Enhanced homogenization technique for magnetomechanical systems using the generalized finite element method

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Abstract

Purpose – The purpose of this paper is to present a homogenization approach to model mechanical structures with multiple scales and periodicity, as they occur, e.g. in power transformer windings, subjected to magnetic forces.

Design/methodology/approach – The idea is based on the framework of generalized finite element methods (GFEM), where the normal polynomial finite element basis functions are enriched by problem dependent basis functions, which are, in this case, the eigenmodes of a quasi-periodic unit cell setup. These eigenmodes are used to enrich the standard polynomial basis functions of higher order on a coarse grid modeling the whole periodic structure.

Findings – It is shown that heterogeneous magnetomechanical structures can be homogenized with the developed method, as demonstrated by homogenization of a transformer coil setup.

Originality/value – An efficient homogenization procedure is proposed on the basis of the GFEM, which is extended using a special set of enrichment functions, i.e. the mechanic eigenmodes of a generalized eigenvalue problem.

Keywords Generalized finite element method, Homogenization technique, Magnetomechanical systems

Paper type Research paper

1. Introduction

A typical example for a magnetomechanical system is an electric power transformer. Its complex core and winding structures with many small-scale features (laminated core and insulated conductors) exhibit problems for the geometric modeling and mesh generation. The main problems stem from the fact, that the overall dimension of a power transformer is typically in the range of a few meters, whereas the single strands in the coils have only a crosssection of a few millimeters. In Figure 1, a typical high-voltage (HV) coil can be seen, which consists of 80 layers of disc-type windings with 22 conductors each. Here, the width of one conductor is about 1.5 mm, whereas the overall height is about 1.2 m. The conductors itself (copper) are wrapped with insulating material (paper and epoxy). The single layers are separated by wooden spacer blocks, distributed in circumferential direction.
In pure magneto-dynamic simulation, especially the efficient treatment of the laminated core is already addressed quite well (Chiampi and Chiarabaglio, 1992). Here, it is important to account for the correct distribution of eddy currents. However, the efficient and reliable computation of winding and tank vibrations due to Lorentz forces is still a major problem: here the accurate modeling of the complete transformer and especially the winding structure is of importance. The key issue lies in reducing the model complexity in the transformer coil by not resolving the single strands and isolation, but to take this small scale structure into account in a suitable way in a very coarse simulation model. As the permeability of those materials is in a similar range, the simulation model for the magnetic field does not have to resolve these structures in detail. However, the mechanical properties (stiffness and density) of the materials differ significantly, which demands for an accurate and efficient homogenization technique.

In the present work, we focus on this issue by utilizing the so-called generalized finite element method (GFEM) (Babuška and Melenk, 1997) to perform a two-scale homogenization of the mechanical winding structure of the HV coil of a power transformer. Here, the fact is exploited, that the coil exhibits a periodic structure, which allows to enrich the normal FEM by spectral information from a single periodic unit cell (Matache et al. 2000).

The outline of this work is as follows: in Section 2, we will introduce the very universal framework of the GFEM, based on the partition of unity method (PUM). The applicability of this method for homogenization using the scale separation property is presented in the subsequent Section 3, together with some convergence properties for a 2D model problem. In the last part in Section 4, we utilize the method to homogenize the HV winding of a power transformer and compare the results with a heterogeneous solution.

2. Framework of generalized finite elements

2.1 Theoretical considerations

In standard finite element methods, polynomial shape functions are used to approximate the unknown solution, which are known to approximate well smooth functions. If the used polynomials are of order \( p \) we can represent also polynomial functions of order \( p \) exactly. However, if the solution sought for has singularities or strong varying coefficients, it is well known that the FE mesh must resolve these small scale features near the singularities, resulting in a large number of unknowns. In contrast to this, the idea of the GFEM is to augment the standard polynomial basis functions of the traditional FEM by problem dependent, so-called enrichment...
functions. This method was first introduced in Strouboulis et al. (2000) and extends the more general idea of the PUM (Babuška and Melenk, 1997).

The PUM states, that if we consider a general simulation domain $\Omega$ we need to select the interpolation functions $\varphi$ according to:

$$\sum_i \varphi_i = 1 \quad \text{on} \quad \Omega. \quad (1)$$

In general, $\varphi$ can be arbitrary functions (e.g. harmonic functions, radial basis functions or even meshless methods) (Babuška and Melenk, 1997). However, in the GFEM method we use the standard piecewise polynomial shape functions $N$. The polynomial basis is then augmented with suitable, problem dependent functions $\psi$ using a tensor product approach:

$$u^h(x) = \sum_{i=1}^{n_n} \sum_{j=1}^{\mu} \hat{u}_{ij} N_i(x) \psi_j(x) = \sum_{i=1}^{n_n} \sum_{j=1}^{\mu} \hat{u}_{ij} \tilde{N}_{ij}(x) \quad \text{with} \quad \psi_1 \equiv 1 \quad \text{in} \quad \Omega, \quad (2)$$

where $u^h$ is a discretized function, $\hat{u}_{ij}$ the function coefficients, $n_n$ the number of standard polynomial basis functions and $\mu$ the number of problem dependent shape functions. Since we enforce $\psi_1 \equiv 1$, one can see that the GFEM reduces to the classical FEM when $\mu$ is set to 1. Note also, that the concrete choice of $\psi$ is not fixed by the GFEM itself. The functions could be either analytical solutions for the singularities or pre-calculated functions, which are themself obtained from a FE simulation.

One has to note, that the resulting modified shape functions $\tilde{N}_{ij}(x)$ do not necessarily form an orthonormal basis anymore. This may lead to severe problems in the process of solving the resulting system of equations, which may be ill-conditioned and semi-definite. If iterative solvers are used, one needs to apply special techniques (Tian et al. 2006). In our case, however, we use the direct solver 	extit{pardiso} (Schenk and Gaertner, 2004), which is capable of solving even indefinite systems robustly.

2.2 Two-scale homogenization of periodic mechanical setups

To apply the GFEM method for the homogenization of mechanical systems, we need to choose, which enrichment functions $\psi$ to take. We have already stated, that we consider problems with a periodic, multiscale character, i.e. the global problem domain $\Omega^L$ consists of $n_u$ unit cells $\Omega^l$ of size $l$, periodically arranged:

$$\Omega^L := \bigcup_{i} \Omega^l_i. \quad (3)$$

The overall size $L$ of the global problem domain here is much larger than the dimension of one unit cell $L >> l$. We consider here meshes, where the mesh size $h \approx l$, so one unit cell gets only resolved by a few finite elements.

The idea for the choice of $\psi$ is motivated by Matache et al. (2000): we solve a generalized eigenvalue problem on one unit cell $\Omega^l$ with periodic boundary conditions and take the first $\mu$ eigensolutions as enrichment functions for the homogenized problem. These eigenmodes contain the spectral information and the small scale features of the unit cell and are therefore suited to enlarge the function space on the macro mesh.
For the unit cell problem we solve the generalized eigenvalue problem:

\[
(K_{uu} - \omega^2 M_{uu})\mathbf{u}^h = 0 \text{ on } \Omega^f \quad \omega = 2\pi f,
\]

where $K_{uu}$ is the mechanical stiffness matrix and $M_{uu}$ represents the mass matrix. In addition we need to apply periodic boundary conditions to yield a set of conforming $l$-periodic enrichment functions.

Using the first $\mu$ vector-valued eigenmodes from equation (4), we arrive at the following approximation of $\mathbf{u}^h$ for the coarse scale:

\[
\mathbf{u}^h(x) = \sum_{i=1}^{n_u} \sum_{j=1}^{\mu} \hat{u}_y N_i(x) \psi_j(x) = \sum_{i=1}^{n_u} \sum_{j=1}^{\mu} \hat{u}_y \tilde{N}_i(x) \quad \text{with } \psi_1 = 1 \text{ in } \Omega.
\]

Note that we have now vectorial shape functions $N_i(x), \psi_j(x)$ and $\tilde{N}_i(x)$:

\[
N_i = \begin{pmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{pmatrix}
\]

\[
\psi_j = \begin{pmatrix} \psi_j^x & 0 & 0 \\ 0 & \psi_j^y & 0 \\ 0 & 0 & \psi_j^z \end{pmatrix}
\]

\[
\tilde{N}_i = \begin{pmatrix} \psi_j^x & 0 & 0 \\ 0 & \psi_j^y & 0 \\ 0 & 0 & \psi_j^z \end{pmatrix}
\]

as $\psi$ is obtained from the mechanical displacements, where each component $(x, y, z)$ has different values compared to the scalar functions $N_i$. With the modified basis from equation (5) we can solve, e.g. for the mechanical deformation by a given harmonic excitation with a known force $f$ on the global problem domain $\Omega^L$. The discrete problem reads as:

\[
K_{uu}\mathbf{u}^h + M_{uu}\ddot{\mathbf{u}}^h = f \text{ on } \Omega^L.
\]

The single entries of the mass matrix and stiffness matrix look like:

\[
m_{pq} = \int_{\Omega_e} \rho \left( \sum_{j=0}^{\mu} N_p \psi_j \right)^T \left( \sum_{j=0}^{\mu} N_q \psi_j \right) d\Omega
\]

\[
k_{pq} = \int_{\Omega_e} \left( \sum_{j=0}^{\mu} B_{ij}^T \right) [c] \left( \sum_{j=0}^{\mu} B_{ij} \right) d\Omega,
\]

where $\Omega_e$ is the domain of one finite element. Owing to the directional dependency of $\psi_j$ we need to calculate the $B_{ij}$ operator in matrix form as:
In the calculation of the $\mathcal{B}$ operator we need to take care of mixed derivatives:

$$
\frac{\partial \tilde{N}_i(x)}{\partial x} = \frac{\partial N_i(x)}{\partial x} \psi_j(x) + N_i \frac{\partial \psi_j(x)}{\partial x}.
$$

(11)

Note, that the density $\rho$ and the tensor of stiffness moduli $[c]$ are sampled at each integration point from the underlying unit cell problem.

With the modified basis from equation (5) we can choose a very coarse grid to capture only the overall geometry of $\Omega^L$, without resolving the small scale features. However, as pointed out in Matache et al. (2000) it might be advantageous not only to increase $\mu$ but also the degree $p$ of the standard polynomial shape functions $N_i$ to achieve uniform convergence. In this work we make use of the hierarchical, Legendre-based interpolation functions:

$$
N_1(\xi) = \frac{1}{2} (1 - \xi); \quad N_2(\xi) = \frac{1}{2} (1 + \xi); \quad N_i(\xi) = \phi_{i-1}(\xi), \quad i = 3, \ldots, p + 1,
$$

where $\phi_i$ denotes the integrated Legendre polynomials $L_i$:

$$
\phi_i(\xi) = \sqrt{\frac{2i - 1}{2}} \int_{-1}^{\xi} L_{i-1}(x) dx,
$$

(12)

$$
L_i(x) = \frac{1}{2^i i!} \frac{d^i}{dx^i} (x^2 - 1)^i.
$$

(13)

Explicit expressions for $\phi_i$ can be found, e.g. in Szabó and Babuška (1991). Note that only the first two functions $N_1$ and $N_2$ contribute to the value at the ends of the unit interval $[-1, 1]$, whereas all other functions $N_i$ of higher order $i > 2$ give only a non-zero value within the interval. Therefore, they are also called internal modes or bubble modes and vanish on the element boundaries in 1D.

3. 2D model problem

To test the convergence behavior of the method, we investigate a 2D setup of a coil-like periodic structure, consisting of two materials (Figure 2). Each periodic unit cell has a dimension of $4 \times 4$ m, made of two materials: Material 1 has a Young’s modulus of
$E = 1.5 \times 10^9$ N/m$^2$, a Poisson ratio of $\nu = 0.3$ and a density $\rho = 230$ kg/m$^3$, whereas the stiffer Material 2 has $E = 1.5 \times 10^{10}$ N/m$^2$, $\nu = 0.3$ and $\rho = 2,300$ kg/m$^3$.

For the reference solution we compare once a setup with 10 unit cells and a second one with 100. For the discretization of the former one we use 4,700 quadrilateral elements of polynomial order $p = 8$. We compare two analysis types:

1. Static simulation with an applied load of 1 kN in vertical direction. The structure is clamped on the left and right side. As convergence criterion we take the mechanical energy of the deformed structure.

2. Eigenfrequency simulation of the unclamped structure. The convergence of the first eigenmode (no rigid body mode) is compared.

For the calculation of the enrichment functions we model one unit cell using a very fine mesh with 1,500 quadrilateral elements of second order (Figure 3).

In addition we need to apply periodic boundary conditions on $\Gamma_l$ and $\Gamma_r$:

$$u(x)|_{\Gamma_l} = u(x)|_{\Gamma_r}$$

(14)

using nodal constraints to obtain a conforming set of enrichment basis functions. The resulting eigenmodes – excluding the two rigid body modes – can be seen in Figure 4. The jump in the nodal displacement at the boundary between the two materials is clearly visible in all six eigenmodes.

For the homogenized setup, we discretize each unit cell only with $2 \times 2$ quadrilateral elements (Figure 5) and keep the macro-mesh aligned with the boundaries of the unit cell.
cell problem. The alignment of both models is not strictly necessary but accelerates the calculation, as our implementation makes use of a caching mechanism, which reuses the mapping from integration points of the macro- to unit-cell elements. To test the convergence, we vary the polynomial degree on the macro mesh $p = 1, \ldots, 5$ and also the number of enrichment functions $\mu = 1, \ldots, 5$. In the first case, we compare the relative error in mechanical energy for the static load applied. The results can be seen in Figures 6 and 7 for 10 and 100 periods, where we compare the relative error in mechanical energy against the number of degrees of freedom (DoF) needed. It can be observed, that when the polynomial degree is chosen $p = 1$, the additional enrichment functions do not increase the convergence in both cases. Only for $\mu > 3$ the convergence increases a little. In contrast, if we increase $p$ and $\mu$ simultaneously, the error decreases much more rapidly. It is also evident, that the combined increase of $p$ and $\mu$ is superior to a pure $p$-refinement, where no additional shape functions are used. A similar result can be seen when the relative error in the first eigenfrequency is compared (Figures 8 and 9). Also here only the combined increase of $p$ and $\mu$ leads to a fast convergence.

4. 3D model of transformer coil
In the following we extend our homogenization approach to a fully 3D model of a transformer winding. Here, we apply the homogenization only on the HV coil, which consists of 80 layers with 22 conductors each.

In the reference model we restrict ourselves to a quarter-symmetric model, where the fine-grained winding structure is resolved with finite elements of first order, resulting in approximately 770,000 nodes. The model can be seen in Figure 10. It is clear, that especially the HV-winding is responsible for the large number of nodes and therefore limits the simulation of a complete transformer with three limbs. Using the reference model, we perform an eigenfrequency simulation and determine the first three
eigenmodes, which are shown in Figure 11. We clearly can see, that the modes exhibit no axial symmetry, which justifies the use of a 3D model. In addition, these modes are located near the critical frequency of the Lorentz forces (100/120 Hz depending on the line frequency), which might amplify the radial displacement drastically.

Figure 6.
Error in mechanical energy of homogenized model (10 periods)

Figure 7.
Error in mechanical energy of homogenized model (100 periods)
For the unit cell, we now have a single coil layer, i.e. one layer of conductors and paper isolation, with half a layer of spacer blocks on bottom and top (Figure 12). We also need to include the surrounding air as pseudo material for which we choose a very small Young’s modulus $E = 10 \text{ N/m}^2$ and a density of $\rho = 1 \text{ kg/m}^3$. In addition, we have to
Figure 10.
Quarter-symmetric reference model of transformer coil

Figure 11.
First three eigenmodes of reference coil model
apply periodic boundary conditions in axial direction to get conforming \( l \)-periodic enrichment functions. The whole model consists of approximately 155,000 nodes and 34,000 hexahedral elements with second order Lagrange polynomials.

The resulting first six eigenmodes can be seen in Figure 13, which can be separated in purely in-plane modes (modes 1, 4 and 6) and out-of-plane modes (2, 3 and 5). Note, that especially in the out-of-plane modes the distribution of the radial spacer blocks is visible.

In the homogenized model (Figure 14), we model the HV coil as one single solid geometric entity, without resolution of the winding and isolation details, which reduces the total number of nodes in the model to approximately 208,000 nodes.

In contrast to the 2D model problem we have here to consider the transition from the homogenized HV coil to the clamping structure, where \( \mu = 1 \). In our approach, we simply set the higher order modes to 0 (Dirichlet boundary condition) at the interface nodes between the homogenized coil \( \Omega_{\text{hom}} \), i.e. the HV coil, and the heterogeneous part \( \Omega_{\text{hetero}} \), i.e. the spacer blocks and the rest of the coils:

\[
\psi_j(x) = 0 \text{ for } x \in \Omega_{\text{hom}} \cap \Omega_{\text{hetero}} \text{ and } j > 1. 
\]  

The details can be seen in Figure 14, where the nodes at the interface are emphasized.

Using the eigenmodes from the previous unit cell simulation we perform an eigenfrequency simulation of the homogenized model and compare the convergence for...
the first three eigenfrequencies, which lie in the range of interest (Table I). When no additional enrichment functions are used ($\mu = 1$), none of the eigenmodes from Figure 11 are shown. In this case, the underlying small-scale structure of the HV-winding is not resolved. When we now successively add the first three eigenmodes from Figure 13 we initially observe convergence of the first eigenmode ($\mu = 2$) and for $\mu = 3, 4$ also for the eigenmodes 2 and 3. The overall number of DoF could be reduced here to approximately one third compared to the reference model. In order to further increase the accuracy, one could raise the macro polynomial degree $p$, in accordance with the results from Section 3.

5. Conclusion
In this work, the framework of the generalized finite element and its most important properties are introduced. The applicability of this method to homogenization of elastic systems is shown by selecting the enrichment functions as eigenmodes of a suitable chosen, periodic unit cell problem. Some general convergence studies are performed for

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<th>Table I.</th>
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<td>Convergence of first three eigenfrequencies of the homogenized coil model</td>
</tr>
<tr>
<td>No. of dofs</td>
</tr>
<tr>
<td>2,269,010</td>
</tr>
<tr>
<td>615,380</td>
</tr>
<tr>
<td>667,589</td>
</tr>
<tr>
<td>719,798</td>
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<td>772,007</td>
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a 2D model problem. Most remarkably we show that also for a complex 3D setup of a transformer winding with mixed homogeneous and heterogeneous regions the proposed method can reduce and simplify the modeling work drastically.

References


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