16 and 7.18.

Finally, we consider the impact of the fusion length on the detector requirement for FRFDK and FHFDK. Recall from Section 7.3.1 that we assume the detector is wide enough to avoid transverse data truncation. Hence, here we only consider the detector requirement in the \( z \) direction. By construction, sufficient data acquisition for FRFDK and FHFDK requires the maximum detector height (in \( z \)) at the top and bottom vertex points of the reverse helix (the case for the bottom vertex point is shown in Figure 7.21). Let \( W \) be the minimum height requirement of the detector to guarantee sufficient data for FRFDK and FHFDK, it is straightforward to show that

\[ W = \frac{2D(H_F/2 + H)}{R - r}. \]  

(7.21)

The above equation indicates that the detector requirement is proportional to \((H_F/2 + H)\). Therefore, a larger fusion length demands a larger detector. In other words, for a fixed
detector height, a larger fusion length ($H_F$) will cause a decrease in the height of the reverse helix ($H$).

Figure 7.21: Illustration of detector requirement for FRFDK and FHFDK. We assume the detector is wide enough to avoid transverse data truncation and only consider the detector requirement in the z direction. The scan geometry for the bottom vertex point of the lower helix, $a(-\lambda_m)$, is illustrated. This vertex point requires the maximum detector height for FRFDK and FHFDK. Here, $W$ is the minimum height (along the z-axis) of the detector so that sufficient projection data can be acquired for FRFDK and FHFDK.

Overall, the fusion process can help reduce image bias and image noise significantly. The larger the fusion length is, the more image bias and image noise will be suppressed. On the other hand, for a fixed height of the reverse helix, a larger fusion length requires a larger detector; whereas for a fixed detector size, a larger fusion length indicates a reverse helix of reduced height. In practice, for a given detector, it is important to find a balance among the fusion length, the image quality, and the height of the reverse helix. This balance may be expected to be application-dependent.

### 7.5 Conclusion and Discussion

We have proposed three approximate reconstruction methods for extended-volume C-arm CT using projection data acquired from the reverse helical trajectory. All three methods are capable of producing 3D long-object images of acceptable image quality.

Regarding implementation complexity, FRFDK is the most practical method because it does not involve view-dependent differentiation, whereas VDHFDK is the most complex because it uses a voxel-dependent weighting and backprojection scheme. Although the implementation complexities are different, all three methods have comparable computational complexity that is mainly characterized by the backprojection step.

Regarding overall performance, FHFDK may be preferred over the other two reconstruction methods. On one hand, FHFDK produced less amount of image bias and image noise than FRFDK while keeping the computation efficient. On the other hand, FHFDK employs a much less complex implementation scheme than VDHFDK while allowing less image bias and image noise in most cases.
7.5. Conclusion and Discussion

Theoretically, VDHFDK is capable of producing more exact reconstructions in the \( \pi \)-line regions. Take Figure 7.14 as an example. For \( \lambda_m = 240^\circ \), the FORBILD head phantom in the slice at \( z = 3 \) cm was partially within the \( \pi \)-line region, whereas for \( \lambda_m = 360^\circ \), the phantom in the same slice was entirely within the \( \pi \)-line region. As a result, VDHFDK produced more CB artifacts in the former case than in the latter case.

However, in Figure 7.15, it is observed that VDHFDK produced more image bias than FHFDK in the bottom \( \pi \)-line region (\( \Omega_\pi^l \)) for \( \lambda_m = 300^\circ \) and \( \lambda_m = 360^\circ \). We believe this contradictory behavior was caused by discretization errors, which were entangled in the CB artifacts. For the FORBILD head phantom, more sub-objects are present in the bottom \( \pi \)-line region than in the top \( \pi \)-line region. Hence, the impact of discretization errors became dominant in the bottom \( \pi \)-line region.

The angular length of the reverse helix, \( \lambda_m \), plays an important role for all three reconstruction methods. For a reverse helix of fixed height and fixed number of views, the larger the angular length is, the more noise these reconstruction methods may present (Figure 7.15, Tables 7.5 and 7.7). For FRFDK and FHFDK, the larger angular length can introduce more data redundancy, and the Parker-like weighting does not address this redundancy in an optimal way in favor of noise. For VDHFDK, a larger angular length indicates fewer points within a \( \pi \)-segment, and thus induces more noise.

In the literature, F. Noo et al. [Noo 09] and S. Cho et al. [Cho 10] have demonstrated that exact long-object imaging using the reverse helix is possible. However, these two methods need additional forward projection and backprojection. Hence, they are more computationally costly than the three proposed FDK-type methods. Also, these two exact reconstruction methods require combining volumes with different shift-variant resolutions, which tends to yield undesirable artifacts. This resolution incompatibility problem does not exist in any of the three approximate methods.

The difficulty encountered in [Noo 09, Cho 10] stems from the fact that the reverse helix does not have sufficient R-line coverage. Recall from Chapter 6 that the ELE trajectory is capable of providing sufficient R-line coverage in the ROI. In the next chapter, we present an R-line based reconstruction scheme for the ELE trajectory, which allows efficient TES image reconstruction using axially truncated projection data.

On the other hand, the reverse helix is a practical trajectory for extended-volume C-arm CT. We have conducted real data acquisition on a laboratory floor-mounted Artis zeego C-arm system (Siemens AG, Healthcare, Erlangen, Germany). These first experimental results are provided in Chapter 9. In particular, image reconstructions from FRFDK using the real data will be presented.
Chapter 7. Practical Image Reconstruction Methods for the Reverse Helical Trajectory
Chapter 8

Exact Image Reconstruction for the Ellipse-Line-Ellipse Trajectory

It’s the little details that are vital.

JOHN WOODEN

The ELE trajectory introduced in Chapter 6 is an attractive data acquisition geometry for extended-volume C-arm CT, because it fulfills all the constraints that were mentioned in Section 1.3.2. In particular, this trajectory is able to provide sufficient R-line coverage for a typical ROI, hence efficient TES image reconstruction from axially truncated data is possible. Up to this point, the R-line coverage of the ELE trajectory is well understood, but how to select R-lines for efficient TES reconstruction remains an open question.

In this chapter, we propose a practical scheme to answer the above question. We evaluate this scheme by reconstructions from the DBP method [Pack 05b, Zou 04b, Ye 05b]. In the rest of this chapter, we will first review the DBP method, then a brief description of the ELE trajectory will be offered as a reminder. Afterwards, a scheme to select R-lines for the DBP method will be introduced, and the corresponding implementation steps will be clarified. Finally, we demonstrate this scheme with computer-simulated data of a modified FORBILD head phantom.

8.1 The DBP Method

The DBP method is an efficient TES reconstruction technique that is suitable for both fanbeam and cone-beam projection data. This technique was introduced by several groups in 2004 and 2005. The 2D version is described in [Noo 04] and the 3D version can be found in [Pack 05b, Zou 04b, Ye 05b].

The DBP method performs view-dependent differentiation to the projection data first, then backprojection. Finally, reconstruction results can be obtained by calculating the finite inverse Hilbert transform [Mikl 57, pages 126-31] of the backprojected data. There are two
ways to arrange data for the finite inverse Hilbert transform, one is using data along an R-line and the other along an M-line. An M-line is a measured line that intersects the source trajectory at least once. It is important to note that both ways require the POI to be on an R-line. In this chapter, the finite inverse Hilbert transform along an R-line will be used, and the corresponding formulae will be described in the rest of this section. For details on how to perform the DBP method along M-lines, we refer to [Pack 05b, Scho 10].

To describe the DBP method, besides the notations that were introduced in Chapter 3, we will need the additional notations below. Let \( n \in S^2 \), and let \( x \) be a POI. We define \( L(x, n) \) as the line that goes through \( x \) with direction \( n \). We denote \( f_H(x, n) \) as the Hilbert transform of \( f(x) \) along \( L(x, n) \) at location \( x \). Mathematically, this quantity can be expressed as

\[
f_H(x, n) = -\int_R \frac{f(x + tn)}{\pi t} dt.
\]  

(8.1)

Because of the singularity that appears in the above equation at \( t = 0 \), we introduce the notation, \( f \), which means that the integral is calculated as a Cauchy principal value. Recall from Chapter 3 that we denote projection data by \( g(\lambda, u, v) \), \( g(\lambda, x) \) or \( g(\lambda, \alpha) \). In this section, we introduce one more notation for projection data, i.e., \( g(x, n) \), which is the line integral of \( f(x) \) along \( L(x, n) \).

Figure 8.1: Illustration of the DBP method for an R-line. The solid red curve is the source trajectory, along which the X-ray source travels from \( a(\lambda_s) \) to \( a(\lambda_e) \). The red dashed line connecting \( a(\lambda_s) \) to \( a(\lambda_e) \) is an R-line that intersects the object over a line segment (the black solid line segment). Here, \( n^* \) is the unit vector pointing from \( a(\lambda_s) \) to \( a(\lambda_e) \), whereas \( t_{\min} \) and \( t_{\max} \) are signed distances from \( x \) along \( n^* \), respectively. Also, we have \( f(x + tn^*) = 0 \) for \( t \in [t_{\min}, t_{\min} + \varepsilon_0] \cup [t_{\max} - \varepsilon_0, t_{\max}] \), where \( \varepsilon_0 \) is a small positive scalar.

Now, we are ready to describe the DBP method that employs the finite inverse Hilbert transform along an R-line; see Figure 8.1. Let \( \lambda_s \) and \( \lambda_e \) be two angular positions on a
source trajectory, and let $\mathbf{x}$ be a POI on the R-line that connects $\mathbf{a}(\lambda_s)$ and $\mathbf{a}(\lambda_c)$. We define $\mathbf{n}^* = \mathbf{a}(\lambda_s, \mathbf{x})$, which is the unit vector pointing from $\mathbf{a}(\lambda_s)$ to $\mathbf{x}$. Line $\mathcal{L}(\mathbf{x}, \mathbf{n}^*)$ intersect the object over a line segment, as indicated by the solid black line in Figure 8.1. We define two functions, $t_{\text{min}}(\mathbf{x}, \mathbf{n}^*)$ and $t_{\text{max}}(\mathbf{x}, \mathbf{n}^*)$, such that

$$f(\mathbf{x} + t\mathbf{n}^*) = 0 \quad \text{for} \quad t \leq t_{\text{min}} + \varepsilon_0 \quad \text{and} \quad t \geq t_{\text{max}} - \varepsilon_0,$$

where $\varepsilon_0$ is a small positive scalar.

The Hilbert transform of $f(\mathbf{x})$ along $\mathcal{L}(\mathbf{x}, \mathbf{n}^*)$ can be obtained by a differentiated-backprojection process that is expressed as

$$f_H(\mathbf{x}, \mathbf{n}^*) = -\frac{1}{2\pi} \int_{\lambda_s}^{\lambda_c} \frac{1}{||\mathbf{x} - \mathbf{a}(\lambda)||} g'(\lambda, \mathbf{x}) d\lambda,$$

where $g'(\lambda, \mathbf{x})$ is the view-dependent differentiation that can be formulated as

$$g'(\lambda, \mathbf{x}) = \frac{\partial}{\partial \lambda} g(\lambda, \mathbf{x}).$$

Finally, $f(\mathbf{x})$ can be recovered by performing the finite inverse Hilbert transform along $\mathcal{L}(\mathbf{x}, \mathbf{n}^*)$, using the formula

$$f(\mathbf{x}) = \frac{-1}{\sqrt{-t_{\text{min}}(\mathbf{x}, \mathbf{n}^*):t_{\text{max}}(\mathbf{x}, \mathbf{n}^*)}} (f_c(\mathbf{x}, \mathbf{n}^*) + C_0),$$

where

$$f_c(\mathbf{x}, \mathbf{n}^*) = -\int_{t_{\text{min}}(\mathbf{x}, \mathbf{n}^*)}^{t_{\text{max}}(\mathbf{x}, \mathbf{n}^*)} \sqrt{(t - t_{\text{min}}(\mathbf{x}, \mathbf{n}^*)) (t_{\text{max}}(\mathbf{x}, \mathbf{n}^*) - t)} \frac{f_H(\mathbf{x} + t\mathbf{n}^*)}{\pi t} dt.$$

The constant $C_0$ in Equation 8.5 needs to be calculated. Three approaches can be used to obtain this quantity [Noo 04, Pack 05b, Yu 07, Scho 10]. The first approach uses Equation 8.2 as prior information. Let $t_0 \in [t_{\text{min}}, t_{\text{min}} + \varepsilon_0] \cup [t_{\text{max}} - \varepsilon_0, t_{\text{max}}]$, then we have

$$C_0 = -f_c(\mathbf{x} + t_0\mathbf{n}^*, \mathbf{n}^*).$$

The second approach utilizes $g(\mathbf{x}, \mathbf{n}^*)$, which is the line integral of $f(\mathbf{x})$ along $\mathcal{L}(\mathbf{x}, \mathbf{n}^*)$. By calculating this line integral with $f(\mathbf{x})$ being replaced by the right side of Equation 8.5, we obtain $C_0$ as

$$C_0 = -g(\mathbf{x}, \mathbf{n}^*) \int_{t_{\text{min}}(\mathbf{x}, \mathbf{n}^*) + \varepsilon_0}^{t_{\text{max}}(\mathbf{x}, \mathbf{n}^*) - \varepsilon_0} \frac{f_c(\mathbf{x} + t\mathbf{n}^*, \mathbf{n}^*)}{\sqrt{(t - t_{\text{min}}(\mathbf{x}, \mathbf{n}^*)) (t_{\text{max}}(\mathbf{x}, \mathbf{n}^*) - t)}} dt$$

$$+ \int_{t_{\text{min}}(\mathbf{x}, \mathbf{n}^*) + \varepsilon_0}^{t_{\text{max}}(\mathbf{x}, \mathbf{n}^*) - \varepsilon_0} \int_{t_{\text{min}}(\mathbf{x}, \mathbf{n}^*) + \varepsilon_0}^{t_{\text{max}}(\mathbf{x}, \mathbf{n}^*) - \varepsilon_0} \frac{1}{\sqrt{(t - t_{\text{min}}(\mathbf{x}, \mathbf{n}^*)) (t_{\text{max}}(\mathbf{x}, \mathbf{n}^*) - t)}} dt.$$
The third approach is the most straightforward, and $C_0$ can be obtained as below.

$$C_0 = \frac{g(x, n^*)}{\pi}.$$  \hfill (8.9)

The first approach to $C_0$ is solely reliant on the prior information. To obtain a more stable and accurate $C_0$, we may use the average of the results that is calculated from several $t_0$ along $\mathcal{L}(x, n^*)$. The second and third approaches utilize the projection data, and thus enforces the reconstruction results to be consistent with the projection data. Although Equation 8.7 looks simpler than Equation 8.8, they are not much different in terms of implementation complexity. This is because Equation 8.6 is a convolution process, and $f_c(x, n^*)$ is usually calculated for each point of the object along $\mathcal{L}(x, n^*)$, which significantly eases the calculation of Equation 8.8.

As described above, the DBP method consists of three steps, i.e., (i) view-dependent differentiation, (ii) backprojection, and (iii) the finite inverse Hilbert transform. The first two steps are local, and the third step is global. Therefore, as long as the R-lines used for the finite inverse Hilbert transform are well arranged, both axial and transverse data truncation may be allowed.

### 8.2 The ELE Trajectory

In this section, we will first briefly review the ELE trajectory (for more details, see Chapter 6), then we will establish a scheme to select R-lines for the DBP method. The ELE trajectory is composed of duplicates of a fundamental trajectory (Figure 6.1), which consists of the upper T-arc, the T-line and the lower T-arc (Figure 6.2). As explained in Chapter 6, it suffices to focus on this fundamental trajectory, and we will simply call this fundamental trajectory the ELE trajectory.

#### 8.2.1 Trajectory Configuration

We assume that we know the radius of the ROI, $r$, the radius of the trajectory cylinder, $R$, and the $z$ coordinate of the center of the upper T-arc, $H$. We also assume that $r/R < 1/2$, and define $d = r/R$. The condition $r/R < 1/2$ is valid for most medical X-ray imaging devices, if not all. Recall from Chapter 6 that $\gamma_m$ and $2\Delta H$ are the fan-angle and the axial height of each T-arc, respectively. Following Section 6.4, to guarantee sufficient R-line coverage in the ROI, we set

$$\gamma_m = \arcsin d \quad \text{and} \quad 2\Delta H = H \cdot d.$$

A vertex point on the T-arcs can be identified by its polar angle, $\lambda$, whereas a point on the T-line can be identified by its $z$ coordinate. Let $a_u(\lambda_u)$ and $a_l(\lambda_d)$ be the vertex points
8.2. The ELE Trajectory

on the upper and lower T-arcs at angular positions \( \lambda_u \) and \( \lambda_l \), respectively. According to Section 6.1, we have

\[
\begin{align*}
a_u(\lambda_u) &= (R \cos \lambda_u, R \sin \lambda_u, H(\lambda_u)), \\
a_l(\lambda_l) &= (R \cos \lambda_l, R \sin \lambda_l, -H(\lambda_l)),
\end{align*}
\]

where \( \{\lambda_u, \lambda_l\} \in [-\gamma_m, \pi + \gamma_m] \), and \( H(\lambda) = H + \Delta H \cos \lambda \). We denote \( \mathbf{b}(z) \) as a point on the T-line with the expression

\[
\mathbf{b}(z) = (R \cos \gamma_m, -R \sin \gamma_m, z),
\]

where \( z \in [-H(\gamma_m), H(\gamma_m)] \).

As mentioned in Chapter 6, it suffices to consider the portion of the ROI that is completely within the convex hull of the ELE trajectory, and we denote this portion of the ROI \( \Omega_\Lambda \). Therefore, \( \Omega_\Lambda \) is a cylinder of radius \( r \) centered on the \( z \)-axis, and it is bounded by the planes where the upper and lower T-arcs lie. Also, note that the ELE trajectory is mirror symmetric relative to the \((x, y)\)-plane and its R-line coverage is mirror symmetric as well. Hereafter in this chapter, our study of this trajectory as well as of the corresponding image reconstruction will concentrate on the ROI that is within the convex hull of the ELE trajectory and above the \((x, y)\)-plane; we denote this portion of the ROI as \( \Omega_\Lambda^+ \).

8.2.2 Selection of R-lines

Similar to the \( \pi \)-line surfaces that were introduced in Chapter 4, we now introduce R-line surfaces of the ELE trajectory. We call the surface that connects a point on the T-line to all the points on the upper T-arc the LUA R-line surface, and call the surface that connects a point on the lower T-arc to all the points on the upper T-arc the AUA R-line surface. Here, LUA stands for the T-line to the upper T-arc, whereas AUA stands for the lower T-arc to the upper T-arc. Therefore, the convergent point of each R-line surface is either a vertex point on the T-line or on the lower T-arc. Some examples of the R-line surfaces are shown in Figure 8.2, the blue R-line surfaces are from the trajectory that is composed of the T-line and the upper T-arc, whereas the green R-line surfaces are from the trajectory that is composed of the two T-arcs.

Now, we consider two sets of R-line surfaces as follows. First, by moving the convergent point of the LUA R-line surface from the top endpoint to the bottom endpoint along the T-line, we create the set of LUA R-line surfaces; see the blue surfaces in the left of Figure 8.2. This set of R-line surfaces occupies a continuous region called the blue region, which will cover a large portion of \( \Omega_\Lambda^+ \). This coverage of the blue region within the ROI in the \((x, y)\)-plane is shown in the middle of Figure 8.2. Next, we continue to move the convergent point of the AUA R-line surface along the lower T-arc from \( \lambda_u = -\gamma_m \) to \( \lambda_u = \lambda_{stop} \), and obtain the set of AUA R-line surfaces, which occupies a continuous region called the green region; see the green surfaces in the left of Figure 8.2. As explained in Chapter 4 there exists a \( \lambda_{stop} \) such that the green region can cover the portion of \( \Omega_\Lambda^+ \)
Chapter 8. Exact Image Reconstruction for the Ellipse-Line-Ellipse Trajectory

Figure 8.2: Selection of the R-lines of the ELE trajectory for the DBP method. Left: LUA (blue) and AUA (green) R-line surfaces. Middle: the R-line coverage in the ROI in the \((x, y)\)-plane. Right: R-line coverage along \(L_1\) and \(L_2\).

is not covered by the blue region, given the configuration in Section 8.2.1 and \(r/R < 1/2\). This coverage within the ROI in the \((x, y)\)-plane is illustrated in the middle of Figure 8.2. Note that the T-line and the lower T-arc are connected, and hence the blue and green regions are connected, and their union is a continuous space.

The above statement can be better understood using the process in Section A.3. We provide two examples as below. Consider the ROI in the \((x, y)\)-plane, let \(Q_1\) and \(Q_2\) be points in the blue and green regions, respectively; see the middle of Figure 8.2. For convenience, we assume that these two points and the intersection between the T-line and the \((x, y)\)-plane are collinear (the red line in the middle of Figure 8.2). We denote \(L_1\) and \(L_2\) as the vertical lines that go through \(Q_1\) and \(Q_2\), respectively.

As illustrated in the right of Figure 8.2, \(L_1\) intersects the blue region along the line segment \(AB\) that contains \(Q_1\). Therefore, the intersection between \(L_1\) and \(\Omega_{\lambda}^+\), i.e., the line segment \(AQ_1\), is fully covered by the blue region. On the other hand, \(L_2\) intersects the blue region along the line segment \(CE\), which is strictly above \(Q_2\). Therefore, \(Q_2\) is not covered by the blue region. Let \(L_2\) intersect the green region along the line segment \(ED\), with \(D\) being the lowest. Note that \(D\) is not necessarily on the R-line that connects \(\omega_{\mu}(\pi + \gamma_m)\) on the upper T-arc. According to Chapter 6, the configuration in Section 8.2.1 ensures that \(Q_2\) is within the line segment \(ED\).

For \(r/R < 1/2\), we found that \(\lambda_{\text{stop}} = 0\) is sufficient and necessary to guarantee full R-line coverage in \(\Omega_{\lambda}^+\), given the configuration in Section 8.2.1. The rest of this subsection is devoted to clarify this statement. We first recall some notations from Chapter 6. We denote \(\Pi_z\) as the horizontal plane that goes through \((0, 0, z)\), and denote the vertical plane that goes through all endpoints of both T-arcs as the cutting plane. The intersection between \(\Pi_z\) and the cutting plane is called the cutting line. We call the negative partial cone surface that connects \(\omega_{\mu}(\lambda)\) to all the points on the upper T-arc as \(S_{\lambda}^\Lambda\), and denote the intersection between \(\Pi_z\) and \(S_{\lambda}^\Lambda\) as R-arc \((O_{\Lambda}^{\lambda,z})\). Also, we denote the ellipse where R-arc \((O_{\Lambda}^{\lambda,z})\) lies
8.2. The ELE Trajectory

Ell\((O_{\lambda, z}^{\lambda_0})\). In this context, we focus on the \((x, y)\)-plane, i.e., \(z = 0\); see Figure 8.3. Note that all the R-arcs (or the corresponding ellipses) in the \((x, y)\)-plane intersect at a common point \(I_1\).

According to Chapter 6, the blue region fills up the area that is inside R-arc \((O_{-\gamma, 0}^{\lambda})\) and above the cutting line in the \((x, y)\)-plane. When \(\omega_3^{\lambda}(t)\) moves counter-clockwise from \(-\gamma_m\), the corresponding R-arc \((O_{\lambda, 0}^{\lambda_0})\) rotates counter-clockwise about \(I_1\), and gradually covers the green region in the ROI. Therefore, if the ROI is completely within Ell\((O_{\lambda, z}^{\lambda})\), the ROI is fully covered by R-lines.

We now show that \(\lambda_{\text{stop}} = 0\) is necessary. Because of the symmetry, the formula for Ell\((O_{\lambda, 0}^{\lambda_0})\) is equivalent to that of Ell\((O_{\lambda, 0}^{\lambda_0})\), which can be obtained according to Equation \(A.34\) and can be expressed as

\[
ay - bx + bR \cos \lambda - aR \sin \lambda = 0.
\]

Replacing \(a\) and \(b\) in the above equation by Equation \(A.33\) we obtain the quadratic expression for Ell\((O_{\lambda, 0}^{\lambda_0})\)

\[
x^2 + xR(d - \cos \lambda) + xyd \sin \lambda - yR \sin \lambda + y^2(1 + d \cos \lambda) - R^2d \cos \lambda = 0, \quad (8.10)
\]

where \(d = r/R\). Let \(p(x, y)\) be the function for the level set of the above equation, then

\[
p(x, y) = x^2 + xR(d - \cos \lambda) + xyd \sin \lambda - yR \sin \lambda + y^2(1 + d \cos \lambda) - R^2d \cos \lambda.
\]

Let \(n_{I_1}\) be the normal to Ell\((O_{\lambda, 0}^{\lambda_0})\) at \(I_1\), then \(n_{I_1}\) can be expressed as

\[
n_{I_1} = \left(\frac{\partial p(x, y)}{\partial x}, \frac{\partial p(x, y)}{\partial y}\right)_{|x=-Rd,y=0} = (-R(d + \cos \lambda), -R(1 + d^2) \sin \lambda).
\]

For \(\lambda \in (-\gamma_m, 0)\), we have \(\cos \lambda > 0\) and \(\sin \lambda < 0\), which indicates that \(n_{I_1}\) belongs to the second quadrant. Also note that the normal of the ROI circle in the \((x, y)\)-plane at \(I_1\) is
Therefore, for $\lambda_l \in (-\gamma_m, 0)$, Ell($O_{\lambda_l}^0$) goes through the region that is inside the ROI and above Ell($O_{-\gamma_m}^0$) (the green area in Figure 8.3). In this situation, the ROI in the $(x, y)$-plane cannot be fully covered by R-line. On the other hand, when $\lambda_l = 0$, we have $u_l = (-R(1 + d), 0)$, which is parallel to the normal of the ROI circle at $I_1$. Therefore, $\lambda_{stop} = 0$ is necessary to ensure that the ROI in the $(x, y)$-plane is fully covered by R-lines.

Next, we show that $\lambda_{stop}$ is sufficient for $r/R < 1/2$. Note that the expression for the ROI circle is $x^2 + y^2 = R^2 d^2$. (8.11)

We consider the intersection between the ROI circle and Ell($O_{0}^0$) using Equations 8.10 and 8.11. Replacing $y^2$ in Equation 8.10 by $R^2 d^2 - x^2$, we get the following equation

$$x^2 d + (1 - d)Rx + R^2 d(1 - d - d^2) = 0,$$ (8.12)

whose solutions, denoted as $x_1$ and $x_2$, are as follows

$$\begin{align*}
x_1 &= -R d \\
x_2 &= (d + 1 - 1/d)R.
\end{align*}$$ (8.13)

These two solutions have to be examined by Equation 8.11 for a real solution of $y$. For $x = x_1$, there is only one solution to $y$, i.e., $y_1 = 0$. Note that $(x_1, y_1)$ are the coordinates of $I_1$. For $x = x_2$, a solution of $y$ is only possible when

$$x_2^2 \leq R^2 d^2.$$ (8.14)

Given that $d \in (0, 1)$, the above relation is equivalent to $1/2 \leq d < 1$.

The above results indicate that, when $0 < d < 1/2$, there is only one intersection between Ell($O_{0}^0$) and the ROI circle in the $(x, y)$-plane, i.e., $I_1$, and the ROI is entirely within Ell($O_{-\gamma_m}^0$). Therefore, for $r/R < 1/2$, $\lambda_{stop} = 0$ ensures that the ROI in the $(x, y)$-plane is fully covered by R-lines.

According to Theorem 4, Corollary 4 and Section A.3, the above analysis demonstrates that, given $r/R < 1/2$, $\lambda_{stop} = 0$ is necessary and sufficient to guarantee full R-line coverage in $\Omega^\lambda_{\Lambda}$.

### 8.3 Reconstruction Scheme and Implementation Details

In this section, we describe the reconstruction scheme for the ELE trajectory using the DBP method. This scheme is composed of two phases: (i) image reconstruction on R-line surfaces using the DBP method; (ii) rebinning of the reconstruction results on R-line surfaces to the $(x, y, z)$-coordinate system. Recall from Section 8.1 that the DBP method is composed of three steps. Hence, the reconstruction scheme consists of four steps. The diagram of this scheme is shown in Figure 8.4. The rest of this section describes the implementation details of each step.
8.3. Reconstruction Scheme and Implementation Details

8.3.1 View-Dependent Differentiation

View-dependent differentiation is an important step for most efficient TES reconstruction algorithms, such as the Katsevich’s algorithm [Kats 02], the FBP style efficient TES reconstruction algorithms [Zou 04a, Pack 05c, Ye 05a] and the DBP method [Pack 05b, Zou 04b, Ye 05b]. The mathematical expression for this operation is available in Equation 8.4.

Numerical implementation of the view-dependent differentiation is challenging and several schemes have been proposed [Noo 07, Fari 08, Kats 11]. In this chapter, we adopt the scheme that was proposed by F. Noo et al in 2007, because it provides a parameter that can tune spatial resolution. We describe this scheme as follows.

First, we introduce some notations. We assume that the source trajectory is parameterized by $\lambda$ with step size $\Delta \lambda$, and its vertex point is denoted as $a(\lambda)$. We also assume that the object of interest is globally centered around an axis $L$ with direction of unit vector $n_L$. Let $L^\alpha_{\lambda}$ be the line that goes through $a(\lambda)$ with direction of unit vector $\alpha$, and let $c(\lambda, \alpha)$ be the point on $L^\alpha_{\lambda}$ that is closest to $L$. Let $x_0$ be an arbitrary point on $L$, according to [Noo 07], $c(\lambda, \alpha)$ can be obtained by

$$c(\lambda, \alpha) = a(\lambda) + \frac{(x_0 - a(\lambda)) \cdot (\alpha - (\alpha \cdot n_L)n_L)}{1 - (\alpha \cdot n_L)^2} \alpha.$$  

(8.14)

For the description of the scheme, we refer to Figure 8.5. We define

$$\alpha_1^+ = \frac{c(\lambda + \epsilon \Delta \lambda, \alpha) - a(\lambda)}{|c(\lambda + \epsilon \Delta \lambda, \alpha) - a(\lambda)|}, \quad \alpha_2^+ = \frac{c(\lambda + \epsilon \Delta \lambda, \alpha) - a(\lambda + \Delta \lambda)}{|c(\lambda + \epsilon \Delta \lambda, \alpha) - a(\lambda + \Delta \lambda)|},$$

and

$$\alpha_1^- = \frac{c(\lambda - \epsilon \Delta \lambda, \alpha) - a(\lambda)}{|c(\lambda - \epsilon \Delta \lambda, \alpha) - a(\lambda)|}, \quad \alpha_2^- = \frac{c(\lambda - \epsilon \Delta \lambda, \alpha) - a(\lambda - \Delta \lambda)}{|c(\lambda - \epsilon \Delta \lambda, \alpha) - a(\lambda - \Delta \lambda)|}.$$
By definition, $\alpha_1^+$ and $\alpha_2^+$ are the unit vectors pointing to $c(\lambda + \epsilon\Delta\lambda, \alpha)$ from $a(\lambda)$ and $a(\lambda + \Delta\lambda)$, respectively; whereas $\alpha_1^-$ and $\alpha_2^-$ are the unit vectors pointing to $c(\lambda - \epsilon\Delta\lambda, \alpha)$ from $a(\lambda)$ and $a(\lambda - \Delta\lambda)$, respectively.

Let $\epsilon$ be a scalar between 0 and 1, $g'(\lambda, \alpha)$ can now be calculated as

$$g'(\lambda, \alpha) \approx \frac{g(\lambda + \epsilon\Delta\lambda) - g(\lambda - \epsilon\Delta\lambda)}{2\epsilon\Delta\lambda},$$

(8.15)

where

$$g(\lambda + \epsilon\Delta\lambda) \approx (1 - \epsilon) g(\lambda, \alpha_1^+) + \epsilon g(\lambda + \Delta\lambda, \alpha_2^+),$$

and

$$g(\lambda - \epsilon\Delta\lambda) \approx (1 - \epsilon) g(\lambda, \alpha_1^-) + \epsilon g(\lambda - \Delta\lambda, \alpha_2^-).$$

Note that $\epsilon$ is the parameter that can tune spatial resolution in the reconstruction. According to [Noo07], in general, a smaller $\epsilon$ produces higher spatial resolution, except that the change of spatial resolution is insignificant when $\epsilon$ is too small.

### 8.3.2 Backprojection on R-line Surfaces

We perform backprojection on R-line surfaces so as to improve computational efficiency. To identify a point on an R-line surface in a more natural way, we introduce two coordinate systems, one for the LUA R-line surfaces (blue region), and the other for the AUA R-line surfaces (green region).

We use the $(h, \gamma, t)$-coordinate system for the LUA R-line surfaces. To describe this coordinate system, we refer to Figure 8.6(left). Let $s_1 = (x_{s_1}, y_{s_1}, z_{s_1})$ be a point in the ROI.
that belongs to the AL R-line connecting \( g_u(\lambda^+_1) \) and \( b(h) \), with the former located on the upper T-arc and the latter on the T-line. We denote \( \mathbf{v} \) as the vector pointing from \( b(h) \) to \( s_1(h, \gamma, t) \), and refer to \( \mathbf{v}_{xy} \) as the orthogonal projection of \( \mathbf{v} \) onto the \((x,y)\)-plane. We call the polar angle and the magnitude of \( \mathbf{v}_{xy} \) as \( \gamma \) and \( t \), respectively. By definition, \( s_1 \) is uniquely identified by \( h, \gamma \) and \( t \).

Geometrically, \( \gamma \) defines the plane that goes through the T-line and \( s_1 \), and \( h \) defines the arc-line (AL) R-line that goes through \( s_1 \) in this plane. Finally, \( t \) defines the location of \( s_1 \) along this R-line. By construction, \( h \in [-H - \Delta H \cos \gamma_m, H + \Delta H \cos \gamma_m] \). Also, because the object is assumed to be completely within the ROI, it is sufficient to consider \( \gamma \in [\pi - 2\gamma_m, \pi] \) and \( t \in [R - r, R + r] \) for backprojection.

Mathematically, if \((h, \gamma, t)\) is given, then \((x_{s_1}, y_{s_1}, z_{s_1})\) can be obtained using the following equations

\[
\begin{align*}
x_{s_1} &= R \cos \gamma_m + t \cos \gamma, \\
y_{s_1} &= -R \sin \gamma_m + t \sin \gamma, \\
z_{s_1} &= t (H + \Delta H \cos \lambda^+_1 - h)/d_1^* + h,
\end{align*}
\]

where

\[
d_1^* = -2R \cos(\gamma + \gamma_m) \quad \text{and} \quad \lambda^+_1 = \text{atan2}(-R \sin \gamma_m + d_1^* \sin \gamma, R \cos \gamma_m + d_1^* \cos \gamma),
\]

with \( \text{atan2}(v, u) \) being the four-quadrant inverse tangent function that gives the polar angle of point \((u,v)\) in the \((x,y)\)-plane in the range \([-\pi, \pi]\). Note that, by construction, \( \pi/2 < \gamma + \gamma_m < 3\pi/2 \), and hence \( d_1^* > 0 \).

![Figure 8.6: Left: the \((h, \gamma, t)\)-coordinate system for the R-line surface that connects one point on the T-line to all the points on the upper T-arc. Right: the \((\omega, \gamma, t)\)-coordinate system for the R-line surface that connects one point on the lower T-arc to all the points on the upper T-arc. All geometries are shown as their orthogonal projections onto the \((x,y)\)-plane.](image-url)
this coordinate system, $\omega$ is used to indicate a vertex point on the lower T-arc, whereas $h$ was used to indicate a vertex point on the T-line. For illustration of this coordinate system, we refer to Figure 8.6 (right).

Let $\mathbf{s}_2 = (x_{s_2}, y_{s_2}, z_{s_2})$ be a point in the ROI that belongs to the arc-to-arc (AA) R-line connecting $\mathbf{u}_i(\lambda_2^*)$ and $\mathbf{u}_i(\omega)$, with the former located on the upper T-arc and the latter on the lower T-arc. We denote $\mathbf{u}$ as the vector pointing from $\mathbf{u}_i(\omega)$ to $\mathbf{s}_2$, and refer to $\mathbf{u}_{xy}$ as its orthogonal projection onto the $(x, y)$-plane. Then, $\gamma$ and $t$ are the polar angle and magnitude of $\mathbf{u}_{xy}$, respectively. By definition, $\mathbf{s}_2$ is uniquely defined by $\omega$, $\gamma$ and $t$.

As explained in Section 8.2.2, for $r/R < 1/2$, the range for $\omega$ is $[-\gamma_m, 0]$. Because the object is entirely inside the ROI, we have $t \in [R - r, R + r]$. Regarding the range of $\gamma$, if only the ROI were considered as restriction, then $\gamma \in [\pi + \omega - \gamma_m, \pi + \omega + \gamma_m]$. However, the AA R-lines corresponding to $\gamma \in (\pi + \gamma_m/2 + \omega/2, \pi + \omega + \gamma_m]$ do not exist. Therefore, the practical range for $\gamma$ is $[\pi + \omega - \gamma_m, \pi + \omega/2 + \gamma_m/2]$.

Mathematically, if $(\omega, \gamma, t)$ is given, then $(x_{s_2}, y_{s_2}, z_{s_2})$ can be obtained using the following equations

\[
\begin{align*}
x_{s_2} &= R \cos \omega + t \cos \gamma, \\
y_{s_2} &= R \sin \omega + t \sin \gamma, \\
z_{s_2} &= t (2H + \Delta H \cos \omega + \Delta H \cos \lambda_2^*) / d_2^* - H - \Delta H \cos \omega,
\end{align*}
\]

where $d_2^* = -2R \cos(\gamma - \omega)$ and $\lambda_2^* = \text{atan2}(R \cos \omega + d_2^* \sin \gamma, R \cos \omega + d_2^* \cos \gamma)$. Note that, by construction, $\pi/2 < \gamma - \omega < 3\pi/2$, and hence $d_2^* > 0$.

Note that each pair of $h$ and $\gamma$ (or $\omega$ and $\gamma$) will uniquely specify an R-line, we perform the backprojection for each point along this R-line using the view-dependent differentiated projection data that belong to the corresponding R-line segment. The backprojection process is an integral, and it can be approximated by the trapezoidal rule for numerical implementation. However, the endpoints of an R-line do not usually hit the vertex points of the source trajectory. Therefore, particular attention is needed when calculating the integral near the endpoints of an R-line. In this work, we adopt the scheme that is presented in [Scho10, Section 4.3.3].

### 8.3.3 Finite Inverse Hilbert Transform

We calculate the finite inverse Hilbert transform using Equation 8.5, in which $f_c(x, y^*)$ is a Hilbert transform. We can treat this integral as a discrete convolution between the signal $f_H$ and the band limited Hilbert kernel with frequency range $[-\frac{1}{2\Delta t}, \frac{1}{2\Delta t}]$, where $\Delta t$ is the step size of $t$. We denote this band limited Hilbert kernel as $\hat{h}_H(t)$, which is calculated as

\[
\hat{h}_H(t) = \int_{-\frac{1}{2\Delta t}}^{\frac{1}{2\Delta t}} i \text{sign}(\sigma) e^{i2\pi t \sigma} d\sigma = \frac{1}{\pi t} (1 - \cos \frac{\pi t}{\Delta t}).
\]

When $t = 0$, we have $\hat{h}_H(t) = 0$, which can be calculated using a limit. According to [Noo04], a half pixel shift to the Hilbert transform is beneficial for ring artifact reduction.
To calculate $C_0$ in Equation 8.5, we use Equation 8.8, which forces the inverse Hilbert transform to be consistent with the backprojection data. For implementation, two aspects deserve attention. First, the integral in the numerator in Equation 8.8 should be carefully calculated. For each R-line, there is an acute angle, $\iota$, between this R-line and the $(x, y)$-plane. Because $t$ is defined as a variable in the $(x, y)$-plane for the R-line surface coordinate systems, the results of the integral in the numerator of Equation 8.8 should be multiplied by a factor of $1/\cos \iota$. Also, to improve accuracy, $\Delta t$ should be small enough so that the projection data are sufficiently used.

Second, to calculate $g(x, n^*)$ in the numerator of Equation 8.8, we developed a special interpolation scheme. We describe this scheme below using an R-line that connects the T-line and the upper T-arc, but it is straightforward to extend this scheme to an R-line that connects the T-arcs.

For illustration, we refer to Figure 8.7. Given an R-line that connects $b(h)$ and $a_u(\lambda)$, let $a_u(\lambda_-)$ and $a_u(\lambda_+)$ be the closest vertex points to $a_u(\lambda)$ on the upper T-arc, and let $b(h_-)$ and $b(h_+)$ be the closest vertex point to $b(\lambda)$ on the T-line. Let $x$ be a point on this R-line, and let $n^*$ be the unit vector pointing from $b(h)$ to $a_u(\lambda)$. Also, we denote $g_b(h, \lambda)$ as the line integral along the X-ray that starts from $b(h)$ and goes through $a_u(\lambda)$, and refer to $g_a(\lambda, h)$ as the line integral along the X-ray that starts from $a(\lambda)$ and goes through $b(h)$.

We calculate $g(x, n^*)$ using projection data from both the T-line and the upper T-arc. First, we calculate the line integral along the R-line using projection data from the T-line by linear interpolation with respect to $\lambda$. This quantity is denoted as $g_1(x, n^*)$, and can be obtained using the following equation

$$g_1(x, n^*) = \frac{\lambda - \lambda_-}{\lambda_+ - \lambda_-} g_b(h_+, \lambda_+) + \frac{\lambda_+ - \lambda}{\lambda_+ - \lambda_-} g_b(h, \lambda_-).$$
Second, we calculate the line integral along the R-line using projection data from the upper T-arc by linear interpolation with respect to $h$. This quantity is denoted as $g_2(x, n^*)$, and can be calculated using the equation

$$g_2(x, n^*) = \frac{h - h_-}{h_+ - h_-} g_a(\lambda, h_+) + \frac{h_+ - h}{h_+ - h_-} g_a(\lambda, h_-).$$

Finally, the average of $g_1(x, n^*)$ and $g_2(x, n^*)$ is used for $g(x, n^*)$, i.e.,

$$g(x, n^*) = \frac{g_1(x, n^*) + g_2(x, n^*)}{2}.$$

### 8.3.4 Rebinning

Rebinning is the final step of the reconstruction scheme; it rebins the DBP reconstruction results from the R-line surface coordinate systems to the $(x, y, z)$-coordinate system. In the following, we will show how to identify the $(h, \gamma, t)$ or $(\omega, \gamma, t)$ coordinates for a given point $x = (x, y, z)$ with $z \geq 0$. We obtain the final reconstruction result using triple linear interpolation.

First, we test whether $x$ belongs to an LUA R-line or an AUA R-line. This can be done by the following steps. Draw a plane that goes through $x$ and the T-line. This plane intersects the upper T-arc at $a_u(\lambda^*)$. Draw a line through $a_u(\lambda^*)$ and $x$. If this line intersects the segment of the T-line, $x$ belongs to an LUA R-line; otherwise, $x$ belongs on an AUA R-line.

If $x$ belongs to an LUA R-line, then $\gamma$ and $t$ can be calculated using the equations below

$$\gamma = \text{mod} \left( \text{atan2}(y + R \sin \gamma_m, x - R \cos \gamma_m) + \frac{\pi}{2}, 2\pi \right) - \frac{\pi}{2},$$

$$t = \sqrt{(y + R \sin \gamma_m)^2 + (x - R \cos \gamma_m)^2},$$

where mod($u, 2\pi$) is identical to $u$ modulo $2\pi$. The mod function is needed so that $\gamma \in [-\pi/2, 3\pi/2)$. Let

$$d^* = -2R \cos (\gamma + \gamma_m),$$

and

$$\lambda^* = \text{atan2}(-R \sin \gamma_m + d^* \sin \gamma, R \cos \gamma_m + d^* \cos \gamma).$$

We then are able to calculate $h$ as follows

$$h = z - \frac{t(H + \Delta H \cos \lambda^* - z)}{d^* - t}.$$

Note that, by definition, $d^* > t$, and hence the above equation is stable.

If $x$ belongs to an AUA R-line, the $(\omega, \gamma, t)$ coordinates should be calculated. In this case, we need to identify the corresponding R-line, i.e., given $(x, y, z)$, find $\lambda_u^*$ and $\lambda_l^*$ such that the R-line connecting $a_u(\lambda_u^*)$ and $a_l(\lambda_l^*)$ goes through $x$. This process can be completed using the inversion formulae presented in Section 6.3.1. We may find two qualified...
pairs of $\lambda_u^*$ and $\lambda_l^*$, but we only choose the pair with the smaller $\lambda_l^*$, which belongs to $[-\gamma_m, 0]$. Once the AUA R-line is identified, the $(\omega, \gamma, t)$ coordinates can be obtained as

$$\omega = \lambda_l^*,$$
$$\gamma = \text{mod} \left( \text{atan2}(y - R \cos \omega, x - R \sin \omega) + \frac{\pi}{2}, 2\pi \right) - \frac{\pi}{2},$$
$$t = \sqrt{(x - R \sin \omega)^2 + (y - R \cos \omega)^2}. $$

Due to the nature of the reconstruction scheme, it is very easy to get streaking artifacts in the direction of the finite inverse Hilbert transform. To combat these artifacts, two aspects deserve particular attention, i.e., the accuracy of the backprojection and the accuracy of the finite inverse Hilbert transform. To improve the precision of the backprojection, on one hand, we can reduce the step size of $\lambda$ or $h$, and on the other hand, we should carefully address the integral around the endpoints of each R-line segment. To improve the precision of the finite inverse Hilbert transform, the accuracy of $C_0$ is the key. We have chosen Equation 8.8 for the implementation of $C_0$. Therefore, a sophisticated interpolation scheme for calculating $g(x, n^*)$ and a smaller step size of $t$ will yield a more accurate $C_0$. If the above two aspects are carefully addressed, the streaking artifacts can be largely suppressed.

### 8.4 Simulations

In this section, we present numerical results using computer-simulated data. A modified FORBILD head phantom was adopted for data simulation. This phantom was obtained by stretching the FORBILD head phantom, except the spheres (object 1, 2, 3 and 4 in [1]), in the $x$-direction by a factor of 1.25, so that the horizontal slices of this phantom is circular, as shown in Figure 8.8. This modification was designed for a better illustration of the reconstruction results from the AUA R-line surfaces.

Regarding the ELE trajectory, we chose $H = 5$ cm and $\Delta H = 2$ cm so that R-line coverage was sufficient in the ROI. For each of the upper and lower T-arcs, 500 CB projections were generated over the angular range $[-24^\circ, 204^\circ]$, whereas for the T-line, 51 CB projections were generated. Note that the $z$ range of the T-line was automatically generated using $H, \Delta H$ and the angular range of the T-arcs. All CB projections were acquired using a quarter detector pixel shift.

Regarding the DBP method, we used $\varepsilon = 0.001$ for Equation 8.15 and $\gamma_0 = (0, 0, 0)$ and $n_L = (0, 0, 1)$ for Equation 8.14.

Reconstructions on the LUA and AUA R-line surfaces are shown in Figure 8.9. For the AL trajectory, we selected 200 R-line surfaces with their convergent points evenly distributed along the T-line. For each LUA R-line surface, 900 points were sampled evenly for $\gamma$ over the range $[132^\circ, 180^\circ]$, whereas 900 points were sampled for $t$ over the range $[16.5 \text{ cm}, 33.5 \text{ cm}]$. For the AA trajectory, we selected 225 R-line surfaces with their convergent points uniformly distributed along the lower T-arc over the range $[\pm24^\circ, 0^\circ]$. For

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1http://www.imp.uni-erlangen.de/phantoms/head/head.html
Figure 8.8: The modified FORBILD head phantom. Left: sagittal slice at $x = 0$ cm with $z \in [0, 7]$ cm. Middle: transverse view at $z = 0$ cm. Right: coronal view at $y = 6$ cm with $z \in [0, 7]$ cm. Display window: $[0, 100]$ HU.

Figure 8.9: Left: the reconstruction on the LUA R-line surface at $h = 0.0343$ cm. Right: the reconstruction on the AUA R-line surface at $\omega = -22.4^\circ$. Display window: $[0, 100]$ HU.

each $\omega$, we evenly sampled 900 points for $\gamma$ over the range $[\pi - \gamma_m + \omega, \pi + \gamma_m + \omega]$, with $\gamma_m = 24\pi/180$. The grid for $t$ was the same as that of the R-line surfaces from the AL trajectory.
8.4. Simulations

We obtained the final image reconstruction through rebinnig using triple linear interpolation with isotropic voxels of size $0.02\,\text{cm} \times 0.02\,\text{cm} \times 0.02\,\text{cm}$. A large portion of the final image was rebinned from the LUA R-line surfaces, and the rest was rebinned from the AUA R-line surfaces. The rebinned images at $z = 0\,\text{cm}$ from the LUA and AUA surfaces are shown in the left and right of Figure 8.10, respectively. The final rebinned result, using both the LUA and AUA R-line surfaces, is shown in Figure 8.11 and the profile of this result along the $y$-axis is shown in Figure 8.12. As illustrated, the final reconstruction results have good image quality with smooth transitions in all coronal, sagittal and transverse directions.

We also performed image reconstruction of the FORBILD head phantom using projection data with Poisson noise. We have used exactly the same configurations for both data simulation and image reconstruction as above, except that Poisson noise was added for each X-ray beam using 150,000 input photons. The final reconstruction result is shown in Figure 8.13.
Chapter 8. Exact Image Reconstruction for the Ellipse-Line-Ellipse Trajectory

Figure 8.11: Final image reconstruction rebinned through triple linear interpolation using isotropic voxels of size $0.02\,\text{cm} \times 0.02\,\text{cm} \times 0.02\,\text{cm}$. Left: sagittal view at $x = 0\,\text{cm}$. Middle: transverse view at $z = 0$. Right: coronal view at $y = 6\,\text{cm}$. Display window: $[0, 100] \, \text{HU}$.

Figure 8.12: Profile of the modified head phantom along the $y$-axis. LAC: linear attenuation coefficient. Red dashed: the ground truth. Solid blue: the reconstruction results.
8.5 Conclusion and Discussion

We have presented a practical scheme to use the R-lines of the ELE trajectory for TES image reconstruction. Reconstruction results from the DBP method demonstrate that this scheme is efficient and stable for extended-volume C-arm CT.

We have demonstrated that the ROI that is above the \((x,y)\)-plane and inside the convex hull of the ELE trajectory can be reconstructed using projection data from the upper T-arc, the T-line and a small continuous portion of the lower T-arc that is connected to the T-line. Similarly, the other half of the ROI inside the convex hull of the ELE trajectory can be reconstructed using projection data from the lower T-arc, the T-line and a small continuous portion of the upper T-arc that is connected to the T-line. The above reconstruction process can be easily extended to other duplicates of the ELE trajectory along the \(z\)-axis, so that long-object imaging using the ELE trajectory is achieved.

Although we adopted the DBP method for image reconstruction in this chapter, other efficient TES image reconstruction algorithms, such as the Katsevich’s algorithm and the FBP style R-line based TES reconstruction algorithms, are also applicable.

Figure 8.13: Reconstruction results from the ELE trajectory using projection data with Poisson noise. The number of input photons per X-ray beam is 150,000. Reconstruction results use a voxel size of 0.02 cm \(\times\) 0.02 cm \(\times\) 0.02 cm. Left: sagittal view at \(x = 0\) cm. Middle: transverse view at \(z = 0\). Right: coronal view at \(y = 6\) cm. Display window: [0, 100] HU.
Chapter 9

The First Experimental Results of Extended-Volume C-arm CT

*The true method of knowledge is experiment.*

*WILLIAM BLAKE*

Figure 9.1: Data acquisition for extended-volume C-arm CT using a laboratorial floor-mounted Artis zeego system (Siemens AG, Healthcare, Erlangen, Germany).

In this chapter, we present a prototype for extended-volume C-arm CT using the reverse helical trajectory. We demonstrate this prototype by experimental results obtained from a laboratorial floor-mounted Artis zeego system (Siemens AG, Healthcare, Erlangen, Germany; see Figure 9.1).
Chapter 9. The First Experimental Results of Extended-Volume C-arm CT

This chapter is organized as follows. First, we provide an overview of the prototype. Then, we describe the experimental setup that was used for data acquisition. Next, we clarify the techniques that were used for data preprocessing. Finally, we present the reconstruction results obtained from Fusion-RFDK.

### 9.1 Prototype Overview

We now provide a brief description of our prototype for extended-volume C-arm CT. Overall, this prototype involves the following aspects.

- **C-arm system**: For data acquisition, we selected a laboratorial floor-mounted Artis zeego C-arm system (Siemens AG, Healthcare, Erlangen, Germany). This system is equipped with a state-of-the-art flat panel detector. More importantly, this system is installed with a multi-axis robot so that the reverse helix can be implemented easily.

- **Trajectory calibration**: Because of the open design, a source trajectory produced by a C-arm system cannot be as precise as an input geometry. Therefore, to assess the real geometrical information of the reverse helix, a process for trajectory calibration is required. In our experiments, we have extended the scheme in [Stro03] for the reverse helix, and designed a new axially long calibration phantom for long-object imaging purpose.

- **Data preprocessing**: Raw projection data obtained from the calibrated trajectory needs to be preprocessed for noise and artifact reduction. In our experiments, the following classical corrections were performed: overexposure correction, scatter correction and beam hardening correction. These corrections were based on [Zell05].

- **Image reconstruction**: Fusion-RFDK was selected for image reconstruction; see Chapter 7. To accommodate the reconstruction method, an ideal reverse helix was obtained from the calibrated trajectory by geometry registration and fitting, and the projection data for this ideal reverse helix were obtained from the preprocessed projection data through a rebinning process.

We provide details for all the above aspects in the rest of this chapter.

### 9.2 Experimental Setup

Projections were obtained from a laboratorial floor-mounted Artis zeego system (Siemens AG, Healthcare, Erlangen, Germany). Detailed parameters of this system are presented in Table 9.1. Note that the detector bin size of the flat panel detector is 0.154 mm. However, in our experiments, the 2-by-2 binning mode was selected. Hence, the actual detector bin size used for our experiments was 0.308 mm. The detector was set in the landscape
9.2. Experimental Setup

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius of the reverse helix ($R$)</td>
<td>785 mm</td>
</tr>
<tr>
<td>Source-to-detector distance ($D$)</td>
<td>1200 mm</td>
</tr>
<tr>
<td>Isotropic detector bin size</td>
<td>0.308 mm</td>
</tr>
<tr>
<td>Detector size</td>
<td>300 mm×400 mm</td>
</tr>
<tr>
<td>Detector A/D converter</td>
<td>16 bits</td>
</tr>
</tbody>
</table>

Table 9.1: Parameters of the floor-mounted Artis zeego system.

mode, in which a side of 400 mm was used for the transverse direction. This configuration maximized the radius of the FOV, which was about 130 mm.

We implemented a five-turn reverse helix on this C-arm system. The five-turn reverse helix was solely realized by translating and rotating the C-arm gantry, while the patient table was kept stationary. The parameters for this five-turn reverse helix are presented in Table 9.2. Note that the listed parameters were our input values to the C-arm control system. The exact positions of the vertex points of this reverse helix had to be estimated using a calibration process. Some systematic deviations and some statistical fluctuations were expected between the input and calibrated parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial heights of each helical turn ($H$)</td>
<td>60 mm</td>
</tr>
<tr>
<td>Angular length per turn ($\lambda_m$)</td>
<td>240°</td>
</tr>
<tr>
<td>Step size of $\lambda$ ($d\lambda$)</td>
<td>0.35°</td>
</tr>
<tr>
<td>Number of samples per turn</td>
<td>681</td>
</tr>
</tbody>
</table>

Table 9.2: Parameters of the five-turn reverse helix.

Three phantoms were selected for data acquisition, i.e., the anthropomorphic torso with a SAWBONES spine\(^1\), the CATPHAN phantom\(^2\) and the Siemens CB phantom (QRM, Möhrendorf, Germany\(^3\); see Figure 9.2. Details about these phantoms are given below.

- The torso phantom was 550 mm long, 400 mm wide and 200 mm thick, and the inserted SAWBONES spine was of length 360 mm and diameter less than 100 mm. This phantom was placed parallel to the long axis of the patient table so that the entire spine was within the FOV. Due to the large size, projection data were truncated in the transverse direction. However, the impact of this data truncation was expected to be small, because other than the spine, the rest of the torso phantom was made of foam. Projection data of this phantom were obtained using 90.0 kVp and 43 mA.

\(^1\)http://www.sawbones.com/products/ortho/category.aspx?9
\(^2\)http://www.phantomlab.com/products/catphan.php
\(^3\)http://www.qrm.de/content/products/imagequality/conebeam.htm
Because the CATPHAN phantom and the Siemens CB phantom were too short, they were placed in-line parallel to the patient table to simulate a long object. Note that the radii of the CATPHAN and the Siemens CB phantom were small, and data truncation in the transverse direction did not occur. The two phantoms were scanned using 91.2 kVp and 123 mA.

Figure 9.2: Physical phantoms for data acquisition. From left to right: the torso phantom with a SAWBONES spine, the CATPHAN phantom and the Siemens CB phantom. The CATPHAN phantom and the Siemens CB phantom were placed in-line to simulate a single long object.

9.3 Trajectory Calibration

Due to its open design, a C-arm system is not capable of producing a perfect source trajectory. To assess the geometrical information of the reverse helix and the orientation of the flat panel detector, a calibration process is required. In this work, we proposed a practical calibration method, which is an extension to the method presented in [Stro03]. In addition, an axially long calibration phantom was designed for long-object imaging purpose. We provide a brief description of this method as follows.

The description uses four coordinate systems, i.e., the world coordinate system, the X-ray source coordinate system, the detector coordinate system and the image coordinate system, as shown in Figure 9.3. The world coordinate system is attached to the calibration phantom. In other words, once the calibration phantom is placed, the world coordinate system is defined.

Let $\mathbf{x}$ be a point of the phantom in the world coordinate system, and let $\hat{\mathbf{x}}$ be its projection in the image coordinate system; both $\mathbf{x}$ and $\hat{\mathbf{x}}$ are expressed using homogeneous coordinates. Then $\mathbf{x}$ and $\hat{\mathbf{x}}$ can be linked by a projection matrix called $P$. This relation can be expressed as

$$\hat{\mathbf{x}} = P \mathbf{x}.$$ (9.1)

The goal of the calibration process is to find the matrix $P$ for each vertex point, because $P$ contains the information about the position of the X-ray source and the orientation of
9.3. Trajectory Calibration

The trajectory calibration involves aligning the world coordinate system with the detector coordinate system, the X-ray coordinate system, and the image coordinate system. This alignment is crucial for accurate 3D reconstruction of objects from their X-ray projections.

The goal is to achieve the least square error between the X-ray projections and the actual object positions. This is done by identifying enough pairs of known positions.

The projection matrix $P$ is decomposed into three matrices $P_1$, $P_2$, and $P_3$:

$$P = P_3 P_2 P_1.$$  

Matrix $P_1$ transforms a point from the world coordinate system to the X-ray coordinate system, characterized by rotation and translation. Matrix $P_2$ models the CB projection from the X-ray to the detector coordinate system, typically defined by the source-to-detector distance. Finally, $P_3$ is an affine transform from the detector to the image coordinate system, defined by detector offset, bin size, and skew angle.

The degrees of freedom of $P_1$, $P_2$, and $P_3$ are 6, 1, and 5, respectively. Theoretically, $P$ should have 12 unknowns, but in practice, $P$ usually only has 9 unknowns because the detector bin size and skew angle are usually known.

To identify pairs of known positions, a modified PDS-2 phantom was used. This phantom consists of a 450 mm long cylindrical wall embedded with 400 spherical balls. Half of the balls have a diameter of 3.2 mm, and the other half have a diameter of 1.6 mm. A large ball indicates a logic 1, whereas a small ball indicates a logic 0. The balls are arranged on a helix such that any subsequence of length 10 defines a unique number. This number uniquely defines the position of each ball through this 10-bit sequence.

To make proficient use of the 10-bit encoding feature, the modified PDS-2 phantom should be placed in the center of the FOV such that its centerline is more or less parallel to the X-ray beam axis.
to the patient table (Figure 9.1). Using this configuration, several 10-bit sequences can be identified in the projection data for each X-ray source, as shown in Figure 9.4 (right). Therefore, for each X-ray source, enough pairs of $x$ and $\hat{x}$ can be identified for a stable solution to the projection matrix $P$.

Figure 9.4: Left: the modified Siemens PDS-2 phantom. Right: projection of the modified Siemens PDS-2 phantom.

Using the above method, we obtained the calibrated five-turn reverse helix in the world coordinate system, denoted as $(x_c, y_c, z_c)$; see Figure 9.5. Note that in the world coordinate system, this calibrated trajectory moves downwards opposite to the $z_c$-axis.

Figure 9.5: The calibrated five-turn reverse helical trajectory. Top left: orthogonal projection onto the $(x_c, y_c)$-plane; bottom left: orthogonal projection onto the $(x_c, z_c)$-plane; right: 3D view.
9.4 Data Preprocessing

In practice, the geometry and physics of X-ray attenuation are complex and the Beer-Lambert law is not a perfect model for the raw data. For example, projection data may be truncated and the X-ray energy is polychromatic. Therefore, various data corrections are needed for a better image reconstruction. These data corrections should be applied to the approximation of the line integral of the linear attenuation coefficients, which can be obtained from the raw data through a logarithm operation. More specifically, these corrections include overexposure correction, scatter correction, beam hardening correction and truncation correction.

In our experiments, we have performed the first three corrections and discarded the truncation correction. Data truncation was discarded for the following reasons. For the CATPHAN phantom and the Siemens CB phantom, both phantoms were entirely in the FOV, hence no data truncation occurred. For the torso phantom, the SAWBONES spine was entirely within the FOV, and the rest of the torso was made of foam, hence data truncation was negligible.

The first data correction applied to the approximation of the line integral was overexposure correction. Overexposure occurs when a high level of radiation saturates the 16-bit flat panel detector. This incident produces low frequency artifacts such as capping and cupping artifacts, which reduce the visibility of low contrast objects. Overexposure correction aims to reduce these low frequency artifacts and improve the image quality.

Following overexposure correction, scatter correction was applied. Scatter happens when an X-ray photon interacts with matter and deviates from its transmission line. This is another physical phenomenon that violates the Beer-Lambert law. X-ray scatter can be generated in two fundamental ways: the Compton effect and coherent scattering. Scatter can introduce several artifacts to image reconstruction results, such as cupping artifacts, streaks and shadows. More importantly, scatter increases noise and reduces contrast in regions of soft tissue. Hence, scatter correction can significantly improve image quality. Typically, to perform scatter correction, a scatter estimation model and a scatter correction algorithm are needed. As mentioned previously, our raw data were preprocessed using the Siemens pipeline, and the corresponding estimation model and correction algorithm are similar to that described in [Zell05].

Finally, beam hardening correction was performed. In practice, X-rays are polychromatic, and an attenuation coefficient of any material varies for different levels of energy. Moreover, for such a material, the attenuation coefficient is larger at low energy than at high energy. Therefore, the longer the distance that X-ray photons travel, the more X-ray photons of low energy will be absorbed; in other words, the X-ray beam becomes harder. Typically, beam hardening can produce cupping and streak artifacts. Several approaches are available for beam hardening correction, such as “water correction” and the “dual-energy approach”. For details of the approach that was used in our experiments, see [Zell05].
In our experiments, the above three data corrections were applied to all the projection data using the Siemens pipeline. Several corrected projections of the torso phantom with a SAWBONES spine are illustrated in Figure 9.6.

Figure 9.6: Illustration of the preprocessed projections of the torso phantom. The preprocessing involved three steps, i.e., overexposure correction, scatter correction and beam hardening correction. From top to bottom: the first to the fifth helical turn. Each column presents projections of the same polar angular position. The angular distances from the second, third and fourth columns to the first column are 80°, 160° and 230°, respectively.
9.5 Image Reconstruction

In this section, we describe how we perform image reconstruction using the calibrated trajectory and the preprocessed projection data. As mentioned at the beginning of this chapter, we selected Fusion-RFDK for image reconstruction. To apply this reconstruction method, the calibrated trajectory should be registered to a more convenient Cartesian coordinate system and the preprocessed data should be rebinned onto an ideal reverse helical trajectory. We call these two steps trajectory registration and trajectory fitting, respectively. In the rest of this section, we will first clarify these two steps, then briefly review the Fusion-RFDK method.

9.5.1 Trajectory Registration

Trajectory registration aims to transform the calibrated trajectory from the world coordinate system to another Cartesian coordinate system called the T coordinate system, in which the Fusion-RFDK can be easily implemented using the expression in Chapter 7.

This transformation between the two Cartesian coordinate systems is necessary. Recall from Chapter 7 that the formulae of Fusion-RFDK is defined for the reverse helical trajectory whose rotational axis is centered on one of the axes of the Cartesian coordinate system. However, the rotational axis of the calibrated trajectory was impossible to be centered on the \( z \)-axis of the world coordinate system, which was defined by the manually placed calibration phantom. Hence, the Fusion-RFDK method could not be directly used for the calibrated reverse helix in the world coordinate system, and a trajectory registration is needed.

Let \((x, y, z)\) and \(x_0\) be the axes and the origin of the T coordinate system, respectively. Let \(e_x, e_y\), and \(e_z\) be the orthonormal vectors that span this coordinate system. Using the calibrated trajectory, we try to find \(e_x, e_y, e_z\) and \(x_0\) that satisfy three requirements in the T coordinate system: (i) the rotational axis of the reverse helix lies on the \( z \)-axis, i.e., the projections of the vertex points onto the \((x, y)\)-plane form a curve that is as close to a circular arc as possible, (ii) the translation of the reverse helix is along the \(e_z\) direction, and (iii) the first source position of the first helical turn lies on the \(x\)-axis. In the rest of this subsection, we explain how we obtained \(e_x, e_y, e_z\) and \(x_0\).

First, we explain how to obtain \(e_z\). To start, we introduce some notations. Let \(a_i(\lambda_k)\) be the vertex point on the \(i\)-th helical turn at angular position \(\lambda_k\), with \(i = 1, 2, 3, 4, 5\) and \(k = 1, \ldots, N\). Here, \(N\) is the number of views per helical turn. In our experiments, \(N\) is equal to 681. Next, we define \(w_{ijk} = (w_{ijkx}, w_{ijky}, w_{ijkz})^T\) as the vector that points from \(a_j(\lambda_k)\) to \(a_j(\lambda_k)\), i.e.,

\[
    w_{ijk} = a_j(\lambda_k) - a_j(\lambda_k).
\]

It can be observed that, if the \(i\)-th and \(j\)-th helical turns have the same rotational direction, then \(w_{ijk}\) is expected to be parallel to the rotation axis of the reverse helix. Mathematically, this observation means that the cross product of \(w_{ijk}\) and \(e_z\) is expected to be 0. Also note
that $\mathbf{e}_z$ is a unit vector, and we have the constraint that $\mathbf{e}_z \cdot \mathbf{e}_z = 1$. Based upon the above two observations, $\mathbf{e}_z$ can be obtained using the following expression

$$
\mathbf{e}_z = \arg\min_{\mathbf{e}_z} \left\{ \sum_{(i,j)} \sum_{k=1}^{N} \left| \mathbf{w}_{ijk} \times \mathbf{e}_z \right|^2 - \chi \left( \mathbf{e}_z \cdot \mathbf{e}_z - 1 \right) \right\}, \quad (9.2)
$$

where $\chi$ is a Lagrange multiplier, and $\Psi = \{(1,3), (1,5), (3,5), (2,4)\}$. The choice on $\Psi$ is based on the fact that each pair of helical turns used for Equation (9.2) should have the same rotational direction. Then, $\mathbf{e}_z$ is defined as the unit vector that minimizes the above objective function. By taking the gradient of the expression inside the big curly bracket of Equation (9.2) with respect to the components of $\mathbf{e}_z$ and setting the result to zero, we obtain the relation below

$$
A \mathbf{e}_z = \chi \mathbf{e}_z,
$$

where

$$
A = \sum_{(i,j)} \sum_{k=1}^{N} \begin{bmatrix}
    w_{ijk}^2 + w_{ijk}^2 & -w_{ijk} w_{ijk} & -w_{ijk} w_{ijk} \\
    -w_{ijk} w_{ijk} & w_{ijk}^2 + w_{ijk}^2 & -w_{ijk} w_{ijk} \\
    -w_{ijk} w_{ijk} & -w_{ijk} w_{ijk} & w_{ijk}^2 + w_{ijk}^2
\end{bmatrix}.
$$

(9.3)

Therefore, $\mathbf{e}_z$ is an eigenvector of $A$.

There are three eigenvectors of $A$. Among the three vectors, we should choose the one that minimizes the right side of Equation (9.2) For each eigenvector, we have the freedom to choose its sign. In our experiments, we select $\mathbf{e}_z$ such that it is the direction of translation of the reverse helix. Mathematically, we require $\mathbf{e}_z$ to satisfy the following relation:

$$
\sum_{(i,j)} \sum_{k=1}^{N} w_{ijk} \cdot \mathbf{e}_z > 0.
$$

Next, we explain how to calculate $\mathbf{e}_x$ and $\mathbf{e}_y$. Recall that the first source position of the first helical turn should lie on the $x$-axis. Therefore, the unit vector for the $x$-axis can be obtained as below

$$
\mathbf{e}_x = \frac{\mathbf{a}_1(\lambda_1) - \left( \mathbf{a}_1(\lambda_1) \cdot \mathbf{e}_z \right) \mathbf{e}_z}{|| \mathbf{a}_1(\lambda_1) - \left( \mathbf{a}_1(\lambda_1) \cdot \mathbf{e}_z \right) \mathbf{e}_z ||}.
$$

Also, because $\mathbf{e}_x, \mathbf{e}_y$ and $\mathbf{e}_z$ are orthonormal, we have

$$
\mathbf{e}_y = \mathbf{e}_z \times \mathbf{e}_x.
$$

Finally, we demonstrate how to get the origin of the T coordinate system, $\mathbf{x}_0$. Let $\mathbf{x}_0 = (x_0, y_0, z_0)^T$ and let $\mathbf{a}_l(\lambda_k) = (x_{ik}, y_{ik}, z_{ik})^T$. To make sure that the reverse helix starts from the $x$-axis of the T-coordinate, we define

$$
z_0 = \mathbf{a}_1(\lambda_1) \cdot \mathbf{e}_z.$$

Also recall that the projections of the vertex points onto the \((x, y)\)-plane should form a curve that is, as much as possible, close to a circular arc. Let \(R\) be the radius of the reverse helix. Ideally, we would expect

\[(x_{ik} - x_0)^2 + (y_{ik} - y_0)^2 = R^2.\]

Let \(R' = R^2 - x_0^2 - y_0^2\) and apply the above equation to all the vertex points, we obtain the following equation

\[
\begin{bmatrix}
2x_{11} & 2y_{11} & 1 \\
2x_{12} & 2y_{12} & 1 \\
\vdots & \vdots & \vdots \\
2x_{21} & 2y_{21} & 1 \\
\vdots & \vdots & \vdots \\
2x_{31} & 2y_{31} & 1 \\
\vdots & \vdots & \vdots \\
\end{bmatrix}
\begin{bmatrix}
x_0 \\
y_0 \\
R' \\
\end{bmatrix}
= 
\begin{bmatrix}
x_{11}^2 + y_{11}^2 \\
x_{12}^2 + y_{12}^2 \\
\vdots \\
x_{21}^2 + y_{21}^2 \\
\vdots \\
x_{31}^2 + y_{31}^2 \\
\vdots \\
\end{bmatrix}.
\]

Therefore, \(x_0, y_0\) and \(R'\) can be obtained by the least-squares approach, which can be implemented using the singular-value-decomposition technique.

Now that we have clarified how to calculate \(e_x, e_y, e_z\) and \(x_0\), we are able to perform trajectory registration in two steps.

- Registration of the vertex points: Let \(a\) be the coordinates of a vertex point in the world coordinate system, and let \(b\) be the corresponding coordinates of \(a\) in the T coordinate system. Then \(b\) can be obtained by a translation and a rotation, as expressed below

\[
b = Q^T(a - x_0),
\]

where \(Q = [e_x, e_y, e_z]\).

- Registration of the detector orientation: Unlike the vertex points, the orientation of the detector in the T coordinate system can be obtained by applying a rotation, \(Q^T\), to the orientation of the detector in the world coordinate system. The translation vector \(x_0\) is not needed.

It is important to note that the registration process is an Euclidean transformation and does not change the relative position between detector and X-ray source. Therefore, the center of the detector and the source-to-detector distance for each view will be the same before and after the registration process.

The above method for trajectory registration fulfills the three requirements that were introduced at the beginning of this subsection; we illustrate the registered trajectory in Figure 9.7. In Figure 9.7(left), visual inspection indicates that the projection of the reverse helix onto the \((x, y)\)-plane forms a fine circular arc. Also, it can be observed that the center of this circular arc is very close to the center of the \((x, y)\)-plane, which means that the rotational axis of the reverse helix is close to the \(z\)-axis.
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Figure 9.7: Registered reverse helix. Left: projection of the registered trajectory onto the \((x, y)\)-plane. Right: projection of the registered trajectory onto the \((x, z)\)-plane.

Figure 9.8: Illustration of some parameters of the registered trajectory. The horizontal axis indicates the index of the source positions along the reverse helix. Top left: scan radius (mm); top right: source-to-detector distance (mm); bottom left: polar angles of source positions (degrees); bottom right: \(z\) coordinates of x-ray sources (mm).
The registered trajectory is close to a conventional reverse helix. To demonstrate that, we plot some parameters of the registered trajectory quantitatively in Figure 9.8. These parameters include the scan radius, the source-to-detector distance, the polar angle and the $z$ coordinate of each source position. It is observed that both the scan radius and the source-to-detector distance contain systematic deviations and statistical fluctuations. We believe that the statistical fluctuations came from our calibration process, whereas the systematic deviations stemmed from gravity. The bottom row of figure 9.8 demonstrates a good linear relation between the rotation and translation of the reverse helix.

Some inconsistencies were observed between Table 9.2 and Figures 9.7 and 9.8. Recall from Table 9.2 that the axial height of each helical turn was set to 60 mm. However, Figure 9.7 (right) shows that the axial heights of the first, third and fifth turns are larger than 60 mm, whereas those of the second and fourth turns are smaller than 60 mm. Also, in Figure 9.8 (bottom right), although the $z$ coordinate increases linearly with the source index, slight slope deviations can still be observed. These inconsistencies may be attributed to imperfection in our trajectory registration process. Indeed, we have observed in a numerical experiment that a deviation as small as 0.2° in the altitude angle of the rotational axis can yield such inconsistencies. Fortunately, these inconsistencies have little impact on the quality of this experiment.

### 9.5.2 Trajectory Fitting

To make the Fusion-RFDK method directly applicable, we performed trajectory fitting. First, we will find an ideal reverse helix such that it is as close as possible to the registered trajectory. Second, we will create the projections of the ideal reverse helix from the registered trajectory through a rebinning process.

Recall from Section 9.2 that the odd helical turns and the even helical turns were designed to have different axial heights. Therefore, it is not realistic to find an ideal reverse helix of constant pitch that fits the entire registered trajectory. Instead, we perform trajectory fitting for each helical turn individually. In the rest of this subsection, we will clarify the trajectory fitting procedure.

To start, we introduce some notations. For a given registered helical turn that is composed of $N$ vertex points, we denote the scan radius, the source-to-detector distance, the angular position and the $z$ coordinate of the $k$-th vertex point as $R_k$, $D_k$, $\lambda_k$ and $z_k$, respectively. Here, $k$ can be any integer between 1 and $N$. For the ideal helical turn, which is to be fitted onto the given registered helical turn, we denote the scan radius and source-to-detector distance as $R$ and $D$, respectively. We assume the ideal helical turn starts at angular position $\lambda_s$ with a uniform step size $d\lambda$. Moreover, we assume the ideal helical turn starts at axial position $z_s$ and translates $p$ mm per degree of rotation.

Now, we clarify how to perform trajectory fitting for a given registered helical turn. More specifically, we explain how to obtain $R$, $D$, $\lambda_s$, $d\lambda$, $z_s$ and $p$ in three steps.
• $R$ and $D$: We consider the scan radius as the average of scan radii of all the vertex points. Let $b_k = (b_{kx}, b_{ky}, b_{kz})^T$ be the $(x, y, z)$ coordinates of the $k$-th vertex position. Then $R$ can be calculated using the following equation

$$R = \frac{1}{N} \sum_{k=1}^{N} \sqrt{b_{kx}^2 + b_{ky}^2}. \quad (9.4)$$

Similarly, the source-to-detector distance is calculated as the average of the source-to-detector distances of all the vertex points. Mathematically, we have

$$D = \frac{1}{N} \sum_{k=1}^{N} D_k. \quad (9.5)$$

• $\lambda_s$ and $d\lambda$: The figure of merit to obtain these two parameters is the least square errors. Mathematically, we have

$$\{\lambda_s, d\lambda\} = \text{argmin}_{\{\lambda_s, d\lambda\}} \left\{ \sum_{k=1}^{N} \left( \lambda_k - \lambda_s - (k - 1) d\lambda \right)^2 \right\}. \quad (9.6)$$

By taking the partial derivatives of the right side of the above equation with respect to $\lambda_s$ and $d\lambda$ and setting the results to zero, we obtain the equation below

$$\begin{bmatrix} N & F_1 \\ F_1 & F_2 \end{bmatrix} \begin{bmatrix} \lambda_s \\ d\lambda \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix},$$

where

$$F_1 = \sum_{k=1}^{N} (k - 1), \quad F_2 = \sum_{k=1}^{N} (k - 1)^2, \quad T_1 = \sum_{k=1}^{N} \lambda_k \quad \text{and} \quad T_2 = \sum_{k=1}^{N} ((k - 1)\lambda_k).$$

It can be shown that

$$NF_2 - F_1^2 = \frac{N^2(N^2 - 1)}{12}. \quad (9.6)$$

Therefore, as long as $N > 1$, stable solutions of $\lambda_s$ and $d\lambda$ are assured.

• $z_s$ and $p$: We obtain $z_s$ and $p$ using the following equation

$$\{z_s, p\} = \text{argmin}_{\{z_s, p\}} \sum_{k=1}^{N} (z_k - z_s - p\lambda_k)^2. \quad (9.7)$$

By taking the partial derivatives of the right side of the above equation with respect to $z_s$ and $p$ and setting the results to zero, we obtain the equation below

$$\begin{bmatrix} N & H_1 \\ H_1 & H_2 \end{bmatrix} \begin{bmatrix} z_s \\ p \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix},$$
where
\[ H_1 = \sum_{k=1}^{N} \lambda_k, \quad H_2 = \sum_{k=1}^{N} \lambda_k^2, \quad G_1 = \sum_{k=1}^{N} z_k \quad \text{and} \quad G_2 = \sum_{k=1}^{N} z_k \lambda_k. \]

According to the Cauchy-Schwarz inequality [Stee04, Page 1], \( NH_2 \geq H_1^2 \). The equality only occurs when all \( \lambda_k \) are equal, which is not true in our case. Therefore, stable solutions of \( z_s \) and \( p \) are guaranteed.

Once these parameters, i.e., \( R, D, \lambda_s, d \lambda, z_s \) and \( p \), are found, the trajectory fitting is accomplished. In Figure 9.9 we plot the fitted trajectory against the registered trajectory for the first helical turn. As can be observed, the fitted trajectory and the registered trajectory are close to each other.

![Figure 9.9: Comparison between the fitted and registered trajectories using the first helical turn. Left: 3D view; right: \( z \) coordinate.](image)

Next, to create projection data for the fitted trajectory, we introduce a rebinning scheme, as shown in Figure 9.10. We start with some notations. Let \( c \) be a vertex point of the fitted trajectory, and let \( \alpha \) be a vector that belongs to the unit sphere. We denote the X-ray beam that points from \( c \) to the detector with direction \( \alpha \) as \( L(c, \alpha) \). This X-ray beam intersects the FOV, \( \Omega \), along a segment of line. We denote the middle point of this line segment as \( M \). Our goal is to create the projection datum for \( L(c, \alpha) \).

Let \( b_1 \) and \( b_2 \) be the two vertex points on the registered trajectory that are closest to \( c \). Also, let \( \alpha_1 \) and \( \alpha_2 \) be the unit vectors that point from \( b_1 \) and \( b_2 \) to \( M \), respectively. We denote the X-ray beams that point from \( b_1 \) and \( b_2 \) to \( M \) as \( L(b_1, \alpha_1) \) and \( L(b_2, \alpha_2) \), respectively. As we know, the line integrals along \( L(b_1, \alpha_1) \) and \( L(b_2, \alpha_2) \) can be obtained via bilinear interpolation in the detector. These two line integrals can be further linearly interpolated with respect to angular position to obtain the line integral along \( L(c, \alpha) \). One rebinned slice from the first helical turn of the fitted trajectory is shown in Figure 9.11. Note that the preprocessed projection is presented in the world coordinate system and the rebinned projection is presented in the T coordinate system. Therefore, the orientation of the spine in the left of Figure 9.11 is different from the right.
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Figure 9.10: Rebinning scheme for the fitted trajectory. The vertex point \( \mathbf{c} \) is located on the fitted trajectory. The X-ray beam that points from \( \mathbf{c} \) to the detector with direction \( \mathbf{\alpha} \) is denoted \( \mathcal{L}(\mathbf{c}, \mathbf{\alpha}) \). This beam intersects the FOV, \( \Omega \), along a segment of line whose middle point is denoted as \( M \). Here, \( \mathbf{b}_1 \) and \( \mathbf{b}_2 \) are two points on the registered trajectory that are closest to \( \mathbf{c} \), and \( \mathbf{\alpha}_1 \) and \( \mathbf{\alpha}_2 \) are unit vectors that point from \( \mathbf{b}_1 \) and \( \mathbf{b}_2 \) to \( M \), respectively.

Figure 9.11: Projection of the torso phantom from the 50-th view point of the first helical turn. Left: preprocessed projection of the calibrated trajectory; right: rebinned projection of the fitted trajectory.

9.5.3 The Fusion-RFDK method

For the fitted trajectory, the Fusion-RFDK method can be directly applied to the rebinned data. In this subsection, we briefly review the Fusion-RFDK method; for more details, see Chapter 7.

The Fusion-RFDK method consists of 4 steps, including (i) CB length correction and Parker-like weighting, (ii) ramp filtering along the \( u \)-axis, (iii) backprojection, and (iv) fusion. For a given fitted reverse helix composed of 5 turns, image reconstruction can be obtained as follows. First, for each helical turn, one volume should be reconstructed using the ramp-kernel-based FDK method, i.e., using the first three steps of the Fusion-RFDK method. As a result, 5 volumes will be reconstructed. These 5 volumes are aligned along the \( z \)-axis in order, and each pair of successive volumes share a common region called the...
fusion zone. Finally, these 5 volumes can be combined by a fusion process; as shown in Figure 7.1.

In practice, two aspects deserve special attention. First, for any two adjacent helical turns, the kink plane needs to be specified. By definition, the kink plane is perpendicular to the \( z \)-axis and goes through the intersection of these two adjacent helical turns. However, this intersection does not exist due to finite sampling. To overcome this problem, we select the closest two endpoints of these two adjacent helical turns, and define the average of the \( z \) coordinates of these two endpoints as the \( z \) position of the kink plane.

Second, the fusion length also needs to be specified. As a reminder, we restate some notations as follows. Let \( R \) and \( r \) be the radii of the reverse helix and the FOV, respectively. Let \( D \) be the source-to-detector distance, and let \( H_F \) be the fusion length. Also, we refer to \( H_{\text{max}} \) as the maximum axial height of all the helical turns that form the reverse helix, and refer to \( H_d \) as the height of the detector along the patient table. According to the CB scan geometry (Chapter 5), it is not difficult to find the restriction of \( H_F \) as below

\[
H_F \leq \frac{(R - r)H_d}{D} - 2H_{\text{max}}. \tag{9.8}
\]

### 9.6 Results

In this section, we present the reconstruction results obtained from Fusion-RFDK using the rebinned projections in the T coordinate system. To verify the accuracy of these results, we will also display reconstruction results obtained from a commercial CT system as reference images.

For implementation of the Fusion-RFDK, we employed the hamming window for apodization in the ramp filter and utilized a fusion length of 30 mm. This fusion length was calculated using Equation 9.8. Recall from Figure 9.8 that the scan radius, \( R \), and the source-to-detector distance, \( D \), were about 784 mm and 1195 mm, respectively. Figure 9.7 indicates that the maximum axial height of all 5 helical turns, \( H_{\text{max}} \), was about 65 mm and, as listed in Table 9.1, the axial height of the detector, \( H_d \) was 300 mm. By construction, the radius of the FOV was about 130 mm. Replacing these values in Equation 9.8, the fusion length had to be less than 32.2 mm. We chose the fusion length to be 30 mm.

We reconstructed three volumes for the torso phantom, the CATPHAN phantom and the Siemens CB phantom, respectively. These volumes were of the same voxel sizes as those of the CT images. In particular, the reconstruction of the torso phantom consists of \( 200 \times 200 \times 1021 \) voxels of size 0.7910 mm \( \times \) 0.7910 mm \( \times \) 0.3 mm; see Figure 9.12. The reconstruction of the CATPHAN phantom is composed of \( 478 \times 478 \times 1021 \) voxels of size 0.3770 mm \( \times \) 0.3770 mm \( \times \) 0.3 mm (Figure 9.13), whereas the reconstruction of the Siemens CB phantom is composed of \( 512 \times 512 \times 1021 \) voxels of size 0.3770 mm \( \times \) 0.3770 mm \( \times \) 0.3 mm (Figure 9.14). Note that, although the CATPHAN phantom and the Siemens CB phantom were combined to form a long object for scanning, the reconstructions were performed using different voxel sizes in order to match the voxel sizes of the CT images.
Chapter 9. The First Experimental Results of Extended-Volume C-arm CT

Figure 9.12: Reconstruction results of the torso phantom. Left column: reconstruction results obtained from the laboratorial floor-mounted Artis zeego C-arm system using the Fusion-RFDK method; right column: reconstruction results obtained from the Somatom Definition AS CT system using the Syngo CT software of version 2011A. Top row: sagittal view; bottom row: transverse view. Display window: $[-1000, -200]$ HU.

The CT reference images were generated by the software called Syngo CT 2011A using projection data from the Somatom Definition AS CT system (Siemens AG, Healthcare, Erlangen, Germany). Note that, for a fair visual inspection, the linear attenuation coefficients of the reconstructions from the Fusion-RFDK method were brought to the same level as that of the CT images using a linear mapping. Be aware that the images from the two modalities are not registered to the same coordinate system; we only use CT images for verification of the anatomy of the phantoms.

Overall, Figures 9.12, 9.13 and 9.14 demonstrate good agreement between our reconstruction results and the CT images. In Figure 9.12 fine details of the torso phantom were well preserved. Figure 9.13 shows high image resolution, whereas Figure 9.14 demonstrates good low-contrast detectability. On the other hand, we found that all reconstruction results from the Fusion-RFDK overall contained more CB artifacts and noise than the CT images. In Figures 9.13 and 9.14, we also observe capping and cupping artifacts.
Figure 9.13: Reconstruction results of the CATPHAN phantom. Left column: reconstruction results obtained from the laboratorial floor-mounted Artis zeego C-arm system using the Fusion-RF DK method; right column: reconstruction results obtained from the Somatom Definition AS CT system using the Syngo CT software of version 2011A. Display window for the top row: $[-500, 500]$ HU. Display window for the bottom row: $[-500, 2000]$ HU.
Figure 9.14: Reconstruction of the Siemens CB phantom. Left column: reconstruction results obtained from the laboratorial floor-mounted Artis zeego C-arm system using the Fusion-RFDK method; right column: reconstruction results obtained from the Somatom Definition AS CT system using the Syngo CT software of version 2011A. Display window: $[-300, 500]$ HU.
9.7 Conclusion and Discussion

We have demonstrated that extended-volume imaging using a reverse helix in the interventional room is feasible on a C-arm system with large motion flexibility such as the laboratorial Artis zeego system. The preliminary reconstruction results show good agreement with the images obtained from a commercial CT system, and hence are generally encouraging. Note that the laboratorial Artis zeego system is capable of producing a reverse helix consisting of more than 5 turns, and thus imaging a volume of longer axial length is possible.

To address the real data for image reconstruction, we have provided a practical procedure. First, the raw projection data were preprocessed by the Siemens software, and the geometrical information of the reverse helix was obtained through a calibration process. Second, the calibrated trajectory was registered to the T coordinate system by an Euclidean transformation. Third, an ideal reverse helix was fitted onto the registered trajectory and the corresponding projection data were obtained through a rebinning process. Finally, reconstruction results were obtained with the Fusion-RFDK method.

Reconstructions obtained from the above procedure are encouraging. They demonstrate high image resolution and good low-contrast detectability. These positive properties are partially due to the flat panel detector. On the other hand, because the flat panel detector only employs a collimator in one dimension for scatter suppression, our reconstruction results look noisier than the CT reference images. We also observed several types of artifacts. The CB artifacts were present due to the nature of the FDK-type reconstruction algorithms, which only produce approximate results whenever a cone angle exists. For the capping and cupping artifacts, we believe they can be suppressed by a better experimental setup and a more dedicated data correction process.

The proposed procedure in this chapter addresses the real data in such a way that the Fusion-RFDK method can be applied without any change. This procedure requires a rebinning process of the projection data from the registered trajectory to the fitted ideal trajectory, which can cause resolution loss. To overcome this problem, an alternative approach would be to modify Fusion-RFDK such that it allows direct usage of the preprocessed projection data for image reconstruction. Also, recall from Chapter 7 that we have another candidate for image reconstruction, i.e., Fusion-HFDK. Comparing the reconstruction results of Fusion-RFDK and Fusion-HFDK would be an interesting topic for future research.
Chapter 10

Summary and Outlook

*If we knew what we were doing, it wouldn’t be called research, would it?*

Albert Einstein

10.1 Summary

In this thesis, we have performed an extensive study on a new imaging technique: extended-volume C-arm CT. This study aimed to develop a practical solution for the long-object imaging problem using a C-arm system. In particular, the study consisted of three parts: (i) design and analysis of novel data acquisition geometries, (ii) development of original trajectory-specific reconstruction schemes, and (iii) the first experimental demonstration of extended-volume C-arm CT using a state-of-the-art C-arm system.

In the first part, we studied three source trajectories with respect to their geometry and R-line coverage: the reverse helical trajectory, the arc-extended-line-arc trajectory (AELA) and the ellipse-line-ellipse (ELE) trajectory. AELA and ELE are novel trajectories that were proposed for the first time in this thesis. Several criteria were used for the source trajectory design, and two of them deserve particular attention. First, a C-arm system cannot rotate non-stop in a single direction, because it has an open design and is lack of the slipping technology. Hence, the source trajectory should have a reverse pattern in its rotational direction. Second, for an efficient theoretical-exact and -stable (TES) image reconstruction using axially truncated projection data, it is preferred that the source trajectory has sufficient R-line coverage in its ROI. The first criterion provided a direct guide for creating the trajectory geometry, whereas the second criterion required thorough investigations on the spatial distribution of R-lines once the trajectory geometry was created.

We started by investigating the reverse helical trajectory, which possesses a reverse pattern in its direction of rotation. This property makes the reverse helix feasible on a C-arm system and thus a competitive candidate for extended-volume C-arm CT. On the other hand, to perform an efficient TES image reconstruction with axially truncated data is
challenging for the reverse helix, because this trajectory does not provide sufficient R-line coverage around the center of its convex hull. From a practical viewpoint, an approximate reconstruction algorithm with acceptable CB artifacts might be an alternative solution.

To pursue a trajectory that has sufficient R-line coverage in the ROI, we then proposed a new source trajectory called AELA. We performed a thorough analysis on the R-line coverage of this trajectory, and demonstrated that sufficient R-line coverage in the ROI can be guaranteed by a simple and practical geometrical configuration. Moreover, the AELA trajectory has a reverse pattern in the rotational direction and can be repeated along the center axis. However, the AELA trajectory is not smooth and continuous because the endpoints of the extended line are not connected to the endpoints of the circular arcs. This shortcoming requires long pauses in X-ray exposure and increases scan time, which is sub-optimal, for example, in vascular interventions.

To further improve the data acquisition geometry for efficient TES reconstruction with axially truncated data, we later proposed the ELE trajectory. The ELE trajectory not only preserves all the advantages of the AELA trajectory, but is also smooth and continuous since the endpoints of the line are connected to one of the endpoint of each elliptical arc. Similar to the AELA trajectory, we also investigated the R-line coverage of the ELE trajectory thoroughly. This investigation offered full understanding about the spatial distribution of the R-lines, and identified conditions that guarantee full R-line coverage in the ROI.

In the second part, we considered trajectory-specific reconstruction methods. The reverse helix and the ELE trajectories were considered: the former was chosen due to its attractive mechanical motion for the C-arm system, and the latter was selected because it is smooth and continuous and has sufficient R-line coverage in the ROI. For the reverse helical trajectory, we proposed three approximate, FDK-type reconstruction algorithms, whereas for the ELE trajectory, we suggested a practical, TES reconstruction scheme using the differentiated-backprojection (DBP) method, which efficiently utilizes the R-lines of the ELE trajectory. The above four reconstruction methods were validated using computer-simulated data.

The three FDK-type reconstruction methods for the reverse helix are called Fusion-RFDK, Fusion-HFDK and Voxel-Dependent-HFDK (VD-HFDK). Although these methods are approximate, they are efficient, stable and straightforward to implement. Computer simulation results demonstrated that these methods are capable of producing acceptable reconstructions for extended-volume C-arm CT. Compared to the exact methods that were separately proposed by F. Noo et al. [Noo09] and S. Cho et al. [Cho10], these approximate reconstruction algorithms are less computationally costly and do not suffer from resolution compatibility issues. The resolution compatibility problem arises for the exact reconstruction algorithms because they require combining volumes of different shift-variant resolutions, which tends to yield undesirable artifacts.

Among the three approximate reconstruction algorithms, VD-HFDK is less preferred. VD-HFDK makes proficient use of the $\pi$-lines of the reverse helix, and hence is capable of producing almost-exact image reconstruction in the $\pi$-line regions. However, these $\pi$-line regions only take a small portion of the target volume and thus do not overall con-
tribute much for CB artifacts suppression. On the other hand, to make proficient use of the π-lines, the VD-HFDK method requires a voxel-dependent weighting scheme and a voxel-dependent backprojection range, which significantly increase the computational complexity.

Both Fusion-RFDK and Fusion-HFDK have their own advantages. Fusion-RFDK is efficient and easy to implement, because it employs the classical ramp filter that has a mature implementation in industry. In contrast, Fusion-HFDK decomposes the ramp filter into the view-dependent differentiation and the Hilbert transform. The view-dependent differentiation is relatively new and difficult to implement, but it induces an oblique component in filtering, which can suppress CB artifacts. Additionally, in the backprojection step, Fusion-RFDK uses the second order of the distance weighting, whereas Fusion-HFDK uses the first order of the distance weighting. The latter is more effective in noise suppression. In short, Fusion-RFDK is efficient and relatively easier to implement, but Fusion-HFDK produces slightly less image bias and image noise.

Fusion-RFDK and Fusion-HFDK share the same fusion concept, which has a significant impact on image quality. In particular, increasing the axial length of the fusion zone can help reduce image bias and image noise. However, it also increases detector requirement. Therefore, a balance needs to be achieved between image quality and detector requirement when it comes to the selection of the fusion length. This balance should be application-dependent.

For the ELE trajectory, we proposed a practical scheme for efficient TES reconstruction. This scheme employs the DBP method and provides an efficient methodology for R-line selection. This methodology was designed based upon the thorough knowledge of the spatial distribution of the ELE R-lines.

In particular, the R-line selection methodology chooses a unique R-line for each point of interest (POI) such that the backprojection range is the shortest. Using this criteria, the largest backprojection range for the entire volume of interest only contains one T-arc, one T-line and a small portion of the other T-arc that is connected to the T-line. Although it might not be optimal for noise suppression, this R-line selection methodology does not require a weighting function for redundancy handling, and helps reduce detector requirement and sensitivity of object motion.

For an efficient implementation, the selected R-lines are grouped in subsets called R-line surfaces. The DBP method then performs image reconstruction on these R-line surfaces. We obtain the final reconstruction results from these R-line surfaces by a rebinning process. Note that, the concepts of the R-line surfaces and the rebinning process significantly improve computational efficiency.

We evaluated the reconstruction scheme of the ELE trajectory using computer simulations with a modified version of the FORBILD head phantom. Except for some aliasing errors and non-uniform noise, the reconstruction results were of high quality.

In the third part, we presented the first experimental results for extended-volume C-arm CT using a five-turn reverse helix that was successfully implemented on a laboratorial Artis zeego C-arm system (Siemens AG, Healthcare, Erlangen, Germany). First, to assess the
geometrical information for each vertex point on the reverse helix, a practical calibration method was proposed. This trajectory calibration can be viewed as an extension of the method that was designed for a circular trajectory [Stro03]. For noise and artifact reduction, the raw projection data were preprocessed using classical data corrections. Second, we found an ideal reverse helix that best fitted the calibration trajectory through an optimization procedure, then we created the projection data for this ideal reverse helix through a rebinning process. Finally, reconstruction results were obtained from the ideal reverse helix using the Fusion-RFDK method. These reconstruction results show good agreement with images obtained from a commercial CT system. These encouraging experimental results strongly demonstrate feasibility of extended-volume C-arm CT in the interventional room.

In summary, this thesis addressed a significant number of problems that covers a wide range of aspects for extended-volume C-arm CT. More specifically, this thesis presented two novel and attractive data acquisition geometries, four effective and geometry-specific reconstruction algorithms, and a successful real data demonstration with a laboratorial C-arm system. We believe all these original contributions are an important step forward for extended-volume C-arm CT to be installed in the interventional room.

10.2 Outlook

As could be expected from the nature of scientific work, the results in this thesis open a lot of opportunities for future research. In the following, insight will be provided on two aspects, which we think may be exciting and valuable for further investigation.

First, for the ELE trajectory, the reconstruction scheme proposed in Chapter 8 has a great potential for improvement. On one hand, this scheme only employs one R-line for image reconstruction and ignores other R-lines, if there is any. A more sophisticated scheme addressing the redundancy of the R-lines of the ELE trajectory may be beneficial for noise suppression. Moreover, the corresponding effects on detector requirement and motion artifacts are also of great interest. On the other hand, the reconstruction scheme only uses the DBP method, but other R-line based reconstruction algorithms such as the Katsevich’s algorithm [Kats02] are also attractive. It is important to perform image reconstructions using these candidates and compare their results with that presented in Chapter 8.

Second, it has been shown that the ELE trajectory is an exciting geometry for extended-volume C-arm CT. It will be valuable to perform a real data experiment on a commercial C-arm using this geometry, and compare the results with that from the reverse helix. To perform such an experiment, several aspects have to be addressed. The first question is how to implement this trajectory. A decision will have to be taken on whether or not using the patient table for realization of the line segment. Next, a geometry-specific calibration method will be needed. This method may be obtained by extending the method that was used for the reverse helix. At the end, a practical reconstruction method has to be chosen. This choice may be better made when the comparison of reconstruction algorithms for the ELE trajectory is available, as suggested in the previous paragraph.
Appendix A

Proofs and Derivations

A.1 Proof for Theorem 1

This section is dedicated to prove Theorem 1 which appears in Section 5.1.2. To prove the theorem, we will need the following lemma. Because this is trivial, we will omit the proof.

Lemma 3. Let $\Pi_1$ and $\Pi_2$ be two parallel planes, and let $\Pi_3$ be another plane that is not parallel to $\Pi_1$. Then the intersection between $\Pi_3$ and $\Pi_1$ is parallel to the intersection between $\Pi_3$ and $\Pi_2$.

Now we prove Theorem 1. As illustrated in the left of Figure 5.2, the intersecting plane is denoted as $\Pi_z$, the vertex of the partial cone surface as $A$, and the center of the base arc as $O_b$. Furthermore, let $B$ and $C$ be the endpoints of the base arc, and let $O'_b$ be the point where $\Pi_z$ intersects the line from $A$ to $O_b$. From basics of Euclidean geometry, we know that the intersection between $\Pi_z$ and the partial cone surface is an arc (which may be either circular or elliptical). This arc is denoted as $\text{Arc}(O'_b)$. Also, any arbitrary point, $D$, on the base arc can be mapped to a unique point, $F'$, on $\text{Arc}(O'_b)$, and vice-versa. Point $F'$ is simply found as the intersection between $\Pi_z$ and the line from $A$ to $D$. Again, see left of Figure 5.2.

By construction, points $A$, $F'$, $D$, $O_b$ and $O'_b$ are coplanar and define a plane $\Pi$, whose intersections with the plane of the base arc and $\Pi_z$ are the lines $O_bD$ and $O'_bF'$, respectively. By Lemma 3 the line $O_bD$ is parallel to the line $O'_bF'$, i.e.,

$$O_bD \parallel O'_bF'.$$  \hspace{1cm} (A.1)

Therefore, the triangle $AO_bD$ is similar to the triangle $AO'_bF'$, and hence,

$$\frac{|O'_bF'|}{|O_bD|} = \frac{|AO'_b|}{|AO_b|}. \hspace{1cm} (A.2)$$

Now, we isotropically scale the base arc by a scaling factor $|AO'_b|/|AO_b|$ relative to its center, $O_b$, to get the arc connecting the points $G$, $F$ and $E$, with the scaling mapping the points $B$, $D$ and $C$ to the points $G$, $F$ and $E$, respectively. Hence, the point $F$
lies on the line $O_bD$, and Equation A.1 yields the relation $O_bF \parallel O'_bF'$. Since, by definition, $|O_bF| = |O_bD| \cdot |AO'_b|/|AO_b|$, we also have $|O_bF| = |O'_bF'|$ from Equation A.2. Consequently, $O_bF = O'_bF'$, which implies that $FF' = O_bO'_b$. Recall that $D$ was chosen arbitrarily. Therefore, any point on $Arc(O'_b)$ can be obtained as a translation of a point on the arc connecting $G$, $F$ and $E$ by the vector $O_bO'_b$. Because the arc connecting $G$, $F$ and $E$ is a scaled copy of the base arc, with the scaling being isotropically applied relative to $O_b$, the theorem is proven.

A.2 Proof for Lemma II

In this section, we offer a proof for Lemma II, which appears in Section 5.2.2. For this proof, we refer to Figure A.1.

Let $Arc(O)$ be a circular or elliptical arc centered at $O$ in plane $\Pi$, and $A$ is a point outside this plane. Let $C$ be an endpoint of $Arc(O)$. We consider a trajectory that consists of $Arc(O)$ and the line that connects $A$ and $C$, and a partial cone surface that is formed by connecting $A$ to all the points on $Arc(O)$. The convex hull of this partial cone surface is bounded by the partial cone surface, plane $\Pi$ as well as the plane that goes through $A$, $B$ and $C$.

Let $P$ be an arbitrary point within the convex hull of the partial cone surface from $A$. Points $A$, $C$ and $P$ define a plane $\zeta$ that intersects $Arc(O)$ at the point $W$. Thus, the triangle defined by points $A$, $C$ and $W$ is the intersection between the convex hull and $\zeta$, and this triangle contains $P$. Consequently, the line through $W$ and $P$ must intersect the line connecting $A$ to $C$ at some point $V$ which is between points $A$ and $C$. Since the points $V$ and $W$ both belong to the source trajectory, $P$ lies on an R-line, and the lemma is proven.
A.3 Proof for Theorem 2

This appendix provides a proof for Theorem 2, which was stated in Section 5.2.3. For this proof, refer to Figure A.2. Consider an arbitrary pair of \((x, y)\) coordinates, \((x_0, y_0)\), such that \(x_0^2 + y_0^2 \leq R^2_m\), and let \(L(x_0, y_0)\) be the line parallel to the z-axis through \((x_0, y_0, 0)\). We define a function \(z(\delta)\) as follows. First, let \(A(\lambda e - \delta)\) be the point at polar angle \(\lambda e - \delta\) on the upper T-arc, and let \(\Pi\) be the plane defined by \(A(\lambda e - \delta)\) and \(L(x_0, y_0)\). Second, let the point \(A(\lambda e - \delta)\) be the intersection of \(\Pi\) with the lower T-arc. Denote the R-line connecting \(A(\lambda e - \delta)\) and \(A(\lambda e - \delta)\) as \(L\). Because \(L\) and \(L(x_0, y_0)\) are coplanar and not parallel, they have an intersection at some point \((x_0, y_0, z(\delta))\).

As \(\delta\) increases from zero, \(A(\lambda e - \delta)\) moves towards \(A(\lambda e)\), so that \(z(\delta)\) is a well-defined function only for \(\delta \in [0, \delta_m]\), where \(\delta_m\) is such that \(A_A(\lambda e - \delta_m)\) is the same point as \(A(\lambda e)\). Note that \((x_0, y_0, z(\delta_m))\) is thus the intersection of \(L(x_0, y_0)\) with the R-line connecting \(A(\lambda e)\) and \(A_A(\lambda e - \delta_m)\). Because the upper and lower T-arcs are connected curves, function \(z(\delta)\) is continuous.

Let \(\tilde{z} = \min_{\delta \in [0, \delta_m]} z(\delta)\). It follows that all points \((x_0, y_0, z)\) with \(z \in [\tilde{z}, z(\delta_m)]\) are covered by R-lines with endpoints on the upper T-arc and the lower T-arc. Now, observe that all points \((x_0, y_0, z)\) with \(z \in [\delta_m, H]\) are covered by R-lines with endpoints on the upper T-arc and the portion of the T-line that is between the two T-arcs, the endpoint on the upper T-arc being always \(A_A(\lambda e - \delta_m)\). Therefore, all points \((x_0, y_0, z)\) with \(z \in [\tilde{z}, H]\) are covered by R-lines. By the symmetry of the ALA trajectory, all points \((x_0, y_0, z)\) with \(z \in [-\tilde{z}, -H]\) are also covered by R-lines.
A.4 Derivation of the Minimum T-Line Extension for the AELA Trajectory

In this section, we derive the results in Section 5.3 expressed by Equations 5.8-5.12 for the minimum T-line extension, $\Delta H$, needed to guarantee that the AELA trajectory covers the ROI with R-lines.

We start by proving Equation 5.8 for the case $\lambda^*_m \leq \lambda_m \leq \lambda_c$; see Figure 5.13(a). Consider the plane, $\Pi$, containing the z-axis and the point $A^\lambda_{0s}$. As depicted in the left of Figure A.3, this plane contains the start points of the T-arcs, $A^\lambda_{s-}$ and $A^\lambda_{s+}$, together with the points $A^\lambda_{0s}$, $O$, and $D$ that are shown in Figure 5.13(a). Also, Figure A.3 labels the intersection of $\Pi$ with the lower T-arc as $A^\lambda_{s-+\pi}$ and the upper endpoint of the extended T-line as $E_+$. Since the triangle $A^\lambda_{s-+\pi}O_+A^\lambda_{s-}$ is similar to the triangle $DE_+A^\lambda_{0s}$, we have

$$\frac{|A^\lambda_{s-+\pi}E_+|}{|A^\lambda_{s-+\pi}A^\lambda_{s-}|} = \frac{|A^\lambda_{0s}E_+|}{|DA^\lambda_{0s}|},$$

which simplifies to

$$\frac{2H + \Delta H}{2R} = \frac{H + \Delta H}{r + R}.$$

Some algebra then yields

$$\frac{\Delta H}{2H} = \frac{r}{R - r},$$

which is the desired equation. The second case in Equation 5.8 for $\lambda_c < \lambda_m \leq 2\pi$ is proven similarly, with $D$ replaced by $F$ in the left of Figure A.3.

Next, we derive Equations 5.9-5.12 which describe $r_x = |OF|$. For this derivation, see the right of Figure A.3 which refers to several points in Figure 5.13(c), and also labels the
angles $\alpha$, $\beta$, $\gamma$, and $\eta$. First, note that $r_x = |A_0^\lambda F| - |A_0^\lambda O| = |A_0^\lambda F| - R$. Substituting the relation $|A_0^\lambda F| = |A_0^\lambda K|/\cos \gamma$ and defining $d = |A_0^\lambda K|$, we obtain

$$r_x = d/\cos \gamma - R.$$ 

By the law of cosines,

$$|A_0^\lambda K|^2 = |OK|^2 + |A_0^\lambda O|^2 - 2|OK||A_0^\lambda O|\cos (\alpha + \beta). \quad (A.3)$$

Recalling that $d = |A_0^\lambda K|$, $|OK| = r$, and $|A_0^\lambda O| = R$, and using the fact that $\eta = 2\pi - (\alpha + \beta)$, Equation (A.3) yields

$$d = \sqrt{r^2 + R^2 - 2Rr\cos \eta}.$$ 

Observe that $\alpha = \arccos(r/R)$ and $\beta = 2\pi - \lambda_m$. So $\eta$ may be expressed as

$$\eta = \lambda_m - \arccos(r/R).$$

Finally, application of the law of sines gives $|OK|/\sin \gamma = |A_0^\lambda K|/\sin(\alpha + \beta)$, which implies $\sin \gamma = -(r \sin \eta)/d$. Hence,

$$\gamma = -\arcsin\left((r \sin \eta)/d\right). \quad (A.4)$$

### A.5 Proof for Lemma [2]

This section is dedicated to prove Lemma [2] which appears in Section 6.2. Let $x \cdot \alpha = d$ be the equation for the plane parallel to $\Pi(A, \alpha)$. Recall that $O$ is the origin of the $(x,y,z)$-coordinate system. Because the plane does not contain $A$, we have $OA \cdot \alpha \neq d$. Let $C$ be an arbitrary point on the ellipse. Let $\mathcal{L}$ be the line connecting $A$ to $C$; this line can be parameterized as $OA + tAC$ and will intersect the plane at $t^*$ such that

$$(OA + t^*AC) \cdot \alpha = d.$$ 

Therefore, we have

$$t^* = \frac{d - OA \cdot \alpha}{AC \cdot \alpha} \neq 0.$$ 

Because the ellipse has no intersection with $\Pi(A, \alpha)$, i.e., $AC \cdot \alpha \neq 0$, $t^*$ is well-defined for any $C$, i.e., $t^*$ is bounded.

Now recall that the point $A$ and the ellipse define an quadratic cone. The intersection between a quadratic cone and a plane that does not go through the vertex of the cone can be only three kinds [Silv01], i.e., parabola, hyperbola, and ellipse (consider circle as a special case of ellipse). Note that ellipse is the only bounded curve among all three curves. In our case, because $t^*$ is bounded, the intersection is an ellipse, and the lemma is proven.
Appendix A. Proofs and Derivations

Figure A.4: Illustration for Lemma A.4. ABCD is a rectangle inscribed in an ellipse that is centered at O. Here, \( \theta \) is orthogonal to AB and CD, whereas \( \theta^\perp \) is orthogonal to BC and AD.

A.6 Proof for Theorem 3

This section is dedicated to prove Theorem 3 which appears in Section 6.2.2. To prove this theorem, we will need Lemma 4.

**Lemma 4.** If a rectangle is inscribed in an ellipse, then the center of the rectangle is identical to the center of the ellipse, and each side of this rectangle is either parallel to the major or parallel to the minor axis of the ellipse.

**Proof of Lemma 4.** We first prove the the centers of the rectangle and the ellipse are identical. Without loss of generality, we define the ellipse that is centered at O, denoted as E(O), as below:

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{A.5}
\]

ABCD is a rectangle that is inscribed in E(O) (Figure A.4). Let \( \theta = (\cos \theta, \sin \theta) \) be orthogonal to AB and CD. Thus \( \theta^\perp = (-\sin \theta, \cos \theta) \) is orthogonal to BC and AD. We define \( \theta \in [0, \pi) \). Let \( x_A, x_B, x_C, \) and \( x_D \) be the coordinates of A, B, C and D, respectively. Using the Cartesian coordinate system that is spanned by \( \theta \) and \( \theta^\perp \), these coordinates can be expressed as following

\[
\begin{align*}
x_B &= s_1 \theta + s_3 \theta^\perp \\
x_C &= s_2 \theta + s_3 \theta^\perp \\
x_A &= s_1 \theta + s_4 \theta^\perp \\
x_D &= s_2 \theta + s_4 \theta^\perp.
\end{align*}
\]

Consider the line that goes through A and B, any point \( \mathbf{x} = (x, y) \) on this line satisfies the equation below

\[
\mathbf{x} = s_1 \theta + t \theta^\perp. \tag{A.6}
\]

Replace \((x,y)\) in Equation A.5 by Equation A.6 we obtain the following equation:

\[
C_1 t^2 + C_2 t + C_3 = 0, \tag{A.7}
\]
where
\[ C_1 = a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad C_2 = 2s_1 \cos \theta \sin(\theta(a^2 - b^2)), \quad C_3 = s_1^2(a^2 \sin^2 \theta + b^2 \cos^2 \theta) - a^2 b^2. \]

By definition, \( s_3 \) and \( s_4 \) are two solutions to Equation \( \text{A.7} \) and hence we have
\[ s_3 s_4 = \frac{C_3}{C_1} = \frac{s_1^2(a^2 \sin^2 \theta + b^2 \cos^2 \theta) - a^2 b^2}{a^2 \cos^2 \theta + b^2 \sin^2 \theta}. \]  
(A.8)

Analogously, \( s_3 s_4 \) can also be obtained using the line that goes through \( C \) and \( D \), which yields
\[ s_3 s_4 = \frac{s_1^2(a^2 \sin^2 \theta + b^2 \cos^2 \theta) - a^2 b^2}{a^2 \cos^2 \theta + b^2 \sin^2 \theta}. \]  
(A.9)

Equations \( \text{A.8} \) and \( \text{A.9} \) indicate that \( s_1^2 = s_2^2 \). By definition, \( s_1 \neq s_2 \). Therefore, we conclude that \( s_1 = -s_2 \).

Using similar approach to the lines that are orthogonal to \( \theta \perp \), we can obtain \( s_3 = -s_4 \). Because \( s_1 = -s_2 \) and \( s_3 = -s_4 \), we have \( x_A = -x_C \) and \( x_B = -x_D \). Therefore, the center of the rectangle is identical to the center of the ellipse.

Next we prove that each side of the rectangle is either parallel to the major axis or parallel to the minor axis. Recall that \( s_3 \) and \( s_4 \) are two solutions to Equation \( \text{A.7} \). Thus we have
\[ s_3 + s_4 = -\frac{C_2}{C_1} = -\frac{2s_1 \cos \theta \sin(\theta(a^2 - b^2))}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} = 0. \]  
(A.10)

Because \( \theta \in [0, \pi] \) and \( s_1 \neq 0 \) (otherwise the rectangle degenerates to a line), we have three solutions to Equation \( \text{A.10} \) i.e., 1) \( a = b \), 2) \( \theta = 0 \), and 3) \( \theta = \pi/2 \). For the case of \( a = b \), the ellipse becomes a circle and the major and minor axes of the ellipse become diameters of the circle. In this case, there is always a diameter that is parallel to each side of the rectangle. Solution \( \theta = 0 \) indicates that \( AB \) is parallel to the \( y \)-axis, whereas solution \( \theta = \pi/2 \) indicates that \( AB \) is parallel to the \( x \)-axis. Hence, the lemma is proven.

Now we are ready for the proof of Theorem 3. Recall from Theorem 1 and Corollary 3 that, for a given partial cone surface and a given R-arc on this surface, the centers of the R-arcs from this cone surface are colinear, and all the other R-arcs from this cone surface can be obtained by translating an isotropically scaled copy of the given R-arc (with respect to the center of the given R-arc). Therefore, the proof for Theorem 3 can be achieved by proving it for any selected R-arc.

In this context, we focus on R-arc \( (\Omega_+^{\lambda, z_0}) \), which is the R-arc of \( S_+^{\lambda} \) in plane \( \Pi_{z_0} \), with \( \lambda \in [\lambda_0, \lambda_c] \) and \( z_0 = -H - \Delta H \). By definition, \( \Pi_{z_0} \) is perpendicular to the \( z \)-axis and goes through the lowest point of the ELE trajectory, \( A_0^0 \). We denote the ellipses where the lower T-arc and R-arc \( (\Omega_+^{\lambda, z_0}) \) lie as Ell(\( O_- \)) and Ell(\( O_+^{\lambda, z_0} \)), respectively. For convenience, we denote \( A^0 \) and \( A^\pi \) of the lower T-arc as \( D \) and \( E' \), respectively; see Figure A.5. Also, let \( O_{z_0} \) be the intersection between the \( z \)-axis and \( \Pi_{z_0} \), and refer to \( \text{Cir}(O_{z_0}) \) as the intersection between \( S_c \) and \( \Pi_{z_0} \).
Appendix A. Proofs and Derivations

Now, we construct a quadrilateral \(ABCD\) in \(\Pi_{z_0}\); see Figure A.5. By definition, \(A_+^\lambda\) is the vertex of \(S_+^\lambda\) (see Section 6.2.1). Through \(A_+^\lambda\), we draw a line parallel to the \(z\)-axis that intersects \(\text{Ell}(O_-)\) and \(\text{Cir}(O_{z_0})\) at \(A_-^\lambda\) and \(A\), respectively. We draw another line through \(A_+^\lambda\) and \(E'\), and refer to the intersection between this line and \(\Pi_{z_0}\) as \(B\). We denote the the middle point of line segment \(BD\) as \(O'\), and extend \(AO'\) to \(C\) such that \(|AO'| = |CO'|\). By the preceding construction, \(ABCD\) is a quadrilateral in \(\Pi_{z_0}\).

![Figure A.5: Illustration for the proof of Theorem 3.](image)

To simplify notation, we denote \(A^0_-\) as \(D\), and denote \(A_+^\pi\) as \(E'\). We define \(\Pi_{z_0}\) as the horizontal plane that goes through \(D\), and it intersects the extension of \(A_+^\lambda A_-^\lambda\) at \(A\). We connect \(A_+^\lambda\) and \(E'\) to obtain a line that intersects \(\Pi_{z_0}\) at \(B\). We denote the intersection between \(\text{Cir}(O_{z_0})\) and the line through \(A\) and \(B\) as \(E\), and refer to \(O'\) as the middle point of the line segment \(BD\), which intersects \(\text{Cir}(O_{z_0})\) at \(G\). Extend \(AO'\) to \(C\) such that \(|AO'| = |CO'|\), and denote the intersection between \(\text{Cir}(O_{z_0})\) and the line through \(A\) and \(C\) as \(F\). Draw a vertical line through \(F\), and this line intersect \(A_+^\lambda C\) at \(F'\). We define \(\angle BAC = \eta\). Left: 3D view. Right: projection onto the \((x,y)\)-plane.

![Figure A.5: Illustration for the proof of Theorem 3.](image)

First, we prove that \(ABCD\) is a rectangle. Let \(E\) be the intersection between \(\text{Cir}(O_{z_0})\) and \(AB\). By construction, \(E\) is the orthogonal projection of \(E'\) onto \(\text{Cir}(O_{z_0})\), which indicates that \(DE\) is a diameter of \(\text{Cir}(O_{z_0})\). According to \[Eucl07, Proposition 31, Book 3\], \(\angle BAD = \angle EAD = \pi/2\). Because \(\angle BAD\) is the right angle and \(O'\) is the middle point of \(BD\), according to \[Dunh91, Proposition 3, pp. 120\], we have \(|AO'| = |BO'| = |CO'| = |DO'|\), given that \(|AO'| = |CO'|\). Hence, we have \(AB \parallel CD\) and \(AD \parallel BC\). Together with the factor that \(\angle BAD = \pi/2\), we can conclude that \(ABCD\) is a rectangle.

Next, we prove that \(A, B, C,\) and \(D\) are on \(\text{Ell}(O_{z_0}^\lambda)\). By the preceding construction, it is trivial that \(A, B, D\) are on \(\text{Ell}(O_{z_0}^\lambda)\). We need to prove that \(C\) is also on \(\text{Ell}(O_{z_0}^\lambda)\). Let \(F\) be the intersection between \(\text{Cir}(O_{z_0})\) and \(AC\). Through \(F\), we draw a line parallel to the \(z\)-axis that intersects \(A_+^\lambda C\) and \(\text{Ell}(O_-)\) at \(F'\) and \(F''\), respectively. To prove that \(C\) is on \(\text{Ell}(O_{z_0}^\lambda)\), it is sufficient to prove that \(F'\) and \(F''\) are identical, which is equivalent to prove that \(|FF'| = |FF''|\).
A.6. Proof for Theorem 3

We now calculate $|FF'|$. Consider $\triangle ABA^\lambda_+$, by construction, we have $FF' \parallel AA^\lambda_+$, which indicates:

$$\frac{|FF'|}{|AA^\lambda_+|} = \frac{|CF|}{|CA|}. \quad (A.11)$$

Let $G$ be the intersection between $BD$ and $\text{Cir}(O_{z_0})$. Note that $\angle FGD$ and $\angle FAD$ share the same arc segment $FD$ of $\text{Cir}(O_{z_0})$. According to [Eucl 07, Proposition 21, Book 3], we have:

$$\angle FGD = \angle FAD. \quad (A.12)$$

Also, recall from the paragraph before last paragraph that $|AO'| = |DO'|$, we have:

$$\angle FAD = \angle GDA. \quad (A.13)$$

Equations $A.12$ and $A.13$ imply that $\angle FGD = \angle GDA$, and thus $GF \parallel AD \parallel BC$. Therefore, we have:

$$\frac{|CF|}{|CA|} = \frac{|BG|}{|BD|}. \quad (A.14)$$

Now consider $\triangle BGE$, because $ED$ is a diameter, $\angle EGD = \pi/2$. This indicates that $\triangle BGE$ is an right triangle, and hence $|BG| = |BE| \cos \eta$, with $\eta = \angle ABD$. Now consider $\triangle BAD$, we have $|BA| = |BD| \cos \eta$. Therefore, we have:

$$\frac{|BG|}{|BD|} = \frac{|BE|}{|BA|} \cos^2 \eta. \quad (A.15)$$

In $\triangle BAA^\lambda_+$, because $EE' \parallel AA^\lambda_+$, we have:

$$\frac{|BE|}{|BA|} = \frac{|EE'|}{|AA^\lambda_+|}. \quad (A.16)$$

According to Equations $A.11$ $A.14$ $A.15$ and $A.16$ we conclude that:

$$|FF'| = |EE'| \cos^2 \eta. \quad (A.17)$$

We then calculate $|FF''|$. Because, $|BO'| = |AO'|$, we have $\angle EAF = \angle ABD = \eta$. According to [Eucl 07, Proposition 20, Book 3], $\angle EO_{z_0}F = 2\eta$, which indicates that $F$ possesses a polar angle of $\pm(\pi + 2\eta)$ (positive when $\lambda \in [0, \pi)$, negative when $\lambda \in [\lambda_x, 0) \cup [\pi, \lambda_e]$). Because $F''$ and $F$ share the same polar angle, the polar angle of $F''$ is also $\pm(\pi + 2\eta)$. According to Equation 6.1 and recall from the Equation 6.2 that $\mathcal{H}(\lambda) = \mathcal{H}(-\lambda)$, we have:

$$|FF''| = -\mathcal{H}(\pi + 2\eta) - (-\mathcal{H}(0)) = 2\Delta H \cos^2 \eta.$$

By construction, we have $|EE'| = 2\Delta H$, and hence:

$$|FF''| = |EE'| \cos^2 \eta. \quad (A.18)$$
Appendix A. Proofs and Derivations

According to Equations A.17 and A.18 we conclude that $|FF''| = |FF'|$, and hence $F'$ is on $E(O_-)$.

We have proved that $ABCD$ is a rectangle inscribed in $\text{Ell}(O_{+}^{\lambda, z_0})$. According to Lemma 4, $O' = O_{+}^{\lambda, z_0}$; further more, we define the major axis as the one that is parallel to $AB$, and the minor axis as the one that is parallel to $AD$. Now, consider $\triangle AO'D$ and $\triangle AO_{z_0}D$, because $|AO'| = |DO'|$ and $|AO_{z_0}| = |DO_{z_0}|$, we have $O'O_{z_0} \perp AD$, which indicates that $O'O_{z_0}$ is parallel to $AB$ and is the major axis. Because $|AO_{z_0}| = |DO_{z_0}|$, $O'O_{z_0}$ is the angular bisector of $\angle AO_{z_0}D$, which concludes that the polar angle from the $x$-axis to the major axis of $\text{Ell}(O_{+}^{\lambda, z_0})$ is half of the polar angle of $A$, i.e., $\lambda/2$.

The preceding proof together with Theorem 1 conclude this proof.

### A.7 Inversion of Equation 6.6

In this appendix, we derive the inversion of Equation 6.6, which appears in Section 6.3.1. First, we rewrite Equation 6.6 into the following form

\[
\begin{align*}
\sin \alpha 
\cos \alpha + y \sin \alpha &= R \cos \beta \\
\cos \alpha - x \sin \alpha &= R \left( e \sin \beta + d \sin(1 - \cos \beta^2) \right) \\
1 + d \cos \alpha \cos \beta
\end{align*}
\]  

(A.19) 

(A.20)

Next, we replace $R \cos \beta$ in Equation A.20 by the left side of Equation A.19, which yields the equation below:

\[R e \sin \beta = a \cos \alpha + b \sin \alpha,\]

(A.21)

where

\[a = y + \frac{xyd}{R} \quad \text{and} \quad b = \frac{y^2d}{R} - x - Rd.\]

Multiple $e$ on both sides of Equation A.19 we get

\[R e \cos \beta = ex \cos \alpha + ey \sin \alpha.\]

(A.22)

Adding the square of both Equation A.21 and Equation A.22 we have

\[R^2e^2 = (a \cos \alpha + b \sin \alpha)^2 + (ex \cos \alpha + ey \sin \alpha)^2.\]

(A.23)

After some calculation, the above equation can be rewritten as below

\[c_1 = c_2 \cos 2\alpha + c_3 \sin 2\alpha,\]

with

\[
\begin{align*}
c_1 &= R^2e^2 - (a^2 + b^2 + e^2x^2 + e^2y^2)/2 \\
c_2 &= (a^2 - b^2 + e^2x^2 - e^2y^2)/2 \\
c_3 &= ab + e^2xy
\end{align*}
\]
Now, let
\[ \sin \theta = c_3 / \sqrt{c_2^2 + c_3^2} \quad \text{and} \quad \cos \theta = c_2 / \sqrt{c_2^2 + c_3^2}, \]
so that
\[ \cos(2\alpha - \theta) = \frac{c_1}{\sqrt{c_2^2 + c_3^2}}, \]
Also, let
\[ \omega = \arccos \left( \frac{c_1}{\sqrt{c_2^2 + c_3^2}} \right), \]
then, we have
\[ \alpha = \frac{\theta \pm \omega}{2} + k\pi, \quad (A.24) \]
where \( \arccos \) is the inverse cos function that returns a polar angle in the range \([0, \pi]\), and \( k \) is an integer. Recall that \( \alpha \in [-\pi/2, 3\pi/2] \) and \( \beta \in [-\pi, \pi] \). Therefore, according to Equations \( A.19 \) and \( A.24 \), all possible \((\alpha, \beta)\) pairs can be expressed as below.

\[
\begin{align*}
\alpha_1 &= \frac{\theta + \omega}{2} + n_1\pi, \\
\beta_1 &= \arccos \left( \frac{x \cos \alpha_1 + y \sin \alpha_1}{R} \right), \\
\alpha_2 &= \frac{\theta + \omega}{2} + n_2\pi, \\
\beta_2 &= -\arccos \left( \frac{x \cos \alpha_2 + y \sin \alpha_2}{R} \right), \\
\alpha_3 &= \frac{\theta - \omega}{2} + m_1\pi, \\
\beta_3 &= \arccos \left( \frac{x \cos \alpha_3 + y \sin \alpha_3}{R} \right), \\
\alpha_4 &= \frac{\theta - \omega}{2} + m_2\pi, \\
\beta_4 &= -\arccos \left( \frac{x \cos \alpha_4 + y \sin \alpha_4}{R} \right),
\end{align*}
\]
where \( \{n_1, n_2\} = \{-1, 0, 1\} \), \( \{m_1, m_2\} = \{0, 1, 2\} \). Note that the above possible solutions need to be examined using Equations \( A.19 \) and \( A.20 \) because fake solutions are introduced due to Equation \( A.23 \).

### A.8 Calculation of the Parameters of an ELE R-arc

In this appendix, we provide a method to calculate the semi-major and -minor axes of an R-arc for the ELE trajectory, which is needed in Section 6.3.2. We will explain this method using the semi-minor axis as an example, but the same method can be applied to the semi-major axis.

To describe this method, we will need Figure A.6 in which, the first row are from the bottom row of Figure 6.5 and the second row are extractions of \( \triangle A^\lambda_+ A^\pi A^\lambda_- \) and \( \triangle A^\lambda_0 A^\pi A^0_- \) from the top left. For the meaning of the labels in this figure, please see the last paragraph of Section 6.2.2. The goal here is to calculate \(|O^\lambda_{+,-} Q|\).

To start, we introduce some notation. For simplification, we denote \( A^\lambda_+ \), \( A^\lambda_\pi \), \( A^\lambda_- \), \( A^\lambda_0 \), \( A^\pi_- \), \( A^0_- \), \( A^\lambda_+ \) as \( N, M, L, G, K, I \) and \( J \), respectively. We refer to \( \theta_1, \theta_2 \) and \( \theta_3 \),
Figure A.6: Method for calculating the semi-minor axis for R-arc \( (O^\lambda_+)^z \). The top row are replottings of the bottom row of Figure 6.5. The bottom left and right are the extractions of \( \triangle A_+^\lambda A^\pi A_+^0 \) and \( \triangle A_+^\lambda A^\pi A_0^- \), respectively.

\[ |NM| = \frac{|MP|}{\sin \theta_3} \quad \text{and} \quad |NL| = \frac{|LP|}{\sin(\pi - \theta_3)} \]
Because \(|MP| = |LP|\) and \(\sin \theta_3 = \sin(\pi - \theta_3)\), the above equations yield

\[
\frac{\sin \theta_1}{\sin \theta_2} = \frac{|NL|}{|NM|}.
\] (A.25)

Now consider \(\triangle NKJ\), using the same procedure as that was used for \(\triangle NML\), we get

\[
\frac{|KP'|}{|JP'|} = \frac{\sin \theta_1 \ |NK|}{\sin \theta_2 \ |NJ|}.
\] (A.26)

Combining Equations A.25 and A.26, we obtain

\[
\frac{|KP'|}{|JP'|} = \frac{|NL| \ |NK|}{|NM| \ |NJ|}.
\]

Let \(c_p = \frac{|KP'|}{|JP'|}\), from basic Euclidean geometry, we have \(x_{P'}\) as below

\[
\begin{align*}
x_{P'} &= (x_K + c_p x_J)/(1 + c_p) \\
y_{P'} &= (y_K + c_p y_J)/(1 + c_p) \\
z_{P'} &= (z_K + c_p z_J)/(1 + c_p)
\end{align*}
\] (A.27)

Similarly, by considering \(\triangle NMG\) and \(\triangle NKI\) (bottom right of Figure A.6), \(x_{O'}\) can be obtained from \(K\) and \(I\), using the the expressions below

\[
\begin{align*}
x_{O'} &= (x_K + c_o x_J)/(1 + c_o) \\
y_{O'} &= (y_K + c_o y_J)/(1 + c_o) \\
z_{O'} &= (z_K + c_o z_J)/(1 + c_o)
\end{align*}
\] (A.28)

where

\[
c_o = \frac{|KO'|}{|IO'|} = \frac{|NG| \ |NK|}{|NM| \ |NI|}.
\]

Next, we calculate the coordinates of \(Q'\), i.e., \(x_{Q'} = (x_{Q'}, y_{Q'}, z_{Q'})\). Let \(\hat{u} = (n_x, n_y, n_z)\) be the unit vector pointing from \(O'\) to \(P'\). Because \(Q'\) is on both the lower T-arc as well as the line that goes through \(O'\) and \(P'\), \(x_{Q'}\) and \(y_{Q'}\) satisfy the following equations

\[
\begin{align*}
x_{Q'} &= x_{O'} + t^* n_x, \\
y_{Q'} &= y_{O'} + t^* n_y, \\
x_{Q'}^2 + y_{Q'}^2 &= R^2.
\end{align*}
\] (A.29-31)

By construction, \(t^* > 0\). Also, because \(\hat{u}\) is parallel to the plane where the lower T-arc lies, \(\hat{u}\) is not parallel to the \(z\)-axis, which indicates that \(n_x^2 + n_y^2 \neq 0\). With the preceding two conditions, we have

\[
t^* = \frac{\sqrt{R^2(n_x^2 + n_y^2) - (n_x y_{Q'} - n_y x_{Q'})^2 - (n_x x_{Q'} + n_y y_{Q'})}}{n_x^2 + n_y^2}.
\]
Now, \( x_Q' \) and \( y_Q' \) can be obtained by replacing \( t^* \) in Equations A.29 and A.30.

Finally, we are ready for the semi-minor axis of R-arc \((O_{\lambda,0}^z)\). Let \( \lambda_Q' \) be the polar angle of \( Q' \), we have
\[
\lambda_Q' = \text{atan2}(y_Q', x_Q'),
\]
where \( \text{atan2}(v,u) \) returns the polar angle of point \((u,v)\) in the \((x,y)\)-plane in the range \([-\pi, \pi]\). Using \( \lambda_Q' \), we are able to get the \((x,y)\) coordinates of \( Q \) according to Equation 6.4.

Therefore, the semi-minor axis, i.e., \(|O_{\lambda,0}^z Q|\), can be calculated. Note that when \( \lambda = 0 \), \( P' \) is identical to \( O' \), and \( \hat{n} \) is not well defined. In this extreme case, \( \hat{n} \) is parallel to the \( y \)-axis according to the middle row of Figure 6.5 i.e., \( \hat{n} = (0,1,0) \). The same \( \hat{n} \) can be used for the extreme case for the semi-major axis, which appears when \( \lambda = \pi \).

A.9 Proof for Theorem 5

This appendix provides a proof for Theorem 5, which appears in Section 6.4.2. For this proof, all the analysis will be performed in the \((x,y)\)-plane, and only 2D notations will be used, i.e., we will omit the \( z \) component for all the vectors and points. Let \( x = (x,y) \) be the point of interest. We denote \( \text{Ell}(O_{\lambda,u}^0) \) as the ellipse where R-arc \((O_{\lambda,u}^0)\) lies, and refer to \( a(\lambda) = (R \cos \lambda, R \sin \lambda) \) as a point on \( D_0 \) of polar angle \( \lambda \).

To start, we present the mathematical expression of \( \text{Ell}(O_{\lambda,u}^0) \) and \( \text{Ell}(O_{\lambda,l}^0) \). By definition, \( \text{Ell}(O_{\lambda,u}^0) \) and \( \text{Ell}(O_{\lambda,l}^0) \) are in the central plane \( \Pi_0 \) and we have \( e = z = 0 \). After some calculation (see Section A.7), Equation 6.13 can be rewritten as below
\[
\begin{align*}
  x \cos \alpha + y \sin \alpha &= R \cos \beta, \\
  a \cos \alpha + b \sin \alpha &= 0,
\end{align*}
\]
where
\[
a = y + \frac{xyd}{R} \quad \text{and} \quad b = \frac{y^2d}{R} - x - Rd.
\]
We replace \((\alpha, \beta)\) by \((\lambda_u, \lambda_l)\) according to Equation 6.5 After some calculation, we get
\[
\begin{align*}
  ay - bx + bR \cos \lambda_u - aR \sin \lambda_u &= 0, \\
  ay - bx + bR \cos \lambda_l - aR \sin \lambda_l &= 0.
\end{align*}
\]
Let
\[
f_e(\lambda, x, y) = ay - bx + bR \cos \lambda - aR \sin \lambda.
\]
As we can see, \( \lambda_l \) and \( \lambda_u \) are both solutions to \( f_e(\lambda, x, y) = 0 \). For a given \( \lambda \), Equation \( f_e(\lambda, x, y) = 0 \) is the general form for both \( \text{Ell}(O_{\lambda,u}^0) \) and \( \text{Ell}(O_{\lambda,l}^0) \). Note that \( a \) and \( b \) can be replaced by expressions of \( x \) and \( y \) using Equation A.33 Hence, function \( f_e(\lambda, x, y) \) is quadratic in both \( x \) and \( y \).
A.9. Proof for Theorem 5

Figure A.7: Illustration for the proof of the sufficiency of Theorem 5. The big circle is $D_0$, and the solid portion indicates the angular range of the T-arcs of the ELE trajectory. For a given point of interest, $x = (x, y)$, we can obtain $\alpha$ by inverting Equation A.32. We found that $\alpha$ is admissible only when $\rho = (\cos \alpha, \sin \alpha)$ is in the fan angle that is bounded by $\alpha_s$ and $\alpha_e$, which are defined in Equations A.40 and A.41.

First, we prove that Region I and Region II are sufficient for the AA R-line coverage in $\Pi_0$. Let $\alpha^*$ and $\beta^*$ be a pair of solutions of Equation A.32 and refer to $\lambda^*_u$ and $\lambda^*_l$ be their change of variables following Equation 6.5. We now study the admissible condition for $\alpha^*$; see Figure A.7. We define

$$\alpha = (\cos \alpha^*, \sin \alpha^*),$$  \hspace{1cm} (A.36)

$$\hat{\alpha} = (-b, a).$$  \hspace{1cm} (A.37)

According to Equation A.32, we know that $\alpha$ is parallel to $\hat{\alpha}$. Next, we define two vectors $\nu_s$ and $\nu_e$, with the former pointing from $a(\lambda_s)$ to $x$, and the latter pointing from $a(\lambda_e)$ to $x$, i.e.,

$$\nu_s = (x - R \cos \lambda_s, \ y - R \sin \lambda_s),$$  \hspace{1cm} (A.38)

$$\nu_e = (x - R \cos \lambda_e, \ y - R \sin \lambda_e).$$  \hspace{1cm} (A.39)

Also, we define

$$\alpha_s = (y - R \sin \lambda_s, \ R \cos \lambda_s - x),$$  \hspace{1cm} (A.40)

$$\alpha_e = (R \sin \lambda_e - y, \ x - R \cos \lambda_e).$$  \hspace{1cm} (A.41)

Note that $\alpha_s$ is a vector that is rotated from $\nu_s$ clockwise by 90°, whereas $\alpha_e$ is a vector that is rotated from $\nu_e$ counter-clockwise by 90°.
Now recall from Equation 6.5 that $\alpha^* = (\lambda_{u_k}^* + \lambda_{l_k}^*)/2$, and $\{\lambda_{u_k}^*, \lambda_{l_k}^*\} \in [\lambda_s, \lambda_e]$. Therefore, $\alpha$ is perpendicular to the line that connects $a(\lambda_{u_k}^*)$ and $a(\lambda_{l_k}^*)$, and $\alpha^*$ is only admissible when $\alpha$ is between $\alpha_s$ and $\alpha_e$, i.e., $\alpha$ belongs to the fan angle that is bounded by $\alpha_s$ and $\alpha_e$. Mathematically, this condition is equivalent to the following relations

$$\alpha \cdot \nu_s > 0 \quad \text{and} \quad \alpha \cdot \nu_e > 0.$$

Because $\hat{\alpha}$ is parallel to $\alpha$, the above relations yield the relation below

$$(\hat{\alpha} \cdot \nu_s)(\hat{\alpha} \cdot \nu_e) > 0. \quad (A.42)$$

Using Equations A.37, A.38, and A.39, the above relation becomes

$$(ay - bx + bR \cos \lambda_e - aR \sin \lambda_e)(ay - bx + bR \cos \lambda_s - aR \sin \lambda_s) > 0. \quad (A.43)$$

According to Equation A.34, Relation A.43 indicates that only the points of the regions that are both outside or both inside Ell$(O_{+}^{k,0})$ and Ell$(O_{+}^{k,0})$ are possibly covered by R-lines. Also, because our R-line coverage belongs to the convex hull of the ELE trajectory, the R-line coverage in $\Pi_0$ is inside $D_o$ and above the cutting line. Therefore, Region I and Region II are sufficient for the R-line coverage in $\Pi_0$.

Next, we prove that Region I and Region II are necessary for the AA R-line coverage in $\Pi_0$. Mathematically, we try to prove that for any given point, $x = (x_s, y_s)$, that belongs to Region I or II, there are at least two solutions $\{\lambda_1, \lambda_2\} \in [\lambda_s, \lambda_e]$ such that $f_e(\lambda_k, x_s, y_s) = 0$ for $k = 1, 2$. For the proof, we will need the following quantities. Let

$$a_s = y_s + \frac{x_s y_s d}{R}, \quad b_s = \frac{y_s^2 d}{R} - x_s - Rd, \quad \text{and} \quad \hat{\alpha}_s = (-b_s, a_s).$$

We also define $\nu_s(\lambda) = (x_s - R \cos \lambda_s, y_s - R \sin \lambda_s)$.

We now consider function $f_e(\lambda, x_s, y_s)$. This function can be rewritten in two forms. The first form is

$$f_e(\lambda, x_s, y_s) = c_1 \cos(\lambda + \theta) + c_2,$$

where

$$c_1 = R \sqrt{a_s^2 + b_s^2}, \quad \theta = \text{atan2}(a_s, b_s), \quad \text{and} \quad c_2 = a_s y_s - b_s x_s.$$

Here, atan2($a_s, b_s$) is the four-quadrant inverse tangent function, which gives the polar angle of point $(b_s, a_s)$ in the $(x, y)$-plane in the range $[-\pi, \pi]$. The second form is

$$f_e(\lambda, x_s, y_s) = \hat{\alpha}_s \cdot \nu_s(\lambda).$$

The first form can be understood as a scaled and shifted cos function, whereas the second form can be understood as a scalar product of two vectors.

We present the proof in two cases: (i) $\bar{x}_s$ belongs to Region I and (ii) $\bar{x}_s$ belongs to Region II. We now present the proof in case (i). Because $\bar{x}_s$ is located in Region I, by definition, we have

$$\begin{align*}
    f_e(\lambda_e, x_s, y_s) &= \hat{\alpha}_e \cdot \nu_s(\lambda_e) < 0, \\
    f_e(\lambda_s, x_s, y_s) &= \hat{\alpha}_s \cdot \nu_s(\lambda_s) < 0. \quad (A.44)
\end{align*}$$
Note that $v_\ast(\lambda_e)$ is the vector that points from $a(\lambda e)$ to $x_\ast$; see Figure \ref{fig:proof}. We draw a line through $a(\lambda e)$ and $x_\ast$, and denote the other intersection between this line and $D_0$ as $a(\lambda_\ast)$. By construction, $v_\ast(\lambda_\ast)$ has the opposite direction of $v_\ast(\lambda_e)$. Therefore, we have

$$f_e(\lambda_\ast, x_\ast, y_\ast) = \hat{\alpha}_\ast \cdot v_\ast(\lambda_\ast) > 0.$$  \hfill (A.45)

![Illustration](image.png)

Figure A.8: Illustration for the proof of the necessity of Theorem 5. The big circle is $D_0$, and the solid portion indicates the angular range of the T-arcs of the ELE trajectory. Here, the point of interest, $x_\ast$, belongs to Region I or II, and $a(\lambda_\ast)$ is the other intersection between $D_0$ and the line that connects $a(\lambda_e)$ and $x_\ast$.

Note that, by construction, $\lambda_\ast \in (\lambda_s, \lambda_e)$. Now, we interpret $f_e(\lambda, x_\ast, y_\ast)$ in its first form with $\lambda$ being variable and $(x_\ast, y_\ast)$ fixed. According to Equations A.44 and A.45, the plot of function $f_e(\lambda, x_\ast, y_\ast)$ against $\lambda$ has to be in the shape as illustrated in Figure A.9. Therefore, there exist two solutions, $\{\lambda_1, \lambda_2\} \in (\lambda_s, \lambda_e)$ such that

$$f_e(\lambda_k, x_\ast, y_\ast) = 0, \quad \text{for} \quad k = \{1, 2\}. \hfill (A.46)$$

The above equation indicates that for any $x_\ast$ in Region I, there are two R-lines going through $x_\ast$: one connects $a_\ast(\lambda_1)$ to $a_\ast(\lambda_2)$ and the other connects $a_\ast(\lambda_2)$ to $a_\ast(\lambda_1)$.

For the case when $x_\ast$ is located in Region II, a similar proof can show that there exist two AA R-lines going through $x_\ast$. We now can conclude that any point in Region I and II are covered by two AA R-lines, which proves that Region I and II are necessary for the AA R-line coverage in $\Pi_0$.

We have shown that Region I and II are necessary and sufficient for the AA R-line coverage in $\Pi_0$, and hence Theorem 5 is proven.
A.10 Proof for the Statement Made in Section 6.4.3

This appendix provides proof for the statement made in Section 6.4.3. For this proof, refer to the middle of Figure 6.14. Extend the line segment $A_0^{\lambda} I_1$ until it intersects $D_0$. We denote the polar angle of this intersection as $\lambda^*$. By construction, for $\gamma_m \in (0, \pi/2)$, we have $\lambda^* \in (\pi - \gamma_m, \pi)$. Because $(x_3, y_3)$ is on R-arc $(O_+^{\lambda,0})$, $x_3$ and $y_3$ can be expressed using Equation 6.13, with

\[
\begin{align*}
\alpha &= \frac{(\lambda_s + \lambda_l) / 2}{(\lambda_s - \lambda_l) / 2}, \\
\beta &= \frac{(\cos \beta + d \cos \alpha)}{(\cos \beta + d \cos \alpha)}.
\end{align*}
\]

where $\lambda_l \in [\lambda^*, \pi + \gamma_m]$. Because $\lambda^* \in (\pi - \gamma_m, \pi)$, we have $\lambda_l \in (\pi - \gamma_m, \pi + \gamma_m]$, and the ranges of $\alpha$ and $\beta$ are

\[
\alpha \in (\pi/2 - \gamma_m, \pi/2] \quad \text{and} \quad \beta \in [-\pi/2 - \gamma_m, -\pi/2).
\]

Let $A = 1 + d \cos \alpha \cos \beta$, $B = \cos \alpha \cos \beta + d(\cos^2 \alpha - \sin^2 \beta)$ and $C = \sin \alpha \cos \beta + d \sin \alpha \cos \alpha$. According to Equations 6.13 and 6.15 we have

\[
x_3^2 + y_3^2 - r_m^2 = \frac{R^2}{A^2}(B^2 + C^2 - d^2 A^2).
\]

Let $f = B^2 + C^2 - d^2 A^2$, because $R^2/A^2 > 0$, the sign of $x_3^2 + y_3^2 - r_m^2$ is determined by $f$. Now, we expand $f$ and replace $d$ by $\sin \gamma_m$, after some calculation, we get

\[
f = f_1 \times (\cos \beta + d \cos \alpha),
\]

where $f_1 = \cos \beta (\cos^2 \gamma_m - \sin^2 \gamma_m \sin^2 \beta) + \sin \gamma_m \cos \alpha (\cos^2 \gamma_m \cos^2 \beta - \sin^2 \beta)$. Now consider the factor $(\cos \beta + d \cos \alpha)$ of $f$, when $\lambda_l$ increases from $\lambda^*$ to $\pi + \gamma_m$, $\alpha$ stays in the first quadrant and $\cos \alpha$ decreases, whereas $\beta$ stays in the third quadrant and $\cos \beta$ decreases. Because $d > 0$, $\cos \beta + d \cos \alpha$ reaches maximum at $\lambda_l = \lambda^*$, i.e.,

\[
\max_{\lambda_l \in [\lambda^*, \pi + \gamma_m]} (\cos \beta + d \cos \alpha) = (\cos \beta + d \cos \alpha) |_{\lambda_l = \lambda^*} = 0.
\]
Therefore, for $\lambda_l \in [\lambda^*, \pi + \gamma_m]$, $\cos \beta + d \cos \alpha \leq 0$.

Next, we consider $f_1$. According to Equation A.47, we replace $\alpha$ by $-\beta - \gamma_m$. After some calculation, we get

$$f_1 = c_1 \sin^3 \beta + c_2 \sin^2 \beta \cos \beta + c_3 \sin \beta \cos^2 \beta + c_4 \cos^3 \beta$$

$$= \cos^3 \beta \left( c_1 \tan^3 \beta + c_2 \tan^2 \beta + c_3 \tan \beta + c_4 \right)$$

where

$$c_1 = \sin^2 \gamma_m, \quad c_2 = \cos^2 \gamma_m - \sin^2 \gamma_m - \sin \gamma_m \cos \gamma_m, \quad c_3 = -\sin^2 \gamma_m \cos^2 \gamma_m, \quad c_4 = \cos^2 \gamma_m + \sin \gamma_m \cos^3 \gamma_m.$$ 

Note that, according to the range of $\beta$, $\cos \beta \neq 0$, the factorization of $\cos^3 \beta$ is stable for $f_1$. We perform a change of variable, $t = \tan \beta$, and let

$$f_2 = c_1 t^3 + c_2 t^2 + c_3 t + c_4.$$ 

Now, we assume $\gamma_m$ is fixed and study the shape of the cubic function $f_2$. First, we observe that, because $c_1 > 0$, when $t$ increases from infinitely negative to infinitely positive, $f_2$ starts from infinitely negative and ends at infinitely positive. Also, note that $c_4$ is positive, and hence $f_2$ is positive when $t = 0$. Secondly, we consider the two local extremes of $f_2$. We have

$$\frac{\partial f_2}{\partial t} = 3c_1 t^2 + 2c_2 t + c_3 = 0,$$ 

and

$$\Delta = 4 (c_2^2 - 3c_1 c_3).$$

Because of the range of $\gamma_m$, we have $c_1 c_3 < 0$ and $\Delta > 0$. Hence, we have two solutions for Equation A.49; they are

$$\begin{cases} 
  t_1 = \frac{-c_2 + \sqrt{c_2^2 - 3c_1 c_3}}{3c_1}, \\
  t_2 = \frac{-c_2 - \sqrt{c_2^2 - 3c_1 c_3}}{3c_1}.
\end{cases}$$

Because, $c_1 > 0$ and $c_3 < 0$, we have $t_1 > 0$ and $t_2 < 0$. According to the first observation, we know that $f_2$ achieves the local minimum at $t_1$ and reaches the local maximum at $t_2$. According to the above analysis, we can draw the shape of $f_2$ as a function of $t$, as illustrated in the left of Figure A.10.

Recall that $\beta$ belongs to the third quadrant, and thus $\tan \beta > 0$, which indicates that i) $t$ is positive for $f_2$, and ii) $f_2$ reaches minimum at $t_1$. We define

$$f_3 = c_1 t_1^3 + c_2 t_1^2 + c_3 t_1 + c_4,$$

which is a function of $\gamma_m$ with $\gamma_m \in (0, \pi/2)$. We now plot $f_3$ with $\gamma_m$ as a variable changing from 0 to $\pi/2$ (the right of Figure A.10). As illustrated, $f_3$ is always positive for $\gamma_m \in$
Figure A.10: Left: Shape of $f_2$ as a function of $t$. When $t = 0$, $f_2$ is positive, whereas at $t_1$ and $t_2$, $f_2$ achieves the local minimum and maximum, respectively. Right: Plot of $f_3$ with $\gamma_m$ as a variable changing from 0 to $\pi/2$. When $\gamma_m \in [0, 1.033]$, $f_3 \geq 0$.

Because $d = \sin \gamma_m$ and $\sin(1.033) = 0.8588$, the previous result indicates that for $d \in (0, 0.8588]$, we have $f_3$ being nonnegative.

Finally, we conclude the proof. For $d \in (0, 0.8588]$, we have $f_3$ being nonnegative, which guarantees that $f_2$ is nonnegative. Because $f_1$ is the product of $\cos^3 \beta$ and $f_2$, and $\cos^3 \beta$ is negative, we have $f_1$ being nonpositive. Also, because $f$ is the product of $f_1$ and $(\cos \beta + d \cos \alpha)$ with the latter being nonpositive, we have $f$ being nonnegative. Therefore, for $d \in (0, 0.8588]$, $x_3^2 + y_3^2 - r_m^2$ is nonnegative, and the statement is proven.
List of Acronyms and Terminologies

Acronyms

CT: computed tomography
HU: Hounsfield units
PB: parallel-beam
FB: fan-beam
CB: cone-beam
POI: point-of-interest
ROI: region-of-interest
FOV: field-of-view
TES: theoretically-exact and -stable
FBP: filtered-backprojection
DBP: differentiated-backprojection
FDK: Feldkamp-Davis-Kress
RFDK: ramp-filter-based-FDK
HFDK: Hilbert-transform-based-FDK
FRFDK: Fusion-RFDK
FHFDK: Fusion-HFDK
VDHFDK: voxel-dependent-HFDK
ALA: arc-line-arc
AELA: arc-extended-line-arc
Appendix A. List of Acronyms and Terminologies

ELE: ellipse-line-ellipse
LUA: T-line to the upper T-arc
AUA: the lower T-arc to the upper T-arc
MTF: modulation transfer function
FWHM: full width at half maximum
MAIB: mean of the absolute image bias

**Terminologies**

\[\pi\text{-line:}\] line segment that connects two vertex points of the helical trajectory that has angular distance less than 360°

\[R\text{-line:}\] line segment that connects two vertex points of a source trajectory

\[C\text{-line:}\] line segment that connects one point on the upper helix to another point on the lower helix for the reverse helix

\[T\text{-arc:}\] circular arc of the ALA/AELE trajectory or orthogonal projection of the lower/upper T-arc of the ALA/AELA trajectory onto a horizontal plane or the elliptical arc of the ELE trajectory;

\[T\text{-line:}\] line segment of the AELA or the ELE trajectory

\[R\text{-arc:}\] intersection between a horizontal plane and a partial cone surface that is composed of R-lines

\[\text{Arc}(O):\] arc that is unambiguously identified by its center \(O\)

\[\text{Circ}(O):\] circle that is unambiguously identified by its center \(O\)
List of Notations

Scalars

$R$: radius of a source trajectory

$R_m$: radius of the largest cylinder within the convex hull of the source trajectory

$D$: source-to-detector distance

$r$: radius of the field-of-view or of the region-of-interest

$u, v$: detector coordinates

$h, \gamma, t$: coordinates for the LUA R-line surfaces

$\omega, \gamma, t$: coordinates for the AUA R-line surfaces

$\gamma_m$: fan angle

$\lambda$: angular position describing a source trajectory

$\lambda_s$: angular position of the start point of a vertex path

$\lambda_e$: angular position of the end point of a vertex path

$\lambda_m$: angular length of a T-arc or a helical turn

$z_k$: $z$-coordinate of the kink point of the reverse helix

$2H$: axial height of one iteration of a source trajectory

$\Delta H$: length of the T-line extension beyond the T-arc of the AELA trajectory; or half of the axial height (along the $z$-axis) of the elliptical arc of the ellipse-line-ellipse trajectory

$H_F$: axial height of the fusion zone
Appendix A. List of Notations

Vectors

\( \overrightarrow{x} \): position vector, could be either 2D or 3D

\( \overrightarrow{\theta} \): 2D unit vector with polar angle \( \theta \)

\( \overrightarrow{\alpha} \): 3D unit vector of an X-ray beam

\( \overrightarrow{a} \): vertex point indicating a source position

\( \overrightarrow{e}_u, \overrightarrow{e}_v, \overrightarrow{e}_w \): orthonormal unit vectors that span the detector coordinate system

Functions

\( f \): X-ray linear attenuation coefficient

\( f_R^+ \): reconstruction result obtained by the principles of RFDK using the projection data from the upper helical turn of the reverse helix

\( f_R^- \): reconstruction result obtained by the principles of RFDK using the projection data from the lower helical turn of the reverse helix

\( f_{FR} \): reconstruction result obtained by Fusion-RFDK using the projection data from the reverse helix

\( f_H^+ \): reconstruction result obtained by the principles of HFDK using the projection data from the upper helical turn of the reverse helix

\( f_H^- \): reconstruction result obtained by the principles of HFDK using the projection data from the lower helical turn of the reverse helix

\( f_{FH} \): reconstruction result obtained by Fusion-HFDK using the projection data from the reverse helix

\( g \): cone beam projection data

\( g' \): view-dependent differentiation of cone beam projection data

\( h_R \): apodized ramp filter function

\( h_H \): Hilbert kernel function

\( \hat{h}_H \): band limited Hilbert kernel function

\( \omega_V \): voxel-dependent weighting function for the VDHFDK method

\( \omega_{\pi} \): weighting function for the VDHFDK method when the point-of-interest lies on a \( \pi \)-line
\( \omega_c \): weighting function for the VDHFDK method when the point-of-interest does not lie on a \( \pi \)-line

\( \omega_F^+, \omega_F^- \): weighting functions used for the fusion process in Fusion-RFDK and Fusion-HFDK

\( \omega_P \): Parker-like weighting function

\( \Phi \): voxel-dependent angular range for the backprojection of the VDHFDK method

\( \Phi_\pi \): backprojection angular range for the VDHFDK method when the point-of-interest lies on a \( \pi \)-line

\( \Phi_c \): backprojection angular range for the VDHFDK method when the point-of-interest does not lie on a \( \pi \)-line

\( u^*, v^* \): detector coordinates of the intersection between the detector plane and the line that connects a source position and a given point-of-interest

**Operators**

\[ ||x|| \]: Euclidean norm of \( x \)

\[ x \cdot y \]: inner product of \( x \) and \( y \)

\[ x \times y \]: cross product of \( x \) and \( y \)

\[ f_1 * f_2 \]: convolution of \( f_1 \) and \( f_2 \)

\[ Q^T \]: transpose of matrix \( Q \)

\[ AB \]: vector pointing from point \( A \) to point \( B \)

\[ |AB| \]: Euclidean distance between points \( A \) and \( B \)

**Geometrical Symbols**

\( \Pi \): plane that is parallel to the \((x, y)\)-plane and goes through \((0, 0, z)\).

\( \mathcal{L}(x, y) \): line composed of all the points that share the same \((x, y)\) coordinates

\( \mathcal{L}(a, \alpha) \): line that goes through \( a \) with direction \( \alpha \)

\( \Lambda \): convex hull of the reverse helix

\( \Omega \): field-of-view or region-of-interest
Appendix A. List of Notations

\( \Omega_A \): portion of \( \Omega \) within the convex hull of the source trajectory

\( \Omega_A^+ \): portion of \( \Omega \) that is above the \((x,y)\)-plane and within the convex hull of the source trajectory

\( \Omega_A^\pi_u \): portion of \( \Omega_A \) covered by \( \pi \)-lines of the upper turn of the reverse helix

\( \Omega_A^\pi_l \): portion of \( \Omega_A \) covered by \( \pi \)-lines of the lower turn of the reverse helix

\( \Omega_A^\pi_c \): portion of \( \Omega_A \) not covered by \( \pi \)-lines of the reverse helix

\( \Omega_A^R \): portion of \( \Omega_A \) not covered by \( R \)-lines of the reverse helix

\( \Omega^+ \): volume to be reconstructed by Fusion-RFDK or Fusion-HFDK using projection data from the upper helical turn of the reverse helix

\( \Omega^- \): volume to be reconstructed by Fusion-RFDK or Fusion-HFDK using projection data from the lower helical turn of the reverse helix

\( \Omega_R \): volume to be reconstructed by Fusion-RFDK or Fusion-HFDK using projection data from the reverse helix

\( \Omega_F \): fusion zone

\( S_c \): cylindrical surface where source trajectories are located

\( S_R \): portion of the bounding surface of \( \Omega_R^c \) that is completely within the convex hull of the reverse helix

\( S_1^+ \): \( \pi \)-line surface that connects the top point of the reverse helix to all the points of the upper helical turn

\( S_1^- \): \( \pi \)-line surface that connects the bottom point of the reverse helix to all the points of the lower helical turn

\( S_0^+ \): \( \pi \)-line surface that connects the kink point to all the points of the upper helical turn

\( S_0^- \): \( \pi \)-line surface that connects the kink point to all the points of the lower helical turn

\( S^\lambda_+ \): positive partial cone surface with its vertex on the upper T-arc at angular position \( \lambda \)

\( S^\lambda_- \): negative partial cone surface with its vertex on the lower T-arc at angular position \( \lambda \)

\( O \): origin of the \((x,y,z)\) coordinate system
$O_d$: origin of the detector coordinate system

$O^\lambda_{+ z}$: center of the R-arc, which is the intersection between $S^\lambda_+$ and $\Pi_z$

$O^\lambda_{- z}$: center of the R-arc, which is the intersection between $S^\lambda_-$ and $\Pi_z$

$A^\lambda_+$: vertex point on the upper T-arc at angular position $\lambda$

$A^\lambda_-$: vertex point on the lower T-arc at angular position $\lambda$

$A^\lambda_z$: intersection between $\Pi_z$ and the T-line

$A^\lambda_{z_1, z_2}$: intersection between $\Pi_z$ and the line that connects $A^\lambda_{z_1}$ and $A^\lambda_{z_2}$
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