On the field-induced transport
of magnetic nanoparticles
in incompressible flow

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Über die feldinduzierte Bewegung magnetischer Nanopartikel in inkompressiblen Strömungen

Zusammenfassung


Anders als in den meisten Arbeiten, die man in der Literatur findet, lassen wir uns bei der Wahl der Randbedingungen von physikalischen Überlegungen und nicht so sehr von mathematisch pragmatischen Erwägungen leiten. So implizieren die Randbedingungen der Magnetisierungsgleichung in „Modell GW“ bei Magnetisierung \( m \) und Gesamtmagnetfeld \( h \) zwar \( H(\text{div, rot}) \)-Regularität, möglicherweise jedoch keine \( \mathcal{H}^1 \)-Regularität. Unter geeigneten Annahmen, darunter eine spezielle nichtlineare Diffusion und Einschränkung auf zwei Raumdimensionen, wird globale Existenz einer distributionellen Lösung in der Zeit gezeigt. Dazu wird ein neuartiger Approximationsansatz entwickelt, der es erlaubt die Kelvin-Kraft \( (m \cdot \nabla) h \) im distributionellen Sinn zu definieren. Im dreidimensionalen Fall wird ein Existenzresultat für ein regularisiertes Modell bewiesen unter allgemeinen Rahmenbedingungen.

Zu allen drei Modellen wird ein uneingeschränkt (energetisch) stabiles Finite-Element-Verfahren entwickelt. Die numerische Methode für „Modell GW“ wurde bereits in [39] veröffentlicht. Ähnlich wie in [51,52, Nochetto (2015)] werden nicht-konforme finite Elemente zur Approximation der Magnetisierung verwendet. Erstens gelingt eine geeignete Diskretisierung der Differentialoperatoren \( \nabla \text{div} \) und \( \text{rot}^2 = \text{rot} \text{rot} \). Voraussetzung dafür ist die variationelle Definition diskreter Operatoren \( \text{div}_h \) und \( \text{rot}_h \), die \( \mathcal{H}^1 \)-konforme Approximationen von Divergenz und Rotation eines Vektorfeldes liefern. Die Wohlgestelltheit der numerischen Schemata, d.h. die Existenz diskreter Lösungen, ist unabhängig von den Diskretisierungsparametern für alle drei Modelle dieser Dissertation garantiert. Hierfür wurde der Schaefer’sche Fixpunktsatz verwendet.

Abstract

Based on Onsager’s variational principle [55,56], the motion of superparamagnetic nanoparticles – which are suspended in an incompressible carrier fluid and subjected to an external magnetic field – is described by systems of partial differential equations. Various modeling assumptions lead to three different systems denoted by 'model GW', 'model W' and 'model B'. When proposed in our joint work [39, 2019], 'model GW' was – to the best of our knowledge – the first one to include evolution equations for the magnetization field and for the magnetic particle density – all of them are nonlinearly coupled to the magnetostatic and the Navier-Stokes equations. The other two models use algebraic equations to determine the magnetization based on the linearized Langevin formula and are derived by us for the purpose of comparison. In case of 'model W', we assume that the suspension yields a single-phase flow with negligible mass of magnetic nanoparticles. The other model is derived under the assumption that fluid particles and magnetic particles set up a two-phase fluid. It shows similarities to the model in [41, Himmelsbach (2017)]. All three models follow a two-domain approach – considering the magnetic field on a possibly larger domain compared to the fluid domain – of which we expect higher accuracy in determining the total magnetic field. The case when the two domains coincide requires some slight changes which are discussed when needed.

In contrast to other works in the mathematical literature, the boundary conditions of the magnetization equation in 'model GW' are motivated by physical arguments only, not by practical aspects with respect to mathematical analysis. They entail $H(\text{div}, \text{curl})$-regularity of the magnetic quantities – magnetization and (total) magnetic field – but possibly not $H^1$-regularity. To establish existence of solutions to this model, the absence of $H^1$-regularity is compensated by an intricate approximation procedure. For this, we give a meaning to the Kelvin force $(\mathbf{m} \cdot \nabla)\mathbf{h}$ in the distributional sense. Existence of distributional global-in-time solutions is guaranteed under appropriate assumptions. The latter include nonlinear diffusion to be used in the evolution equation for the magnetic particles’ density and the restriction to the two-dimensional setting. An existence result of global-in-time weak solutions to a regularized model is presented also in the three-dimensional setting.

To each of the three models, we propose an unconditionally energy stable finite element scheme. The discretization of 'model GW' has already been published in our joint work [39]. Here, non-conforming finite elements are used to approximate the magnetization – similar as in [52, Nochetto (2015)]. For the first time, however, the second order differential operators $\nabla \text{div}$ and $\text{curl}^2 = \text{curl} \text{curl}$ in the magnetization equation have been discretized by introducing the operators $\text{div}_h$ and $\text{curl}_h$ – discrete versions of $\text{div}$ and $\text{curl}$. The latter are defined by duality and yield $H^1$-conforming approximations of the divergence and curl of a vector field. By means of Schaefer’s fixed point theorem, existence of discrete solutions is guaranteed for all three schemes. The existence results are independent of the discretization parameters.

The thesis concludes with simulations that serve as a proof of concept for the models and their numerical schemes. The three models are compared in the case of linear diffusion and the case of nonlinear diffusion which has been used in the analysis part of this thesis. For simplicity, most simulations are performed under the assumption that the domain of the magnetic field coincides with the fluid domain. Using 'model W' – for practical reasons – the effects of multiple different external magnetic fields are examined as well as the impact of using a strictly larger domain for the magnetic field.
1 Introduction

Multiple different magnetic materials are used in modern applications ranging from solids to liquids of varying magnetic properties. Recent research is especially interested in colloidal suspensions of magnetic nanoparticles which typically are coated with polymers in order to "stabilize" the suspension (cf. [74]). As the size of these nanoparticles is sufficiently small, particle-particle interactions are such low (further reduced by the coating) that they are superparamagnetic, i.e. they show no remnant magnetization (cf. hysteresis effects) in absence of external magnetic fields – due to thermal fluctuations. Within the fluid they move around freely and can be controlled via external magnetic fields. They differ from other popular magnetic materials as follows.

- In colloidal suspensions, superparamagnetic nanoparticles form no agglomerates, have variable macroscopic magnetization and their concentration in the fluid may be inhomogeneous.

- In ferrofluids [54], superparamagnetic nanoparticles are suspended in an oil-based carrier liquid in sufficiently high concentration, this way keeping them close together leading to an approximately homogeneous particle concentration.

- Magnetorheological fluids are suspensions involving larger magnetic particles with diameters in the range of micrometers. Particle-particle interactions are strong which is why these fluids change their viscosity in external magnetic fields (used e.g. for damping).

- Magnetoeelastic materials typically are solids (which can come in a variety of sizes) such that the magnitude of their magnetization remains constant but their shape can change in magnetic fields (e.g. magnetic actuators).

The above features with matching colors roughly play the role of opposites. The first type of magnetic material in the list above has the smallest number of specific features but this is exactly why they are of growing interest. The superparamagnetic nanoparticles can be seen as little agents performing specific tasks without changing the properties of the surrounding fluid too much. This is important in medical applications, where e.g. the blockage of blood vessels could lead to a fatal incident. Magnetic nanotechnology is a rapidly growing field of research finding new ways to exploit magnetic properties and at the same time avoiding unwanted side effects.

In fact, superparamagnetic iron oxide nanoparticles (SPIONs) show high biocompatibility, are already clinically approved as contrast agents for magnetic resonance imaging and their design is still being improved by active research [69]. Recent research suggests new promising applications for SPIONs as drug delivery systems (magnetic drug targeting [67]). Here, the magnetic particles carry drug molecules and flow through the vascular system without clogging the blood vessels. Being controlled by magnetic fields they hold on to precisely defined locations in the human body where they are needed. The goal of this procedure is to reduce side effects and increase performance of the medical treatment, e.g. in tumor therapy.

However, nanotechnology based on magnetic particles is not restricted to medical sciences. Studies of magnetic nanoparticles as message carriers are made in the context of molecular communication systems. Molecular systems are of such small size that communication based on electromagnetic waves is not feasible anymore (cf. [43][70][73]). Magnetic fields might be used to steer magnetic particles through channel systems and to improve sharpness of signals. Another application is concerned with remediation of nanoplastics from water [61]. Here, the magnetic particles are treated in such a way that they act as a glue between the plastic particles. After larger agglomerates are formed, the plastic particles can be removed from the water together with the magnetic particles by an external magnet.
Despite the growing interest from the application side, the mathematical understanding of such magnetic fluids is still in its early stages. In this thesis we derive a new system of partial differential equations (cf. (1.1)) via principles of non-equilibrium thermodynamics (see also [39]) which model the field-induced transport of superparamagnetic nanoparticles in flow and examine the latter analytically and numerically. Concerning the related mathematical literature about superparamagnetic particles, mainly two previous lines of research can be distinguished so far.

(Hom-compl): In the first line of research, classical ferrofluids are considered such that a homogeneous distribution of magnetic particles can be assumed both in the case of single-phase flows [3, 52] (consequently the density of magnetic particles is constant in space and time) and two-phase flows [51] (the density of magnetic particles is approximately constant in space for each phase and contained between designated values in a diffuse interface). Rosensweig [60] derived a complex system of equations for this setting. However, mathematical literature is based on simplifications thereof, all of them quite similar in spirit, which at least include the Navier-Stokes equations, a relaxation law for the magnetization – developed by Shliomis [63] – and the magnetostatic equations.

(Inhom-simpl): In a second line of research, the transport of dilute magnetic nanoparticles with inhomogeneous particle densities varying in space and time have been investigated [41]. The models used for this purpose have in common that the magnetization is assumed to be directly proportional to particle density and magnetic field, which simplifies the magnetic modeling. Moreover, other simplifications may apply, e.g. assuming a fixed velocity flow field to be given which clearly shows that the inhomogeneous particle density is the main interest.

State of the art. The following is an excerpt of the available results in mathematical literature which show the scientific context of this thesis.

ad (Hom-compl) Nochetto considered a model which additionally includes angular momentum [52], [53], in the case of single-phase flows. Also, a simple two-phase flow model [51] was examined (mostly) numerically and the famous Rosensweig-instability (cf. Figure 1.1) has been recaptured numerically in two space dimensions. Amirat and Hamdache considered single-phase flows [3–8] analytically and established an elaborate existence theory. They used a regularization in the magnetization equation and assumed mathematically favorable boundary conditions for the magnetization – particularly vanishing normal trace – which admitted \( H^1 \)-regularity of the magnetic quantities. The more recent result of Nochetto [53] uses a different approach by means of renormalization in spirit of DiPerna Lions’ theory for compressible fluids [26, 47]. The latter approach avoids the aforementioned regularization term in the magnetization equation by exploiting a clever way to rewrite the magnetic body force in the setting of homogeneously distributed magnetic particles.

Figure 1.1: Snapshot of the surface instability, the so-called 'Rosensweig instability'. License free image – https://www.pikist.com/free-photo-xdaiu/de
1.1 Achievements of this thesis

Models with non-homogeneous and non-steady distribution of magnetic particles are mainly considered numerically. Existence of solutions is only guaranteed in very special situations, to the best of our knowledge. Himmelsbach, Neuss-Radu and Neuß [sic] have shown existence and uniqueness of radially symmetric solutions in two space dimensions for their new model proposed in [41]. It features an evolution equation for the particle density coupled to the magnetostatic equations. A flow field is assumed to be given and numerical experiments are carried out. One of the main goals of numerical experiments is the simulation of magnetic drug targeting, albeit other investigations are going on, too. For example, Polevikov and Tobiska [57] were numerically interested in a steady-state diffusion problem for particles in a ferrofluid. General existence theory of such models is still an open field of research.

As the inhomogeneous case is the more general and complex case, one might want to look at the techniques from (Hom-compl) first to find suitable techniques for the inhomogeneous case (Inhom-simpl). Especially in view of the many numerical simulations in the latter case, it would be most welcome if one could find ways to transfer the results from the homogeneous to the inhomogeneous case. However, no pathways emerge from previous literature.

In order to close the gaps between the two research lines, one could think of two more lines of research denoted by (Hom-simpl) and (Inhom-compl), respectively. The former is concerned with the limit of the relaxation time \( \tau_{rel} \to 0 \) in case of homogeneous distribution of magnetic particles [53] – arriving at a simplified magnetization law as in (Inhom-simpl). However, in this thesis we want to develop the latter research line (Inhom-compl) and strive to generalize the complex ferrofluid models of (Hom-compl) to the case of inhomogeneous particle densities (cf. our joint works [39,40]). For future research, it might also be interesting to investigate if the methods of [53] (transition from (Hom-compl) to (Hom-simpl)) can be adapted to the inhomogeneous case in order to obtain analytical results for (Inhom-simpl) via the techniques of this thesis (Inhom-compl). After all, the absence of a separate equation for the magnetization in the second line of research (Inhom-simpl) might be the reason why existence results are so difficult to obtain. For further information we refer the reader to Section 2.4.3.

1.1 Achievements of this thesis

We connect the two lines of research, (Hom-compl) and (Inhom-simpl), which were mentioned earlier, i.e. we examine the transport of suspended superparamagnetic particles (which can move freely) numerically and establish a corresponding mathematical foundation. To put it briefly, our achievements are summarized as follows.

- First results in research line (Inhom-compl) by deriving a novel model including evolution equations for both (inhomogeneous) particle density and magnetization, denoted by "model GW".
  - Using a new type of boundary conditions for the magnetization – not prescribing tangential or normal components thereof.
  - Improving the stray field’s (consequently the total magnetic field’s) accuracy by a two-domain approach, i.e. modeling of the magnetic field on a possibly larger domain.

- Proof of first existence results of global-in-time weak/distributional solutions to "model GW" under appropriate assumptions – nonlinear diffusion and two space dimensions.

- Further derivation of reduced models, "model W" and "model B", which are used to compare models from (Inhom-compl) to models from (Inhom-simpl) numerically – improving the understanding on the advantages and disadvantages of the model types.
• Design of energy stable numerical schemes and running proof-of-concept simulations.

The works of Nochetto et al. [51–53], Amirat and Hamdache [3,4] and Himmelsbach and Neuss-Radu and Neuß [41] seem to be most relevant and inspiring for this thesis. In Figure 1.2, the analytical results of this thesis (green boxes) will be put into context of the latter references. Note that in this sketch no chronological or causal order is implied. There are other references which are not included in this comparison and the numerical perspective has been omitted entirely.

Emerging from the first line of research (Hom-compl) our main concern is the transition from homogeneous distributions of magnetic particles to variable distributions while avoiding the simplifications of the second line of research (Inhom-simpl). In Figure 1.2. Our "model GW" couples an evolution equation for the magnetic particle density $c$ to a system of equations of similar complexity as in [3], involving the hydrodynamic equations, an evolution equation for the magnetization $m$ (inspired by [63]) and the magnetostatic equations for the magnetic field $h$ which include the effects of an applied external field $h_a$ and of the particle-induced magnetic field $h$.

Figure 1.2: Selected analytical results put into context.
1.1 Achievements of this thesis

Stray field \( h_d \). As far as we know, our model, see in (2.89) (see also (1.1)), is the first one to account for both non-constant magnetic susceptibilities – a consequence of the inhomogeneous particle density – and the evolution of magnetization fields. The derivation is based on Onsager’s variational principle for which we assumed the magnetic contribution to the energy to consist of the system’s classical magnetic field energy,

\[ (h + m) \cdot h, \]

a term of the type \((h + m) \cdot h\),

as well as Zeeman-type energies describing the particle-field interaction,

\[ -m \cdot h_a \text{ and } -m \cdot h_d, \]

and a decay energy reflecting the superparamagnetic behavior of the particles,

\[ |m|^2. \]

A clever reformulation of the total magnetic energy shows coercivity of the respective energy functional, see Section A.3.

In earlier analytical works [3], global \( H^1 \)-regularity of the magnetization has been achieved via the diffusive term

\[ -\sigma \Delta m = -\sigma \nabla \text{div} m + \sigma \text{curl curl} m \]

in the magnetization equation and by exploiting the embedding of \( H(\text{div, curl}) \)-regular functions with vanishing normal trace into \( H^1(\Omega)^d \). This thesis follows a similar principle, however, we avoid the latter boundary condition to justify our model in the case when magnetic particles accumulate at the boundary and non-negligible normal traces are expected, see e.g. the discussion in Section A.3. By modifying typical no-flux conditions in order to account for the additional flux of magnetization which is induced by the diffusive term, one arrives at the boundary conditions

\[ ((v_{\text{part}} \cdot \nu) m_{\text{nor}} - \sigma \text{div} m \nu) |_{\partial \Omega} = 0, \]
\[ ((v_{\text{part}} \cdot \nu) m_{\text{tan}} - \sigma \text{curl} m \times \nu) |_{\partial \Omega} = 0, \]

where \( m_{\text{nor}} \) and \( m_{\text{tan}} \) denote the normal and tangential components of \( m \), respectively, and \( v_{\text{part}} \) is the convective velocity of the particles due to diffusive and magnetic effects. Notice that in case of homogeneous particle densities (hence no particle flux) the conditions (2.90c), (2.90d) boil down to the natural boundary conditions of the \( \nabla \text{div} \)-operator and the \( \text{curl curl} \)-operator, respectively, also appearing as suitable candidates in the discussion about boundary conditions in [52]. However, as a drawback, \( H^1 \)-estimates can only hold locally as \( H(\text{div, curl})(\Omega) \) does not embed compactly into \( L^2(\Omega)^d \), see [9]. This particularly complicates the treatment of the Kelvin force \((m \cdot \nabla) h\) where a gradient is applied to the magnetic field \( h \). Nevertheless, we were able to give a meaning to the latter and to prove existence of global solutions in the sense of distributions in two space dimensions (see also [40]).

For further physical improvements, we choose the fluid domain \( \Omega \) to be compactly contained in the magnetic field domain \( \Omega' \), because the magnetic particles, while aligned in a magnetic field, induce another magnetic field themselves – often called stray field or demagnetizing field – raising the question of suitable boundary conditions for the total magnetic field. As far as we know, both domains have been chosen to be identical in the related literature so far which might lead to unphysical boundary behavior of the magnetic field. In our framework, the two domains \( \Omega' \) and \( \Omega' \setminus \Omega \) are linked via transmission conditions, see (2.90f) (see also (1.3e)). The impact of the two-domain approach will be examined numerically, see Figure 1.3 for a preview. Moreover, while comparing ’model GW’ to our reduced ’model W’ on basis of finite simulations, we identified distinguishing features which we briefly summarized in Table 1.1. Hence, based on our numerical experiments (see Section 5) we recommend to use a strictly larger magnetic field domain \( \Omega' \supset \Omega \) for maximum physical accuracy and we recommend ’model GW’ if one strives for good numerical accuracy. On the other hand, if one is interested in short computation times, a reduced model is recommended.
1.2 Outline of this thesis

This section gives an overview on the structure of this thesis which can roughly be split into the parts 'modeling', 'mathematical analysis', 'numerical analysis' and 'simulation results'.

**Modeling.** In the modeling part,[Section 2] Onsager’s variational principle \[55,56\] is used to derive three different models. Application of this method is divided into three steps. The governing equations will be introduced first. Afterwards, energy and dissipation which are needed for the variational approach will be discussed. Lastly, the specific model equations will be derived. At the end of the modeling part, weak formulations and a priori estimates will be presented formally.

The main model will be denoted by "model GW" and has already been derived (up to a change of boundary conditions) in \[39\]. Using generic parameters \( \alpha_1, \alpha_3 > 0, \beta \in \mathbb{R} \) and mobility \( f_p(c) \), the equations read

\[
\begin{align*}
\rho_0 u_t + \rho_0 (u \cdot \nabla) u + \nabla p - \text{div}(2\eta D u) &= \mu_0 (m \cdot \nabla)(\alpha_1 h + \frac{\beta}{2} h_a) + \frac{\mu_0}{2} \text{curl}(m \times (\alpha_1 h + \frac{\beta}{2} h_a)), \\
\text{div } u &= 0, \\
\alpha_1 u_t + u \cdot \nabla c + \text{div}(cv_{\text{part}}) &= 0, \\
v_{\text{part}} &= -KD \frac{f_p(c)}{c} \nabla g'(c) + K\mu_0 \frac{f_p(c)}{c^2} (\nabla (\alpha_1 h + \frac{\beta}{2} h_a) - \alpha_3 m))^{T} m, 
\end{align*}
\]

\( \Omega \subset \Omega', \ t = 14. \)

Figure 1.3: Visualization of the particle induced stray field (normalized arrows), density of magnetic particles (black color levels) and magnetization (red/green cylinders mimicking a bar magnet). For further information see Section 5.1.

<table>
<thead>
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<th>( \Omega = \Omega' )</th>
<th>Numerical accuracy</th>
<th>Physical accuracy</th>
<th>Computational costs</th>
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<tr>
<td>GW</td>
<td>+</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>W</td>
<td>-</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>( \Omega \subset \Omega' )</td>
<td>(+)</td>
<td>+</td>
<td>(+)</td>
</tr>
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</table>

Table 1.1: Positive (+), negative (-) and neutral (0) rating of complex 'model GW' and reduced 'model W' in direct comparison. Concerned with computational costs, a positive rating means lower costs, i.e. faster computation. Ratings in brackets have not been discussed explicitly in this thesis but are very plausible from the contents of [Section 5.1.4] and [Section 5.2]. Particularly in light of Figure 5.23 and Table 5.4 one can expect in case of \( \Omega \subset \Omega' \) that the numerical accuracy improves and the computational costs increase (more degrees of freedom).
\[ -\Delta R = \text{div} \, m, \quad (1.1e) \]
\[ m_t + \text{div}(m \otimes (u + v_{\text{part}})) - \sigma \Delta m = \frac{1}{2} \text{curl} \, u \times m - \frac{1}{\tau_{\text{rel}}} (m - \chi(c, h)h) \quad (1.1f) \]
in \( \Omega \times (0, T) \) and
\[ -\Delta R = 0 \quad (1.1g) \]
in \( (\Omega' \setminus \Omega) \times (0, T) \). Here, the magnetic field is given by its potential,
\[ h := \nabla R. \quad (1.2) \]

The system is completed by boundary conditions
\[ u = 0 \quad \text{on } \partial \Omega \times [0, T], \quad (1.3a) \]
\[ c v_{\text{part}} \cdot \nu = 0 \quad \text{on } \partial \Omega \times [0, T], \quad (1.3b) \]
\[ (v_{\text{part}} \cdot \nu)(m - (m \cdot \nu)\nu) - \sigma \text{curl} \, m \times \nu = 0 \quad \text{on } \partial \Omega \times (0, T), \quad (1.3c) \]
\[ (v_{\text{part}} \cdot \nu)(m \cdot \nu) - \sigma \text{div} \, m = 0 \quad \text{on } \partial \Omega \times (0, T), \quad (1.3d) \]
transmission conditions
\[ [\nabla R + m] \cdot \nu = 0 \quad \text{on } \partial \Omega \times [0, T], \quad (1.3c) \]
\[ \nabla R \cdot \nu = h_a \cdot \nu \quad \text{on } \partial \Omega' \times [0, T], \quad (1.3f) \]
and initial conditions. Here, \( u \) is the velocity field of the fluid, \( p \) is its pressure, \( c \) denotes the density of magnetic particles, \( m \) is the magnetization and \( h = \nabla R \) describes the total magnetic field. The vector field \( v_{\text{part}} \) denotes the convective velocity corresponding to the flux. Detailed explanations are in Section 2 or [39].

The other two models are derived for comparison and fit into the second line of research [Inhom, simpl] mentioned before. The second model, denoted by 'model W', is a simplification of 'model GW' where the magnetization equation is replaced by an explicit expression for the magnetization. These two models have in common that they assume the magnetic particles’ momentum to be negligible. The third model is devoted to the case when the latter momentum is not negligible. It will be denoted by 'model B'. For this, a two-phase flow approach is used to account for the momentum of magnetic particles. It turns out that 'model B' resembles the model of [41] with regard to magnetic aspects and extends it by coupling it to the momentum equation for incompressible flow. The motivation for the comparison of these models has its origin in a discussion between ourselves, E. Bänsch, G. Grün, M. Neuss-Radu and R.H. Nochetto.

Mathematical analysis. In Section 3, existence of solutions to 'model GW' is examined via a Galerkin approximation. In less detail, this was already done in [40]. The analysis part of this thesis starts with the construction of discrete ansatz spaces, see Section 3.1. Afterwards we pass to the limit in two steps.

- First, additional regularizations will be introduced which include the diffusive term \(-\sigma_c \Delta c\) in the density equation of magnetic particles as well as cut-offs from below near zero, because the particle density is not guaranteed to be non-negative on the level of Galerkin solutions. Keeping them fixed, we pass to the limit from the discrete Galerkin setting to the continuous setting. Therefore, existence of weak solutions to a regularized model is obtained as an intermediate step in Section 3.3.
• Afterwards, the additional regularizations will be dropped in the two-dimensional case assuming nonlinear diffusion. The details are discussed in Section 3.4. Notice that – similar to [3] – we do not consider $\sigma \to 0$ in the second passage to the limit as the recent tricks of [53] which handle the case of negligible spin magnetic moment, i.e. $\sigma = 0$, are not directly applicable in the inhomogeneous setting.

The main result will be briefly introduced here and will reappear and be proven in Section 3.4. For this, we introduce the notation $L^p(I; W^{k,q}_0(\Omega))$ which means the following. A function $f$ is in aforementioned space if for any $\Omega \subset \subset \Omega$ the restriction $f|_{\Omega}$ is in the Bochner-space $L^p(I; W^{k,q}_0(\hat{\Omega}))$ (see also Section 5.3). For the definition of the space $\mathcal{R} \subset H^1_{\text{mean}}(\Omega')$ we refer the reader to (3.8).

**Main theorem.** Let the spatial dimension be $d = 2$ and consider the mobility $f_p(c) = c^2$ for the particle density. Let initial data

$$ u^{\text{init}} \in H_{n0}(\text{div}_0)(\Omega), \quad c^{\text{init}} \in L^2(\Omega; \mathbb{R}^+), \quad m^{\text{init}} \in L^2(\Omega)^d $$

be given. Under general assumptions on the boundary regularity of the domain $\Omega \subset \subset \Omega$, external fields $\hat{\mathbf{H}}^{1.2}$ as well as a growth condition (like a square root) on the susceptibility $\hat{\mathbf{H}}^{3.3}$, there are functions

$$ u \in L^2(I; H^1_0(\Omega)^d \cap H(\text{div}_0)(\Omega)) \cap L^\infty(I; L^2(\Omega)^d \cap H^1(\Omega; (H^2(\Omega)^d \cap H(\text{div}_0)(\Omega))'),

$$ $c \in L^{4/3}(I; W^{1,4/3}_0(\Omega)) \cap L^\infty(I; L^1(\Omega)) \cap L^2(I; L^2_{\text{loc}}(\Omega))$, $R \in L^2(I; \mathcal{R})$ with $R_{\Omega} \in L^\infty(I; H^1(\Omega)) \cap L^2(I; H^2_{\text{loc}}(\Omega))$, $m \in L^2(I; H(\text{div}, \text{curl})(\Omega)) \cap L^\infty(I; L^2(\Omega)^d) \cap L^2(I; H^1_{\text{loc}}(\Omega)^d) \cap H^1(I; (H^2(\Omega)^d)'),$

such that for all

$$ \mathbf{v} \in L^2(I; (H^2(\Omega)^2 \cap H(\text{div}_0)(\Omega))), \quad \psi \in C^1([0, T]; C^1_0(\Omega)) \text{ with } \psi(T) \equiv 0, \quad S \in \mathcal{R}, \quad \Psi \in L^2(I; H^3_0(\Omega)^2),

$$

the equations

$$ \rho_0 \int_0^T \langle \partial_t u, \mathbf{v} \rangle (H^2_0(\Omega)^d \cap H(\text{div}_0)(\Omega))' \times H^3_0(\Omega)^d \cap H(\text{div}_0)(\Omega) \text{ d}t + \int_0^T \int_\Omega 2\eta \mathbf{D} u \cdot \mathbf{D} \mathbf{v} \text{ d}x \text{ d}t

$$ $+ \frac{\rho_0}{2} \int_0^T \int_\Omega (u \cdot \nabla) u \cdot \mathbf{v} \text{ d}x \text{ d}t - \frac{\rho_0}{2} \int_0^T \int_\Omega (u \cdot \nabla) \mathbf{v} \cdot u \text{ d}x \text{ d}t

$$ $= -\mu_0 \int_0^T \int_\Omega (m \cdot \nabla) \mathbf{v} \cdot (\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a) \mathbf{d}x \text{ d}t - \mu_0 \int_0^T \int_\Omega \text{div } m \mathbf{v} \cdot (\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a) \text{ d}x \text{ d}t

$$ + \frac{\mu_0}{2} \int_0^T \int_\Omega (m \times (\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a)) \cdot \text{curl } \mathbf{v} \text{ d}x \text{ d}t,

$$ - \int_0^T \int_\Omega c \partial_t \psi \text{ d}x \text{ d}t - \int_0^T \int_\Omega c^{\text{init}} \psi(0) \text{ d}x - \int_0^T \int_\Omega \mathbf{u} \cdot \nabla \psi \text{ d}x \text{ d}t

$$ + \int_0^T \int_\Omega K c \left( D \nabla c - \mu_0 (\nabla (\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a - \alpha_3 m))^T m \right) \cdot \nabla \psi \text{ d}x \text{ d}t = 0,

$$ \int_{\Omega'} \nabla R \cdot \nabla S \text{ d}x = \int_{\Omega'} \mathbf{h}_a \cdot \nabla S \text{ d}x - \int_{\Omega} \mathbf{m} \cdot \nabla S \text{ d}x \quad \text{almost everywhere in } I = [0, T],

1.2 Outline of this thesis
\[
\int_0^T \langle \partial_t \mathbf{m}, \Psi \rangle_{(H^3_0(\Omega))^d \times H^3_0(\Omega)^d} \, dt \\
- \int_0^T \int_\Omega \left( \left( u + K \left[ -D\nabla c + \mu_0 (\nabla (\alpha_1 h + \frac{\beta}{2} h_a - \alpha_3 \mathbf{m}))^T \mathbf{m} \right] \right) \cdot \nabla \right) \Psi \cdot \mathbf{m} \, dx \, dt \\
+ \sigma \int_0^T \int_\Omega \nabla \mathbf{m} \cdot \nabla \Psi \, dx \, dt + \sigma \int_0^T \int_\Omega \nabla \mathbf{m} \cdot \nabla \Psi \, dx \, dt \\
= \frac{1}{2} \int_0^T \int_\Omega (\mathbf{m} \times \Psi) \cdot \nabla \mathbf{u} \, dx \, dt - \frac{1}{\tau_{\text{rel}}} \int_0^T \int_\Omega (\mathbf{m} - \chi(c, \mathbf{h}) \cdot \Psi) \cdot \mathbf{d} \, dx \, dt,
\]

where the magnetic field is given as
\[
\mathbf{h} := \nabla R,
\]

are satisfied and the initial data of \( \mathbf{u} \) and \( \mathbf{m} \) is attained in a distributional sense, \( \mathbf{u}(0), \Psi)_{u' \times \Omega} = \int_\Omega \mathbf{u}^{\text{init}} \cdot \Psi \, dx \, \forall \Psi \in H^3_0(\Omega)^d \cap H(\text{div})(\Omega), \)

\[
\langle \mathbf{m}(0), \Psi \rangle_{(H^3_0(\Omega))^d \times H^3_0(\Omega)^d} = \int_\Omega \mathbf{m}^{\text{init}} \cdot \Psi \, dx \, \forall \Psi \in H^3_0(\Omega)^d.
\]

The solution satisfies the a priori estimate (3.171) which reflects the regularity of the weak solution mentioned above.

Let us comment on the importance of the first aforementioned step. On the one hand, the magnetic quantities are \( H(\text{div}, \text{curl}) \)-regular and thereby allow only for local \( H^1 \)-estimates. On the other hand, the Kelvin force \( (\mathbf{m} \cdot \nabla) \mathbf{h} \) requires to control \( (\nabla \mathbf{h})^T = (\nabla \nabla R)^T = \nabla \nabla R = \nabla \mathbf{h} \). Because of that, the passage to the limit relies on higher regularity estimates which are only available locally. Hence, localization via test functions with compact support is necessary. Projecting them onto Galerkin ansatz spaces does not preserve their compact support, thereby we choose to treat a regularized model first. It is worth mentioning that results on compensated compactness (see global div-curl lemma [62]) are not applicable under the assumptions of this thesis – see e.g. the boundary conditions of the magnetization. If it was applicable, it might be used to improve local compactness results in space for the magnetic field \( \mathbf{h} \) (cf. [3] see equations (28) and (29)) to be valid globally on \( \Omega \). However, there is no hint on how to control \( \nabla \mathbf{h} \) globally or how to obtain global \( L^p \)-regularity for \( \mathbf{h} \) when \( p > 2 \). Therefore, it seems indispensable to consider the solutions \( \mathbf{m} \) and \( \mathbf{h} \) in the sense of distributions. Typically, higher regularity for the particle density \( c \) is achieved via testing \([1.1c]\) by \( c \). In the light of \([1.1d]\), this requires \( L^2 \)-estimates of the Kelvin force \( (\mathbf{m} \cdot \nabla) \mathbf{h} \) which are out of reach. Pursuing a different approach in Section 3.3, \( L^p \)-estimates on \( \nabla \mathbf{c} \) depend on the integrability of \( (\mathbf{m} \cdot \nabla) \mathbf{h} \). Consequently, the solution \( c \) will be considered in the sense of distributions, too. For this approach it will be advantageous to choose the nonlinear mobility in the particle evolution in such a way that the convective velocity \( \mathbf{v}_{\text{part}} \) of the particles has \( L^2 \)-regularity. It turns out that this can be achieved by choosing \( f_p(c) = c^2 \), cf. \([1.1d]\).

Our ansatz functions for the magnetization \( \mathbf{m} \) are required to satisfy the natural boundary conditions of the \( (-\nabla \text{div} + \text{curl} \text{curl}) \)-operator which is non-conforming in view of \([1.3c], [1.3d]\). However, it is worth mentioning in points \((t, \mathbf{x}) \in (0, T) \times \partial \Omega \) where \( c(t, \mathbf{x}) \neq 0 \) the condition \([1.3b]\) reduces \([1.3c]\) and \([1.3d]\) exactly to the aforementioned natural boundary conditions. A priori estimates for Galerkin solutions are obtained among other things by testing the magnetization equation \([1.7]\) by the magnetization \( \mathbf{m} \) itself and by the magnetic field \( \mathbf{h} = \nabla R \) (cf. \([1.8]\) and \([1.9]\)). The latter is a gradient field and is defined on a possibly larger domain \( \Omega' \supset \Omega \). Hence, special attention had to be paid to construct suitable ansatz spaces (Section 3.1) which fit to the
structural coupling between the two equations (1.7) and (1.6). For this, a decomposition of the ansatz space

\[ M = \{ \Psi \in H^3(\Omega)^d | \text{curl} \Psi \times \nu_{\partial \Omega} = 0, \text{div} \Psi|_{\partial \Omega} = 0 \} = \nabla[H^1(\Omega)] \cap M \oplus V \]

for the magnetization \( m \) into gradient fields and other fields – following the spirit of the Helmholtz decomposition – plays an essential role. Note that the \( H^3 \)-regularity allows approximations of test functions to compensate the low regularity of the magnetization, see e.g. (3.152).

**Numerical analysis.** The numerical section starts with introducing the discrete numerical schemes of the three models of this thesis. Energy estimates and existence of discrete solutions will be established in a straightforward way by using Schaefer’s fixed point theorem. Moreover, some practical aspects will be discussed in Section 4.3 including the procedure to handle the convection-dominant nature of the model (the main cause of motion are the magnetic forces – not the diffusion).

Similar to [51,52], non-conforming finite elements are used for the magnetization equation in *model GW* (see also [39]). In contrast to [51,52] an energy stable scheme has been designed even for the case \( \sigma > 0 \) (cf. (1.1f)). For this, a novel duality argument [39] is used to obtain \( H^1 \)-conforming approximations of divergence and curl of a vector field. For instance, the discrete divergence \( \text{div}_h m \) of the discrete magnetization \( m \), which is not differentiable, can be defined via the variational problem

\[ \int_{\Omega} \text{div}_h m \varphi_0 \, dx = -\int_{\Omega} m \cdot \nabla \varphi_0 \, dx \quad \forall \varphi_0 \in P^0_2(\Omega), \]

see (4.32). With this technique, discretization of the second order differential operators \( \nabla \text{div} \) and \( \text{curl} \text{curl} \) is straightforward. So far, choosing the parameter \( \sigma > 0 \) strictly positive seems to be the most viable approach to obtain compactness results. Convergence of discrete solution, however, is still an open problem for such types of models – also in case of the homogeneous ferrofluid models from [Hom-compl].

**Simulation results.** The three types of models of this thesis will be compared with each other in a fixed scenario, involving an external magnetic field generated by a magnetic dipole located to the right of the simulation domain. Different choices of \( f_p \) and the effect of \( \Omega' \) being strictly larger than \( \Omega \) will be examined as well. The results are presented in Section 5.1. They show that the magnetization field is strongly coupled to the support of the superparamagnetic particles – and such details like, how streamlines of the total magnetic field are influenced by the magnetization, become clearly visible. We evaluate the models based on physical considerations and numerical robustness. We also briefly discuss mixed schemes as in [39] with finite volume based discretizations of the convection-dominant evolution of the density of magnetic particles and comment on ways to reduce the computational time. After the comparison, we use *model W*, for practical reasons, to examine different external magnetic fields. Among those are fields generated by dipoles of different distance from the simulation domain and superpositions of dipole fields.

**Notation and appendix.** Treating the continuous models, an intricate Galerkin approximation setting and the numerical analysis all in one work requires extensive notation. Particularly, the construction of suitable Galerkin ansatz spaces requires a lot of careful crafting and notation. Hence, at the end of this thesis (see Section 5.3) a comprehensive list of the most important or widely used notation is given. First occurrences or important appearances will be stated, too.
This thesis relies on concepts that are not taught in the classical PDE courses – particularly the spaces of functions whose distributional divergence or distributional curl is $L^2$-regular. Such functions have $H^{-\frac{1}{2}}$-regular normal or tangential traces, respectively. More information about this topic is available in the appendix – Section A.1. Furthermore, jumps along boundary curves appear in the modeling part of this thesis through transmission conditions as well as in the discrete setting due to non-conforming approximations. In this regard, the interested reader is referred to [24]. Moreover, we encourage to have a look at the Helmholtz decomposition as this thesis, especially in Section 3.1.1 has been influenced by the latter and shares similarities with it, see Section A.2. Section A.4 includes definitions of curl and $\times$ in the two-dimensional setting.
2 Modeling

Three models will be derived in this section. All of them aim to describe the motion of suspended superparamagnetic nanoparticles subjected to magnetic fields. Here, the particles and their magnetic moments (strength and orientation of magnetic dipoles) are modeled by continuous quantities, the particle density and the magnetization. Hence, this section is devoted to derive a system of equations for

- the density of magnetic particles \(c\),
- the velocity field \(u\) of the fluid and its hydrodynamic pressure \(p\),
- the total magnetic field \(h\),
- the magnetization \(m\) of the fluid.

Onsager’s variational principle \([55,56]\) will be used for the derivation. This is a powerful tool for adapting general balance laws to explicit situations, solely based on the energy \(E\) and dissipation \(D\) of a thermodynamic system. The method is an extension of Rayleigh’s principle of the least energy dissipation as illustrated in e.g. \([27]\) and holds for general irreversible processes. Hence, let \(X\) be a list of all unknown quantities (i.e. fluxes or stress tensors) of the general equations and let the dissipation and time derivative of energy be written in terms of those, \(D = D(X)\) and \(E_t = E_t(X)\). Then,

\[
\delta_X (E_t(X) + D(X)) = 0
\]  

(2.1)

is solved for \(X\), where \(\delta_X\) is the first variation with respect to \(X\). The variational principle particularly dictates the dissipation functional to be a quadratic form which is a consequence of Onsager’s reciprocal relations.

After discussing the choice of general laws in Section 2.1, the energy and the dissipation of the system are considered in Section 2.2. The main work is carried out in Section 2.3 i.e. the computation of \(E_t\) and solving the variational equation of type (2.1).

The main interest of this thesis is a system of partial differential equations featuring intricate coupling of magnetic field \(h\) and magnetization \(m\) to the remaining equations. The derivation of such a model has already been done in \([39]\) and will be presented in this thesis with more details. Let this model be called "model GW". The derivations of the other two models used in this thesis are new, to the best of our knowledge, and are more complex than other similar models in the literature. They are based on a simplified law for the magnetization, i.e. an algebraic equation. One of them is analogous to "model GW" apart from the magnetization law and will be called "model W". The other one uses a slightly different approach for the total momentum. To be precise, a two-phase flow model is used as a foundation. It turns out that the model to be obtained this way is similar to the model of \([41]\) in regards to magnetism and will be denoted by "model B". The introduction and comparison of these models is inspired by discussions with E. Bänsch, M. Neuss-Radu and R.H. Nochetto.

2.1 Modeling assumptions and governing equations

Fix \(T > 0\) as final time. Let \(d \in \{2, 3\}\) be the spatial dimension. While \(d = 3\) is the physically relevant choice, the two-dimensional case is considered as well as it eases the complexity of numerical simulations. The results of this thesis are valid for both cases unless stated otherwise. The curl-operator can be defined in two dimensions, too, cf. \([18]\) or appendix A.4.
Let $\Omega, \Omega' \subset \mathbb{R}^d$ be bounded Lipschitz domains with the properties

\begin{align}
\Omega &\subset\subset \Omega' \quad \text{or} \quad \Omega = \Omega', \quad (2.2a) \\
\Omega' &\text{ is simply connected.} \quad (2.2b)
\end{align}

The domain $\Omega$ will be called "fluid domain" occasionally. With the exception of the magnetic field $h$ all quantities are defined on the fluid domain only. The magnetic field is defined on the possibly larger domain $\Omega'$ in order to describe the boundary behavior of the magnetic variables more realistically. Assumption (2.2a) simplifies the setup in the sense that if $\Omega \neq \Omega'$, then their boundaries have a positive distance from each other.

**Magnetic nanoparticles.** The magnetic nanoparticles will be considered first. The choice of macroscopic modeling motivates the simplifying assumption that

\[
\text{all magnetic nanoparticles are spherical and identical.} \quad (2.3)
\]

Additionally, it is assumed that

\[
\text{particle-to-particle interactions,}
\]

\[
\text{like agglomeration phenomena or anisotropic behavior, do not to occur.} \quad (2.4)
\]

The latter can be achieved by coating (cf. [71]) and affects the magnetization of the magnetic particles which measures their alignment with the magnetic field lines. In this thesis, only paramagnetic particles will be considered, i.e. it is energetically favorable to have magnetic moments pointing in the direction of the magnetic field. Moreover, the particles’ magnetization changes randomly under the influence of temperature (thermal fluctuation). Hence, in absence of external magnetic fields the magnetization is expected to decay to zero on average as there are no other influences that maintain the orientation of the magnetic moments (see (2.4)). In other words, hysteresis effects are negligible [66] (unless the temperature is extremely low) and the assumption that

\[
\text{the magnetic nanoparticles are superparamagnetic} \quad (2.5)
\]

is justified. For simplification, assume that

\[
\text{magnetic fields which are induced by the alignment of the magnetic particles}
\]

\[
\text{do not influence applied external fields.} \quad (2.6)
\]

The continuity equation

\[
c_t + \div (c u + c v_{\text{part}}) = 0 \quad (2.7)
\]

on $(0, T) \times \Omega$ will be used to describe the particle density $c : [0, T] \times \Omega \to \mathbb{R}$. The type of density is arbitrary as e.g. volume concentration (used in "model B") or number density are directly proportional to each other. A volume concentration is dimensionless and thereby has the advantage that nonlinear terms like $\log(c)$ do not need any normalization parameters. Usually, an unknown flux

\[
J := c v_{\text{part}} \quad (2.8)
\]

is used in (2.7), but the convective velocity $v_{\text{part}} : [0, T] \times \Omega \to \mathbb{R}^d$ will be needed later to determine the convection of the magnetization field, whence the choice. But keep in mind that this quantity is an artificial quantity that is not physical when $c = 0$. 


2.1 Modeling assumptions and governing equations

There will be two different assumptions affecting the momentum of the fluid, both of them leading to different models. Either

the contribution of magnetic particles to the gross momentum of the fluidic system is negligible (2.9a)

or

magnetic particles and fluid particles develop a two-phase flow. (2.9b)

The first variant is reasonable due to their very small size. The second variant naturally indicates a bounded volume concentration of the magnetic particles. As this affects the momentum equation only, details follow later in this subsection. As the flow will be assumed to satisfy the usual no-slip boundary condition, see (2.54) later on, the no-flux condition

\[ c \nu_{\text{part}} \cdot \nu |_{\partial \Omega} = 0 \] (2.10)

on \( [0,T] \times \partial \Omega \) guarantees conservation of mass of the magnetic particles. In order to clarify, this condition should not be interpreted to force \( c = 0 \). In contrast, it has to be seen as a condition that is only valid for \( c \neq 0 \) and is automatically satisfied when coincidentally \( c = 0 \).

**Magnetic field.** The dilute magnetic particles in \( \Omega \) are treated as a magnetic material, while the outside of \( \Omega \) is a non-magnetic material. When the material changes, the well-known transmission conditions (cf. e.g. [21]) are used. Introducing \( \mathbb{1}_\Omega m : [0,T] \times \Omega' \rightarrow \mathbb{R}^d \) to be the extension of \( m : [0,T] \times \Omega \) onto \( [0,T] \times \Omega' \),

\[ \mathbb{1}_\Omega m(\cdot, x) := \begin{cases} m(\cdot, x) & \text{if } x \in \Omega, \\ 0 & \text{else} \end{cases} \] (2.11)

they read

\[ [h + \mathbb{1}_\Omega m] \cdot \nu |_{\partial \Omega} = 0, \] (2.12a)
\[ [h] \times \nu |_{\partial \Omega} = 0, \] (2.12b)

on \( [0,T] \times \partial \Omega \), where \([\cdot] \cdot \nu\) denotes the normal jump and \([\cdot] \times \nu\) a collection of tangential jumps, shortly called tangential jump (see Section A.1 for details).

**Remark 2.1.** Additionally, note that in the two-dimensional setting the vector-valued \( 0 \) on the right-hand side of (2.12b) changes to a scalar valued \( 0 \). The same is true for the upcoming equations (2.13a), (2.14a) and (2.17b) of this paragraph.

The following discussion about the magnetic variables is based on [39] which was inspired by [52]. The system is assumed to be triggered by an external magnetic field \( h_a : [0,T] \times \Omega' \rightarrow \mathbb{R}^d \) which satisfies the magnetostatic equations in the absence of matter, i.e.

\[ \text{curl } h_a = 0, \] (2.13a)
\[ \text{div } h_a = 0, \] (2.13b)

on \( [0,T] \times \Omega' \), without free currents. For the magnetic variables \( h : [0,T] \times \Omega' \rightarrow \mathbb{R}^d \) and \( m : [0,T] \times \Omega \rightarrow \mathbb{R}^d \) the analogous equations are posed, i.e. magnetostatic equations with matter,

\[ \text{curl } h = 0, \] (2.14a)
\[ \text{div}(h + \mathbb{1}_\Omega m) = 0, \] (2.14b)
on \([0, T] \times \Omega\). Due to the possibly sudden change of the magnetic properties after crossing \(\partial \Omega\), differentiability as indicated above needs to be discussed. As an example consider \(\text{curl} \ h = 0\). It is reasonable to assume the equation to be well-defined on \(\Omega\) and \(\Omega' \setminus \overline{\Omega}\) individually. Now, pick an arbitrary \(\Phi \in C^\infty_0(\Omega')^d\) and test the equations by \(\Phi|_\Omega\) or \(\Phi|_{\Omega' \setminus \overline{\Omega}}\), respectively, sum up the resulting terms and integrate by parts.

\[
\int_\Omega \text{curl} \ h \cdot \Phi \, dx + \int_{\Omega' \setminus \overline{\Omega}} \text{curl} \ h \cdot \Phi \, dx = \int_\Omega h \cdot \text{curl} \Phi \, dx + \int_{\Omega' \setminus \overline{\Omega}} h \cdot \text{curl} \Phi \, dx - \int_{\partial \Omega} h \times \nu \cdot \Phi \, d\sigma - \int_{\Omega' \setminus \overline{\Omega}} h \times \nu \cdot \Phi \, d\sigma = 0 \text{ on } \partial \Omega'
\]

(2.15)

The boundary integral on \(\partial \Omega\) vanishes due to the transmission condition (2.12b) and thereby one finds \(h \in H(\text{curl})(\Omega')\) where \(h = 0\). Note that generalized boundary values for \(H(\text{curl})\)-regular functions are well-defined on Lipschitz domains (see also Section A.1). An analogous result holds true for (2.14b).

This way, \(\text{curl} \ h\) and \(\text{div}(h + \mathbf{1}_\Omega m)\) are explained in a weak sense, i.e. \(h \in H(\text{curl})(\Omega')\) and \((h + \mathbf{1}_\Omega m) \in H(\text{div})(\Omega')\). However, while \(\text{div}(h + \mathbf{1}_\Omega m)\) is well-defined (in aforementioned weak sense) on space domain \(\Omega'\), neither \(\text{div}(\mathbf{1}_\Omega m)\) nor \(\text{div} \ h\) are defined on \(\Omega'\). Otherwise, \(m \cdot \nu|_{\partial \Omega} = 0\) would be imposed implicitly, see Section A.3. Such a condition has been used in [4] to improve regularity, but is ignored in this thesis, in which the choice of boundary conditions is based on physical justification only. In order to avoid confusion, (2.14a) and (2.14b) will be separated into two individual equations, one of which is defined on \(\Omega\) and the other one on \(\Omega' \setminus \overline{\Omega}\).

On simply connected domains, \(\text{curl}\)-free functions can be written by means of a potential (cf. Poincaré’s Lemma [18] for smooth vector fields or Lemma A.10 for \(H(\text{curl})\)-regular functions). Hence there is a function \(R : [0, T] \times \Omega' \to \mathbb{R}\) such that

\[
h = \nabla R.
\]

(2.16)

The latter automatically guarantees (2.12b) to be satisfied which can be inferred from (2.15) by applying \(\text{curl} \ \nabla \equiv 0\) to its left-hand side and integration by parts, i.e.

\[
\int_{\partial \Omega} [\nabla R] \times \nu \cdot \Phi \, d\sigma = \int_{\Omega'} \nabla R \cdot \text{curl} \Phi \, dx = \int_{\Omega'} \text{curl} \nabla R \cdot \Phi \, dx - \int_{\partial \Omega'} \Phi \times \nu \cdot \nabla R \, d\sigma = 0.
\]

Therefore, transmission condition (2.12b) and equation (2.14a) can be discarded from now on. Notice that the existence of a potential is also true in two dimensions provided \(\text{curl} \ h\) (A.23) \(\partial_{x_1} h_2 - \partial_{x_2} h_1 = 0\). This can be inferred from the relations in Remark A.11.

On account of (2.12), in absence of matter \((m = 0)\) the external field is assumed to have no jumps across \(\partial \Omega\), hence

\[
|h_a| \cdot \nu|_{\partial \Omega} = 0, \quad (2.17a)
\]

\[
|h_a| \times \nu|_{\partial \Omega} = 0, \quad (2.17b)
\]

on \([0, T] \times \partial \Omega\). The magnetic particles seek alignment – parallel to the magnetic field lines – such that they form a magnetized material inducing their own magnetic field. This field is often called 'stray field' or 'demagnetizing field', \(h_d : [0, T] \times \Omega' \to \mathbb{R}^d\). It can leave the fluid domain and superposes the external field. Hence, the total magnetic field \(h : [0, T] \times \Omega' \to \mathbb{R}^d\) is assumed to decompose into two parts,

\[
h = h_a + h_d. \quad (2.18)
\]
There, assumption (2.6) was used, i.e. the external field \( h_a \) is generated in a way that is not influenced by the stray field (or magnetization). The field \( h_d \) is named "demagnetizing field" as it tends to reduce the magnetization of the magnetized material. The term 'stray field' reflects the part of \( h_d \) on the outside of the magnetized material (i.e. outside \( \text{supp} \, c \)).

Be aware that all magnetic field quantities in (2.18) above can be written in terms of potentials, i.e. there exist potentials \( R_a, R_d : [0, T] \times \Omega' \to \mathbb{R} \) such that
\[
\nabla R_a = h_a, \quad \text{and} \quad \nabla R_d = h_d
\]
and also \( R = R_a + R_d \).

In order to close the system that will be obtained later, another boundary condition is needed. It is assumed that the effects of the stray field are negligible at a sufficient distance from the fluid domain, i.e.
\[
\mathbf{h} \cdot \nu|_{\partial \Omega'} = h_a \cdot \nu|_{\partial \Omega'}
\]
on \([0, T] \times \partial \Omega'\). For physically realistic modeling of the magnetic variables, \( \partial \Omega' \) is supposed to be sufficiently far away from \( \partial \Omega \). However, in the literature the simplification \( \Omega' = \Omega \) is widely used, see e.g. [4,52]. In [41] the part \( \mathbf{h}^{\text{out}}|_{\partial \Omega} = (\mathbf{h}|_{\Omega', \Gamma})|_{\partial \Omega} \) is assumed to be given boundary data. Within this thesis, in case of \( \Omega' = \Omega \) the boundary condition is obtained as follows. Start with \( \Omega' \supset \Omega \) and consider
\[
(\mathbf{h} - h_a) \cdot \nu|_{\partial \Omega'} =: \mathbf{h}^{\text{out}}_d \cdot \nu|_{\partial \Omega'} \quad \text{(2.20)}
\]
Also, let \( \mathbf{m}^{\text{out}} \cdot \nu = 0 \) be the magnetization’s normal component from the perspective of \( \Omega' \setminus \Omega \). When \( \Omega' \to \Omega \) it is assumed that \( \mathbf{h}^{\text{out}}_d \cdot \nu|_{\partial \Omega} \) and \( \mathbf{m}^{\text{out}} \cdot \nu \) remain to vanish at \( \partial \Omega \). The latter and (2.17a) simplify the jump in (2.12a) which therefore will be replaced with
\[
(\mathbf{h}_d + \mathbf{m}) \cdot \nu|_{\partial \Omega} = 0
\]
on \([0, T] \times \partial \Omega \). This influences how the stray field is allowed to leave the fluid domain \( \Omega \), e.g. it is not allowed to leave the domain in normal direction if \( \mathbf{m} = 0 \). This boundary condition is consistent with [51,52]. The main focus of this thesis, however, is on the case \( \Omega' \supset \Omega \).

In view of (2.16), the magnetostatic equations (2.14) read
\[
\begin{align*}
-\Delta R &= \text{div} \, \mathbf{m} & \text{on} \ [0, T] \times \Omega, \\
-\Delta R &= 0 & \text{on} \ [0, T] \times (\Omega' \setminus \Omega)
\end{align*}
\]
which will be the preferred formulation for the rest of this thesis. In case of \( \Omega' = \Omega \), equation (2.22b) is simply discarded. In the same way as in (2.22), the transmission (2.12a) and boundary conditions (2.20) or (2.21) will be rewritten for future reference, i.e.
\[
\begin{align*}
[\nabla R + \mathbf{I}_\Omega \mathbf{m}] \cdot \nu|_{\partial \Omega} &= 0 & \text{on} \ [0, T] \times \partial \Omega, \\
\nabla R \cdot \nu|_{\partial \Omega'} &= 0 & \text{on} \ [0, T] \times \partial \Omega'
\end{align*}
\]
in case of \( \Omega \subset \subset \Omega' \) and
\[
(\nabla R + \mathbf{m}) \cdot \nu|_{\partial \Omega} = h_a \cdot \nu|_{\partial \Omega} \quad \text{on} \ [0, T] \times \partial \Omega
\]
in case of \( \Omega' = \Omega \). The boundary condition at \( \partial \Omega' \) can be rewritten for both cases as
\[
(\mathbf{h} + \mathbf{I}_\Omega \mathbf{m}) \cdot \nu|_{\partial \Omega'} = \mathbf{h}_a \cdot \nu|_{\partial \Omega'},
\]
where \( (\mathbf{h} + \mathbf{I}_\Omega \mathbf{m}) =: \mathbf{b} \) is called "magnetic flux density". This makes it more evident that the case \( \Omega \subset \subset \Omega' \) is more physical, as \( \mathbf{b} \) includes effects of the magnetic particles while \( \mathbf{h}_a \) does not. Hence, equality is more realistic when further away from the fluid domain.

Also observe that (2.22) does not include a time derivative. Therefore, an initial magnetic field could easily be computed for a given initial magnetization.
Magnetization. The magnetization \( \mathbf{m} : [0, T] \times \Omega \to \mathbb{R}^d \) will be modeled phenomenologically by a relaxation equation. Combining (2.3) with (2.5), a simple relaxation law will be chosen, inspired by [63]. Every magnetic particle aligns with the magnetic field by Brownian relaxation or Néel-relaxation. Each effect has its own relaxation time \( \tau_B \) or \( \tau_N \), respectively. However, a so-called "reduced relaxation time", cf. [59,63], will be used here,

\[
\tau_{\text{rel}} = \frac{\tau_N \tau_B}{\tau_N + \tau_B} > 0. \tag{2.25}
\]

With this, a phenomenological equation of the type

\[
\frac{d}{dt} \mathbf{m} = -\frac{1}{\tau_{\text{rel}}} (\mathbf{m} - \mathbf{m}_{\text{eq}}) \tag{2.26}
\]

on \((0, T) \times \Omega\) will be used, where \(\frac{d}{dt}\) is the time derivative in a suitable reference frame \(\Sigma'\) and \(\mathbf{m}_{\text{eq}}\) is the desired equilibrium state of the magnetization. The details will be indicated here and can further be looked up in [63].

The following is written for the three-dimensional setting. Nevertheless, the analogous results are true in the two-dimensional setting as well. The subsequent changes are annotated wherever needed.

Convection and rotation of magnetic particles affect the desired alignment of the particles, therefore \(\Sigma'\) is chosen to be the local reference frame which moves and rotates with the magnetic particles. Adopting the notation of Shliomis [63], let \(\omega_p\) be the macroscopic angular velocity of the magnetic particles and \(\Omega\) be the fluid’s angular velocity, which is assumed to be given by the approximation \(\Omega = \frac{1}{2} \text{curl} \mathbf{u}\). Other models — including the evolution of angular momentum — are considered in [52,60]. With the help of the relation \(\mathbf{v} = \omega \times \mathbf{r}\) for any angular velocity \(\omega\), position vector \(\mathbf{r}\) and corresponding tangential velocity \(\mathbf{v}\), the formula

\[
\frac{d}{dt} \mathbf{m} = \partial_t \mathbf{m} + \text{div}(\mathbf{m} \otimes \mathbf{v}_m) - \omega_p \times \mathbf{m} \tag{2.27}
\]

on \((0, T) \times \Omega\) is evident, where the convection velocity \(\mathbf{v}_m : [0, T] \times \Omega \to \mathbb{R}^d\) is determined by the same effects as those which drive the magnetic particles, i.e. the fluid’s velocity field \(\mathbf{u} : [0, T] \times \Omega \to \mathbb{R}^d\) and the convective velocity \(\mathbf{v}_{\text{part}} : [0, T] \times \Omega \to \mathbb{R}^d\) of the magnetic particles, which is yet to be determined,

\[
\mathbf{v}_m = \mathbf{u} + \mathbf{v}_{\text{part}}. \tag{2.28}
\]

Remark 2.2. In two space dimensions the angular velocity is given by a scalar value which determines the speed and direction (clockwise or counter-clockwise) of a position vector \(\mathbf{r}\) in a plane. Therefore, \(\omega\), \(\omega_p\) and \(\Omega\) turn into scalar quantities. The relation \(\Omega = \frac{1}{2} \text{curl} \mathbf{u}\), where by (A.23) the right-hand side indeed is a scalar, remains meaningful which can be seen by the considerations in Remark A.11 (the fact that the two-dimensional curl can be seen as a three-dimensional curl where missing entries in the vector fields are filled up with zeros).

Analogously, the notions \(\mathbf{v} = \omega \times \mathbf{r}\) and \(\omega_p \times \mathbf{m}\) make sense, when the angular velocities are scalar values. This way, (2.27) and the upcoming equations (2.29), (2.30) remain valid in the two-dimensional setting.

The difference between the magnetic particles’ angular velocity and the fluid’s angular velocity is given by a magnetic torque

\[
(\omega_p - \Omega) \approx \frac{1}{4\eta_r(c)}(\mathbf{m} \times \mathbf{h}), \tag{2.29}
\]
see \[63\], where \(\eta_r > 0\) is the rotational viscosity. Relaxation times for magnetic nanoparticles are such small (\(\tau_{\text{rel}} \sim 10^{-4}\) and lower \[30, 63\]) that the additional magnetic torque above will be neglected – notice that \(\frac{m}{\eta_r(c)} \sim \frac{m}{c}\) is assumed to be bounded even for \(c \to 0\) as \(|m| \to 0\) simultaneously and \(m\) tends to be parallel to \(h\) (see (2.31) later on). Hence,

\[
\omega_p = \Omega = \frac{1}{2} \text{curl } u
\]  

(2.30)

is assumed for small \(\tau_{\text{rel}} > 0\).

As in our previous work \[39\], the equilibrium magnetization \(m_{\text{eq}}\) will be described by the Langevin formula \[49\]. For this, denote the number density of magnetic particles by \(\tilde{\chi}(c)\). Then,

\[
L(x) = \begin{cases} \coth(x) - \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}
\]

\[
m_{\text{eq}} = \tilde{\chi}(c)m_0L(\xi|h|)\frac{h}{|h|},
\]

which follows from Boltzmann statistics, where \(\xi|h| := \frac{\mu_0m_0|h|}{k_B T} \geq 0\). Here, \(m_0 > 0\) is the magnetic moment of a single magnetic particle, \(\mu_0\) is the vacuum permeability, \(k_B\) is Boltzmann’s constant and \(T\) is the temperature. The value \(\xi|h|\) represents the ratio between the energy of a single magnetic dipole with magnetic moment \(m_0\) positioned inside the magnetic field \(h\) and the thermal fluctuation energy at temperature \(T\). Analogous approaches are made in \[63\] and \[41\], too. The Langevin-function \(L\) describes how much of the alignment is perturbed by thermal fluctuations. The range of values is \(L(\xi|h|) \in [0, 1]\), where 1 is approached for \(\xi|h| \to \infty\), see Figure 2.1 and \(L(0) = 0\). Also, \(L\) is continuous, which can e.g. be verified by L’Hospital’s rule (applied twice). As the function \(\tilde{\chi}\) gives the number density of magnetic particles in dependence of \(c\), the type of density used for the magnetic particles is left open. However, one can always think of \(\tilde{\chi} \equiv \text{id}\) and \(c\) being the number density.

We rewrite the equilibrium magnetization in (2.31) in terms of the susceptibility (denoted by \(\chi\)) which describes to which extent the particles align parallel to the magnetic field, i.e.

\[
m_{\text{eq}} := \chi(c, h)h,
\]

\[
\chi(c, h) := \tilde{\chi}(c)m_0\frac{L(\xi|h|)}{|h|}.
\]

(2.32)

Figure 2.1: Graphs of the Langevin function \(L\) (normal line style) at different scalings of the argument and their respective linearizations (dashed line style) on the left side. The quotients \(\frac{L(\cdot)}{\cdot}\) depicted at differently scaled arguments on the right.
In the regime of magnetic nanoparticles, \( m_0 \) and therefore the argument \( \xi|h| \) of the Langevin-function is expected to be such small that linearization (with respect to \( h \)) of the ansatz in (2.31) is widely used in literature, i.e.

\[
m_{eq} \approx \bar{\chi}(c) \frac{\mu_0 m_0^2}{3k_BT} h = \frac{m_0 \xi}{3} h.
\]

(2.33)

This motivates the definition

\[
\chi_{\text{lin}}(c) := \chi_0 c := \frac{\mu_0 m_0^2}{3k_BT} c,
\]

(2.34)

which resembles the usual formula, see e.g. [41]. The value \( \xi \) is the limit point of \( L(\xi|h|)|_h \) for \( |h| \to 0 \) (see Remark 2.3). The linearized version will be used for simplification of proof-of-concept simulations (Section 5) and for the simplified models, i.e. "model W" and "model B".

Remark 2.3. The Langevin-function is globally Lipschitz-continuous. It is worth mentioning that for fixed \( c \) the function \( h \mapsto \frac{L(\xi|h|)}{h} \) is bounded as indicated in Figure 2.1. For values \( |h| \geq \varepsilon \) for some \( \varepsilon > 0 \) this is obvious due to the boundedness of the Langevin function \( L \). For \( |h| \to 0 \) one can use the fact that the Langevin function is approximately linear for small arguments. For this consider \( \xi = 1, x > 0 \) and use L’Hospital’s rule to obtain the limit

\[
\lim_{x \to 0} \frac{L(x)}{x} = \lim_{x \to 0} \frac{L'(x)}{1} = \lim_{x \to 0} \left( \frac{1}{x^2} - \frac{4}{(e^x - e^{-x})^2} \right) = \frac{(e^x - e^{-x})^2}{x^2(e^x - e^{-x})^2} - 4x^2
\]

L’Hospital’s rule applied another four times yields the limit \( \frac{1}{3} \).

Following the suggestions of mathematical literature, e.g. [4], the diffusive term

\[-\Delta m = -\nabla \text{div} m + \text{Curl curl} m\]

is added to (2.26) which otherwise would be of hyperbolic nature. While this is clearly beneficial from a mathematical point of view, Torrey [68] suggests the presence of a diffusive term also from a physical perspective in situations where the diffusion of the spin magnetic moment is not negligible. Above identity (2.35a) is easy to verify by computing and interchanging derivatives. See Section A.4 for insight on the two-dimensional version,

\[-\Delta m = -\nabla \text{div} m + \text{Curl curl} m.\]

(2.35b)

In the magnetostatic equations (2.14) the operators \( \text{div} \) and \( \text{curl} \) appear – hinting at \( H(\text{div}, \text{curl}) \)-regularity of the magnetic quantities. The term \(-\sigma \nabla \text{div} m + \sigma \text{curl curl} m\) resembles this kind of regularity, too, which is the reason why the reformulation (2.35) of the diffusive term \(-\sigma \Delta m\) has been used.

Collecting the ingredients (2.26), (2.27), (2.28), (2.30), (2.32) and (2.35), the final equation on \((0, T) \times \Omega\) reads

\[
\begin{align*}
\mathbf{m}_t + \text{div} (\mathbf{m} \otimes (\mathbf{u} + \mathbf{v}_{\text{part}})) - \sigma \nabla \text{div} m + \sigma \text{curl curl} m &= \frac{1}{2} \text{curl} \mathbf{u} \times \mathbf{m} - \frac{1}{\tau_{\text{rel}}} (\mathbf{m} - \chi(c, \mathbf{h}) \mathbf{h}), \\
\end{align*}
\]

(2.36)

which is identical to the equation in [39]. Here, \( \sigma > 0 \) is a regularization parameter. The equation above can be seen as generalization of Shliomis’ equation [63], which is widely used as a basis for modeling (see e.g. [4], [52], [60]), to the case of inhomogeneous susceptibility.
By the same logic why \( \mathbf{v}_{\text{part}} \) appears in (2.36), the analogous boundary condition will be imposed to prevent flux of magnetization out of or into the fluid domain \( \Omega \). In case of \( \sigma = 0 \) those read

\[
\mathbf{m}_i \mathbf{v}_{\text{part}} \cdot \nu |_{\partial \Omega} = 0 \quad \forall i \in \{1, \cdots, d\}
\]

on \( (0, T] \times \partial \Omega \). Equation (2.37) can equivalently be written as two conditions

\[
(\mathbf{v}_{\text{part}} \cdot \nu) \mathbf{m}_{\text{nor}} = 0 \quad \text{and} \quad (\mathbf{v}_{\text{part}} \cdot \nu) \mathbf{m}_{\text{tan}} = 0,
\]

where \( \mathbf{m}_{\text{nor}} \) and \( \mathbf{m}_{\text{tan}} \) are the normal or tangential components of \( \mathbf{m} \), respectively. Analogously to (2.10), this has to be interpreted as a condition that is automatically satisfied if \( \mathbf{m} = \mathbf{0} \) but is not supposed to force \( \mathbf{m} \) to vanish. Also note that \( \mathbf{u} \) does not play a role in (2.37) due to the no-slip condition, see (2.54) later on.

For \( \sigma > 0 \) a combination of (2.37) and the natural boundary conditions associated with the\((-\nabla \text{div} + \text{curlcurl})\)-operator, i.e.

\[
\begin{align*}
\text{div} \mathbf{m}|_{\partial \Omega} &= 0, \\
\text{curl} \mathbf{m} \times \nu|_{\partial \Omega} &= 0,
\end{align*}
\]

on \( (0, T] \times \partial \Omega \) will be used. The first condition in (2.38) can equivalently be seen as \( \text{div} \mathbf{m} \nu|_{\partial \Omega} = 0 \) which indicates a flux in normal direction. The term in the second condition of (2.38) is tangential to \( \nu \) and indicates a tangential flux. Hence, the terms on the left-hand side of (2.38) indicate a potential decrease or increase of magnetization by flux over the boundary. Therefore, the final boundary conditions in the case \( \sigma > 0 \) are

\[
\begin{align*}
(\mathbf{v}_{\text{part}} \cdot \nu)(\mathbf{m} - \sigma \text{div} \mathbf{m})|_{\partial \Omega} &= 0, \\
(\mathbf{v}_{\text{part}} \cdot \nu)\left(\mathbf{m} - (\mathbf{m} \cdot \nu)\nu - \sigma \text{curl} \mathbf{m} \times \nu\right)|_{\partial \Omega} &= 0,
\end{align*}
\]

on \( (0, T] \times \partial \Omega \). The total flux

\[
((\mathbf{v}_{\text{part}} \cdot \nu) \mathbf{m} - \sigma \text{div} \mathbf{m} \nu - \sigma \text{curl} \mathbf{m} \times \nu)|_{\partial \Omega} = 0
\]

is given by summing up all normal fluxes and tangential fluxes, i.e. it is given by multiplying (2.39a) by the normal \( \nu \) and adding the result to (2.39b). The notation in (2.39b) naturally is defined in the two-dimensional setting, too, see \[\text{Section A.4}\].

**Remark 2.4.** On points \((t, \mathbf{x}) \in (0, T] \times \partial \Omega \) where \( c(t, \mathbf{x}) \neq 0 \) (implying \( \mathbf{v}_{\text{part}}(t, \mathbf{x}) = \mathbf{0} \) by (2.10)) or \( \mathbf{m}(t, \mathbf{x}) = \mathbf{0} \) the boundary conditions (2.39) boil down to the natural boundary conditions of the\((-\nabla \text{div} + \text{curlcurl})\)-operator, i.e.

\[
\begin{align*}
(\text{div} \mathbf{m}|_{\partial \Omega})(t, \mathbf{x}) &= 0, \\
(\text{curl} \mathbf{m} \times \nu|_{\partial \Omega})(t, \mathbf{x}) &= 0.
\end{align*}
\]

From a physical perspective, supp \( c = \text{supp} |\mathbf{m}| \) should at least be satisfied approximately, which means that – to some extent – the natural conditions above are the relevant conditions. This is reflected by the choice of non-conforming boundary conditions for Galerkin type approximations in \[\text{Section 3}\] see e.g. (3.7). Similarly, the discrete divergence and curl of \( \mathbf{m} \) in \[\text{Section 4.1}\] satisfy these conditions, too.

Those conditions in (2.39) do not force tangential or normal traces to vanish, the latter of which has been used in [4] in order to increase regularity. By avoiding such conditions, we try to increase the applicability of our model from a physical perspective. In \[\text{Section A.3}\] we discuss why we consider \( \mathbf{m} \cdot \nu|_{\partial \Omega} = 0 \) to be unphysical.
This concludes the modeling of the magnetization via an intricate partial differential equation, which will be used for "model GW". Alternatively, a much simpler law is suggested in mathematical literature. Inspired by [41], the equation
\[ m = \chi_{\text{lin}}(c) h \] (2.41)
will be considered in this thesis as well, leading to different and simplified models ("model W" and 'model B'). Here, the linearized Langevin-formula (2.33) has been used. The use of \( \chi(c, h) \) is possible as well – leading to a more involved model – but will not be examined within the scope of this thesis. Evidently, (2.41) is not supplemented with any boundary conditions. Moreover, no initial condition needs to be imposed on \( m \). Plugging (2.41) into (2.22), an initial magnetic field can be computed for a given initial particle density and therefore an initial magnetization is available, too.

**Velocity field.** The fluid is assumed to be incompressible, i.e. let the flow velocity field \( u : [0, T] \times \Omega \rightarrow \mathbb{R}^d \) satisfy
\[ \text{div} \, u = 0 \] (2.42)
in \( (0, T) \times \Omega \). Therefore, in case of negligible momentum of magnetic particles (2.9a), the Navier-Stokes equations,
\[ \rho_0 u_t + \rho_0 (u \cdot \nabla) u + \nabla p - \text{div} \, T = k \] (2.43)
in \( (0, T) \times \Omega \), are used to describe the conservation of momentum, where \( \rho_0 > 0 \) is the fluid’s density. The force density \( k : (0, T) \times \Omega \rightarrow \mathbb{R}^d \) and the stress tensor \( T : (0, T) \times \Omega \rightarrow \mathbb{R}^{d \times d} \) are to be determined by Onsager’s variational principle. Furthermore, the fluid particles are assumed to be without microstructure such that the usual symmetry condition of the stress tensor holds, i.e.
\[ T^T = T. \] (2.44)
Alternatively, in case of non-negligible momentum of magnetic particles (2.9b), a two-phase flow with volume averaged velocity field \( u := \varphi_1 u_1 + c u_2 \) is considered, where \( u_1 \) and \( u_2 \) are the velocity fields of the fluid particles or the suspended magnetic particles, respectively. Hence (2.43) is subject to changes. However, assumptions (2.44) and (2.42) stay valid by following the approach of [1].

Let \( \varphi : [0, T] \times \Omega, \varphi := c - \varphi_1 \) be a phase field parameter describing the mixture of the fluid particles’ volume density \( \varphi_1 : [0, T] \times \Omega \rightarrow [0, 1] \) and the magnetic particles’ volume density \( c : [0, T] \times \Omega \rightarrow [0, 1] \) which are assumed to satisfy
\[ \varphi_1 + c = 1. \] (2.45)
The density is given by
\[ \tilde{\rho}(\varphi) := \frac{\tilde{\rho}_2 + \tilde{\rho}_1}{2} + \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} \varphi \] (2.46)
or alternatively
\[ \rho(c) := \hat{\rho}_1 + (\hat{\rho}_2 - \hat{\rho}_1)c, \] (2.47)
where the parameters \( \hat{\rho}_1 > 0 \) and \( \hat{\rho}_2 > 0 \) are the mass densities of the pure fluid particles’ phase or pure magnetic particles’ phase, respectively. The relations
\[ \tilde{\rho}(\varphi) = \rho(c) \text{ and } \tilde{\rho}' = \frac{1}{2} \rho' \] (2.48)
hold true and will be used later.
From [1], the governing equations of the two-phase flow are

\[
\hat{\rho}(\varphi)\mathbf{u}_t + ((\hat{\rho}(\varphi)\mathbf{u} + \hat{\rho}'(\varphi)\mathbf{J}_\varphi) \cdot \nabla)\mathbf{u} + \nabla p - \text{div}(\mathbf{T}) = \mathbf{k},
\]

\[
\text{div} \mathbf{u} = 0,
\]

\[
\varphi_t + \nabla \varphi \cdot \mathbf{u} + \text{div} \mathbf{J}_\varphi = 0,
\]

in \((0, T) \times \Omega\), where \(\mathbf{J}_\varphi : [0, T] \times \Omega\) is a flux that is yet to be determined. In the context of this thesis, (2.49c) will be rewritten as follows. From (2.45) one gets

\[
\varphi = 2c - 1.
\]

Consequently,

\[
\mathbf{c}_t + \nabla \mathbf{c} \cdot \mathbf{u} + \text{div}(\frac{1}{2} \mathbf{J}_\varphi) = 0
\]

in \((0, T) \times \Omega\) replaces (2.49c). In comparison with (2.7), the identity

\[
\frac{1}{2} \mathbf{J}_\varphi = \mathbf{c}_\text{part}
\]

is posed in order to be able to recycle the formulation of (2.7). With (2.52) and (2.48), equation (2.49a) turns into

\[
\rho(\mathbf{c})\mathbf{u}_t + ((\rho(\mathbf{c})\mathbf{u} + \rho'(\mathbf{c})\mathbf{c}_\text{part}) \cdot \nabla)\mathbf{u} + \nabla p - \text{div}(\mathbf{T}) = \mathbf{k}
\]

in \((0, T) \times \Omega\). In both cases, single-phase flow (2.9a) and two-phase flow (2.9b), the momentum equation is supplemented with

\[
\mathbf{u}|_{\partial \Omega} = \mathbf{0}
\]

on \([0, T] \times \Omega\), the usual no-slip boundary condition.

### 2.2 Energy and dissipation

Having all governing equations ready (cf. Section 2.1), the next step is to discuss energy and dissipation of the models. The energy of the system consists of kinetic energy, magnetic energy and a mixture energy for the magnetic particles. Define

\[
\mathcal{E}_\text{kin}^1 := \frac{\rho_0}{2} \int_\Omega \mathbf{|u|^2} \, dx,
\]

\[
\mathcal{E}_\text{kin}^2 := \frac{1}{2} \int_\Omega \rho(\mathbf{c})\mathbf{|u|^2} \, dx,
\]

where the former is the kinetic energy in the single-phase setting (2.9a) and the latter is the kinetic energy in the two-phase setting (2.9b). By setting \(\hat{\rho}_1 = \hat{\rho}_2 = \rho_0\), the second expression could be used for the first case, too.

The mixture energy is as follows. Let

\[
g(c) := c \log c - c
\]

be the entropic term as it was e.g. discussed in [10]. The term \(-c\) plays a normalizing role of the classical entropic term of the kind

\[
c \log \left( \frac{c}{c_0} \right) = c(\log c - \log c_0) \doteq c \log c - c
\]
for appropriate normalization parameter $c_0$. Due to conservation of mass of magnetic particles, the term $-c$ and the parameter $c_0$ do not play a role in Onsager’s variational principle as $\partial_t \int_\Omega c \, d\mathbf{x} = 0$. Therefore,

$$
\mathcal{E}^{1}_{\text{mix}} := D \int_{\Omega} g(c) \, d\mathbf{x},
$$

$$
\mathcal{E}^{2}_{\text{mix}} := D \int_{\Omega} g(c) \, d\mathbf{x} + D \int_{\Omega} g(1 - c) \, d\mathbf{x}
$$

are the mixture energies under the assumption (2.9a) or (2.9b), respectively. The parameter $D > 0$ controls the diffusivity of the magnetic particles. In the latter case, both $c = 0$ as well as $c = 1$ are considered to be unlikely states, hence the entropic ansatz has been chosen for both $c$ and $\varphi_1 = 1 - c$. Also in the latter case, $c$ must be a volume concentration in order to make sense of the expression $1 - c = \varphi_1$, see (2.45).

There will be two individual magnetic energies depending on which of the two magnetization equations, (2.36) or (2.41), is used. In case of (2.36), the ansatz for the magnetic energy follows [39]. The energy density of the magnetic field is classically given by

$$
\mathcal{E}_{\text{field}} := \frac{1}{2} \int_{\Omega'} \mathbf{b} \cdot \mathbf{h} \, d\mathbf{x},
$$

where

$$
\mathbf{b} := \mathbf{h} + \mathbf{I}_\Omega \mathbf{m}
$$

is the magnetic flux density. Further possible magnetic contributions to the total energy are the interaction of the magnetizable particles with the magnetic fields $\mathbf{h}_a$ and $\mathbf{h}_d = \mathbf{h} - \mathbf{h}_a$. This may be described by Zeeman-type terms [12] which are proportional to $-\mathbf{m} \cdot \mathbf{h}_a$ or $-\mathbf{m} \cdot \mathbf{h}_d$, respectively, i.e.

$$
\mathcal{E}_{\text{Zeeman}} := -\alpha_1 \mu_0 \int_{\Omega} |\mathbf{h}_d| \, d\mathbf{x} - \alpha_2 \mu_0 \int_{\Omega} \mathbf{h}_a \cdot \mathbf{m} \, d\mathbf{x}
$$

for some parameters $\alpha_1, \alpha_2 > 0$. Additionally, the magnetization’s decay in absence of (external) magnetic fields, and consequently the decay of stray fields, will be accounted for by a term like

$$
\mathcal{E}_{\text{decay}} := \alpha_3 \frac{\mu_0}{2} \int_{\Omega} |\mathbf{m}|^2 \, d\mathbf{x}.
$$

The following generic magnetic energy, which represents the sum of the above components $\mathcal{E}_{\text{field}}$, $\mathcal{E}_{\text{Zeeman}}$ and $\mathcal{E}_{\text{decay}}$, was proposed in [39],

$$
\mathcal{E}^{1}_{\text{mag}} := \alpha_0 \frac{\mu_0}{2} \int_{\Omega'} (|\mathbf{h}|^2 + \mathbf{h} \cdot \mathbf{I}_\Omega \mathbf{m}) \, d\mathbf{x} - \alpha_1 \frac{\mu_0}{2} \int_{\Omega} \mathbf{h}_d \cdot \mathbf{m} \, d\mathbf{x}
$$

$$
- \alpha_2 \frac{\mu_0}{2} \int_{\Omega} \mathbf{h}_a \cdot \mathbf{m} \, d\mathbf{x} + \alpha_3 \frac{\mu_0}{2} \int_{\Omega} |\mathbf{m}|^2 \, d\mathbf{x},
$$

(2.59a)

with generic energy parameters $\alpha_0, \alpha_1, \alpha_2, \alpha_3 > 0$. A model can be derived for any combination of those parameters. See [39] or Section A.3 for insight on coercivity of the magnetic energy (2.59a) with respect to $\mathbf{h}$ and $\mathbf{m}$, which is not immediately obvious in above formulation. Indeed, it can be shown that the identity

$$
\mathcal{E}^{1}_{\text{mag}} = \alpha_0 \frac{\mu_0}{2} \int_{\Omega'} \mathbf{h} \cdot \mathbf{h}_d \, d\mathbf{x} + \alpha_1 \frac{\mu_0}{2} \int_{\Omega'} |\mathbf{h} - \mathbf{h}_d|^2 \, d\mathbf{x}
$$

$$
- \alpha_2 \frac{\mu_0}{2} \int_{\Omega'} \mathbf{h}_a \cdot \mathbf{m} \, d\mathbf{x} + \alpha_3 \frac{\mu_0}{2} \int_{\Omega} |\mathbf{m}|^2 \, d\mathbf{x}
$$

(2.59b)

holds true – using the magnetostatic equations.
In the case of (2.41), energy-minimizing processes based on particle-particle or particle-field interaction are assumed to cease, as the behavior of the magnetization is already fully regulated by (2.41) for given particle distribution and magnetic field. Therefore, only the magnetic field energy remains, which can be expressed as follows,

\[ \mathcal{E}^{2\text{mag}} := \frac{\mu_0}{2} \int_{\Omega} |\mathbf{h}|^2 \, dx + \frac{\mu_0}{2} \int_{\Omega} \chi_{\text{lin}}(c) |\mathbf{h}|^2 \, dx. \]  

(2.59c)

No generic parameter is used here, as there is only one type of energy involved here.

**Remark 2.5.** For simplicity, effects of gravity will not be considered. However, it is straightforward to model those by including gravity's potential energy in the total energy.

In a standard way, dissipation functionals will be chosen. Before explaining the choices, let them be listed here first,

\[ D_{\text{kin}}(\mathbf{T}) := \int_{\Omega} \frac{\mathbf{T}^2}{4\eta} \, dx, \]  

(2.60)

\[ D^1_{\text{mix}}(v_{\text{part}}) := \int_{\Omega} \frac{c^2}{2f_p(c)K} |v_{\text{part}}|^2 \, dx, \]  

(2.61)

\[ D^2_{\text{mix}}(v_{\text{part}}) := \int_{\Omega} \frac{c^2}{2f_p(c)(1-c)K} |v_{\text{part}}|^2 \, dx. \]  

(2.62)

Here, \( K > 0 \) denotes the general mobility of the magnetic particles (then \( KD \) is the actual diffusion constant), \( \eta > 0 \) is the dynamic viscosity and

\[ f_p \text{ controls the propagation} \]  

(2.63)

is reasonable in view of the prominent choices.

The first functional (2.60) is being proposed for both cases, single-phase flow or two-phase flow, and will lead to the typical dissipation due to friction in the fluid. In the single-phase setting (2.9a), the functional (2.61) will be used, in the two-phase setting (2.62) will be used. Notice that with the choice \( m = 1 \) (i.e. \( f_p(c) = c^m \) for \( m \in \{1, 2\} \), the former entailing linear diffusion, the latter contributing to finite speed of propagation of the magnetic particles (cf. \[33\]). Assuming that

\[ f_p \text{ is locally Lipschitz-continuous} \]  

(2.64)

is recovered in (2.61), which has, for instance, been used in [1] for soluble species. Hence, for the suspended magnetic particles the same type of functional seems reasonable. The choice (2.62) is based on a similar approach. In this case the flux variable is denoted with \( J_\varphi \), see (2.49c). Due to its generic nature, it can formally be required to be of the type "density" times "convective velocity" as specified in (2.52), i.e. \( \frac{1}{2} J_\varphi = c v_{\text{part}} \), similar to (2.8) \( (J = c v_{\text{part}}) \), see also (2.51). Based on [1], a functional of the type

\[ \int_{\Omega} \frac{|J_\varphi|^2}{2M(\varphi)} \, dx \]

will be used as starting point, where \( M \) is the phase field’s mobility. Be aware that \( 4c^2 |v_{\text{part}}|^2 = |J_\varphi|^2 \), see (2.49c) and (2.52). Consistent to the ansatz that \( c = 0 \) and \( c = 1 \) are equally significant
and unlikely states in the two-phase setting, large dissipation will be assumed in those cases. This property will be translated to the phase field parameter’s mobility which motivates the choice

\[ M(\varphi) = K(1 + \varphi)(1 - \varphi). \]

By using (2.50),

\[ K(1 + \varphi)(1 - \varphi) = K(2c)(2 - 2c) = 4Kc(1 - c) \quad (2.65) \]

is the analogous expression in terms of the variable \( c \). Hence,

\[ \int_{\Omega} \frac{|J_{\varphi}|^2}{2M(\varphi)} \, dx \stackrel{(2.65), (2.52)}{=} \int_{\Omega} \frac{4c^2|\text{v}_{\text{part}}|^2}{8Kc(1 - c)} \, dx \]

which yields (2.62) by replacing the factor \( c \) in the denominator with \( f_p(c) \) in order to control the propagation behavior as in (2.61). This way, consistency of notation has been achieved which particularly enables to discuss ‘model W’ and ‘model B’ simultaneously in Section 4.

2.3 Model derivation by Onsager’s variational principle

In this section, Onsager’s variational method is used to derive three different models, based on various assumptions and equations from the preceding sections. The following sketches in Figure 2.2, Figure 2.3 and Figure 2.4 are used to show which equations, boundary conditions, energy functionals and dissipation functionals will be used in the derivation processes of the respective models. As usual, it is assumed that all quantities are sufficiently regular to allow for the following computations and particularly to give meaning to the governing equations and boundary conditions. For further insight on conducting Onsager’s variational method see [19, 28, 58].

2.3.1 Model GW

The model from [39], denoted by 'model GW' in this thesis, will be derived here. In case of \( \Omega' = \Omega \) the boundary conditions (2.23a) and (2.23) will be replaced with (2.24) and equation (2.22b) is dismissed.
The following computations regarding "model GW" are based on \[39\]. Using equations (2.43), (2.54), (2.42) and integration by parts, the time-derivative of \(E^1_{\text{kin}}\) (cf. (2.55a)) is easily computed to be

\[
\partial_t E^1_{\text{kin}} = \partial_t \left( \frac{p_0}{2} \int_{\Omega} |u|^2 \, dx \right) = p_0 \int_{\Omega} u \cdot u \, dx - \int_{\Omega} k \cdot u \, dx + \int_{\Omega} \text{div} \mathbf{T} \cdot u \, dx - \int_{\Omega} \nabla p \cdot u \, dx - \int_{\Omega} \nabla \cdot (u \cdot \nabla) u \cdot u \, dx
\]

(2.43)

Analogously, integration by parts, (2.42), (2.7), (2.54) and (2.10) yield

\[
\partial_t E^1_{\text{mix}} \quad \text{cf. (2.57a)} \quad \partial_t \left( D \int_{\Omega} g(c) \, dx \right) = D \int_{\Omega} g'(c) c_t \, dx - D \int_{\Omega} \text{div} (c u + c \nu_{\text{part}}) g'(c) \, dx
\]

(2.44)

(2.42)

(2.54)

(2.10)

\[
D \int_{\Omega} \mathbf{c} \cdot \nabla g'(c) \, dx + D \int_{\Omega} c \nu_{\text{part}} \cdot \nabla g'(c) \, dx - \int_{\Omega} c \nabla g'(c) \cdot \nu_{\text{part}} \, dx
\]

(2.54)

(2.49)

(2.70a)

(2.70b)

The magnetic terms require some preparations. A weak formulation of (2.22a) and (2.22b) will be presented, first. For the ease of presentation consider the equation

\[
\text{div} (h + \mathbf{I}_\Omega \mathbf{m}) = 0 \quad \text{on } [0, T] \times \Omega',
\]

which is a roundup of both equations in one single equation, see (2.14b), which is well-posed for \((h + \mathbf{I}_\Omega \mathbf{m}) \in H(\text{div})(\Omega')\). Testing above identity by a test function \(S \in H^1_{\text{mean}}(\Omega')\) and integrating by parts yields

\[
\int_{\Omega} (h + \mathbf{I}_\Omega \mathbf{m}) \cdot \nabla S \, dx = \int_{\partial \Omega'} (h + \mathbf{I}_\Omega \mathbf{m}) \cdot \nu S \, d\sigma - \int_{\partial \Omega} h_a \cdot \nu S \, d\sigma.
\]

(2.20)

In Section 2.4.1, this formulation will be derived in more detail based on the individual equations (2.22a) and (2.22b) where integration by parts is done on \(\Omega\) and \(\Omega' \setminus \Pi\) separately. Exploiting (2.13b) and Gauss’s divergence theorem,

\[
\int_{\Omega'} \nabla R \cdot \nabla S \, dx = \int_{\Omega'} h_a \cdot \nabla S \, dx - \int_{\Omega} \mathbf{m} \cdot \nabla S \, dx \quad \forall S \in H^1_{\text{mean}}(\Omega').
\]

(2.68)

The time-derivative of (2.68) is

\[
\int_{\Omega'} \nabla R_t \cdot \nabla S \, dx = \int_{\Omega'} \partial_t h_a \cdot \nabla S \, dx - \int_{\Omega} \mathbf{m}_t \cdot \nabla S \, dx \quad \forall S \in H^1_{\text{mean}}(\Omega').
\]

(2.69)

The analogous computation in the case \(\Omega' = \Omega\) leads to the same result, as has been carried out in [51,52], too. Recalling \(\nabla R = \mathbf{h}\) (cf. (2.16)) as well as (2.19), one gets the auxiliary identities

\[
\int_{\Omega'} |h|^2 \, dx = \int_{\Omega'} h_a \cdot h \, dx - \int_{\Omega} \mathbf{m} \cdot h \, dx,
\]

(2.70a)

\[
\int_{\Omega'} h \cdot h_d \, dx = \int_{\Omega'} h_a \cdot h_d \, dx - \int_{\Omega} \mathbf{m} \cdot h_d \, dx.
\]

(2.70b)
and
\[
\begin{align*}
\int_{\Omega'} \mathbf{h}_t \cdot \mathbf{h}_a \, dx &= \int_{\Omega'} \partial_t \mathbf{h}_a \cdot \mathbf{h}_a \, dx - \int_{\Omega} \mathbf{m}_t \cdot \mathbf{h}_a \, dx, \\
\int_{\Omega'} \mathbf{h}_t \cdot \mathbf{h}_d \, dx &= \int_{\Omega'} \partial_t \mathbf{h}_a \cdot \mathbf{h}_d \, dx - \int_{\Omega} \mathbf{m}_t \cdot \mathbf{h}_d \, dx
\end{align*}
\]
(2.71a) (2.71b)

by setting \( S := R \) or \( S := R_d \) in (2.68) and \( S := R_a \) or \( S := R_d \) in (2.69). By means of the last four identities (2.70a)–(2.71b) and \( \mathbf{h} = \mathbf{h}_a + \mathbf{h}_d \) (cf. (2.18)) one obtains for the time derivative of the magnetic energy (cf. (2.59a)) the identity
\[
\partial_t \mathcal{E}_{\text{mag}}^1 = \partial_t \left( \frac{\alpha_0 \mu_0}{2} \int_{\Omega'} (|\mathbf{h}|^2 + \mathbf{h} \cdot \mathbf{I}_\Omega \mathbf{m}) \, dx \right) - \partial_t \left( \frac{\alpha_1 \mu_0}{2} \int_{\Omega} \mathbf{h}_d \cdot \mathbf{m} \, dx \right)
\]
(2.70a) (2.70b)
\[
= \frac{\alpha_0 \mu_0}{2} \partial_t \left( \int_{\Omega'} \mathbf{h}_a \cdot \mathbf{h} \, dx + \int_{\Omega'} \mathbf{h}_a \cdot \mathbf{h}_t \, dx \right) + \frac{\alpha_1 \mu_0}{2} \partial_t \left( \int_{\Omega} |\mathbf{m}|^2 \, dx \right)
\]
(2.71a) (2.71b)
\[
- \frac{\alpha_0 \mu_0}{2} \partial_t \left( \int_{\Omega} \mathbf{h}_a \cdot \mathbf{m} \, dx + \int_{\Omega} \mathbf{h}_a \cdot \mathbf{m}_t \, dx \right) + \frac{\alpha_0 \mu_0}{2} \partial_t \left( \int_{\Omega} |\mathbf{m}|^2 \, dx \right)
\]
(2.72)

Observe that the remaining terms in (2.72) contain time derivatives only of \( \mathbf{h}_a \). Therefore, they are not relevant with respect to model derivation by Onsager’s principle. Sorting the terms of the first integral on the right-hand side of (2.72), one gets
\[
\begin{align*}
(a_0 + a_2) \mathbf{h}_a + 2a_1 \mathbf{h}_d - 2a_3 \mathbf{m} \\
= 2a_1 \mathbf{h} + (a_0 + a_2 - 2a_1) \mathbf{h}_a - 2a_3 \mathbf{m}
\end{align*}
\]
(2.73)

For the ease of notation, let
\[
\hat{\mathbf{h}} := \alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a \quad \text{and} \quad \hat{\mathbf{b}} := \hat{\mathbf{h}} - \alpha_3 \mathbf{m}.
\]
(2.73)

In a first step, formulas for the terms \(-\alpha_1 \mu_0 \int_{\Omega} \mathbf{m}_t \cdot \mathbf{h} \, dx\) and \(\alpha_3 \mu_0 \int_{\Omega} \mathbf{m}_t \cdot \mathbf{m} \, dx\) will be derived. For this, integration by parts will be used, see Section A.4 (A.29) or (A.30), for formulas involving the curl-operator in three or two dimensions, respectively. The three-dimensional formula will be repeated here for convenience,
\[
\int_{\Omega} \text{curl} \mathbf{a} \cdot \mathbf{b} \, dx = \int_{\Omega} \mathbf{a} \cdot \text{curl} \mathbf{b} \, dx - \int_{\partial \Omega} (\mathbf{a} \times \mathbf{\nu}) \cdot \mathbf{b} \, d\sigma.
\]
Another integration by parts formula is given by

\[ \int_\Omega \text{div}(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} \, d\mathbf{x} = \int_\Omega \sum_{i,j,k=1}^d \partial_{x_i}(a_j b_i)c_j \, d\mathbf{x} \]

\[ = - \int_\Omega \sum_{i,j,k=1}^d b_i \partial_{x_i}(c_j a_j) \, d\mathbf{x} + \int_{\partial\Omega} (\mathbf{b} \cdot \mathbf{\nu}) c_j a_j \, d\sigma \]

\[ = - \int_\Omega (\mathbf{b} \cdot \nabla) \mathbf{c} \cdot \mathbf{a} \, d\mathbf{x} + \int_{\partial\Omega} (\mathbf{b} \cdot \mathbf{\nu})(\mathbf{c} \cdot \mathbf{a}) \, d\sigma. \tag{2.74} \]

It is known (see (A.26) and (A.22) if \(d = 2\)) that

\[(\mathbf{a}, \mathbf{b}, \mathbf{c}) \mapsto (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\text{ is an alternating trilinear form}. \tag{2.75} \]

Moreover (cf. (A.25)), \(\text{curl} \nabla\) vanishes in three or two dimensions. By using (2.36) as well as integration by parts combined with the boundary conditions (2.54), (2.37), (2.39) it is now straightforward to compute

\[ \alpha_3 \mu_0 \int_\Omega \mathbf{m} \cdot \mathbf{m} \, d\mathbf{x} \tag{2.36} \]

\[ = - \alpha_3 \mu_0 \int_\Omega \text{div}(\mathbf{m} \otimes (\mathbf{u} + \mathbf{v}_{\text{part}})) \cdot \mathbf{m} \, d\mathbf{x} \]

\[ + \alpha_3 \mu_0 \sigma \int_\Omega \nabla \text{div} \mathbf{m} \cdot \mathbf{m} \, d\mathbf{x} + \alpha_3 \mu_0 \sigma \int_\Omega \text{curl} \text{curl} \mathbf{m} \cdot \mathbf{m} \, d\mathbf{x} \]

\[ + \frac{\alpha_3 \mu_0}{2} \int_\Omega \text{curl} \mathbf{u} \times \mathbf{m} \cdot \mathbf{m} \, d\mathbf{x} + \frac{\alpha_3 \mu_0}{\tau_{\text{rel}}} \int_\Omega |\mathbf{m}|^2 \, d\mathbf{x} + \frac{\alpha_3 \mu_0}{\tau_{\text{rel}}} \int_\Omega \chi(\mathbf{c}, \mathbf{h}) \mathbf{h} \cdot \mathbf{m} \, d\mathbf{x} \]

\[ = \alpha_3 \mu_0 \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{m} \cdot \mathbf{m} \, d\mathbf{x} + \alpha_3 \mu_0 \int_\Omega (\mathbf{v}_{\text{part}} \cdot \nabla) \mathbf{m} \cdot \mathbf{m} \, d\mathbf{x} \]

\[ - \alpha_3 \mu_0 \int_{\partial\Omega} (\mathbf{u} \cdot \mathbf{\nu}) (\mathbf{m} \cdot \mathbf{m}) \, d\sigma - \alpha_3 \mu_0 \int_{\partial\Omega} (\mathbf{v}_{\text{part}} \cdot \mathbf{\nu})(\mathbf{m} \cdot \mathbf{m}) \, d\sigma \]

\[ - \alpha_3 \mu_0 \sigma \int_{\partial\Omega} |\text{div} \mathbf{m}|^2 \, d\mathbf{x} - \alpha_3 \mu_0 \sigma \int_{\partial\Omega} |\text{curl} \mathbf{m}|^2 \, d\mathbf{x} \]

\[ + \alpha_3 \mu_0 \sigma \int_{\partial\Omega} \text{div} \mathbf{m} \mathbf{m} \cdot \mathbf{\nu} \, d\sigma + \alpha_3 \mu_0 \sigma \int_{\partial\Omega} \text{curl} \mathbf{m} \times \mathbf{\nu} \cdot \mathbf{m} \, d\sigma \]

\[ - \frac{\alpha_3 \mu_0}{\tau_{\text{rel}}} \int_\Omega |\mathbf{m}|^2 \, d\mathbf{x} + \frac{\alpha_3 \mu_0}{\tau_{\text{rel}}} \int_\Omega \chi(\mathbf{c}, \mathbf{h}) \mathbf{h} \cdot \mathbf{m} \, d\mathbf{x} \]

\[ = \alpha_3 \mu_0 \int_\Omega (\mathbf{v}_{\text{part}} \cdot \nabla) \mathbf{m} \cdot \mathbf{m} \, d\mathbf{x} - \alpha_3 \mu_0 \int_{\partial\Omega} |\text{curl} \mathbf{m}|^2 \, d\mathbf{x} \]

\[ - \alpha_3 \mu_0 \sigma \int_{\partial\Omega} |\text{div} \mathbf{m}|^2 \, d\mathbf{x} - \alpha_3 \mu_0 \sigma \int_{\partial\Omega} |\text{curl} \mathbf{m}|^2 \, d\mathbf{x} \]

\[ + \alpha_3 \mu_0 \sigma \int_{\partial\Omega} \text{div} \mathbf{m} \mathbf{m} \cdot \mathbf{\nu} \, d\sigma + \alpha_3 \mu_0 \sigma \int_{\partial\Omega} \text{curl} \mathbf{m} \times \mathbf{\nu} \cdot \mathbf{m} \, d\sigma \]

\[ - \frac{\alpha_3 \mu_0}{\tau_{\text{rel}}} \int_\Omega |\mathbf{m}|^2 \, d\mathbf{x} + \frac{\alpha_3 \mu_0}{\tau_{\text{rel}}} \int_\Omega \chi(\mathbf{c}, \mathbf{h}) \mathbf{h} \cdot \mathbf{m} \, d\mathbf{x} \]

\[ - \alpha_3 \mu_0 \int_{\partial\Omega} ((\mathbf{v}_{\text{part}} \cdot \mathbf{\nu})(\mathbf{m} \cdot \mathbf{m}) - \sigma \text{div} \mathbf{m} \mathbf{\nu} \cdot \mathbf{m} - \sigma \text{curl} \mathbf{m} \times \mathbf{\nu} \cdot \mathbf{m}) \, d\sigma \]

\[ = 0 \text{ due to (2.40)} \tag{2.76} \]

and analogously

\[ - \alpha_1 \mu_0 \int_\Omega \mathbf{m} \cdot \mathbf{h} \, d\mathbf{x} \tag{2.36} \]

\[ = \alpha_1 \mu_0 \int_\Omega \text{div}(\mathbf{m} \otimes (\mathbf{u} + \mathbf{v}_{\text{part}})) \cdot \mathbf{h} \, d\mathbf{x} \]

\[ - \alpha_1 \mu_0 \sigma \int_{\partial\Omega} \nabla \text{div} \mathbf{m} \cdot \mathbf{h} \, d\mathbf{x} + \alpha_1 \mu_0 \sigma \int_{\partial\Omega} \text{curl} \text{curl} \mathbf{m} \cdot \mathbf{h} \, d\mathbf{x} \]

\[ \text{Curl if } d = 2 \]
Again, a closer look on the boundary integrals related to the magnetization equation is available in the preceding computation (2.76). The arguments here are analogous. The magnetostatic equations (2.22a), (2.22b) as well as (2.70a) admit some simplifications, i.e.

\[
\frac{\alpha_1 \mu_0}{\tau_{\text{rel}}} \int_{\Omega} \mathbf{m} \cdot \mathbf{h} \, dx - \frac{\alpha_1 \mu_0}{\tau_{\text{rel}}} \int_{\Omega'} \mathbf{h} \cdot \mathbf{h} \, dx - \frac{\alpha_1 \mu_0}{\tau_{\text{rel}}} \int_{\Omega'} (\mathbf{v}_{\text{part}} \cdot \nabla) \mathbf{h} \cdot \mathbf{m} \, dx
\]

More insight into the integration by parts of \( \int_{\Omega} \nabla \cdot (\mathbf{m} \otimes (\mathbf{u} + \mathbf{v}_{\text{part}})) \cdot \mathbf{m} \, dx \) and the diffusive term in the magnetization equation is given in the computations in (2.76). The arguments here are analogous. The magnetostatic equations (2.22a), (2.22b) as well as (2.70a) admit some simplifications, i.e.

\[
\alpha_1 \mu_0 \sigma \int_{\Omega} \text{div} \mathbf{m} \, dx - \alpha_1 \mu_0 \sigma \int_{\Omega} |\text{div} \mathbf{h}|^2 \, dx,
\]

while recalling \( \nabla R = \mathbf{h} \) (cf. (2.16)). Moreover,

\[
\frac{\alpha_1 \mu_0}{\tau_{\text{rel}}} \int_{\Omega} \mathbf{m} \cdot \mathbf{h} \, dx - \frac{\alpha_1 \mu_0}{\tau_{\text{rel}}} \int_{\Omega'} \mathbf{h} \cdot \mathbf{h} \, dx - \frac{\alpha_1 \mu_0}{\tau_{\text{rel}}} \int_{\Omega'} (\mathbf{v}_{\text{part}} \cdot \nabla) \mathbf{h} \cdot \mathbf{m} \, dx
\]

Therefore,

\[
- \alpha_1 \mu_0 \int_{\Omega} \mathbf{m} \cdot \mathbf{h} \, dx - \alpha_1 \mu_0 \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{h} \cdot \mathbf{m} \, dx - \alpha_1 \mu_0 \int_{\Omega} (\mathbf{v}_{\text{part}} \cdot \nabla) \mathbf{h} \cdot \mathbf{m} \, dx
\]

\[
- \alpha_1 \mu_0 \sigma \int_{\Omega} |\text{div} \mathbf{h}|^2 \, dx - \alpha_1 \mu_0 \sigma \int_{\Omega'} |\text{div} \mathbf{h}|^2 \, dx - \frac{\alpha_1 \mu_0}{\tau_{\text{rel}}} \int_{\Omega'} \text{div} \mathbf{m} \cdot \mathbf{h} \, dx
\]

Now, a formula for \(- \frac{\beta \mu_0}{2} \int_{\Omega} \mathbf{m} \cdot \mathbf{h} \, dx\) will be computed. Using \( \text{div} \mathbf{h}_a = 0 \), \( \text{curl} \mathbf{h}_a = 0 \), cf. (2.13), the formula is

\[
- \frac{\beta \mu_0}{2} \int_{\Omega} \mathbf{m} \cdot \mathbf{h}_a \, dx - \frac{\beta \mu_0}{2} \int_{\Omega} \text{div} (\mathbf{m} \otimes (\mathbf{u} + \mathbf{v}_{\text{part}})) \cdot \mathbf{h}_a \, dx
\]

Again, a closer look on the boundary integrals related to the \( \text{div} (\mathbf{m} \otimes [\ldots]) \cdot \mathbf{h}_a \)-term and the diffusive term in the magnetization equation is given in the computations in (2.76).
Next, the results so far will be put together. Combining the previous results in (2.76), (2.80), (2.81) with (2.72) yields

\[
\partial_t \mathcal{E}^{1}_{\text{mag}} = -\mu_0 \int_{\Omega} m \cdot \left( \alpha_1 h + \frac{\beta}{2} h_a - \alpha_3 m \right) \, dx \\
+ \frac{\alpha_0 \mu_0}{2} \int_{\Omega'} \partial_t h_a \cdot h_a \, dx + \frac{\alpha_0 \mu_0}{2} \int_{\Omega'} \partial_t h_a \cdot h \, dx - \frac{\alpha_2 \mu_0}{2} \int_{\Omega} \partial_t h_a \cdot m \, dx
\]

(2.82)

Introducing the abbreviations

\[
\mathcal{D}^{1}_{\text{mag}} := \frac{\alpha_3 \mu_0}{\tau_{\text{rel}}} \int_{\Omega} |\text{curl} \, \mathbf{m}|^2 \, dx + \frac{\alpha_1 \mu_0}{\tau_{\text{rel}}} \int_{\Omega} \chi(c, h)|\mathbf{h}|^2 \, dx + \frac{\alpha_1 \mu_0}{\tau_{\text{rel}}} \int_{\Omega'} |\mathbf{h}|^2 \, dx \\
+ \alpha_3 \mu_0 \sigma \int_{\Omega} \frac{1}{\Omega'} |\text{curl} \, \mathbf{m}|^2 \, dx + \alpha_3 \mu_0 \sigma \int_{\Omega} |\text{div} \, \mathbf{m}|^2 \, dx + \alpha_1 \mu_0 \sigma \int_{\Omega \setminus \partial \Omega} |\text{div} \, \mathbf{h}|^2 \, dx
\]

(2.83)

\[
\mathbf{k}^{1}_{\text{mag}} := -\mu_0 \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{h} \cdot \mathbf{m} \, dx - \frac{\mu_0}{2} \int_{\Omega} \text{curl}((\mathbf{m} \times \hat{h}) \cdot \mathbf{u}) \, dx,
\]

(2.73)

\[
\mathbf{w}^{1}_{\text{mag}} := -\int_{\Omega} \mathbf{T} : \nabla \mathbf{u} \, dx,
\]

(2.73)

\[
\mathbf{F}^{1}_{\text{mag}} := \frac{\alpha_1 \mu_0}{\tau_{\text{rel}}} \int_{\Omega'} h_a \cdot h \, dx + \frac{\beta \mu_0}{2 \tau_{\text{rel}}} \int_{\Omega} m \cdot h_a \, dx - \frac{\beta \mu_0}{2 \tau_{\text{rel}}} \int_{\Omega} \chi(c, h) h \cdot h_a \, dx \\
+ \frac{\alpha_0 \mu_0}{2} \int_{\Omega'} \partial_t h_a \cdot h_a \, dx + \frac{\alpha_0 \mu_0}{2} \int_{\Omega'} \partial_t h_a \cdot h \, dx - \frac{\alpha_2 \mu_0}{2} \int_{\Omega} \partial_t h_a \cdot m \, dx,
\]

as well as summing up (2.82) with (2.66) and (2.67) from before, the time derivative of the energy is obtained, i.e.

\[
\partial_t \mathcal{E}^{\text{GW}} := \partial_t \mathcal{E}^{1}_{\text{kin}} + \partial_t \mathcal{E}^{1}_{\text{mix}} + \partial_t \mathcal{E}^{1}_{\text{mag}}
\]

(2.84)

By solving

\[
\delta (T, \mathbf{v}_{\text{part}})(\mathcal{E}^{1}_{\text{kin}}(T) + \mathcal{E}^{1}_{\text{mix}}(\mathbf{v}_{\text{part}}) + \mathcal{E}^{1}_{\text{mag}}(\mathbf{v}_{\text{part}}) + \mathcal{D}_{\text{kin}}(T) + \mathcal{D}_{\text{mix}}(\mathbf{v}_{\text{part}})) = 0,
\]
As external magnetic fields input energy into the system, the right-hand side does not necessarily vanish. Besides of the term \( -\mu_0 \int_{\Omega} (\hat{v} \cdot \nabla) \hat{h} \cdot \mathbf{m} \, dx + D \int_{\Omega} c \nabla q(c) \cdot \hat{v} \, dx = \int_{\Omega} \frac{c^2}{f_p(c) K} v_{part} \cdot \hat{v} \, dx \quad \forall \hat{v} \in L^2(\Omega; \mathbb{R})^d, \)

implying

\[
T \equiv 2 \eta D u, \quad (2.85)
\]

and (see the definition of \( w_{\text{mag}} \))

\[
-\mu_0 \int_{\Omega} (\hat{v} \cdot \nabla) \hat{h} \cdot \mathbf{m} \, dx + D \int_{\Omega} c \nabla q(c) \cdot \hat{v} \, dx = \int_{\Omega} \frac{c^2}{f_p(c) K} v_{part} \cdot \hat{v} \, dx \quad \forall \hat{v} \in L^2(\Omega; \mathbb{R})^d,
\]

in spirit of energy conservation, consider

\[
\partial_t \mathcal{E}_{GW} + \mathcal{D}_{\text{kin}}(T) + \mathcal{D}_{\text{mix}}^1(v_{\text{part}}) + \mathcal{D}_{\text{mag}}^1 \nabla_{\mathcal{F}_{\text{mag}}}^1 + \frac{\alpha_3 \mu_0}{\tau_{\text{rel}}} \int_{\Omega} \chi(c, \mathbf{h}) \hat{h} \cdot \mathbf{m} \, dx + \int_{\Omega} k \cdot u \, dx + k_{\text{mag}}^1. \quad (2.87)
\]

As external magnetic fields input energy into the system, the right-hand side does not necessarily vanish. Besides of \( \mathcal{F}_{\text{mag}}^1 \), one can interpret the term

\[
\int_{\Omega} \chi(c, \mathbf{h}) \hat{h} \cdot \mathbf{m} \, dx = \int_{\Omega} \chi(c, \mathbf{h}) h_a \cdot \mathbf{m} \, dx + \int_{\Omega} \chi(c, \mathbf{h}) h_d \cdot \mathbf{m} \, dx
\]

to describe the increase of energy due to magnetizing the magnetic particles and its decreasing counter part which is prevalent when the external field \( h_a \) is switched off. This effect is not to be mistaken for decrease of magnetization by thermodynamic fluctuations which is reflected in the energy term \( |\mathbf{m}|^2 \) (see (2.59a)). In fact, this effect is a side-effect of the assumption that (2.89e) holds. The latter equation induces this change of energy by its source/sink-term on the right-hand side.

However, regarding the remaining terms involving the flow field \( u \), the usual force balance

\[
\int_{\Omega} k \cdot u \, dx + k_{\text{mag}}^1 = 0
\]

will be assumed, which implies

\[
k := \mu_0 (\nabla (\alpha_1 h + \frac{\beta}{2} h_a))^T \mathbf{m} + \frac{\mu_0}{2} \text{curl}(\mathbf{m} \times (\alpha_1 h + \frac{\beta}{2} h_a)). \quad (2.88)
\]

Due to (A.25) the term \( \text{curl} \hat{h} \) vanishes and consequently the matrix \( \nabla \hat{h} \) is symmetric. Therefore, the term \( (\nabla \hat{h})^T \mathbf{m} = (\mathbf{m} \cdot \nabla) \hat{h} \) represents the Kelvin force.

The final system of equations reads

\[
\rho_0 u_t + \rho_0 (u \cdot \nabla) u + \nabla p - \text{div}(2\eta D u) = \mu_0 (\mathbf{m} \cdot \nabla) (\alpha_1 h + \frac{\beta}{2} h_a) + \frac{\mu_0}{2} \text{curl}(\mathbf{m} \times (\alpha_1 h + \frac{\beta}{2} h_a)) + \text{Curl if } d = 2 \quad (2.89a)
\]

\[
\text{div } u = 0, \quad (2.89b)
\]
2.3 Model derivation by Onsager’s variational principle

\[ \frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c + \text{div}(c \mathbf{v}_{\text{part}}) = 0, \]  
\[ \mathbf{v}_{\text{part}} = K \frac{f_p(c)}{c^2} (-D \nabla c + \mu_0 (\nabla (\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_d) - \alpha_3 \mathbf{m}))^T \mathbf{m}), \]  
\[ \mathbf{m}_t + \text{div} (\mathbf{m} \otimes (\mathbf{u} + \mathbf{v}_{\text{part}})) - \sigma \nabla \text{div} \mathbf{m} + \sigma \text{curl} \text{curl} \mathbf{m} \\ = \frac{1}{2} \text{curl} \mathbf{u} \times \mathbf{m} - \frac{1}{\tau_{\text{rel}}} (\mathbf{m} - \chi(c, \mathbf{h}) \mathbf{h}), \]  
\[ - \Delta R = \text{div} \mathbf{m}, \]

on \((0, T) \times \Omega\) and

\[ -\Delta R = 0, \]

on \((0, T) \times (\Omega' \setminus \Omega')\), where

\[ h := \nabla R. \]

It is supplemented with boundary conditions

\[ \mathbf{u} = 0 \quad \text{on } [0, T] \times \partial \Omega, \]  
\[ c \mathbf{v}_{\text{part}} \cdot \mathbf{\nu} = 0 \quad \text{on } (0, T) \times \partial \Omega, \]  
\[ (\mathbf{v}_{\text{part}} \cdot \mathbf{\nu})(\mathbf{m} \cdot \mathbf{\nu}) - \sigma \text{div} \mathbf{m} = 0 \quad \text{on } (0, T) \times \partial \Omega, \]
\[ (\mathbf{v}_{\text{part}} \cdot \mathbf{\nu})(\mathbf{m} - (\mathbf{m} \cdot \mathbf{\nu}) \mathbf{\nu}) - \sigma \text{curl} \mathbf{m} \times \mathbf{\nu} = 0 \quad \text{on } (0, T) \times \partial \Omega, \]
\[ \nabla R \cdot \mathbf{\nu} = \mathbf{h}_a \cdot \mathbf{\nu} \quad \text{on } [0, T] \times \partial \Omega', \]

the transmission condition

\[ [\nabla R + \mathbf{I}_\Omega \mathbf{m}] \cdot \mathbf{\nu} = 0 \quad \text{on } [0, T] \times \partial \Omega, \]

and initial conditions

\[ \mathbf{u}(0, \cdot) = \mathbf{u}^{\text{init}}, \]  
\[ c(0, \cdot) = c^{\text{init}}, \]  
\[ \mathbf{m}(0, \cdot) = \mathbf{m}^{\text{init}}. \]

Recall, in case of \(\Omega' = \Omega\) the boundary conditions \((2.90c)\) and \((2.90d)\) are replaced with the condition

\[ (\nabla R + \mathbf{m}) \cdot \mathbf{\nu}|_{\partial \Omega} = \mathbf{h}_a \cdot \mathbf{\nu}|_{\partial \Omega} \]

from \((2.24)\).

2.3.2 Model W

In contrast to "model GW", we will be concise in our presentation of "model W" (and also "model B" later on), by referring to similar steps which have already been carried out before. First, notice that the case \(\Omega' = \Omega\) is treated analogously as in the derivation of "model GW", i.e. \((2.23a)\) and \((2.23b)\) will be replaced with \((2.24)\) and equation \((2.22b)\) is dismissed. The kinetic energy and mixture energy are identical to the case of "model GW". Therefore, the results from the derivation of "model GW" can be used and only the time derivative of the magnetic energy needs to be considered.
Inserting (2.41) into (2.69) yields
\[
\int_{\Omega'} \nabla R_t \cdot \nabla S \, dx = \int_{\Omega'} \partial_t h_n \cdot \nabla S \, dx - \int_{\Omega} \chi_0 c_t \nabla R \cdot \nabla S \, dx - \int_{\Omega} \chi_{\text{lin}}(c) \nabla R_t \cdot \nabla S \, dx \tag{2.92}
\]
for all \( S \in H^1_{\text{mean}}(\Omega') \). Therefore, setting \( S := R \) in (2.92) and recalling \( h = \nabla R \) (cf. (2.16)) yields
\[
\int_{\Omega} h_t \cdot h \, dx + \int_{\Omega} \chi_{\text{lin}}(c) h_t \cdot h \, dx + \int_{\Omega} \chi_0 c_t |h|^2 \, dx = \int_{\Omega'} \partial_t h_n \cdot h \, dx. \tag{2.93}
\]
With this at hand, the time derivative of the magnetic energy (cf. (2.59c)) is
\[
\partial_t \mathcal{E}^2_{\text{mag}} = \partial_t \left( \frac{\mu_0}{2} \int_{\Omega'} |h|^2 \, dx + \frac{\mu_0}{2} \int_{\Omega} \chi_{\text{lin}}(c) |h|^2 \, dx \right)
\]
\[
= \frac{\mu_0}{2} \int_{\Omega'} h_t \cdot h \, dx + \mu_0 \int_{\Omega} \chi_{\text{lin}}(c) h_t \cdot h \, dx + \frac{\mu_0}{2} \int_{\Omega} \chi_0 c_t |h|^2 \, dx - \frac{\mu_0}{2} \int_{\Omega} \chi_0 c_t |h|^2 \, dx + \mu_0 \int_{\Omega} \partial_t h_n \cdot h \, dx \tag{2.94}
\]
where in the last step integration by parts has been used. Observe that (for vector fields \( v \))
\[
\frac{\mu_0 \chi_0}{2} c \nabla |h|^2 \cdot v \tag{2.95}
\]
\[
= \mu_0 \chi_{\text{lin}}(c) \left( (\nabla h)^T h \right) \cdot v = \mu_0 \chi_{\text{lin}}(c)(v \cdot \nabla) h \cdot h
\]
\[
\text{\( h = \nabla R \) is symmetric } \implies \mu_0 \chi_{\text{lin}}(c) h \cdot \nabla) h \cdot v = \mu_0 (m \cdot \nabla) h \cdot v, \tag{2.41}
\]
hence the first term of the right-hand side in (2.94) represents the Kelvin force.
With the computations from the derivation of 'model GW' the time derivative of the total energy is evident.

\[ \partial_t E^W := \partial_t E^{1}_{\text{kin}} + \partial_t E^{1}_{\text{mix}} + \partial_t E^{2}_{\text{mag}} \]

\[ \begin{aligned}
&= \int_{\Omega} T : \nabla u \, dx + \int_{\Omega} k \cdot u \, dx + D \int_{\Omega} \nabla g'(c) \cdot v_{\text{part}} \, dx \\
&\quad - \mu_0 \int_{\Omega} (m \cdot \nabla) h \cdot (u + v_{\text{part}}) \, dx + \mu_0 \int_{\Omega} \partial_t h_a \cdot h \, dx
\end{aligned} \]  

(2.96)

Solving

\[ \delta (T, v_{\text{part}})(E^{1}_{\text{kin}}(T) + E^{1}_{\text{mix}}(v_{\text{part}}) + E^{2}_{\text{mag}}(v_{\text{part}}) + D_{\text{kin}}(T) + D^{1}_{\text{mix}}(v_{\text{part}})) = 0 \]

yields for fixed time the following equations,

\[ \int_{\Omega} \hat{T} : \nabla u \, dx = \int_{\Omega} \frac{2\eta}{\kappa} \hat{T} \, dx \quad \forall \hat{T} \in L^2(\Omega; \mathbb{R})^{d \times d}, \]

\[ \mu_0 \int_{\Omega} (m \cdot \nabla) h \cdot v_{\text{part}} \, dx = \int_{\Omega} \frac{c^2}{f_p(c)K} v_{\text{part}} : \hat{v} \, dx \quad \forall \hat{v} \in L^2(\Omega; \mathbb{R})^d. \]

Hence,

\[ T \overset{(2.94)}{=} 2\eta D u, \]

(2.97)

\[ v_{\text{part}} = - \frac{K D f_p(c)}{c^2} \nabla c + \frac{K \mu_0 f_p(c)}{c^2} (m \cdot \nabla) h. \]  

(2.98)

The force density is obtained by considering

\[ \partial_t E^W + D_{\text{kin}}(T) + D^{1}_{\text{mix}} = \mu_0 \int_{\Omega} \partial_t h_a \cdot h \, dx + \int_{\Omega} k \cdot u \, dx - \mu_0 \int_{\Omega} (m \cdot \nabla) h \cdot u \, dx. \]

In a state without external forces \((h_a = 0)\), conservation of energy suggests the force balance

\[ \int_{\Omega} k \cdot u \, dx = \mu_0 \int_{\Omega} (m \cdot \nabla) h \cdot u \, dx. \]

Hence,

\[ k = \mu_0 (m \cdot \nabla) h = \mu_0 (\chi_{\text{lin}}(c) h \cdot \nabla) h \]  

(2.99)

is the force density.

The final system of equations reads

\[ \begin{align*}
\rho_0 u_t + \rho_0 (u \cdot \nabla) u + \nabla p - \text{div}(2\eta D u) &= \mu_0 (\chi_{\text{lin}}(c) h \cdot \nabla) h, \\
\text{div} u &= 0,
\end{align*} \]

(2.100a)

(2.100b)

\[ \begin{align*}
c_t + u \cdot \nabla c + \text{div}(c v_{\text{part}}) &= 0, \\
v_{\text{part}} &= \frac{f_p(c)}{c^2} \left( -D \nabla c + \mu_0 (\chi_{\text{lin}}(c) h \cdot \nabla) h \right), \\
- \text{div}(1 + \chi_{\text{lin}}(c)) \nabla R &= 0,
\end{align*} \]

(2.100c)

(2.100d)

(2.100e)

\[ \begin{align*}
\text{on } (0, T) \times \Omega \text{ and } \Delta R &= 0,
\end{align*} \]

(2.100f)
on \((0, T) \times (\Omega' \setminus \overline{\Omega})\), where
\[ h := \nabla R. \]

It is supplemented with boundary conditions
\[
\begin{align*}
\mathbf{u} &= 0 & \text{on } [0, T] \times \partial \Omega, \\
c\nu_{\text{part}} &\cdot \nu = 0 & \text{on } (0, T] \times \partial \Omega, \\
\nabla R \cdot \nu &= h_a \cdot \nu & \text{on } [0, T] \times \partial \Omega', \\
\end{align*}
\]
the transmission condition
\[
[\nabla R + \mathbf{1}_{\Omega} \chi_{\text{lin}}(c) h] \cdot \nu = 0 \\
\text{on } [0, T] \times \partial \Omega,
\]
and initial conditions
\[
\begin{align*}
\mathbf{u}(0, \cdot) &= \mathbf{u}^{\text{init}}, \\
c(0, \cdot) &= c^{\text{init}}.
\end{align*}
\]

Here, \(\mathbf{1}_{\Omega} \chi_{\text{lin}}(c)\) is analogously defined as is \((2.11)\) as the extension of \(\chi_{\text{lin}}(c)\) onto \(\Omega'\) by zero outside of \(\Omega\).

In case of \(\Omega' = \Omega\) the boundary conditions \((2.101c), (2.101d)\) are replaced on basis of \((2.24)\), which yields
\[
(1 + \chi(c)) \nabla R \cdot \nu|_{\partial \Omega} = h_a \cdot \nu|_{\partial \Omega}.
\]

### 2.3.3 Model B

The last model to be considered is based on the two-phase flow assumption \((2.9b)\). As before, in the case \(\Omega' = \Omega\), \((2.23a)\) and \((2.23)\) will be replaced with \((2.24)\) and equation \((2.22b)\) is dismissed. The time derivative of the kinetic energy and the mixture energy needs to be revised, but the magnetic energy can be treated as in the derivation of 'model W'.

**Figure 2.4:** Overview of governing equations and energies of "model B". Differences to "model GW" are colored in blue, differences to "model W" are colored in green.
Using \(2.53, 2.7\) along with their boundary conditions \(2.54, 2.10\) and integration by parts, the time derivatives of the kinetic and mixture energies (cf. \(2.56\), \(2.57\)) are given by

\[
\partial_t \mathcal{E}_{\text{kin}}^2 = \partial_t \left( \frac{1}{2} \int k \cdot u \, dx \right) = \int k \cdot u \, dx - \int \left( (\rho(c) u + \rho'(c) c v_{\text{part}}) \cdot \nabla \right) u \cdot u \, dx
\]

and

\[
\partial_t \mathcal{E}_{\text{mix}}^2 = D \partial_t \left( \int g(c) \, dx + \int g(1-c) \, dx \right) = D \int g'(c) - g'(1-c) c \, dx
\]

Reusing the computation in \(2.94\), the time derivative of the total energy follows to be

\[
\partial_t \mathcal{E} = \partial_t \mathcal{E}_{\text{kin}}^2 + \partial_t \mathcal{E}_{\text{kin}}^2 + \partial_t \mathcal{E}_{\text{mag}}^2
\]

From the requirement

\[
\delta(T, v_{\text{part}})(\partial_t \mathcal{E}^2_{\text{kin}}(T) + \partial_t \mathcal{E}^2_{\text{mix}}(v_{\text{part}}) + \partial_t \mathcal{E}_{\text{mag}}^2(v_{\text{part}}) + D_{\text{kin}}(T) + D_{\text{mix}}^2(v_{\text{part}}) = 0
\]

it follows for fixed time that

\[
\int T : \nabla u \, dx = \int T : \hat{T} \, dx \quad \forall T \in L^2(\Omega; \mathbb{R})^{d \times d}
\]
and it follows with the help of (2.95) and $c \nabla (g'(c) - g'(1-c)) = \frac{\nabla c}{1-c}$ (see the auxiliary computations in (2.105)) that

$$\mu_0 \int _\Omega \left( \chi_{\text{lin}}(c) \mathbf{h} \cdot \nabla \right) \mathbf{h} \cdot \mathbf{v}_{\text{part}} \, dx + D \int _\Omega \mathbf{v}_{\text{part}} \cdot \frac{\nabla c}{1-c} \, dx$$

$$= \int _\Omega \frac{c^2}{f_p'(c)(1-c)K} \mathbf{v}_{\text{part}} : \nabla \mathbf{h} \, dx \quad \forall \mathbf{h} \in L^2(\Omega; \mathbb{R}^d).$$

One easily obtains

$$T \overset{(2.106)}{=} 2 \eta \mathbf{D} \mathbf{u},$$
$$\mathbf{v}_{\text{part}} = - \frac{K D f_p'(c)}{c^2} \nabla c + \frac{K \mu_0 f_p'(c)}{c^2} (1-c)(\chi_{\text{lin}}(c) \mathbf{h} \nabla) \mathbf{h}. \quad (2.107)$$

The green-colored part is the only difference between (2.108) and (2.98). In case of $f_p(c) = c$ (the linear diffusion case) the convective velocity (2.108) can be written as follows,

$$\mathbf{v}_{\text{part}} = - K D \frac{\nabla c}{c} + K \mu_0 \left( 1 - \frac{c}{c} \right) (\chi_{\text{lin}}(c) \mathbf{h} \cdot \nabla) \mathbf{h}, \quad (2.108)$$

and (2.7) turns out to be

$$c_t + \text{div}(c \mathbf{u} - K D \nabla c + K \mu_0 ((1-c) \chi_{\text{lin}}(c) \mathbf{h} \cdot \nabla) \mathbf{h}) = 0.$$

which coincides with the equation in [41]. The force density is given by

$$k \overset{(2.95)}{=} \mu_0 (\chi_{\text{lin}}(c) \mathbf{h} \cdot \nabla) \mathbf{h} \quad (2.109)$$

which follows in spirit of energy conservation from

$$\partial_t \mathcal{E}^B + D_{\text{lin}}(\mathbf{T}) + D_{\text{mix}}^2(\mathbf{v}_{\text{part}}) = \mu_0 \int _\Omega \partial_t \mathbf{h}_a \cdot \mathbf{h} \, dx + \int _\Omega \mathbf{k} \cdot \mathbf{u} \, dx - \mu_0 \int _\Omega (\chi_{\text{lin}}(c) \mathbf{h} \cdot \nabla) \mathbf{h} \cdot \mathbf{u} \, dx \quad (2.106) \quad (2.107) \quad (2.108)$$

analogously as in case of "model W*.

The final system is given by

$$\rho(c) \mathbf{u}_t + ((\rho(c) \mathbf{u} + \rho'(c)c \mathbf{v}_{\text{part}}) \cdot \nabla) \mathbf{u} + \nabla p - \text{div}(2 \eta \mathbf{D} \mathbf{u}) = \mu_0 (\chi_{\text{lin}}(c) \mathbf{h} \cdot \nabla) \mathbf{h}, \quad (2.110a)$$
$$\text{div} \mathbf{u} = 0, \quad (2.110b)$$
$$c_t + \mathbf{u} \cdot \nabla c + \text{div}(c \mathbf{v}_{\text{part}}) = 0, \quad (2.110c)$$
$$\mathbf{v}_{\text{part}} = - K D \frac{\nabla c}{c} + K \mu_0 (1-c)(\chi_{\text{lin}}(c) \mathbf{h} \cdot \nabla) \mathbf{h}, \quad (2.110d)$$
$$- \text{div}((1 + \chi_{\text{lin}}(c)) \nabla R) = 0, \quad (2.110e)$$

on $(0, T) \times \Omega$ and

$$- \Delta R = 0, \quad (2.110f)$$

on $(0, T) \times (\Omega' \setminus \overline{\Omega})$, where

$$\mathbf{h} := \nabla R.$$
2.4 Formal aspects

It is supplemented with boundary conditions
\[ u = 0 \quad \text{on } [0, T] \times \partial \Omega, \]  
\[ c v_{\text{part}} \cdot \nu = 0 \quad \text{on } (0, T] \times \partial \Omega, \]  
\[ \nabla R \cdot \nu = h_a \cdot \nu \quad \text{on } [0, T] \times \partial \Omega', \]
the transmission condition
\[ [\nabla R + \mathbb{I}_\Omega \chi_{\text{lin}}(c) h] \cdot \nu = 0 \quad \text{on } [0, T] \times \partial \Omega, \]
and initial conditions
\[ u(0, \cdot) = u^{\text{init}}, \]
\[ c(0, \cdot) = c^{\text{init}}, \]
Here, \( \mathbb{I}_\Omega \chi_{\text{lin}}(c) \) is analogously defined as in (2.11) as extension of \( \chi_{\text{lin}}(c) \) by zero onto \( \Omega' \). The boundary and initial conditions are identical to those of "model W" but have been repeated for easier readability.

In case of \( \Omega' = \Omega \) the boundary conditions (2.101c), (2.101d) are replaced on basis of (2.24), which yields
\[ (1 + \chi(c)) \nabla R \cdot \nu|_{\partial \Omega} = h_a \cdot \nu|_{\partial \Omega}. \]

2.4 Formal aspects

2.4.1 Formal weak formulations

In order to motivate the variational formulations used in the analysis part of this thesis, a weak formulation of 'model GW' will be discussed here. For simplicity, consider a fixed time \( t \in [0, T] \) only as well as \( \alpha_1 = 1, \beta = 0, d = 3 \) and \( \Omega \subset \subset \Omega' \).

Assume all unknowns to be sufficiently smooth.

Multiply (2.89a) by test functions \( v \in H^1_0(\Omega)^d \) and integrate over \( \Omega \). Integration by parts and (2.90a) yield
\[ \rho_0 \int_{\Omega} u_t \cdot v \, dx + \rho_0 \int_{\Omega} (u \cdot \nabla) u \cdot v \, dx - \int_{\Omega} p \, \text{div} v \, dx + 2\eta \int_{\Omega} Du : Dv \, dx \]
\[ = \mu_0 \int_{\Omega} (m \cdot \nabla) h \cdot v \, dx + \frac{\mu_0}{2} \int_{\Omega} (m \cdot h) \cdot \text{curl} v \, dx \]
In a standard way – recalling the integration by parts formula (A.29) for the curl-operator. Based on (38), the nonlinear convective part will be rewritten – allowing for linearization in the discrete scheme (with respect to the flow velocity) – via integration by parts and (2.89b) (\( \text{div} u = 0 \)),
\[ \rho_0 \int_{\Omega} (u \cdot \nabla) u \cdot v \, dx = \frac{\rho_0}{2} \int_{\Omega} (u \cdot \nabla) u \cdot v \, dx - \frac{\rho_0}{2} \int_{\Omega} (u \cdot \nabla) v \cdot u \, dx. \]
The magnetic body force will be rewritten by means of symmetry of \( \nabla h = \nabla \nabla R \) (cf. (2.16)),
\[ \mu_0 \int_{\Omega} (m \cdot \nabla) h \cdot v \, dx = \mu_0 \int_{\Omega} (v \cdot \nabla) h \cdot m \, dx. \]
In case of \( \beta \neq 0 \) the same computation can be done because \( h_a = \nabla R_a \) is a gradient field as well (cf. (2.19)).
In the numerics section, further stabilization terms need to be added to the right-hand side of the momentum equation. These can be included in the continuous setting as well by reformulating the pressure. Set
\[ \hat{p} := p + \alpha_3 \frac{\mu_0}{2} |\mathbf{m}|^2 + Dc. \]

By writing \( \alpha_3 \frac{\mu_0}{2} \nabla(|\mathbf{m}|^2) = \alpha_3 \mu_0 (\nabla \mathbf{m})^T \mathbf{m} \) and \( D \nabla c = Dc \nabla g'(c) \) (cf. \(2.56\)), the following weak formulation is obtained for the momentum equation \(2.89a\),
\[
\rho_0 \int_\Omega \mathbf{u}_t \cdot \mathbf{v} \, dx + \frac{\rho_0}{2} \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} \, dx - \frac{\rho_0}{2} \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{u} \, dx
- \int_\Omega \hat{p} \text{div} \mathbf{v} \, dx + 2\eta \int_\Omega \text{D} \mathbf{u} : \text{D} \mathbf{v} \, dx
= -D \int_\Omega c \nabla g'(c) \cdot \mathbf{v} \, dx + \mu_0 \int_\Omega (\mathbf{v} \cdot \nabla)(\mathbf{h} - \alpha_3 \mathbf{m}) \cdot \mathbf{m} \, dx
+ \frac{\mu_0}{2} \int_\Omega (\mathbf{m} \times \mathbf{h}) \cdot \text{curl} \mathbf{v} \, dx.
\]

The numerical scheme for 'model GW', see \(4.37\), is based on this formulation.

In the analysis section of this thesis (Section 3), the test functions will be chosen to be solenoidal such that the third term on the left-hand side of \(2.113\) will disappear and \(2.89b\) will be redundant. Moreover, the additional stabilization terms on the right-hand side of \(2.115\) are not needed. A weak form of the magnetic Kelvin force which gets along with less regular magnetic variables is given via integration by parts by
\[
\int_\Omega (\mathbf{m} \cdot \nabla) \mathbf{h} \cdot \mathbf{v} \, dx = -\int_\Omega (\mathbf{m} \cdot \nabla) \mathbf{v} \cdot \mathbf{h} \, dx
- \int_\Omega \text{div} \mathbf{m} \, \mathbf{v} \cdot \mathbf{h} \, dx.
\]

Hence,
\[
\rho_0 \int_\Omega \mathbf{u}_t \cdot \mathbf{v} \, dx + \frac{\rho_0}{2} \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} \, dx
- \frac{\rho_0}{2} \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{u} \, dx + 2\eta \int_\Omega \text{D} \mathbf{u} : \text{D} \mathbf{v} \, dx
= -\int_\Omega (\mathbf{m} \cdot \nabla) \mathbf{v} \cdot \mathbf{h} \, dx
- \int_\Omega \text{div} \mathbf{m} \, \mathbf{v} \cdot \mathbf{h} \, dx
+ \frac{\mu_0}{2} \int_\Omega (\mathbf{m} \times \mathbf{h}) \cdot \text{curl} \mathbf{v} \, dx
\]

for all \(\mathbf{v} \in H_0^1(\Omega)^d \cap H(\text{div}0)(\Omega)\) is another weak formulation which is used for the existence theory in Section 3, see e.g. \(3.140a\), \(3.167\).

The weak formulation of \(2.89c\) is given by
\[
0 = \int_\Omega c \psi \, dx
- \int_\Omega (c \mathbf{u} + c \mathbf{v}_{\text{part}}) \cdot \nabla \psi \, dx
(2.89d)
\int_\Omega c \psi \, dx
- \int_\Omega c \mathbf{u} \cdot \nabla \psi \, dx
- \int_\Omega \frac{Kf_\sigma(c)}{c} (-D \nabla c + \mu_0(\nabla(\mathbf{h} - \alpha_3 \mathbf{m}))^T \mathbf{m}) \cdot \nabla \psi \, dx
(2.118)
\]

for \(\psi \in H^1(\Omega)\), having used \(2.90a\), \(2.90b\) and integration by parts.

The weak formulation of the magnetostatic problem \(2.89f\), \(2.89g\) is obtained easily by exploiting the transmission conditions. Pick \(S \in H_{\text{mean}}^1(\Omega')\) and multiply \(2.89f\) by \(S|_\Omega\), \(2.89g\) by \(S|_{\Omega' \setminus \overline{\Omega}}\), integrate with respect to the spatial variable and sum up both equations. This way one obtains
\[
- \int_{\Omega' \setminus \partial \Omega} \Delta RS \, dx = \int_\Omega \text{div} \mathbf{m} S \, dx.
\]

In above identity, integration by parts on \(\Omega\) and \(\Omega' \setminus \overline{\Omega}\) individually results in
\[
\int_{\Omega'} \nabla R \cdot \nabla S \, dx = \int_{\partial \Omega} \nabla R \cdot \nu S \, d\sigma + \int_{\partial(\Omega' \setminus \overline{\Omega})} \nabla R \cdot \nu S \, d\sigma
- \int_\Omega R \cdot \nabla S \, dx + \int_{\partial \Omega} \mathbf{m} \cdot \nabla S \, d\sigma + \int_{\partial \Omega} \mathbf{m} \cdot \nu S \, d\sigma
\]
The weak formulation of the magnetization equation (2.89c) is fairly straightforward. Recall the inequality and some regularization parameter which restricts the growth of the susceptibility. In case of "model GW" one can formally expect the regularity because the choice of test functions does not change and the two-phase flow related computations under understanding. The energy estimate of "model B" is evident from the estimate for "model W" which was implicitly proven via the above computation.

Due to (2.40) the boundary integral above vanishes. Therefore, the weak formulation of (2.89c) is given by

\[
\int_{\Omega} \nabla R \cdot \nabla S \, dx = \int_{\Omega} h_a \cdot \nabla S \, dx - \int_{\Omega} m \cdot \nabla S \, dx.
\] (2.119)

Notice that the derivation is much easier, see the derivation of (2.68), if one already knows that \((h + 1_{\Omega} m) \in H(\text{div})(\Omega')\) which was implicitly proven via the above computation.

The weak formulation of the magnetization equation (2.89c) is fairly straightforward. Recall the integration by parts formulas (A.29) for the curl-operator and (2.74) for the convective part. Then, it follows from (2.90a) that

\[
\begin{align*}
\int_{\Omega} \operatorname{div}(m \otimes (u + v_{\text{part}})) \cdot \theta \, dx & - \sigma \int_{\Omega} \nabla \operatorname{div} m \cdot \theta \, dx + \sigma \int_{\Omega} \operatorname{curl} \operatorname{curl} m \cdot \theta \, dx \\
& = - \int_{\Omega} ((u + v_{\text{part}}) \cdot \nabla) \theta \cdot m \, dx + \sigma \int_{\Omega} \operatorname{div} m \operatorname{div} \theta \, dx + \sigma \int_{\Omega} \operatorname{curl} m \cdot \operatorname{curl} \theta \, dx \\
& \quad + \int_{\partial \Omega} ((v_{\text{part}} \cdot \nu) (m \cdot \theta) - \sigma \operatorname{div} m (\theta \cdot \nu) - \sigma \operatorname{curl} m \times \nu \cdot \theta) \, d\sigma.
\end{align*}
\]

Due to (2.40) the boundary integral above vanishes. Therefore, the weak formulation of (2.89c) is given by

\[
\begin{align*}
\int_{\Omega} m_t \cdot \theta \, dx - \int_{\Omega} ((u + v_{\text{part}}) \cdot \nabla) \theta \cdot m \, dx \\
& - \sigma \int_{\Omega} \operatorname{div} m \operatorname{div} \theta \, dx + \sigma \int_{\Omega} \operatorname{curl} m \cdot \operatorname{curl} \theta \, dx \\
& = \frac{1}{2} \int_{\partial \Omega} (m \times \theta) \cdot \text{curl} u \, dx - \frac{1}{\tau_{\text{rel}}} \int_{\Omega} (m - \chi(c, h) h) \cdot \theta \, dx,
\end{align*}
\] (2.120)

where the formula \((a \times b) \cdot c = (b \times c) \cdot a\) was applied to the first term on the right-hand side.

### 2.4.2 Formal a priori estimates

Formally, energy estimates to "model GW" and "model W" will be presented here. For simplicity, choose \(f_p(c) = c^m, m \geq 1\). In case of "model GW", the scenario will be simplified further to ease understanding. The energy estimate of "model B" is evident from the estimate for "model W" because the choice of test functions does not change and the two-phase flow related computations can be found in [1] or extracted from the computations in Section 2.3 see (2.104). First, the a priori estimates will be introduced. The formal computations follow afterwards.

In case of "model GW" one can formally expect the regularity

\[
\begin{align*}
\|u\|_{L^\infty(I; L^2(\Omega)^d)} + \|u\|_{L^2(I; H^1(\Omega)^d)} + \|c\|_{L^\infty(I; L^1(\Omega))} + \|c^{\frac{2-m}{2}} v_{\text{part}}\|_{L^2(I \times \Omega)^d} \\
+ \|h\|_{H(\text{div, curl})(\Omega')} + \|h\|_{L^\infty(I; L^2(\Omega)^d)} + \|\chi(c, h) h\|_{L^2(I \times \Omega)^d} \\
+ \|m\|_{H(\text{div, curl})(\Omega)} + \|m\|_{L^\infty(I; L^2(\Omega)^d)} \leq C,
\end{align*}
\]

where \(C > 0\) depends on initial data, external data (related to \(h_a\)), final time \(T\) (due to Gronwall’s inequality) and some regularization parameter which restricts the growth of the susceptibility \(\chi(\cdot, \cdot)\) with respect to the variable \(c\).
In case of 'model W' or 'model B', Gronwall's inequality is not needed. Unfortunately, the \(H(\text{div, curl})\)-regularity is lost, see Section 2.4.3. One can formally expect the regularity

\[
\|u\|_{L^\infty(I; L^2(\Omega)^d)} + \| \nabla u \|_{L^2(I; H^1(\Omega)^d)} + \| c \|_{L^\infty(I; L^1(\Omega)^d)} + \| c \|_{L^2(I; \text{part})^d} \geq \| \chi(c, h) h \|_{L^\infty(I; L^2(\Omega)^d)} \leq C',
\]

where \(C' > 0\) only depends on initial data and external data.

**Model GW.** Assume for the ease of presentation

\[ \alpha_1 = 1, \beta = 0, d = 3 \text{ and } \Omega \subset \subset \Omega'. \]

The abbreviation \( \hat{b} = h - \alpha_3 \mu \) will be used, see (2.73).

Inserting \( v := u \) in (2.115) gives

\[
\frac{\mu_0}{2} \partial_t \int_\Omega |u|^2 \, dx + 2\eta \int_\Omega |D u|^2 \, dx = -D \int_\Omega c \nabla g'(c) \cdot u \, dx + \mu_0 \int_\Omega (u \cdot \nabla) \hat{b} \cdot m \, dx + \frac{\mu_0}{2} \int_\Omega (m \times h) \cdot \text{curl} u \, dx.
\]

Choosing \( \psi := D g'(c) = D \log(c) \) in (2.117) and application of the chain rule yields

\[
0 = \partial_t D \int_\Omega g(c) \, dx - D \int_\Omega c \nabla g'(c) \cdot u \, dx - D \int_\Omega c \nabla_{\text{part}} \cdot \nabla c \, dx
\]

\[
\text{testing } (2.89d) \quad \partial_t D \int_\Omega g(c) \, dx + \int_\Omega \frac{c^2}{K_p(c)} |\nabla_{\text{part}}|^2 \, dx - \mu_0 \int_\Omega (\nabla_{\text{part}} \cdot \nabla) \hat{b} \cdot m \, dx.
\]

Inserting \( S := \frac{\mu_0}{\tau_{\text{rel}}} \) into (2.119) yields

\[
\frac{\mu_0}{\tau_{\text{rel}}} \int_{\Omega'} |h|^2 \, dx = \frac{\mu_0}{\tau_{\text{rel}}} \int_{\Omega'} h_a \cdot h \, dx - \frac{\mu_0}{\tau_{\text{rel}}} \int_{\Omega} m \cdot h \, dx.
\]

Choosing \( S := \mu_0 R \) in the time derivative of (2.119) yields

\[
\frac{\mu_0}{2} \partial_t \int_{\Omega'} |h|^2 \, dx = \mu_0 \int_{\Omega'} \partial_t h_a \cdot h \, dx - \mu_0 \int_{\Omega} m_t \cdot h \, dx.
\]

Moreover, by using the strong form of the magnetostatic problem (cf. (2.89f) and (2.89g)) one gets

\[
- \text{div}(h|_{\Omega'}) = \text{div} m \quad \text{and} \quad \text{div}(h|_{\Omega' \setminus \Omega}) = 0.
\]

Inserting the test function \( \theta := -\mu_0 \hat{b} = -\mu_0 h + \alpha_3 \mu_0 m \) in (2.120) yields

\[
- \mu_0 \int_{\Omega} m_t \cdot h \, dx + \frac{\alpha_3 \mu_0}{2} \partial_t \int_{\Omega} |m|^2 \, dx + \mu_0 \int_{\Omega} ((u + \nabla_{\text{part}}) \cdot \nabla) \hat{b} \cdot m \, dx
\]

\[
- \sigma \mu_0 \int_{\Omega} \text{div} m \text{div} h \, dx + \sigma \mu_0 \alpha_3 \int_{\Omega} |\text{div} m|^2 \, dx
\]
Exploiting that \((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\) is an alternating trilinear form, \(\mathbf{m} \times \mathbf{m} = 0\) and the computation (2.125), one arrives – partially expanding the abbreviation \(\mathbf{b} – \) at

\[
\begin{align*}
- \sigma \mu_0 \int_\Omega \text{curl} \mathbf{m} \cdot \text{curl} \mathbf{h} \, dx + \sigma \mu_0 \alpha_3 \int_\Omega |\text{curl} \mathbf{m}|^2 \, dx \\
= - \frac{\mu_0}{2} \int_\Omega \text{curl} \mathbf{u} \times \mathbf{m} \cdot \mathbf{b} \, dx + \frac{\mu_0}{\tau_{\text{rel}}} \int_\Omega (\mathbf{m} - \chi(c, \mathbf{h}) \mathbf{h}) \cdot \mathbf{b} \, dx.
\end{align*}
\]

Adding up (2.121)-(2.122), (2.123)-(2.124) and (2.126) yields

\[
\begin{align*}
\rho_0 \partial_t \int_\Omega |\mathbf{u}|^2 \, dx + 2 \eta \int_\Omega |\nabla \mathbf{u}|^2 \, dx + \partial_t D \int_\Omega G(c) \, dx + \int_\Omega \frac{c^2}{K f_p(c)} |\mathbf{v}_{\text{part}}|^2 \, dx \\
+ \frac{\mu_0}{\tau_{\text{rel}}} \int_{\Omega'} |\mathbf{h}|^2 \, dx + \frac{\mu_0}{2} \partial_t \int_{\Omega'} |\mathbf{h}|^2 \, dx + \frac{\alpha_3 \mu_0}{2} \partial_t \int_{\Omega'} |\mathbf{m}|^2 \, dx \\
+ \sigma \mu_0 \int_{\Omega', \partial \Omega} |\text{div} \mathbf{h}|^2 \, dx + \sigma \mu_0 \alpha_3 \int_{\Omega'} |\text{div} \mathbf{m}|^2 \, dx + \sigma \mu_0 \alpha_3 \int_{\Omega'} |\text{curl} \mathbf{m}|^2 \, dx \\
+ \frac{\mu_0}{\tau_{\text{rel}}} \int_{\Omega'} \chi(c, \mathbf{h}) |\mathbf{h}|^2 \, dx + \frac{\mu_0 \alpha_3}{\tau_{\text{rel}}} \int_{\Omega'} |\mathbf{m}|^2 \, dx \\
= \frac{\mu_0 \alpha_3}{\tau_{\text{rel}}} \int_{\Omega'} \chi(c, \mathbf{h}) \mathbf{h} \cdot \mathbf{m} \, dx + \mu_0 \int_{\Omega'} \partial_t \mathbf{h} \cdot \mathbf{h} \, dx + \frac{\mu_0}{\tau_{\text{rel}}} \int_{\Omega'} \mathbf{h} \cdot \mathbf{h} \, dx.
\end{align*}
\]

Integrating this identity with respect to time, an a priori estimate can be achieved by using Young’s inequality, absorption and Gronwall’s lemma. For more details, the reader is referred to the detailed derivation in the framework of the Galerkin approximation – see the proof of Lemma 3.31.

**Model W.** For the a priori estimate of 'model W' the weak formulation of 'model GW' is reused but (2.120) is replaced with

\[
\mathbf{m} = \chi_{\text{lin}}(c) \mathbf{h}.
\]

Be aware that terms of the type \(\mathbf{m} \times \mathbf{h} = \chi_{\text{lin}}(c) \mathbf{h} \times \mathbf{h} \equiv 0\) will vanish now.

Choosing \(\mathbf{v} := \mathbf{u}\) in (2.113), one obtains by exploiting (2.42) the identity

\[
\frac{\rho_0}{2} \partial_t \int_\Omega |\mathbf{u}|^2 \, dx + 2 \eta \int_\Omega |\nabla \mathbf{u}|^2 \, dx = \mu_0 \int_\Omega (\chi(c) \mathbf{h} \cdot \nabla) \mathbf{h} \cdot \mathbf{u} \, dx,
\]

where the convective term vanishes due to \((\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{u} = \frac{1}{2} \mathbf{u} \cdot \nabla(|\mathbf{u}|^2)\) and integration by parts.
Recall \( \chi_{\text{lin}}(c) = \chi_0 c \), see (2.34). Choosing \( \psi = Dg'(c) - \frac{\mu_0 \chi_0}{2} |h|^2 \) in (2.117), application of the chain rule and symmetry of \( \nabla h = \nabla \nabla R \) (cf. (2.15)) yield

\[
0 = \partial_t D \int_{\Omega} g(c) \, dx - D \int_{\Omega} c u \cdot \nabla g'(c) \, dx - D \int_{\Omega} c v_{\text{part}} \cdot \nabla g'(c) \, dx = v_{\text{part}} \times c \partial_c u.
\]

Testing (2.100) by \( c^2 \partial_{\chi_{\text{lin}}}(c) \partial_t D \int_{\Omega} g(c) \, dx - \mu_0 \int_{\Omega} \partial_t \chi_{\text{lin}}(c) |h|^2 \, dx = \mu_0 \int_{\Omega} \chi_{\text{lin}}(c) v_{\text{part}} \cdot \nabla (|h|^2) \, dx + \mu_0 \int_{\Omega} \chi_{\text{lin}}(c) v_{\text{part}} |v_{\text{part}}|^2 \, dx \quad (2.130)

Choose \( S := \mu_0 R \) both in (2.119) and the time derivative thereof. Using (2.128), this yields the identities

\[
\mu_0 \int_{\Omega'} |h|^2 \, dx + \mu_0 \int_{\Omega} \chi_{\text{lin}}(c) |h|^2 \, dx = \mu_0 \int_{\Omega'} h_n \cdot h \, dx.\quad (2.131)
\]

and

\[
\frac{\mu_0}{2} \partial_t \int_{\Omega'} |h|^2 \, dx + \mu_0 \int_{\Omega} \partial_t \chi_{\text{lin}}(c) |h|^2 \, dx = \mu_0 \int_{\Omega'} \partial_t h_n \cdot h \, dx - \mu_0 \int_{\Omega} \chi_{\text{lin}}(c) h_t \cdot h \, dx = \frac{\mu_0}{2} \int_{\Omega} \chi_{\text{lin}}(c) \partial_t (|h|^2) \, dx \quad (2.132)
\]

by the product rule. Summing up (2.129)-(2.132) and applying the product rule to the terms \( \partial_t \chi_{\text{lin}}(c) |h|^2 \) and \( \chi_{\text{lin}}(c) \partial_t (|h|^2) \) yields

\[
\frac{\rho_0}{2} \partial_t \int_{\Omega} |u|^2 \, dx + 2\eta \int_{\Omega} |D u|^2 \, dx + \partial_t D \int_{\Omega} g(c) \, dx + \int_{\Omega} \frac{c}{K f_p(c)} |v_{\text{part}}|^2 \, dx + \frac{\mu_0}{2} \partial_t \int_{\Omega} \chi_{\text{lin}}(c) |h|^2 \, dx + \mu_0 \int_{\Omega'} \chi_{\text{lin}}(c) |h|^2 \, dx + \mu_0 \int_{\Omega'} |h|^2 \, dx
\]

\[
= \mu_0 \int_{\Omega'} h_n \cdot h \, dx + \mu_0 \int_{\Omega'} \partial_t h_n \cdot h \, dx.
\]

After application of Young's inequality and absorption, the a priori estimate is evident.

### 2.4.3 Lack of regularity in "model W" and "model B"

Looking at (2.14b) on the fluid domain \( \Omega \), i.e.

\[
\text{div } h = - \text{div } m \quad \text{in } [0, T] \times \Omega,
\]

it is evident that the distributional divergence of \( h \) can only be \( L^2 \)-regular if \( \text{div } m \) is \( L^2 \)-regular as well. In case of "model GW" the \( H(\text{div}) \)-regularity of \( h \) is directly linked to the \( H(\text{div}) \)-regularity of \( m \). The latter is assured by \( \sigma > 0 \) in (2.120). Once this equation is replaced with the simple algebraic expression \( m = \chi_{\text{lin}}(c) h \), the \( H(\text{div}) \)-regularity of \( m \) is not obvious anymore. The question arises if the \( H(\text{div}) \)-regularity of \( h \) can be established another way.
\[ -\text{div}((1 + \chi_0 c)\nabla) = 0, \]
\[ (1 + \chi_0 c)\nabla \cdot \nu|_{\partial \Omega} \overset{(\text{cf. } (2.103))}{=} 0. \]

Putting the magnetization \( m = \chi_0 c\nabla \) to the right-hand side instead is not frugal as the regularity of the product \( c\nabla \) is unknown. Hence, one has to deal with non-constant coefficients on the left-hand side. However, not much is known about the regularity of \( c \) either. In case of "model B" one might expect \( L^\infty \)-estimates of \( c \) due to the mobility \( (1 - c) \) in front of the magnetic Kelvin force, see (2.110d).

Several problems arise.

- The boundary condition \( (1 + \chi_0 c)\nabla \cdot \nu|_{\partial \Omega} = 0 \) is complicated and of unknown regularity.
- To the best of our knowledge, regularity results are only available for discontinuous bounded coefficients if they satisfy some additional condition, i.e. if they have vanishing mean oscillation. More details can be found in the introduction of [15,22] and the references therein.
- Non-negativity of \( c \) is not a priori given in rigorous computations.

For these reasons, this thesis does not include existence theory for "model W" or "model B", respectively.

The more involved model, i.e. "model GW", is more difficult to handle numerically and consumes more computational resources. However, it admits existence of weak solutions to some extent, see Section 3. Existence theory for models similar to "model W" and "model B" is still an open problem in mathematical literature.
3 Existence of weak solutions to "model GW"

This section is concerned with existence of global weak solutions of "model GW", see (2.89), (2.90), (2.91), i.e. weak solutions on the time interval $I := [0, T]$ for any $T > 0$. The results are based on our work [40] enriched with details and slightly modified. As usual, a Galerkin-approximation is used to obtain solutions of a system of ordinary differential equations by means of Picard-Lindelöf’s theorem. In this first step, additional regularizations will be used in order to be able to pass to the limit, which yields solutions to a regularized model. Those additional regularizations are concerned with the particle density $c$ only and compensate the low regularity of the magnetic variables the particle density is coupled with.

It turns out that the choice of finite dimensional ansatz spaces is quite involved. Therefore Section 3.1 is devoted to construct suitable basis functions which are used to span the finite dimensional spaces. The results of this step are true in both two and three space dimensions. In case of two spatial dimensions, $d = 2$, and $f_p(c) = c^2$, an additional approximation process shows that the regularizations from the first step can be dropped. See Section A.4 or [18] for the treatment of the curl-operator and the cross product $\times$ in two dimensions.

For the first approximation process "discrete to regularized continuous", additional assumptions on the spatial domains and on the external data have to be made.

(H1) Let $\Omega, \Omega'$ be bounded domains which satisfy (2.2), i.e.

$$\Omega \subset\subset \Omega' \text{ or } \Omega = \Omega', \text{ where } \Omega' \text{ is simply connected.}$$

Let $\Omega'$ be of class $C^{1,1}$.

and let $\Omega$ be of class $C^{3,1}$.

(H2) Assume

$$h_0 \in H^1([0, T]; H^3(\Omega)^d \cap H^1(\Omega') \cap H(\text{div}_0, \text{curl}_0)(\Omega')).$$

(H3) Further, it is assumed that $\tilde{\chi}$ is Lipschitz-continuous and has at most growth like a square root function,

$$|\tilde{\chi}(c)| \leq \tilde{K}_1 + \tilde{K}_2 \sqrt{|c|}$$

for some $\tilde{K}_1, \tilde{K}_2 > 0$, which effectively regularizes the susceptibility $\chi(\cdot, \cdot)$ (cf. (2.32)) with respect to the particle density $c$.

The latter implies that $\chi$ satisfies

$$|\chi(c, h)| \leq K_1 + K_2 \sqrt{|c|} \quad \text{and} \quad |\chi(c, h)| \leq K_1 + K_2 \sqrt{|c|},$$

for all scalars $c$, vectors $h$ and some fixed $K_1, K_2 > 0$. This can be deduced from the boundedness of $L$ (cf. (2.31)) and $L(\cdot, \cdot)$, respectively, see also Remark 2.3. The regularization of the susceptibility $\chi$ facilitates a priori estimates. From a physical perspective the magnetic particle density $c$ is expected to be bounded and consequently (H3 3.3) seems acceptable. However, this thesis does not include an $L^\infty$-estimate.

One might want to assert above assumption by introducing a regularization parameter $r > 0$ such that $\tilde{\chi}$ is linear up to $r$ and from there one continues like a square root function. Such an approach was used in the numerics section (see (N2 4.2)) and guarantees that the regularization does not impact the simulation results as long as the particle density remains sufficiently small.
3.1 Construction of discrete spaces

For this section the assumptions in (H1 3.1) are of great importance. Hence, we kindly remind the reader of these.

This section is devoted to the construction of approximation spaces for the Faedo-Galerkin approach of Section 3.2. The construction proceeds as follows. Pick a suitable (e.g. sufficiently regular) function space and construct a basis. The linear hull of finitely many basis functions will yield the discrete approximation spaces.

Parabolic equations often are treated by using eigenfunctions of their elliptic operators as foundation for the discrete approximation spaces. Properties like regularity and convergence of orthogonal projections which are needed for technical reasons will then be proven as needed. The advantage of eigenfunctions is their comprehensive structure, i.e. orthogonality properties. However, (2.89e) and (2.89f) are coupled while being of quite different structure – and in the case \( \Omega \subset \subset \Omega' \) even defined on different domains. As already seen in Section 2.3, the magnetization equation’s test functions should admit gradient fields, i.e. the magnetic field \( \mathbf{h} = \nabla R \) (cf. (2.16)). This strongly couples the basis functions used to approximate the magnetization \( \mathbf{m} \) with those being used to approximate the magnetic potential \( R \).

Moreover, the \((-\nabla \operatorname{div} + \text{curl curl})\)-operator in (2.89e) only accounts for \( H(\operatorname{div}, \text{curl})(\Omega) \)-regularity. It is known [9, Proposition 2.7] that \( H(\operatorname{div}, \text{curl})(\Omega) \) does not embed continuously into \( H^1(\Omega)^d \) which makes it either impossible or at least very difficult to draw conclusions about the convergence of approximations based on eigenfunctions in the \( H^1(\Omega)^d \)-norm. The convective term in (2.89e) (second term on the left-hand side) includes a gradient on the test function in the weak formulation of this equation, though, see (2.120). In (3.152), it will become evident why this is troublesome, as it turns out that the basis functions need to be able to approximate test functions in very strong norms. For those reasons a different ansatz will be used for the magnetic quantities.

First, one needs to know which kind of properties are desirable. Let \((\mathcal{M}_n)_{n \in \mathbb{N}} \) and \((\mathcal{R}_n)_{n \in \mathbb{N}} \) be families of discrete spaces for \( \mathbf{m} \) and \( R \), then

\[
\nabla[\mathcal{R}_n]|_{\Omega} \subset \mathcal{M}_n \quad \forall n \in \mathbb{N}
\]

(3.5) is supposed to hold for stability reasons. Moreover, time-compactness of discrete solutions usually is based on \( L^2 \)-orthogonality of basis functions and stable orthogonal projection operators onto the discrete spaces. The utility of those properties is as follows. For time-compactness of an approximation \( \mathbf{m}_n \in \mathcal{M}_n \), terms like

\[
\int_{\Omega} \partial_t \mathbf{m}_n \cdot \Phi \, dx
\]

need to be estimated while \( \Phi \notin \mathcal{M}_n \) is not included in a discrete space, hence it is not a valid test function. Therefore, by means of \( L^2 \)-orthogonality an identity like

\[
\int_{\Omega} \partial_t \mathbf{m}_n \cdot \Phi \, dx = \int_{\Omega} \partial_t \mathbf{m}_n \cdot \Pi_n \Phi \, dx,
\]

where \( \Pi_n \) is a \( L^2 \)-orthogonal projection operator onto \( \mathcal{M}_n \) is needed. Later on, this operator in front of the test function must be got rid of again, for which a stability estimate of the type

\[
\| \Pi_n \Phi \| \lesssim \| \Phi \|
\]

for all \( n \in \mathbb{N} \) and all test functions \( \Phi \) will be used.
3.1 Construction of discrete spaces

Stability can easily be proven by exploiting orthogonality (if available). As this stability most probably is needed in a stronger norm as the $L^2$-norm, another orthogonality is desired besides of $L^2$-orthogonality. The specific spaces involved depend on the problem, of course. For reference, it is desirable to have

$$L^2$$-orthogonality and orthogonality with respect to some other scalar product. \((3.6)\)

The regularity which is needed due to technical reasons is not evident yet at this stage of this thesis. However, it turns out to be beneficial to have $H^3$-regular basis functions for the magnetization. More insight on how this is helpful can be seen in \((3.152)\) which affects the desired regularity of the magnetization’s basis functions most severely. Implicitly, due to the coupling of \((2.89e)\) and \((2.89a)\) the analogous regularity is proposed for the basis functions of the flow velocity field $\mathbf{u}$.

Based on this observation, consider the following Hilbert spaces.

Let

$$M := \left\{ \Psi \in H^3(\Omega)^d \ | \ \text{curl} \Psi \times \nu|_{\partial \Omega} = 0, \ \text{div} \Psi|_{\partial \Omega} = 0 \right\} \subset H^3(\Omega)^d \quad (3.7)$$

be the function space related to the magnetization $\mathbf{m}$. Notice that the boundary conditions in this space are non-conforming to those of "model GW". However, they allow for energy-estimates and in spirit of Remark 2.4 yield a reasonable non-conforming approximation. The space of magnetic potentials $\mathcal{R}$ is given by

$$\mathcal{R} := \left\{ S \in H_{\text{mean}}^1(\Omega') \ | \ \nabla S|_{\Omega} \in H(\text{div})(\Omega), \ \nabla S|_{\Omega' \setminus \overline{\Omega}} \in H(\text{div}_0)(\Omega' \setminus \overline{\Omega}) \right\}, \quad (3.8)$$

taking into consideration that the corresponding magnetic field is $H(\text{div})$-regular on $\Omega$ (as indicated by dissipative terms in Section 2.3, last term in \((2.83)\)) and has vanishing divergence on $\Omega' \setminus \overline{\Omega}$ (cf. \((2.89g)\)).

**Remark 3.1.** The smooth and compactly supported functions of $C_0^\infty(\Omega)^d$ clearly are included in $M$. Hence,

$$M \subset L^2(\Omega)^d \text{ densely.} \quad (3.9)$$

Moreover, in context of the $H^3$-norm the curl-operator and the div-operator used in the definition of $M$ are continuous operators which is why $M$ is the preimage of closed sets under continuous mappings ($M$ is a zero-set). Therefore,

$$M \text{ is a closed subspace of } H^3(\Omega)^d. \quad (3.10)$$

With respect to the scalar product

$$\langle A, B \rangle_\mathcal{R} := \langle \nabla A, \nabla B \rangle_{L^2(\Omega)^d} + \langle \Delta A, \Delta B \rangle_{L^2(\Omega)} \quad (3.11)$$

which induces the norm

$$\| \cdot \|_\mathcal{R} := \| \nabla (\cdot) \|_{L^2(\Omega)^d} + \| \Delta (\cdot) \|_{L^2(\Omega)} \quad (3.12)$$

the space

$$\mathcal{R} \text{ is a Hilbert-space.} \quad (3.13)$$

The expression in \((3.12)\) indeed is a norm due to vanishing mean values and Poincaré’s inequality. Due to completeness of $H^1_{\text{mean}}(\Omega')$ a Cauchy-sequence $(R_k)_{k \in \mathbb{N}}$ in $\mathcal{R}$ has a limit $R$ in $H^1_{\text{mean}}(\Omega')$. Similarly, the sequences $(\nabla R_k|_{\Omega})_{k \in \mathbb{N}}$ and $(\nabla R_k|_{\Omega' \setminus \overline{\Omega}})_{k \in \mathbb{N}}$ have limits $\mathbf{x}$ and $\tilde{\mathbf{x}}$ in $H(\text{div})(\Omega)$ or $L^2(\Omega' \setminus \overline{\Omega})^d$, respectively. By means of integration by parts it is straightforward to verify $\mathbf{x} = \nabla R|_{\Omega}$ and $\tilde{\mathbf{x}} = \nabla R|_{\Omega' \setminus \overline{\Omega}}$ with vanishing divergence.
Consistent with \(2.54\) and \(2.42\), let the space related to the velocity field be given by
\[
\mathcal{U} := H^3(\Omega)^d \cap H(\text{div} \nu)(\Omega) \cap H_0^1(\Omega)^d. \tag{3.14}
\]
Here, \(2.42\) is already embedded into the space \(\mathcal{U}\). Due to the nature of the Faedo-Galerkin argument, this implicitly dictates the use of solenoidal test functions in the weak formulation of \(2.43\) which is a standard way of dealing with the Navier-Stokes equations. The connection between such a problem and the original saddle-point formulation is well-known, see e.g. [32, section IV.1]. For the particle density \(c\), choose
\[
\mathcal{C} := H^2(\Omega) := \left\{ \psi \in H^2(\Omega) \mid \nabla \psi \cdot \nu|_{\partial \Omega} = 0 \right\}. \tag{3.15}
\]

**Remark 3.2.** Evidently \(\mathcal{U}\) and \(\mathcal{C}\) are zero-sets of continuous operators, hence they are closed subsets of \(H^3(\Omega)^d\) or \(H^2(\Omega)\), respectively.

Before proceeding, the following auxiliary result will be stated as it helps to generate bases with some structure. It will be used particularly for the construction of a basis of the space \(\mathcal{M}\).

**Lemma 3.3.** Let \(X, Y\) be separable Hilbert spaces and assume \(X\) to be densely and compactly embedded into \(Y\). Then there exists a basis \(\{b_i\}_{i \in N}, N \subset \mathbb{N}\), such that
\[
Y = \text{span}\{b_i\}_{i \in N}^Y, \quad \langle b_i, b_j \rangle_Y = \delta_{ij} \quad \forall i, j \in N,
\]
\[
X = \text{span}\{b_i\}_{i \in N}^X, \quad \langle b_i, b_j \rangle_X = 0 \quad \forall i, j \in N \text{ with } i \neq j.
\]

**Proof:** Define \(T : Y \to Y\) by \(Tf := x\), where \(x \in X\) is the solution of
\[
\langle x, \phi \rangle_X = \langle f, \phi \rangle_Y \quad \forall \phi \in X. \tag{3.16}
\]

First, well-posedness will be addressed. On the left-hand side clearly is an \(X\)-elliptic bilinear form and \(f\) induces an element of \(Y' \subset X'\). By an application of Lax-Milgram’s theorem, the operator \(T\) is well-defined. Moreover, \(T\) is linear and its image is in \(X \hookrightarrow Y\), hence \(T\) is compact. Moreover, it is self-adjoint due to symmetry of the scalar products.

By the spectral theorem one obtains an orthonormal set \(\{b_i\}_{i \in N}, N \subset \mathbb{N}\), of eigenvectors to eigenvalues \(\{\lambda_i\}_{i \in N} \subset \mathbb{R} \setminus \{0\}\) such that
\[
Y = \text{span}\{b_i\}_{i \in N}^Y \oplus N(T),
\]
\[
\langle b_i, b_j \rangle_Y = \delta_{ij},
\]
and \(Tb_i = \lambda_i b_i\) for all \(i \in N\). In \(N(T)\) are such \(f \in Y\) such that
\[
0 = \sum_{a=0}^{3.16} \langle Tf, \phi \rangle_X = \langle f, \phi \rangle_Y \quad \forall \phi \in X.
\]

Easily one deduces \(N(T) \subset \langle X^Y \rangle^\perp = Y^\perp = \{0\}\), where the orthogonal complement is taken in \(Y\). Similarly, if there was an element \(x \in X \setminus \text{span}\{b_i\}_{i \in N}^X\), then without loss of generality \(x \in \langle \text{span}\{b_i\}_{i \in N}^X \rangle^\perp\) (where the orthogonal complement is taken in \(X\)) which implies the identities
\[
0 = \langle \lambda_i b_i, x \rangle_X = \langle Tb_i, x \rangle_X = \langle b_i, x \rangle_Y \quad \forall i \in \mathbb{N} \text{ and therefore } x = 0.
\]
Hence,
\[
X = \text{span}\{b_i\}_{i \in N}^X, \quad \lambda_i \langle b_i, b_j \rangle_X = \delta_{ij}.
\]

From this the result follows.
3.1 Construction of discrete spaces

3.1.1 Construction of a basis of \( \mathcal{M} \)

The goal of this subsection is to construct a basis of \( \mathcal{M} \) which is orthonormal with respect to the \( L^2 \)-scalar product. In particular, substructures need to be found which can be exploited to be the starting point to construct a basis of the set \( \mathcal{R} \), which is supposed to facilitate (3.5).

As \( \nabla[\mathcal{R}|_{\Omega}] \) consists of gradient fields, let

\[
\mathcal{H} := \nabla[H^1(\Omega)] \cap \mathcal{M} = \nabla[H^1(\Omega) \cap H^4(\Omega)] \cap \{ \Psi \in H^3(\Omega)^d | \text{div } \Psi|_{\partial\Omega} = 0 \} \tag{3.17}
\]

Indeed, above representation in the second line is correct and can easily be verified comparing (3.7) and (3.17), i.e. \( H^3 \)-gradient fields have a \( H^4 \)-potential and \( \text{curl } \nabla = 0 \) (in the case \( d = 2 \), analogously \( \text{curl } \nabla \) vanishes, cf. (A.25)) which implies \( \text{curl}(\nabla(\cdot)) \times \nu|_{\partial\Omega} = 0 \).

For technical reasons, see for instance the proof of the identity in (3.110) or the proof of Lemma 3.43, another subspace of gradient fields which contains potentials with constant trace will be introduced,

\[
S := \nabla[H^1_0(\Omega)] \cap \mathcal{H} = \{ \eta \in \mathcal{H} | \eta = \nabla S \text{ for a } S \in H^1_0(\Omega) \}. \tag{3.18}
\]

Note that \( S \) is not trivial as all gradients of homogeneous Dirichlet-Laplace eigenfunctions (cf. e.g. (3.43)) are elements of this space.

Lemma 3.4. i) \( \mathcal{H} \) is a closed subspace of \( \mathcal{M} \) with respect to the \( H^3 \)-norm.

ii) \( S \) is a closed subspace of \( \mathcal{H} \) with respect to the \( H^3 \)-norm.

Proof: Ad i):
Closedness of \( \mathcal{A}_1 \subset H^4(\Omega)^d \) is obvious as it is a zero set of a continuous operator. The range of the gradient operator on the domain \( H^1_{\text{mean}}(\Omega) \cap H^4(\Omega) \) is closed as well, as can be seen by standard arguments.

Ad ii):
By the closedness of the gradient operator on \( H^1_0(\Omega) \) the result immediately follows.

Now, starting with the most specialized space, \( S \subset \mathcal{H} \subset \mathcal{M} \), suitable complements need to be found to fill up the remainder that is not covered by \( S \) or \( \mathcal{H} \), respectively. In order to guarantee \( L^2 \)-orthogonality, orthogonal complements with respect to the \( L^2 \)-scalar product will be defined,

\[
S^\parallel := (S^{L^2(\Omega)^d})^\perp \cap \mathcal{H}, \tag{3.19}
\]

\[
\mathcal{V} := (\mathcal{H}^{L^2(\Omega)^d})^\perp \cap \mathcal{M}. \tag{3.20}
\]

Concerning the spaces \( \mathcal{M} \) (cf. (3.7)), \( \mathcal{H} \) (cf. (3.17)), \( \mathcal{V} \) (cf. (3.20)), \( S \) (cf. (3.18)) and \( S^\parallel \) (cf. (3.19)), the subsequent lemma holds true.

Lemma 3.5. The following sums are direct.

i) \( \mathcal{H} = S \oplus S^\parallel \),

ii) \( \mathcal{M} = \mathcal{H} \oplus \mathcal{V} = S \oplus S^\parallel \oplus \mathcal{V} \).

Moreover, the above identities hold algebraically and topologically, i.e. there exists a constant \( C > 0 \) such that for any \( f \in \mathcal{M} \) with \( f = s + s^\parallel + v \), where \( s \in S \), \( s^\parallel \in S^\parallel \) and \( v \in \mathcal{V} \), it follows that

\[
\|f\|_{H^3(\Omega)^d} \leq \|s\|_{H^3(\Omega)^d} + \|s^\parallel\|_{H^3(\Omega)^d} + \|v\|_{H^3(\Omega)^d} \leq C\|f\|_{H^3(\Omega)^d}. \tag{3.21}
\]
The Helmholtz decomposition holds true for higher regular spaces, see \((A.16)\). Use the decomposition of \(H^3(\Omega)^d\) and intersect both sides with \(\mathcal{H} = A_0 \cap A_1\) (cf. \((3.17)\)).

\[
\mathcal{H} = \mathcal{H} \cap H^3(\Omega)^d = \left( \nabla [H^1_0(\Omega)] \cap H^3(\Omega)^d \oplus L^2(\Omega)^d \right) H(\text{div})_0(\Omega) \cap H^3(\Omega)^d \cap \mathcal{H} \\
\subset A_0 \quad (\text{notice } \nabla [H^1(\Omega)] = \nabla [H^1_{\text{mean}}(\Omega)])
\]

\[
= \left( \nabla [H^1_0(\Omega)] \cap A_0 \oplus L^2(\Omega)^d \right) H(\text{div})_0(\Omega) \cap A_1 \cap A_1
\]

\[
= \left( \nabla [H^1_0(\Omega)] \cap \mathcal{H} \oplus L^2(\Omega)^d \right) H(\text{div})_0(\Omega) \cap \mathcal{H},
\]

\((3.22)\)

see Lemma A.9. The first summand on the right-hand side of \((3.22)\) is \(\mathcal{S}\) which follows from its definition \((3.18)\). Elements of the second summand, \(f \in H(\text{div})_0(\Omega) \cap \mathcal{H}\), satisfy \(f \in \mathcal{S}\) and due to orthogonality of the sum in \((3.22)\) they satisfy \((f, \Phi)_{L^2(\Omega)^d} = 0\) for all \(\Phi \in \mathcal{S}\) (and – by density – all \(\Phi \in \mathcal{S}^{L^2(\Omega)^d}\), i.e. \(f \in \mathcal{S}^o\). Since \(\mathcal{S}\) and \(\mathcal{S}^o\) are subsets of \(\mathcal{H}\),

\[
\mathcal{H} \supset \mathcal{S} \oplus L^2(\Omega)^d \mathcal{S}^o \supset \mathcal{S} \oplus L^2(\Omega)^d H(\text{div})_0(\Omega) \cap \mathcal{H} \supset \mathcal{S}^o \cap \mathcal{H}
\]

Therefore, \(\mathcal{H} = \mathcal{S} \oplus L^2(\Omega)^d \mathcal{S}^o\) which proves \([i]\). Additionally, it follows that \(\mathcal{H} = \mathcal{S} \oplus H(\text{div})_0(\Omega) \cap \mathcal{H}\). Hence, an element \(f \in \mathcal{H}\) can be decomposed in two ways. However, due to \(H(\text{div})_0(\Omega) \cap \mathcal{H} \subset \mathcal{S}^o\) both ways are a valid decomposition in view of \(\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^o\) and consequently

\[
\mathcal{S}^o = H(\text{div})_0(\Omega) \cap \mathcal{H},
\]

\((3.23)\)
as the sum is direct (uniqueness of decomposition of \(f\)).

Ad \([ii]\):
The second part of the claimed identity is just an application of \([i]\). The proof of the first part is analogous to the proof of \([i]\). By use of the Helmholtz decomposition from \((A.15)\) one obtains

\[
H^3(\Omega)^d = \nabla [H^4(\Omega)] \oplus L^2(\Omega)^d H_{n0}(\text{div})_0(\Omega) \cap H^3(\Omega)^d
\]

and by intersecting with

\[
\mathcal{M} = \{ \Psi \in H^3(\Omega)^d | \text{curl } \Psi \times \nu|_{\partial \Omega} = 0 \} \cap \{ \Psi \in H^3(\Omega)^d | \text{div } \Psi|_{\partial \Omega} = 0 \} =: B_0 \cap B_1
\]

one gets

\[
\mathcal{M} = \mathcal{M} \cap H^3(\Omega)^d = \left( \nabla [H^4(\Omega)] \oplus L^2(\Omega)^d \right) H_{n0}(\text{div})_0(\Omega) \cap H^3(\Omega)^d \cap \mathcal{M} \cap \mathcal{B}_0 \cap \mathcal{B}_1
\]

\[
= \left( \nabla [H^1_0(\Omega)] \cap B_0 \oplus L^2(\Omega)^d \right) H_{n0}(\text{div})_0(\Omega) \cap B_0 \cap B_1 \cap \mathcal{M} \cap \mathcal{M}
\]

\((3.24)\)

The first summand clearly is \(\mathcal{H}\), see its definition \((3.17)\). Clearly, as in "Ad \([i]\)"*, the second summand on the right-hand side of \((3.24)\) satisfies \(H_{n0}(\text{div})_0(\Omega) \cap \mathcal{M} \subset \mathcal{H}^{L^2(\Omega)^d} \cap \mathcal{M} = \mathcal{V}\), which implies

\[
\mathcal{M} \supset \mathcal{H} \oplus L^2(\Omega)^d \mathcal{V} \supset \mathcal{H} \oplus L^2(\Omega)^d H_{n0}(\text{div})_0(\Omega) \cap \mathcal{M} \supset \mathcal{M}
\]

\((3.24)\)
The claim follows. Moreover, with the same argument as in the proof of i),

\[ \mathcal{V} = \mathcal{H}_1(\text{div}_0)(\Omega) \cap \mathcal{M} \quad (3.25) \]

is evident.

The Helmholtz decompositions of $H^3(\Omega)^d$ which have been used so far rely on regularity theory of the Poisson problem. The latter not only guarantees the weak differentiability of the solution – the gradient of which is the respective element in the space of gradient fields $\nabla[H^1_0(\Omega)] \cap H^3(\Omega)^d$ or $\nabla[H^4(\Omega)]$, respectively – but a stability estimate is obtained as well – see for instance Lemma A.8.

In fact, the latter stability estimate transfers to the new type of decomposition. For instance, $\mathcal{H} \subset H^3(\Omega)^d$, $S \subset \nabla[H^1_0(\Omega)] \cap H^3(\Omega)^d$ and $S^o \subset H(\text{div}_0)(\Omega) \cap H^3(\Omega)^d$ imply that a decomposition in view of [3], i.e.

\[ \mathcal{H} \ni v^1 = s + s^1, \quad s \in S, \ s^1 \in S^o, \]

is a valid decomposition in view of [A.17]. By uniqueness of the latter decomposition, $v^1$, $s$ and $s^1$ satisfy the stability estimate of the Helmholtz decomposition [A.18] as well, i.e. there is a constant $\tilde{C} > 0$ such that

\[ \|s\|_{H^3(\Omega)^d} + \|s^1\|_{H^3(\Omega)^d} \leq \tilde{C}\|v^1\|_{H^3(\Omega)^d}, \]

where $\tilde{C}$ is independent of $v^1$, $s$ and $s^1$. Analogously, let $f = s + s^1 + v$ be as in the assumptions of this lemma and set $v^1 := s + s^1 \in \mathcal{H}$. Then regularity theory of the Neumann-Laplace problem (used in the proof of [A.15], similarly to the proof of Lemma A.7) yields $\tilde{C} > 0$ such that

\[ \|v^1\|_{H^3(\Omega)^d} \leq \tilde{C}\|f\|_{H^3(\Omega)^d} \]

and consequently

\[ \|v\|_{H^3(\Omega)^d} = \|f - v^1\|_{H^3(\Omega)^d} \leq (1 + \tilde{C})\|f\|_{H^3(\Omega)^d}. \]

Combining the stability estimates yields the norm equivalence estimate (3.21).

\[ \square \]

**Lemma 3.6.** i) $S^o$ is a closed subspace of $\mathcal{H}$ with respect to the $H^3(\Omega)^d$-norm.

ii) $\mathcal{V}$ is a closed subspace of $\mathcal{M}$ with respect to the $H^3(\Omega)^d$-norm.

**Proof:** This is a straightforward consequence of the representations [3.23] and [3.25] combined with continuity of the divergence operator and the normal trace. Notice that in [40] a proof is given solely based on the definitions of $S^o$ and $\mathcal{V}$. \[ \square \]

**Lemma 3.7.** There are sets \( \{s_i\}_{i \in \mathbb{N}}, \{s^1_i\}_{i \in \mathbb{N}}, \{m_i\}_{i \in \mathbb{N}}, \) such that

\[ \mathcal{H} = \overline{\text{span}}\{s_i\}_{i \in \mathbb{N}} H^3(\Omega)^d, \quad S^o = \overline{\text{span}}\{s^1_i\}_{i \in \mathbb{N}} H^3(\Omega)^d, \quad (3.26) \]

\[ \mathcal{V} = \overline{\text{span}}\{m_i\}_{i \in \mathbb{N}} H^3(\Omega)^d, \quad (3.27) \]

\[ \mathcal{M} = \overline{\text{span}}\{s_i\}_{i \in \mathbb{N}} \oplus \overline{\text{span}}\{s^1_i\}_{i \in \mathbb{N}} \oplus \overline{\text{span}}\{m_i\}_{i \in \mathbb{N}} H^3(\Omega)^d. \quad (3.28) \]

In particular, the family \( \{s_i\}_{i \in \mathbb{N}} \cup \{s^1_j\}_{j \in \mathbb{N}} \cup \{m_k\}_{k \in \mathbb{N}} \) is orthonormal in $L^2(\Omega)^d$. Each of the families \( \{s_i\}_{i \in \mathbb{N}}, \{s^1_j\}_{j \in \mathbb{N}}, \{m_k\}_{k \in \mathbb{N}} \) is orthogonal in $H^3(\Omega)^d$.\[ \square \]
Proof: By Lemma 3.3 applied to \((\mathcal{S}, \mathcal{S}^{L^2(\Omega)^d})\) and \((\mathcal{S}^o, \mathcal{S}^{oL^2(\Omega)^d})\), one obtains bases
\[
\{s_i\}_{i \in \mathbb{N}} \text{ of } \mathcal{S}, \quad \{s_i^+\}_{i \in \mathbb{N}} \text{ of } \mathcal{S}^o
\] (as subsets of \(H^3(\Omega)^d\)) that are \(L^2\)-orthonormal and orthogonal with respect to the inner product of \(H^3(\Omega)^d\). Note that the dense and compact embeddings needed for Lemma 3.3 are obvious. Indeed, \(\mathcal{S}\) and \(\mathcal{S}^o\) are separable Hilbert-spaces as they are closed subsets of the separable Hilbert-space \(H^3(\Omega)^d\), see Lemma 3.4 ii) and Lemma 3.6 i). Hence, the first claim is proven. Their union \(\{h_i\}_{i \in \mathbb{N}}\),
\[
h_{2i-1} := s_i, \quad h_{2i} := s_i^+, \quad \forall i \in \mathbb{N},\tag{3.31}
\] generates the space \(\mathcal{H}\) due to the following argument. By using basis functions with even indices only or odd indices only, it is evident that \(\mathcal{S}^{(3.30)}:\) span\(\{s_i\}_{i \in \mathbb{N}}\) \(H^3(\Omega)^d\) and \(\mathcal{S}^o^{(3.30)}:\) span\(\{s_i^+\}_{i \in \mathbb{N}}\) \(H^3(\Omega)^d\) are subsets of \(\text{span}\{h_i\}_{i \in \mathbb{N}}\) \(H^3(\Omega)^d\). Then clearly \(\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^o\) is a subset thereof as well. This implies the equality \(\text{span}\{h_i\}_{i \in \mathbb{N}} = \mathcal{H}\).

The representation of elements of \(\mathcal{H}\) as \(H^3\)-convergent sum \(\sum_{i=1}^{\infty} \beta_i h_i\) is unique as can be seen by the following argument. Due to linearity, it suffices to consider \(\sum_{i=1}^{\infty} \beta_i h_i = 0\) with convergence in the norm of \(H^3(\Omega)^d\) and to conclude that all coefficients above have to vanish. If the series vanishes with respect to the \(H^3\)-norm, it also vanishes with respect to the \(L^2\)-norm. But the set \(\{h_i\}_{i \in \mathbb{N}}\) obviously is an orthonormal basis of \(\text{span}\{h_i\}_{i \in \mathbb{N}}\) \(L^2(\Omega)^d\) and therefore is a (Schauder-)basis, hence all coefficients vanish.

Analogously, proceed with the third claim and the fourth claim. By another application of Lemma 3.3 to \((\mathcal{V}, \mathcal{V}^{L^2(\Omega)^d})\) an additional basis
\[
\{m_i\}_{i \in \mathbb{N}} \text{ of } \mathcal{V}\tag{3.32}
\] is obtained – applicability is guaranteed as \(\mathcal{V}\) is a closed subspace of \(H^3(\Omega)^d\), cf. Lemma 3.6 ii) – and the union of all three bases generates the space \(\mathcal{M}\) due to Lemma 3.3 as before while the basis representation is unique.

The orthogonality properties are evident by construction. \(\Box\)

For later use, relabel the basis functions of \(\mathcal{M}\),
\[
\mathcal{M} := \text{span}\{\Psi_{i,j}^m\}_{i \in \mathbb{N}}\] \(H^3(\Omega)^d\), \(\Psi_{2i}^m := h_i, \quad \Psi_{2i+1}^m := m_i \forall i \in \mathbb{N}.\tag{3.33}
\]

By construction the following properties are satisfied,
\[
\Psi_{2i}^m \perp \Psi_{2j}^m \quad \text{for } i, j \in \mathbb{N}, \quad i \neq j, \quad \text{with respect to the } \mathcal{L}^2 \text{ and } H^3 \text{ scalar products},
\Psi_{2i-1}^m \perp \Psi_{2j-1}^m \quad \text{for } i, j \in \mathbb{N}, \quad i \neq j, \quad \text{with respect to the } \mathcal{L}^2 \text{ and } H^3 \text{ scalar products},
\Psi_{4i}^m \perp \Psi_{4j}^m \quad \text{for } i, j \in \mathbb{N}, \quad i \neq j, \quad \text{with respect to the } \mathcal{L}^2 \text{ and } H^3 \text{ scalar products},
\Psi_{4i-1}^m \perp \Psi_{4j-1}^m \quad \text{for } i, j \in \mathbb{N}, \quad i \neq j, \quad \text{with respect to the } \mathcal{L}^2 \text{ scalar product.} \tag{3.34}
\]

Corollary 3.8. The basis in (3.33) satisfies
\[
L^2(\Omega)^d = \text{span}\{\Psi_{i,j}^m\}_{i \in \mathbb{N}}\] \(L^2(\Omega)^d\). \tag{3.35}
3.1 Construction of discrete spaces

Proof: Recalling Remark 3.1, elements in \( L^2(\Omega)^d \) can be approximated by elements in \( M \) in the \( L^2 \)-norm. Moreover, elements in \( M \) can be approximated by functions in \( \text{span}\{\Psi_i\} \) in the \( H^1 \)-norm, hence in the \( L^2 \)-norm as well. Combining these results proves the claim.

Remark 3.9. Along the lines of the proof of Lemma 3.5 it is evident that the decomposition of \( \mathcal{M} = S \oplus S^o \oplus \mathcal{V} \) can be seen as a special case of the Helmholtz type decomposition

\[
L^2(\Omega)^d = \nabla[H_0^1(\Omega)] \oplus \nabla[H^1(\Omega) \cap H(\text{div})(\Omega) \oplus H_{n0}(\text{div})(\Omega)].
\]

This decomposition is readily available from (A.12) combined with an application of Lemma A.9 to (A.13), i.e. intersecting both sides of (A.13) with \( \nabla[H^1(\Omega)] \). From [18] it is also known that for simply connected \( \Omega \) the space \( \nabla[H^1(\Omega)] \cap H(\text{div})(\Omega) \subset H(\text{div}, \text{curl})(\Omega) \) coincides with \( H(\text{div}, \text{curl})(\Omega) \) and \( H_{n0}(\text{div})(\Omega) \) turns into \( \text{curl}[H_{10}(\text{div}, \text{curl})(\Omega)] \) which yields (A.14).

The new decomposition from Lemma 3.5 then is the special case of above decomposition where all involved spaces inherit the regularity and boundary conditions of \( \mathcal{M} \).

3.1.2 Construction of a basis of \( \mathcal{R} \)

This section caters to the setting \( \Omega \subset \subset \Omega' \) as the other case, \( \Omega = \Omega' \), is straightforward. Whenever necessary, remarks about the latter case will be made. The analytical results of Section 3.2 hold true in case of \( \Omega = \Omega' \) if the basis functions from Remark 3.20 are used. Consequently, the results in Section 3.3 and Section 3.4 remain unchanged as they are solely based on the results from Section 3.2. Let the reader be reminded of the assumptions in (H1 3.1) which are mandatory for this section.

Recalling the requirement that approximation spaces \( \mathcal{R}_n \) and \( \mathcal{M}_n \) should satisfy \( \nabla[\mathcal{R}_n|\Omega] \subset \mathcal{M}_n \) (cf. (3.5)), it is natural to extend – as a first step – potentials of the gradient fields, which are contained in \( \mathcal{H} \), to whole \( \Omega' \), while being consistent with the definition of \( \mathcal{R} \). In a second step, the set of obtained potentials needs to be augmented to account for the missing information on \( \Omega' \setminus \Omega \) which cannot be provided by the gradient fields of \( \mathcal{H} \), which are only defined on \( \Omega \). The overall structure of those two steps is depicted in Figure 3.1 and Figure 3.2.

![Figure 3.1: Sketch of the first step to obtain a generating set of potentials for \( \mathcal{R} \) based on the gradient fields of \( \mathcal{H} \).](image)

For this, choose the uniquely determined mean-value-free potentials

\[
\phi^\Omega_i \in H^1_{\text{mean}}(\Omega) \cap H^4(\Omega) \quad \text{such that} \quad \nabla \phi^\Omega_i = h_i = \begin{cases} s_{i/2}, & i \text{ even}, \\ s_{(i-1)/2}^\perp, & i \text{ odd} \end{cases}, \quad \forall i \in \mathbb{N},
\]

(3.36)
Note that the functions \( \phi^\Omega_{2i}, \ i \in \mathbb{N} \), correspond to the potentials of gradient fields in \( S \) which admit potentials with constant trace, hence
\[
\phi^\Omega_{2i}|_{\partial \Omega} \equiv \text{const.} = \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \phi^\Omega_{2i} \, d\sigma \quad \forall i \in \mathbb{N}.
\]
(3.37)

The following operator will help to construct the aforementioned extensions of \( \phi^\Omega_i \) onto \( \Omega' \setminus \overline{\Omega} \).

**Definition 3.10.** On a bounded Lipschitz domain \( V \subset \mathbb{R}^d \) let \( L_V^{-1} : H^\frac{1}{2}(\partial V) \to H^1(V) \) be defined pointwise by the following problem,
\[
-\Delta L_V^{-1} f = 0,
L_V^{-1} f|_{\partial V} = f,
\]
(3.38)
for any \( f \in H^\frac{1}{2}(\partial V) \).

Well-posedness in the weak sense is obvious and standard stability estimates state
\[
\| \nabla L_V^{-1} f \|_{L^2(V)} \leq C \| f \|_{H^\frac{1}{2}(\partial V)}.
\]
(3.39)

Moreover, for any constant \( C \in \mathbb{R} \),
\[
\nabla L_V^{-1} C = 0.
\]
(3.40)

It is worth mentioning that by definition
\[
\text{Rng}(\nabla L_V^{-1}) \subset H(\text{div}_0, \text{curl}_0)(V),
\]
(3.41)
which is why this operator can be used to obtain extensions of potentials onto \( \Omega' \setminus \overline{\Omega} \) while respecting the definition of \( \mathcal{R} \) which requires vanishing \( \text{div} \) and \( \text{curl} \) of gradients on \( \Omega' \setminus \overline{\Omega} \), see (3.8). For more details, see Section A.5.

**Definition 3.11.** On a bounded domain \( V \subset \mathbb{R}^d \), let \( \{u^V_i\}_{i \in \mathbb{N}} \) denote the eigenfunctions to positive eigenvalues \( (\lambda^V_i)_{i \in \mathbb{N}} \) of the homogeneous Neumann-Laplace problem,
\[
-\Delta u^V_i = \lambda^V_i u^V_i \quad \text{in } V,
\nabla u^V_i \cdot \nu|_{\partial V} = 0,
\int_V u^V_i \, dx = 0.
\]
(3.42)

Augmented with the constant function \( u^V_0 := |V|^{-\frac{1}{2}} \), the set of all functions \( \{u^V_i\}_{i \in \mathbb{N}_0} \) is dense in \( H^1(V) \) and their traces \( \{u^V_i|_{\partial V}\}_{i \in \mathbb{N}_0} \) are dense in \( H^\frac{1}{2}(\partial V) \).

**Remark 3.12.** The homogeneous Neumann-Laplace eigenfunctions will be used below in (3.44), but any set of functions with traces that are dense in the space of \( H^\frac{1}{2}\)-regular traces would be equally helpful.

**Definition 3.13.** Let \( \{\psi^\text{dir}_i\}_{i \in \mathbb{N}} \) denote the eigenfunctions associated with positive eigenvalues \( (\mu^\text{dir}_i)_{i \in \mathbb{N}} \) of the homogeneous Dirichlet-Laplace problem on \( \Omega \), i.e.
\[
-\Delta \psi^\text{dir}_i = \mu^\text{dir}_i \psi^\text{dir}_i \quad \text{in } \Omega,
\psi^\text{dir}_i|_{\partial \Omega} = 0.
\]
(3.43)

**Remark 3.14.** The eigenfunctions of the homogeneous Dirichlet-Laplace operator will be used later in order to prove that the extension procedure in (3.44) is sufficient to generate the whole space \( \mathcal{R} \).
3.1 Construction of discrete spaces

Figure 3.2: Sketch of the extension and augmentation needed to obtain potentials on \( \Omega' \). Green lines indicate extensions, brown lines indicate augmentation of the set of basis functions. The latter type of functions can attain arbitrary boundary values and all extensions' gradients have vanishing \( \text{div} \) and \( \text{curl} \) by construction, see (3.41).

The potentials obtained from \( \mathcal{H} \) will be combined with zero boundary data on \( \partial \Omega' \), see the green lines of Figure 3.2 and arbitrary boundary data on \( \partial \Omega' \) will be combined with vanishing potentials on \( \Omega \), see brown lines in Figure 3.2. By using the operator in Definition 3.10, it is guaranteed that the extensions have vanishing \( \text{div} \) and \( \text{curl} \), see (3.41). With the help of (3.42), define

\[
\tilde{R}_{2i+1} := \begin{cases} 
0 & \text{in } \Omega \\
L_{\Omega' \setminus \Omega}^{-1} \begin{cases} 
0 & \text{on } \partial \Omega \\
u_{i}^\Omega & \text{on } \partial \Omega'
\end{cases} & \text{in } \Omega' \setminus \Omega
\end{cases} \quad \forall i \in \mathbb{N}_0,
\]

\[
\tilde{R}_{2i} := \begin{cases} 
\phi_i^\Omega - c_i & \text{in } \Omega \\
L_{\Omega' \setminus \Omega}^{-1} \begin{cases} 
\phi_i^\Omega & \text{on } \partial \Omega \\
0 & \text{on } \partial \Omega'
\end{cases} & \text{in } \Omega' \setminus \Omega
\end{cases} \quad \forall i \in \mathbb{N},
\]

\[
c_i := \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \phi_i^\Omega \, d\sigma
\]

and normalize them by

\[
R_i := \tilde{R}_i - \int_{\Omega'} \tilde{R}_i \, dx \quad \forall i \in \mathbb{N}.
\]

By construction, those functions are elements of \( \mathcal{R} \), because for any arbitrary \( i \in \mathbb{N} \) one can conclude that \( R_i|_{\Omega} \) either equals a constant function or a potential \( \phi_j^\Omega \) up to a constant for some \( j \in \mathbb{N} \). Therefore, \( \nabla R_i|_{\Omega} \in H^3(\Omega)^d \subset H(\text{div})(\Omega) \). By the use of \( L_{\Omega' \setminus \Omega}^{-1} \) (cf. (3.38)) one has \( \nabla R_i|_{\Omega' \setminus \Omega} \in H(\text{div}_0)(\Omega' \setminus \Omega) \). Additionally, \( R_i \in H^1_{\text{mean}}(\Omega') \) where the vanishing mean value is due to (3.45) and the weak differentiability is due to matching traces on \( \partial \Omega \) between the inner part \( \Omega \) and the outer part \( \Omega' \setminus \Omega \). Particularly, from

\[
\tilde{R}_{2i+1} \quad \begin{cases} 
\phi_{2i}^\Omega - c_{2i} & \text{in } \Omega \\
L_{\Omega' \setminus \Omega}^{-1} \begin{cases} 
\phi_{2i}^\Omega & \text{on } \partial \Omega \\
0 & \text{on } \partial \Omega'
\end{cases} & \text{in } \Omega' \setminus \Omega
\end{cases}
\]

\[
\tilde{R}_{2i} \quad \begin{cases} 
\phi_{2i}^\Omega - c_{2i} & \text{in } \Omega \\
L_{\Omega' \setminus \Omega}^{-1} \begin{cases} 
\phi_{2i}^\Omega & \text{on } \partial \Omega \\
0 & \text{on } \partial \Omega'
\end{cases} & \text{in } \Omega' \setminus \Omega
\end{cases}
\]
one obtains
\[ R_{4i}|_{\Omega' \setminus \Omega} \equiv \text{const.} \quad \forall i \in \mathbb{N}. \] (3.46)

As a first step to find suitable approximation spaces for the magnetic potential, the space
\[ \mathcal{R}_{\text{temp}} := \text{span}\{R_i\}_{i \in \mathbb{N}} \subset \mathcal{R} \] (3.47)
will be chosen. The goal is to prove the identity \( \mathcal{R}_{\text{temp}} = \mathcal{R} \). For this, as usual, the existence of functions in the orthogonal complement of \( \mathcal{R}_{\text{temp}} \) (as subset of \( \mathcal{R} \)) is assumed. By use of suitable test functions it will be shown that such functions need to vanish. Three types of test functions will be used in the proof of Lemma 3.17. However, before proceeding, it has to be shown that those test functions are admissible, which is the task of the following two results.

Lemma 3.15. Using the eigenfunctions \( \{\psi_i^{\text{dir}}\}_{i \in \mathbb{N}} \) of the homogeneous Dirichlet-Laplace problem from (3.43), define
\[ \tilde{p}_i := \begin{cases} \psi_i^{\text{dir}} & \text{in } \Omega, \\ 0 & \text{in } \Omega' \setminus \Omega \end{cases}, \quad p_i := \tilde{p}_i - \int_{\Omega'} \tilde{p}_i \, dx \in \mathcal{R}, \] (3.48)
for all \( i \in \mathbb{N} \). The statement
\[ p_i \in \mathcal{R}_{\text{temp}} \] (3.49)
is true for all \( i \in \mathbb{N} \), where \( \mathcal{R}_{\text{temp}} \) was defined in (3.47).

**Proof:** Notice that the eigenfunctions \( \{\psi_i^{\text{dir}}\}_{i \in \mathbb{N}} \) are \( H^4 \)-regular which results from standard regularity theory [34] of the Poisson-problem – provided sufficient regular boundary which is given by assumption (H1 3.1c). Thereby the functions \( p_i|_{\Omega} \) are \( H^4 \)-regular, too.

Due to
\[ \Delta \psi_i^{\text{dir}}|_{\partial \Omega} = -\mu \psi_i^{\text{dir}}|_{\partial \Omega} = 0 \quad \text{and} \quad \left( \psi_i^{\text{dir}} - \int_{\Omega} \psi_i^{\text{dir}} \, dx \right) \in H_1^1(\Omega) \cap H^4(\Omega), \]
the gradients \( \nabla p_i|_{\Omega} \) are in \( \mathcal{H} \) (cf. (3.17)). Hence, there exists a sequence \( (\tilde{h}_n)_{n \in \mathbb{N}} \subset \text{span}\{h_i\}_{i \in \mathbb{N}} \) (see (3.31) and (3.27)) such that
\[ \tilde{h}_n \to \nabla (\psi_i^{\text{dir}} - \int_{\Omega} \psi_i^{\text{dir}} \, dx) = \nabla p_i|_{\Omega} \quad \text{in } H^3(\Omega)^d. \] (3.50)

Due to (3.36) one can write
\[ \tilde{h}_n = \sum_{i=1}^{N_n} \alpha_{i}^{n} \nabla \phi_i^{\Omega}, \quad \text{for some } N_n \in \mathbb{N}, \quad \alpha_i^{n} \in \mathbb{R}, \quad i = 1, \ldots, N_n, \quad \forall n \in \mathbb{N}. \] (3.51)

Based on this, the following sequence will be chosen which is expected to approximate \( p_i \).
\[ Q_n := \sum_{i=1}^{N_n} \alpha_{i}^{n} \frac{\Omega}{|\Omega|^{\frac{1}{2}}} \phi_i^{\Omega}, \quad \text{for some } N_n \in \mathbb{N}, \quad \alpha_i^{n} \in \mathbb{R}, \quad i = 1, \ldots, N_n, \quad \forall n \in \mathbb{N}. \] (3.52)
where the $c_i$ are the constants defined in (3.44). Recalling the definition of the $\mathcal{R}$-norm, see (3.12),
\[
\|Q_n - p_i\|_\mathcal{R} \leq \|\nabla Q_n - \nabla p_i\|_{H(\text{div})(\Omega)} + \|\nabla Q_n - \nabla p_i\|_{L^2(\Omega')^d} =: I_n + II_n.
\]
On $\Omega$ one obtains $Q_n(3.54) \sum_{i=1}^{N_n} \alpha_i^\Omega (\phi_i^\Omega - c_i)$ up to a constant, hence the gradient $\nabla Q_n$ converges in $H^3(\Omega)^d$ towards $\nabla p_i$, see
\[
\nabla Q_n|_\Omega = \sum_{i=1}^{N_n} \alpha_i^\Omega (\phi_i^\Omega - c_i) = \sum_{i=1}^{N_n} \alpha_i^\Omega \nabla \phi_i^\Omega \tilde{\mathbf{n}}_n(3.56) \nabla p_i|_\Omega. \tag{3.53}
\]
This implies $I_n \to 0$. For the second term all definitions will be expanded. One ends up with
\[
II_n = \|\nabla Q_n - \nabla p_i\|_{L^2(\Omega')(\Omega')^d} \tag{3.57}
\]
\[
\leq C \left\{ \sum_{i=1}^{N_n} \alpha_i^\Omega (\phi_i^\Omega|_{\partial \Omega} - c_i) + \left( \int_{\Omega'} \nabla \phi_i^\Omega \, dx \right) \right\}_{H^1(\Omega')} + \|0\|_{H^1(\partial \Omega')}
\]
\[
= C \left\{ \sum_{i=1}^{N_n} \alpha_i^\Omega (\phi_i^\Omega|_{\partial \Omega} - \left( \psi_i^\text{dir} - \int_{\Omega'} \psi_i^\text{dir} \, dx \right)|_{\partial \Omega'} \right\}_{H^1(\partial \Omega')}
\]
\[
\leq C \left\{ \sum_{i=1}^{N_n} \alpha_i^\Omega (\phi_i^\Omega - \left( \psi_i^\text{dir} - \int_{\Omega'} \psi_i^\text{dir} \, dx \right) \right\}_{H^1(\Omega)}
\]
\[
\leq C \left\| \tilde{\mathbf{h}}_n - \nabla \left( \psi_i^\text{dir} - \int_{\Omega'} \psi_i^\text{dir} \, dx \right) \right\|_{L^2(\Omega)} \tag{3.58}
\]
where the trace theorem and Poincaré’s inequality for mean value free functions have been used in the last two steps. Also note that for $\Omega \subset \subset \Omega'$ the boundaries $\partial \Omega$ and $\partial \Omega'$ have a positive distance from each other which allows to conclude
\[
\| \cdot \|_{H^\frac{1}{2}(\partial \Omega')^d} = \| \cdot \|_{H^\frac{1}{2}(\partial \Omega)} + \| \cdot \|_{H^\frac{1}{2}(\partial \Omega')}
\]
due to the nature of the definition of the $H^\frac{1}{2}$-norm, see e.g. [46]. Therefore, $p_i$ can be approximated by span$\{R_i\}_{i \in \mathbb{N}}$ with respect to the norm of $\mathcal{R}$ which implies $p_i \in \mathcal{R}_{\text{temp}}$. \qed
Lemma 3.16. Using the set of functions \( \{ u_i^\Omega \}_{i \in \mathbb{N}_0} \) given in (3.42) as eigenfunctions of the homogeneous Neumann-Laplace operator on \( \Omega \), define

\[
\hat{q}_i := \begin{cases} 
L_\Omega^{-1} u_i^\Omega |_{\partial \Omega} & \text{in } \Omega \\
L_{\Omega' \setminus \Pi}^{-1} \begin{cases} u_i^\Omega |_{\partial \Omega} & \text{on } \partial \Omega \\
0 & \text{on } \partial \Omega' \end{cases} & \text{in } \Omega',
\end{cases}
\quad \forall i \in \mathbb{N}_0,
\tag{3.55}
\]

and \( q_i := \hat{q}_i - \int_{\Omega'} \hat{q}_i \, dx \in \mathcal{R} \quad \forall i \in \mathbb{N}_0, \)

where \( L_{\cdot}^{-1} \) is defined in (3.38). For all \( i \in \mathbb{N}_0 \) the statement

\[ q_i \in \mathcal{R}_{\text{temp}} \tag{3.56} \]

holds true, where \( \mathcal{R}_{\text{temp}} \) was defined in (3.47).

**Proof:** In order to guarantee \( H^4 \)-regularity of \( q_i \) on \( \Omega \), the homogeneous Neumann-Laplace eigenfunctions (cf. (3.42)) need to have \( H^2 \)-regular traces on \( \partial \Omega \) or by the trace theorem they have to be \( H^4 \)-regular on the volume \( \Omega \), see Section A.5 for information on how this relates to the operator \( L_{\cdot}^{-1} \). Indeed, they are \( H^4(\Omega) \)-regular due to H1 3.1c, see [34, Theorem 9.19].

By definition,

\[ \Delta(q_i|_\Omega) = 0 \quad \text{and} \quad \left( \hat{q}_i - \int_{\Omega} \hat{q}_i \, dx \right) \in H^{1}_{\text{mean}}(\Omega) \cap H^4(\Omega), \]

hence \( \nabla q_i \in \mathcal{H} \) (cf. (3.17)). Similarly as in the proof of Lemma 3.15 by using the same notation, there exists a sequence

\[ \tilde{f}_n \to \nabla(\hat{q}_i - \int_{\Omega} \hat{q}_i \, dx) = \nabla q_i|_\Omega \quad \text{in } H^3(\Omega)^d, \tag{3.57} \]

which can be written as in (3.51). Now choose the sequence

\[
Q_n := \sum_{i=1}^{N_n} \alpha_i^n R_{2i} + \hat{c}_n R_1 \quad \in \text{span}\{R_i\}_{i \in \mathbb{N}},
\quad \hat{c}_n = -|\Omega'|^{\frac{1}{2}} \left( \int_{\Omega'} \hat{q}_i \, dx + \sum_{i=1}^{N_n} \alpha_i^n c_i \right),
\tag{3.58}
\]

where the \( c_i \) are the constants defined in (3.44). By means of analogous arguments as in (3.53), it is easy to compute,

\[ \| \nabla Q_n - \nabla q_i \|_{\mathcal{H}} \leq \| \nabla Q_n - \nabla q_i \|_{L^2(\Omega')} + \| \nabla Q_n - \Delta q_i \|_{L^2(\Omega)} \]

\[ \leq \left\| \tilde{f}_n - \nabla q_i \right\|_{H(\text{div})(\Omega)} + \| \nabla Q_n - \nabla q_i \|_{L^2(\Omega') \setminus \Pi}. \quad \text{Lemma A.5} \]

Now we consider

\[ \left\| \nabla Q_n - \nabla q_i \right\|_{L^2(\Omega' \setminus \Pi)} \leq \sum_{i=1}^{N_n} \alpha_i^n \nabla L_{\Omega' \setminus \Pi}^{-1} \begin{cases} \phi_i^\Omega |_{\partial \Omega} - c_i & \text{on } \partial \Omega \\
0 & \text{on } \partial \Omega' \end{cases} \]

\[ -|\Omega'|^{\frac{1}{2}} \left( \int_{\Omega'} \hat{q}_i \, dx + \sum_{i=1}^{N_n} \alpha_i^n c_i \right) \nabla L_{\Omega' \setminus \Pi}^{-1} \begin{cases} 0 & \text{on } \partial \Omega \\
|\Omega'|^{-\frac{1}{2}} & \text{on } \partial \Omega' \end{cases} \]

\[ \nabla L_{\Omega' \setminus \Pi}^{-1} \begin{cases} u_i^\Omega |_{\partial \Omega} & \text{on } \partial \Omega \\
0 & \text{on } \partial \Omega' \end{cases} \| \|_{L^2(\Omega' \setminus \Pi)^d} \]
holds true. According to Lemma 3.15, the functions in (3.48) are admissible test functions, i.e.

\[
\text{on } \partial \Omega
\]

\[
\text{on } \partial \Omega'
\]

\[= 0 \text{ due to } (3.40)
\]

\[
+ \nabla L_{\Omega \setminus \Pi}^{-1} \left( f_{\Omega} \tilde{\varphi}_i \, dx + \sum_{i=1}^{N_n} \alpha_i^n c_i \right)
\]

\[L^2(\Omega \setminus \Pi)^d
\]

\[\leq C \left\{ \sum_{i=1}^{N_n} \alpha_i^n \phi_i^\Omega |_{\partial \Omega} - \left( \tilde{\varphi}_i - f_{\Omega} \tilde{\varphi}_i \, dx \right) \right\}_{H^{\frac{1}{2}}(\partial \Omega)}^2 + \|0\|_{H^\frac{1}{2}(\partial \Omega')}
\]

\[\leq C \left\{ \sum_{i=1}^{N_n} \alpha_i^n \phi_i^\Omega \right\}_{H^1(\Omega)}^2
\]

\[\leq C \| \tilde{\varphi}_n - \nabla (\tilde{\varphi}_i - f_{\Omega} \tilde{\varphi}_i \, dx) \|_{L^2(\Omega)^d} (3.57)
\]

where in the last two steps the trace theorem and Poincaré’s inequality (for mean value free functions) have been used. Also recall (3.54), which has previously been used in the proof of Lemma 3.15. Hence, \( \varphi_i \in \mathcal{R}_{\text{temp}} \). 

\[\mathcal{R}_{\text{temp}} = \mathcal{R}
\]

(3.59)

**Lemma 3.17.** The identity

\[\mathcal{R}_{\text{temp}} = \mathcal{R}
\]

holds true.

**Proof:** By construction, \( \mathcal{R}_{\text{temp}} \subset \mathcal{R} \). Assume by contradiction the existence of a function \( S \) in \( \mathcal{R} \setminus \mathcal{R}_{\text{temp}} \). Without loss of generality, \( S \) can be chosen orthogonal to \( \mathcal{R}_{\text{temp}} \), hence

\[
\langle S, R \rangle_{\mathcal{R}} (3.14) \| \nabla S, \nabla R \|_{L^2(\Omega)^d} + \| \Delta S, \Delta R \|_{L^2(\Omega)} \geq 0 \quad \forall R \in \mathcal{R}_{\text{temp}}.
\]

It will be shown that \( S \) has to be constant. For this, three types of test functions will be considered.

According to Lemma 3.15, the functions in (3.48) are admissible test functions, i.e. \( p_i \in \mathcal{R}_{\text{temp}} \) for all \( i \in \mathbb{N} \). By Lemma 3.16 the functions in (3.55) are admissible test functions, i.e. elements of \( \mathcal{R}_{\text{temp}} \). The last set of functions to be considered is \( \{ \mathcal{R}_{2i+1} \}_{i \in \mathbb{N}_0} \). Obviously, those functions are in \( \mathcal{R}_{\text{temp}} \).

Now, plugging in those test functions, starting with the functions in (3.48), for all \( i \in \mathbb{N} \) one obtains

\[0 = \langle \nabla S, \nabla p_i \rangle_{L^2(\Omega)^d} + \langle \Delta S, \Delta p_i \rangle_{L^2(\Omega)} (3.48)
\]

\[\langle \nabla S, \nabla \psi^\text{dir} \rangle_{L^2(\Omega)^d} + \langle \Delta S, \Delta \psi^\text{dir} \rangle_{L^2(\Omega)} (3.43)
\]

\[-(1 + \mu_i) \langle \Delta S, \psi^\text{dir} \rangle_{L^2(\Omega)}]

which implies \( \nabla S \in H(\text{div}_0, \text{curl}_0)(\Omega) \). So far one has

\[\Delta S|_{\Omega} = 0 \quad \text{and} \quad \Delta S|_{\Omega \setminus \Pi} = 0.
\]

(3.60)
Consequently, for all \( i \in \mathbb{N}_0 \), using the functions in \( (3.55) \),
\[
0 = \langle \nabla S, \nabla q_i \rangle_{L^2(\Omega')}^{d} + \langle \Delta S, \Delta q_i \rangle_{L^2(\Omega)} \quad \text{(3.55)}
\]
\[
= \langle \nabla S, \nabla q_i \rangle_{L^2(\Omega)} + \langle \nabla S, \nabla q_i \rangle_{L^2(\Omega' \setminus \tilde{\Omega})^{d}}
\]
\[
0 + \langle \nabla S \cdot \nu, u^0_i \rangle_{(H^1(\partial \Omega))' \times H^1(\partial \Omega)}
\]
\[
0 + \langle \nabla S \cdot \nu, \{ u^0_i \text{ on } \partial \Omega, 0 \text{ on } \partial \Omega' \} \rangle_{(H^1(\partial (\Omega' \setminus \tilde{\Omega}))') \times H^1(\partial (\Omega' \setminus \tilde{\Omega}))}
\]
\[
= \langle [\nabla S] \cdot \nu, u^0_i \rangle_{(H^1(\partial \Omega))' \times H^1(\partial \Omega)}
\]
where \( \langle [\nabla S] \cdot \nu, u^0_i \rangle_{(H^1(\partial \Omega))' \times H^1(\partial \Omega)} \) resembles a distributional variant of the normal jump of \( \nabla S \) on \( \partial \Omega \), see \( (A.11) \). As the functions \( \{ u^0_i \} \) \( i \in \mathbb{N}_0 \), cf. \( (3.42) \), generate \( H^1(\partial \Omega) \), this implies the normal jump to vanish. Therefore, one can furthermore conclude via integration by parts that for all \( \phi \in H^1(\Omega') \) the identity
\[
\int_{\Omega'} \nabla S \cdot \nabla \phi \, dx = - \int_{\Omega} \Delta S \phi \, dx - \int_{\Omega' \setminus \tilde{\Omega}} \Delta S \phi \, dx + \langle [\nabla S] \cdot \nu, \phi \rangle_{(H^1(\partial \Omega))' \times H^1(\partial \Omega)} = 0
\]
holds. This implies that \( \nabla S \) is in \( H(\text{div}_0)(\Omega') \) globally and therefore \( \nabla S \in H(\text{div}_0, \text{curl}_0)(\Omega') \).

By the last class of functions one obtains
\[
0 = \langle \nabla S, \nabla R_{2i+1} \rangle_{L^2(\Omega')}^{d} \quad \text{(3.45), (3.44)} \quad \langle \nabla S, \nabla R_{2i+1} \rangle_{L^2(\Omega')}^{d}
\]
\[
0 + \langle \nabla S \cdot \nu, u^0_i \rangle_{(H^1(\partial \Omega))' \times H^1(\partial \Omega)}
\]
for all \( i \in \mathbb{N}_0 \). One finally can conclude – on simply connected \( \Omega' \) with \( C^{1,1} \)-boundary \( \text{(3.5) Remark 3.5} \) – that \( \nabla S \in H_{\partial \Omega}(\text{div}_0, \text{curl}_0)(\Omega') = \{0\} \) and therefore \( S \in \mathcal{R} \) is constant with zero mean, i.e. \( S = 0 \). Notice that the condition on the boundary is satisfied, see \( (H1 \text{ 3.1b}) \).

In Section 3.3 Galerkin approximate solutions will be used to obtain a solution to 'model GW'. The usual approach is to define a space given as the linear hull of only finitely many elements of the generating sets for \( \mathcal{M} \) and \( \mathcal{R} \). For this, it is crucial that those sets are linearly independent. The generating set of \( \mathcal{M} \) already is a basis. In the case of \( \mathcal{R} \) linear independence is not evident yet. Moreover, for later purposes, it is mandatory to orthogonalize parts of the already established generating set \( \{ R_i \} \) \( i \in \mathbb{N} \). Those issues are addressed in the following lemma.

**Lemma 3.18.** There exists a basis \( \{ \psi^R_i \} \) \( i \in \mathbb{N} \) of \( \mathcal{R} \), i.e.
\[
\mathcal{R} = \overline{\text{span}\{ \psi^R_i \} \mathcal{R}}, \quad (3.61)
\]
which satisfies the following.

i) The set \( \{ \nabla \psi^R_{2i} | \Omega \} \) \( i \in \mathbb{N} \) is orthonormal in \( L^2(\Omega)^d \).

ii) The sets \( \{ \nabla \psi_{4i} | \Omega \} \) \( i \in \mathbb{N} \) and \( \{ \nabla \psi_{4i-2} | \Omega \} \) \( i \in \mathbb{N} \) are orthogonal in \( H^3(\Omega)^d \).

iii) The set \( \{ \nabla \psi_{2i-1} | \Omega \} \) \( i \in \mathbb{N} \) is orthonormal in \( L^2(\Omega')^d \).

**Proof:** The basis will be constructed from a linearly independent selection of functions from \( \{ R_i \} \) \( i \in \mathbb{N} \). Therefore, by construction i) and ii), see \( (3.36) \) and Lemma 3.7 are satisfied.
Consider the set \( \{ R_{2i} \}_{i=1,...,n} \). This set is linearly independent as it is linearly independent on \( \Omega \) due to \( \nabla R_{2i}|_{\Omega} = \nabla \delta_{2i}^\Omega = \delta_i \) and the set \( \{ \delta_i \}_{i \in \mathbb{N}} \) is already linearly independent (it is a basis).

The set \( \{ R_{2i-1} \}_{i \in \mathbb{N}} \) is clearly linearly independent from \( \{ R_{2i} \}_{i \in \mathbb{N}} \) due to \( R_{2i-1} \mid_{\Omega} \equiv 0 \) for all \( i \in \mathbb{N} \). From those functions with odd index one can pick \( n \) elements by induction, starting with \( R_1 := R_1 \). If \( j \in \{ 1, ..., n-1 \} \) elements \( R_1, ..., R_j \) have been picked, then add \( R_{ij+1} := R_{K} \), where \( K \geq ij \) is the lowest odd integer such that \( \{ R_1, ..., R_{ij}, R_{K} \} \) is linearly independent. If this procedure fails for some \( n \in \mathbb{N} \), the space \( H^1(\partial \Omega') \) was finite dimensional as the traces \( R_{2i-1}|_{\partial \Omega'} \), see (3.44), generate the latter. Therefore, infinitely many functions need to be chosen. Let

\[
\psi_{2k}^R := R_{2k}, \quad \psi_{2k-1}^R := R_{2k-1} \quad \forall k \in \mathbb{N}.
\]

(3.62)

It remains to find functions \( \psi_{2i-1}^R, \ i \in \mathbb{N} \), such that iii) is satisfied.

By applying Gram-Schmidt-orthogonalization to the functions \( \{ \psi_{2i-1}^R \}_{i \in \mathbb{N}} \), a new set of functions \( \{ \psi_{2i-1}^R \}_{i \in \mathbb{N}} \) will be obtained, which are required to satisfy

\[
\langle \nabla \psi_{2i-1}^R, \nabla \psi_{2j-1}^R \rangle_{L^2(\Omega')^d} = \delta_{ij} \quad \forall i, j \in \mathbb{N}.
\]

(3.63)

Due to Poincaré’s inequality the bilinear form above is an inner product and the Gram-Schmidt procedure is well-posed.

The condition in (3.61) is satisfied due to Lemma 3.17 and the fact that the selection of linearly independent vectors and orthogonalization do not change the linear hull. By construction it is expected that the basis representation is unique. Due to linearity it suffices to consider \( \sum_{i=1}^{\infty} \beta_i \psi_i^R = 0 \) with convergence in the \( \mathcal{R} \)-norm and to conclude that the coefficients above have to vanish. Above convergent series implies

\[
\left\| \sum_{i=1}^{\infty} \beta_i \nabla \psi_i^R \right\|_{L^2(\Omega)^d} \leq \left\| \sum_{i=1}^{\infty} \beta_i \nabla \psi_i^R \right\|_{L^2(\Omega')^d} \stackrel{(3.12)}{\leq} \left\| \sum_{i=1}^{\infty} \beta_i \psi_i^R \right\|_{\mathcal{R}} = 0.
\]

The first term above yields \( \sum_{i=1}^{\infty} \beta_i \nabla \psi_i^R \mid_{\Omega} = 0 \) in \( L^2(\Omega)^d \) as \( \psi_{2i-1}^R \mid_{\Omega} \) is constant (cf. (3.44), (3.45)). As the set \( \{ \nabla \psi_{2i-1}^R \mid_{\Omega} \}_{i \in \mathbb{N}} \) is \( L^2 \)-orthogonal (see [1]), it is a (Schauder-)basis of its linear hull’s \( L^2 \)-completion and thereby the coefficients with even indices have to vanish, i.e. \( \beta_{2i} = 0, \ i \in \mathbb{N} \).

Consequently, from the second term in the inequality above one obtains \( \sum_{i=1}^{\infty} \beta_{2i-1} \nabla \psi_{2i-1}^R = 0 \) in \( L^2(\Omega')^d \), because the coefficients with even indices are already zero. Using the \( L^2 \)-orthogonality of \( \{ \nabla \psi_{2i-1}^R \}_{i \in \mathbb{N}} \), see (3.63), the same argument as before implies that the coefficients with odd indices have to vanish. Therefore, \( \{ \psi_i^R \}_{i \in \mathbb{N}} \) is a basis, indeed.

**Lemma 3.19.** The basis \( \{ \psi_i^R \}_{i \in \mathbb{N}} \) from Lemma 3.18 satisfies

\[
\nabla \psi_{4i}^R|_{\Omega} = s_{2i}, \quad \nabla \psi_{4i-2}^R|_{\Omega \setminus \Omega'} = 0,
\]

\[
\nabla \psi_{4i-2}^R|_{\Omega \setminus \Omega'} = s_{2i-1}, \quad \nabla \psi_{2i-1}^R|_{\Omega} = 0,
\]

(3.64)

for all \( i \in \mathbb{N} \).

**Proof:** The application of Gram-Schmidt-orthogonalization in the proof of Lemma 3.18 only orthogonalizes the basis functions with odd indices, therefore the properties of \( \nabla R_{2i}|_{\Omega} = \delta_i \) do not change, see the definition (3.44) for even indices and particularly (3.46). Also, \( \psi_{2i-1}^R \mid_{\Omega} \) is still constant after the orthogonalization from the proof of Lemma 3.18 as it is a linear combination of constants, see (3.44) for odd indices.
As announced at the beginning of this section, the required changes in case of \( \Omega = \Omega ' \) will be discussed next. Extension of potentials onto \( \Omega ' \) and augmentation of the basis functions to account for missing boundary data on \( \partial \Omega ' \) are not required anymore.

**Remark 3.20.** Consider the case \( \Omega = \Omega ' \). For convenience, in this case the space of potentials is given by

\[
\mathcal{R} = \{ S \in H^1_{\text{mean}}(\Omega) \mid \nabla S \in H(\text{div})(\Omega) \}
\]

as \( \Omega ' \setminus \Omega = \emptyset \). From (3.44) or (3.45) only take the functions with even indices, i.e.

\[
\tilde{R}_2i := \phi_i^\Omega - c_i,
\]

\[
c_i := \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \phi_i^\Omega \, d\sigma,
\]

for all \( i \in \mathbb{N} \) and

\[
\psi^R_{2i} := R_{2i} := \tilde{R}_{2i} - \frac{1}{\Omega} \int_{\Omega} \tilde{R}_{2i} \, dx
\]

as basis functions. In order to have consistent numbering (with the case \( \Omega \subset \subset \Omega ' \)) the result of Lemma 3.16 remains true with a much shorter proof as \( \tilde{S}_n = 0 \) (in the notation of the proof thereof). Following the lines of the proof of Lemma 3.16 it is evident that the result still remains true.

The functions \( \psi^\text{dir}_i \) and \( L_\Omega^{-1} u_j^\Omega |_{\partial \Omega} \) are elements of \( \mathcal{R}_{\text{temp}} \) up to a constant for all \( i \in \mathbb{N}, j \in \mathbb{N}_0 \).

Following the lines of the proof of Lemma 3.17, let

\[
(\nabla S, \nabla R)^{L^2(\Omega)^d} + (\Delta S, \Delta R)^{L^2(\Omega)} = 0 \quad \forall R \in \mathcal{R}_{\text{temp}}.
\]

Setting \( R := (\psi^\text{dir}_i - \int_{\Omega} \psi^\text{dir}_i \, dx), i \in \mathbb{N}, \) one obtains \( \nabla S \in H(\text{div}_0, \text{curl}_0)(\Omega) \). By setting \( R := (L_\Omega^{-1} u_j^\Omega |_{\partial \Omega} - \int_{\Omega} L_\Omega^{-1} u_j^\Omega |_{\partial \Omega} \, dx) \) one obtains due to \( \Delta S = 0 \) the identity

\[
(\nabla S \cdot \nu, u_i^\Omega)_{(H^1/2(\Omega))'} \times H^{1/2}(\Omega) = 0 \quad \forall i \in \mathbb{N}_0
\]

and consequently \( \nabla S \in H_{\text{div}_0}(\text{curl}_0)(\Omega) = \{0\} \) as \( \Omega = \Omega ' \) is simply connected (cf. (H1.3)). One now easily concludes that [Lemma 3.18] remains true for the set \( \{ \psi_{2i}^R \}_{i \in \mathbb{N}} \) with \( \psi_{2i}^R := R_{2i}, \ i \in \mathbb{N}, \) where iii) is skipped. The first three claims from (3.64) remain true as well per construction.

### 3.1.3 Construction of a basis of \( \mathcal{U} \)

In this section, the choice of Galerkin functions to approximate the flow field is briefly discussed. Observing that \( \mathcal{U} \) (cf. (3.14)) is a dense subset of \( H_{\text{div}_0}(\text{curl}_0)(\Omega) \subset L^2(\Omega)^d \) (see 32, Theorem III.2.3] and a closed subset of \( H^3(\Omega)^d \), Lemma 3.3 implies the existence of a basis \( \{ \Psi^u_i \}_{i \in \mathbb{N}} \) such that

\[
(\Psi^u_i, \Psi^u_j)^{L^2(\Omega)^d} = \delta_{ij} \quad \forall i, j \in \mathbb{N},
\]

\[
(\Psi^u_i, \Psi^u_j)^{H^3(\Omega)^d} = 0 \quad \forall i, j \in \mathbb{N} \text{ with } i \neq j.
\]

\[
\mathcal{U} := \text{span} \{ \Psi^u_i \}_{i \in \mathbb{N}} \subset H^3(\Omega)^d \cap H(\text{div}_0)(\Omega) \quad \text{and} \quad H_{\text{div}_0}(\text{curl}_0)(\Omega) = \text{span} \{ \Psi^u_i \}_{i \in \mathbb{N}} \subset L^2(\Omega)^d
\]

**Remark 3.21.** There is no further structure needed for those basis functions, hence for simplicity a general approach based on [Lemma 3.3] has been used instead of using eigenfunctions of the Stokes-operator.
3.1.4 Construction of a basis of $\mathcal{C}$

For the particle density, nonlinear test functions will be expected (the entropic term $g'$) as can be seen in Section 2.3. Therefore, following the usual approach, the complete set $\{\psi_i^c\}_{i \in \mathbb{N}}$ of eigenfunctions of the Laplacian on $\Omega$ subjected to homogeneous Neumann boundary conditions – augmented with a normalized constant function – will be used. Those are identical to the functions $\{u_{i0}\}_{i \in \mathbb{N}_0}$ from (3.42). However, for convenience they will be relabeled (and numbered starting with index 1) here.

By standard regularity results [34], they are $H^4$-regular under the assumption in (H1 3.1c). Moreover, from [37] and the spectral theorem it is known that

$$\langle \psi_i^c, \psi_j^c \rangle_{L^2(\Omega)} = \delta_{ij} \quad \forall i, j \in \mathbb{N},$$

(3.69)

$$\langle \nabla \psi_i^c, \nabla \psi_j^c \rangle_{L^2(\Omega)} = 0 \quad \forall i, j \in \mathbb{N} \text{ with } i \neq j,$$

(3.70)

$$\mathcal{C} \quad (\text{cf. (3.15)})$$

$$H^2_s(\Omega) = \text{span}\{\psi_i^c\}_{i \in \mathbb{N}}$$

(3.71)

$$L^2(\Omega) = \text{span}\{\psi_i^c\}_{i \in \mathbb{N}}.$$ (3.72)

3.2 Construction of approximate solutions

The results of this section are written for the case $\Omega \subsetneq \Omega'$. However, those results remain true in the case $\Omega = \Omega'$ in a straightforward way. The required changes to be made for the latter setting will be annotated whenever needed and supplementary remarks will clarify the differences.

Moreover, the assumptions (H1)-(H3) from the beginning of Section 3 are still in use.

The approximate solutions will be used to prove existence of solutions to a regularized model, see Section 3.3. For this, the following definitions are needed. The regularized entropy is given for $0 < s < e < L \leq \infty$ ($e$ is Euler’s number) by

$$-1 \leq g^L_s(c) := \begin{cases} \frac{c^2}{2} + (\log s - 1)c - \frac{s}{2} & \text{for } c \leq s, \\ c \log c - c & \text{for } s < c < L, \\ \frac{c^2}{2e} + (\log L - 1)c - \frac{L}{2} & \text{for } L \leq c. \end{cases}$$

(3.73)

The choice $L = \infty$ is allowed in (3.73) where the impossible condition $\infty \leq c$ is skipped. Set

$$g_s := g^\infty_s.$$

The lower bound of $g^L_s$ in (3.73) is obvious for the non-regularized part $c \log c - c$. For more details about the regularized parts, see Section A.5. Obviously,

$$(g^L_s)'(c) = \begin{cases} \frac{c}{s} + \log s - 1 & \text{for } c \leq s, \\ \log c & \text{for } s < c < L, \\ \frac{c}{L} + \log L - 1 & \text{for } L \leq c, \end{cases}$$

(3.74)

$$(g^L_s)''(c) = \begin{cases} \frac{1}{s} & \text{for } c \leq s, \\ \frac{1}{L} & \text{for } L \leq c. \end{cases}$$

Hence, $g^L_s$ is convex. In particular, $s < e$, implies $(\log s - 1) < 0$. The regularized entropy will help to obtain uniform estimates by means of the following lemma.

**Lemma 3.22.** Consider $0 < s < e < L$, where $e$ is Euler’s number. For all $c \in \mathbb{R}$ the estimates

$$|c|^2 \leq 2(L + e)g^L_s(c) + 2L^2 + e^2,$$

$$|c| \leq (2e + 1)g_s(c) + 2e^2,$$

hold. The last estimate remains true if $g_s$ is replaced with $g^L_s$. 

3.2 Construction of approximate solutions

**Proof:** Assume \( c \geq L \), then
\[
g^L_s(c) = \frac{c^2}{2L} + (\log L - 1)c - \frac{L}{2} \geq \frac{1}{2L} c^2 - \frac{L}{2}
\]
and therefore \(|c|^2 \leq 2Lg^L_s(c) + L^2\). If \( c < 0 \), then analogously
\[
g^L_s(c) = \frac{c^2}{2s} + (\log s - 1)c - \frac{s}{2} \geq \frac{1}{2s} c^2 - \frac{s}{2},
\]
implying \(|c|^2 \leq 2sg^L_s(c) + s^2 \leq 2eg^L_s(c) + e^2\). If \( c \in (0, L) \), then
\[
e^2 \leq L^2.
\]
Altogether, \(|c|^2 \leq 2(L + e)g^L_s(c) + 2L^2 + e^2\).

For the other estimate consider \( c > e^2 \) first. Then,
\[
g_s(c) = c \log(c) - c = c(\log(c) - 1) \geq c = |c|.
\]
For \( c \in [-1, e^2] \) one obviously obtains
\[
e^2 \geq |c|.
\]
And for \( c < -1 \) one obtains from (3.74) as before
\[
|c| \leq |c|^2 \leq 2eg_s(c) + e^2.
\]
In total, \(|c| \leq (2e + 1)g_s(c) + 2e^2\). Also observe that \( g^L_s(c) \geq g_s \).

A cut-off from below for \( s > 0 \),
\[
\mathbb{R} \ni x \mapsto (x)_s := \max(s, x),
\]
is necessary to prevent division by zero in terms like \( \frac{f^L_s(c)}{e^2} \). The regularized model is then given by replacing (2.89c), (2.89d), (2.90b) with
\[
\nabla(c) = -KD\frac{f^L_p(c)}{c_s} \nabla(g^L_s(c)) + K\mu_0\frac{f^L_p(c)}{c_s^2} (\nabla(\alpha_1 h + \frac{3}{2}\beta h_n - \alpha_3 m))^T m, \text{ (3.76)}
\]
\[
c_l + u \cdot \nabla c + \text{div}(c_s v_{\text{part}}) = \sigma_c \Delta c, \text{ (3.77)}
\]
and
\[
(c)_s v_{\text{part}} \cdot \nu|_{\partial \Omega} = \sigma_c \nabla c \cdot \nu|_{\partial \Omega}, \text{ (3.78)}
\]
while keeping the remaining equations from *model GW* (2.89) unchanged.

The ingredients for the discrete Galerkin scheme follow next. Recall (H1 3.1) which allows to use the functions from Section 3.1. Let
\[
\mathcal{U}_n := \text{span}\{\Psi_1^u, ..., \Psi_{2n}^u\},
\]
\[
\mathcal{C}_n := \text{span}\{\psi_1^c, ..., \psi_{2n}^c\},
\]
\[
\mathcal{R}_n := \text{span}\{\psi_1^R, ..., \psi_{2n}^R\},
\]
\[
\mathcal{M}_n := \text{span}\{\Psi_1^m, ..., \Psi_{2n}^m\},
\]
\[
\mathcal{H}_n := \text{span}\{\Psi_2^m, \Psi_4^m, ..., \Psi_{2n}^m\},
\]
with the functions \( \Psi_i^u \) from (3.67), \( \psi_i^c \) from (3.72), \( \psi_i^R \) from (3.61) and \( \Psi_i^m \) from (3.33). Moreover,
\[
\mathcal{X}_n := C^1(I; \mathcal{U}_n) \times C^1(I; \mathcal{C}_n) \times H^1(I; \mathcal{R}_n) \times C^1(I; \mathcal{M}_n) \text{ (3.80)}
\]
will be used for the Galerkin scheme \(3.28\). Let

\[
\Pi_{\mathcal{H}_n} : \nabla [\mathcal{R}] \to \mathcal{H}_n \subset \mathcal{M}_n
\]

be the \((L^2-)\)orthogonal projection onto \(\mathcal{M}_n\) defined by

\[
\Pi_{\mathcal{H}_n} h = \Pi_{\mathcal{H}_n} \left( \sum_{i=1}^{\infty} \beta_i \nabla \psi_i^R \right) := \sum_{i=1}^{2n} \beta_i \nabla \psi_i^R_{\Omega} \quad \text{(3.64), (3.33)}
\]

The infinite sum above is well-defined as the basis representation \(R = \sum_{i=1}^{\infty} \beta_i \psi_i^R\) for the mean value free potential, satisfying \(\nabla R = h\), converges with respect to the \(R\)-norm (cf. (3.12)) which dominates the \(H^1\)-norm. Moreover let

\[
\begin{align*}
\Pi_{\mathcal{C}_n} : & \mathcal{C}_L^{L^2(\Omega)} = L^2(\Omega) \to \mathcal{C}_n, \\
\Pi_{\mathcal{U}_n} : & \mathcal{U}_L^{L^2(\Omega)^d} = H_{n0}(\text{div}_0)(\Omega) \to \mathcal{U}_n, \\
\Pi_{\mathcal{R}_n} : & \mathcal{R} \to \mathcal{R}_n, \\
\Pi_{\mathcal{M}_n} : & \mathcal{M}_L^{L^2(\Omega)^d} = L^2(\Omega)^d \to \mathcal{M}_n
\end{align*}
\]

be the projections defined by

\[
\begin{align*}
\Pi_{\mathcal{C}_n} g = & \Pi_{\mathcal{C}_n} \left( \sum_{i=1}^{\infty} \beta_i \psi_i^C \right) := \sum_{i=1}^{2n} \beta_i \psi_i^C, \\
\Pi_{\mathcal{U}_n} u = & \Pi_{\mathcal{U}_n} \left( \sum_{i=1}^{\infty} \beta_i \Psi_i^u \right) := \sum_{i=1}^{2n} \beta_i \Psi_i^u, \\
\Pi_{\mathcal{R}_n} S = & \Pi_{\mathcal{R}_n} \left( \sum_{i=1}^{\infty} \beta_i \psi_i^R \right) := \sum_{i=1}^{2n} \beta_i \psi_i^R, \\
\Pi_{\mathcal{M}_n} \Phi = & \Pi_{\mathcal{M}_n} \left( \sum_{i=1}^{\infty} \beta_i \Psi_i^m \right) := \sum_{i=1}^{2n} \beta_i \Psi_i^m.
\end{align*}
\]

Notice that \(\Pi_{\mathcal{C}_n}, \Pi_{\mathcal{U}_n}, \Pi_{\mathcal{M}_n}\) are also \(L^2\)-orthogonal projection operators. The projections satisfy the following convergence and stability results.

**Lemma 3.23.** The following convergences hold true.

\[
\begin{align*}
\forall c \in L^2(\Omega) : & \Pi_{\mathcal{C}_n} c \xrightarrow{n \to \infty} c \quad \text{in } L^2(\Omega), \\
\forall c \in H^1(\Omega) : & \Pi_{\mathcal{C}_n} c \xrightarrow{n \to \infty} c \quad \text{in } H^1(\Omega), \\
\forall c \in H^2_c(\Omega) : & \Pi_{\mathcal{C}_n} c \xrightarrow{n \to \infty} c \quad \text{in } H^2(\Omega), \\
\forall u \in \mathcal{U} : & \Pi_{\mathcal{U}_n} u \xrightarrow{n \to \infty} u \quad \text{in } H^3(\Omega)^d, \\
\forall R \in \mathcal{R} : & \Pi_{\mathcal{R}_n} R \xrightarrow{n \to \infty} R \quad \text{in } \mathcal{R}, \\
\forall m \in \mathcal{M} : & \Pi_{\mathcal{M}_n} m \xrightarrow{n \to \infty} m \quad \text{in } H^3(\Omega)^d.
\end{align*}
\]

**Proof:** The first claim is a standard result. The eigenfunctions are \(H^4\)-regular due to \([H1 3.1c]\), are elements of \(H^2_c(\Omega)\) and are a complete orthogonal set in \(H^1(\Omega)\) as well as in \(H^2_c(\Omega)\), which was proven in \([37]\). The last three claims are true by construction, see Section 3.1.

Moreover, for orthogonal projections one has stability estimates in a standard way.
Lemma 3.24. Independently of $n \in \mathbb{N}$,

$$\|\Pi_{e_n} c\|_{L^2(\Omega)} \leq \|c\|_{L^2(\Omega)} \quad \forall c \in L^2(\Omega),$$

$$\|\nabla \Pi_{e_n} c\|_{L^2(\Omega)^d} \leq \|\nabla c\|_{L^2(\Omega)^d} \quad \forall c \in H^1(\Omega),$$

(3.84)

$$\|\Pi_{u_n} u\|_{H^3(\Omega)^d} \leq \|u\|_{H^3(\Omega)^d} \quad \forall u \in \mathcal{U},$$

$$\exists C > 0 : \|\Pi_{\mathcal{M}_n} m\|_{H^3(\Omega)^d} \leq C \|m\|_{H^3(\Omega)^d} \quad \forall m \in \mathcal{M}$$

and

$$\exists C > 0 : \|\Pi_{e_n} \psi\|_{H^2(\Omega)} \leq C \|\psi\|_{H^2(\Omega)} \quad \forall \psi \in H^2(\Omega),$$

(3.85)

as well as

$$\|\Pi_{\mathcal{M}_n} h\|_{L^2(\Omega)^d} \leq \|h\|_{L^2(\Omega)^d} \quad \forall h \in \nabla [\mathcal{H}],$$

$$\exists C > 0 : \|\Pi_{\mathcal{M}_n} h\|_{H^3(\Omega)^d} \leq C \|h\|_{H^3(\Omega)^d} \quad \forall h \in \nabla [\mathcal{H}] \text{ with } h|_{\Omega} \in \mathcal{H}.$$  

(3.86)

Proof: The stability (3.85) was proven in [37]. The stability estimates in (3.84) are all proven similarly. Let $c \in L^2(\Omega)$ be given as $c = \sum_{i=1}^{\infty} \beta_i \psi_i^c$ (with respect to the $H^1$-norm). By orthogonality (3.69),

$$\|\Pi_{e_n} c\|_{L^2(\Omega)}^2 = \langle \Pi_{e_n} c, \Pi_{e_n} c \rangle_{L^2(\Omega)} \quad \text{Parseval} = \sum_{i=1}^{2n} \beta_i^2 \leq \sum_{i=1}^{\infty} \beta_i^2 \quad \text{Parseval} = \|c\|_{L^2(\Omega)}^2.$$ 

For the gradients of $c \in H^1(\Omega)$ one similarly obtains

$$\|\nabla \Pi_{e_n} c\|_{L^2(\Omega)^d}^2 = \langle \nabla \Pi_{e_n} c, \nabla \Pi_{e_n} c \rangle_{L^2(\Omega)^d} \quad \text{(3.70)}$$

$$= \sum_{i=1}^{2n} \beta_i^2 \| \nabla \psi_i^c \|_{L^2(\Omega)^d}^2 \leq \sum_{i=1}^{\infty} \beta_i^2 \| \nabla \psi_i^c \|_{L^2(\Omega)^d}^2 \quad \text{(3.70)}$$

$$\leq \sum_{i=1}^{\infty} \beta_i^2 \| \nabla \psi_i^c \|_{L^2(\Omega)^d}^2 \quad \text{(3.70)}$$

$$\leq \sum_{i=1}^{\infty} \beta_i^2 \| \nabla \psi_i^c \|_{L^2(\Omega)^d}^2 \quad \text{(3.70)}$$

For $\Pi_{u_n}$ the result is obtained in the same way. For the last one of those stability estimates the triangle inequality is applied first before the orthogonality properties of the bases of $\mathcal{S}$, $\mathcal{S}^o$, and $\mathcal{V}$ in $H^3(\Omega)^d$ will be used (cf. Lemma 3.7). In detail, let $m = \sum_{i=1}^{\infty} \beta_i \Psi_i^m$ in $H^3(\Omega)^d$, then one can obtain (cf. (3.33), (3.31)) representations of the individual summands of $m = m^s + m^{s\perp} + m^{h\perp}$ as follows,

$$m^s = \sum_{i=1}^{\infty} \gamma_i \Psi_i^m, \quad m^{s\perp} = \sum_{i=1}^{\infty} \gamma_{i-2} \Psi_{i-2}^\perp, \quad \text{and} \quad m^{h\perp} = \sum_{i=1}^{\infty} \gamma_{2i-1} \Psi_{2i-1}^\perp.$$  

(3.87)

Hence by adding up, $\sum_{i=1}^{\infty} \gamma_i \Psi_i^m = m$, but the basis representation of $m$ is unique which implies $\gamma_i = \beta_i$ for all $i \in \mathbb{N}$. From there,

$$\|\Pi_{\mathcal{M}_n} m\|_{H^3(\Omega)^d} = \left\| \sum_{i=1}^{2n} \beta_i \Psi_i^m \right\|_{H^3(\Omega)^d}$$

$$\leq \sum_{i=1}^{2n} \| \beta_i \Psi_i^m \|_{H^3(\Omega)^d} + \sum_{i=1}^{2n} \| \beta_{i-2} S_i^\perp \|_{H^3(\Omega)^d} + \sum_{i=1}^{n} \| \beta_{2i-1} M_i \|_{H^3(\Omega)^d}$$
\[ \gamma_i = \beta_i \leq H^3\text{-orthogonality} \]
\[ \| \mathbf{m}^a \|_{H^3(\Omega)^d} + \| \mathbf{m}^{a^+} \|_{H^3(\Omega)^d} + \| \mathbf{m}^{a^-} \|_{H^3(\Omega)^d} \leq \]
\[ 3.2 \]
\[ \| \mathbf{m} \|_{H^3(\Omega)^d} \]

follows.

The first claim in (3.86) follows by \( L^2\)-orthogonality (3.34). The remaining claim follows in a similar way as the stability of \( \Pi_{M_n} \) above by means of the triangle inequality and \( H^3\)-orthogonality of the bases of \( \mathcal{S} \) and \( \mathcal{S}' \).

The next step is to establish results for space-time convergence of the projections. For convenience, recall the definitions of \( \mathcal{C} \) (cf. (3.15)), \( \mathcal{U} \) (cf. (3.14)) and \( \mathcal{M} \) (cf. (3.7)).

**Lemma 3.25.** For arbitrary \( p \in [1, \infty) \) the following holds.
\[ \forall c \in L^p(I; L^2(\Omega)) : \Pi_{e_n} c \xrightarrow{n \to \infty} c \text{ in } L^p(I; L^2(\Omega)), \]
\[ \forall c \in L^p(I; H^1(\Omega)) : \Pi_{e_n} c \xrightarrow{n \to \infty} c \text{ in } L^p(I; H^1(\Omega)), \]
\[ \forall c \in L^p(I; \mathcal{C}) : \Pi_{e_n} c \xrightarrow{n \to \infty} c \text{ in } L^p(I; H^2(\Omega)), \]
\[ \forall u \in L^p(I; \mathcal{U}) : \Pi_{u_n} u \xrightarrow{n \to \infty} u \text{ in } L^p(I; H^3(\Omega)^d), \]
\[ \forall \mathbf{m} \in L^p(I; \mathcal{M}) : \Pi_{M_n} \mathbf{m} \xrightarrow{n \to \infty} \mathbf{m} \text{ in } L^p(I; H^3(\Omega)^d). \]

**Proof:** Use the dominated convergence theorem for which the stability estimates from Lemma 3.24 yield the majorant. As an example, consider the first claim. For almost all \( t \in I \) the pointwise convergence \( \Pi_{e_n} c(t) - c(t) \to 0 \) in \( L^2(\Omega) \) follows from Lemma 3.23.

\[ \| \Pi_{e_n} c - c \|_{L^2(\Omega)} \leq \| \Pi_{e_n} c \|_{L^2(\Omega)} + \| c \|_{L^2(\Omega)} \leq 2 \| c \|_{L^2(\Omega)} \in L^p(I) \]

shows the existence of an integrable majorant. Hence, the space-time convergence follows. All other claims follow by the same procedure.

**Lemma 3.26.** From assumption (H2 3.2), i.e. \( \mathbf{h}_a \in H^1(I; H^3(\Omega)^d \cap H^1(\Omega)^d \cap H(\text{div}_0, \text{curl}_0)(\Omega')) \), it follows for almost all \( t \in I \) that

i) \( \mathbf{h}_a(t) \in \nabla[\mathcal{R}] \) and \( \mathbf{h}_a(t)|_{\partial \Omega} \in \partial \mathcal{K}, \)

ii) \( \Pi_{\mathcal{K}_t} \mathbf{h}_a(t) \to \mathbf{h}_a(t)|_{\partial \Omega} \text{ in } H^3(\Omega)^d. \)

Moreover,

iii) \( \Pi_{\mathcal{H}_t} \mathbf{h}_a \to \mathbf{h}_a|_{\partial \Omega} \text{ in } L^2(I; H^3(\Omega)^d). \)

**Proof:** Let \( t \in I \) be fixed such that \( \mathbf{h}_a(t) \in H^3(\Omega)^d \cap H^1(\Omega)^d \cap H(\text{div}_0, \text{curl}_0)(\Omega^t) \). For the ease of notation the time argument will not be denoted in the following. Due to \( \text{curl} \mathbf{h}_a = \mathbf{0} \) (on simply connected \( \Omega^t \), cf. (H1 3.1)) there exists a potential \( \mathbf{h}_a = \nabla R_a \), see Lemma A.10. Recalling the definition of \( \mathcal{R} \), see (3.8), the first claim of i) follows.

For the second claim see that in particular \( \mathbf{h}_a|_{\partial \Omega} \in H^3(\Omega)^d \cap H(\text{div}_0)(\Omega) \cap \nabla[H^1(\Omega)] \). From there it is evident that \( \mathbf{h}_a|_{\partial \Omega} \in \nabla[H^1(\Omega)] = \nabla[H^1_{\text{mean}}(\Omega) \cap H^1(\Omega)] \) and \( \text{div} \mathbf{h}_a|_{\partial \Omega} = 0 \), which implies \( \mathbf{h}_a|_{\partial \Omega} \in \mathcal{K} \) (cf. (3.17)). Now by Lemma 3.5 the gradient field \( \mathbf{h}_a|_{\partial \Omega} \) decomposes into two parts
\[ \mathbf{h}_a|_{\partial \Omega} = \mathbf{s} + \mathbf{s}^\perp, \quad \mathbf{s} \in \mathcal{S}, \quad \mathbf{s}^\perp \in \mathcal{S}', \quad \text{where } \mathbf{s}^\perp L^2(\Omega)^d \mathbf{s}^\perp \text{ (see (3.19)).} \]

As \( \mathbf{s} = \nabla \varphi_0 \) for some \( \varphi_0 \in H^1_0(\Omega) \) (cf. (3.18)) one infers
\[ \| \mathbf{s} \|_{L^2(\Omega)^d}^2 = \int_{\Omega} \mathbf{h}_a \cdot \mathbf{s} \, dx = -\int_{\Omega} \text{div} \mathbf{h}_a \varphi_0 \, dx + \int_{\partial \Omega} \mathbf{h}_a \cdot \mathbf{n} \varphi_0 \, d\sigma = 0. \]

The second claim of i) follows.
Consider \( h_a = \nabla R_a \in \nabla[R] \) with basis representation \( \| R_a - \lim_{n \to \infty} \sum_{i=1}^{2n} \alpha_i^R \psi_i^R \|_R \to 0 \). Then by \( \| \cdot \|_R \approx \| \cdot \|_{H(\text{div})(\Omega)} + \| \cdot \|_{L^2(\Omega)^d} \) (cf. (3.12)) one obtains

\[
\| h_a \|_\Omega = \lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i^R \Psi_i^m \|_{H(\text{div})(\Omega)} \leq \| \nabla R_a \|_\Omega = \lim_{n \to \infty} \sum_{i=1}^{2n} \alpha_i^R \psi_i^R \|_{H(\text{div})(\Omega)} \leq \| R_a - \lim_{n \to \infty} \sum_{i=1}^{2n} \alpha_i^R \psi_i^R \|_R = 0.
\]

From \( h_a \in H \) one obtains another basis representation \( \| h_a \|_\Omega = \lim_{n \to \infty} \sum_{i=1}^{n} \beta_i \Psi_i^m \|_{H^3(\Omega)^d} \to 0 \) in terms of the basis of \( H \) with convergence in the \( H^3 \)-norm. Both representations of \( h_a \) converge with respect to the \( L^2 \)-norm (at least). Due to \( L^2 \)-orthogonality (cf. (3.34)) the coefficients must coincide, i.e. the projections \( \Pi_{h_a} \) converge strongly in \( H^3(\Omega)^d \). This concludes the proof of ii). Finally, the same technique which was used in the proof of Lemma 3.25 combined with (3.86) yields iii).

**Remark 3.27.** This is a supplementary remark about the projection operators in (3.82) in the case \( \Omega = \Omega' \) and the results about them. In (3.81) a basis expansion was used, i.e.

\[
\Pi_{h_a} \left( \sum_{i=1}^{\infty} \beta_i \psi_i^R \right) = \sum_{i=1}^{2n} \beta_i \psi_i^R.
\]

In the setting \( \Omega = \Omega' \) the functions \( \psi_i^R \), \( i \in \mathbb{N} \), vanish (cf. Remark 3.26) and the respective coefficients are not determined uniquely. At the same time, those functions vanish on the right-hand side above, too, which guarantees the operator to be well-defined. The results in (3.83) remain true as \( \Pi_{h_a} R \to R \) in \( R \) per construction. In Lemma 3.26 the first claim in i) is obvious by definition, and for the other claims only functions \( \psi_i^R \), \( i \in \mathbb{N} \), with even index play a role. Therefore all definitions or results related to \( R \) so far remain valid.

The Galerkin scheme for approximate solutions will be introduced next. How a reasonable weak formulation might look like can be seen in Section 2.4.1. The scheme will be based on that and the ansatz

\[
\begin{align*}
    u_n(t, x) &:= \sum_{i=1}^{2n} \alpha_i^u(t) \Psi_i^u(x) \forall (t, x) \in I \times \Omega, \\
    c_n(t, x) &:= \sum_{i=1}^{2n} \alpha_i^c(t) \psi_i^c(x) \forall (t, x) \in I \times \Omega, \\
    R_n(t, x) &:= \sum_{i=1}^{2n} \alpha_i^R(t) \psi_i^R(x) \forall (t, x) \in I \times \Omega', \\
    m_n(t, x) &:= \sum_{i=1}^{2n} \alpha_i^m(t) \Psi_i^m(x) \forall (t, x) \in I \times \Omega,
\end{align*}
\]

where for all \( i \in \mathbb{N} \), the coefficients are supposed to satisfy \( \alpha_i^u, \alpha_i^c, \alpha_i^m \in C^1(I) \) and \( \alpha_i^R \in H^1(I) \). Notice that the equations for the magnetic potential, (2.89f) and (2.89g), do not include a time derivative. This is why no classical differentiability is needed. This allows for weak differentiability of the external magnetic field \( h_a \) with respect to time, see (H2 3.2). Above representation can
also be seen as a mapping of the coefficients, i.e.

$$u_f : \mathbb{R}^{2n} \to L^\infty(\Omega)^d, \quad \beta = (\beta_i)_{i=1,\ldots,2n} \mapsto \sum_{i=1}^{2n} \beta_i \Psi_i^u,$$

$$c_f : \mathbb{R}^{2n} \to L^\infty(\Omega), \quad \beta = (\beta_i)_{i=1,\ldots,2n} \mapsto \sum_{i=1}^{2n} \beta_i \psi_i^c,$$

$$R_f : \mathbb{R}^{2n} \to L^\infty(\Omega'), \quad \beta = (\beta_i)_{i=1,\ldots,2n} \mapsto \sum_{i=1}^{2n} \beta_i \psi_i^R,$$

$$m_f : \mathbb{R}^{2n} \to L^\infty(\Omega)^d, \quad \beta = (\beta_i)_{i=1,\ldots,2n} \mapsto \sum_{i=1}^{2n} \beta_i \Psi_i^m,$$

yields

$$u_f((\alpha^u_i(t))_{i=1,\ldots,2n}) = u_n(t, \cdot),$$

$$c_f((\alpha^c_i(t))_{i=1,\ldots,2n}) = c_n(t, \cdot),$$

$$R_f((\alpha^R_i(t))_{i=1,\ldots,2n}) = R_n(t, \cdot),$$

$$m_f((\alpha^m_i(t))_{i=1,\ldots,2n}) = m_n(t, \cdot).$$

(3.91)

Naturally, due to the density results

$$H^{1/2}_0(\Omega)^d \equiv \mathcal{H}^{1/2}(\Omega)^d, \quad H^1(\Omega) L^2(\Omega) = L^2(\Omega), \quad \mathcal{M}^{L^2(\Omega)^d} \equiv L^2(\Omega)^d,$$

it is possible to choose discrete data $u_n^0 \in U_n$, $e_n^0 \in C_n$ and $m_n^0 \in M_n$ for each $n \in \mathbb{N}$ in such a way that

$$u_n^0 \to u^\text{init} \text{ in } L^2(\Omega)^d, \quad c_n^0 \to c^\text{init} \text{ in } L^2(\Omega), \quad m_n^0 \to m^\text{init} \text{ in } L^2(\Omega)^d,$$

(3.92)

for $n \to \infty$, e.g. via the $L^2$-orthogonal projection operators, for any initial data $u^\text{init} \in H^{1/2}_0(\Omega)^d$, $c^\text{init} \in L^2(\Omega)$ and $m^\text{init} \in L^2(\Omega)^d$ from the continuous setting.

Recalling [Lemma 3.26] the applied magnetic field $h_n(t)$ at time $t \in I$ is in the domain of the projection operator $\Pi_{g_n}$. Hence, we can propose the following Galerkin scheme.

**Definition 3.28.** Let $\sigma > 0$ and $0 < s < e < L < \infty$, where $e$ is Euler’s number. For arbitrary $n \in \mathbb{N}$, functions $(u_n, c_n, R_n, m_n) \in X_n$ (cf. (3.80)) are called a Galerkin solution if

$$\rho_0 \int_\Omega \partial_t u_n(t) \cdot \Psi_j^u \, dx + \int_\Omega 2\eta \nabla u_n(t) \cdot D\Psi_j^u \, dx$$

$$+ \frac{\rho_0}{2} \int_\Omega (u_n(t) \cdot \nabla) u_n(t) \cdot \Psi_j^u \, dx - \frac{\rho_0}{2} \int_\Omega (u_n(t) \cdot \nabla) \Psi_j^u \cdot u_n(t) \, dx$$

$$= \mu_0 \int_\Omega (\Psi_j^u \cdot \nabla)(\alpha_1 \nabla R_n(t) + \frac{\beta}{2} \Pi_{g_n} h_n(t)) \cdot m_n(t) \, dx$$

$$+ \frac{\mu_0}{2} \int_\Omega (m_n(t) \times (\alpha_1 \nabla R_n(t) + \frac{\beta}{2} \Pi_{g_n} h_n(t))) \cdot \text{curl} \Psi_j^u \, dx$$

$$- D \int_\Omega (c_n(t))^s \nabla \Pi_{e_n}(g_n^L)(c_n(t)) \cdot \Psi_j^u \, dx,$$

$$\int_\Omega \partial_t c_n(t) \psi_j^c \, dx - \int_\Omega (c_n(t))^s u_n(t) \cdot \nabla \psi_j^c \, dx + \sigma_c \int_\Omega \nabla c_n(t) \cdot \nabla \psi_j^c \, dx$$

$$- \int_\Omega K_f \partial_t (c_n(t)) \left( - D \nabla \Pi_{e_n}(g_n^L)(c_n(t)) ight) \cdot \nabla \psi_j^c \, dx,$$

$$+ \frac{\rho_0}{(c_n(t))^s} (\nabla (\alpha_1 \nabla R_n(t) + \frac{\beta}{2} \Pi_{g_n} h_n(t) - \alpha_3 m_n(t))^T m_n(t)) \cdot \nabla \psi_j^c \, dx = 0,$$

(3.93a)

(3.93b)
\[ \int_\Omega \nabla R_n(t) \cdot \nabla \psi_j^R \, dx = \int_\Omega h_n(t) \cdot \nabla \psi_j^R \, dx - \int_\Omega m_n(t) \cdot \nabla \psi_j^R \, dx, \]  
\tag{3.93c}
\]
\[
\int_\Omega \partial_t m_n(t) \cdot \Psi_j^m \, dx
- \int_\Omega \left( \left( u_n(t) + \frac{Kf(c_n(t))}{(c_n(t))^2} \right) - D \nabla \Pi_{c_n} \left( g_s^L \right) \right) \cdot \nabla \nabla \psi_j^R \, dx
+ \frac{\mu_m}{(c_n(t))^r} \left( \nabla (c_n(t))^2 \right) \cdot \nabla \nabla \psi_j^R \, dx
+ \sigma \int_\Omega \text{div} \, m_n(t) \, dx + \sigma \int_\Omega \text{curl} \, m_n(t) \, dx
\]
\[
= \frac{1}{2} \int_\Omega (m_n(t) \times \Psi_j^m) \cdot \text{curl} \, u_n(t) \, dx
- \frac{1}{\tau_{rel}} \int_\Omega (m_n(t) - \chi(c_n(t), \nabla R_n(t)) \nabla R_n(t)) \cdot \Psi_j^m \, dx,
\tag{3.93d}
\]
for all \( j = 1, \ldots, 2n \) and
\[ u_n(0) = u_n^0 \in \mathcal{U}_n, \quad c_n(0) = c_n^0 \in \mathcal{C}_n, \quad m_n(0) = m_n^0 \in \mathcal{M}_n. \tag{3.93e} \]

Remark 3.29. The terms in (3.93) are well-defined under the assumptions (H1 3.1) and (H2 3.2) from the beginning of Section 3.

Recall that by (H1 3.1) the basis functions used to define the Galerkin ansatz spaces (3.79) are well-defined, see also Section 3.1. First, check the domains of the projection operators. Due to Lemma 3.26, \( h_n \) is in the domain of \( \Pi_{c_n} \). As \( c_n \) is a linear combination of \( H^4 \)-regular functions, \( (g_s^L)'(c_n) \) is clearly a \( L^2 \)-regular function, i.e. it is in the domain of \( \Pi_{c_n} \).

As the space dimension is at most \( d \leq 3 \), by Sobolev’s embedding all terms are well-defined in terms of integrability. In detail, the regularity of the basis functions implies that all test functions \( \nabla \Psi_j^m, \nabla \psi_j^R, \nabla \psi_j^R \mid \Omega, \nabla \Psi_j^m \) are \( L^\infty \)-functions and consequently the unknowns \( \nabla u_n, \nabla c_n, \nabla m_n, \nabla \nabla R_n \mid \Omega \) are \( L^\infty \)-functions for any fixed time. Additionally, (H2 3.2) guarantees sufficient regularity for \( h_n \).

In fact, (3.93) is a system of ordinary differential equations which can be written in explicit form. This can easily be verified by using (3.91).

Hence, find
\[
\alpha^u := (\alpha^u_1, \ldots, \alpha^u_{2n}) \in C(I)^{2n},
\]
\[
\alpha^c := (\alpha^c_1, \ldots, \alpha^c_{2n}) \in C(I)^{2n},
\]
\[
\alpha^m := (\alpha^m_1, \ldots, \alpha^m_{2n}) \in C(I)^{2n},
\]
\[
\alpha^R := (\alpha^R_1, \ldots, \alpha^R_{2n}) \in H(I)^{2n},
\]
such that

the equations from (3.93) hold for all \( j = 1, \ldots, 2n \)
where \( u_n, c_n, R_n, m_n \) will be replaced
with \( u_f(\alpha^u), c_f(\alpha^c), R_f(\alpha^R), m_f(\alpha^m) \), respectively. \tag{3.94}

Keep in mind that (3.93c) is not part of a differential equation as there is no time derivative. By using the stiffness matrix \( L^R \), where \( (L^R)_{i,j} := (\nabla \psi_i^R, \nabla \psi_j^R)_{L^2(\Omega)^d}, i, j = 1, \ldots, 2n \), one can
rewrite the equation via (3.91) and exploiting the basis representation (3.90) as follows,

\[
L^R \begin{pmatrix}
\alpha_1^R(t) \\
\vdots \\
\alpha_{2n}^R(t)
\end{pmatrix} = F^R(t, \alpha^m(t))
\]

(3.95)

for some \( F^R : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) which resembles the right-hand sides of the 2n equations obtained from (3.93c) for \( j = 1, \ldots, 2n \) in view of (3.94). By construction, the basis functions \( \{ \psi^R \}_i \in \mathbb{N} \) have vanishing mean value. Hence, Poincaré’s inequality is available to prove that \( L^R \) is positive definite, i.e.,

\[
(\alpha^R)^T L^R \alpha^R = \int_{\Omega} |\nabla R_f(\alpha^R)|^2 \, dx = 0
\]

if and only if \( R_f(\alpha^R) = 0 \), which is equivalent to \( \alpha^R = 0 \). Therefore, for any \( \alpha \in \mathbb{R}^{2n} \) a solution \( \tilde{\alpha}^R \in \mathbb{R}^{2n} \) of \( L^R \tilde{\alpha}^R = F^R(t, \alpha) \) exists. Hence, by replacing \( R_n(t) \) in (3.93) with \( \tilde{R}(t, m_n(t)) \), where

\[
\tilde{R} : \mathbb{R} \times M_n \to \mathcal{R}_n
\]

(3.96)

the magnetic potential does not need to be considered as a variable of the system (3.93) of ordinary differential equations anymore.

The mass matrix \( M^u \) with \( M^u_{i,j} := \langle \psi^i_i, \psi^j_j \rangle_{L^2(\Omega)^d} \), \( i, j = 1, \ldots, 2n \), is invertible. Therefore, (3.93a) can be written in the context of (3.94) as a system of 2n ordinary differential equations in explicit form. In detail, one has

\[
\rho_0 \partial_t M^u \begin{pmatrix}
\alpha_1^u(t) \\
\vdots \\
\alpha_{2n}^u(t)
\end{pmatrix} + F^u(t, \alpha^u(t), \alpha^c(t), \alpha^m(t)) = 0
\]

for some \( F^u : \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) which reflects the remaining terms of the 2n equations that are given by (3.93a) for \( j = 1, \ldots, 2n \). Notice that \( \alpha^R \) which can be seen as a function depending on \( t \) and \( \alpha^m \) (cf. (3.95)), does not need to appear in \( F^u \) explicitly. By inverting the mass matrix and dividing by \( \rho_0 \), the explicit form is obtained,

\[
\partial_t \alpha^u(t) = -\frac{1}{\rho_0} (M^u)^{-1} F^u(t, \alpha^u(t), \alpha^c(t), \alpha^m(t)).
\]

The analogous procedure can be done in all remaining equations of (3.94). Hence, theory of ordinary differential equations can be applied.

**Lemma 3.30.** The system of ordinary differential equations in (3.94), supplemented by an arbitrary initial condition, has a unique local solution.

**Proof:** By (H1 3.1) the terms of the scheme are well-defined, see Remark 3.29. First, observe that, due to \( H^1(I) \hookrightarrow C(I) \) (applied to \( h_a \)-terms) and all nonlinear components being continuous, the right-hand side of (3.94) is continuous with respect to space and time. The continuity in time particularly holds true for terms containing \( \Pi_{X_h} h_a \) due to the following reasons. For continuity estimates, the choice of norms is arbitrary because \( \dim X_h < \infty \). By choosing \( L^2 \)-norms whenever time differences of \( \Pi_{\mathcal{X}_h} h_a \) appear in the proof of continuity, stability of the projector \( \Pi_{\mathcal{X}_h} \) (cf. (3.86)) can be exploited to get rid of the projector and employ the time regularity of \( h_a \) again.
The overall goal is to prove a local Lipschitz-condition for the right-hand side of the system \((3.94)\) in explicit form. Inverted mass matrices (or stiffness matrices as in \((3.95)\)) do not hinder this condition. Hence, for simplicity, it suffices to examine the original right-hand side of \((3.94)\). Moreover, due to linearity and uniqueness of the basis representations \((3.90)\) and finite dimensionality of the Galerkin ansatz spaces, it suffices to consider the function-based scheme \((3.93)\) which significantly eases the notation of this proof’s steps – in contrast to the scheme based on the degrees of freedom – and each equation thereof can be treated individually. Further simplifications arise due to the following straightforward auxiliary results.

- Locally Lipschitz-continuous mappings are locally bounded.
- The product of two locally Lipschitz-continuous mappings is locally Lipschitz-continuous.
- The sum of two locally Lipschitz-continuous mappings is locally Lipschitz-continuous.

One can deduce that each summand on the right-hand side of \((3.93)\) can be treated separately. As the dimensions of the Galerkin ansatz spaces is finite, the integrals do not pose any problems as one can just choose the \(L^\infty\)-norm for Lipschitz-continuity in order to extract the integrands from the integrals in the respective estimates. If those integrands are products of locally Lipschitz-continuous functions, the whole term will be locally Lipschitz-continuous.

As an example, consider for fixed \(j \in \{1, \ldots, 2n\}\) the mapping

\[
\mathcal{U}_n \ni \tilde{u} \mapsto \frac{\rho_0}{2} \int_\Omega (\tilde{u} \cdot \nabla) \tilde{u} \cdot \Phi_j^u \, dx
\]

and estimate the difference of two function values

\[
\left| \frac{\rho_0}{2} \int_\Omega (\tilde{u}_1 \cdot \nabla) \tilde{u}_1 \cdot \Phi_j^u \, dx - \frac{\rho_0}{2} \int_\Omega (\tilde{u}_2 \cdot \nabla) \tilde{u}_2 \cdot \Phi_j^u \, dx \right|
\]

\[
\leq \frac{\rho_0}{2} \| \Phi_j^u \|_{L^\infty(\Omega)^d} \Omega \| \nabla \tilde{u}_1 - \nabla \tilde{u}_2 \|_{L^\infty(\Omega)}
\]

\[
\leq \frac{\rho_0}{2} \| \Phi_j^u \|_{L^\infty(\Omega)^d} \left( \| \tilde{u}_1 \|_{L^\infty(\Omega)^d} \| \nabla \tilde{u}_1 \|_{L^\infty(\Omega)^{d \times d}} + \| \nabla \tilde{u}_2 \|_{L^\infty(\Omega)^{d \times d}} \right)
\]

From there, local Lipschitz-continuity is evident as the terms \(\| \tilde{u}_1 \|_{L^\infty(\Omega)^d}\) and \(\| \nabla \tilde{u}_2 \|_{L^\infty(\Omega)^{d \times d}}\) will be bounded locally – on bounded balls for instance – and the choice of norms is arbitrary in finite dimensions. If on the right-hand side was a term of the type \(\| f_{\text{nonlinear}}(\tilde{u}_1) - f_{\text{nonlinear}}(\tilde{u}_2) \|_{L^\infty(\Omega)}\) for some nonlinear but locally Lipschitz-continuous \(f_{\text{nonlinear}}\), then the latter local Lipschitz-continuity can be exploited and the local Lipschitz-continuity of the total term would follow. Hence, the above example serves as a blueprint how to treat all terms which have integrands that are the product of locally Lipschitz-continuous functions. Additionally, this method extends to the case when terms depend on the time variable \(t\) explicitly and satisfy a local Lipschitz-condition. Then, in the same way a local Lipschitz-condition as required by the Picard-Lindelöf theorem is satisfied by confining oneself to a suitable time domain. Moreover, it is easy to see that the function \(R_n(t) = \tilde{R}(t, m_n(t)), \text{ see } (3.96)\), satisfies a Lipschitz condition as the explicitly time depending term \(h_n(t)\) in \((3.93c)\) is detached from the magnetization term.

The only terms that require further investigation are

- terms containing projection operators

as continuity properties need to be verified. All other terms can be treated as explained before, because all occurring nonlinear mappings are at least locally Lipschitz-continuous.
Actually, there are terms containing $h_a$, i.e. being non-autonomous, and a projection operator simultaneously, e.g.

$$(t,\tilde{m}) \mapsto \int_{\Omega} (\Psi^u_j \cdot \nabla) \Pi_{\mathcal{H}_n} h_a(t) \cdot \tilde{m} \, dx,$$

which is an extract from the right-hand side of (3.93). However, due to stability estimates, one can get rid of the projection operator and the regularity of $h_a$ with respect to time yields the Lipschitz-condition. In detail, for a fixed $j \in \{1, \ldots, 2n\}$ see

$$\left| \int_{\Omega} (\Psi^u_j \cdot \nabla) \Pi_{\mathcal{H}_n} h_a(t) \cdot \tilde{m}_1 \, dx - \int_{\Omega} (\Psi^u_j \cdot \nabla) \Pi_{\mathcal{H}_n} h_a(t) \cdot \tilde{m}_2 \, dx \right| \lesssim \|\Psi^u_j\|_{L^\infty(\Omega)} \|\Pi_{\mathcal{H}_n} h_a(t)\|_{L^2(\Omega)^d} \|\tilde{m}_1 - \tilde{m}_2\|_{L^\infty(\Omega)^d},$$

where the norm of $\nabla \Pi_{\mathcal{H}_n} h_a(t)$ has been exchanged as $\dim \mathcal{H}_n < \infty$. Keep in mind that for existence of discrete solutions for fixed $n \in \mathbb{N}$ the estimates do not need to be uniform. Exploiting stability (3.86), the projector can be discarded and Sobolev’s embedding guarantees boundedness of the remaining $h_a$-term with respect to time. Hence, the Lipschitz-condition is satisfied (globally on $I$). Other terms of this type can be treated analogously.

It remains to discuss terms which contain the projector $\Pi_{\mathcal{G}_n}$. The most complicated occurrence will be discussed in detail. The treatment of the other candidates is analogous. From the left-hand side of (3.93) extract for fixed $j \in \{1, \ldots, 2n\}$ the mapping

$$(\tilde{c}, \tilde{m}) \mapsto \int_{\Omega} \left( K f_p(\tilde{c}) \left( \frac{\Pi_{\mathcal{G}_n}}{c_n(t)} \right) s \nabla \Pi_{\mathcal{G}_n} (g^L_s)'(\tilde{c}) \cdot \nabla \right) \Psi^m_j \cdot \tilde{m} \, dx.$$

Note that $\hat{c} \mapsto \frac{f_p(\hat{c})}{c_n(t)}$ is locally Lipschitz-continuous due to $s > 0$ being fixed. Moreover, the last factor above, i.e. $\tilde{m} \mapsto (\nabla \Psi^m_j)' \tilde{m}$, is Lipschitz-continuous. Hence, it suffices to prove local Lipschitz-continuity of $\hat{c} \mapsto \nabla \Pi_{\mathcal{G}_n} (g^L_s)'(\hat{c})$ in order to conclude the local Lipschitz-continuity of the product of the three aforementioned factors. As $\dim \mathcal{G}_n < \infty$ the choice of norms is arbitrary. It holds

$$\|\nabla \Pi_{\mathcal{G}_n} (g^L_s)'(\hat{c}_1) - \nabla \Pi_{\mathcal{G}_n} (g^L_s)'(\hat{c}_2)\|_{L^2(\Omega)^d} \overset{\text{stability}}{\lesssim} \|\nabla (g^L_s)'(\hat{c}_1) - \nabla (g^L_s)'(\hat{c}_2)\|_{L^2(\Omega)^d} \overset{\dim \mathcal{G}_n < \infty}{\lesssim} \|g^L_s)'(\hat{c}_1) - (g^L_s)'(\hat{c}_2)\|_{L^2(\Omega)}.$$

By local Lipschitz-continuity of $(g^L_s)'$ the result follows.

By Picard-Lindelöf’s theorem, the system (3.94) above has a unique local solution that attains any prescribed initial data. Let the domain of the solution be $[0, \hat{T})$. Notice that due to weak differentiability of $h_a$ in time and $C^1$-regularity (on $[0, \hat{T})$) of the solution $m_n$, the magnetic potential $R_n = R_n(\cdot, \cdot, m_n(\cdot))$ is at least $H^1$-regular on $[0, \hat{T})$ for any $\hat{T} < \hat{T}$. Here, $\hat{T}$ needs to be less than $\hat{T}$ to guarantee boundedness of $m_n$ and $\partial_t m_n$ with respect to time (implying $L^2$-integrability) and conclude $H^1$-regularity of $m_n$ in time. 

For global existence, Gronwall’s lemma will be used to obtain a priori estimates which allow to prove boundedness of the solution from Lemma 3.30. Hence, the following result on the existence of Galerkin solutions is obtained.

**Lemma 3.31.** Let $\sigma_\sigma > 0$, $0 < s < e < L < \infty$ ($e$ is Euler’s number). Under the assumptions [H1 3.1], [H2 3.2] and [H3 3.3] the following is true.
For all \( n \in \mathbb{N} \) the system (3.93), supplemented with initial data \( u_0^m \in U_n, c_0 \in \mathbb{C}_n, m_0^m \in M_n \) as in (3.92), has a unique solution \( (u_n, c_n, R_n, m_n) \in \mathcal{X}_n \) globally on \( I \). The sequence of Galerkin solutions \( (u_n, c_n, R_n, m_n)_{n \in \mathbb{N}} \) satisfies the a priori estimate

\[
\frac{\mu_0}{2} \sup_{t \in I} \| u_n(t) \|^2_{L^2(\Omega)} + D \sup_{t \in I} \int_{\Omega} g^L_n(c_n(t)) \, dx + \frac{\mu_0 \alpha_3}{4} \sup_{t \in I} \| m_n(t) \|^2_{L^2(\Omega)} d^d + \frac{\mu_0 \alpha_1}{2} \sup_{t \in I} \| R_n(t) \|^2_{L^2(\Omega)} d + \mu_0 \alpha_0 \| \nabla R_n \|^2_{L^2(\Omega)} d + D \sigma_c \| \nabla (g^L_n)''(c_n) \|_{L^2(\Omega)} d + K \| f_p(c_n) \|_{L^2(\Omega)} d + 2 \eta \| D\Pi \|_{L^2(\Omega)} d + \sigma \alpha_3 \mu_0 \| \nabla \|_{L^2(\Omega)} d + \sigma \alpha_3 \mu_0 \| \nabla \|_{L^2(\Omega)} d + \frac{\mu_0 \alpha_1}{4 \tau_{rel}} \| \nabla R_n \|^2_{L^2(\Omega)} d + \frac{\mu_0 \alpha_3}{4 \tau_{rel}} \| m_n \|^2_{L^2(\Omega)} d \leq C
\]

uniformly for \( n \in \mathbb{N} \) for some \( C > 0 \) which does only depend on \( T, |\Omega| \), initial data and \( h_n \) (as well as the fixed constants from the growth constraint of the susceptibility (3.4)).

**Proof:** Of course, a local solution of (3.94) is a local Galerkin solution of (3.93) as well via (3.91). Let

\[
(u_n, c_n, R_n, m_n) \in C^1([0, \hat{T}); \mathcal{U}) \times C^1([0, \hat{T}); \mathcal{C}) \times H^1([0, \hat{T}); \mathcal{R}) \times C^1([0, \hat{T}); \mathcal{M})
\]

be a local solution of (3.93) for some \( 0 < \hat{T} < T \). Its existence is guaranteed by Lemma 3.30. The local solution is indeed a global solution on \( I \) due to a priori estimates which will be proven next.

The general procedure is to multiply (3.93) for each \( t \in [0, \hat{T}) \) by degrees of freedom, i.e. real numbers \( a^m_n(t), \ldots, a_m^m(t) \), and sum up over all \( j = 1, \ldots, 2n \) for each set of \( 2n \) equations in (3.93a), (3.93b), (3.93c) or (3.93d), respectively. This way, a linear combination of the corresponding basis functions is obtained which can turn into any element of the discrete ansatz spaces \( \mathcal{U}_n, \mathcal{C}_n, \mathcal{R}_n \) or \( \mathcal{M}_n \), respectively. Conveniently, we will call this procedure 'testing' (3.93) with 'test functions'. Particularly, the local solutions are admissible test functions.

**Step 1: Basic integral identities.** Observe that \( \nabla [\mathcal{R}_n] \subset \mathcal{M}_n \) is satisfied by construction (cf. Section 3.1.2), which makes it possible to take the magnetic field \( h|_{\Omega} = \nabla R|_{\Omega}, R \in \mathcal{R}_n \), as test function for the magnetization equation (3.93d). As a first step, it will be tested

- (3.93a) by \( u_n \),
- (3.93b) by \( D\Pi \|_{\mathcal{C}_n} (g^L_n)'(c_n) \),
- (3.93c) by \( \mu_0 \alpha_1 R_n \),
- (3.93d) by \( -\mu_0 (\alpha_1 \nabla R_n|_{\Omega} + \frac{\sigma}{2} \Pi \|_{\mathcal{R}_n} h_n - \alpha_3 m_n) \)

and – noting that the local solutions are differentiable and \( h_n \) is weakly differentiable (cf. (H2 3.2)) in time – the weak time derivative of (3.93c) by \( \mu_0 \alpha_1 R_n \).

For the ease of presentation, the abbreviations

\[
\tilde{h}_n = \alpha_1 \nabla R_n|_{\Omega} + \frac{\sigma}{2} \Pi \|_{\mathcal{R}_n} h_n - \alpha_3 m_n \quad \text{and} \quad \tilde{b}_n = \tilde{h}_n - \alpha_3 m_n
\]

will be used. A rather involved computation is related to the nonlinear test function \( g'_n(c_n) \) which has to be projected onto \( \mathcal{C}_n \). By (at least) \( L^2 \)-regularity of \( c_n \) and linear growth of \( (g^L_n)''(c_n) \) (see
the term \(\Pi \xi_n(g^L_s)'(c_n)\) is well defined (see also Remark 3.29). One obtains from (3.93b) the identity

\[
D \int_{\Omega} \xi_n^c \Pi \xi_n(g^L_s)'(c_n) \, dx - D \int_{\Omega} (c_n)_a u_n \cdot \nabla \Pi \xi_n(g^L_s)'(c_n) \, dx \quad \text{(3.73)}
\]

\[
\begin{align*}
&+ D \sigma_c \int_{\Omega} \nabla \xi_n \cdot \nabla \Pi \xi_n(g^L_s)'(c_n) \, dx + K D^2 \int_{\Omega} f_p(c_n) |\nabla \Pi \xi_n(g^L_s)'(c_n)|^2 \, dx \\
&- KD \mu_0 \int_{\Omega} \frac{f_p(c_n)}{(c_n)_a} (\nabla b_n)^T m_n \cdot \nabla \Pi \xi_n(g^L_s)'(c_n) \, dx = 0
\end{align*}
\]

which turns by means of the chain-rule into

\[
D \partial_t \int_{\Omega} g^L_s(c_n) \, dx - D \int_{\Omega} (c_n)_a u_n \cdot \nabla \Pi \xi_n(g^L_s)'(c_n) \, dx
\]

\[
+ D \sigma_c \int_{\Omega} (g^L_s)'(c_n) |\nabla c_n|^2 \, dx + K D^2 \int_{\Omega} f_p(c_n) |\nabla \Pi \xi_n(g^L_s)'(c_n)|^2 \, dx
\]

\[
- KD \mu_0 \int_{\Omega} \frac{f_p(c_n)}{(c_n)_a} (\nabla b_n)^T m_n \cdot \nabla \Pi \xi_n(g^L_s)'(c_n) \, dx = 0
\]

where the differentiation with respect to time and the integration were interchanged at the first term of the left-hand side. It is straightforward to see why the latter interchanging is valid as the ansatz space \(\xi_n\) guarantees \(c_n(t) \in L^\infty(\Omega)\) and \(\partial_t c_n(t) \in L^\infty(\Omega)\) for fixed time \(t \in [0, \bar{T}]\). With that, also the time derivative \(\partial_t c_n(t)(g^L_s)'(c_n(t))\) of \((g^L_s)(c_n)\) is bounded, too. Hence, the pointwise in \(x \in \Omega\) converging difference quotient, for \(h \neq 0\), is bounded by

\[
\left| (g^L_s)(c_n(t+h, x)) - (g^L_s)(c_n(t, x)) \right| \leq \frac{1}{h} \int_{t}^{t+h} |\partial_t c_n(t,x)(g^L_s)'(c_n(t,x))| \, dt
\]

\[
\leq \left\| \partial_t c_n(t,x)(g^L_s)'(c_n(t,x)) \right\|_{L^\infty(\Omega)} \quad t \leq C < \infty
\]

uniformly in \(x \in \Omega\). The constant \(C\) is an (on bounded \(\Omega\)) integrable function. Hence by the dominated convergence theorem, the limit process \(h \to 0\) and spatial integration can be interchanged.

During the computations related to the other equations, integration and time differentiation need to be interchanged, too. In those cases, the same type of argument as above can be applied. Hence, the computations for the other equations are straightforward and one easily arrives at

\[
\frac{\rho_0}{2} \partial_t \int_{\Omega} |u_n|^2 \, dx + 2 \eta \int_{\Omega} [D u_n]^2 \, dx + \frac{\rho_0}{2} \int_{\Omega} (u_n \cdot \nabla) u_n \cdot u_n \, dx - \frac{\rho_0}{2} \int_{\Omega} (u_n \cdot \nabla) u_n \cdot u_n \, dx = 0
\]
3.2 Construction of approximate solutions

\[ \int_{\Omega} (\mathbf{u}_n \cdot \nabla) \tilde{\mathbf{b}}_n \cdot \mathbf{m}_n \, dx + \frac{\mu_0}{2} \int_{\Omega} (\mathbf{m}_n \times \tilde{\mathbf{h}}_n) \cdot \text{curl} \mathbf{u}_n \, dx \]
\[ = \int_{\Omega} (\mathbf{u}_n \cdot \nabla) \tilde{\mathbf{b}}_n \cdot \mathbf{m}_n \, dx + \frac{\alpha}{2} \int_{\Omega} \mathbf{u}_n \cdot \nabla |\mathbf{m}_n|^2 \, dx \]
\[ = 0 \text{ (integration by parts, div } \mathbf{u}_n = 0) \]
\[ - D \int_{\Omega} (c_n \sigma) \nabla \mathbf{c}_n (g^L_{n}) (c_n) \cdot \mathbf{u}_n \, dx, \]
\[ = \mu_0 \int_{\Omega} (\mathbf{u}_n \cdot \nabla) \tilde{\mathbf{b}}_n \cdot \mathbf{m}_n \, dx + \frac{\mu_0}{2} \int_{\Omega} (\mathbf{m}_n \times \tilde{\mathbf{h}}_n) \cdot \text{curl} \mathbf{u}_n \, dx \]
\[ - D \int_{\Omega} (c_n \sigma) \nabla \mathbf{c}_n (g^L_{n}) (c_n) \cdot \mathbf{u}_n \, dx, \]
\[ \text{and} \]
\[ \frac{\mu_0 \alpha_1}{\tau_{rel}} \int_{\Omega'} |\nabla R_n|^2 \, dx, \]
\[ \text{with } (3.99) \text{ and canceling out coupling terms, i.e.} \]
\[ \text{• the second term on the left-hand side of } (3.99) \text{ with the third term on the right-hand side of } (3.100), \]
\[ \text{• the first and second on the right-hand side of } (3.100) \text{ with the second on the left-hand side of } (3.102) \text{ and the first on the right-hand side of } (3.102), \]

yields
\[ \frac{D}{\omega} \int_{\Omega} g^L_{n} (c_n) \, dx + \frac{\rho_0}{2} \frac{\partial}{\partial t} \int_{\Omega} |\mathbf{u}_n|^2 \, dx + \frac{\mu_0 \alpha_1}{2} \frac{\partial}{\partial t} \int_{\Omega'} |\nabla R_n|^2 \, dx \]
\[ + D \sigma_\alpha \int_{\Omega} (g^L_{n})'' (c_n) |\nabla c_n|^2 \, dx + KD^2 \int_{\Omega} f_p(c_n) |\nabla \mathbf{c}_n (g^L_{n}) (c_n)|^2 \, dx \]
\[ - 2KD \mu_0 \int_{\Omega} \frac{f_p(c_n) K}{(c_n)^2} |(\nabla \tilde{\mathbf{b}}_n) (\mathbf{u}_n)^T \mathbf{m}_n \cdot \nabla \mathbf{c}_n (g^L_{n}) (c_n) \, dx + \mu_0^2 \int_{\Omega} \frac{f_p(c_n) K}{(c_n)^2} |(\nabla \tilde{\mathbf{b}}_n) (\mathbf{u}_n)^T \mathbf{m}_n|^2 \, dx \]
\[ + 2 \eta \int_{\Omega} |\text{Div } \mathbf{u}_n|^2 \, dx + \frac{\mu_0 \alpha_1}{\tau_{rel}} \int_{\Omega'} |\nabla R_n|^2 \, dx \]
\[ - \mu_0 \int_{\Omega} \frac{\partial}{\partial t} \mathbf{m}_n \cdot \tilde{\mathbf{b}}_n \, dx - \sigma \mu_0 \int_{\Omega} \text{ Div } \mathbf{m}_n \cdot \text{Div } \tilde{\mathbf{b}}_n \, dx - \sigma \mu_0 \int_{\Omega} \text{curl } \mathbf{m}_n \cdot \text{curl } \tilde{\mathbf{b}}_n \, dx \]
The third, fourth and fifth term on the left-hand side above are a square, i.e.

\[
K \int_{\Omega} f_p(c_n) \left| D \nabla \Pi_{c_n} (g_s^L)'(c_n) - \frac{\mu_0}{(c_n)^{3}} (\nabla b)^T m_n \right|^2 \, dx
= KD^2 \int_{\Omega} f_p(c_n) \left| \nabla \Pi_{c_n} (g_s^L)'(c_n) \right|^2 \, dx
- 2KD\mu_0 \int_{\Omega} \frac{f_p(c_n)}{(c_n)^{3}} (\nabla \hat{b}_n)^T m_n \cdot \nabla \Pi_{c_n} (g_s^L)'(c_n) \, dx
+ \mu_0^2 \int_{\Omega} \frac{f_p(c_n)K}{(c_n)^{2}} |(\nabla \hat{b}_n)^T m_n|^2 \, dx.
\]  

(3.105)

By expanding the abbreviations \((3.98)\) one gets

\[
- \mu_0 \int_{\Omega} \partial_t m_n \cdot \hat{b}_n \, dx = \frac{\mu_0\alpha_3}{2} \frac{1}{t} \int_{\Omega} \left| m_n \right|^2 \, dx
- \mu_0\alpha_1 \int_{\Omega} \partial_t m_n \cdot \nabla R_n \, dx
\]

(3.106)

and

\[
\frac{\mu_0}{\tau_{rel}} \int_{\Omega} \left( m_n - \chi(c_n, \nabla R_n) \nabla R_n \right) \cdot \hat{b}_n \, dx
= \frac{\mu_0}{\tau_{rel}} \int_{\Omega} m_n \cdot \hat{b}_n \, dx
- \frac{\mu_0}{\tau_{rel}} \int_{\Omega} \chi(c_n, \nabla R_n) \nabla R_n \cdot \hat{b}_n \, dx
\]

\[
= -\frac{\mu_0\alpha_3}{\tau_{rel}} \int_{\Omega} \left| m_n \right|^2 \, dx
+ \frac{\mu_0\alpha_1}{\tau_{rel}} \int_{\Omega} m_n \cdot \nabla R_n \, dx
+ \frac{\mu_0\beta}{2\tau_{rel}} \int_{\Omega} m_n \cdot \Pi_{3\alpha_n} h_n \, dx
\]

\[
+ \frac{\mu_0\alpha_3}{\tau_{rel}} \int_{\Omega} \chi(c_n, \nabla R_n) \nabla R_n \cdot m_n \, dx
- \frac{\mu_0\alpha_1}{\tau_{rel}} \int_{\Omega} \chi(c_n, \nabla R_n) \nabla R_n \left| m_n \right|^2 \, dx
\]

(3.107)

and

\[
- \sigma\mu_0 \int_{\Omega} \text{div} m_n \text{div} \hat{b}_n \, dx = \sigma\mu_0\alpha_3 \int_{\Omega} \left| \text{div} m_n \right|^2 \, dx
- \sigma\mu_0\alpha_1 \int_{\Omega} \text{div} m_n \text{div} \nabla R_n \, dx
\]

(3.108)

and

\[
- \sigma\mu_0 \int_{\Omega} \text{curl} m_n \cdot \text{curl} \hat{b}_n \, dx
= 0, \text{ also vanishes in case } d = 2, \text{ see } (A.25)
\]

\[
= \sigma\mu_0\alpha_3 \int_{\Omega} \left| \text{curl} m_n \right|^2 \, dx
- \sigma\mu_0\alpha_1 \int_{\Omega} \text{curl} m_n \cdot \text{curl} \nabla R_n \, dx
\]

vanishes as well

\[
\in \mathcal{K} \subset \nabla[H^1(\Omega)] \text{ (cf. } (3.17))
\]
Combining (3.104), (3.105), (3.106), (3.107), (3.108) and (3.109), one arrives at

\[
\frac{\rho_0}{2} \frac{\partial_t}{\partial t} \| \mathbf{u}_n \|_{L^2(\Omega)}^2 + D \partial_t \int g_s^L(c_n) \, dx + \frac{\mu_0 \alpha_3}{2} \partial_t \| \mathbf{m}_n \|_{L^2(\Omega)}^2 + \frac{\mu_0 \alpha_1}{2} \partial_t \| \nabla R_n \|_{L^2(\Omega')}^2 + 2\eta \| \mathbf{D} \mathbf{u}_n \|_{L^2(\Omega)}^2 + D \sigma \| \sqrt{(g_s^L(c_n))^\nu (c_n) \nabla c_n} \|_{L^2(\Omega)}^2 + \frac{\mu_0 \alpha_1}{\tau_{rel}} \| \nabla R_n \|_{L^2(\Omega')}^2 + K \sqrt{f_p(c_n)} |D \nabla \Pi_{\mathbb{H}}(g_s^L(c_n)) - \beta \Pi_{\mathbb{H}}(h_a) - \alpha_3 \nabla \mathbf{m}_n \|_{L^2(\Omega)}^2 + \frac{\mu_0 \alpha_1}{\tau_{rel}} \| \mathbf{m}_n \|_{L^2(\Omega)}^2 + \sigma \alpha_3 \mu_0 \| \nabla \mathbf{m}_n \|_{L^2(\Omega')}^2 + \sigma \beta \mu_0 \int \mathbf{m}_n \, d\mathbf{R}_n \, dx - \frac{\sigma \beta \mu_0}{2} \int \mathbf{m}_n \, d\Pi_{\mathbb{H}}(h_a) \, dx + \frac{\mu_0 \alpha_1}{\tau_{rel}} \int \chi(c_n, h_a) \nabla R_n \cdot \mathbf{m}_n \, dx + \frac{\mu_0 \beta}{2 \tau_{rel}} \int \chi(c_n, h_a) \nabla R_n \cdot \mathbf{m}_n \, dx + \frac{\mu_0 \beta}{2 \tau_{rel}} \int \chi(c_n, h_a) \nabla R_n \cdot \mathbf{m}_n \, dx + \frac{\mu_0 \beta}{2 \tau_{rel}} \int \chi(c_n, h_a) \nabla R_n \cdot \mathbf{m}_n \, dx,
\]

where

- the second term on the right-hand side of (3.106) together with the last term in (3.104),
- as well as the second term on the right-hand side of (3.107) together with the second term of the right-hand side in (3.104),

canceled out.

**Step 2: Auxiliary identity.** The identity

\[
- \int \mathbf{m}_n \, d\mathbf{R}_n \, dx = \| \nabla \mathbf{R}_n \|_{L^2(\Omega' \setminus \partial \Omega)} \quad (3.110)
\]

holds true. The proof is as follows.

First, one has to obtain more information about the term \( \nabla \mathbf{R}_n \). Notice that this notion is only defined on \( \Omega \) and \( \Omega' \setminus \overline{\Omega} \) separately, as \( \Delta \mathbf{R}_n \) is not necessarily defined on volumes including parts of \( \partial \Omega \). However, it can be identified with an \( L^2 \)-function, of course. Evidently,

\[
\nabla \mathbf{R}_n \big|_{\Omega' \setminus \overline{\Omega}} \equiv 0 \quad \text{and} \quad \nabla \mathbf{R}_n \big|_{\Omega} \in H^1_0(\Omega) \quad (3.111)
\]

which is a consequence of the choice of the basis functions for \( \mathcal{R} \), see (3.61) and (3.8), see also (3.64), (3.7) and (3.17). Obviously, in case of \( \Omega = \Omega' \) the first claim in (3.111) is meaningless and the second claim remains true.

It will be proven that \( \nabla \mathbf{R}_n \big|_{\Omega} \) can be represented with the help of the potentials from the basis of \( \mathcal{S} \), (3.18), which correspond to the basis of \( \mathcal{R} \) as described in (3.64). The gradients of eigenfunctions \( \{ \psi_i^{\text{dir}} \}_{i \in \mathbb{N}} \) from the homogeneous Dirichlet-Laplace operator, see (3.43), are members of \( \mathcal{S} \). Their \( L^2 \)-closure is \( \nabla \big[ \mathcal{H}_0^1(\Omega) \big] \) which can be proven as follows. They are elements of \( \mathcal{S} \), i.e.

\[
\{ \nabla \psi_i^{\text{dir}} \}_{i \in \mathbb{N}} \subset \mathcal{S} = \nabla \big[ \mathcal{H}_0^1(\Omega) \big] \cap \mathcal{R} \subset \nabla \big[ \mathcal{H}_0^1(\Omega) \big], \quad (3.112)
\]
and if there was an element that could not be approximated, one could find $\nabla S \in \nabla [H^1_0(\Omega)]$ such that for all $i \in \mathbb{N}$ the identity

$$0 = \langle \nabla \psi_i^{\text{dir}}, \nabla S \rangle_{L^2(\Omega)^d}$$

holds. The eigenfunctions are a basis of $H^1_0(\Omega)$, hence $\Delta S = 0$. Together with $S_{|\partial \Omega} = 0$ one gets $S = 0$ as unique weak solution of the homogeneous Dirichlet-Laplace problem, hence the claim and consequently

$$\mathcal{S} \in L^2(\Omega)^d = \nabla [H^1_0(\Omega)]$$

via (3.112).

Now, there is a sequence $(\psi_k)_{k \in \mathbb{N}}$,

$$\mathcal{R} \ni \psi_k = \sum_{i=1}^{2k} a_i \psi_i^R, \quad \text{for given real numbers } (a_j)_{j \in \mathbb{N}}, \text{ where } a_j = 0 \quad \forall \mathbb{N} \ni j \notin 4\mathbb{N},$$

such that

$$\mathcal{R}_k \ni \psi_k |_{\Omega} \mapsto \nabla \nabla R_n |_{\Omega} \text{ in } L^2(\Omega)^d,$$

(3.115)

where $\nabla \psi_i^R |_{\Omega} = s_i$, see (3.64), was exploited. Moreover, by construction (cf. (3.64))

$$\nabla \psi_k |_{\Omega \cap \mathcal{R}} \equiv 0.$$  

(3.116)

This still holds true in case of $\Omega = \Omega'$ as only basis functions with even index have been used to proof this (see Remark 3.20).

Due to the boundary conditions of (3.61), the $L^2$-orthogonality of (3.33) and the considerations above, one obtains

$$- \sigma_1 \mu_0 \int_{\Omega} \nabla m_n \cdot \nabla R_n \, dx$$

$$= \sigma_1 \mu_0 \int_{\Omega} m_n \cdot \nabla \nabla R_n \, dx = \sigma_1 \mu_0 \int_{\Omega} m_n \cdot \lim_{k \to \infty} \nabla \psi_k |_{\Omega} \, dx$$

$$= \sigma_1 \mu_0 \lim_{k \to \infty} \int_{\Omega} m_n \cdot \nabla \psi_k |_{\Omega} \, dx,$$  

(3.117)

$L^2$-orthogonality

$$= \sigma_1 \mu_0 \int_{\Omega'} h_n \cdot \nabla \psi_n |_{\Omega} \, dx - \sigma_1 \mu_0 \int_{\Omega'} \nabla R_n \cdot \nabla \psi_n |_{\Omega} \, dx$$

$$= \sigma_1 \mu_0 \int_{\Omega'} h_n \cdot \nabla \psi_n |_{\Omega} \, dx - \sigma_1 \mu_0 \int_{\Omega} \nabla R_n \cdot \nabla \psi_n |_{\Omega} \, dx$$

$$= \int_{\partial \Omega} h_n \cdot \nu \psi_n |_{\partial \Omega} \, d\sigma - \sigma_1 \mu_0 \int_{\Omega} \nabla R_n \cdot \nabla \psi_n |_{\Omega} \, dx.$$  

(3.118)

By construction, $\nabla \psi_k |_{\Omega} \in \mathcal{S}$, which implies $\psi_k |_{\partial \Omega} \equiv \text{const.} = C_k$, see (3.37) which transfers to the functions $\psi_i^R$, $i \in \mathbb{N}$. Therefore,

$$J_1 = \sigma_1 \mu_0 C_k \int_{\partial \Omega} h_n \cdot \nu \, d\sigma = \sigma_1 \mu_0 C_k \int_{\Omega} \nabla h_n \, dx = 0.$$  

and

$$J_2 \text{ L}^2\text{-orthogonality} = \sigma_1 \mu_0 \lim_{k \to \infty} \int_{\Omega} \nabla R_n \cdot \nabla \psi_k |_{\Omega} \, dx$$

(3.119)
\[ -\sigma_1 \mu_0 \int_{\Omega} \nabla R_n \cdot \nabla (\text{div } R_n) \, dx \]
\[ = \sigma_1 \mu_0 \int_{\Omega} \nabla R_n \cdot \nabla (\text{div } R_n) \, dx \]
\[ = \sigma_1 \mu_0 \int_{\Omega} \nabla R_n \cdot \nabla (\text{div } R_n) \, dx \]
\[ + \sigma_1 \mu_0 \int_{\partial \Omega} \nabla R_n \cdot \nu (\text{div } R_n) \, d\sigma \]
\[ = \sigma_1 \mu_0 \int_{\Omega} \nabla R_n \cdot \nabla (\text{div } R_n) \, dx . \]

Combining the computations one obtains (3.110). No adaptations are needed in case of \( \Omega = \Omega' \). As a quick side note, recall that \( R_n|_{\Omega} \) is \( H^2 \)-regular and thereby the boundary integral above is well-defined.

**Step 3: Integration in time and further estimates.** Young’s inequality will be applied to terms involving the external field, i.e.

\[
\frac{\mu_0 \alpha_1}{\tau_{\text{rel}}} \int_{\Omega'} \nabla R_n \, dx \leq \frac{\mu_0 \alpha_1}{\tau_{\text{rel}}} \int_{\Omega'} |\nabla R_n|^2 \, dx,
\]

\[
\frac{\mu_0 \alpha_1}{\tau_{\text{rel}}} \int_{\Omega'} |\nabla R_n|^2 \, dx \leq \frac{\mu_0 \alpha_1}{\tau_{\text{rel}}} \int_{\Omega'} |\nabla R_n|^2 \, dx,
\]

\[
\frac{\mu_0 \beta}{2\tau_{\text{rel}}} \int_{\Omega} |m_n|^2 \, dx \leq \frac{\mu_0 \alpha_3}{\tau_{\text{rel}}} \int_{\Omega} |m_n|^2 \, dx + \frac{\mu_0 \beta}{2\tau_{\text{rel}} \alpha_3} \int_{\Omega} |\Pi_{\beta n} h_n|^2 \, dx \leq \int_{\Omega} |h_n|^2 \, dx \text{ (cf. (3.86))}
\]

which occur on the right-hand side of the basic integral identity from the first step of the proof.

Also, the identity to be obtained with the help of the preceding steps will be integrated in time. Actually, integration in time can only be done until some arbitrary \( T_{\text{end}} \in [0, T) \) which lies in the existence interval of the local solution. From 0 up to \( T_{\text{end}} \) the solution’s coefficient functions \( t \mapsto \alpha^{(1)}(t) \), cf. (3.89), are (at least) continuous and bounded (otherwise they could not exist), hence integrable. The same holds for time derivatives thereof. Later on, it will be deduced that the solutions of (3.94) will be bounded in time and therefore still exist for even larger times allowing integration in time up to \( T \) and thereby guaranteeing existence of global solutions on \( I \).

Let \( I_{\text{end}} := [0, T_{\text{end}}] \).

Combining step 1 and step 2 with integration in time and the estimates above yields

\[
\frac{\rho_0}{2} \|u_n(T_{\text{end}})\|^2_{L^2(\Omega)^d} + D \int_{\Omega} g_{k_n}^{(c_n)(T_{\text{end}})} \, dx + \frac{\mu_0 \alpha_3}{2} \|m_n(T_{\text{end}})\|^2_{L^2(\Omega)^d} + \frac{\mu_0 \alpha_1}{2} \|\nabla R_n(T_{\text{end}})\|^2_{L^2(\Omega)^d}
\]

\[
+ 2\eta \|D u_n\|^2_{L^2(L^2(\Omega)^d)^d} + D \sigma_c \|\sqrt{(g_{k_n}^{(c_n)^2}(\nabla c_n)\nabla c_n\|^2_{L^2(L^2(\Omega)^d)^d}}
\]

\[
+ K \|f_p(c_n)\|_{L^2(L^2(\Omega)^d)^d} + \sigma_3 \|\nabla R_n\|^2_{L^2(\Omega)^d} + \sigma_3 \mu_0 \|\nabla R_n\|^2_{L^2(\Omega)^d} + \sigma_1 \mu_0 \|\nabla R_n\|^2_{L^2(\Omega)^d}
\]

\[
+ \frac{\mu_0 \alpha_1}{\tau_{\text{rel}}} \|\nabla (c_n, \nabla R_n)\|^2_{L^2(L^2(\Omega)^d)^d} + \frac{\mu_0 \alpha_1}{\tau_{\text{rel}}} \|\nabla R_n\|^2_{L^2(L^2(\Omega)^d)^d} + \frac{3\mu_0 \alpha_3}{4\tau_{\text{rel}}} \|m_n\|^2_{L^2(L^2(\Omega)^d)^d}
\]
The other term is absorbed in the left-hand side of (3.117). Analogously, with the help of the

Note that the term $h(0)$ is well-defined, as the space of the time variable is one-dimensional and $h_n$ is weakly differentiable ($h_n(T_{\text{end}})$ is well-defined analogously) in time. Multiplying the above by $\alpha_j^R(0)$ and summing up over all $j = 1, \ldots, 2n$ (i.e. testing by $R_n(0, \cdot)$) gives

by means of Young’s inequality. Due to the convergence $\mathbf{m}_n^0 \to \mathbf{m}^{\text{init}}$ in $L^2(\Omega)^d$ one can easily bound $\mathbf{m}_n^0$ in $L^2(\Omega)^d$. Analogously, $\mathbf{u}_n^0$ is bounded in $L^2(\Omega)^d$. The regularized entropy $g_L^e$ has at most quadratic growth, therefore $g_L^e(c_n^0)$ is bounded in $L^1(\Omega)$ because $c_n^0$ is bounded in $L^2(\Omega)$.

It remains to estimate the right-hand side. Recall, the solution of (3.93c) is just a function of the other unknowns, i.e. $R_n(0) \in \mathcal{R}_n$ is defined as mean value free solution of

Note that the term $\mathbf{h}_n(0)$ is well-defined, as the space of the time variable is one-dimensional and $\mathbf{h}_n$ is weakly differentiable ($\mathbf{h}_n(T_{\text{end}})$ is well-defined analogously) in time. Multiplying the above by $\alpha_j^R(0)$ and summing up over all $j = 1, \ldots, 2n$ (i.e. testing by $R_n(0, \cdot)$) gives

by means of Young’s inequality. Due to the convergence $\mathbf{m}_n^0 \to \mathbf{m}^{\text{init}}$ in $L^2(\Omega)^d$ one can easily bound $\mathbf{m}_n^0$ in $L^2(\Omega)^d$. Analogously, $\mathbf{u}_n^0$ is bounded in $L^2(\Omega)^d$. The regularized entropy $g_L^e$ has at most quadratic growth, therefore $g_L^e(c_n^0)$ is bounded in $L^1(\Omega)$ because $c_n^0$ is bounded in $L^2(\Omega)$.

It remains to deal with the first four terms of the right hand side of (3.117). Therefore, we compute

The $\mathbf{h}_n$-term is bounded due to convergence of the projection in $L^2(I; H^3(\Omega)^d)$, see Lemma 3.26. The other term is absorbed in the left-hand side of (3.117). Analogously, with the help of the growth constraint (3.4) of the susceptibility $\chi(c, \mathbf{h})\mathbf{h}$ one obtains

In the last step, $L^2$-stability of $\Pi_{\mathcal{R}_n}$ has been used (3.86). For the term of the type $\partial_t \mathbf{m}_n \cdot \Pi_{\mathcal{R}_n} \mathbf{h}_n$, rewriting

rewriting
will lead to an estimate. Integration of (3.119) in time yields
\[
\int_0^{T_{\text{end}}} \int_{\Omega} \partial_t \mathbf{m}_n \cdot \mathbf{h}_a \, dx \, dt = \int_0^{T_{\text{end}}} \int_{\Omega} \mathbf{m}_n(T_{\text{end}}) \cdot \mathbf{h}_a(T_{\text{end}}) \, dx - \int_0^{T_{\text{end}}} \int_{\Omega} \mathbf{m}_n(0) \cdot \mathbf{h}_a(0) \, dx - \int_0^{T_{\text{end}}} \int_{\Omega} \mathbf{m}_n \cdot \partial_t \mathbf{h}_a \, dx \, dt. \tag{3.120}
\]

The second term on the right-hand side of (3.120) is bounded due to Young’s inequality,
\[
-\frac{\mu_0 \beta}{2} \int_0^{T_{\text{end}}} \int_{\Omega} \mathbf{m}_n(0) \cdot \mathbf{h}_a(0) \, dx \leq \frac{\mu_0 \beta}{4} \| \mathbf{m}_n(0) \|^2_{L^2(\Omega)^d} + \frac{\mu_0 \beta}{4} \| \mathbf{h}_a(0) \|^2_{L^2(\Omega)^d}, \tag{3.121}
\]
and convergence of \( \mathbf{m}_n(0) = \mathbf{m}_0^n \) in \( L^2(\Omega)^d \). For the first term on the right-hand side of (3.120) the estimate
\[
\frac{\mu_0 \beta}{2} \int_0^{T_{\text{end}}} \int_{\Omega} \mathbf{m}_n(T_{\text{end}}) \cdot \mathbf{h}_a(T_{\text{end}}) \, dx \leq \frac{\mu_0 \alpha_3}{4} \| \mathbf{m}_n(T_{\text{end}}) \|^2_{L^2(\Omega)^d} + \frac{\mu_0 \beta^2}{4 \alpha_3} \| \mathbf{h}_a(T_{\text{end}}) \|^2_{L^2(\Omega)^d} \tag{3.122}
\]
will be used. Furthermore, by Young’s inequality,
\[
-\frac{\mu_0 \beta}{2} \int_0^{T_{\text{end}}} \int_{\Omega} \mathbf{m}_n \cdot \partial_t \mathbf{h}_a \, dx \, dt \leq \frac{\mu_0 \alpha_3}{4 \tau_{\text{rel}}} \int_0^{T_{\text{end}}} \int_{\Omega} |\mathbf{m}_n|^2 \, dx \, dt + \frac{\mu_0 \beta^2 \tau_{\text{rel}}}{4 \alpha_3} \int_0^{T_{\text{end}}} \int_{\Omega} |\partial_t \mathbf{h}_a|^2 \, dx \, dt, \tag{3.123}
\]
where the last term is bounded due to the regularity of \( \mathbf{h}_a \) (cf. (H2.3.2)). Combining (3.119)-3.123 yields
\[
\frac{\mu_0 \beta}{2} \int_0^{T_{\text{end}}} \int_{\Omega} \partial_t \mathbf{m}_n \cdot \Pi_{\Omega^n} \mathbf{h}_a \, dx \, dt \leq \frac{\mu_0 \alpha_3}{4} \| \mathbf{m}_n(T_{\text{end}}) \|^2_{L^2(\Omega)^d} + \frac{\mu_0 \beta^2}{4 \alpha_3} \| \mathbf{h}_a(T_{\text{end}}) \|^2_{L^2(\Omega)^d} + \frac{\mu_0 \alpha_3}{4 \tau_{\text{rel}}} \| \mathbf{m}_n \|^2_{L^2(\Omega^d)} + \frac{\mu_0 \beta^2 \tau_{\text{rel}}}{4 \alpha_3} \| \partial_t \mathbf{h}_a \|^2_{L^2(\Omega^d \times T_{\text{end}} \times \Omega^d)}.
\]

The first and the fifth terms on the right-hand side will be absorbed later and the remaining terms are bounded.

Young’s inequality will be used in combination with (3.4) which yields
\[
\frac{\mu_0 \alpha_3}{\tau_{\text{rel}}} \int_0^{T_{\text{end}}} \int_{\Omega} \chi(c_n, \nabla R_n) \nabla R_n \cdot \mathbf{m}_n \, dx \, dt \\
\leq \frac{\mu_0 \alpha_3}{\tau_{\text{rel}}} \int_0^{T_{\text{end}}} \int_{\Omega} \left( K_1 + K_2 \sqrt{|c_n|} \right) |\mathbf{m}_n| \, dx \, dt \\
\leq \frac{\mu_0 \alpha_3}{\tau_{\text{rel}} D} \int_0^{T_{\text{end}}} \int_{\Omega} \left( K_1 + K_2 \sqrt{|c_n|} \right) \, dx \, dt + \frac{\mu_0 \alpha_3}{4 \tau_{\text{rel}}} \int_0^{T_{\text{end}}} \int_{\Omega} |\mathbf{m}_n|^2 \, dx \, dt, \tag{3.124}
\]

The estimate in Lemma 3.22 applied to (3.118) and (3.124) yields
\[
-\frac{\mu_0 \beta}{2 \tau_{\text{rel}}} \int_0^{T_{\text{end}}} \int_{\Omega} \chi(c_n, \nabla R_n) \nabla R_n \cdot \Pi_{\Omega^n} \mathbf{h}_a \, dx \, dt + \frac{\mu_0 \alpha_3}{\tau_{\text{rel}}} \int_0^{T_{\text{end}}} \int_{\Omega} \chi(c_n, \nabla R_n) \nabla R_n \cdot \mathbf{m}_n \, dx \, dt \\
\leq \frac{\mu_0 \beta^2}{16 \tau_{\text{rel}} D} \int_0^{T_{\text{end}}} \int_{\Omega} |\mathbf{h}_a|^2 \, dx \, dt + \frac{\mu_0 \alpha_3}{4 \tau_{\text{rel}}} \int_0^{T_{\text{end}}} \int_{\Omega} |\mathbf{m}_n|^2 \, dx \, dt + C_0 \left( \| \Omega \| + D \int g^T(c_n) \, dx \right)
\]
for some \( C_0 > 0 \), which makes it possible to achieve an estimate using Gronwall’s inequality later on.
In conclusion, starting with \[3.117\] and applying all discussed steps, one obtains

\[
\frac{\rho_0}{2} \| u_n(T_{\text{end}}) \|_{L^2(\Omega)^d}^2 + D \int_\Omega g_n^L(c_n(T_{\text{end}})) \, dx + \frac{\mu_0 \alpha_3}{4} \| m_n(T_{\text{end}}) \|_{L^2(\Omega)^d}^2 + \frac{\mu_0 \alpha_1}{2} \| \nabla R_n(T_{\text{end}}) \|_{L^2(\Omega)^d}^2 \\
+ 2\eta \| D u_n \|_{L^2(I_{\text{end}} \times \Omega)^d}^2 + D \sigma_c \| \sqrt{(g_n^L)'}(c_n) \|_{L^2(I_{\text{end}} \times \Omega)^d}^2 \\
+ K \| f_p(c_n) \|_{L^2(\Omega)} \| \Pi \nabla c_n \|_{L^2(I_{\text{end}} \times \Omega)^d}^2 \\
+ \frac{\sigma \alpha_3 \mu_0}{2} \| \text{div } m \|_{L^2(I_{\text{end}} \times \Omega)^d}^2 + \sigma \alpha_3 \mu_0 \| \text{curl } m \|_{L^2(I_{\text{end}} \times \Omega)^d}^2 \\
+ \frac{\mu_0 \alpha_1}{2 \tau_{\text{rel}}} \| \chi(c_n, \nabla R_n) \|_{L^2(I_{\text{end}} \times \Omega)^d}^2 + \mu_0 \alpha_1 \| \partial t h_n \|_{L^2(I_{\text{end}} \times \Omega)^d}^2 \\
+ \frac{\mu_0 \alpha_1}{2 \tau_{\text{rel}}} \| \chi(c_n, \nabla R_n) \|_{L^2(I_{\text{end}} \times \Omega)^d}^2 + \mu_0 \alpha_1 \| \partial t h_n \|_{L^2(I_{\text{end}} \times \Omega)^d}^2 \\
+ \frac{\mu_0 \beta^2}{4 \alpha_3} \| h_n \|_{L^2(I_{\text{end}} \times \Omega)^d}^2 + C_0 |\Omega| + C_0 D \int_{T_{\text{end}}}^1 \int_\Omega g_n^L(c_n) \, dx \, dt \\
+ \frac{\mu_0 \alpha_1}{\tau_{\text{rel}}} \| h_n \|_{L^2(I_{\text{end}} \times \Omega)^d}^2 + \mu_0 \alpha_1 \| \partial t h_n \|_{L^2(I_{\text{end}} \times \Omega)^d}^2 \\
+ \frac{\rho_0}{2} \| u_n \|_{L^2(\Omega)^d}^2 + D \int_\Omega g_n^L(c_n^0) \, dx + \frac{\mu_0 \alpha_3}{2} \| m_n^0 \|_{L^2(\Omega)^d}^2 \\
+ \mu_0 \alpha_1 \| h_n(0) \|_{L^2(\Omega)^d}^2 + \| m_n \|_{L^2(\Omega)^d}^2 \\
\leq C \| \Pi \xi_n h_n \|_{L^2(I_{\text{end}}; L^2(\Omega)^d)}^2 + \| h_n(T_{\text{end}}) \|_{L^2(\Omega)^d}^2 + \| h_n(0) \|_{L^2(\Omega)^d}^2 \\
+ \| h_n \|_{L^2(I_{\text{end}} \times \Omega)^d}^2 + \| \partial t h_n \|_{L^2(I_{\text{end}} \times \Omega)^d}^2 \\
+ C(\| u_n \|_{L^2(\Omega)^d}^2 + \| m_n \|_{L^2(\Omega)^d}^2 + \int_\Omega g_n^L(c_n^0) \, dx) \\
+ C_0 |\Omega| + C_0 D \int_{T_{\text{end}}}^1 \int_\Omega g_n^L(c_n) \, dx \, dt \\
=: C_{h_n, \text{initial}, |\Omega|} + C_0 D \int_{T_{\text{end}}}^1 \int_\Omega g_n^L(c_n) \, dx \, dt \\
(3.125)
\]

for some constant \( C > 0 \). The constant \( C_{h_n, \text{initial}, |\Omega|} > 0 \) consists of

- terms depending on \( |\Omega| \),
- terms related to the external magnetic field, which are bounded altogether by

\[
\| h_n \|_{H^1(I_{\text{end}}; L^2(\Omega)^d)} + \| \Pi \xi_n h_n \|_{L^2(I_{\text{end}}; L^2(\Omega)^d)},
(3.126)
\]

by using the continuous embedding from \( H^1 \)-in-time functions into the \( C^0 \)-in-time functions in order to show boundedness of \( h_n(T_{\text{end}}) \) and \( h_n(0) \) by the \( H^1(I; L^2(\Omega)^d) \)-norm of \( h_n \),

- and terms related to initial data.

All of the above are bounded uniformly for \( n \in \mathbb{N} \) due to convergence of the projections \( \Pi \xi_n h_n \) in \( L^2(I; H^3(\Omega)^d) \) (cf. Lemma 3.26) and convergence of discrete initial data (cf. (3.92)).

Above estimate implies

\[
D \int_\Omega g_n^L(c_n(T_{\text{end}})) \, dx \leq C_{h_n, \text{initial}, |\Omega|} + C_0 D \int_{T_{\text{end}}}^1 \int_\Omega g_n^L(c_n) \, dx \, dt
\]
and by Gronwall’s inequality,
\[
D \int_0^{T_{\text{end}}} \int_{\Omega} g^T_n(c_n) \, dx \, dt \leq C_{h_n, \text{initial}, [\Omega]}(1 + e^{C_0 T_{\text{end}}}) (1 + e^{C_0 T}) \quad \text{for} \quad T_{\text{end}} \leq T.
\] (3.127)

This yields a bound for the right-hand side of (3.125). Note that the final time \( T_{\text{end}} \) could have been chosen arbitrarily within the existence interval of the local solution. Hence, \( L^\infty \)-in-time-estimates for \( u_n, c_n, m_n \) can be obtained as the right-hand side in (3.127) does not depend on \( T_{\text{end}} \in [0, T] \).

**Boundedness of the functions**\( u_n, c_n \) and \( m_n \) \( (R_n \text{ does always exist}) \) yields global existence as follows. It is well-known that the solution \( (\alpha^u, \alpha^c, \alpha^m) \) to (3.94) from Lemma 3.30, which corresponds to \((u_n, c_n, m_n)\) via (3.91), continues to exist for larger times than \( T_{\text{end}} \) if it is bounded in time. For any \( k \in \{1, \ldots, 2n\} \) it follows via \( L^2 \)-orthogonality of the basis of \( \mathcal{U}_n \) that
\[
|\alpha^u_k(t)| = \left| \sum_{j=1}^{2n} \langle \alpha^u_j \Psi_j^u, \Psi^u_k \rangle_{L^2(\Omega)} \right| \leq \left\| (u_n(t), \Psi^u_k)_{L^2(\Omega)} \right\| \cdot \left\| \Psi^u_k \right\|_{L^\infty(\Omega)} < \infty.
\]

Therefore, the part \( \alpha^u \) of the solution of (3.94) is bounded. As the bases of \( c_n \) and \( m_n \) also are \( L^2 \)-orthonormal, the analogous result follows.

Therefore, there exists a global solution of (3.93) on \( I \). And the estimate in (3.125) combined with (3.127) yields (for \( T_{\text{end}} \) being replaced with \( T \)) the a priori estimate (3.97).

**Remark 3.32.** This is the final remark about the case \( \Omega = \Omega' \) which summarizes all considerations made before. From here on, all analytical results within Section 3 will be based on the energy estimate for the Galerkin solutions (cf. Lemma 3.31). Hence, the upcoming results turn out to remain true in the case \( \Omega = \Omega' \).

In contrast to the setting \( \Omega \subset \Omega' \), the basis functions of \( \mathcal{R} \) with odd index were set to zero (cf. Remark 3.20) without removing them from the generating set to achieve consistency in notation. By this approach the same results as in the case \( \Omega \subset \Omega' \) remain valid as they are currently printed – see also Remark 3.27 – with only one modification, see the stiffness matrix \( L^R \) in (3.95) which needs to be reduced to be associated with evenly indexed functions only to admit an inverse, i.e. consider \((L^R)_{i,j} := (L^R)_{2i,2j} \) and \((F^R)_{i,j} := (F^R)_{2i,2j} \) as well as the system \( L^R(\alpha^u_1, \ldots, \alpha^u_{2n}) = F^R(t, \alpha^m(t)) \). By this modification, the Lipschitz-condition for \( R_n(t) = \hat{R}(t, m_n(t)) \) (cf. (3.96)) and thereby the proof of Lemma 3.30 remains valid.

There also are some instances of notation which are correct but include trivial parts, see the definitions of \( \mathcal{R} \) (see Remark 3.20), the last claim in (3.64) (see Remark 3.20) or the first claim in (3.111). Here, trivial assumptions posed on the empty set are dropped. The results originally following those claims remain true. The operator \( \Pi_{\mathcal{R}_n} \) in (3.82) (see Remark 3.27) remains well-defined.

Finally, other instances which relate to \( \mathcal{R} \) or \( \Omega' \) remain to be valid automatically, e.g. the weak formulation of the magnetostatic problem, see Section 2.4.1 which always is valid for both cases. There are no trivial parts to be discarded and no modifications to be made.

### 3.3 Existence result for a regularized model

This section covers the passage to the limit of the discrete Galerkin solutions which have been obtained in Lemma 3.31. This way, a solution of a regularized model will be obtained as the regularization parameter \( \sigma_c > 0 \) in (3.93b) stays fixed as well as \( s, L > 0 \). This way, one can cope with the low regularity of the magnetic variables. Having only \( H(\text{div}, \text{curl}) \)-regularity, higher
The discrete convective velocity of the magnetic particles is defined as (see (3.73)) such that its data, \( |_T \) and the magnetic field as uniformly for \( \Omega \) in the sense of Definition 3.28 with initial data as specified in Lemma 3.31) such that for any throughout this section the sequence of Galerkin solutions will have a specific notation which will be stated as definition here for convenience.

**Definition 3.33.** Let \((u_n, c_n, R_n, m_n)_{n\in \mathbb{N}}\) denote the sequence of Galerkin solutions (provided by Lemma 3.31) such that for any \( n \in \mathbb{N} \) one finds \((u_n, c_n, R_n, m_n) \in X_n\) to be a Galerkin solution in the sense of Definition 3.28 with initial data as specified in (3.92).

The discrete convective velocity of the magnetic particles is defined as

\[
(v_{\text{part}})_n := \frac{K f_p (c_n)}{(c_n)_s} [-D\nabla \epsilon_n (g_s L^r)' (c_n) + \mu_0 \frac{1}{(c_n)_s} (\alpha_1 \nabla h_n + \frac{\beta}{2} \nabla \Pi \epsilon_n h_n - \alpha_3 \nabla m_n)^T m_n] \tag{3.128}
\]

and the magnetic field as

\[
h_n := \nabla R_n \tag{3.129}
\]

for all \( n \in \mathbb{N} \).

**Corollary 3.34.** The sequences defined in Definition 3.33 satisfy

\[
\| u_n \|_{L^\infty(I; L^2(\Omega)^d)} + \| u_n \|_{L^2(I; H^1(\Omega))} + \| g_s L^r (c_n) \|_{L^\infty(I; L^1(\Omega))} + \frac{\sigma_c}{L} \| \nabla c_n \|_{L^2(I; L^2(\Omega)^d)} \\
+ \frac{(c_n)_s}{\sqrt{f_p (c_n)}} \| v_{\text{part}} \|_{L^2(I; L^2(\Omega)^d)} + \| m_n \|_{L^\infty(I; L^2(\Omega)^d)} + \| h_n \|_{L^\infty(I; L^2(\Omega)^d)} \tag{3.130}
\]

uniformly for \( n \in \mathbb{N} \) for a constant \( C > 0 \) which does not depend on \( \sigma_c, L, s \) but only on \( h, \) initial data, \( |\Omega| \) and \( T \).

**Proof:** The estimate follows directly from the a priori estimate in Lemma 3.31 after applying the following considerations.

For the fourth term on the left-hand side, (3.73) will be used to determine \( \frac{1}{L} \leq |(g_s L^r)'' (c)|. \) The \( H^1 \)-norm of \( u_n \) is bounded due to Korn’s inequality (see e.g. [17]). The \( H(\text{curl}) \)-norm of \( h_n \) is bounded as \( \text{curl} h_n = \text{curl} \nabla R_n \equiv 0. \) The entropic term \((g_s L^r)' (c_n)\) is bounded from below by \(-1\) (see (3.73)) such that its \( L^1 \)-norm with respect to space can be bounded by adding (a multiple of) \( |\Omega| \) to the constant on the right-hand side.

### 3.3.1 Compactness in space

By the use of embedding results the regularity of Galerkin solutions can be improved. For this, Sobolev’s embedding, some \( L^p \)-interpolation inequalities as well as local \( H^1 \)-estimates for \( H(\text{div}, \text{curl}) \)-regular functions will be used.

**Lemma 3.35.** Let \( 1 \leq p < \infty \) and \( 1 \leq q < r < \infty \). Then

\[
L^\infty(I; L^p(\Omega)) \cap L^q(I; L^r(\Omega)) \hookrightarrow L^\sigma(I \times \Omega),
\]

where \( \sigma := q + \frac{p(r-q)}{r} \in (q, q + p) \).
Proof: Using Hölder’s inequality one obtains for arbitrary \( \alpha \in (0,1) \) and \( \gamma := \frac{\sigma - p\alpha}{1-\alpha} \) the estimate
\[
\int_0^T \int_\Omega |c|^{(\sigma - p\alpha) + p\alpha} \, dx \, dt \leq \int_0^T \left( \int_\Omega |c|^\gamma \, dx \right)^{1-\alpha} \left( \int_\Omega |c|^p \, dx \right)^\alpha \, dt 
\leq \|c\|_{L^\infty(I; L^p(\Omega))} \|c\|^\gamma_{L^{\gamma - p\alpha}(I; L^\gamma(\Omega))}.
\]
Choose \( \alpha = \frac{\gamma}{r} \), then \( \gamma = r \) and \( p\alpha = \sigma - q \) which yields
\[
\|c\|_{L^\sigma(I \times \Omega)} \leq \|c\|^q_{L^{\gamma - p\alpha}(I; L^\gamma(\Omega))} \|c\|_{L^\sigma(I; L^q(\Omega))}.
\]
Noting \( \frac{\sigma}{q} > 1 \) and \( \frac{\sigma}{\sigma - q} = 1 \), Young’s inequality yields the continuity of the embedding, i.e.
\[
\|c\|_{L^\sigma(I \times \Omega)} \leq \frac{\sigma}{\sigma - q} \|c\|_{L^\infty(I; L^p(\Omega))} + \frac{q}{\gamma} \|c\|_{L^q(I; L^\gamma(\Omega))}.
\]

Corollary 3.36. Let \( g_\delta^L(c) \in L^\infty(I; L^1(\Omega)) \) with \( \|g_\delta^L(c)\|_{L^\infty(I; L^1(\Omega))} \leq C, 0 < s < c < L \). Then \( c \) is in \( L^\infty(I; L^2(\Omega)) \) as well, where the norm is bounded by a constant depending on \( C, L \) and \( |\Omega| \). Moreover, \( g_\delta^L(c) + c \) \((\approx c \log(c))\) as well as \( c \) are in \( L^\infty(I; L^1(\Omega)) \), where their norms are bounded by a constant depending only on \( C \) and \( |\Omega| \).

Proof: This is an easy consequence of the estimates in Lemma 3.22.

Lemma 3.37. A bounded sequence \((f_n)_{n \in \mathbb{N}}\) in \( H(\text{div}, \text{curl})(\Omega) \) is also bounded in \( H^1_{\text{loc}}(\Omega) \), i.e. it is bounded in \( H^1(\hat{\Omega})^d \) for any \( \hat{\Omega} \subset \subset \Omega \) where the bound depends on \( \hat{\Omega} \).

Proof: The proof is based on
\[
H_{10}(\text{curl})(\Omega) \cap H_{n0}(\text{div})(\Omega) = H^1_0(\Omega)^d, \tag{3.131}
\]
which holds algebraically and topologically, see [33, Lemma 2.5]. Using the formulas
\[
\text{div}(\phi f) = \nabla \phi \cdot f + \phi \text{div} f, \\
\text{curl}(\phi f) = \nabla \phi \times f + \phi \text{curl} f,
\]
(3.132)
for any scalar \( \phi \in C^\infty_0(\Omega) \) and \( f \in H(\text{div}, \text{curl})(\Omega) \), one finds \((\phi f) \in H^1_0(\Omega)^d\) according to (3.131). For any \( \hat{\Omega} \subset \subset \Omega \) one can find a cut-off function \( \phi \in C^\infty_0(\Omega; [0,1]) \) such that \( \phi \equiv 1 \) on \( \hat{\Omega} \), hence
\[
\|f\|_{H^1(\hat{\Omega})^d} \leq \|\phi f\|_{H^1(\Omega)^d} \lesssim \|\phi f\|_{H^1_0(\Omega)^d} \leq \|\phi f\|_{L^2(\Omega)^d} + \|\text{div}(\phi f)\|_{L^2(\Omega)^d} + \|\text{curl}(\phi f)\|_{L^2(\Omega)^d},
\]
which is according to (3.132) bounded by \( \tilde{C}(\phi)\|f\|_{H(\text{div}, \text{curl})(\Omega)} \), where \( \tilde{C}(\phi) \) only depends on the cut-off function \( \phi \) or, equivalently, on \( \hat{\Omega} \).

Corollary 3.38. The Galerkin solutions from Definition 3.33 satisfy the following.

Case \( d = 3 \): The sequences \((u_n)_{n \in \mathbb{N}}\) and \((c_n)_{n \in \mathbb{N}}\) are bounded in \( L^{10/3}(I; L^{10/3}(\Omega)^d) \). The sequences \((h_n)_{n \in \mathbb{N}}\) and \((m_n)_{n \in \mathbb{N}}\) are bounded in \( L^{10/3}(I; L^{10/3}_{\text{loc}}(\Omega)^d) \).

Case \( d = 2 \): The sequences \((u_n)_{n \in \mathbb{N}}\) and \((c_n)_{n \in \mathbb{N}}\) are bounded in \( L^4(I; L^4(\Omega)^d) \). The sequences \((h_n)_{n \in \mathbb{N}}\) and \((m_n)_{n \in \mathbb{N}}\) are bounded in \( L^4(I; L^4_{\text{loc}}(\Omega)^d) \).

Proof: Let \( d = 3 \). First, consider \((u_n)_{n \in \mathbb{N}}\). By (3.130) and Sobolev’s embedding theorem one obtains boundedness in \( L^\infty(I; L^2(\Omega)^d) \cap L^2(I; L^6(\Omega)^d) \) and by Lemma 3.35 the result follows. Now, consider \((c_n)_{n \in \mathbb{N}}\). After application of (Corollary 3.36) the sequence \((c_n)_{n \in \mathbb{N}}\) admits the exact same treatment as \((u_n)_{n \in \mathbb{N}}\) before. The magnetic variables have local \( H^1 \)-regularity with respect to the spatial variable due to Lemma 3.37. Again, the same arguments as before apply.

In the case \( d = 2 \) a classic interpolation result from [25] yields the result for \((u_n)_{n \in \mathbb{N}}\). The other claims follow by the same argument.
3.3.2 Compactness in time

In this section, estimates for $\partial_t u_n$, $\partial_t c_n$ and $\partial_t m_n$ will be established. They will be required for application of Aubin-Lion’s lemma later.

**Lemma 3.39.** Let $(c_n, u_n, \frac{(c_n)}{\sqrt{f_p(c_n)}} (v_{\text{part}})_n)_{n \in \mathbb{N}}$, see Definition 3.33, be bounded in

$$L^{10/3}(I \times \Omega) \cap L^2(I; H^1(\Omega)) \times L^{10/3}(I \times \Omega)^d \times L^2(I \times \Omega)^d$$

such that additionally

$$\|\sqrt{f_p(c_n)}\|_{L^{10/3}(I \times \Omega)} \leq \tilde{C}$$

uniformly for $n \in \mathbb{N}$, for some $\tilde{C} > 0$. Then, there is a constant $C > 0$ such that for all $n \in \mathbb{N}$

$$\|\partial_t c_n\|_{L^{5/4}(I; (H^2_n(\Omega)))^d} = \|\partial_t c_n\|_{L^{5/4}(I; (H^2_n(\Omega)))^d} \leq C.$$

**Proof:** The $L^2$-orthogonality of the basis functions from (3.72) will be used. This gives for any $\psi \in L^5(I; H^2_n(\Omega))$ the estimate

$$\int_0^T \int_\Omega \partial_t c_n \psi \, dx \, dt \leq \int_0^T \int_\Omega (c_n)_s \|\nabla \Pi c_n \psi\| \, dx \, dt + \int_0^T \int_\Omega (c_n) (v_{\text{part}})_n \|\nabla \Pi c_n \psi\| \, dx \, dt$$

$$\leq \| (c_n)_s \|_{L^{10/3}(I \times \Omega)} \| u_n \|_{L^{10/3}(I \times \Omega)} \|\nabla \Pi c_n \psi\|_{L^{5/2}(I \times \Omega)}$$

$$+ \| \sqrt{f_p(c_n)} \|_{L^{10/3}(I \times \Omega)} \| (v_{\text{part}})_n \|_{L^2(I \times \Omega)} \|\nabla \Pi c_n \psi\|_{L^5(I \times \Omega)}$$

$$+ \| \nabla c_n \|_{L^2(I \times \Omega)} \|\nabla \Pi c_n \psi\|_{L^2(I \times \Omega)}.$$

From this the result follows easily using $H^2(\Omega) \hookrightarrow W^{1,5}(\Omega)$ and the $H^2_n$-stability (3.85) of $\Pi c_n$. \qed

**Remark 3.40.** For any fixed cut-off function $\phi \in C_0^\infty(\Omega; \mathbb{R}_0^+)$ one can prove analogously to Lemma 3.34 that $(\partial_t (\phi c_n))_{n \in \mathbb{N}}$ is bounded in $L^{5/4}(I; (H^2_n(\Omega)))^d$. For the proof use

$$\int_0^T \int_\Omega \partial_t (\phi c_n) \psi \, dx \, dt = \int_0^T \int_\Omega \partial_t c_n (\phi \psi) \, dx \, dt$$

and $\| (\phi \psi) \|_{H^2(\Omega)} \leq C(\phi) \| \psi \|_{H^2(\Omega)}$, the latter of which follows from the chain rule.

**Lemma 3.41.** Let $(u_n, c_n, \frac{(c_n)}{\sqrt{f_p(c_n)}} (v_{\text{part}})_n, m_n)_{n \in \mathbb{N}}$, see Definition 3.33, be bounded in

$$L^{10/3}(I \times \Omega)^d \cap L^2(I; H^1(\Omega))^d \times L^\infty(I; L^1(\Omega)) \times L^2(I \times \Omega)^d \times L^\infty(I; L^2(\Omega)^d) \cap H(\text{div, curl})(\Omega)$$

such that additionally

$$\| \sqrt{f_p(c_n)} \|_{L^\infty(\Omega \times \Omega)} \leq \tilde{C}$$

uniformly for $n \in \mathbb{N}$, for some $\tilde{C} > 0$. Then, there is a constant $C > 0$ such that for all $n \in \mathbb{N}$

$$\|\partial_t m_n\|_{(L^2(I; M_r))^d} = \|\partial_t m_n\|_{L^2(I; M_r)} \leq C.$$
**Proof:** Let \( \Psi \in L^2(I; M) \). Then
\[
\int_0^T \int_\Omega \partial_t m_n \cdot \Psi \, dx \, dt = \int_0^T \int_\Omega \partial_t m_n \cdot \Pi_{M_n} \Psi \, dx \, dt =: J
\]
due to \( L^2 \)-orthogonality of \( \Pi_{M_n} \), see \[3.34\].

\[
|J| \leq \int_0^T \int_\Omega |(u_n) + |(v_{part})_n)| |\nabla \Pi_{M_n} \Psi||m_n| \, dx \, dt
+ \sigma \int_0^T \int_\Omega \| \text{div} \, m_n \|\text{div} \Pi_{M_n} \Psi \| \, dx \, dt + \sigma \int_0^T \int_\Omega \| \text{curl} \, m_n \|\text{curl} \Pi_{M_n} \Psi \| \, dx \, dt
+ C \int_0^T \int_\Omega \|m_n\|\Pi_{M_n} \Psi\| \, dx \, dt + \frac{1}{\tau_{rel}} \int_0^T \int_\Omega \left[ |u_n| + |\chi(c_n, h_n) h_n| \right] \Pi_{M_n} \Psi \| \, dx \, dt
\leq K_1 + K_2 \sqrt{|c|}; \text{ see \[3.4\]}
\]
for some \( C' > 0 \). The claim follows as \( H^3(\Omega)^d \hookrightarrow W^{1,\infty}(\Omega)^d \) and
\[
\|\Pi_{M_n} \Psi\|_{H^3(\Omega)^d} \lesssim \|\Psi\|_{H^3(\Omega)^d}
\]
due to \[3.34\]. \( \Box \)

**Lemma 3.42.** Let \((u_n, \frac{(c_n)}{\sqrt{f_p(c_n)}}, (v_{part})_n, m_n, h_n)_{n \in \mathbb{N}}\), see \textit{Definition 3.33} be bounded in
\[
L^\infty(I; L^2(\Omega)^d) \cap L^2(I; H^1(\Omega)^d) \times L^2(I \times \Omega)^d \times L^\infty(I; L^2(\Omega)^d) \times L^\infty(I; L^2(\Omega)^d)
\]
such that additionally
\[
\|\frac{(c_n)}{\sqrt{f_p(c_n)}}\|_{L^\infty(I; L^2(\Omega))} \leq \bar{C}
\]
uniformly for \( n \in \mathbb{N} \), for some \( \bar{C} > 0 \). Then, there is a constant \( C > 0 \) such that for all \( n \in \mathbb{N} \)
\[
\|\partial_t u_n\|_{(L^2(\Omega))'} = \|\partial_t u_n\|_{L^2(I; \mathcal{W})} \leq C.
\]

**Proof:** Let \( \Psi \in L^2(I; \mathcal{W}) \). Due to \( L^2 \)-orthogonality of \( \Pi_{\mathcal{W}_n} \), see \[3.65\], one infers
\[
\int_0^T \int_{\Omega} \partial_t u_n \cdot \Psi \, dx \, dt = \int_0^T \int_{\Omega} \partial_t u_n \cdot \Pi_{\mathcal{W}_n} \Psi \, dx \, dt =: J.
\]
Now, notice that due to \( (\nabla m_n)^T m_n = \frac{1}{2} \nabla |m_n|^2 \) one has
\[
\mu_0 \int_0^T \int_{\Omega} (\Pi_{\mathcal{W}_n} \Psi \cdot \nabla)(\alpha_1 h_n + \frac{\beta}{2} \Pi_{\mathcal{W}_n} h_n) \cdot m_n \, dx \, dt
- D \int_0^T \int_{\Omega} (c_n) \nabla \Pi_{\mathcal{W}_n} (g L)'(c_n) \cdot \Pi_{\mathcal{W}_n} \Psi \, dx \, dt
\]
3.3 Existence result for a regularized model

\[ \text{div} \Pi_{\text{fin}} \Psi = 0, \quad \Pi_{\text{fin}} \Psi \big|_{\Omega \equiv 0} = 0, \quad \int_0^T \int_{\Omega} (\Pi_{\text{fin}} \Psi \cdot \nabla) (\alpha_1 h_n + \frac{\rho}{2} \Pi_{\text{fin}} h_n - \alpha_3 m_n) \cdot m_n \, dx \, dt \]

\[ - D \int_0^T \int_{\Omega} (c_n) s \nabla \Pi_{\text{fin}} (g_s') \cdot \Pi_{\text{fin}} \Psi \, dx \, dt \]

\[ \int_0^T \int_{\Omega} \frac{(c_n)^2}{2} \frac{\nabla \Pi_{\text{fin}} (v_{\text{part}} \dot{n}) \cdot \Pi_{\text{fin}} \Psi }{K f_p(c_n)} \, dx \, dt. \]

(3.128)

Then, compute

\[ |J| \leq \int_0^T \int_{\Omega} 2\eta |D u_n||D \Pi_{\text{fin}} \Psi | \, dx \, dt + \frac{\rho_0}{2} \int_0^T \int_{\Omega} |u_n| ||\nabla u_n|| \Pi_{\text{fin}} \Psi | \, dx \, dt \]

\[ + \frac{\eta_0}{2} \int_0^T \int_{\Omega} |u_n||\nabla \Pi_{\text{fin}} \Psi | ||u_n|| \Pi_{\text{fin}} \Psi | \, dx \, dt \]

\[ + \frac{\eta_0}{2} \int_0^T \int_{\Omega} |m_n||\nabla \Pi_{\text{fin}} \Psi | ||u_n|| \Pi_{\text{fin}} \Psi | \, dx \, dt \]

Due to \( H^3 \)-stability of \( \Pi_{\text{fin}} \), cf. (3.84), and \( L^2 \)-stability of \( \Pi_{h_n} \), cf. (3.86), the claim follows. \( \square \)

Lemma 3.43. Under the assumptions of Lemma 3.41, there exists a constant \( C > 0 \) such that for all \( V \subset \Omega \) and for all \( n \in \mathbb{N} \)

\[ ||\partial_t h_n|V||_{L^2(I; L^2(\Omega)')} = ||\partial_t h_n|V||_{L^2(I; \nabla H^{3}_0(V))'} \leq C, \]

where the functions \( h_n \) are defined as in Definition 3.33

Proof: Equation (3.93c) can be differentiated with respect to time weakly, which entails

\[ \int_{\Omega'} \partial_t h_n \cdot \nabla \psi_i^R \, dx = \int_{\Omega'} \partial_t h_n \cdot \nabla \psi_i^R \, dx - \int_{\Omega} \partial_t m_n \cdot \nabla \psi_i^R \, dx \quad \forall i = 1, \ldots, 2n. \]

Let \( \nabla \psi \in L^2(I; H(\nu)) \), where

\[ H(V) := \left\{ \nabla \psi : \Omega' \rightarrow \mathbb{R}^d \mid \psi|_{V} \in H^3_0(V), \psi|_{\Omega \setminus V} \equiv 0 \right\}. \]

(3.133)

It follows for almost all \( t \in I \) that \( \nabla \psi(t)|_{\Omega} \in H^3(\Omega)^d \subset H(\text{div}, \text{curl})(\Omega) \) as well as \( \nabla \psi(t)|_{\Omega \setminus \Omega} \equiv 0 \in H(\text{div}0, \text{curl})(\Omega' \setminus \Omega) \) and thereby \( \nabla \psi \in L^2(I; \nabla \mathbb{R}^d) \) (cf. (3.8)). From the definitions of \( \Pi_{\text{fin}} \) and \( \Pi_{h_n} \) (cf. (3.81)) it follows that for \( \psi(t) \in \mathbb{R} \) one has \( (\nabla \Pi_{h_n} \psi(t))|_{\Omega} = \Pi_{h_n} \nabla \psi(t) \). Moreover, \( \nabla \psi(t)|_{\Omega} \in S \) for almost all \( t \in I \) by definition, see (3.133) and (3.15). Therefore, one can expect that only such basis functions for the representation of \( \psi(t) \) are needed whose gradients are elements of \( S \), i.e. basis functions with indices \( 4i, i \in \mathbb{N} \), see (3.64), (3.30). In fact, this can be proven as follows.
Fix a time $t \in I$ and let $\psi(t) = \sum_{i=1}^{\infty} \beta_i \psi_i^R$. By the same argument as in the proof of Lemma 3.24 concerned with the stability of $\Pi_{\varphi_n}$ (see (3.87)), it follows that $\beta_{4i-2} = 0$ for all $i \in \mathbb{N}$, i.e. basis functions $\nabla \psi_{4i-2}^R = s_i^+$, $i \in \mathbb{N}$, are not needed to represent $\nabla \psi(t)|_{\Omega}$. It remains to rule out basis functions with odd indices (which have their support in $\Omega' \setminus \overline{\Omega}$). Observe that
\[
\nabla \psi(t)|_{\Omega' \setminus \overline{\Omega}} = \sum_{i=1}^{\infty} \beta_{2i-1} \nabla \psi_{2i-1}^R|_{\Omega' \setminus \overline{\Omega}}
\]
as the coefficients $\beta_{4i-2}$, $i \in \mathbb{N}$, vanish as shown before and $\nabla \psi_{4i-2}^R \equiv 0$, $i \in \mathbb{N}$, on $\Omega' \setminus \overline{\Omega}$ (cf. (3.64)). Due to (3.63), those basis functions with odd index are a basis in the Hilbert space $\text{span}\{\nabla \psi_{2i-1}^R\}_{i \in \mathbb{N}}$ and thereby their coefficients must vanish in order to assert $\nabla \psi(t)|_{\Omega' \setminus \overline{\Omega}} = 0$ (see (3.133)). This proves the auxiliary claim.

In particular, from there one can conclude $\nabla \Pi_{\psi_n}|_{\Omega' \setminus \overline{\Omega}} \equiv 0$, cf. (3.64). Hence,
\[
\int_0^T \int_{\Omega'} \partial_t h_n \cdot \nabla \psi \; dx \; dt = \int_0^T \int_{\Omega} \partial_t h_n \cdot \Pi_{\psi_n} \nabla \psi \; dx \; dt = \int_0^T \int_{\Omega} \partial_t h_n \cdot \nabla \Pi_{\psi_n} \psi \; dx \; dt =: J.
\]
By means of (3.93c) the estimate
\[
|J| \leq \left| \int_0^T \int_{\Omega} \partial_t h_n \cdot \nabla \Pi_{\psi_n} \psi \; dx \; dt - \int_0^T \int_{\Omega} \partial_t m_n \cdot \nabla \Pi_{\psi_n} \psi \; dx \; dt \right| \\
\leq \left\| \partial_t h_n \right\|_{L^2(I \times \Omega)^d} \left\| \nabla \Pi_{\psi_n} \psi \right\|_{L^2(I \times \Omega)^d} + \left\| \partial_t m_n \right\|_{L^2(I \times \Omega)^d} + \left\| \nabla \Pi_{\psi_n} \psi \right\|_{L^2(I \times \Omega)^d}
\]
bounded due to (H2.3.2)
\[
\leq C \left( \left\| \nabla \Pi_{\psi_n} \psi \right\|_{L^2(I \times \Omega)^d} + \left\| \nabla \Pi_{\psi_n} \psi \right\|_{L^2(I, L^3(\Omega)^d)} \right)
\]
is obtained, where Lemma 3.41 was used in order to bound $\partial_t m_n$. The first term on the right-hand side is bounded by the other one and the proof is finished by using stability of the projections due to $H^3$-orthogonality of the corresponding basis functions of $\mathcal{S}$. Recalling $\text{supp} \nabla \psi \subset V$ the claim follows.

Remark 3.44. This remark specifies an assumption on $f_p(c_n)$ which guarantees that the conditions in Lemma 3.39, Lemma 3.41 and Lemma 3.42 are satisfied. Based on the ansatz $f_p(c) = ((c)_s)^m$ (for fixed $s$ and $L$), one infers the condition $m \in [0, 2]$. Here, it was exploited that $(c_n)_{n \in \mathbb{N}}$ is bounded in $L^\infty(I; L^2(\Omega))$ for fixed $s, L > 0$, see Corollary 3.36. Note that if only the $L$-independent $L^\infty(I; L^1(\Omega))$-bound is used (see Corollary 3.36), then the condition $m \in [1, 2]$ is required.

As the gradients of $m_n$ and $h_n$ are only locally bounded in $\Omega$, see Lemma 3.37 for an application of the Aubin-Lions lemma the functions $m_n|_{\Omega}$ and $h_n|_{\Omega}$, where $\Omega \subset \subset \Omega$, will be considered. Global estimates (in $\Omega$) for the time derivatives of $m_n$ and $h_n$ have been established before in Lemma 3.41 and Lemma 3.43. In fact, the latter lemma already includes local estimates by choosing $V \subset \subset \Omega$. Local estimates for $\partial_t m_n$ hold true as well, i.e. estimates for $\partial_t (m_n|_{\Omega})$, see
the subsequent corollary. As the "global" time-compactness estimates already allow to extract weakly (or weakly-*) converging subsequences, the dual space in Corollary 3.45 cannot impact the properties of the limit formulation (cf. (3.140d)) negatively. Hence, the dual space may be a very strict subspace of $\mathcal{M}$.

**Corollary 3.45.** Under the assumptions of Lemma 3.41 there is a constant $C > 0$ such that for all $V \subset \Omega$ and for all $n \in \mathbb{N}$

$$\|\partial_t u_n(t)\|_{L^2(I;\nabla[H^1_0(V)])} = \|\partial_t u_n(t)\|_{L^2(I;\nabla[H^1_0(V)])} \leq C.$$  

**Proof:** Let $V \subset \Omega$ be fixed and set

$$H(V) := \{\nabla \psi : \Omega \to \mathbb{R}^d|\psi|_V \in H^1_0(V), \psi|_{\Omega \setminus V} \equiv 0\}. \quad (3.134)$$

Proceed as in the proof of Lemma 3.41 but take $\Phi \in L^2(I;H(V)) \subset L^2(I;\mathcal{M})$ as test function to obtain the estimate. 

**Remark 3.46.** The time-compactness estimates serve two purposes,

- extracting weakly or weakly-* converging subsequences out of sequences of time derivatives
- and meeting the requirements of the Aubin-Lions lemma.

As an example, consider Lemma 3.42. The identification of the dual spaces of Bochner spaces was used, see e.g. [46]. Reflexivity of $X$ is sufficient such that $(L^p(I;X)) \cong L^p(I;X')$ holds for $1 < p < \infty$, where the dual exponent is given by $p' = \frac{p}{p-1}$.

The time derivative is identified with a functional $u'_n \in (L^2(I;\mathcal{U}))'$ by

$$u'_n(\Psi) := \int_0^T \int_\Omega \partial_t u_n(t) \cdot \Psi(t,x) \, dx \, dt \quad \forall \Psi \in L^2(I;\mathcal{U}).$$

The estimate in the proof bounds $|u'_n(\Psi)|$ in terms of $\|\Psi\|_{L^2(I;\mathcal{U})}$ which allows to obtain an estimate in the norm of the dual space $(L^2(I;\mathcal{U}))'$.

However, the space $(L^2(I;\mathcal{U}))'$ can be identified with $L^2(I;\mathcal{U}')$, see e.g. [46]. This way, the time derivative $\partial_t u_n$ is identified with a function $(\frac{d}{dt} u_n) \in L^2(I;\mathcal{U}')$ by

$$\langle (\frac{d}{dt} u_n)(t), \Psi \rangle_{\mathcal{U}' \times \mathcal{U}} := \int_\Omega \partial_t u_n(t) \cdot \Psi(x) \, dx \quad \forall \Psi \in \mathcal{U}$$

and almost all $t \in I$. Notice the relation

$$u'_n(\Phi) = \int_0^T \langle (\frac{d}{dt} u_n)(t), \Phi(t) \rangle_{\mathcal{U}' \times \mathcal{U}} \, dt \quad \forall \Phi \in L^2(I;\mathcal{U})$$

which resembles the isometry which is used to show the identification mentioned before. The uniform estimate for $(\frac{d}{dt} u_n)$ is needed in order to apply the theorem of Aubin and Lion later, which guarantees for $(u_n)_{n \in \mathbb{N}}$ the existence of a strongly converging subsequence, see Corollary 3.47.

In regards to weakly converging subsequences for $(\partial_t u_n)_{n \in \mathbb{N}}$ both estimates lead to the same conclusion. One the one hand, one can view $u'_n$ to be uniformly bounded in a dual space of a separable space. Then a subsequence exists such that $u'_n$ converges weakly-* towards some $u' \in (L^2(I;\mathcal{U}))'$ which can be represented by some $(\frac{d}{dt} u) \in L^2(I;\mathcal{U}')$ via

$$u'(\Phi) = \int_0^T \langle (\frac{d}{dt} u)(t), \Phi(t) \rangle_{\mathcal{U}' \times \mathcal{U}} \, dt,$$
hence
\[ \int_0^T \langle (\frac{d}{dt} u_n)(t), \Phi(t) \rangle_{W \times U} dt = u'_n(\Phi) \rightarrow u'(\Phi) = \int_0^T \langle (\frac{d}{dt} u)(t), \Phi(t) \rangle_{W \times U} dt, \]

On the other hand, \((\frac{d}{dt} u_n)\) is uniformly bounded in a reflexive space. Then there is a weakly converging subsequence towards \((\frac{d}{dt} u)\) in \(L^2(I; U')\), i.e.
\[ \int_0^T \langle (\frac{d}{dt} u_n)(t), \Phi(t) \rangle_{W \times U} dt \rightarrow \int_0^T \langle (\frac{d}{dt} u)(t), \Phi(t) \rangle_{W \times U} dt, \]

because the operator \(\int_0^T \langle \cdot, \Phi(t) \rangle_{W \times U} dt\) is in \((L^2(I; U'))'\) for a fixed \(\Phi \in L^2(I; U)\).

In fact, the reflexivity and separability of \(L^2(I; U)\) which was used within this remark follows from the fact that \(U\) is a closed subspace of \(H^3(\Omega)^d\). Concerning the other Bochner spaces occurring in Lemma 3.39, Lemma 3.41, Corollary 3.49 and Lemma 3.49 reflexivity and separability boils down to reflexivity and separability of well-known spaces.

### 3.3.3 Passage to the limit

In this section, the terms of (3.93) will be identified with their continuous counterparts by passing to the limit when \(\delta, \theta, \alpha_c > 0\) remain fixed. This way a weak formulation will be obtained corresponding to a regularized version of 'model GW'. At the beginning of this section, auxiliary results will be stated determining the convergence behavior of the sequence of Galerkin solutions (cf. Definition 3.33). For simplicity, \(f_p\) is assumed to satisfy
\[ 0 < a_0 \leq f_p(c) \leq a_1 |c|^m + a_2, \quad m \in [0, 2] \quad \text{for some } a_0, a_1, a_2 > 0, \]

\[ f_p \text{ is continuous.} \quad (3.135) \]

At the current state of progress, regularization from below (see \(a_0 > 0\)) is still necessary. The choice of the exponent \(m \in [0, 2]\) guarantees \(f_p(c_n)\) to be uniformly bounded in the spaces required by the previous results on time compactness, see Section 3.3.2.

**Corollary 3.47.** Under the assumption (3.135), there exists a subsequence of Galerkin solutions (cf. Definition 3.33), which will not be relabeled for the ease of notation, and functions

- \(u \in L^\infty(I; L^2(\Omega)^d) \cap L^2(I; H^1_0(\Omega)^d \cap H(\text{div})(\Omega)) \cap L^{q_0}(I; L^{q_0}(\Omega)^d), \)
- \(c \in L^\infty(I; L^2(\Omega)^d) \cap L^2(I; H^1(\Omega)^d) \cap L^{q_0}(I; L^{q_0}(\Omega)) , \)
- \(m \in L^\infty(I; L^2(\Omega)^d) \cap L^2(I; H(\text{div}, \text{curl})(\Omega)) \cap L^2(I; H^{1\text{loc}}(\Omega)^d) \cap L^{q_0}(I; L^{q_0}(\Omega)^d), \)
- \(h \in L^\infty(I; L^2(\Omega)^d) \cap L^2(I; H(\text{div}, \text{curl})(\Omega' \setminus \partial \Omega)) \) with \(h|_{\Omega} \in L^2(I; H^{1\text{loc}}(\Omega)^d) \cap L^{q_0}(I; L^{q_0}(\Omega)^d), \)

such that
\[
\begin{align*}
\{ u_n \rightharpoonup u \text{ in } L^\infty(I; L^2(\Omega)^d) \} & \cup \{ c_n \rightharpoonup c \text{ in } L^\infty(I; L^2(\Omega)) \}, \\
\{ u_n \rightarrow u \text{ in } L^2(I; H^1(\Omega)^d) \} & \cup \{ c_n \rightarrow c \text{ in } L^2(I; H^1(\Omega)) \}, \\
\{ u_n \rightarrow u \text{ in } L^{q_0}(I; L^{q_0}(\Omega)^d) \} & \cup \{ c_n \rightarrow c \text{ in } L^{q_0}(I; L^{q_0}(\Omega)) \}, \\
\{ h_n \rightharpoonup h \text{ in } L^\infty(I; L^2(\Omega)^d) \} & \cup \{ m_n \rightharpoonup m \text{ in } L^\infty(I; L^2(\Omega)^d) \}, \\
\{ h_n \rightarrow h \text{ in } L^2(I; H(\text{div}, \text{curl})(\Omega')) \} & \cup \{ m_n \rightarrow m \text{ in } L^2(I; H(\text{div}, \text{curl})(\Omega')) \}, \\
\{ h_n|_{\Omega} \rightarrow h|_{\Omega} \text{ in } L^2(I; H^{1\text{loc}}(\Omega)^d) \} & \cup \{ m_n \rightarrow m \text{ in } L^2(I; H^{1\text{loc}}(\Omega)^d) \}, \\
\{ h_n \rightarrow h \text{ in } L^{q_0}(I; L^{q_0}(\Omega)^d) \} & \cup \{ m_n \rightarrow m \text{ in } L^{q_0}(I; L^{q_0}(\Omega)^d) \}.
\end{align*}
\]

(3.136)
Moreover, there exists a function \( R \in H^1_{\text{mean}}(\Omega') \) such that the magnetic field
\[
h = \nabla R
\]
can be identified with a gradient field.

**Proof:** By (3.130), Corollary 3.36 and Lemma 3.37 all weak convergences from above follow (for a subsequence). Also note the \( L^{q_d} \)-regularities of the limit functions are a consequence of Lemma 3.35. The general approach for the strong convergences is to use the Aubin-Lions Lemma in order to find a strongly converging subsequence in some Lebesgue-space and improve the convergence with the help of uniform estimates and Vitali’s convergence theorem.

First, consider the sequence \((u_n)_{n \in \mathbb{N}}\). It is bounded in \( L^2(I; H^1(\Omega)^d) \), see (3.130), with \((\partial_t u_n)_{n \in \mathbb{N}}\) being bounded in \( L^2(I; U')\), see Lemma 3.42. As \( U \subset H^d(\Omega) \hookrightarrow L^2(\Omega)^d \) it is easy to see \( H^1(\Omega)^d \hookrightarrow L^2(\Omega)^d \hookrightarrow U' \). Hence, Aubin-Lion’s lemma implies the existence of a strongly converging subsequence in \( L^2(I; L^2(\Omega)^d) \). From the already obtained subsequence another subsequence can be extracted which converges pointwise almost everywhere. The uniform \( L^{q_d} \)-estimates from Corollary 3.38 imply uniform integrability of \( |u_n|^p \) for any \( 1 \leq q < q_d \). Hence, Vitali’s convergence theorem implies strong convergence in \( L^{q_d}(I; L^{q_d}(\Omega)^d) \). As \( I \) and \( \Omega \) are bounded it is evident that all convergences imply at least weak convergence in \( L^2(I; L^2(\Omega)^d) \) such that by uniqueness of weak limits the various limit functions of \( u_n \) must coincide.

In case of the sequence \((c_n)_{n \in \mathbb{N}}, (3.130), Corollary 3.36 and Lemma 3.39\) yield boundedness in \( L^2(I; H^1(\Omega)) \) with time derivatives uniformly bounded in \( L^{5/4}(I; (H^1_0(\Omega))') \). Obviously, the chain of embeddings \( H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow (H^1_0(\Omega))' \) holds and by the Aubin-Lions lemma there is a strongly converging subsequence in \( L^2(I; L^2(\Omega)) \). With the same techniques as before, there even is a subsequence strongly converging in \( L^{q_d}(I; L^{q_d}(\Omega)) \). For this, uniform estimates were used, see Corollary 3.38. As before, the limits of \( c_n \) in the different spaces must coincide.

From the proof of (3.130), Lemma 3.37 and Corollary 3.45 one obtains uniform boundedness of \( m_{|\Omega_i} \) in \( L^2(I; H^1(\Omega)^d) \) and of \( \partial_t m_{|\Omega_i} \) in \( L^2(I; (\nabla H^1_0(\Omega))') \) for any \( \Omega \subset \subset \Omega \). Using Corollary 3.38 together with the techniques from before, the strong convergence (for a subsequence) in \( L^{q_d}(I; L^{q_d}(\Omega)^d) \) can be achieved. The only difficulty is to guarantee the exact same subsequence to be strongly converging for all domains \( \hat{\Omega} \subset \subset \Omega \). For this, one can use an exhaustion argument combined with a diagonal argument. In detail, there exist sets \( \Omega_i, i \in \mathbb{N} \) such that
\[
\Omega = \bigcup_{i \in \mathbb{N}} \Omega_i \quad \text{with} \quad \Omega_i \subset \subset \Omega_{i+1} \subset \subset \Omega \quad \text{for all} \quad i \in \mathbb{N}.
\]

For each \( i \in \mathbb{N} \) one obtains a strictly monotonic mapping \( \phi_i : \mathbb{N} \to \mathbb{N} \) such that \( \phi_{i+1}(\mathbb{N}) \subset \subset \phi_i(\mathbb{N}) \) and the sequence \((m_{\phi_i(n)}|\Omega_i)_{n \in \mathbb{N}}\) converges strongly when \( n \to \infty \). Then with the choice \( \phi : \mathbb{N} \to \mathbb{N}, \phi(i) := \phi_i(i) \), the diagonal sequence \((m_{\phi(n)}|\Omega)_{n \in \mathbb{N}}\) converges strongly on any \( \Omega_i \). Moreover, note that all convergences that hold on some \( \Omega_i \subset \subset \Omega \) remain true on any subset thereof due to Hölder’s inequality. Let us prove that the various limit functions (associated with different \( \Omega_i \)) can be expressed by a single function. From the weak-* convergence in \( L^\infty(I; L^2(\Omega)^d) \) one obtains a weak limit (for another subsequence) in \( L^2(I; L^2(\Omega)^d) \) (due to \( I \) being bounded and Hölder’s inequality), and by a localization argument the restriction of the aforementioned weak limit to \( \Omega_i \) is a weak limit in \( L^2(I; L^2(\Omega_i)^d) \) for any \( i \in \mathbb{N} \). Hence the \( L^{q_d} \)-limit (which also is a weak \( L^2 \)-limit due to \( 2 \leq q_d \)) of the diagonal sequence on any \( \Omega_i \) is given by the restriction of the weak limit in \( L^2(I; L^2(\Omega)^d) \) due to uniqueness of weak limits. Therefore, it is evident that there exists exactly one function \( m \) as specified in (3.136).
The treatment of the magnetic field is analogous to the treatment of the magnetization. Moreover, as \( h_n = \nabla R_n \) with mean value free potentials \( R_n \in \mathcal{R}_n \), weak \( L^2 \)-convergence of \( h_n \) implies \( H^1 \)-boundedness of \( (R_n)_{n \in \mathbb{N}} \) by Poincaré’s inequality. By compact embedding, there is a subsequence (not relabeled) and a function \( R \in L^2(\Omega') \) such that \( R_n \to R \) in \( L^2(\Omega') \). By standard arguments, i.e. integration by parts, it can be deduced that \( R \in H^1(\Omega') \) with weak derivative \( \nabla R = h \).

**Lemma 3.48.** Under the assumption (3.135), there is a subsequence of Galerkin solutions \( (c_n)_{n \in \mathbb{N}} \) from Definition 3.33 (which will not be relabeled for the ease of notation) and a function \( w_{\text{part}} \in L^2(I \times \Omega)^d \) such that

\[
\frac{(c_n)_s}{\sqrt{f_p(c_n)}} (v_{\text{part}})_n \to w_{\text{part}} \text{ in } L^2(I \times \Omega)^d,
\]

\[
(v_{\text{part}})_n \to \frac{\sqrt{f_p(c)}}{(c)_s} w_{\text{part}} \text{ in } L^2(I \times \Omega)^d,
\]

\[
\frac{(c_n)_s}{\sqrt{f_p(c_n)}} \to \frac{(c)_s}{\sqrt{f_p(c)}} \text{ and } \frac{\sqrt{f_p(c_n)}}{f_p(c)} \to \frac{\sqrt{f_p(c)}}{f_p(c)} \text{ in } L^{q_d}(I \times \Omega),
\]

\[
\left( g_s^L \right)'(c_n) \to \left( g_s^L \right)'(c) \text{ in } L^{q_d}(I \times \Omega),
\]

where \( q_d \) is defined as in (3.137).

**Proof:** From the uniform estimate (3.130) one immediately obtains the first weak convergence

\[
\frac{(c_n)_s}{\sqrt{f_p(c_n)}} (v_{\text{part}})_n \to w_{\text{part}} \text{ in } L^2(I \times \Omega)^d.
\]

As \( c_n \) converges strongly in \( L^{q_d}(I \times \Omega) \) (for a subsequence) due to (3.136), there is a subsequence (not relabeled) that converges pointwise almost everywhere, and so does

\[
\frac{(c_n)_s}{\sqrt{f_p(c_n)}} \to \frac{(c)_s}{\sqrt{f_p(c)}} \text{ pointwise almost everywhere in } I \times \Omega,
\]

because \((\cdot)_s\) and \( f_p \) are continuous as well as \( f_p \geq a_0 > 0 \), see (3.135). The same argument yields the pointwise convergences

\[
\sqrt{f_p(c_n)} \to \sqrt{f_p(c)} \quad \text{and} \quad \frac{\sqrt{f_p(c_n)}}{(c)_s} \to \frac{\sqrt{f_p(c)}}{(c)_s} \quad \text{and} \quad \left( g_s^L \right)'(c_n) \to \left( g_s^L \right)'(c)
\]

almost everywhere in \( I \times \Omega \). By uniform bounds – based on (3.135) and Corollary 3.38 – uniform integrability follows. Therefore, by means of Vitali’s convergence theorem one can deduce the strong convergences of this lemma’s claim. For instance,

\[
\left| \frac{(c_n)_s}{\sqrt{f_p(c_n)}} \right| \leq \frac{|(c_n)_s|}{\sqrt{a_0}} \leq \frac{s}{\sqrt{a_0}} + \frac{|c_n|}{\sqrt{a_0}}
\]

where the right-hand side has linear growth in terms of \( |c_n| \) and is bounded with respect to the \( L^{q_d} \)-norm. Hence, uniform integrability of \( |c_n|^q \) for any \( 1 \leq q < q_d \) and Vitali’s convergence theorem yield the convergence \( \frac{(c_n)_s}{\sqrt{f_p(c_n)}} \to \frac{(c)_s}{\sqrt{f_p(c)}} \) in \( L^{q_d}(I \times \Omega) \). The other terms are treated
analogously, where for \((g_n^L)'(c_n)\) the definition \(3.73\) clearly shows that \(|(g_n^L)'(c_n)|\) has linear growth in terms of \(|c_n|\) such that the same treatment as before can be applied. The term

\[
0 \leq \frac{\sqrt{f_p(c_n)}}{(c_n)_s} \leq \sqrt{\frac{a_1|c_n|^m + a_2}{(c_n)_s}} \leq \frac{\sqrt{a_1|c_n|^{m-1}}}{(c_n)_s} + \frac{\sqrt{a_2}}{(c_n)_s}
\]

is bounded entirely due to \(3.135\) combined with \(\frac{1}{(c_n)_s} \leq \frac{1}{\frac{1}{4}}\). The term \(0 \leq \frac{\sqrt{f_p(c_n)}}{(c_n)_s}\) is easily estimated from above by \(\sqrt{\frac{a_1|c_n|^{m}}{(c_n)_s}} + \frac{\sqrt{a_2}}{(c_n)_s}\) which has at most linear growth in \(|c_n|\).

Combining strong convergence of \(\frac{\sqrt{f_p(c_n)}}{(c_n)_s} \times n \in \mathbb{N}\) with weak convergence of \(\frac{(c_n)_s}{\sqrt{f_p(c_n)}}(v_{\text{part}})_n\), one obtains weak convergence \((v_{\text{part}})_n \to \sqrt{\frac{f_p(c)}{(c)_s}}w_{\text{part}}\) in \(L^2(I \times \Omega)^d\). However, weak convergence in \(L^2(I \times \Omega)^d\) can be achieved, because the convective velocities

\[
(v_{\text{part}})_n = \frac{\sqrt{f_p(c_n)}}{(c_n)_s} \left[ \frac{(c_n)_s}{\sqrt{f_p(c_n)}}(v_{\text{part}})_n \right]
\]

are bounded uniformly in \(L^2(I \times \Omega)^d\). Hence, a weak limit in \(L^2(I \times \Omega)^d\) exists for a subsequence. This subsequence then also converges weakly in \(L^2(I \times \Omega)^d\) and by uniqueness of weak limits the weak limit in \(L^2(I \times \Omega)^d\) is identified with \(\sqrt{\frac{f_p(c)}{(c)_s}}w_{\text{part}}\). This concludes the proof. □

**Lemma 3.49.** Under the assumption \(3.135\), there exist distributional derivatives of \(u, c\) and \(m\) from Corollary 3.47 with respect to time,

\[
\partial_t u \in L^2(I; \mathcal{U}), \quad \partial_t c \in L^{5/4}(I; \{H^2_4(\Omega)\}'), \quad \partial_t m \in L^2(I; \mathcal{M}'),
\]

such that the Galerkin solutions from Definition 3.33 satisfy

\[
\begin{align*}
\int_0^T \int_\Omega \partial_t u_n \cdot v \, dx \, dt & \to \int_0^T \langle \partial_t u, v \rangle_{W \times \mathcal{U}} \, dt \quad \forall v \in L^2(I; \mathcal{U}), \\
\int_0^T \int_\Omega \partial_t c_n \cdot \psi \, dx \, dt & \to \int_0^T \langle \partial_t c, \psi \rangle_{(H^2_4(\Omega)') \times H^2_4(\Omega)} \, dt \quad \forall \psi \in L^5(I; H^2_4(\Omega)), \\
\int_0^T \int_\Omega \partial_t m_n \cdot \theta \, dx \, dt & \to \int_0^T \langle \partial_t m, \theta \rangle_{\mathcal{M} \times \mathcal{M}'} \, dt \quad \forall \theta \in L^2(I; \mathcal{M}),
\end{align*}
\]

for some subsequence which will not be relabeled for the ease of notation. Moreover, for all \(v \in H^1(I; \mathcal{U})\) with \(v(T) \equiv 0\), \(\psi \in W^{1,5}(I; H^2_4(\Omega))\) with \(\psi(T) \equiv 0\) and \(\theta \in H^1(I; \mathcal{M})\) with \(\theta(T) \equiv 0\) the limits can be identified with

\[
\begin{align*}
\int_0^T \langle \partial_t u, v \rangle_{W \times \mathcal{U}} \, dt & = - \int_0^T \int_\Omega u \cdot \partial_t v \, dx \, dt - \int_\Omega u_{\text{init}} \cdot v(0) \, dx, \\
\int_0^T \langle \partial_t c, \psi \rangle_{(H^2_4(\Omega)') \times H^2_4(\Omega)} \, dt & = - \int_0^T \int_\Omega c \partial_t \psi \, dx \, dt - \int_\Omega c_{\text{init}} \psi(0) \, dx, \\
\int_0^T \langle \partial_t m, \theta \rangle_{\mathcal{M} \times \mathcal{M}'} \, dt & = - \int_0^T \int_\Omega m \cdot \partial_t \theta \, dx \, dt - \int_\Omega m_{\text{init}} \cdot \theta(0) \, dx
\end{align*}
\]

and for any arbitrary cut-off function \(\phi \in C_0^\infty(\Omega; \mathbb{R}^+_0)\) the distributional time derivative \(\partial_t (\phi c)\) of \((\phi c)\) is in \(L^{5/4}(I; \{H^2_4(\Omega)\}')\) and satisfies the identity

\[
\int_0^T \langle \partial_t (\phi c), \psi \rangle_{(H^2_4(\Omega)') \times H^2_4(\Omega)} \, dt = \int_0^T \langle \partial_t c, \phi \psi \rangle_{(H^2_4(\Omega)') \times H^2_4(\Omega)} \, dt \quad \forall \psi \in L^5(I; H^2_4(\Omega)). \quad (3.138)
\]
Proof: One might check the proof of Lemma 3.48 in order to see that

- $\left| \sqrt{f_p(c_n)} \right|$ has at most linear growth in $|c_n|$ which itself is uniformly bounded in $L^q(I \times \Omega)$ for $n \in \mathbb{N}$ as required by Lemma 3.39
- $\frac{\sqrt{f_p(c_n)}}{(c_n)}$ is uniformly bounded in $L^\infty(I \times \Omega)$ for $n \in \mathbb{N}$ as required by Lemma 3.41 and
- $\left| \frac{(c_n)}{\sqrt{f_p(c_n)}} \right|$ has at most linear growth in $|c_n|$ which itself is uniformly bounded in $L^\infty(I; L^2(\Omega))$ for $n \in \mathbb{N}$ as required by Lemma 3.42.

Combined with (3.130), (H2 3.2) and Corollary 3.38, it is evident that Lemma 3.39, Lemma 3.42, and Lemma 3.41 can be applied. Hence, a function $\partial_t u \in L^2(I; \mathcal{U}')$ exists such that

$$\int_0^T \int_\Omega \partial_t u_n \cdot \Phi \, dx \, dt \to \int_0^T \langle \partial_t u(t), \Phi(t) \rangle_{\mathcal{U}' \times \mathcal{U}} \, dt \quad \forall \Phi \in L^2(I; \mathcal{U})$$  \hspace{1em} (3.139)

for a subsequence, either by weak convergence in $L^2(I; \mathcal{U}')$ or weak-* convergence in the dual space $(L^2(I; \mathcal{U}'))'$ and a standard representation theorem, see e.g. [46] (see also Remark 3.46). The notation is already chosen to be $\partial_t u$ as it turns out that this function is the distributional derivative of $u$ with respect to time. This will be evident later on.

The limit satisfies the following relation. Let $\Phi \in H^1(I; \mathcal{U})$ with $\Phi(T) \equiv 0$.

$$\int_0^T \int_\Omega \partial_t u_n \cdot \Phi \, dx \, dt$$

$$= - \int_0^T \int_\Omega u_n \cdot \partial_t \Phi \, dx \, dt + \int_\Omega u_n(T) \cdot \Phi(T) \, dx - \int_\Omega u_0 \cdot \Phi(0) \, dx$$

$$\to - \int_0^T \int_\Omega u \cdot \partial_t \Phi \, dx \, dt - \int_\Omega u^\text{init} \cdot \Phi(0) \, dx$$

$$\equiv \int_0^T \langle \partial_t u, \Phi \rangle_{\mathcal{U}' \times \mathcal{U}} \, dt.$$  \hspace{1em} (3.139)

From here – using smooth compactly supported test functions in space and time – it immediately follows that $\partial_t u$ is the distributional derivative. This concludes the claim for the time derivatives of $u_n$. The treatment of $\partial_t c_n$ and $\partial_t m_n$ is completely analogous – using Lemma 3.39 and Lemma 3.41.

The last part of the proof is concerned with (3.138). The analogous arguments as before can be applied to $\partial_t (\phi c_n)$ due to boundedness, see Remark 3.40. Hence, by extracting yet another subsequence, one obtains (without relabeling) a sequence $(c_n)_{n \in \mathbb{N}}$ of Galerkin solutions such that both

$$\partial_t c_n \xrightarrow{\ast} \partial_t c \quad \text{in} \quad L^{5/4}(I; (H^2_0(\Omega))') \quad \text{and} \quad \partial_t (\phi c_n) \xrightarrow{\ast} \partial_t (\phi c) \quad \text{in} \quad L^{5/4}(I; (H^2_0(\Omega))').$$

By the approximation argument

$$\int_0^T \langle \partial_t (\phi c), \psi \rangle_{(H^2_0(\Omega))' \times H^2_0(\Omega)} \, dt = \int_0^T \langle \partial_t c, \phi \psi \rangle_{(H^2_0(\Omega))' \times H^2_0(\Omega)} \, dt$$

$$\int_0^T \int_\Omega \partial_t (\phi c_n) \psi \, dx \, dt = \int_0^T \int_\Omega \partial_t c_n (\phi \psi) \, dx \, dt,$$

the identity (3.138) follows. This concludes the proof. \hfill \square

With those convergence results, passing to the limit is straightforward. For convenience, recall the definition of $\mathcal{R}$ in (3.8).
Then, there exist functions
\[ u \in L^2(I; H^1_0(\Omega)^d \cap H(div)(\Omega)) \cap L^\infty(I; L^2(\Omega)^d) \cap H^1(I; (H^1_0(\Omega)^d \cap H(div)(\Omega))^s), \]
\[ c \in L^2(I; H^1(\Omega)) \cap L^\infty(I; L^2(\Omega)) \cap W^{1,5/4}(I; (H^2_0(\Omega))^s), \]
\[ R \in L^2(I; \mathcal{R}) \text{ with } R|_0 \in L^\infty(I; H^1(\Omega)) \cap L^2(I; H^{1,0}_{loc}(\Omega)), \]
\[ m \in L^2(I; H(div, curl)(\Omega)) \cap L^\infty(I; L^2(\Omega)^d) \cap L^2(I; H^{1,0}_{loc}(\Omega)^d) \cap H^1(I; (H^2_0(\Omega)^d)^s), \]

where the magnetic field is given as \( h := \nabla R \), such that for all
\[ \psi \in L^2(I; H^1_0(\Omega)^d \cap H(div)(\Omega)), \]
\[ S \in \mathcal{R}, \]
\[ \theta \in L^2(I; H^1_0(\Omega)^d), \]

the equations
\[ \rho_0 \int_0^T \langle \partial_t u, \psi \rangle_{(H^2_0(\Omega)^d \cap H(div)(\Omega))^s \times H^2_0(\Omega)^d \cap H(div)(\Omega))} \, dt + \int_0^T \int_\Omega 2\eta D\!u \cdot D\!v \, dx \, dt \]
\[ + \frac{\rho_0}{2} \int_0^T \int_\Omega (u \cdot \nabla)u \cdot v \, dx \, dt - \frac{\rho_0}{2} \int_0^T \int_\Omega (u \cdot \nabla)v \cdot u \, dx \, dt \]
\[ = -\mu_0 \int_0^T \int_\Omega (m \cdot \nabla)v \cdot (\alpha_1 \nabla R + \frac{\beta}{2} h_a) \, dx \, dt \]
\[ - \mu_0 \int_0^T \int_\Omega \text{div} \, m \cdot v \cdot (\alpha_1 \nabla R + \frac{\beta}{2} h_a) \, dx \, dt \]
\[ + \mu_0 \frac{\mu_0}{2} \int_0^T \int_\Omega (m \times (\alpha_1 \nabla R + \frac{\beta}{2} h_a)) \cdot \text{curl} \, v \, dx \, dt, \]

\[ \int_0^T \langle \partial_t c, \psi \rangle_{(H^2_0(\Omega))^s \times H^2_0(\Omega))} \, dt - \int_0^T \int_\Omega (c \cdot u \cdot \nabla \psi) \, dx \, dt + \sigma_c \int_0^T \int_\Omega \nabla c \cdot \nabla \psi \, dx \, dt \]
\[ - \int_0^T \int_\Omega \frac{K_{f_0}}{(c)^s} \left( -D(c)_s \nabla (g_s^L)'(c) \right. \]
\[ + \mu_0 (\nabla (\alpha_1 \nabla R + \frac{\beta}{2} h_a - \alpha_3 m)\cdot m) \cdot \nabla \psi \, dx \, dt = 0, \]

\[ \int_{\Omega'} \nabla R \cdot \nabla S \, dx = \int_{\Omega'} h_a \cdot \nabla S \, dx - \int_{\Omega} m \cdot \nabla S \, dx \quad \text{almost everywhere in } I = [0,T], \]

\[ \int_0^T \langle \partial_t m, \theta \rangle_{(H^2_0(\Omega)^d)^s \times H^2_0(\Omega)^d} \, dt \]
\[ - \int_0^T \int_\Omega \left( \left( u + \frac{K_{f_0}}{(c)^s} \right) \left[ -D(c)_s \nabla (g_s^L)'(c) \right. \]
\[ + \mu_0 (\nabla (\alpha_1 \nabla R + \frac{\beta}{2} h_a - \alpha_3 m)\cdot m) \]
hold and the initial data are attained in the sense
\[ (u(0), v) \in L^2(\Omega) \times H^1(\Omega) \]
\[ \langle c(0), \psi \rangle \in L^2(\Omega) \times L^2(\Omega) \]
\[ \langle m(0), \theta \rangle \in L^2(\Omega) \times L^2(\Omega) \]

Moreover, these functions satisfy the energy estimate
\[ \|u\|_{L^2(I;L^2(\Omega)^d)} + \|u\|_{L^2(I;H^1(\Omega)^d)} + \|g^L(c)\|_{L^\infty(I;L^1(\Omega))} + \frac{\sigma_c}{L} \|\nabla c\|_{L^2(I;L^2(\Omega)^d)} \\
+ \left\{ \begin{array}{ll}
\frac{\epsilon}{\sqrt{f_0(c)}} & \text{if } d = 3, \\
\frac{4}{3} & \text{if } d = 2
\end{array} \right. \]
\[ \|\nabla R\|_{L^2(I;H^1(\Omega)^d)} \leq C, \]
where \( q_d = \left\{ \begin{array}{ll}
\frac{10}{7} & \text{if } d = 3, \\
\frac{4}{3} & \text{if } d = 2
\end{array} \right. \).

\[ \|h\|_{L^2(I;L^2(\Omega)^d)} \leq C. \]

as well as the constants from the growth constraint of the susceptibility (3.4).

**Proof:** Recall the definitions of \( \mathcal{U} \) (cf. (3.14)), \( \mathcal{C} \) (cf. (3.15)) and \( \mathcal{M} \) (cf. (3.7)). At first, the following test functions will be considered in (3.140). Let
\[ \begin{aligned}
\bullet & \quad v \in L^\infty(I;C_0^\infty(\Omega)^d \cap \mathcal{U}) \quad \text{for (3.93a),} \\
\bullet & \quad \psi \in L^\infty(I;\mathcal{C}) \quad \text{for (3.93b),} \\
\bullet & \quad S \in \mathcal{R} \quad \text{for (3.93c),} \\
\bullet & \quad \theta \in L^\infty(I;C_0^\infty(\Omega)^d \cap \mathcal{M}) \quad \text{for (3.93d),}
\end{aligned} \]
be chosen arbitrarily. At the end of this proof, by a density argument, the test function spaces will be enlarged as specified by this theorem. Also notice that the choice of boundedness in time of the test functions \( v, \psi \) and \( \theta \) has been made for simplification, allowing to conclude sufficient time regularity without the need of specifying it precisely.

A weak formulation in space and time will be obtained for all equations except for the magnetostatic equation (3.93c), which does not include time derivatives. Hence, the test function \( S \) is not depending on time. Moreover, a notation like \( C_0^\infty(\Omega)^d \cap \mathcal{M} \) was used to emphasize that such a space is considered under the norm of \( \mathcal{M} \) (the \( H^d \)-norm) even though \( C_0^\infty(\Omega)^d \cap \mathcal{M} = C_0^\infty(\Omega)^d \) algebraically. Analogously, \( C_0^\infty(\Omega)^d \cap \mathcal{U} \) is endowed with the norm of \( \mathcal{U} \) (the \( H^3 \)-norm).

The existence result is done via an approximation argument by using the Galerkin approximations, see [Definition 3.33]. The passage to the limit is split into multiple steps. One step for the identification of \( w_{\text{part}} \) from [Lemma 3.48], one step for each equation in (3.93) and one last step for the remaining details. Moreover, let the reader be reminded that the limit of \( h_n = \nabla R_n \) as provided by [Corollary 3.47] is a gradient field \( h = \nabla R \).
3.3 Existence result for a regularized model

Limit of convective fluxes \( (\mathbf{v}_{\text{part}})_n \). One can identify the weak limit of the convective velocities \( (\mathbf{v}_{\text{part}})_n \) if the function \( \mathbf{w}_{\text{part}} \) from Lemma 3.48 is identified. Multiplying the definition of the convective velocities \( (3.128) \) by \( \frac{(c_n)_s}{\sqrt{f_p(c_n)}} \) and testing by \( \Phi \in C_0^\infty((0, T) \times \Omega)^d \) yields

\[
\int_0^T \int_{\Omega} \frac{-(c_n)_s}{\sqrt{f_p(c_n)}} \mathbf{v}_{\text{part}} \cdot \Phi \, dx \, dt = -KD \int_0^T \int_{\Omega} \sqrt{f_p(c_n)} \nabla \Pi_{c_n}(g^L_s)'(c_n) \cdot \Phi \, dx \, dt + \int_0^T \int_{\Omega} \frac{\mu_0 K}{(c_n)_s} (\alpha_1 \nabla h_n + \frac{\beta}{2} \nabla \mathbf{g}_n, h_n - \alpha_3 \nabla m_n)^T m_n \cdot \Phi \, dx \, dt.
\]

(3.144)

The limit of the first term on the right-hand side will be considered first. For this, recall that \( (g^L_s)' \) is differentiable with bounded derivative \( (g^L_s)'' \), see (3.73). Hence, the composition \( (g^L_s)'(c) \) is in \( H^1(\Omega) \) as \( c \in H^1(\Omega) \) as well.

\[
\left| \int_0^T \int_{\Omega} \sqrt{f_p(c_n)} \nabla \Pi_{c_n}(g^L_s)'(c_n) \cdot \Phi \, dx \, dt - \int_0^T \int_{\Omega} \sqrt{f_p(c)} \nabla (g^L_s)'(c) \cdot \Phi \, dx \, dt \right| \leq \left| \int_0^T \int_{\Omega} (\sqrt{f_p(c_n)} - \sqrt{f_p(c)}) \nabla \Pi_{c_n}(g^L_s)'(c_n) \cdot \Phi \, dx \, dt \right| + \left| \int_0^T \int_{\Omega} \sqrt{f_p(c_n)} \nabla \Pi_{c_n}(g^L_s)'(c_n) - \nabla (g^L_s)'(c) \cdot \Phi \, dx \, dt \right| =: J_1 + J_2.
\]

The term \( J_1 \) tends to zero because of the strong \( L^{2d} \)-convergence of \( \sqrt{f_p(c_n)} \), where \( d \geq 2 \), (cf. Lemma 3.48) and the \( L^2 \)-boundedness of \( \nabla \Pi_{c_n}(g^L_s)'(c_n) \). In fact, by

\[
\| \nabla \Pi_{c_n}(g^L_s)'(c_n) \|_{L^2(\Omega)^d} \overset{3.84}{\leq} \| \nabla (g^L_s)'(c_n) \|_{L^2(\Omega)^d} = \| (g^L_s)''(c_n) \nabla c_n \|_{L^2(\Omega)^d} \overset{3.73}{\leq} \frac{1}{d} \| \nabla c_n \|_{L^2(\Omega)^d}
\]

and (3.130) the boundedness follows. Hence, application of Hölder’s inequality proofs \( J_1 \to 0 \).

For the treatment of \( J_2 \) the weak convergence \( \nabla \Pi_{c_n}(g^L_s)'(c_n) \rightharpoonup (g^L_s)'(c) \) (for a subsequence without relabeling) in \( L^2(I \times \Omega)^d \) will be proven. From the boundedness of the term \( \nabla \Pi_{c_n}(g^L_s)'(c_n) \) (see above) one also gets \( \nabla \Pi_{c_n}(g^L_s)'(c_n) \rightharpoonup \mathbf{w}_g \) in \( L^2(I \times \Omega)^d \) for some \( L^2 \)-function \( \mathbf{w}_g \). Before identifying the limit, consider the term \( \Pi_{c_n}(g^L_s)'(c_n) \) for which

\[
\| \Pi_{c_n}(g^L_s)'(c_n) - (g^L_s)'(c) \|_{L^2(I \times \Omega)} \leq \| \Pi_{c_n}(g^L_s)'(c_n) - \Pi_{c_n}(g^L_s)'(c_n) \|_{L^2(I \times \Omega)} + \| \Pi_{c_n}(g^L_s)'(c_n) - (g^L_s)'(c) \|_{L^2(I \times \Omega)}.
\]

(3.145)

The last term on the right-hand side vanishes in the limit due to (3.88) and \( (g^L_s)'(c) \in L^2(I; L^2(\Omega)) \). The latter is due to the linear growth of \( \| (g^L_s)'(c_n) \| \) in terms of \( |c_n| \). For the first term on the right-hand side of (3.145), exploit the \( L^2 \)-stability of \( \Pi_{c_n} \) (cf. (3.84)) in order to estimate it by

\[
\| (g^L_s)'(c_n) - (g^L_s)'(c) \|_{L^2(I \times \Omega)} \overset{\text{Lemma 3.48}}{\leq} q_d > 0.
\]
Hence, $\Pi_{\varepsilon_n}(g_\varepsilon'(c_n)) \to (g_\varepsilon')'(c)$ in $L^2(I \times \Omega)$ and the limit $w_g$ is identified via testing by $\Psi \in C_0^\infty((0,T) \times \Omega)^d$, i.e.

$$
\int_0^T \int_\Omega \nabla \Pi_{\varepsilon_n}(g_\varepsilon'(c_n)) \cdot \Psi \, dx \, dt = - \int_0^T \int_\Omega \Pi_{\varepsilon_n}(g_\varepsilon'(c_n)) \div \Psi \, dx \, dt \\
\downarrow \\
\int_0^T \int_\Omega w_g \cdot \Psi \, dx \, dt = - \int_0^T \int_\Omega (g_\varepsilon')'(c) \div \Psi \, dx \, dt.
$$

Hence, $w_g = \nabla (g_\varepsilon')'(c)$ implying $J_2 \to 0$ as $\sqrt{f_p(c)} \Phi \in L^2(I \times \Omega)^d$ is fixed – for the integrability see (3.135) which guarantees $|\sqrt{f_p(c)}|$ to have at most linear growth in terms of $|c|$. Therefore,

$$- KD \int_0^T \int_\Omega \sqrt{f_p(c_n)} \nabla \Pi_{\varepsilon_n}(g_\varepsilon'(c_n)) \cdot \Phi \, dx \, dt \to - KD \int_0^T \int_\Omega \sqrt{f_p(c)} \nabla (g_\varepsilon')'(c) \cdot \Phi \, dx \, dt$$

follows.

For the second term on the right-hand side of (3.144) the compact support of $\Phi$ needs to be exploited, giving rise to higher regularity/convergence results for the magnetic variables. First, only consider the magnetic contribution of $(\nabla h_n)^T m_n$. The factor $\sqrt{f_p(c)}_{(c_n)_s}$ converges strongly in any $L^r(I \times \Omega), r \in [1, \infty)$ (cf. Lemma 3.48), the magnetic field $\nabla h_n$ converges weakly in $L^2(I; L^2_{loc}(\Omega)^{d \times d})$ (cf. (3.136)) and the magnetization $m_n$ converges strongly in $L^{q_d^{-}}(I; L^{q_d^{-}}(\Omega)^d)$ (cf. (3.136)). Note that $q_d > 2$. In conclusion, the term $\sqrt{f_p(c)}_{(c_n)_s} m_n \otimes \Phi$ converges strongly in $L^{q_d^{-}}(I; L^{q_d^{-}}(\supp \Phi)^d)$. Together with the weak convergence of $\nabla h_n$ in $L^2(I; L^2(\supp \Phi)^{d \times d})$ the passage to the limit is straightforward, i.e.

$$
\int_0^T \int_\Omega \sqrt{f_p(c_n)}_{(c_n)_s} (\nabla h_n)^T m_n \cdot \Phi \, dx \, dt = \int_0^T \int_{\supp \Phi} \sqrt{f_p(c_n)}_{(c_n)_s} (m_n \otimes \Phi) : \nabla h_n \, dx \, dt \\
\to \int_0^T \int_{\supp \Phi} \sqrt{f_p(c)}_{(c)_s} (m \otimes \Phi) : \nabla h \, dx \, dt = \int_0^T \int_\Omega \sqrt{f_p(c)}_{(c)_s} (\nabla h)^T m \cdot \Phi \, dx \, dt.
$$

The magnetic contribution $(\nabla m_n)^T m_n$ is treated in the same way and $(\nabla \Pi_{\varepsilon_n} h_n)^T m_m$ is even simpler given the strong convergence of $\nabla \Pi_{\varepsilon_n} h_n$ towards $\nabla h_n$ in $L^2(I \times \Omega)^{d \times d}$, see Lemma 3.26.

Therefore one obtains from (3.144) the identity

$$
\int_0^T \int_\Omega w_{\text{part}} \cdot \Phi \, dx \, dt = - KD \int_0^T \int_\Omega \sqrt{f_p(c)} \nabla (g_\varepsilon')'(c) \cdot \Phi \, dx \, dt \\
= - KD \int_0^T \int_\Omega \sqrt{f_p(c)} \nabla (g_\varepsilon')'(c) \cdot \Phi \, dx \, dt \\
+ \int_0^T \int_\Omega \mu_0 K \sqrt{f_p(c)}_{(c)_s} (\alpha_1 \nabla h + \frac{\beta}{2} \nabla h_n - \alpha_3 \nabla m)^T m \cdot \Phi \, dx \, dt
$$

for all $\Phi \in C_0^\infty((0,T) \times \Omega)^d$. Hence,

$$w_{\text{part}} = - KD \sqrt{f_p(c)} \nabla (g_\varepsilon')'(c) + K \mu_0 \sqrt{f_p(c)}_{(c)_s} (\alpha_1 \nabla h + \frac{\beta}{2} \nabla h_n - \alpha_3 \nabla m)^T m
$$

and consequently

$$
(v_{\text{part}})_n \to - KD \sqrt{f_p(c)}_{(c)_s} (g_\varepsilon')'(c) + K \mu_0 \sqrt{f_p(c)}_{(c)_s} (\alpha_1 \nabla h_n + \frac{\beta}{2} \nabla h_n - \alpha_3 \nabla m)^T m =: v_{\text{part}}
$$

in $L^2(I \times \Omega)^d$ by means of Lemma 3.48.
Passage to the limit in the momentum equation. From (3.148) one obtains

\[ \rho_0 \int_0^T \int_\Omega \partial_t u_n \cdot \Pi_{\Pi_n} v \, dx \, dt + \int_0^T \int_\Omega 2\eta D u_n \cdot D\Pi_{\Pi_n} v \, dx \, dt \]

\[ + \frac{\rho_0}{2} \int_0^T \int_\Omega (u_n \cdot \nabla) u_n \cdot \Pi_{\Pi_n} v \, dx \, dt - \frac{\rho_0}{2} \int_0^T \int_\Omega (u_n \cdot \nabla) \Pi_{\Pi_n} v \cdot u_n \, dx \, dt \]

\[ = \mu_0 \int_0^T \int_\Omega (\Pi_{\Pi_n} v \cdot \nabla)(\alpha_1 \nabla R_n + \frac{\beta}{2} \Pi_{\Pi_n} h_n) \cdot m_n \, dx \, dt \]

\[ + \mu_0 \int_0^T \int_\Omega (m_n \times (\alpha_1 \nabla R_n + \frac{\beta}{2} \Pi_{\Pi_n} h_n)) \cdot \Pi_{\Pi_n} v \, dx \, dt \]

\[ - D \int_0^T \int_\Omega (c_n) \nabla \Pi_{\Pi_n} (g^L_n)'(c_n) \cdot \Pi_{\Pi_n} v \, dx \, dt . \]  

(3.148)

One can easily identify the limits of the left-hand side due to Lemma 3.49, (3.136) and (3.88). Starting with the first term on the left-hand side one gets

\[ \rho_0 \int_0^T \int_\Omega \partial_t u_n \cdot \Pi_{\Pi_n} v \, dx \, dt \rightarrow \rho_0 \int_0^T \int_\Omega \partial_t u \cdot v \, dx \, dt \]  

The second term on the left-hand side converges as follows.

\[ \int_0^T \int_\Omega 2\eta D u_n \cdot D\Pi_{\Pi_n} v \, dx \, dt \rightarrow \int_0^T \int_\Omega 2\eta D u \cdot D v \, dx \, dt . \]

Noting \( q_d \) from (3.137) particularly satisfies \( q_d > 3 \), the limits of the third and fourth terms are given by

\[ \frac{\rho_0}{2} \int_0^T \int_\Omega (u_n \cdot \nabla) u_n \cdot \Pi_{\Pi_n} v \, dx \, dt \rightarrow \frac{\rho_0}{2} \int_0^T \int_\Omega (u \cdot \nabla) u \cdot \Pi_{\Pi_n} v \, dx \, dt , \]

\[ - \nabla u_n \rightarrow \nabla u \text{ in } L^2([0,T] ; L^2(\Omega)) \]

\[ - \frac{\rho_0}{2} \int_0^T \int_\Omega (u_n \cdot \nabla) \Pi_{\Pi_n} v \cdot u_n \, dx \, dt \rightarrow - \frac{\rho_0}{2} \int_0^T \int_\Omega (u \cdot \nabla) v \cdot u \, dx \, dt . \]

\[ \rightarrow \nabla v \text{ in } L^2([0,T] ; L^2(\Omega)) \]

The right-hand side of (3.148) will be considered next. Following the lines of the proof of Lemma 3.42, rewrite the sum of the first and third term on the right-hand side.

\[ \mu_0 \int_0^T \int_\Omega (\Pi_{\Pi_n} v \cdot \nabla)(\alpha_1 \nabla R_n + \frac{\beta}{2} \Pi_{\Pi_n} h_n) \cdot m_n \, dx \, dt \]

\[ - D \int_0^T \int_\Omega (c_n) \nabla \Pi_{\Pi_n} (g^L_n)'(c_n) \cdot \Pi_{\Pi_n} v \, dx \, dt \]

\[ - \frac{(c_n)}{K \sqrt{f_p(c)}} \text{ in } L^3([0,T] ; L^3(\Omega)) , \text{ see Lemma 3.48} \]

\[ = \int_0^T \int_\Omega \frac{(c_n)}{K \sqrt{f_p(c)}} \left[ \frac{(c_n)}{\sqrt{f_p(c)}} v_{p_{\text{part}}(c)} \right] \Pi_{\Pi_n} v \, dx \, dt \]

\[ \rightarrow \nabla v \text{ in } L^6([0,T] ; L^6(\Omega)) \]

\[ \rightarrow \int_0^T \int_\Omega \frac{(c_n)}{K \sqrt{f_p(c)}} w_{p_{\text{part}}} \cdot v \, dx \, dt \]
where in the last step $L^1$-regularity of $|c_s\nabla (g_{s}^L)'(c)|$ was used in order to split up the integrals in two parts. In fact, it has linear growth each in $|c|$ and $|\nabla c|$ which both are $L^2$-integrable (cf. Corollary 3.47). As $\text{div } \mathbf{v} = 0$ and $\mathbf{v}|_{\partial \Omega} = 0$, some parts vanish by using $(\nabla \mathbf{m})^T \mathbf{m} = \frac{1}{2} \nabla (|\mathbf{m}|^2)$ and

$$
(\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a) \cdot \mathbf{v} = \nabla \cdot \mathbf{g}(c),
$$

where $\mathbf{g}(c) := \begin{cases} 
1 & \text{if } c < L, \\
\frac{c}{2} & \text{if } L \leq c,
\end{cases}$ combined with integration by parts. Hence,

$$
I^u = - D \int_0^T \int_{\Omega} \nabla \cdot \mathbf{g}(c) \cdot \mathbf{v} \, dx \, dt + \mu_0 \int_0^T \int_{\Omega} (\mathbf{v} \cdot \nabla)(\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a) \cdot \mathbf{m} \, dx \, dt
$$

$$
- \mu_0 \alpha_3 \int_0^T \int_{\Omega} \mathbf{v} \cdot \nabla (|\mathbf{m}|^2) \, dx \, dt.
$$

Here, the integrand $\mathbf{v} \cdot \nabla (|\mathbf{m}|^2)$ is integrable because $\mathbf{v}$ has compact support in $\Omega$. Therefore, it could be extracted from the integrand of the second integral on the right-hand side. In case of the momentum equation, integration by parts can be used in order to arrive at a formulation which is well-defined for test functions without compact support as well. Both $\mathbf{h}$ and $\mathbf{h}_a$ are gradient fields. Hence, their gradient (which is the hessian of their potentials) is symmetric. Therefore,

$$
I^u = \mu_0 \int_0^T \int_{\Omega} (\mathbf{v} \cdot \nabla)(\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a) \cdot \mathbf{m} \, dx \, dt = \mu_0 \int_0^T \int_{\Omega} (\mathbf{m} \cdot \nabla)(\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a) \cdot \mathbf{v} \, dx \, dt
$$

$$
= - \mu_0 \int_0^T \int_{\Omega} (\mathbf{v} \cdot \nabla)\mathbf{m} \cdot (\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a) \, dx \, dt - \mu_0 \int_0^T \int_{\Omega} \text{div } \mathbf{m} \cdot \mathbf{v} \cdot (\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a) \, dx \, dt.
$$

The last remaining passage to the limit in the momentum equation is for the second term on the right-hand side of (3.148). Consider $\hat{\mathbf{h}}_n := \alpha_1 \mathbf{h}_n + \frac{\beta}{2} \Pi_{\mathbf{m}_a} \mathbf{h}_a$ which converges weakly to $\hat{\mathbf{h}} := \alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a$ in (at least) $L^2(I \times \Omega)^d$, see weak-$*$ convergence of $\mathbf{h}_n$ in (3.136) and strong convergence of $\Pi_{\mathbf{m}_a} \mathbf{h}_a$ in Lemma 3.26.

$$
\left| \int_0^T \int_{\Omega} (\mathbf{m}_n \times \hat{\mathbf{h}}_n) \cdot \text{curl } \Pi_{\mathbf{m}_a} \mathbf{v} \, dx \, dt - \int_0^T \int_{\Omega} \mathbf{m} \times \hat{\mathbf{h}} \cdot \mathbf{v} \, dx \, dt \right|
$$

$$
\leq \int_0^T \int_{\Omega} (\mathbf{m}_n \times \hat{\mathbf{h}}_n) \cdot (\text{curl } \Pi_{\mathbf{m}_a} \mathbf{v} - \text{curl } \mathbf{v}) \, dx \, dt
$$

$$
+ \int_0^T \int_{\Omega} [(\mathbf{m}_n \times \hat{\mathbf{h}}_n) - (\mathbf{m} \times \hat{\mathbf{h}})] \cdot \text{curl } \mathbf{v} \, dx \, dt = II^a + III^a.
$$

Due to weak convergence of $\hat{\mathbf{h}}_n$ in $L^2(I \times \Omega)^d$ and weak-$*$ convergence of $\mathbf{m}_n$ in $L^\infty(I; L^2(\Omega)^d)$ (see (3.136)) the product $\mathbf{m}_n \times \hat{\mathbf{h}}_n$ is bounded in $L^2(I; L^1(\Omega)^d)$. The projection $\Pi_{\mathbf{m}_a} \mathbf{v}$ converges
strongly towards \( \mathbf{v} \) in \( L^2(I; W^{1,\infty}(\Omega)^d) \) due to the embedding \( H^3(\Omega)^d \to W^{1,\infty}(\Omega)^d \) and (3.88). Hence, \( II^u \to 0 \). In \( III^u \) the compact support of the fixed test function \( \mathbf{v} \) allows to use local strong convergence of \( \mathbf{h}_n \) and \( \mathbf{m}_n \) (at least with respect to the \( L^2 \)-norm each). Therefore, \( III^u \to 0 \). In total,

\[
\frac{\mu_0}{2} \int_0^T \int_\Omega (\mathbf{m}_n \times (\alpha_1 \nabla R_n + \frac{\beta}{2} \Pi_{\Omega_n} \mathbf{h}_n)) \cdot \text{curl} \Pi_{\Omega_n} \mathbf{v} \, dx \, dt
\]

\[
- \frac{\mu_0}{2} \int_0^T \int_\Omega (\mathbf{m} \times (\alpha_1 \nabla R + \frac{\beta}{2} \mathbf{h}_n)) \cdot \text{curl} \mathbf{v} \, dx \, dt.
\]

Combining all convergence results of this paragraph, (3.140a) is obtained for a subset of test functions, see (3.143).

**Passage to the limit in the magnetic particles’ evolution equation.** From (3.93b) combined with (3.128) it follows that

\[
\int_0^T \int_\Omega \partial_t c_n \Pi_{\Omega_n} \psi \, dx \, dt - \int_0^T \int_\Omega (c_n)_s \mathbf{u}_n \cdot \nabla \Pi_{\Omega_n} \psi \, dx \, dt + \sigma_c \int_0^T \int_\Omega \nabla \mathbf{c}_n \cdot \nabla \Pi_{\Omega_n} \psi \, dx \, dt - \int_0^T \int_\Omega (c_n)_s (\mathbf{v}_{\text{part}})_n \cdot \nabla \Pi_{\Omega_n} \psi \, dx \, dt = 0.
\]

(3.149)

Convergence of the first term of (3.149) is evident from Lemma 3.49 and \( L^2 \)-orthogonality of the projection operator \( \Pi_{\Omega_n} \), i.e.

\[
\int_0^T \int_\Omega \partial_t c_n \Pi_{\Omega_n} \psi \, dx \, dt \xrightarrow{\text{3.69}} \int_0^T \int_\Omega \partial_t c_n \psi \, dx \, dt \to \int_0^T \langle \partial_t c, \psi \rangle_{(H^2(\Omega))' \times H^2(\Omega)} \, dt.
\]

The second term of (3.149) converges by using (3.88) combined with Sobolev’s embedding and Corollary 3.47.

\[
- \int_0^T \int_\Omega (c_n)_s \mathbf{u}_n \cdot \nabla \Pi_{\Omega_n} \psi \, dx \, dt \to \int_0^T \int_\Omega (c)_s \mathbf{u} \cdot \nabla \psi \, dx \, dt.
\]

The convergence of the third term (for fixed \( \sigma_c \)) is obvious due to linearity, i.e.

\[
\sigma_c \int_0^T \int_\Omega \nabla \mathbf{c}_n \cdot \nabla \Pi_{\Omega_n} \psi \, dx \, dt \to \sigma_c \int_0^T \int_\Omega \nabla \mathbf{c} \cdot \nabla \psi \, dx \, dt.
\]

Analogously to the second term, the fourth term converges as follows.

\[
- \int_0^T \int_\Omega (c_n)_s (\mathbf{v}_{\text{part}})_n \cdot \nabla \Pi_{\Omega_n} \psi \, dx \, dt \to - \int_0^T \int_\Omega (c)_s \mathbf{v}_{\text{part}} \cdot \nabla \psi \, dx \, dt.
\]

Notice that for the strong convergence of the projection Sobolev’s embedding in at most three dimensions is sufficient. Combining all convergence results of this paragraph and (3.147), (3.140b) is obtained for a subset of test functions, see (3.143).
Passage to the limit in the magnetostatic equation. From (3.93c) one infers for fixed \( t \in I \)
\[
\int_{\Omega'} \nabla R_n(t) \cdot \nabla \Pi_{\mathbb{R}_n} S \, dx = \int_{\Omega'} h_n(t) \cdot \nabla \Pi_{\mathbb{R}_n} S \, dx - \int_{\Omega} m_n(t) \cdot \nabla \Pi_{\mathbb{R}_n} S \, dx. \tag{3.150}
\]
Next, (3.150) will be multiplied by a function \( \varphi \in C^\infty_0((0, T)) \) and integrated in time. As \( \varphi \) does not depend on the spatial variable, one can write
\[
\int_0^T \int_{\Omega'} \nabla R_n \cdot \nabla \Pi_{\mathbb{R}_n}(S\varphi) \, dx \, dt = \int_0^T \int_{\Omega'} h_n \cdot \nabla \Pi_{\mathbb{R}_n}(S\varphi) \, dx \, dt - \int_0^T \int_{\Omega} m_n \cdot \nabla \Pi_{\mathbb{R}_n}(S\varphi) \, dx \, dt.
\]
Taking the limit is straightforward by exploiting weak convergence of \( \nabla R_n, m_n \) in \( L^2(I \times \Omega)^d \) (as a consequence of the weak-\( * \) convergence from (3.136)) and strong convergence of the projections in \( L^2(I \times \Omega)^d \). Indeed, the projections \( \Pi_{\mathbb{R}_n}(S\varphi(t)) = \varphi(t)\Pi_{\mathbb{R}_n} S \) converge for fixed time towards \( \varphi(t)\nabla S \) in \( L^2(\Omega')^d \), see (3.83), and
\[
|\varphi(t)||\Pi_{\mathbb{R}_n} S - \nabla S|_{L^2(\Omega')^d} \leq |\varphi(t)||\Pi_{\mathbb{R}_n} S - S|_R
\]
is uniformly bounded for \( t \in I \) by a constant, which is integrable on the bounded interval \([0, T]\). By Lebesgue’s dominated convergence theorem the projections converge in \( L^2(I \times \Omega)^d \) and therefore
\[
\int_0^T \int_{\Omega'} \nabla R \cdot \nabla S \, dx \, \varphi \, dt = \int_0^T \int_{\Omega'} h_n \cdot \nabla S \, dx \, \varphi \, dt - \int_0^T \int_{\Omega} m \cdot \nabla S \, dx \, \varphi \, dt.
\]
As \( \varphi \in C^\infty_0((0, T)) \) was arbitrary, by the fundamental lemma of calculus of variations, the equations
\[
\int_{\Omega'} \nabla R(t) \cdot \nabla S \, dx = \int_{\Omega'} h_n(t) \cdot \nabla S \, dx - \int_{\Omega} m(t) \cdot \nabla S \, dx \quad \forall S \in \mathbb{R}
\]
hold for almost all \( t \in I \) which is identical to (3.140c).

Passage to the limit in the magnetization equation. It is evident from (3.93d) and Definition 3.33 that
\[
\int_0^T \int_{\Omega} \partial_t m_n \cdot \Pi_{\mathbb{M}_n} \theta \, dx \, dt - \int_0^T \int_{\Omega} (u_n + (v_{\text{part}})_n) \cdot \nabla \Pi_{\mathbb{M}_n} \theta \cdot m_n \, dx \, dt \\
+ \sigma \int_0^T \int_{\Omega} \text{div} \ m_n \, \text{div} \ \Pi_{\mathbb{M}_n} \theta \, dx \, dt + \sigma \int_0^T \int_{\Omega} \text{curl} \ m_n \cdot \text{curl} \ \Pi_{\mathbb{M}_n} \theta \, dx \, dt \\
= \frac{1}{2} \int_{\Omega} \int_{0}^{T} (m_n \times \Pi_{\mathbb{M}_n} \theta) \cdot \text{curl} \ u_n \, dx \, dt \\
- \frac{1}{\tau_{\text{rel}}} \int_{\Omega} \int_{0}^{T} (m_n - \chi(c_n, \nabla R_n) \nabla R_n) \cdot \Pi_{\mathbb{M}_n} \theta \, dx \, dt. \tag{3.151}
\]
The first term on the left-hand side is handled via Lemma 3.49 and \( L^2 \)-orthogonality of the projection operator \( \Pi_{\mathbb{M}_n} \), i.e.
\[
\int_0^T \int_{\Omega} \partial_t m_n \cdot \Pi_{\mathbb{M}_n} \theta \, dx \, dt \overset{3.34}{=} \int_0^T \int_{\Omega} \partial_t m_n \cdot \theta \, dx \, dt \rightarrow \int_0^T \langle \partial_t m, \theta \rangle_{\mathbb{M}' \times \mathbb{M}} \, dt.
\]
Exploiting the local regularity of the magnetic variables, the limit of the second term on the left-hand side of (3.151) is computed as follows. Be aware that the sum \( u_n + (v_{\text{part}})_n \) converges
weakly to \( u + v_{\text{part}} \) in \( L^2(I \times \Omega)^d \), where the convergence is determined by the less regular term \( (v_{\text{part}})_n \), see (3.147). Then,

\[
\left| \int_0^T \int_\Omega \left( (u_n + (v_{\text{part}})_n) \cdot \nabla \right) \Pi_{M_n} \theta \cdot m_n \, dx \, dt - \int_0^T \int_\Omega \left( (u + (v_{\text{part}})) \cdot \nabla \right) \theta \cdot m \, dx \, dt \right|
\]

\[
\leq \left| \int_0^T \int_\Omega \left( (u_n + (v_{\text{part}})_n - u - v_{\text{part}}) \cdot \nabla \right) \theta \cdot m \, dx \, dt \right|
\]

\[
+ \left| \int_0^T \int_\Omega \left( (u_n + (v_{\text{part}})_n) \cdot \nabla \right) \theta \cdot (m_n - m) \, dx \, dt \right|
\]

\[
+ \left| \int_0^T \int_\Omega \left( (u_n + (v_{\text{part}})_n) \cdot \nabla \right) (\Pi_{M_n} \theta - \theta) \cdot m_n \, dx \, dt \right|.
\]

The first term of the right-hand side of (3.152) converges to zero due to weak convergence of \( u_n + (v_{\text{part}})_n \) in \( L^2(I \times \Omega)^d \) and the sufficient integrability of the compactly supported test function \( \theta \) and \( m \in L^{q_d - 1}(I; L^\infty_{\text{loc}}(\Omega)) \). In the second term, higher regularity of \( m_n \) will be used, hence by boundedness of \( (u_n + (v_{\text{part}})_n) \) we have \( L^2(I \times \Omega)^d \) and strong (local) convergence of \( m_n \to m \) in \( L^{q_d - 1}(I; L^\infty_{\text{loc}}(\Omega)^d) \) this term converges to zero. The local strong convergence of \( m_n \) was applicable due to the compact support of \( \theta \). In the last term, boundedness of \( (u_n + (v_{\text{part}})_n) \) in \( L^2(I \times \Omega)^d \) and of \( (m_n) \) in \( L^{\infty} \) will be used. For convergence it is sufficient that \( \nabla \Pi_{M_n} \theta \to \nabla \theta \) in \( L^2(I; L^{\infty}(\Omega)^d) \). But as \( H^3(\Omega)^d \to W^{1,\infty}(\Omega)^d \) this holds true, see (3.88). Therefore,

\[
\left| \int_0^T \int_\Omega \left( (u_n + (v_{\text{part}})_n) \cdot \nabla \right) \Pi_{M_n} \theta \cdot m_n \, dx \, dt - \int_0^T \int_\Omega \left( (u + v_{\text{part}}) \cdot \nabla \right) \theta \cdot m \, dx \, dt \right|
\]

Due to (3.88) and weak convergence of the magnetic variables \( m_n \) and \( h_n \) in \( L^2(I; H(\text{div}, \text{curl})(\Omega)) \) (cf. Corollary 3.47) the third and fourth term on the right-hand side of (3.151) converge obviously,

\[
\sigma \int_0^T \int_\Omega \text{div} m_n \text{div} \Pi_{M_n} \theta \, dx \, dt \to \sigma \int_0^T \int_\Omega \text{div} m \text{div} \theta \, dx \, dt,
\]

\[
\sigma \int_0^T \int_\Omega \text{curl} m_n \cdot \text{curl} \Pi_{M_n} \theta \, dx \, dt \to \sigma \int_0^T \int_\Omega \text{curl} m \cdot \text{curl} \theta \, dx \, dt.
\]

The first term on the right-hand side of (3.151) is treated similarly to the second term of the left-hand side, i.e. the compact support of \( \theta \) will be exploited.

\[
\left| \frac{1}{2} \int_0^T \int_\Omega (m_n \times \Pi_{M_n} \theta) \cdot \text{curl} u_n \, dx \, dt - \frac{1}{2} \int_0^T \int_\Omega (m \times \theta) \cdot \text{curl} u \, dx \, dt \right|
\]

\[
\leq \left| \frac{1}{2} \int_0^T \int_\Omega (m \times \theta) \cdot (\text{curl} u_n - \text{curl} u) \, dx \, dt \right|
\]

\[
+ \left| \frac{1}{2} \int_0^T \int_\Omega ((m_n - m) \times \theta) \cdot \text{curl} u_n \, dx \, dt \right|
\]

\[
+ \left| \frac{1}{2} \int_0^T \int_\Omega (m_n \times (\Pi_{M_n} \theta - \theta)) \cdot \text{curl} u_n \, dx \, dt \right|.
\]

The first term on the right-hand side above converges to zero due to weak convergence of \( \text{curl} u_n \) towards \( \text{curl} u \) in \( L^2(I \times \Omega)^d \) (cf. Corollary 3.47) and sufficient integrability of \( m \) and \( \theta \) (which is even bounded). The terms \( \text{curl} u_n \) are uniformly bounded in \( L^2(I \times \Omega)^d \), hence the second term
vanishes in the limit as \(m_n \to m\) strongly in \(L^2(I \times \text{supp } \theta)^d\). Finally, as the functions \(m_n\) and \(\text{curl } u_n\) are uniformly bounded in \(L^\infty (I; L^2(\Omega)^d)\) or \(L^2 (I \times \Omega)^d\), respectively, the convergence of \(\Pi_{M_n} \theta\) in \(L^2(I; L^\infty(\Omega)^d)\) is sufficient to show that the third term of the right-hand side above converges to zero. Due to \(H^3(\Omega)^d \to L^\infty(\Omega)^d\) this easily follows from (3.88). In conclusion,

\[
\frac{1}{2} \int_0^T \int_\Omega (m_n \times \Pi_{M_n} \theta) \cdot \text{curl } u_n \, dx \, dt \to \frac{1}{2} \int_0^T \int_\Omega (m \times \theta) \cdot \text{curl } u \, dx \, dt.
\]

The last term on the right-hand side of (3.151) will be split into two parts, i.e.

\[
-\frac{1}{\tau_{rel}} \int_0^T \int_\Omega m_n \cdot \Pi_{M_n} \theta \, dx \, dt \to -\frac{1}{\tau_{rel}} \int_0^T \int_\Omega m \cdot \theta \, dx \, dt
\]

and the term

\[
\frac{1}{\tau_{rel}} \int_0^T \int_\Omega \chi(c_n, h_n) h_n \cdot \Pi_{M_n} \theta \, dx \, dt,
\]

which requires more involved treatment. For this, a subsequence will be extracted such that \((h_n)_{n \in \mathbb{N}}\) and \((c_n)_{n \in \mathbb{N}}\) converge pointwise almost everywhere in \(I \times \Omega\). The susceptibility (cf. (2.32)) is a continuous function (it has a continuous extension when the second argument goes to zero), hence for a pointwise almost everywhere in \(I \times \Omega\) converging subsequence (not relabeled) of \(c_n\) and \(h_n\) one easily gets pointwise convergence almost everywhere in \(I \times \Omega\) for \(\chi(c_n, h_n)\). From (3.4) if follows that \(|\chi(c_n, h_n)|^2\) is as regular as \(|c_n|\) and therefore \(\chi(c_n, h_n)\) is bounded in \(L^{2q_d^*}(I \times \Omega)\). One can conclude convergence \(\chi(c_n, h_n) \to \chi(c, h)\) in \(L^4(I \times \Omega)\). The technique which is used to prove this convergence is the same as has already been used in Lemma 3.48, i.e. uniform \(L^{2q_d^*}\)-estimates yield uniform \(L^4\)-integrability (as \(q_d > 2\)) which leads to convergence by Vitali’s theorem. With that at hand, one obtains in a straightforward way,

\[
\frac{1}{\tau_{rel}} \int_0^T \int_\Omega \chi(c_n, h_n) h_n \cdot \Pi_{M_n} \theta \, dx \, dt \to \frac{1}{\tau_{rel}} \int_0^T \int_\Omega \chi(c, h) h \cdot \theta \, dx \, dt.
\]

In total,

\[
-\frac{1}{\tau_{rel}} \int_0^T \int_\Omega (m_n - \chi(c_n, h_n)) h_n \cdot \Pi_{M_n} \theta \, dx \, dt \to -\frac{1}{\tau_{rel}} \int_0^T \int_\Omega (m - \chi(c, h)) h \cdot \theta \, dx \, dt.
\]

By combining this with the previous results, the equation (3.140d) is obtained for a subset of test functions, see (3.143).

**Remaining details.** Finally, another subsequence can be extracted such that for instance \((u_n)_{n \in \mathbb{N}}\) converges weakly in \(L^{q_d}(I; L^{q_d}(\Omega)^d)\). Due to weak lower semi-continuity of norms one obtains

\[
\|u\|_{L^\infty(I; L^2(\Omega)^d)} + \|u\|_{L^2(I; H^1(\Omega)^d)} + \|\partial_t u\|_{L^2(I; W^{1,\infty})} + \|u\|_{L^{q_d}(I; L^{q_d}(\Omega)^d)} \leq C
\]

from Corollary 3.38, Lemma 3.42 and (3.130). Analogously,

\[
\|m\|_{L^\infty(I; L^2(\Omega)^d)} + \|m\|_{L^2(I; H(div, \text{curl} ))(\Omega)} + \|m\|_{L^{q_d}(I; L^{q_d}(\Omega)^d)}
\]

\[
+ \|h\|_{L^\infty(I; L^2(\Omega)^d)} + \|h\|_{L^2(I; H(div, \text{curl} ))(\Omega \setminus \partial \Omega)} + \|h\|_{L^{q_d}(I; L^{q_d}(\Omega)^d)}
\]

\[
+ \frac{\sigma_c}{L} \|\nabla c\|_{L^2(I; L^2(\Omega)^d)} + C_{L, \sigma_c} \|\partial c\|_{L^{q_d}(I; H^1(\Omega)^d)} + C_{L, \sigma_c} \|c\|_{L^{q_d}(I; L^{q_d}(\Omega)^d)} \leq C,
\]

(3.154)

(3.153)
by Corollary 3.38, Lemma 3.39 and (3.130). From Lemma 3.48 (3.147) and (3.146) it follows that
\[ \| \nabla (c_s)_{\sqrt{f(c)}} \|_{L^2(I;L^2(\Omega))} \leq \| g_s \|_{L^2(I;L^2(\Omega))} \leq C, \] where \( C > 0 \) does not depend on \( s, L \) or \( \sigma_c \). Moreover, \( g_s \) is continuous. For a subsequence, pointwise convergence \( g_s(c_n) \rightarrow g_s(c) \) almost everywhere in \( I \times \Omega \) is evident and – by uniform estimates (yielding uniform integrability) and Vitali’s convergence theorem – strong convergence in e.g. \( L^1(I \times \Omega) \) is obvious. Therefore,
\[ \int_0^T \int_\Omega g_s(c_n) \varphi \, dx \, dt \rightarrow \int_0^T \int_\Omega g_s(c) \varphi \, dx \, dt \quad \forall \varphi \in C^\infty(0,T \times \Omega). \]
By the uniform bound \( \| g_s(c_n) \|_{L^\infty(I;L^2(\Omega))} \leq C \) from (3.130), a weakly-\( * \) converging subsequence exists, i.e. there is a function \( y \in L^\infty(I; L^2(\Omega)) \) such that
\[ \int_0^T \int_\Omega g_s(c_n) \varphi \, dx \, dt \rightarrow \int_0^T \int_\Omega y \varphi \, dx \, dt \quad \forall \varphi \in C^\infty(0,T \times \Omega) \subset L^1(I; L^2(\Omega)). \]
Hence, by the fundamental lemma of variational calculus,
\[ \| g_s(c) \|_{L^\infty(I;L^2(\Omega))} = \| y \|_{L^\infty(I;L^2(\Omega))} \leq C, \] where weak lower semi-continuity of norms was used as well. The constant \( C > 0 \) does not depend on the regularization parameters \( (s, L, \sigma_c) \). The estimates for \( \partial_t \mathbf{m}_n \) and \( \partial_t \mathbf{h}_n \) follow easily from Lemma 3.42, Lemma 3.41 and weak lower semi-continuity of norms. Collecting all estimates from (3.153)-(3.156), one obtains (3.141).

The test functions \( \theta \) are in \( L^\infty([0,T]; C^\infty_c(\Omega)^d) \) which is dense in \( L^2(I; H^3_0(\Omega)^d) \). The latter can easily be verified by choosing an approximation with tolerance \( \varepsilon > 0 \) by a simple function in time (cf. e.g. [46]) where for the \( \varepsilon \)-th step of the simple function one can additionally approximate in space with a tolerance of the type \( \varepsilon 2^{-i} \) due to the dense embedding \( C^\infty_c(\Omega)^d \subset H^3_0(\Omega)^d \). Hence, there is an approximation of tolerance \( 2\varepsilon \) which lies in the space \( L^\infty([0,T]; C^\infty_c(\Omega)^d) \) as the simple function is bounded in time due to its finitely many steps. The analogue arguments apply for the continuous extensions of the other spaces of test functions.

Therefore, there exists a continuous extension of \( \partial_t \mathbf{m} \) onto \( L^2(I; H^3_0(\Omega)^d) \) which will be denoted by \( \partial_t \mathbf{m} \) again. This finally yields (3.140b). The continuous extension of \( \partial_t c \) is straightforward – leading to (3.140a). For the momentum equation use the continuous extension of the domain \( C^\infty_c(\Omega)^d \) to \( H^3(\Omega)^d \) onto its \( H^3 \)-closure – given by \( H^3_0(\Omega)^d \cap H(\text{div}\, v)(\Omega) \) to obtain (3.140a). The latter can be proven by adapting the corresponding result for the closure in \( H^1(\Omega)^d \) which can be found in [32, p. 196, Theorem III.4.1]. The approach in [32] is based on star-shaped sets. A Lipschitz-boundary (see [H1 3.1]) is sufficient to apply the arguments from [32] as they can be written as finite union of star-shaped domains. Let \( v \in H^3_0(\Omega)^d \cap H(\text{div}\, v)(\Omega) \) be approximated in the \( H^3 \)-norm by a sequence \( (v_k)_{k \in \mathbb{N}} \in C^\infty_0(\Omega)^d \). Then there is a sequence of solutions \( (w_k)_{k \in \mathbb{N}} \in H^3_0(\Omega)^d \) with compact support to the problem
\[ \text{div} \, w_k = - \text{div} \, v_k, \]
according to [32] Theorem III.3.1 and Remark III.3.2], which satisfies the stability estimate
\[ \| w_k \|_{H^3(\Omega)^d} \lesssim \| \text{div} \, v_k \|_{H^2(\Omega)}. \]
Altogether, \( x_k := v_k + w_k \) is in \( H^3_0(\Omega) \cap H(\text{div}\, v)(\Omega) \) with compact support and
\[ \| x_k - v \|_{H^3(\Omega)^d} \leq \| v_k - v \|_{H^3(\Omega)^d} + \| \text{div} \, v_k \|_{H^2(\Omega)} \rightarrow 0 \]
as \text{div} \, v = 0. Due to compact support of \( x_k \), mollifications thereof converge towards \( v \) as well and are elements of \( C^\infty_0(\Omega)^d \cap H(\text{div}\, v)(\Omega) \) for sufficiently small support of their respective mollifying kernels. Hence, the space \( C^\infty_0(\Omega)^d \cap H(\text{div}\, v)(\Omega) \) is dense in \( H^3_0(\Omega)^d \cap H(\text{div}\, v)(\Omega) \).
The constant $C > 0$ of the stability estimate \((3.141)\) relies on \((3.130)\) only, hence it only depends on initial data, $\Omega$, $T$ and

$$\liminf_{n \to \infty}(\| \text{div} \Pi_{3c} h_a - \| H^2(I; L^2(\Omega)) + \| H^2(I; L^2(\Omega'))^d) \| h_a - \| H^2(I; L^2(\Omega'))^d) \|,$$

see \((3.126)\).

The convergence of solutions at time $t = 0$ towards initial data in the distributional sense is guaranteed in virtue of Sobolev’s embedding. It follows from $u_n \rightarrow u$ in $H^1(I; U')$, see e.g. Lemma \(3.49\) that $u_n \rightarrow u$ in $C([0, T]; U')$. Hence, $u_n(0) \rightarrow u(0)$ in $U'$ and consequently

$$(u(0), \Psi)_{U \times U} \leftarrow (u_n(0), \Psi)_{U \times U} = \int_{\Omega} u_n(0) \cdot \Psi \, dx \rightarrow \int_{\Omega} u_{\text{init}} \cdot \Psi \, dx \quad \forall \Psi \in \mathbb{U}.$$

The other claims about initial data can be proven analogously. □

**Remark 3.51.** In \(\text{Theorem 3.50}\) we stated the regularity $\partial_t m \in L^2(I; H^3_0(\Omega)^d)$). In fact, the regularity is slightly better, i.e. $\partial_t m \in L^2(I; M')$ according to \(\text{Lemma 3.49}\). Analogously, the space of test functions in \((3.140c)\) could be enlarged from $H^3_0(\Omega)^d$ to $M$. On the other hand, this does not yield any advantages and the identification \((3.140d)\) of the distribution $\partial_t m$ is only known on the domain $H^3_0(\Omega)^d$ (see the passage to the limit in the magnetization equation). Therefore, we have chosen to use the simpler space everywhere in order to ease the readability of the theorem. The same applies to $\partial_t u \in L^2(I; U')$ as well.

**Remark 3.52.** The solution concept in \(\text{Theorem 3.50}\) is of distributional nature due to the restrictive space of test functions in \((3.140c)\). Hence, in case of smooth solutions, identification of boundary data is out of reach. However, the strong formulation of the equations on the bulk can be recovered by the analogous procedure as in \(\text{Remark 3.71}\) which discusses this issue on regard of the solution concept of the main existence result, \(\text{Theorem 3.69}\).

**Remark 3.53.** The $H^3$-regularity on $\Omega$ of external magnetic fields $h_a$ (cf. \(H^2 3.2\)) is only needed due to technicalities – strong convergence of $\nabla \Pi_{3c} h_a$ in \(\text{Section 3.3}\) is obscurely linked to $H^3$-regularity of the basis functions in \((3.33)\) (guaranteeing stability \((3.86)\)). However, regarding the external magnetic field, the estimate in \((3.141)\) only depends on the $L^2$-norms of $h_a$ and $\partial_t h_a$. Hence, the existence result might be improved in regards to the assumptions on the data $h_a$.

Finding a $L^2$-converging approximation $h_a^n$ of $h_a$ such that $h_a^n$ satisfies \(H^2 3.2\) is not trivial and does not serve much purpose, though. If one considers the magnetic field to be a superposition of dipole fields, the regularity in \(H^2 3.2\) is satisfied anyway (apart from the respective sources).

### 3.4 The non-regularized case

This sections is concerned with the limit for $s \to 0$, $L \to \infty$ and $\sigma_c \to 0$. While the result in \(\text{Theorem 3.50}\) is rather general, in the case $\sigma_c \to 0$ a special setting will be considered as some uniform bounds will be lost. We choose

$$f_p(c) := c^2 \quad \text{and} \quad d = 2.$$

**Definition 3.54.** Under the assumptions of \(\text{Theorem 3.50}\), consider functions $(u_n, c_n, R_n, m_n)_{n \in \mathbb{N}}$ such that for any $n \in \mathbb{N}$ one finds $(u_n, c_n, R_n, m_n)$ to be the functions as described in \(\text{Theorem 3.50}\) corresponding to $(u, c, R, m)$ in the notation of \(\text{Theorem 3.50}\) for the specific choice of regularization parameters

$$s = s(n) := \frac{1}{n}, \quad L = L(n) := 3n, \quad \text{and} \quad \sigma_c = \sigma_c(n) := \frac{1}{n},$$
external force $\mathbf{h}_n$, as well as

$$f^n_p(c) := (c_n)_{s(n)}(c_n)_{s(n)}^{L(n)},$$

(3.157)

where for $x \in \mathbb{R}$

$$(x)^L := \min(L, x),$$
$$(x)^s := \max(s, x),$$
$$(x)^L_s := ((x)^L)_s = \min(s, \max(L, x)) \quad (\text{for } s < L).$$

The sequence members $(\mathbf{u}_n, c_n, R_n, \mathbf{m}_n)$ will be called "regularized solutions". The "regularized convective velocity" of the magnetic particles is defined as

$$\mathbf{v}_{\text{part}}(n) = -KD \left\{(c_n)^{L(n)}_{s(n)} - (g_{s(n)}^{L(n)})'c_n \nabla c_n \right\}$$

$$= -KD\nabla c_n + \mu_0 K \left((c_n)^{L(n)}_{s(n)}(\alpha_1 \nabla \mathbf{h}_n + \frac{\beta}{2} \nabla \mathbf{h}_n - \alpha_3 \nabla \mathbf{m}_n)\right)^T \mathbf{m}_n$$

(3.159)

for all $n \in \mathbb{N}$. Moreover, the usual notation for the magnetic field,

$$\mathbf{h}_n := \nabla R_n$$

(3.160)

for all $n \in \mathbb{N}$, will be used.

**Corollary 3.55.** A sequence of regularized solutions $(\mathbf{u}_n, c_n, R_n, \mathbf{m}_n)_{n \in \mathbb{N}}$ as described above in Definition 3.53 exists and satisfies

$$\|\mathbf{u}_n\|_{L^\infty(I; L^2(\Omega)^d)} + \|\mathbf{u}_n\|_{L^2(I; H^1(\Omega)^d)} + \|\mathbf{v}_{\text{part}}(n)\|_{L^2(I; \Omega)^d} + \|g_{s(n)}(c_n)\|_{L^\infty(I; L^1(\Omega))}$$

$$+ \|c_n\|_{L^\infty(I; L^1(\Omega))} + \|g_{s(n)}(c_n)\|_{C^0(I; L^1(\Omega))} + \|\mathbf{m}_n\|_{L^\infty(I; L^2(\Omega)^d)} + \|\mathbf{h}_n\|_{L^\infty(I; L^2(\Omega)^d)}$$

$$+ \|\mathbf{m}_n\|_{L^2(I; H^1(\nabla \mathbf{u}_n)(\Omega))} + \|\mathbf{h}_n\|_{L^2(I; H^1_0(\nabla \mathbf{u}_n)(\Omega))}$$

$$+ \|\partial_t \mathbf{u}_n\|_{L^2(I; H^2(\nabla \mathbf{u}_n)(\Omega))} + \|\partial_t \mathbf{m}_n\|_{L^2(I; H^2(\nabla \mathbf{u}_n)(\Omega))} + \|\partial_t \mathbf{h}_n\|_{L^2(I; H^2(\nabla \mathbf{u}_n)(\Omega))}$$

$$+ \|\mathbf{u}_n\|_{L^d(I; \Omega)^d} + \|\mathbf{m}_n\|_{L^d(I; L^q_d(\Omega)^d)} + \|\mathbf{h}_n\|_{L^d(I; L^q_d(\Omega)^d)} \leq C$$

(3.161)

uniformly in $n \in \mathbb{N}$, where

$$q_d = \begin{cases} 10 & \text{if } d = 3, \\ 4 & \text{if } d = 2, \end{cases}$$

and $g_{s(n)} := g_{s(n)}^\infty$ (cf. (3.73) and the text below).

**Proof:** The choices of $s(n), L(n), \sigma_c(n), \mathbf{h}_n$ and $f_p$ satisfy the assumptions of Theorem 3.50. Indeed,

$$0 < s(n)^2 \leq (c)_{s(n)}(c)^{L(n)}_{s(n)} = f_p(c) \leq L(n)(c)_{s(n)} \leq L(n)|c|^1 + L(n)s(n)$$

yields (3.135) for any fixed $n \in \mathbb{N}$.
The regularized solutions satisfy the estimate in (3.141). Omitting all terms on the left-hand side of (3.141) which depend on the regularization parameters \( s(n) \), \( L(n) \) or \( \sigma_c(n) \) one ends up with the estimate

\[
\|u_n\|_{L^\infty(I; L^2(\Omega)^d)} + \|u_n\|_{L^2(I; H^1(\Omega)^d)} + \|g_{s(n)}^{L(n)}(c_n)\|_{L^\infty(I; L^1(\Omega))} \\
+ \left\| \frac{(c_n)}{f_p(c_n)} (\nu_{\text{part}})_n \right\|_{L^2(I; L^2(\Omega)^d)} + \|m_n\|_{L^\infty(I; L^2(\Omega)^d)} + \|h_n\|_{L^\infty(I; L^2(\Omega)^d)} \\
+ \|\nabla u_n\|_{L^2(I; H(\text{curl}, \text{div})(\Omega))} + \|\nabla h_n\|_{L^2(I; H(\text{curl}, \text{div})(\Omega)^d)} \\
+ \|\partial_t u_n\|_{L^2(I; (H^1_0(\Omega)^d)\cap H(\text{div}, \Omega)^d))} + \|\partial_t h_n\|_{L^2(I; (H^1_0(\Omega)^d)\cap H(\text{div}, \Omega)^d))} \\
+ \|\nu_n\|_{L^2(I; \Omega^d)} + \|\nu_{\text{part}}\|_{L^2(I; W^{1,4/3}_\text{loc}(\Omega)^d)} + \|\nu_{\text{part}}\|_{L^2(I; (W^{1,4/3}_\text{loc}(\Omega)^d))} \leq C',
\]

where \( C' > 0 \) does not depend on \( n \in \mathbb{N} \), see (3.142). Recalling (3.157),

\[
1 \leq \frac{(c_n)^2}{f_p(c_n)} = \frac{(c_n)_{s(n)}}{(c_n)_{s(n)}}^{L(n)}
\]

implies

\[
\left\| (\nu_{\text{part}})_n \right\|_{L^2(I; \Omega^d)} \leq \frac{(c_n)_{s(n)}}{f_p(c_n)} (\nu_{\text{part}})_n \left\| L^2(I; \Omega^d) \leq C'.
\]

The estimates related to \( g_{s(n)} \) and \( c_n \) follow from Corollary 3.36. Hence, the left-hand side of (3.161) is bounded by a constant \( C > 0 \) that only depends on \(|\Omega|, T, \) initial data and the external magnetic field \( h_n \).

The passage to the limit of the regularized solutions from Definition 3.54 is similar to the regularized setting in Section 3.3. The main difficulty is to obtain suitable estimates for \( \nabla c_n \) or \( c_n \) independently of \( \sigma_c(n) \).

### 3.4.1 Compactness in space

The regularized convective velocity \( (\nu_{\text{part}})_n \) (cf. (3.159)) couples the magnetic contribution to the particles’ motion to the diffusive motion. Exploiting the \( L^2 \)-boundedness of \( (\nu_{\text{part}})_n \), the regularity of the magnetic Kelvin force can be transferred to \( \nabla c_n \) independently from \( \sigma_c(n) \).

**Lemma 3.56.** For the sequence of Galerkin solutions from Definition 3.54, the following holds true.

1) Case \( d = 2 \).

\[
(c_n)_{n \in \mathbb{N}} \text{ is bounded in } L^{4/3}(I; W^{1,4/3}_\text{loc}(\Omega)) \cap L^2(I; L^2_\text{loc}(\Omega)),
\]

\( (m_n)_{n \in \mathbb{N}} \) and \( (h_n|\Omega)_{n \in \mathbb{N}} \) are bounded in \( L^2(I; H^1_\text{loc}(\Omega)^2) \cap L^4(I; L^2_\text{loc}(\Omega)^2) \).

2) Case \( d = 3 \).

\[
(c_n)_{n \in \mathbb{N}} \text{ is bounded in } L^{5/4}(I; W^{1,5/4}_\text{loc}(\Omega)) \cap L^{5/3}(I; L^{5/3}_\text{loc}(\Omega)),
\]

\( (m_n)_{n \in \mathbb{N}} \) and \( (h_n|\Omega)_{n \in \mathbb{N}} \) are bounded in \( L^2(I; H^1_\text{loc}(\Omega)^3) \cap L^{10/3}(I; L^{10/3}_\text{loc}(\Omega)^3) \).

**Proof:** By boundedness of \( (\nu_{\text{part}})_n \) in \( L^2(I \times \Omega)^d \) (cf. (3.161)) and the identity (3.159) one infers that \( \nabla c_n \) has at least the regularity the three terms

\[
(\nabla m_n)^T m_n, \ (\nabla h_n)^T m_n, \ (\nabla h_n)^T m_n
\]
come along with (or regularity of \((v_{\text{part}})_n\) if the latter is worse). Comparing regularity of \(m_n, h_n, h_a\) it suffices to consider the term \((\nabla m_n)^T m_n\) only. According to \((3.161)\) one obtains
\[
m_n \in L^{q_d}(I; L^{q_d}_{\text{loc}}(\Omega)^d), \quad \text{where } q_d = \begin{cases} 4 & \text{if } d = 2, \\ \frac{10}{3} & \text{if } d = 3. \end{cases}
\]
Due to
\[
\int_{\Omega} (ab)^\gamma \, dx \leq \left( \int_{\Omega} a^2 \, dx \right)^{\frac{\gamma}{2}} \left( \int_{\Omega} b^{\frac{2\gamma}{2-\gamma}} \, dx \right) \frac{2-\gamma}{2\gamma}
\]
for \(\gamma \in (0, 2)\), one obtains, setting \(a := |\nabla m_n|, b := |m_n|\) and choosing \(\gamma = \frac{d+2}{d+1}\) (which implies \(\frac{2\gamma}{2-\gamma} = q_d\)), that
\[
\nabla c_n \in L^\gamma(I; L^{q_d}_{\text{loc}}(\Omega)^d).
\]
By Sobolev’s embedding,
\[
c_n \in L^\gamma(I; L^{q_\gamma}_{\text{loc}}(\Omega)), \quad \text{where } \gamma' = \begin{cases} 4 & \text{if } d = 2, \\ \frac{15}{7} & \text{if } d = 3. \end{cases}
\]
Using the uniform boundedness of \(c_n\) in \(L^\infty(I; L^1(\Omega))\), see \((3.161)\), the claim follows by Lemma 3.35.

In order to identify the limit of the fluxes \((c_n)_{n\in\mathbb{N}}(v_{\text{part}})_n\) via strong convergence of \(c_n\) and weak convergence of \((v_{\text{part}})_n\) in \(L^2(I_\times \Omega)^d\) higher regularity for \(c_n\) is needed.

**Lemma 3.57.** Let \(d = 2\) and set
\[
G(c) := \begin{cases} c^2, & c \leq e, \\ |c^2| \log(c)^2, & c > e. \end{cases} \quad (3.162)
\]

Then, for all \(\hat{\Omega} \subset \subset \Omega\) there exists a constant \(C > 0\) such that the regularized solutions \(c_n\) from Definition 3.54 satisfy
\[
\int_0^T \int_{\hat{\Omega}} G(c_n) \, dx \, dt \leq C \quad (3.163)
\]
uniformly in \(n \in \mathbb{N}\).

**Proof:** By Lemma 3.56 and \((3.161)\) the sequences \((c_n)_{n\in\mathbb{N}}\) and \((g_n(c_n) + c_n)_{n\in\mathbb{N}}\) are bounded in \(L^{4/3}(I; L^{4}_{\text{loc}}(\Omega))\) and \(L^\infty(I; L^1(\Omega))\), respectively. By splitting the integration into two parts, one finds
\[
\int_0^T \int_{\hat{\Omega}} G(c_n) \, dx \, dt = \int_0^T \int_{\{c_n > e\} \cap \hat{\Omega}} c_n^2 |\log(c_n)|^{\frac{\gamma}{2}} \, dx \, dt + \int_0^T \int_{\{c_n \leq e\} \cap \hat{\Omega}} e^2 \, dx \, dt
\]
\[
\leq \int_0^T \int_{\{c_n > e\} \cap \hat{\Omega}} c_n^2 |\log(c_n)|^{\frac{\gamma}{2}} \, dx \, dt + T|\Omega|e^2.
\]
Then,
\[
\int_0^T \int_{\{c_n > e\} \cap \hat{\Omega}} c_n^\gamma |\log(c_n)|^\mu \, dx \, dt
\]
\[
\leq \int_0^T \left( \int_{\{c_n > e\} \cap \hat{\Omega}} |c_n| \log(c_n)^{\frac{\gamma}{2}} \, dx \right)^\alpha \left( \int_{\{c_n > e\} \cap \hat{\Omega}} |c_n|^{\frac{\gamma-\alpha}{\gamma-\alpha}} \, dx \right)^{1-\alpha} \, dt.
\]
By setting \(\alpha = \mu = \frac{d}{2}\) and \(\gamma = 2\)
\[
\int_0^T \int_{\{c_n > e\} \cap \hat{\Omega}} c_n^2 |\log(c_n)|^{\frac{\gamma}{2}} \, dx \, dt \leq \|g_n(c_n) + c_n\|_{L^\infty(I; L^1(\Omega))} \|c_n\|_{L^{4/3}(I; L^{4}_{\text{loc}}(\Omega))}^{\frac{2}{3}}
\]
is obtained which immediately implies the claim of this lemma.
3.4.2 Compactness in time

Due to the local regularity of the magnetic variables, the magnetic particle density has only been bounded locally as well, see [Lemma 3.56]. Hence, a local version of time compactness estimates is needed, too.

**Lemma 3.58.** Let \( d = 2 \) and \( \phi \in C_0^\infty(\Omega, \mathbb{R}_+^d) \) be an arbitrary cut-off function, \( \hat{\Omega} := \text{supp} \phi \subset \subset \Omega \). Then, there is a constant \( 0 < C < \infty \) depending on \( \phi \), such that the regularized solutions \( c_n \) from [Definition 3.54] satisfy

\[
\| \partial_t (\phi c_n) \|_{L^1(I; (H^2_0(\Omega))^2)} \leq C
\]

uniformly in \( n \in \mathbb{N} \).

**Proof:** By [Lemma 3.56] and (3.161), the sequence \( (c_n, u_n, (v_{\text{part}})n)_{n \in \mathbb{N}} \) is bounded in

\[
L^2(I; L^2_{\text{loc}}(\Omega)) \times L^4(I \times \Omega)^2 \times L^2(I \times \Omega)^2.
\]

First, observe that the weak formulation (3.140b) holds pointwise in time for almost all \( t \in [0, T] \). This can be achieved by using test functions of the type \( \psi = \psi_1(t)\psi_2(x) \) for \( \psi_1 \in C_0^\infty((0, T)) \) and \( \psi_2 \in H^2_0(\Omega) \). Then, the first term becomes

\[
\int_0^T \psi_1(t) \langle \partial_t c_n(t), \psi_2(\cdot) \rangle_{(H^2_0(\Omega))^2} \, dt = \int_0^T \psi_1(t) a_n(t) \, dt.
\]

The other terms can be written – for some \( f_n \in L^{5/4}(I \times \Omega)^d \) playing the role of the remaining terms – altogether as

\[
\int_0^T \psi_1(t) \int_\Omega f_n(t) \cdot \nabla \psi_2 \, dx \, dt = \int_0^T \psi_1(t) b_n(t) \, dt.
\]

Hence, \( \forall \psi_1 \in C_0^\infty((0, T)) \) the identity

\[
\int_0^T \psi_1(t)(a_n(t) + b_n(t)) \, dt = 0 \tag{3.164}
\]

holds. As countable unions of sets of measure zero have measure zero, too, locality is established, i.e. for almost all \( t \in [0, T] \) one obtains \( a_n(t) + b_n(t) = 0 \) for all \( n \in \mathbb{N} \).

Now, take \( \psi \in H^2_0(\Omega) \cap W^{1,\infty}(\Omega) \) arbitrarily. Choose \( \hat{\psi} := (\phi \psi) \) as test function in the formulation without time integrals,

\[
\langle \partial_t c_n, (\phi \psi) \rangle_{(H^2_0(\Omega))^2} = - \int_\Omega (c_n)_{s(n)} u_n \cdot \nabla (\phi \psi) \, dx - \int_\Omega (c_n)_{s(n)} (v_{\text{part}})n \cdot \nabla (\phi \psi) \, dx + \sigma_c(n) \int_\Omega \nabla c_n \cdot \nabla (\phi \psi) \, dx \equiv 0,
\]

implying

\[
\left| \langle \partial_t c_n, (\phi \psi) \rangle_{(H^2_0(\Omega))^2} \right| \leq \int_\Omega \|(c_n)_{s(n)}\| (|u_n| + |(v_{\text{part}})n|) + \sigma_c(n)\|\nabla c_n\| (|\nabla \phi| |\psi| + |\phi||\nabla \psi|) \, dx
\]

\[
\leq C_1(\phi) \int_\Omega \|(c_n)_{s(n)}\| (|u_n| + |(v_{\text{part}})n|) + \sigma_c(n)\|\nabla c_n\| (|\psi| + |\nabla \psi|) \, dx
\]

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\[ C_1(\phi) \left[ \| (c_n)_{s(n)} \|_{L^2(\Omega)} \| u_n \|_{L^4(\Omega)^d} + \| (v_{\text{part}})_{n} \|_{L^2(\Omega)^d} + \| \nabla c_n \|_{L^{4/3}(\Omega)^d} \right] \| \psi \|_{W^{1,\infty}(\Omega) \cap H^2(\Omega)}. \]

Taking the supremum over all \( \phi \in H^2(\Omega) \cap W^{1,\infty}(\Omega) \) with norm equal to 1, the dual norm of \( \partial_t c_n \) is bounded in the sense

\[
\| \psi \| \mapsto \langle \partial_t c_n, \phi \psi \rangle_{(H^2(\Omega) \cap W^{1,\infty}(\Omega))'} \times H^2(\Omega) \cap W^{1,\infty}(\Omega) \|_{(H^2(\Omega) \cap W^{1,\infty}(\Omega))'}^d \leq C_1(\phi) \left[ \| (c_n)_{s(n)} \|_{L^2(\Omega)} \| u_n \|_{L^4(\Omega)^d} + \| (v_{\text{part}})_{n} \|_{L^2(\Omega)^d} + \| \nabla c_n \|_{L^{4/3}(\Omega)^d} \right].
\]

The integrability in time of the right-hand side is determined by the least regular term which is \( \| (c_n)_{s(n)} \|_{L^2(\Omega)} \| u_n \|_{L^4(\Omega)^d} \) and which is \( L^1 \)-integrable. By (3.138) one finds that

\[
\langle \partial_t c_n, \phi \psi \rangle_{(H^2(\Omega) \cap W^{1,\infty}(\Omega))'} \times H^2(\Omega) = \langle \partial_t (\phi c_n), \psi \rangle_{(H^2(\Omega) \cap W^{1,\infty}(\Omega))'}. \]

In order to resolve ambiguity regarding the notation, notice that in (3.138) the variable \( c \) was the limit of the Galerkin approximation and here within this section all \( c_n \) are such limits for different Galerkin problems. Hence,

\[
\int_0^T \| \langle \partial_t (\phi c_n), \psi \rangle_{(H^2(\Omega) \cap W^{1,\infty}(\Omega))} \|_{(H^2(\Omega) \cap W^{1,\infty}(\Omega))'} \, dt \leq \mathcal{C},
\]

where \( \mathcal{C} > 0 \) depends only on \( \phi \).

### 3.4.3 Passage to the limit

Combining Simon’s compactness theorem [65, Section 8, Corollary 4] with the uniform regularity of \( (c_n)_{n \in \mathbb{N}} \) established in Lemma 3.56 and appropriate exhaustion and diagonal arguments, strong local convergence can be obtained for a subsequence.

**Lemma 3.59.** Let

\[
\gamma_d := \begin{cases} 
\frac{4}{3} & \text{if } d = 2, \\
\frac{5}{4} & \text{if } d = 3,
\end{cases} \quad \gamma'_d := \begin{cases} 
4 & \text{if } d = 2, \\
\frac{15}{7} & \text{if } d = 3.
\end{cases}
\]

There exist a subsequence of \( (c_n)_{n \in \mathbb{N}} \) from Definition 3.54 (not relabeled for the ease of notation) and a function \( c \in L^{\gamma_d}(\Omega; \mathcal{W}^{1,\gamma_d}_{\text{loc}}(\Omega)) \) such that

i) \( c_n \to c \) strongly in \( L^{\gamma_d}(\Omega; \mathcal{W}^p_{\text{loc}}(\Omega)) \) for any \( 1 \leq p < \gamma'_d \).

ii) \( c_n \to c \) pointwise almost everywhere in \( I \times \Omega \).

**Proof:** Let \( \phi \in C^\infty_0(\Omega) \) be a cut-off function and let \( 1 \leq p < \gamma'_d \). Then by Lemma 3.58 and Lemma 3.56 one has boundedness of \( (\phi c_n)_{n \in \mathbb{N}} \) in \( L^{\gamma_d}(\Omega; \mathcal{W}^{1,\gamma_d}_{\text{loc}}(\Omega)) \) with time derivatives \( (\partial_t (\phi c_n))_{n \in \mathbb{N}} \) bounded in \( L^1(I; (H^2(\Omega) \cap W^{1,\infty}(\Omega))') \). Sobolev’s embedding yields

\[
W^{1,\gamma_d}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow (H^2(\Omega) \cap W^{1,\infty}(\Omega))'.
\]

From Simon’s compactness theorem one can conclude the existence of a strongly converging subsequence in \( L^{\gamma_d}(\Omega; L^p(\Omega)) \). Hence, \( c_n|_{\Omega} \to c \) in \( L^{\gamma_d}(I; L^p(\Omega)) \) for some \( c \in L^{\gamma_d}(I; L^p(\Omega)) \). By choosing a sequence of cut-off functions \( (\phi_i)_{i \in \mathbb{N}} \) whose supports \( \text{supp} \phi_i := \hat{\Omega}_i \) exhaust \( \Omega \) combined with an diagonal argument, there is a subsequence such that the aforementioned strong convergence holds true for all \( \hat{\Omega}_i \subset \subset \Omega \) and a single function \( c \in L^{\gamma_d}(I; L^p(\hat{\Omega})) \). For more details, inspect the proof of Corollary 3.4.7 in which the analogous arguments were applied for the sequence of magnetizations from the Galerkin setting. Extracting possibly another subsequence, the pointwise convergence almost everywhere in \( I \times \Omega \) can be concluded.

\[ \square \]
Corollary 3.60. Let \((c_n)_{n \in \mathbb{N}}\) be the sequence from Definition 3.54 and let \(d = 2\). Then, there exists a subsequence \((c_{n_k})_{n_k} \subset (c_n)_{n \in \mathbb{N}}\) (not relabeled) which converges pointwise almost everywhere in \(I \times \Omega\) and strongly in \(L^2(I; L^2_{\text{loc}}(\Omega))\).

Proof: For any fixed \(\tilde{\Omega} \subset \subset \Omega\) there exists a constant \(C > 0\) (depending on \(\tilde{\Omega}\)) such that \(G\) from (3.162) satisfies

\[
\int_0^T \int_{\Omega} G(c_n) \, dx \, dt \leq C
\]

uniformly in \(n \in \mathbb{N}\), cf. Lemma 3.57. By [20] Chapter 2 and \(\frac{G(x)}{x} \to \infty\), equi-integrability of \((c_n)_{n \in \mathbb{N}}\) in \(L^2(I \times \tilde{\Omega})\) is obtained. Together with pointwise convergence almost everywhere, the result follows from Vitali’s convergence theorem.

Remark 3.61. By collecting previous results on the regularized solutions from Definition 3.54, the following regularity results are obtained. In the case \(d = 3\), the functions

\[
h_n|_{\Omega}, m_n\]

are uniformly bounded in

\[L^\infty(I; L^2(\Omega)^3) \cap L^2(I; H(\text{div}, \text{curl})(\Omega)) \cap L^2(I; H^{1}_{\text{loc}}(\Omega)^3) \cap L^{10/3}(I; L^{10/3}_{\text{loc}}(\Omega)^3),\]

\(c_n\) are uniformly bounded in

\[L^\infty(I; L^2(\Omega')^3) \cap L^2(I; H(\text{div}, \text{curl})(\Omega' \setminus \partial \Omega)),\]

and

\[L^{5/4}(I; W^{1,5/4}_{\text{loc}}(\Omega)) \cap L^{5/4}(L^{15/7}_{\text{loc}}(\Omega)) \cap L^\infty(I; L^1(\Omega)) \cap L^{5/3}(I; L^{5/3}_{\text{loc}}(\Omega)).\]

In the case \(d = 2\), the analogous regularity holds as well and the following holds additionally. The functions

\[
h_n|_{\Omega}, m_n\]

are uniformly bounded in \(L^4(I; L^4_{\text{loc}}(\Omega)^2)\), \(c_n\) are uniformly bounded in \(L^{4/3}(I; W^{1,4/3}_{\text{loc}}(\Omega)) \cap L^{4/3}(L^{4}_{\text{loc}}(\Omega)) \cap L^2(I; L^2_{\text{loc}}(\Omega)).\)

Moreover, for \(d \in \{2, 3\}\), the time derivatives

\[
\partial_t m_n, \partial_t h_n|_{\Omega}\]

are uniformly bounded in \(L^2(I; (H^3_0(\Omega)^d)')\), \(L^2(I; (H^3_0(\Omega)^d) \cap H(\text{div}_0)(\Omega))')\), \(L^2(I; (\nabla H^3_0(\Omega)^d)')\), \(L^2(I; (\nabla H^{1/2}(\Omega)^d)'))\).

Corollary 3.62. Let the spatial dimension be \(d = 2\). There exists a subsequence of regularized solutions – see Definition 3.54 – which will not be relabeled for the ease of notation and functions

- \(\mathbf{u} \in L^\infty(I; L^2(\Omega)^d) \cap L^2(I; H^1_0(\Omega)^d \cap H(\text{div}_0)(\Omega)) \cap L^4(I; L^4(\Omega)^d),\)

- \(\mathbf{c} \in L^\infty(I; L^2(\Omega)) \cap L^{4/3}(I; W^{1,4/3}_{\text{loc}}(\Omega)^d) \cap L^2(I; L^2_{\text{loc}}(\Omega)),\)

- \(\mathbf{m} \in L^\infty(I; L^2(\Omega)^d) \cap L^2(I; H(\text{div}, \text{curl})(\Omega)) \cap L^2(I; H^{1}_{\text{loc}}(\Omega)^d) \cap L^4(I; L^4_{\text{loc}}(\Omega)^d),\)

- \(\mathbf{h} \in L^\infty(I; L^2(\Omega')^d) \cap L^2(I; H(\text{div}, \text{curl})(\Omega' \setminus \partial \Omega))\) with \(h|_{\Omega} \in L^2(I; H^{1}_{\text{loc}}(\Omega)^d) \cap L^4(I; L^4_{\text{loc}}(\Omega)^d),\)

such that

\[
\begin{align*}
\mathbf{u}_n & \xrightarrow{\ast} \mathbf{u} \quad \text{in} \quad L^\infty(I; L^2(\Omega)^d), \\
\mathbf{u}_n & \rightharpoonup \mathbf{u} \quad \text{in} \quad L^2(I; H^1(\Omega)^d), \\
\mathbf{u}_n & \rightarrow \mathbf{u} \quad \text{in} \quad L^4(I; L^4(\Omega)^d), \\
\mathbf{c}_n & \xrightarrow{\ast} \mathbf{c} \quad \text{in} \quad L^\infty(I; L^1(\Omega)), \\
\mathbf{c}_n & \rightharpoonup \mathbf{c} \quad \text{in} \quad L^{4/3}(I; W^{1,4/3}_{\text{loc}}(\Omega)^d), \\
\mathbf{c}_n & \rightarrow \mathbf{c} \quad \text{in} \quad L^2(I; L^2_{\text{loc}}(\Omega)), \\
\mathbf{m}_n & \xrightarrow{\ast} \mathbf{m} \quad \text{in} \quad L^\infty(I; L^2(\Omega)^d), \\
\mathbf{m}_n & \rightharpoonup \mathbf{m} \quad \text{in} \quad L^2(I; H(\text{div}, \text{curl})(\Omega)), \\
\mathbf{m}_n & \rightarrow \mathbf{m} \quad \text{in} \quad L^4(I; L^4(\Omega)^d). \\
\end{align*}
\]
Moreover, there exists a function \( R \in H^1_{\text{mean}}(\hat{\Omega}') \) such that the magnetic field
\[
h = \nabla R
\]
can be identified with a gradient field.

**Proof:** The proof is analogous to the proof of Corollary 3.47. The energy estimate in (3.161) of the regularized solutions is essentially the same as in the Galerkin approximation setting, see (3.130), combined with the respective time compactness estimates Lemma 3.42, Lemma 3.41, and Lemma 3.43 and Lemma 3.39. For the application of the Aubin-Lions lemma, local versions of time compactness of \( m_n \) or \( h_n \) are needed, i.e. boundedness of \( \partial_t m_n |_V \) and \( \partial_t h_n |_V \) for any \( V \subset \subset \Omega \). Those are available from the previous estimates of the Galerkin approximations in Corollary 3.45 or Lemma 3.43 (for a strict subset \( V \subset \subset \Omega \)), respectively, combined with weak lower semi-continuity of norms.

The only noticeable difference in the current setting is the regularity of \( c_n \). Hence, in order to prevent repetition, it is sufficient to discuss convergence of \( c_n \), only. For the other variables the reader is referred to the proof of Corollary 3.47 which can be adapted to this setting in a straightforward way. The results about weak convergence of \( (c_n)_{n \in \mathbb{N}} \) follow immediately from (3.161). The strong convergence is proven in Corollary 3.60.

**Remark 3.63.** In case of \( d = 3 \) the results from Corollary 3.62 are identical to the Galerkin setting (cf. Corollary 3.47) in regard to \( u_n, m_n \) and \( h_n \). For the particle density one would get
\[
\begin{align*}
    c_n & \xrightarrow{\ast} c \quad \text{in } L^\infty(I; L^1(\Omega)), \\
    c_n & \rightharpoonup c \quad \text{in } L^{5/4}(I; W^{1,5/4}_{\text{loc}}(\Omega)), \\
    c_n & \rightarrow c \quad \text{in } L^{5/3}(I; L^{5/3}_{\text{loc}}(\Omega)),
\end{align*}
\]
where the strong convergence is obtained by an argument as in Corollary 3.60 and an appropriate modification of Lemma 3.57. This convergence, however, is not sufficient to pass to the limit in (3.165). The term
\[
\int_0^T \int_{\Omega} (c_n)s(n)(v_{\text{part}})_n \cdot \nabla \psi \, dx \, dt
\]
suggests at least strong (local) \( L^2 \)-convergence of \( c_n \) as we do not expect the convective velocity \( (v_{\text{part}})_n \) to have more than \( L^2 \)-regularity.

Based on the stability estimate (3.161), a non-negativity result for the limit function \( c \) from (3.165) can be obtained.

**Lemma 3.64.** The function \( c \) from (3.165) which is a limit of the sequence of regularized solutions \( \{c_n\}_{n \in \mathbb{N}} \) from Definition 3.54 is non-negative.

**Proof:** One can deduce from (3.161) that \( \|g_s(n)(c_n)\|_{L^\infty(I; L^1(\Omega))} \) is bounded, hence the negative part of the particle density satisfies the estimate
\[
\|c_n \|_{L^\infty(I; L^1(\Omega))} \leq \frac{3}{8} s(n) \|c(n)\|_{L^\infty(I; L^1(\Omega))} + s(n)^2 |\Omega| \leq C s(n) \rightarrow 0.
\]
By Corollary 3.62 (or Remark 3.63) there is a subsequence of \( (c_n)_{n \in \mathbb{N}} \) that is strongly converging on any compact subset \( \hat{\Omega} \subset \subset \Omega \). For instance, there exists a subsequence (not relabeled) such that
\[
(c_n)_+|_{\hat{\Omega}} = c_n|_{\hat{\Omega}} - (c_n)_-|_{\hat{\Omega}} \rightarrow c|_{\hat{\Omega}} + 0
\]
in \( L^1(I \times \hat{\Omega}) \) and pointwise almost everywhere in \( I \times \hat{\Omega} \). As \( (c_n)_+ \geq 0 \) it is evident that its limit for \( n \rightarrow \infty \) is non-negative (almost everywhere in \( I \times \hat{\Omega} \)), too. Therefore, \( c|_{\hat{\Omega}} \geq 0 \) almost everywhere for any \( \hat{\Omega} \subset \subset \Omega \) as \( \hat{\Omega} \subset \subset \Omega \) is arbitrary. In conclusion, \( c \) is non-negative on whole \( I \times \Omega \) (almost everywhere).
Lemma 3.65. Let \((c_n)_{n \in \mathbb{N}} \subset L^1(I \times \Omega)\) be a sequence that converges pointwise almost everywhere to a function \(c \in L^1(I \times \Omega)\) with \(c \geq 0\) almost everywhere. Then,

\[
(c_n)_{s(n)} \to c, \quad (c_n)_{s(n)}^{L(n)} \to c, \quad \frac{(c_n)_{s(n)}^{L(n)}}{(c_n)_{s(n)}} \to 1,
\]

pointwise almost everywhere in \(I \times \Omega\).

**Proof:** Fix \(x \in \Omega\) and \(t \in I\) such that indeed \(c_n(t, x) \to c(t, x)\). In the following – for the ease of presentation – the time variable is dropped from the notation. As \(c(x) \geq 0\),

\[
|c_n(x)_{s(n)} - c(x)| = \begin{cases} |c_n(x) - c(x)| & \text{if } c_n(x) \geq s(n), \\ s(n) - c(x) & \text{if } c_n(x) < s(n), \end{cases} \to 0
\]

which implies the first claim of this lemma.

For the second claim, use

\[
|c_n(x)_{s(n)}^{L(n)} - c(x)| \leq |c_n(x)_{s(n)}^{L(n)} - (c(x)_{s(n)}^{L(n)})| + |(c(x)_{s(n)}^{L(n)}) - c(x)|. \tag{3.166}
\]

Without loss of generality one can assume \(|c(x)| =: b < \infty\). Therefore, there exists sufficiently large \(n_0 \in \mathbb{N}\) such that \(c_n(x)\) stays within \((b-1, b+1)\) and \((b+1 < L(n)\) for all \(n \geq n_0\). Hence, \((c(x)_{s(n)}^{L(n)}) = (c(x)_{s(n)})\) for all \(n \geq n_0\). Hence, the first claim implies that the second term on right-hand side of (3.166) vanishes in the limit, i.e. the second claim has been proven.

Moreover, the third claim immediately follows from \((c(x)_{s(n)}^{L(n)}) = (c(x)_{s(n)})\) for all \(n \geq n_0\) as with that identity the quotient stays equal to 1 for almost all \(n \in \mathbb{N}\). \(\square\)

As a remark, notice that Lemma 3.65 can be applied to \((c_n)_{n \in \mathbb{N}}\) from Definition 3.54 in two or three space dimensions (see Lemma 3.59).

**Corollary 3.66.** In case \(d = 2\), there is a subsequence of \((c_n)_{n \in \mathbb{N}}\) from Definition 3.54 (not relabeled for the ease of presentation) such that

\[
(c_{n})_{s(n)} \to c \text{ in } L^2(I; L^2_{\text{loc}}(\Omega)).
\]

**Proof:** The regularized functions \((c_{n})_{s(n)}\) satisfy an analogous estimate as in (3.163), as the function \(G\) (cf. (3.162)) that was used in Lemma 3.57 satisfies \(G(c_n) = G((c_n)_{s(n)})\) due to \(s(n) < e\). Moreover, the \(L^\infty(I; L^1(\Omega))\)-estimate (cf. (3.161)) as well as the \(L^{1/3}(I; L^1_{\text{loc}}(\Omega))\)-estimate (cf. Lemma 3.56 and Sobolev’s embedding) of \(c_n\) can be carried over to \((c_{n})_{s(n)}\) trivially. With the same arguments as in the proof of Corollary 3.60 the result follows. \(\square\)

**Lemma 3.67.** There exist distributional derivatives of \(u\) and \(m\) from Corollary 3.62 with respect to time,

\[
\partial_t u \in L^2(I; U'), \quad \partial_t m \in L^2(I; M'),
\]

such that the regularized solutions from Definition 3.54 satisfy

\[
\int_0^T \int_{\Omega} \partial_t u_n \cdot v \, dx \, dt \to \int_0^T \langle \partial_t u, v \rangle_{U' \times U} \, dt \quad \forall v \in L^2(I; U),
\]

\[
\int_0^T \int_{\Omega} \partial_t m_n \cdot \theta \, dx \, dt \to \int_0^T \langle \partial_t m, \theta \rangle_{M' \times M} \, dt \quad \forall \theta \in L^2(I; M),
\]
for some subsequence which will not be relabeled for the ease of notation. Moreover, for all \( v \in H^1(I; \mathcal{U}) \) with \( v(T) = 0 \) and \( \theta \in H^1(\mathcal{M}; \mathcal{M}) \) with \( \theta(T) = 0 \) the limits can be identified with

\[
\begin{align*}
\int_0^T \langle \partial_t u, v \rangle_{\mathcal{U}^* \times \mathcal{U}} \, dt &= - \int_0^T \int_{\Omega} u : \partial_t \mathbf{v} \, dx \, dt - \int_{\Omega} u^{\text{init}} : \mathbf{v}(0) \, dx, \\
\int_0^T \langle \partial_t m, \theta \rangle_{\mathcal{M} \times \mathcal{M}} \, dt &= - \int_0^T \int_{\Omega} m : \partial_t \theta \, dx \, dt - \int_{\Omega} m^{\text{init}} : \theta(0) \, dx. 
\end{align*}
\]

**Proof:** The functionals \( \partial_t u_n \) are uniformly bounded in \( L^2(I; \mathcal{U}^*) \) according to (3.161). Then, the argument is similar – even simpler – than in the proof of **Lemma 3.49** because the initial values are already attained exactly. The claim about the magnetization is proven analogously. \( \square \)

**Remark 3.68.** The time compactness estimate from **Lemma 3.59** is sufficient for application of Simon's compactness theorem \((65)\). On the other hand, the time derivatives \( \partial_t c_n \) can only converge in the distributional sense, i.e. the existence of a weak limit \( \partial_t c \) in \( L^1(I; (H^2_0(\Omega) \cap W^{1, \infty}(\Omega))^d) \) can not be guaranteed by the results of this thesis, because that space is not reflexive.

With all those ingredients at hand, passing to the limit is handled in a similar way as in the previous existence result of regularized solutions, see **Theorem 3.50**. The main difference lies in the passage to the limit of the magnetic particles’ evolution equation. For convenience, the definition of the space \( \mathcal{K} \) was given in (3.8) and the setting in the following theorem correlates to the case \( f_p(c) = c^2 \) in (2.89).

**Theorem 3.69.** Assume \((H1 3.1), (H2 3.2)\) as well as \((H3 3.3)\) and \( d = 2 \). Let initial data

\( u^{\text{init}} \in H_{\text{init}}(\text{div}_0)(\Omega), \quad c^{\text{init}} \in L^2(\Omega; \mathbb{R}^+_0), \quad m^{\text{init}} \in L^2(\Omega)^d \)

be given. Then, there exist functions

\[
\begin{align*}
\mathbf{u} &\in L^2(I; H^1_0(\Omega)^d \cap H(\text{div}_0)(\Omega)) \cap L^\infty(I; L^2(\Omega)^d) \cap W^{1,2}(I; (H^3_0(\Omega)^d \cap H(\text{div}_0)(\Omega))^d), \\
c &\in L^{4/3}(I; W^{1,4/3}_{\text{loc}}(\Omega)) \cap L^\infty(I; L^1(\Omega)) \cap L^2(I; L^2_{\text{loc}}(\Omega)), \\
\mathbf{R} &\in L^2(I; \mathcal{K}) \text{ with } R|_\Omega \in L^\infty(I; H^1(\Omega)) \cap L^2(I; H^2_{\text{loc}}(\Omega)), \\
\mathbf{m} &\in L^2(I; H(\text{div}, \text{curl})(\Omega)) \cap L^\infty(I; L^2(\Omega)^d) \cap L^2(I; H^1_{\text{loc}}(\Omega)^d) \cap H^1(I; (H^3_0(\Omega)^d)^d),
\end{align*}
\]

where the magnetic field is given as \( \mathbf{h} := \nabla \mathbf{R} \), such that for all

\[
\begin{align*}
\mathbf{v} &\in L^2(I; H^3_0(\Omega)^2 \cap H(\text{div}_0)(\Omega)), \\
\psi &\in C^1([0, T]; C^0_0(\Omega)) \text{ with } \psi(T) \equiv 0, \\
S &\in \mathcal{K}, \\
\mathbf{\Psi} &\in L^2(I; H^3_0(\Omega)^2),
\end{align*}
\]

the equations

\[
\begin{align*}
\rho_0 &\int_0^T \langle \partial_t \mathbf{u}, \mathbf{v} \rangle_{(H^3_0(\Omega)^d \cap H(\text{div}_0)(\Omega))^d \times H^3_0(\Omega)^d \cap H(\text{div}_0)(\Omega)^d)} \, dt + \int_0^T \int_{\Omega} 2\rho_0 \mathbf{D}\mathbf{u} : \mathbf{D}\mathbf{v} \, dx \, dt \\
&+ \frac{\rho_0}{2} \int_0^T \int_{\Omega} (\mathbf{u} : \nabla) \mathbf{u} \cdot \mathbf{v} \, dx \, dt - \frac{\rho_0}{2} \int_0^T \int_{\Omega} (\mathbf{u} : \nabla) \mathbf{v} : \mathbf{u} \, dx \, dt \\
&= - \mu_0 \int_0^T \int_{\Omega} (\mathbf{m} : \nabla) \mathbf{v} : (\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a) \, dx \, dt \\
&- \mu_0 \int_0^T \int_{\Omega} \text{div} \mathbf{m} \cdot \mathbf{v} : (\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a) \, dx \, dt \\
&+ \frac{\mu_0}{2} \int_0^T \int_{\Omega} (\mathbf{m} \times (\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a)) : \text{curl} \mathbf{v} \, dx \, dt,
\end{align*}
\]
Moreover, these functions satisfy the energy estimate

\[ - \int_0^T \int_\Omega c_\partial \psi \, dx \, dt - \int_0^T \int_\Omega \varepsilon^{init} \psi(0) \, dx - \int_0^T \int_\Omega c \cdot \nabla \psi \, dx \, dt \]

\[ + \int_0^T \int_\Omega Kc \left( D \nabla c - \mu_0 (\nabla (\alpha_1 h + \frac{\beta}{2} h_a - \alpha_3 m)) \right) \cdot \nabla \psi \, dx \, dt = 0, \]

\[ \int_{\Omega'} \nabla R \cdot \nabla S \, dx = \int_{\Omega'} h_a \cdot \nabla S \, dx - \int_\Omega m \cdot \nabla S \, dx \quad \text{almost everywhere in } I = [0, T], \]

\[ \int_0^T \langle \partial_t \mathbf{m}, \mathbf{Ψ} \rangle_{(H^2(\Omega))^d \times H^2(\Omega)} \, dt \]

\[ - \int_0^T \int_\Omega \left( \left( \mathbf{u} + K \left[ -D \nabla c + \mu_0 (\nabla (\alpha_1 h + \frac{\beta}{2} h_a - \alpha_3 m)) \right] \right) \cdot \nabla \right) \mathbf{Ψ} \cdot \mathbf{m} \, dx \, dt \]

\[ + \sigma \int_0^T \int_\Omega \text{div} \mathbf{m} \cdot \text{div} \mathbf{Ψ} \, dx \, dt + \sigma \int_0^T \int_\Omega \text{curl} \mathbf{m} \cdot \text{curl} \mathbf{Ψ} \, dx \, dt \]

\[ = \frac{1}{2} \int_0^T \int_\Omega (\mathbf{m} \times \mathbf{Ψ}) \cdot \text{curl} \mathbf{u} \, dx - \frac{1}{\tau_{rel}} \int_0^T \int_\Omega (\mathbf{m} - \chi(c, h) \mathbf{h}) \cdot \mathbf{Ψ} \, dx \, dt \]

hold and the initial data are attained in the sense

\[ \langle \mathbf{u}(0), \mathbf{Ψ} \rangle_{(H^2(\Omega))^d \cap H(\text{div}\,\Omega))} = \int_\Omega \mathbf{u}^{\text{init}} \cdot \mathbf{Ψ} \, dx \quad \forall \mathbf{Ψ} \in H^2(\Omega)^d \cap H(\text{div}\,\Omega), \]

\[ \langle \mathbf{m}(0), \mathbf{Ψ} \rangle_{(H^2(\Omega))^d \times H^2(\Omega)} = \int_\Omega \mathbf{m}^{\text{init}} \cdot \mathbf{Ψ} \, dx \quad \forall \mathbf{Ψ} \in H^2(\Omega)^d. \]

Moreover, these functions satisfy the energy estimate

\[ \|\mathbf{u}\|_{L^\infty(I;L^2(\Omega)^2)} + \|\mathbf{u}\|_{L^2(I;L^1(\Omega)^2)} + \|c\|_{L^\infty(I;L^1(\Omega))^d} \]

\[ + \|\mathbf{v}_{\text{part}}\|_{L^2(I;L^2(\Omega)^2)} + \|\mathbf{m}\|_{L^\infty(I;L^2(\Omega)^2)} + \|h\|_{L^\infty(I;L^2(\Omega)^2)} \]

\[ + \|\partial_t \mathbf{u}\|_{L^2(I;H(c,\text{div}(\Omega)))} + \|\partial_t \mathbf{m}\|_{L^2(I;H(c,\text{div}(\Omega)^d))} \]

\[ + \|\partial_t \mathbf{h}\|_{L^2(I;H^1(\Omega)^d)} + \|\text{div} \mathbf{m}\|_{L^2(I;L^2(\Omega)^d)} \]

\[ + \|\text{curl} \mathbf{h}\|_{L^2(I;L^4(\Omega)^d)} \]

\[ \leq C. \]

**Proof:** The sequences \((u_n)_{n \in \mathbb{N}}, (m_n)_{n \in \mathbb{N}}\) and \((h_n)_{n \in \mathbb{N}}\) from **Definition 3.54** have convergence properties of at least the same quality as in the passage from Galerkin approximation sequences to regularized solutions, see **Corollary 3.47** and **Corollary 3.62**. There are only three differences between this setting and the former setting.

a) The identification of the limit of convective velocities \((v_{\text{part}})_n\) is more complicated as \(f^a_p\) is not a fixed function anymore.

b) The momentum equation \((3.140a)\) features a different right-hand side compared to \((3.93a)\) in the Galerkin-setting.

c) The convergence of the sequence \((c_n)_{n \in \mathbb{N}}\) of **Definition 3.54** is significantly worse as in the Galerkin setting. The only appearances of \(c_n\) that can not be shadowed by rewriting terms into \((v_{\text{part}})_n\) (which has the same regularity as in the Galerkin setting, see \((3.161)\)) are in the magnetic particles’ evolution equation \((3.140b)\) and in the last term of \((3.140d)\) (which includes the susceptibility \(\chi(c, h)\)).

It is sufficient to only address the differences mentioned in a), b) and c). All other parts of the proof are analogous to the proof of **Theorem 3.50** and are valid even in three space dimensions.
Ad \( b \) First, the weak convergence result of \((v_{\text{part}})_n\) in \(L^2(I \times \Omega)^d\) will be restored. One easily gets \((v_{\text{part}})_n \to w\) in \(L^2(I \times \Omega)^d\) for some \(w \in L^2(I \times \Omega)^d\), see (3.161). Let \( \Phi \in C_0^\infty(I \times \Omega)^d\), then

\[
\int_0^T \int_\Omega (v_{\text{part}})_n \cdot \Phi \, dx \, dt \quad \text{and} \quad -KD \int_0^T \int_\Omega \nabla c_n \cdot \Phi \, dx \, dt + \int_0^T \int_\Omega \mu_0 K \frac{(c_n)_s(n)}{(c_n)_s(\cdot)} (\alpha_1 \nabla h_n + \beta \nabla h_n - \alpha_3 \nabla m_n)^T m_n \cdot \Phi \, dx \, dt.
\]

First, note that the two terms on the right-hand side can be separated into individual integrals due to integrability of \(\nabla c_n\). The second term of the right-hand side converges as the gradients converge weakly and locally in \(L^2(I; L^2_0(\Omega)^{d \times d})\) (see Corollary 3.62) and \(m_n\) converges strongly and locally in \(L^3(I; L^3_{\text{loc}}(\Omega)^d)\) (in any space dimension \(d \in \{2, 3\}\)) while the bounded quotient \(0 < \frac{(c_n)_s(n)}{(c_n)_s(\cdot)} \leq 1\) converges strongly in \(L^6(I \times \Omega)\), due to pointwise convergence (for a non-relabeled subsequence), see Lemma 3.65 and Vitali’s convergence theorem. The first term converges due to local weak convergence of \(\nabla c_n\) in \(L^{5/2}(I; L^{5/2}_{\text{loc}}(\Omega)^d)\) (in any space dimension \(d \in \{2, 3\}\), see Remark 3.63) hence

\[
w = -KD\nabla c + \mu_0 K (\alpha_1 \nabla h + \beta \nabla h_n - \alpha_3 \nabla m)^T m
\]

is the weak limit, as expected.

Ad \(b\) Analogously to the proof of Theorem 3.50, the compactly supported test functions \(v \in L^\infty(I; C_0^\infty(\Omega)^d \cap \mathbb{L}) \subset W^{1, \infty}(I \times \Omega)^d\) will be used as a first step. The compact support of \(v\) is needed in order to exploit local regularity of the magnetic variables. After identifying the limit a density argument allows to enlarge the space of test functions. The third term on the right-hand side of (3.140a) is handled identically as in the proof of Theorem 3.50. The other terms of the right-hand side will be handled in a straightforward way by exploiting the compact support (in space) of the test function as follows. Using Corollary 3.62 one can conclude

\[
- \mu_0 \int_0^T \int_\Omega (m_n \cdot \nabla) v \cdot (\alpha_1 \nabla h_n + \beta \nabla h_n) \, dx \, dt \to - \mu_0 \int_0^T \int_\Omega (m \cdot \nabla) v \cdot (\alpha_1 \nabla h + \beta \nabla h) \, dx \, dt,
\]

\[
- \mu_0 \int_0^T \int_\Omega \text{div} m_n \cdot v \cdot (\alpha_1 \nabla h_n + \beta \nabla h_n) \, dx \, dt \to - \mu_0 \int_0^T \int_\Omega \text{div} m \cdot v \cdot (\alpha_1 \nabla h + \beta \nabla h) \, dx \, dt.
\]

These arguments also work in three space dimensions.

Ad \(c\) First, plug in compactly supported (in space) test functions into (3.140b), which yields combined with the representation in Lemma 3.49 and (3.159) the identities

\[
- \int_0^T \int_\Omega c_n \cdot \partial_t \psi \, dx \, dt - \int_\Omega c^{\text{nint}}(0) \, dx - \int_0^T \int_\Omega (c_n)_s(n) u_n \cdot \nabla \psi \, dx \, dt + \sigma_c(n) \int_0^T \int_\Omega \nabla c_n \cdot \nabla \psi \, dx \, dt - \int_0^T \int_\Omega (c_n)_s(n) (v_{\text{part}})_n \cdot \nabla \psi \, dx \, dt = 0,
\]
for all $\psi \in H^1(I; C^0_\infty(\Omega) \cap H^s_2(\Omega))$ with $\psi(0) \equiv 0$. Let $\psi$ be an arbitrary test function as in (3.168). Using Corollary 3.62, the first term on the left-hand side converges as follows,

$$- \int_0^T \int_\Omega c \nabla \psi \cdot d\mathbf{x} \, dt \to - \int_0^T \int_\Omega c \partial_t \psi \, d\mathbf{x} \, dt.$$ 

There is nothing to do for the second term on the left-hand side of (3.173). The limit of the third term is

$$- \int_0^T \int_\Omega (cn) \nabla \psi \, d\mathbf{x} \, dt \to - \int_0^T \int_\Omega c \mathbf{u} \cdot \nabla \psi \, d\mathbf{x} \, dt,$$

where Remark 3.63 and Lemma 3.65 yield the strong convergence of $(cn)$ in combination with Vitali’s convergence theorem (i.e. suitable uniform estimates imply uniform integrability) even in three space dimensions. The fourth term on the left-hand side of (3.173) vanishes in the limit as the coefficient in front of the integral goes to zero while the integral can be bounded by means of Hölder’s inequality. The last term on the left-hand side of (3.173) is the only one to require higher regularity than the three-dimensional setting can provide. However, in the case $d = 2$, Corollary 3.66 implies

$$- \int_0^T \int_\Omega (cn) \nabla \psi \, d\mathbf{x} \, dt \to - \int_0^T \int_\Omega c \mathbf{w} \cdot \nabla \psi \, d\mathbf{x} \, dt,$$

where

$$\nabla \psi \in L^2(I; \Omega)^d$$

$$\mathbf{w} \in L^2(I; \Omega)^d$$

and

$$\mathbf{w} \in L^2(I; \Omega)^d$$

where $\mathbf{w}$ is given by

$$\mathbf{w} = \frac{1}{\tau_{rel}} \int_0^T \int_\Omega (\chi(c, h)) \nabla \mathbf{h} \cdot \mathbf{\theta} \, d\mathbf{x} \, dt \to \frac{1}{\tau_{rel}} \int_0^T \int_\Omega \chi(c, h) \mathbf{h} \cdot \mathbf{\theta} \, d\mathbf{x} \, dt.$$

Hence, the proof is complete.

Remark 3.70. The magnetic contribution and the diffusive contribution to the particle motion in (3.168) can not be separated into individual integrals as neither of the two parts have sufficient integrability. In detail, $c$ is at most $L^2$-integrable locally while $\nabla c$ and the magnetic part are at most $L^{5/3}$-integrable locally.
Remark 3.71. Assuming sufficient regularity for the weak solutions from Theorem 3.69, they can in a standard way be shown to satisfy the classical solution concept for the specific choice of mobility \( f_p(c) := c^2 \). Particularly, \( c \) solves

\[
ct + \nabla c \cdot u - \text{div}(KD\nabla c - K\mu_0c(\nabla(\alpha_1 h + \frac{\delta}{2} h_a - \alpha_3 m))T m) = 0
\]

which is for \( f_p(c) = c^2 \). For the hydrodynamic equations, the usual pathway to prove existence of a pressure, see e.g. [32, Lemma III.1.1], can be pursued.

Boundary conditions, however, can be identified for the velocity field, but not for the magnetic particle density \( c \) or the magnetization \( m \), as the latter are solutions only in the sense of distributions. This is due to the fact that the integral estimates derived so far do not provide more than \( H(\text{div}, \text{curl})(\Omega) \)-regularity for \( m \) and \( h \). As a consequence, spatial gradients of \( c \) or of \( m \) have only \( L^p_{\text{loc}} \)-type integrability due to their coupling expressed by equation (2.89d). This requires test functions in the weak formulation to be compactly supported.

In the case that non-compactly supported test functions, e.g. of class \( H^1 \) with respect to the spatial variables, were permitted, the identity

\[
\int_{\partial \Omega} c \nu_{\text{part}} \cdot \nu \psi \, d\sigma = 0 \quad \forall \psi \in H^\frac{1}{2}(\partial \Omega)
\]

would be a direct consequence, entailing (2.90b) for positive \( t \). In case of the magnetization equation, one would get the identity

\[
\int_{\Omega} [(v_{\text{part}} \cdot \nu)(m \cdot \theta) - \sigma \text{div} m \theta \cdot \nu - \sigma \text{curl} m \times \nu \cdot \theta] \, d\sigma = 0 \quad \forall \theta \in H^\frac{1}{2}(\partial \Omega)^2 \quad (3.174)
\]

for positive \( t \). Assuming the normal vector field \( \nu \) to be sufficiently regular as well, by surjectivity of the trace operator one could find \( \theta \) such that

\[
\theta|_{\partial \Omega} = ((v_{\text{part}} \cdot \nu)(m \cdot \nu) - \sigma \text{div} m)\nu.
\]

Inserting this into (3.174) would entail

\[
\int_{\partial \Omega} |(v_{\text{part}} \cdot \nu)(m \cdot \nu) - \sigma \text{div} m|^2 \, d\sigma = 0,
\]

hence (2.90c). Condition (2.90c) would follow easily now.

Remark 3.72. The above existence result (Theorem 3.69) suggests that a converging numerical scheme based on conforming finite elements might require \( H^3 \)-regular finite elements. However, in order to reduce complexity, a non-conforming approach was used in [39]. In fact, if the magnetization is discretized by discontinuous elements, the requirement \( \nabla R_h|_{\Omega} \subset M_h \) (cf. (3.5)) for suitable discrete finite element spaces \( R_h \) and \( M_h \) may be satisfied. In [53], strategies have been presented how to prove convergence in the (partially) discontinuous setting of a finite element scheme in the case that \( h = h_a \) is given and \( \sigma \) in (2.89c) is chosen to be zero. In our work [39] and this thesis, an energy stable scheme in case of \( \sigma > 0 \) was designed by introducing divergence and curl operators defined by duality.
4 Numerical analysis

The numerical part of this thesis features unconditionally energy stable finite element schemes, which admit discrete solutions. Proof-of-concept simulations in two space dimensions are presented in Section 5. However, convergence analysis is not part of this thesis, which is why general assumptions on the mesh are sufficient. Before discussing stable numerical schemes for all three models from Section 2, the general framework and notation will be introduced. For simplicity, assume the following.

\((N1)\) Let the function \(f_p\) be of the specific type

\[ f_p(c) := c^m, \quad m \geq 0. \]  

\((N1\ 4.1)\)

\((N2)\) In case of 'model GW', assume \(\hat{\chi}\) (cf. (2.32)) to have at most growth like a square root function, i.e. for given \(r > 0\) let

\[ \chi_r(c, h) := \begin{cases} \max(0, c) & \text{for } c^\frac{m_0}{3} < r, \\ \sqrt{c - \frac{3r}{m_0} + \frac{1}{4} + \frac{3r}{m_0} - \frac{1}{2}} & \text{else.} \end{cases} \]  

\((N2\ 4.2)\)

In case of 'model W' or 'model B' no growth constraints are needed.

\((N3)\) Let the external magnetic field be of the following regularity,

\[ h_a \in H^1(I; H(\text{div}_0, \text{curl}_0)(\Omega')). \]  

\((N3\ 4.3)\)

Depending on the choice of discretization of \(h_a\), see (4.18a) or (4.18b), respectively, further assumptions might apply.

Assumption \((N2)\) admits – in case of 'model GW' – the estimate

\[ 0 \leq |\chi_r(c, h)| \leq K_1 + K_2 \sqrt{|c|} \]  

\((4.4)\)

for some constants \(K_1, K_2 > 0\) (depending on \(r\)). The estimate \((4.4)\) follows immediately from the boundedness of the Langevin function \(L\) and \([\text{Remark } 2.3]\) also see \([3.4]\).

In this section, a uniform decomposition of the time interval is used. However, the numerical theory presented in this section remains valid for schemes which use adaptivity in time, i.e. possibly different time increments in each step. The only difference needed in the theory is to change the notation to denote individual time increments. Fix \(n_T \in \mathbb{N}\) in order to define a time increment \(\tau > 0\) via

\[ \tau := \frac{T}{n_T}, \quad t^k := k\tau \text{ for } k \in \{0, \ldots, n_T\}. \]  

\((4.5)\)

For simplicity, let \(\Omega\) and \(\Omega'\) be polygonal, i.e. they can be decomposed into triangles properly. Fix \(h > 0\) and let \(T_h(\Omega')\) be a simplicial triangulation (in the sense of Brenner and Scott \([13]\)) of \(\Omega'\) with mesh size \(h\) that has a subset \(T_h(\Omega) \subset T_h(\Omega')\) which itself is a simplicial triangulation of \(\Omega\), i.e. for some \(M(\Omega), M(\Omega') \in \mathbb{N}\) and simplices \(\mathcal{K}_i, i = 1, \ldots, M(\Omega')\), one has

\[ \{\mathcal{K}_1, \ldots, \mathcal{K}_{M(\Omega')}\} = T_h(\Omega') \subset T_h(\Omega) = \{\mathcal{K}_1, \ldots, \mathcal{K}_{M(\Omega)}\}, \quad M(\Omega) < M(\Omega'), \]  

\[ \mathcal{K}_i \cap \mathcal{K}_j = \emptyset \quad \forall i, j \in \{1, \ldots, M(\Omega')\} \text{ with } i \neq j, \]  

\[ \Omega' = \bigcup_{\mathcal{K} \in T_h(\Omega')} \mathcal{K}, \quad \Omega = \bigcup_{\mathcal{K} \in T_h(\Omega)} \mathcal{K}. \]  

\((4.6)\)
Naturally,

$$\mathcal{T}_h(\Omega' \setminus \overline{\Omega}) := \mathcal{T}_h(\Omega') \setminus \mathcal{T}_h(\Omega).$$

In regard to the simplices, let them be closed subsets of \( \mathbb{R}^d \). However, we tend to write \( \overline{K} \) or \( \hat{K} \) explicitly in the following definitions for clarification. For some subset \( V \subset \mathbb{R}^d \) let

$$P_k(V) := \{ f : V \to \mathbb{R} | \text{There exist numbers } a_{i_1, \ldots, i_d} \in \mathbb{R}, i_1, \ldots, i_d = 0, \ldots, k \text{ such that } f(x) = \sum_{i_1, \ldots, i_d=0}^k a_{i_1, \ldots, i_d} \Pi_{l=1}^d x_l^{i_l} \quad \forall x \in V \}$$

denote the space of polynomials of order \( k \) on \( V \). Then, define for \( W = \Omega, W = \Omega' \) or \( W = \Omega' \setminus \overline{\Omega} \), respectively, the spaces

$$D_k(\overline{W}) := \left\{ f \in L^2(W) \left| f|_\overline{K} \in P_k(\overline{K}) \quad \forall K \in \mathcal{T}_h(W) \right. \right\},$$

$$D^\text{mean}_k(\overline{W}) := D_k(\overline{W}) \cap L^2_0(W),$$

$$\mathcal{P}_k(\overline{W}) := \left\{ f \in C(\overline{W}) \left| f|_\overline{K} \in P_k(\overline{K}) \quad \forall K \in \mathcal{T}_h(W) \right. \right\},$$

$$\mathcal{P}^\text{mean}_k(\overline{W}) := \mathcal{P}_k(\overline{W}) \cap L^2_0(W),$$

$$\mathcal{P}^\text{zero}_k(\overline{W}) := \mathcal{P}_k(\overline{W}) \cap H^1_0(W).$$

(4.7)

For convenience, \( \mathcal{D} \) denotes the use of discontinuous elements, while \( \mathcal{P} \) denotes the use of continuous elements. The upper indices 'mean' or 'zero' indicate functions with vanishing mean value or zero boundary values, respectively.

Continuous functions \( f \in C(\overline{K}) \) can be projected onto the space of polynomials \( P_k(\overline{K}) \) by Lagrangian interpolation. For instance, in two space dimensions on a triangle \( K \), the function values of \( f \) at the vertices uniquely describe an affine linear function. If one additionally evaluates \( f \) on the midpoints of the edges, a quadratic function will be determined uniquely, see e.g. [13, p. 72-76]. For any \( K \in \mathcal{T}_h(\Omega) \) let

$$I^k_K : C(\overline{K}) \to P_k(\overline{K})$$

be the local Lagrangian interpolation operator which utilizes the typical aforementioned degrees of freedom on \( \partial K \). Then, the global interpolation operator on \( \Omega \) can be defined as follows,

$$\tilde{I}_{h,k} : \{ f : \Omega \to \mathbb{R} | \forall K \in \mathcal{T}_h(\Omega) : f|_\overline{K} \in C(\overline{K}) \} \to D_k(\overline{\Omega}),$$

$$\left( \tilde{I}_{h,k} f \right)|_\overline{K} := \left( I^k_K f|_\overline{K} \right)|_\overline{K}, \quad \forall K \in \mathcal{T}_h(\Omega),$$

(4.8)

where \( f|_\overline{K} \) is the continuous extension of \( f|_\overline{K} \) onto \( \overline{K} \). The definition on the faces is arbitrary as they form a null set. Here, piecewise continuous functions are treated elementwise, which is why the continuous extension is needed in order to obtain the correct nodal values for the Lagrangian interpolation. If one plugs in continuous functions into \( \tilde{I}_{h,k} \), the values on the faces can be chosen in such a way that the interpolant is continuous, as the respective elementwise continuous extensions of the interpolating function and the original function share the same nodal values. Hence, we can define

$$\mathcal{I}_{h,k} : C(\overline{\Omega}) \to \mathcal{P}_k(\overline{\Omega}),$$

$$\mathcal{I}_{h,k} := \tilde{I}_{h,k}|_{C(\overline{\Omega})},$$

(4.9)
Vector-valued analogues, i.e.

\[ \hat{I}_{h,k}^d : \{ f : \Omega \to \mathbb{R}^d \mid \forall \mathcal{K} \in \mathcal{T}_h(\Omega) : f|_{\hat{\mathcal{K}}} \in C(\hat{\mathcal{K}})^d \} \to \mathcal{D}_h(\Omega)^d, \]

\[ I_{h,k}^d : C(\Omega)^d \to \mathcal{P}_h(\Omega)^d, \]

are given by applying the scalar-valued interpolation operators \( \hat{I}_{h,k}^d \) or \( I_{h,k}^d \), respectively, componentwise – for instance \( (\hat{I}_{h,k}^d f)_i = \hat{I}_{h,k} f_i, i = 1, \ldots, d \). In light of Remark 3.72 jumps and mean values along interior faces (edges in case of \( d = 2 \)) need to be defined. Let

\[ \mathcal{E}_{\text{int}}(\Omega) \]

be the set of all interior faces of the simplices from \( \mathcal{T}_h(\Omega) \), i.e. the faces which are not subsets of \( \partial \Omega \). Let these faces be closed subsets of \( \mathbb{R}^d \) (they include their \((d - 2)\)-dimensional boundary). However, we tend to write \( \hat{E} \) or \( \circ E \) explicitly to avoid confusion, where \( \hat{E} \) denotes the face without its \((d - 2)\)-dimensional boundary. Let the union of all interior faces be denoted by

\[ \mathcal{F}_{\text{int}} = \bigcup_{E \in \mathcal{E}_{\text{int}}} \hat{E}. \]

For each \( E \in \mathcal{E}_{\text{int}} \) there are exactly two unit normals, out of which we choose one arbitrarily and denote it by \( \nu_E \). Let

\[ \nu_{\mathcal{F}_{\text{int}}} : \mathcal{F}_{\text{int}} \to \mathbb{R}^d, \]

\[ x \mapsto \nu_{\mathcal{F}_{\text{int}}}(x) := \begin{cases} \nu_E & \text{if } x \in \circ E, \ E \in \mathcal{E}_{\text{int}}, \\ 0 & \text{else}, \end{cases} \] (4.11)

be the unit normal field of the interior faces. Consider simplices \( \mathcal{K}^+, \mathcal{K}^- \in \mathcal{T}_h(\Omega) \) which have a common face \( E \in \mathcal{E}_{\text{int}} \) and let \( \nu_E \) point from simplex \( \mathcal{K}^+ \) to simplex \( \mathcal{K}^- \), then the jump \( [a]_E : \hat{E} \to \mathbb{R}^d \) and the mean value \( \{a\}_E : \hat{E} \to \mathbb{R}^d \) of a vector field \( a \) which is bounded and continuous on \( \mathcal{K}^+ \) and \( \mathcal{K}^- \), are defined by

\[ [a]_E := a^+ - a^- \quad \text{and} \quad \{a\}_E := \frac{1}{2}(a^+ + a^-), \] (4.12)

where \( a^+ \) or \( a^- \) denote the continuous extensions of \( a |_{\mathcal{K}^+} \) onto \( \mathcal{K}^+ \) or of \( a |_{\mathcal{K}^-} \) onto \( \mathcal{K}^- \), respectively. Notice that the sign of the jump depends on the choice of the unit normal \( \nu_E \) which is only unique up to sign. However, in the remainder of this thesis, the jump will always appear together with the corresponding unit normal in a way that gets rid of this ambiguity. For instance, the normal jump \( [a]_E : \nu_E \) of a vector field is independent of the choice of the sign of \( \nu_E \).

The jump \( [a] : \mathcal{F}_{\text{int}} \to \mathbb{R}^d \) and the mean value \( \{a\} : \mathcal{F}_{\text{int}} \to \mathbb{R}^d \) on the \((d - 1)\)-skeleton \( \mathcal{F}_{\text{int}} \) are defined face-wise for any piecewise continuous vector field \( a \in \{ f : \Omega \to \mathbb{R}^d \mid \forall \mathcal{K} \in \mathcal{T}_h(\Omega) : f|_{\hat{\mathcal{K}}} \in C(\hat{\mathcal{K}})^d \}, \) in particular for any \( a \in \mathcal{D}_h(\Omega)^d \). They read

\[ [a](x) := \begin{cases} [a]_E(x) & \text{if } x \in \circ E, \ E \in \mathcal{E}_{\text{int}}, \\ 0 & \text{else}, \end{cases} \]

\[ \{a\}(x) := \begin{cases} \{a\}_E(x) & \text{if } x \in \circ E, \ E \in \mathcal{E}_{\text{int}}, \\ 0 & \text{else}. \end{cases} \]
4.1 Design of energy stable finite element schemes

This section is divided into two parts. The first part is devoted to the design of a numerical scheme for "model GW" (cf. (2.89)) which has already been discussed by us in [39]. In contrast to [39], this thesis includes a proof that discrete solutions thereof exist. The second part handles the other two models of this thesis both at once (cf. (2.100), (2.110)) as they are quite similar.

The finite element spaces associated with the unknowns $(u, p, c, v_{\text{part}}, m, R)$ will be denoted by their respective capital calligraphic letters, i.e. $U_h, P_h, C_h, V_h, M_h, R_h$. Inspired by [52] and [1] we require the following.

(C1) The pair $(U_h, P_h)$ satisfies the inf-sup condition
\[
\inf_{q \in P_h} \sup_{v \in U_h} \frac{|\int_{\Omega} q \, \text{div} \, v \, dx|}{\|v\|_{H^1(\Omega)} \|q\|_{L^2(\Omega)}} \geq \gamma \quad (C1 \ 4.13)
\]
for some $\gamma > 0$, see e.g. [29].

(C2) The spaces $C_h$ and $V_h$ will have polynomial order 1.

(C3) The spaces related to magnetism satisfy
\[
\nabla [R_h|_{\Omega}] \subset M_h. \quad (C3 \ 4.14)
\]

Although no convergence analysis will be provided in this thesis, the first condition above is a standard requirement for convergent schemes. The second condition above ensures monotonicity of the interpolation operator $I_{h,1}$, see Lemma 4.1 for a proof. However, the result also holds true for $\hat{I}_{h,1}$.

Lemma 4.1. For all $f, g \in C(\overline{\Omega})$ satisfying $f \leq g$ the interpolation operator $I_{h,1}$ from (4.9) satisfies $I_{h,1} f \leq I_{h,1} g$.

Proof: From $f \leq g$ it follows that $f(e_i) \leq g(e_i)$ for all nodes $e_1, ..., e_{\dim P_1(\Omega)}$ which are to be used by the Lagrangian interpolation. Let $b_0, ..., b_{\dim P_1(\Omega)}$ be the typical nodal basis functions corresponding to the nodes $e_0, ..., e_{\dim P_1(\Omega)}$. Those are known to be non-negative. Hence,
\[
I_{h,1} f = \sum_{i=1}^{\dim P_1(\Omega)} f(e_i) \frac{b_i}{g(e_i)} \leq \sum_{i=1}^{\dim P_1(\Omega)} g(e_i) b_i = I_{h,1} g.
\]

Moreover, the interpolation operators can be used to define an equivalent norm on the space of piecewise linear functions, i.e. for $W = \Omega$ or $W = \Omega'$ and $f \in D_1(\overline{W})^n$, $n \in \mathbb{N}$, let
\[
\|f\|_{h,W,n} := \left( \int_{\Omega} \hat{I}_{h,1}((f,f)_{\mathbb{R}^n}) \, dx \right)^{\frac{1}{2}}. \quad (4.15)
\]
The interpolation operator $\hat{I}_{h,1}$ can be replaced with $I_{h,1}$ if continuous elements are used. Moreover, let $\|\cdot\|_{h,W,1}$ be abbreviated by $\|\cdot\|_{h,W}$.

Lemma 4.2. Let $W = \Omega$ or $W = \Omega'$ and $n \in \mathbb{N}$. There exists a constant $M_0 > 1$ such that
\[
\frac{1}{M_0} \|\cdot\|_{L^2(W)^n} \leq \|\cdot\|_{h,W,n} \leq M_0 \|\cdot\|_{L^2(W)^n} \quad (4.16)
\]
uniformly for $h > 0$. 
**Proof:** This result is well-known. The sketch of the proof is as follows. Consider an arbitrary simplex $\mathcal{X}$ and do a transformation to a fixed reference simplex $\hat{\mathcal{X}}$. By computing both the local mass matrix and local mass matrix with respect to lumped masses, a comparison of eigenvalues yields an estimate. Transform back to the original simplex $\mathcal{X}$. This process is independent of the mesh size. Sum up over all simplices to obtain the final result. Note that global continuity of finite element functions is not necessary for this procedure. 

Above general assumptions on the discrete ansatz spaces are already sufficient for the design of an energy stable finite element scheme. However, for the sake of simplicity, this section will continue by specifying suitable ansatz spaces explicitly. Let

$$
\begin{align*}
U_h &= \mathcal{P}_2^{\text{zero}}(\hat{\Omega})^d, \\
V_h &= \mathcal{P}_1(\hat{\Omega})^d, \\
\mathcal{M}_h &= D_1(\hat{\Omega})^d, \\
\mathcal{P}_h &= \mathcal{P}_1^{\text{mean}}(\hat{\Omega}), \\
C_h &= \mathcal{P}_1(\hat{\Omega}), \\
R_h &= \mathcal{P}_2^{\text{mean}}(\hat{\Omega}).
\end{align*}
$$

4.1 Design of energy stable finite element schemes

This choice satisfies all conditions (C1) (C3) above.

The discrete quantities corresponding to the unknowns of our models will be denoted by using the same symbol together with an upper index which indicates the discrete time, i.e.

$$
u^k, p^k, c^k, \xi^k_{\text{part}}, R^k \text{ and } m^k.$$

Here, the symbol $u^k$ denotes the discrete velocity field corresponding to the unknown $u$ from (2.89a) ("model GW") or (2.100a) ("model W") or (2.110a) ("model B") at time $t^k$ which corresponds to the time step $k \in \{0, ..., n_T\}$. Analogously, the other symbols depict the discrete pressure, density of magnetic particles, convective velocity of magnetic particles, magnetic potential and magnetization at time $t^k$, respectively.

Let discrete initial data be given, e.g. obtained via $L^2$-projection of the initial data of the continuous setting. For instance, in case of the Navier-Stokes equations consider e.g. $u^{\text{init}} \in H_{n0}(\text{div}\,)_0(\Omega)$ and let $u^0 \in U_h$ be defined via

$$
\int_\Omega u^0 \cdot v \, dx = \int_\Omega u^{\text{init}} \cdot v \, dx \quad \forall v \in U_h.
$$

Alternatively, if $u^{\text{init}}$ is sufficiently smooth, i.e. continuous on $\Omega$, one can take

$$
u^0 := \mathcal{I}_{h,1}(u^{\text{init}})
$$

as discretization of initial data $u^{\text{init}}$. Similarly, for $c^{\text{init}} \in L^2(\Omega)$ and $m^{\text{init}} \in L^2(\Omega)^d$ one can use

$$
\begin{align*}
\int_\Omega c^0 \psi \, dx &= \int_\Omega c^{\text{init}} \psi \, dx \\
\forall \psi &\in C_h, \\
\int_\Omega m^0 \cdot n \, dx &= \int_\Omega m^{\text{init}} \cdot n \, dx
\end{align*}
$$

or alternatively $c^0 = \mathcal{I}_{h,1}(c^{\text{init}})$ and $m^0 = \mathcal{I}_{h,1}(m^{\text{init}})$ if the original initial data are sufficiently regular.

Let $\mathcal{I}_{h,1}^{\hat{\Omega}'}$ be defined as $\mathcal{I}_{h,1}^\Omega$ from (4.10) with $\Omega$ being replaced with $\hat{\Omega}'$. The discrete external magnetic field at time step $k = 0, ..., n_T$ is defined either as

$$
h_a^k := \mathcal{I}_{h,1}^{\hat{\Omega}'}(h_a(\cdot, t^k)) \in \mathcal{P}_1(\hat{\Omega})^d,
$$

assuming sufficient regularity of $h_a$, e.g. $h_a \in H^1(I; H^2(\hat{\Omega}'))^d$, or as $L^2$-projection

$$
\int_{\hat{\Omega}'} (h_a^k) \cdot \theta \, dx := \int_{\hat{\Omega}'} h_a(t^k, \cdot) \cdot \theta \, dx \quad \forall \theta \in D_1(\hat{\Omega})^d
$$

without further assumptions besides of (N3 4.3). The first version is very practical in regards to implementation. However, the second version is not difficult to implement either and naturally is $L^2$-stable.
Recall the definition of the regularized entropy function $g_s$ from (3.73). For convenience, it will be repeated here.

$$-1 \leq g_s(c) := \begin{cases} \frac{c^2}{2s} + (\log s - 1)c - \frac{s}{2} & \text{for } c \leq s, \\ c \log c - c & \text{for } s < c. \end{cases}$$ (4.19)

Here, $0 < s < e$ without loss of generality. The derivatives are given by

$$g'_s(c) := \begin{cases} \frac{c}{s} + (\log s - 1) & \text{for } c \leq s, \\ \log c & \text{for } s < c, \end{cases}$$

$$g''_s(c) := \begin{cases} 1 & \text{for } c \leq s, \\ 1 & \text{for } s < c. \end{cases}$$ (4.20)

Then,

$$g_{k,s,h} := I_{h,1}(g_s((c^k))),$$

$$g'_{k,s,h} := I_{h,1}((g'_s(c^k))),$$

$$g''_{k,s,h} := I_{h,1}((g''_s(c^k))),$$

for all $k \in \{0, ..., n_T\}$. Define

$$(c^{k-1})_{s,h} := I_{h,1}(\max(s, c^{k-1})) = I_{h,1}((c^{k-1})_s)$$ (4.22)

and approximate $f_p$ from (N1.4.1) in two different ways as follows,

$$f_{p,h}^{(s),k-1} := I_{h,1}(\max(s, c^{k-1})^{m-1}c^{k-1}),$$

$$f_{p,h}^{(e),k-1} := I_{h,1}(\max(s, c^{k-1})^m) = I_{h,1}(f_p((c^{k-1})_s)).$$ (4.23)

The two different approximations are needed for stability reasons.

For the ease of notation, a magnetic field $h^k \in D_1(\Omega)^d$ could be defined as $L^2$-projection of $\nabla R^k$ for general finite element spaces, see e.g. (3.12g) in [39]. However, for the choices made in this thesis it is evident that such an approach results directly in

$$h^k := \nabla R^k,$$ (4.25)

where $\nabla$ just is the weak gradient. The following abbreviations will be used later on,

$$\hat{h}^k := \alpha_1 h^k + \frac{\beta}{2} (h_a)^k_h, \quad \hat{b}^k := \hat{h}^k|_{\Pi} - \alpha_3 m^k.$$ (4.26)

4.1.1 Model GW

Recalling (N2) set

$$\chi^{k-1,1}_{r,h} := I_{h,1}(\chi_r((c^{k-1}, h^k)).$$ (4.27)

The Kelvin forces in (2.89a) and (2.89d) will be discretized in a similar way as in [51,52]. Their discretization affects the discretization of the convection terms in the magnetization equation as they are coupling terms of the Kelvin forces. Let $\delta_{sym} \in \{0, 1\}$ and set

$$b_{h,\delta_{sym}}^m : U_h \times M_h \times M_h \rightarrow \mathbb{R},$$

$$b_{h,\delta_{sym}}^m(u, h, m) := \sum_{X \in T_h(\Omega)} \int_X (\mathbf{u} \cdot \nabla) h \cdot m \, dx - \int_{\mathcal{J}_{\text{int}}} [h] \cdot \{m\} (\mathbf{u} \mathbf{\nu}_{\text{int}}) \, d\sigma$$

$$+ \delta_{sym} \frac{1}{2} \int_{\Omega} \text{div} \mathbf{u} \cdot m \, dx.$$ (4.28)

Notice, for the choice $\delta_{sym} = 1$ this is exactly the discretization of the Kelvin force $(m \cdot \nabla) h$ tested by $u$ which has been used in [51,52]. The boundary integral with jumps and mean values
is needed for consistency in regard to $\nabla h|_{\Omega}$ for non-differentiable $h|_{\Omega} \in D_{1}(\Omega)$. For an elaborate introduction to discrete finite element spaces the reader will be referred to [23][24]. The presence of the term $\frac{1}{2} \int_{\Omega} \text{div} u \cdot h \cdot m \, dx$ implies skew-symmetry of $b_{h,\text{sym}}^{m}$ with respect to the last two arguments, see [51][52]. This term introduces an additional error which is expected to vanish in the limit as the velocity field is solenoidal. On the other hand, if $\delta_{\text{sym}} = 0$, a stabilizing term of the type $b_{h,\text{sym}}^{m}(\cdot, m^{k}, m^{k})$ needs to be added to the discretization, see (4.37a) below, which also is expected to vanish in the limit by means of $\frac{1}{2} m \cdot \nabla (|m|^{2})$ and a reformulation of the pressure. In contrast to [51][52], a discretization of the Kelvin force within the convective velocity of magnetic particles (see (2.89d)) is needed, i.e.

$$
\hat{b}_{h}^{m} : V_{h} \times M_{h} \times M_{h} \rightarrow \mathbb{R},
$$

$$
\hat{b}_{h}^{m}(\nu_{\text{part}}, h, m) := \sum_{X \in T(\Omega)} \int_{X} (\nu_{\text{part}} \cdot \nabla)h \cdot m \, dx - \int_{F_{\text{int}}} [h] \cdot \{m\} (\nu_{\text{part}} \cdot \nu^{\text{int}}) \, d\sigma. \quad (4.29)
$$

Evidently, a skew-symmetric version as above is not available as $\text{div} \nu_{\text{part}} \neq 0$ in general. The specific choices of $b_{h,\text{sym}}^{m}$ and $\hat{b}_{h}^{m}$ are irrelevant in regards of existence of stable solutions as long as they provide continuous trilinear forms. Bear in mind that $V_{h}, U_{h} \subset P_{2}(\Omega)^{d}$. Hence, for simplification, this thesis will use the definition

$$
b_{h}^{m} : P_{2}(\Omega)^{d} \times M_{h} \times M_{h} \rightarrow \mathbb{R},
$$

$$
b_{h}^{m}(u, h, m) := \sum_{X \in T(\Omega)} \int_{X} (u \cdot \nabla)h \cdot m \, dx - \int_{F_{\text{int}}} [h] \cdot \{m\} (u \cdot \nu^{\text{int}}) \, d\sigma \quad (4.30)
$$

for the discretization of both Kelvin forces and their corresponding coupling terms. As the finite element spaces are finite dimensional and $b_{h}^{m}$ is a trilinear form, continuity of $b_{h}^{m}$ is guaranteed.

The discretization of the diffusive term $-\Delta m = -\nabla \text{div} m + \text{curl} \text{curl} m$ in the magnetization equation (2.89e) turns out to be tricky and is done as in [30] which – to the best of our knowledge – was the first time this approach appeared. Transferring the local $H^{1}$-regularity, which is based on global $H(\text{div}, \text{curl})$-regularity, to the discrete setting is not trivial, though. Further research is needed.

In the weak formulation of the magnetization equation the product of the type $\text{div} m \text{div} \theta$ or $\text{curl} m \cdot \text{curl} \theta$ appears, see e.g. (3.140). But the ansatz space for $m^{k} \in M_{h} = D_{1}(\Omega)^{d}$ does not yield any differentiability. Typical discontinuous Galerkin type discretizations lead to additional (inner) boundary integrals which might be difficult to balance. Therefore, a duality approach is chosen. First, consider $d = 3$. Let $\text{curl} h m^{k} \in P_{1}(\Omega)^{d} \cap H_{0}(\text{curl})(\Omega)^{d}$ and $\text{div}_{h} m^{k} \in P_{2}(\Omega) \cap H_{1}^{0}(\Omega)$ be determined by the variational problems

\[
\int_{\Omega} I_{h,1}(\text{curl}_{h} m^{k} \cdot n) \, dx = \int_{\Omega} m^{k} \cdot \text{curl} n \, dx \quad \forall n \in P_{1}(\Omega)^{d} \cap H_{0}(\text{curl})(\Omega),
\]

\[
\int_{\Omega} \text{div}_{h} m^{k} S \, dx = - \int_{\Omega} m^{k} \cdot \nabla S \, dx \quad \forall S \in P_{2}^{\text{zero}}(\Omega). \quad (4.31)
\]

By construction, $\text{curl}_{h} m^{k} \times \nu|_{\partial \Omega} = 0$ and $\text{div}_{h} m^{k}|_{\partial \Omega} = 0$, which are the (non-conforming) boundary conditions that have also been used in the Galerkin approximation process in Section 3 see e.g. (3.7). Using the discrete divergence and the discrete curl, terms of the type $\nabla \text{div}_{h} m^{k}$ and $\text{curl}_{h} m^{k}$ are well-defined.

**Remark 4.3.** One might wonder why the lumped-mass scalar product, which was used in (4.31), has not been used in (4.32). In order to use it, the space of test functions has to be chosen of polynomial order 1. However, as it will be evident from the discrete energy estimate in Theorem 4.3, the finite element space here must have the same polynomial order as $\mathcal{R}_{h} = P_{2}^{\text{linear}}(\Omega)$. 
The next question one might ask is why the second order space $R_h$ has been chosen in the first place. For this, the reader is advised to look at (4.30) which is used to discretize the Kelvin force. Due to (3.4.14) a polynomial order of 0 for $M_h$ allows to choose $R_h$ to have polynomial order 1. The definition (4.30) remains valid. However, if $h \in M_h$ in (4.30) was of order 0, then the whole volume integral vanishes and only the (inner) boundary integral remains. This consists of a jump term, which immediately attracts attention in regard to numerical stability concerns – corresponding to cancellation effects. Numerical investigation, see Figure 5.29, supports this concern.

As the discrete analogue of $\text{div } h$ in the energy estimate will be needed, too, the discrete divergence of $h^k$ will be defined by duality as well. There will be two definitions, one for each discrete analogue of $h|_\Omega$ or $h|_{\Omega\setminus\Gamma}$, respectively. Both together will be combined into $\text{div}_h h^k$. Hence,

$$
\int_\Omega \text{div}_h h^k S \, dx = - \int_\Omega h^k \cdot \nabla S \, dx \quad \forall S \in P_2^\text{zero}(\Omega),
$$

$$
\int_{\Omega\setminus\Gamma} \text{div}_h h^k S \, dx = - \int_{\Omega\setminus\Gamma} h^k \cdot \nabla S \, dx \quad \forall S \in P_2^\text{zero}(\Omega') \cap H_0^1(\Omega' \setminus \partial \Omega),
$$

(4.33)

determine $\text{div}_h h^k$ and imply by construction that $\text{div } h^k |_{\partial \Omega} = 0$. Equivalently, one can determine $\text{div}_h h^k$ via

$$
\int_{\Omega'} \text{div}_h h^k S \, dx = - \int_{\Omega'} h^k \cdot \nabla S \, dx \quad \forall S \in P_2(\Omega) \cap H_0^1(\Omega' \setminus \partial \Omega).
$$

However, the approach in (4.33) serves as a reminder that in the continuous setting $h$ does not necessarily have a distributional divergence which is globally $L^2$-regular on $\Omega'$. As $\text{div}_h h^k$ will not appear in the discretization but only in the discrete energy estimate, any conditions that are enforced by the space of test functions can only hurt the resulting regularity, but not the quality of the numerical approximation. Having a look at (2.14), it does not seem peculiar that $\text{div } m$ and $- \text{div } h$ have the same boundary values. The discrete analogue of $\text{curl } h$ is not needed as it vanishes anyway. The latter claim is e.g. evident from the computations in (4.34).

Now, in the two-dimensional case, the definition of the discrete curl operator needs to be modified. The variational problem

$$
\int_\Omega \mathcal{I}_{h\cdot}(\text{curl}_h m^k \varphi) \, dx = \int_\Omega m^k \cdot \text{Curl } \varphi \, dx \quad \forall \varphi \in P_1^\text{zero}(\Omega),
$$

(4.34)

which is based on (A.30), determines $\text{curl}_h m^k \in P_1^\text{zero}(\Omega)$ uniquely.

Let an initial magnetic field potential be determined by

$$
\int_{\Omega'} \nabla R^0 \cdot \nabla S \, dx = \int_{\Omega'} (h_a)^0 \cdot \nabla S \, dx - \int_\Omega m^0 \cdot \nabla S \, dx \quad \forall S \in P_2^\text{mean}(\Omega').
$$

(4.35)

Define

$$
\mathcal{Y}_h := P_2^\text{zero}(\Omega)^d \times P_1^\text{mean}(\Omega) \times P_1(\Omega) \times P_1(\Omega)^d \times P_2^\text{mean}(\Omega) \times D_1(\Omega)^d
$$

$$
= \mathcal{U}_h \times \mathcal{P}_h \times \mathcal{C}_h \times \mathcal{V}_h \times \mathcal{R}_h \times \mathcal{M}_h,
$$

then the discrete numerical scheme is as follows. First, the case $d = 3$ will be considered. The minor adaptations needed for the case $d = 2$ will be mentioned below the scheme.
Starting with
\[ \mathbf{u}^0 \in \mathcal{U}_h, \quad \mathbf{c}^0 \in \mathcal{C}_h, \quad \mathbf{m}^0 \in \mathcal{M}_h, \]
(4.36)
find for each time step \( k = 1, \ldots, n_T \) functions \((\mathbf{u}^k, p^k, \mathbf{c}^k, \mathbf{v}^k_{\text{part}}, R^k, \mathbf{m}^k) \in \mathcal{Q}_h \) such that for all test functions \((\mathbf{v}, q, \psi, \theta, S, \mathbf{n}) \in \mathcal{Q}_h \) the equations
\[
\rho_0 \int_{\Omega} (\mathbf{u}^{k-1} - \mathbf{u}^k) \cdot \mathbf{v} \, dx - \int_{\Omega} p^k \text{div} \mathbf{v} \, dx + 2\eta \int_{\Omega} \mathbf{D}u^k : \mathbf{Dv} \, dx \\
+ \frac{\rho_0}{2} \int_{\Omega} (\mathbf{u}^{k-1} \cdot \nabla) \mathbf{u}^k \cdot \mathbf{v} \, dx - \frac{\eta}{2} \int_{\Omega} (\mathbf{u}^{k-1} \cdot \nabla) \mathbf{v} \cdot \mathbf{u}^k \, dx \\
= -D \int_{\Omega} \nabla g_{s,h}^k \cdot \mathbf{v} \, dx + \mu_0 b_{\Omega}^m (\mathbf{v}, (\alpha_1 h^k|_\Omega + \frac{\beta}{2} (h_a)_{h|\Omega} - \alpha_3 m^k), \mathbf{m}) \\
+ \frac{\mu_0}{2} \int_{\Omega} (\mathbf{m}^k \times (\alpha_1 h^k + \frac{\beta}{2} (h_a)_{h|\Omega})) \cdot \text{curl} \mathbf{v} \, dx,
\]
(4.37a)
\[
\int_{\Omega} \text{div} \mathbf{u}^k q \, dx = 0,
\]
(4.37b)
\[
\int_{\Omega} \mathcal{I}_{p,1} \left( \frac{(c^k - c^{k-1})}{\tau} \right) \psi \, dx = \int_{\Omega} \mathcal{I}_{c,1}^{k-1} \mathbf{u}^k \cdot \nabla \psi \, dx + \int_{\Omega} \mathcal{I}_{p,1} \mathbf{c}^k \cdot \nabla \psi \, dx,
\]
(4.37c)
\[
\int_{\Omega} \mathcal{I}_{p,1} \mathbf{v}^k_{\text{part}} \cdot \theta \, dx = -KD \int_{\Omega} \mathcal{I}_{p,1} \left( \frac{\mathcal{J}_{p,h}^{k-1}}{\mathcal{J}_{c,h}^{k-1} s,h} \nabla g_{s,h}^k \cdot \theta \right) \, dx \\
+ K\mu_0 b_{\Omega}^m \left( \mathcal{I}_{p,1} \left( \frac{\mathcal{J}_{p,h}^{k-1}}{\mathcal{J}_{c,h}^{k-1} s,h} \right)^2 \theta , (\alpha_1 h^k|_\Omega + \frac{\beta}{2} (h_a)_{h|\Omega} - \alpha_3 m^k), \mathbf{m} \right),
\]
(4.37d)
\[
\int_{\Omega} \nabla R^k \cdot \nabla S \, dx = \int_{\Omega} (h_a)^k_{h|\Omega} \cdot \nabla S \, dx - \int_{\Omega} \mathbf{m}^k \cdot \nabla S \, dx,
\]
(4.37e)
\[
\int_{\Omega} \frac{(m^k - m^{k-1})}{\tau} \cdot \mathbf{n} \, dx - b_{\Omega}^m (\mathbf{u}^k, \mathbf{n}, \mathbf{m}^k) - b_{\Omega}^m (\mathbf{v}^k_{\text{part}}, \mathbf{n}, \mathbf{m}^k) \\
= \frac{1}{2} \int_{\Omega} (\text{curl} \mathbf{u}^k \times \mathbf{m}^k) \cdot \mathbf{n} \, dx - \frac{1}{\tau_{\text{rel}}} \int_{\Omega} (\mathbf{m}^k - \chi_{c,h}^{k-1} h^k) \cdot \mathbf{n} \, dx \\
- \sigma \int_{\Omega} \text{curl} h \mathbf{m}^k \cdot \mathbf{n} \, dx + \sigma \int_{\Omega} \nabla \text{div}_h \mathbf{m}^k \cdot \mathbf{n} \, dx
\]
(4.37f)
hold, where \( h^k = \nabla R^k \) (cf. (4.25)).

Note that the first two terms on the right-hand side of (4.37a) contain discrete counterparts of \( Dg'(c) \nabla c \) and \( \mu_0 (\nabla \mathbf{m})^T \mathbf{m} \), two terms which can be written as gradients and which therefore have been absorbed in \( \nabla p \) in the continuous setting (2.89a). In the discrete setting they are needed for stability reasons.

In case of \( d = 2 \) minor modifications are necessary whenever a curl-related or \( \times \)-related term appears. Inspecting Section A.4 it becomes evident that the first term on the right-hand side of (4.37f) is just well-defined as it is. The third term on the right-hand side of (4.37f), however, needs to be replaced with the very similar term
\[
-\sigma \int_{\Omega} \text{Curl} \text{curl}_h \mathbf{m}^k \cdot \mathbf{n} \, dx.
\]
(4.38)
4.1 Design of energy stable finite element schemes

The only other term to be considered is the last one on the right-hand side of (4.37a). However, by viewing \((\mathbf{m}^k \times \hat{h}^k) \cdot \text{curl} \mathbf{v}\) as a scalar-valued product

\[
(\mathbf{m}^k \times \hat{h}^k) \cdot \text{curl} \mathbf{v}
\]

the term is well-defined. For more insight about why those definitions are physically meaningful, the reader is referred to Remark A.11, (A.28), (A.26) and its implications on the discretization terms.

4.1.2 Model W and Model B

The discretizations of 'model W' or 'model B' are based on the discretization of 'model GW' and on the discretization of a two-phase flow model in [38, scheme B]. One of the main differences from 'model GW' is that the magnetostatic equation changes, i.e. in (4.37e) we replace \(\mathbf{m}^k\) with \(\chi_{s,h}^k \nabla R^k\), where \(\chi_{s,h}^k\) is a suitable approximation of the susceptibility to be defined later. Another difference is the discretization of the Kelvin force. The last main difference is the fact that in case of 'model B' the Navier-Stokes equations are discretized in a significantly different way to account for two-phase flows. The discretizations will be defined within one scheme with the help of some "switches".

The function

\[
\Theta(x) := \begin{cases} 
1 & \text{in case of "model W"}, \\
\chi & \text{in case of "model B"}
\end{cases}
\]

plays the role of a switch between the models. Furthermore, the regularized density of the two-phase flow,

\[
\rho_{\text{reg}}(c) := \begin{cases} 
\rho_0 & \text{in case of "model W"}, \\
\hat{\rho}_1 + (\hat{\rho}_2 - \hat{\rho}_1) \max(\min(c, 1), 0) & \text{in case of "model B"},
\end{cases}
\]

naturally acts as switch, too, where the definition for 'model B' matches the definition for 'model W' in case of \(\rho_0 = \hat{\rho}_1 = \hat{\rho}_2\). The regularization from below (in contrast to (2.47)) is sufficient for discrete energy stability. Set for any \(k \in \{0, ..., n_T\}\)

\[
\rho^k_h := I_{h,1}(\rho_{\text{reg}}(c^k)) \in C_h.
\]

Starting with the Navier-Stokes equations for 'model B' one can easily obtain from [38] a first discretization of the left-hand side of (2.110a),

\[
\begin{align*}
\int_\Omega & \frac{\rho_h^{k+1} - \rho_h^k}{\tau} \cdot \mathbf{v} \, dx + \frac{1}{2} \int_\Omega \rho_h^k \mathbf{u}^{k-1} \cdot \mathbf{v} \, dx - \int_\Omega \rho_h^k \nabla \mathbf{v} \cdot \mathbf{v} \, dx + 2\eta \int_\Omega \mathbf{D} \mathbf{u}^k : \mathbf{D} \mathbf{v} \, dx \\
& + \frac{1}{2} \int_\Omega \rho_h^{k-1} \mathbf{u}^{k-1} \cdot \nabla \mathbf{u} \cdot \mathbf{v} \, dx - \frac{1}{2} \int_\Omega \rho_h^{k-1} \mathbf{u}^{k-1} \cdot \nabla \mathbf{v} \cdot \mathbf{u} \, dx \\
& + \frac{1}{2} \int_\Omega \left( \frac{\partial \rho}{\partial c} (c^k \mathbf{v}_\text{part}) \cdot \nabla \right) \mathbf{u}^k \cdot \mathbf{v} \, dx - \frac{1}{2} \int_\Omega \left( \frac{\partial \rho}{\partial c} (c^k \mathbf{v}_\text{part}) \cdot \nabla \right) \mathbf{v} \cdot \mathbf{u}^k \, dx
\end{align*}
\]

for some approximation \(\frac{\partial \rho}{\partial c}\) of \(\rho'\), where \(c^k \mathbf{v}_\text{part}\) plays the role of the flux \(j^{k+1}_z\) in the notation of [38] (where the time index is shifted in contrast to the notation in this thesis). In this case, where the flux \(\mathbf{J}\) is split into the product \(c \mathbf{v}_\text{part}\), a different approach seems reasonable, i.e. the last two terms above are discretized via

\[
\begin{align*}
\frac{1}{2} \int_\Omega (g^{k-1} \mathbf{v}_\text{part} \cdot \nabla) \mathbf{u}^k \cdot \mathbf{v} \, dx - \frac{1}{2} \int_\Omega (g^{k-1} \mathbf{v}_\text{part} \cdot \nabla) \mathbf{v} \cdot \mathbf{u}^k \, dx,
\end{align*}
\]
where \( \hat{\rho}^{k-1} := (\rho^{k-1} - \hat{\rho}_1) \in C_h \), inspired by \( \rho'(c)c = \rho(c) - \rho_1 \), see (2.47). Other choices are conceivable, too. For energy stability the choice is arbitrary – see the proof of Theorem 4.6 where the terms in question just cancel out with each other – and for existence of discrete solutions it may even involve discontinuous expressions (e.g. \( \rho_{\text{reg}} \) in \( \hat{\rho}^{k-1} := I_{h,1}(\rho_{\text{reg}}'(c^{k-1})c^{k-1}) \)) as long as it depends only on the data \( c^{k-1} \) from the previous time step – because then it does not affect continuity of \( \mathcal{L}^{W|B} \) from (4.119) where data from the time step \( k-1 \) are fixed. Rewriting the discretization of the left-hand side of (2.110a) by means of the switch function \( \Theta \), we obtain

\[
\begin{align*}
\int_{\Omega} \rho_0 \hat{\rho}^{k-1} \frac{(u^k - u^{k-1})}{\tau} \cdot v \, dx + \frac{1}{2} \int_{\Omega} \rho_0 \hat{\rho}^{k-1} \cdot \nabla u^k \cdot v \, dx - \int_{\Omega} p^k \, \text{div} \, v \, dx + 2\eta \int_{\Omega} u^k : Dv \, dx \\
+ \frac{1}{2} \int_{\Omega} \rho_h \hat{\rho}^{k-1} (u^k - \nabla u^k) \cdot v \, dx - \frac{1}{2} \int_{\Omega} \rho_h \hat{\rho}^{k-1} (u^k - \nabla v) \cdot v \, dx \\
+ \frac{1}{2} \int_{\Omega} \Theta'[\hat{\rho}^{k-1}] (v_{\text{part}} \cdot \nabla) u^k \cdot v \, dx - \frac{1}{2} \int_{\Omega} \Theta'[\hat{\rho}^{k-1}] (v_{\text{part}} \cdot \nabla) v \cdot u^k \, dx.
\end{align*}
\]

In case of 'model W' the density function is constant (cf. (4.41), (4.40)) and one readily obtains

\[
\begin{align*}
\rho_0 \int_{\Omega} (u^k - u^{k-1}) \cdot v \, dx - \int_{\Omega} p^k \, \text{div} \, v \, dx + 2\eta \int_{\Omega} u^k : Dv \, dx \\
+ \rho_0 \int_{\Omega} (u^{k-1} \cdot \nabla) u^k \cdot v \, dx - \frac{\rho_0}{2} \int_{\Omega} (u^{k-1} \cdot \nabla) v \cdot u^k \, dx
\end{align*}
\]

which is identical to the left-hand side of (4.37a) and serves as discretization of the left-hand side of the momentum equation (2.100a) of 'model W'.

The switch \( \Theta \) can also be used in order to fill in the additional factor \((1 - c)\) in (2.110a) in case of 'model B'. Further regularization is needed in order to make the discrete analogue of \((1 - c)\) strictly positive. Here, choose \( 0 << r < 1 \) and set

\[
c^{k-1,r} := \min(c^{k-1}, r), \quad k = 1, ..., n_T.
\]

For the magnetic forces consider the computation rule

\[
\frac{1}{2} \nabla(|h|^2) = (\nabla h) = (\nabla \nabla R)^T h = (\nabla \nabla R) h = \nabla hh = (h \cdot \nabla) h.
\]

The Kelvin force in the continuous setting can then be written as

\[
\frac{\mu_0}{2} \chi_{\text{lin}}(c) \nabla(|h|^2) = \frac{\mu_0}{2} \nabla(\chi_{\text{lin}}(c) |h|^2),
\]

see (2.34). For the discretization, define the lumped \( P_1 \)-projector \((\cdot)_{1,h} : L^2(\Omega) \to C_h, f \mapsto (f)_{1,h}, \) where

\[
\int_{\Omega} I_{h,1}((f)_{1,h}, \xi) \, dx = \int_{\Omega} f \xi \, dx \quad \forall \xi \in C_h.
\]

This is used to obtain \((|h^k|^2)_{1,h} \in C_h\) which is weakly differentiable. By means of Lemma 4.2 it is easy to obtain

\[
\|((\cdot))_{1,h}\|_{L^2(\Omega)} \leq M_0 \|f\|_{L^2(\Omega)}.
\]

Before the discrete Kelvin force can be defined, a discrete version of the susceptibility is needed.
Choose $\omega \leq 0$, $s := (s, \omega)$. The discrete susceptibility is based on
\[
\chi_{\delta}(c) := \begin{cases} 
\chi_0 c & \text{if } c - \frac{\omega}{\chi_0} \geq s, \\
\chi_0 \left( - \frac{1}{4s^2} (c - \frac{\omega}{\chi_0})^4 + \frac{3}{8s} (c - \frac{\omega}{\chi_0})^2 \right) + \frac{1}{2} (c - \omega) + \frac{3}{16} (c - \omega)^3 & \text{if } -s < \omega - \frac{\omega}{\chi_0} < s, \\
\omega & \text{if } c - \omega \leq -s.
\end{cases}
\] (4.46)

No regularization from above is proposed in contrast to the setting in "model GW". Above function represents a smooth cut-off from below at level $\omega$. The discrete susceptibility then is given for $k \in \{0, \ldots, n_T\}$ by
\[
\chi_{\delta,h}^k := \mathcal{I}_{h,1}(\chi_{\delta}(c^k)) \in C_h
\] (4.47) and $\mathcal{I}_\Omega \chi_{\delta,h}^k$ denotes the extension onto $\Omega'$ by zero.

In case of $\omega = 0$ the function remains non-negative. Values $\omega \leq -s\chi_0$, on the other hand, guarantee the original susceptibility $c \mapsto \chi_0 c$ to be unchanged in the physically reasonable range of arguments ($c \geq 0$). In simulations it turned out to be beneficial if $\omega < -s\chi_0$ such that the numerically critical zero-level of $c$ will be unaffected of the artificial error (overestimated susceptibility) which otherwise would come along with $\chi_{\delta}$ when $\omega > -s\chi_0$. However, as the magnetic energy density on the domain $\Omega$, i.e. $(1 + \chi(c))|\mathbf{h}|^2$, suggests, we assume $\omega > -1$.

The derivative will be approximated by means of
\[
\frac{\delta \chi_{\delta}^{k-1}}{\delta c}(c) := \begin{cases} 
\chi_{\delta}(c) - \chi_0 c & \text{if } c \neq c^{k-1}, \\
\chi_{\delta}(c^{k-1}) & \text{if } c = c^{k-1},
\end{cases}
\] (4.48)
\[
\frac{\delta^2 \chi_{\delta}^{k-1}}{\delta c^2}(c) := \mathcal{I}_{h,1}\left( \frac{\delta^2 \chi_{\delta}^{k-1}}{\delta c^2}(c) \right)
\] (4.49)
for $c \in C_h$. Sufficient differentiability can be proven by tedious but straightforward computations. In fact, the properties
\[
\omega \leq \chi_{\delta} \in C^2(\mathbb{R}), \quad 0 \leq \chi_0' \leq \chi_0, \quad 0 \leq \chi_0'' \leq \frac{3}{4s}, \quad \chi_0 c \leq \chi_{\delta}(c),
\] (4.50)
\[
\frac{\delta \chi_{\delta}^{k-1}}{\delta c} \in C^0(\mathbb{R}), \quad 0 \leq \frac{\delta \chi_{\delta,h}^{k-1}}{\delta c} \leq \chi_0,
\] (4.51)
hold, which follow easily from the definitions and the mean value theorem of differential calculus. For more details, see Section A.3.

Then, the Kelvin force is approximated on basis of (4.43) by
\[
\mathbf{c}^{k-1} \nabla \mathcal{I}_{h,1}\left( \frac{\delta \chi_{\delta}^{k-1}}{\delta c}(c^k)|\mathbf{h}|^2\right)_{1,h}\right).
\]

Actually, this way of exchanging the linear susceptibility from (4.43) with the regularized nonlinear susceptibility from (4.46) is consistent with the model derivation if there had been used a nonlinear susceptibility. For more insight on this regard, see (2.94) and replace $\chi_0$ with $\chi'(c)$ to see in the last step that $\chi'(c)$ would appear together with $|\mathbf{h}|^2$ entailing the term $\nabla(\chi'(c)|\mathbf{h}|^2)$. Let an initial magnetic field potential be determined by
\[
\int_{\Omega'} \nabla R^0 \cdot \nabla \mathbf{S} \, dx + \int_{\Omega} \chi_{\delta,h}^0 \nabla R^0 \cdot \nabla \mathbf{S} \, dx = \int_{\Omega'} (\mathbf{h}_0)_{h}^0 \cdot \nabla \mathbf{S} \, dx \quad \forall \mathbf{S} \in \mathcal{P}_2^{\text{mean}}(\Omega'),
\] (4.52)
where the susceptibility is already given via $c^0$. With those tools at hand, the numerical scheme is given as follows.
Starting with
\[
\mathbf{u}^0 \in \mathcal{U}_h, \quad \mathcal{C}_h, \quad (4.53)
\]
find for each time step \(k = 1, \ldots, n_T\) functions \((\mathbf{u}^k, p^k, c^k, \mathbf{v}_{\text{part}}^k, R^k)\) in
\[
\mathcal{Y}_h := \mathcal{U}_h \times \mathcal{P}_h \times \mathcal{C}_h \times \mathcal{V}_h \times \mathcal{R}_h
\]
such that for all test functions \((\mathbf{v}, q, \psi, \Theta, S) \in \mathcal{Y}_h\) the equations
\[
\int \frac{\rho_h}{2} (\mathbf{u}^k - \rho_h^{-1} (\mathbf{u}^k - \mathbf{u}^{-k-1})) \cdot \mathbf{v} \, dx + \frac{1}{2} \int \frac{\rho_h - \rho_h^{-1}}{\tau} \mathbf{u}^{k-1} \cdot \mathbf{v} \, dx + 2\eta \int \mathbf{D} \mathbf{u}^k : \mathbf{D} \mathbf{v} \, dx - \int \rho_h^k \text{div} \mathbf{v} \, dx
\]
\[
+ \frac{1}{2} \int \rho_h^{-1} (\mathbf{u}^{k-1} \cdot \nabla) \mathbf{u}^k \cdot \mathbf{v} \, dx - \frac{1}{2} \int \rho_h^{-1} (\mathbf{u}^{k-1} \cdot \mathbf{v}) \cdot \mathbf{u}^k \, dx
\]
\[
+ \frac{1}{2} \int (\Theta' [g^{k-1}] \mathbf{v}_{\text{part}}^k \cdot \nabla) \mathbf{u}^k \cdot \mathbf{v} \, dx - \frac{1}{2} \int (\Theta' [g^{k-1}] \mathbf{v}_{\text{part}}^k \cdot \nabla) \mathbf{v} \cdot \mathbf{u}^k \, dx
\]
\[
= -D \int \mathbf{c}^{k-1} \nabla \Theta_{h,1} (g_s' (c^k) - \Theta [g_s' (1 - c^k)]) \cdot \mathbf{v} \, dx
\]
\[
+ \frac{\mu_0}{2} \int \mathbf{c}^{k-1} \mathbf{v} \cdot \nabla \Theta_{h,1} \left( \frac{\delta g_h^{-1}}{c^k} (c^k) \left( |h|^2 \right)_{h,1} \right) \, dx
\]
\[
(4.54a)
\]
\[
\int \text{div} \mathbf{u}^k q \, dx = 0,
\]
\[
\int \mathcal{I}_{h,1} \left( \frac{c^{k-1}}{c^k - r} \right) \, dx = \int \mathbf{c}^{k-1} \mathbf{u}^k \cdot \nabla \psi \, dx + \int \mathcal{I}_{h,1} (\mathbf{c}^{k-1} \mathbf{v}_{\text{part}}^k \cdot \nabla \psi) \, dx,
\]
\[
(4.54b)
\]
\[
\int \mathcal{I}_{h,1} (\mathbf{v}_{\text{part}}^k \cdot \Theta) \, dx
\]
\[
= -KD \int \mathcal{I}_{h,1} \left( \frac{\hat{c}^{p,h} (c^k)}{c^k - r} \right) \left( \frac{c^k - r}{c^{p,h} (c^k)} \right) \Theta [1 - c^{k-1} \mathbf{c}^k \nabla \Theta_{h,1} (g_s' (c^k) - \Theta [g_s' (1 - c^k)]) \cdot \Theta \right) \, dx
\]
\[
+ \frac{K}{2} \int \mathcal{I}_{h,1} \left( \Theta [1 - c^{k-1} \mathbf{c}^k \nabla \Theta_{h,1} (g_s' (c^k) - \Theta [g_s' (1 - c^k)]) \cdot \Theta \right)
\]
\[
\int \nabla R^k \cdot \nabla S \, dx + \int \mathcal{I}_{h,1} \mathbf{c}^k \mathbf{c}^k \mathbf{v} \cdot \nabla S \, dx = \int \Omega^r \left( \mathbf{h}_{h}^k \right) \cdot \nabla S \, dx,
\]
\[
(4.54c)
\]
hold, where \(\mathbf{h}^k = \nabla R^k\) (cf. (4.25)).

**Remark 4.4.** For convenience, the difference between the discretizations (4.37) (for "model GW") and (4.54) (for "model W" or "model B") will be highlighted. Notice that (4.54c) and (4.37c) are identical. In (4.54d) the only differences to (4.37d) are the switch function \(\Theta\) and the Kelvin force. The magnetization equation is missing in (4.54), of course, but could be revitalized as
\[
\mathbf{m}^k := \chi_{\Delta h}^k \mathbf{h}^k,
\]
which renders (4.54e) identical to (4.37e). The Navier-Stokes equation (4.54a) has already been discussed below (4.41). In case of "model W" there was no difference between the left-hand sides of (4.54a) and (4.37a). Terms of the kind \(\mathbf{m} \times \mathbf{h}\) (see e.g. last term in (4.37a) vanish in case of "model W" or "model B" (which use \(\mathbf{m} = \chi_{\text{lin}} (c) \mathbf{h}\)) due to \(\mathbf{h} \times \mathbf{h} = 0\). Besides of that, the Kelvin force in (4.54a) is discretized differently compared to (4.37a).

### 4.1.3 Discrete energy estimates

The following results are concerned with the discrete energy stability of the numerical schemes (4.37) and (4.54).
Theorem 4.5. Let \((u^k, p^k, c^k, v^{\text{part}}_k, R^k, m^k)_{k=0,\ldots,n_T}\) be a discrete solution to the scheme \(4.37\) and let \(f^{(s),k-1}_h, k = 1, \ldots, n_T\), be given by
\[
f^{(s),k-1}_h := \mathcal{I}_{h,1}(\max(s, c^{k-1})^{2-m}).
\]
A positive constant \(C_0 > 0\), depending only on \(h_s\), initial data, ending time \(T\), \(|\Omega|\) and regularization parameter \(r > 0\) (cf. \(N^2 4.2\)), exists such that
\[
\frac{\rho_0}{2} \int_\Omega |u^k|^2 \, dx + D \int \Omega g_{s,h}^k \, dx + \frac{\alpha^3 \mu_0}{4} \int_\Omega |m^k|^2 \, dx + \frac{\mu_0 \alpha_3}{2} \int_{\Omega'} |\nabla R^k|^2 \, dx
+ \mu_0 \alpha_3 \sum_{l=1}^k \int_{\Omega'} |\nabla \rho - \nabla \rho|^2 \, dx
+ \frac{\tau}{2} \sum_{l=1}^k \int_{\Omega'} |\mathbf{D}u^l|^2 \, dx + \frac{\tau}{2} \sum_{l=1}^k \int_{\Omega'} \left|\mathbf{D}u^l - u^{l-1}\right|^2 \, dx
+ \frac{\tau \sigma \mu_0 \alpha_3}{2} \sum_{l=1}^k \int_{\Omega'} \left|\text{div}_h m^l\right|^2 \, dx + \frac{\tau \sigma \mu_0 \alpha_3}{2} \sum_{l=1}^k \int_{\Omega'} \mathcal{I}_{h,1}(\text{curl}_h m^l)^2 \, dx
+ \frac{\tau \sigma \mu_0 \alpha_3}{2} \sum_{l=1}^k \int_{\Omega'} \left|\text{div}_h \mathbf{h}^l\right|^2 \, dx + \frac{\tau \sigma \mu_0 \alpha_3}{2} \sum_{l=1}^k \int_{\Omega'} \left|\nabla \rho\right|^2 \, dx
+ \frac{\tau \mu_0 \alpha_3}{2} \sum_{l=1}^k \int_{\Omega'} \left|\nabla R^l\right|^2 \, dx + \frac{\tau \mu_0 \alpha_3}{2} \sum_{l=1}^k \int_{\Omega'} \left|\nabla R^l\right|^2 \, dx
\leq \frac{\rho_0}{2} \int_\Omega |u^0|^2 \, dx + D \int \Omega g_{s,h}^0 \, dx + \frac{\alpha^3 \mu_0}{2} \int_\Omega |m^0|^2 \, dx + \frac{\mu_0 \alpha_3}{2} \int_{\Omega'} |\nabla R^0|^2 \, dx + C_0.
\]
Proof: Test \(4.37a\) by \(\mathbf{u}^k\) and use \(4.37b\) and \(m = m = 0\) to obtain
\[
\frac{\rho_0}{2} \int_\Omega |u^k|^2 \, dx + \frac{\rho_0}{2} \int_\Omega |u^k - u^{k-1}|^2 \, dx + \frac{\mu_0 \alpha_3}{2} \int_{\Omega'} |\nabla \mathbf{h}_s^k \cdot \mathbf{u}^k|^2 \, dx
+ \frac{\tau \mu_0 \alpha_3}{2} \int_{\Omega'} (\mathbf{m}^k \times \mathbf{h}^k) \cdot \text{curl} \mathbf{u}^k \, dx.
\]
Testing \(4.37c\) by \(D \mathbf{g}_{s,h}^k\) gives
\[
D \int \Omega \mathcal{I}_{h,1} \left( (c^k - c^{k-1}) \mathbf{g}_{s,h}^k \right) \, dx
= \tau D \int \Omega c^{k-1} \mathbf{u}^k \cdot \nabla \mathbf{g}_{s,h}^k \, dx + \tau D \int \Omega \mathcal{I}_{h,1}(c^{k-1} \mathbf{v}_{\text{part}}^k \cdot \nabla \mathbf{g}_{s,h}^k) \, dx.
\]
Equation \(4.37d\) will be tested by \(\tau \mathcal{I}_{h,1}(f^{(s),k-1}_h \mathbf{v}_{\text{part}}^k)\). First, observe that the following identities hold true due to the fact that all interpolation operators inside another interpolation operator can be dropped as they do not change the nodal values which are the only information needed for the outer most interpolation operator.
\[
\int \Omega \mathcal{I}_{h,1} \left( \frac{f^{(s),k-1}_h}{(c^{k-1})_{s,h}} \nabla \mathbf{g}_{s,h}^k \cdot \mathcal{I}_{h,1}(f^{(s),k-1}_h \mathbf{v}_{\text{part}}^k) \right) \, dx
= \int \Omega \mathcal{I}_{h,1}(c^{k-1} \nabla \mathbf{g}_{s,h}^k \cdot \mathbf{v}_{\text{part}}^k) \, dx
\]
and
\[ \mathcal{I}_{h,1}^d \left( \frac{\hat{I}_{p,h}^{(s),k-1}}{((e^{s})^{k-1})_{x,h}} \mathcal{I}_{h,1}^d (f_h^{(s),k-1}, v_{\text{part}}^k) \right) \]
\[ = \mathcal{I}_{h,1}^d (\max(s, e^{k-1}) - 2, \max(s, e^{k-1})^2 - m \cdot v_{\text{part}}^k) = \mathcal{I}_{h,1}^d (v_{\text{part}}^k) = v_{\text{part}}^k. \] (4.61)

Hence, one readily arrives at
\[ \tau \int_{\Omega} \mathcal{I}_{h,1}^d \left( f_h^{(s),k-1} \left| \frac{v_{\text{part}}^k}{K} \right|^2 \right) \, dx \]
\[ = -\tau D \int_{\Omega} \hat{I}_{h,1} (e^{k-1} \nabla g_{s,h}^k \cdot v_{\text{part}}^k) \, dx + \tau \mu_0 \hat{b}_h^m (v_{\text{part}}^k, \hat{b}^k, m^k). \] (4.62)

Using the convexity of \( g_s \), implying \( g_s(x) - g_s(y) \leq (x - y)g'_s(x) \), and Lemma 4.1 one obtains
\[ \mathcal{I}_{h,1}^d \left( \left( e^{k-1} - e^{k-1} \right) g_{s,h}^k \right) \geq \mathcal{I}_{h,1}^d (g_{s,h}^k - g_{s,h}^{k-1}) = g_{s,h}^k - g_{s,h}^{k-1}, \] (4.63)

recalling inner interpolation operators to be negligible, see the paragraph above (4.60). Combining (4.59)-(4.63) yields
\[ \tau \int_{\Omega} \mathcal{I}_{h,1}^d \left( f_h^{(s),k-1} \left| \frac{v_{\text{part}}^k}{K} \right|^2 \right) \, dx + D \int_{\Omega} g_{s,h}^k \, dx \]
\[ \leq D \int_{\Omega} g_{s,h}^{k-1} \, dx + \frac{\tau \mu_0}{2} \hat{b}_h^m (v_{\text{part}}^k, \hat{b}^k, m^k) + \tau D \int_{\Omega} e^{k-1} u^k \cdot \nabla g_{s,h}^k \, dx. \] (4.64)

The magnetization (4.37) will be tested by
\[ -\tau \mu_0 \hat{b}^k = -\tau \mu_0 (\hat{h}^k)_{\Omega} - \alpha_3 m^k = -\tau \mu_0 (\alpha_1 h^k)_{\Omega} + \frac{\tau}{2} (h_{a} h_{h})_{\Omega} - \alpha_3 m^k. \]

This yields
\[ \begin{aligned}
-\mu_0 \int_{\Omega} (m^k - m^{k-1}) \cdot \hat{b}^k \, dx &+ \frac{\tau \mu_0 \alpha_3}{\tau_{rel}} \int_{\Omega} |m^k|^2 \, dx \\
-\frac{\tau \mu_0}{\tau_{rel}} \int_{\Omega} m^k \cdot \hat{h}^k \, dx &+ \frac{\tau \mu_0}{\tau_{rel}} \int_{\Omega} \chi_{s,h}^{k-1} h^k \cdot \hat{h}^k \, dx \\
+ \tau \sigma \mu_0 \alpha_3 \left( \int_{\Omega} \text{curl} \text{curl}_h m^k \cdot m^k \, dx - \int_{\Omega} \nabla \text{div}_h m^k \cdot m^k \, dx \right) \\
- \tau \sigma \mu_0 \left( \int_{\Omega} \text{curl} \text{curl}_h m^k \cdot \hat{h}^k \, dx - \int_{\Omega} \nabla \text{div}_h m^k \cdot \hat{h}^k \, dx \right) \\
= -\tau \mu_0 \hat{b}_h^m (u^k, \hat{b}^k, m^k) - \tau \mu_0 \hat{b}_h^m (v_{\text{part}}^k, \hat{b}^k, m^k) \\
- \frac{\tau \mu_0}{2} \int_{\Omega} (\text{curl} u^k \times m^k) \cdot b^k \, dx + \frac{\tau \mu_0 \alpha_3}{\tau_{rel}} \int_{\Omega} \chi_{s,h}^{k-1} h^k \cdot m^k \, dx. \end{aligned} \] (4.65)

Expanding the abbreviation \( \hat{b}^k \) (cf. (4.26)) and using
\[ -2ab = -a^2 - b^2 + (a - b)^2 \] (4.66)

the first term on the left hand side reads
\[ -\mu_0 \int_{\Omega} (m^k - m^{k-1}) \hat{b}^k \, dx \]
\[ = \frac{\alpha_3 \mu_0}{2} \int_{\Omega} |m^k|^2 \, dx - \frac{\alpha_3 \mu_0}{2} \int_{\Omega} |m^{k-1}|^2 \, dx + \frac{\alpha_3 \mu_0}{2} \int_{\Omega} |m^k - m^{k-1}|^2 \, dx \]
\[ - \mu_0 \alpha_1 \int_{\Omega} m^k \cdot h^k \, dx + \mu_0 \alpha_1 \int_{\Omega} m^{k-1} \cdot h^k \, dx - \frac{\mu_0 \beta}{2} \int_{\Omega} (m^k - m^{k-1}) \cdot (h_a)^k \, dx. \] (4.67)
Testing (4.37c) by $R^k$ yields
\[ \int_{\Omega'} |\nabla R^k|^2 \, dx = \int_{\Omega'} (h_a)^k_h \cdot \nabla R^k \, dx - \int_{\Omega} m^k \cdot \nabla R^k \, dx. \] (4.68)

Testing (4.37c) in time step $k$ and $k-1$ by $\mu_0 \alpha_1 R^k$ gives
\[ -\mu_0 \alpha_1 \int_{\Omega} m^k \cdot h^k \, dx + \mu_0 \alpha_1 \int_{\Omega} m^{k-1} \cdot h^k \, dx \]
\[ = \mu_0 \alpha_1 \int_{\Omega'} |\nabla R^k|^2 \, dx - \mu_0 \alpha_1 \int_{\Omega'} (h_a)^k_h \cdot \nabla R^k \, dx \]
\[ - \mu_0 \alpha_1 \int_{\Omega'} \nabla R^{k-1} \cdot \nabla R^k \, dx + \mu_0 \alpha_1 \int_{\Omega'} h_a^{k-1} \cdot \nabla R^k \, dx \] (4.69)
\[ - \frac{\tau \mu_0}{\tau_{rel}} \int_{\Omega'} \left( (h_a)^k_h - (h_a)^{k-1}_h \right) \cdot \nabla R^k \, dx + \frac{\mu_0 \alpha_1}{2} \int_{\Omega'} |\nabla R^k|^2 \, dx \]
\[ - \frac{1}{2} \mu_0 \alpha_1 \frac{\mu_0 \alpha_1}{2} \int_{\Omega'} |\nabla R^{k-1}|^2 \, dx + \frac{1}{2} \mu_0 \alpha_1 \int_{\Omega'} |\nabla R^k - \nabla R^{k-1}|^2 \, dx. \]

Analogously, in (4.65) the third term of the left hand side can be rewritten to
\[ - \frac{\tau \mu_0}{\tau_{rel}} \int_{\Omega} m^k \cdot \hat{h}^k \, dx \] (4.68)
\[ \frac{\tau \mu_0 \alpha_1}{\tau_{rel}} \int_{\Omega'} |\nabla R^k|^2 \, dx - \frac{\tau \mu_0 \alpha_1}{\tau_{rel}} \int_{\Omega'} (h_a)^k_h \cdot \nabla R^k \, dx - \frac{\beta \tau \mu_0}{2 \tau_{rel}} \int_{\Omega} m^k \cdot (h_a)^k_h \, dx \] (4.70)
by expanding the abbreviation $\hat{h}^k$ (cf. (4.26)) and multiplying (4.68) by $\frac{\tau \mu_0 \alpha_1}{\tau_{rel}}$. The fourth term of the left hand side in (4.65) will be rewritten by expanding the definition of $\hat{h}^k$ (cf. (4.26)) as follows,
\[ \frac{\tau \mu_0}{\tau_{rel}} \int_{\Omega} \chi_{r,h}^{k-1} h^k \cdot \hat{h}^k \, dx = \frac{\tau \mu_0 \alpha_1}{\tau_{rel}} \int_{\Omega} \chi_{r,h}^{k-1} |h^k|^2 \, dx + \frac{1}{2} \frac{\beta \tau \mu_0}{\tau_{rel}} \int_{\Omega} \chi_{r,h}^{k-1} h^k \cdot (h_a)^k_h \, dx. \] (4.71)

Using the definitions (4.32) and (4.31), the fifth and the sixth term (counting each term in the parentheses individually) in the left hand side of (4.65) are
\[ \tau \sigma \mu_0 \alpha_3 \left( \int_{\Omega} \text{curl} \text{curl}_h m^k \cdot m^k \, dx - \int_{\Omega} \nabla \text{div}_h m^k \cdot m^k \, dx \right) \]
\[ = \tau \sigma \mu_0 \alpha_3 \left( \int_{\Omega} I_{h,1}(|\text{curl}_h m^k|^2) \, dx + \int_{\Omega} |\text{div}_h m^k|^2 \, dx \right). \] (4.72)

The seventh and eighth term (counting each term in the parentheses individually) on the left hand side of (4.65) will be treated as follows. First, expand the abbreviation $\hat{h}^k$ (cf. (4.26)),
\[ - \tau \sigma \mu_0 \left( \int_{\Omega} \text{curl} \text{curl}_h m^k \cdot \hat{h}^k \, dx - \int_{\Omega} \nabla \text{div}_h m^k \cdot \hat{h}^k \, dx \right) \]
\[ = - \tau \sigma \mu_0 \alpha_1 \int_{\Omega} \text{curl} \text{curl}_h m^k \cdot h^k \, dx - \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \text{curl} \text{curl}_h m^k \cdot (h_a)^k_h \, dx \] (4.73)
\[ + \tau \sigma \mu_0 \alpha_1 \int_{\Omega} \nabla \text{div}_h m^k \cdot h^k \, dx + \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \nabla \text{div}_h m^k \cdot (h_a)^k_h \, dx. \]

The first term on the right hand side vanishes by the following observations. Piecewise integration by parts (see formulas (A.29), (A.30)), continuity of $R^k \in R_h$ – implying continuity of tangential
derivatives on $\partial \mathcal{K}, \mathcal{K} \in \mathcal{T}_h(\Omega)$ – and the boundary conditions enforced on $\text{curl}_h \mathbf{m}^k$ by the choice of test functions in (4.31) imply in case of $d = 3$

$$\int_{\Omega} \text{curl} \text{curl}_h \mathbf{m}^k \cdot \nabla R^k \, dx = \sum_{\mathcal{K} \in \mathcal{T}_h(\Omega)} \int_{\mathcal{K}} \text{curl} \text{curl}_h \mathbf{m}^k \cdot \nabla R^k \, dx$$

$$= \sum_{\mathcal{K} \in \mathcal{T}_h(\Omega)} \int_{\mathcal{K}} \text{curl}_h \mathbf{m}^k \cdot (\text{curl} \nabla R^k) \, dx - \sum_{\mathcal{K} \in \mathcal{T}_h(\Omega)} \int_{\partial \mathcal{K}} \text{curl}_h \mathbf{m}^k \times \nu \cdot \nabla R^k \, d\sigma$$

$$= \int_{\Omega'} \left[ \nabla R^k \right] \times \nu_{\text{int}} \cdot \text{curl}_h \mathbf{m}^k \, dx - \int_{\partial \Omega} \text{curl}_h \mathbf{m}^k \times \nu \cdot \nabla R^k \, d\sigma = 0. \quad (4.74)$$

In the two-dimensional case the starting line in (4.74) changes accordingly (cf. (4.38)) but already the first computation step, i.e. integration by parts (see (A.30)), yields the exact same result as presented in the second line of (4.74), where some former vector-valued scalar products need to be viewed as scalar-valued products.

The third term on the right-hand side of (4.73) will be treated by using (4.33) and testing (4.37e) by $\text{div}_h \mathbf{h}^k$. One finds

$$\int_{\Omega} \nabla \text{div}_h \mathbf{m}^k \cdot \mathbf{h}^k \, dx = \int_{\Omega} \text{div}_h \mathbf{m}^k \cdot \text{div}_h \mathbf{h}^k \, dx - \int_{\Omega} \mathbf{m}^k \cdot \nabla \text{div}_h \mathbf{h}^k \, dx$$

$$= \int_{\Omega'} \nabla \text{div}_h \mathbf{h}^k \cdot (\mathbf{h}_a)_h \, dx - \int_{\Omega'} \nabla \text{div}_h \mathbf{h}^k \cdot \mathbf{h}^k \, dx$$

$$= - \int_{\Omega'} \text{div}_h \mathbf{h}^k \cdot \text{div}(\mathbf{h}_a)_h \, dx + \int_{\Omega'} |\text{div}_h \mathbf{h}^k|^2 \, dx. \quad (4.75)$$

Notice that the first term on the right-hand side of (4.75) would vanish if (4.18b) was used to discretize the external magnetic field. Indeed, starting at the second last line,

$$\int_{\Omega'} \nabla \text{div}_h \mathbf{h}^k \cdot (\mathbf{h}_a)_h \, dx = \int_{\Omega'} \nabla \text{div}_h \mathbf{h}^k \cdot \mathbf{h}_a (t^k, \cdot) \, dx$$

$$= - \int_{\Omega'} \text{div}_h \mathbf{h}^k \text{div}(\mathbf{h}_a)_{h} (t^k, \cdot) \, dx \quad (4.76)$$

The second and fourth term of the right-hand side of (4.73) need to be treated differently depending on the choice of discretization of $\mathbf{h}_a$, see (4.18a) or (4.18b), respectively. In case of (4.18a) one just integrates by parts in order to obtain

$$- \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \text{curl} \text{curl}_h \mathbf{m}^k \cdot (\mathbf{h}_a)_h \, dx + \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \nabla \text{div}_h \mathbf{m}^k \cdot (\mathbf{h}_a)_h \, dx$$

$$= - \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \text{curl}_h \mathbf{m}^k \cdot \text{curl}(\mathbf{h}_a)_h \, dx - \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \text{div}_h \mathbf{m}^k \text{div}(\mathbf{h}_a)_h \, dx \quad (4.77)$$

in the three-dimensional case and the analogous result

$$- \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \text{curl} \text{curl}_h \mathbf{m}^k \cdot (\mathbf{h}_a)_h \, dx + \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \nabla \text{div}_h \mathbf{m}^k \cdot (\mathbf{h}_a)_h \, dx$$

$$= - \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \text{curl}_h \mathbf{m}^k \text{curl}(\mathbf{h}_a)_h \, dx - \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \text{div}_h \mathbf{m}^k \text{div}(\mathbf{h}_a)_h \, dx \quad (4.78)$$
in two spatial dimensions, see formula (A.30). The boundary integrals vanish due to the imposed boundary conditions on \( \text{curl}_h \mathbf{m}^k \) and \( \text{div}_h \mathbf{m}^k \), respectively, see (4.31) or (4.34) in case of \( d = 2 \) and (4.32). In the other case, use (4.18b) and integration by parts to reformulate in case of \( d = 3 \)

\[
- \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \text{curl} \text{curl}_h \mathbf{m}^k \cdot (\mathbf{h}_a)_h^k \, dx + \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \text{div}_h \mathbf{m}^k \cdot (\mathbf{h}_a)_h^k \, dx
\]

\[
- \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \text{curl} \text{curl}_h \mathbf{m}^k \cdot (\mathbf{h}_a)_h^k \, dx + \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \text{div}_h \mathbf{m}^k \cdot (\mathbf{h}_a)_h^k \, dx
\]

\[
= - \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \text{curl} \text{curl}_h \mathbf{m}^k \cdot (\mathbf{h}_a)_h^k \, dx + \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \text{div}_h \mathbf{m}^k \cdot (\mathbf{h}_a)_h^k \, dx
\]

\[
= - \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \text{curl} \text{curl}_h \mathbf{m}^k \cdot (\mathbf{h}_a)_h^k \, dx + \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \text{div}_h \mathbf{m}^k \cdot (\mathbf{h}_a)_h^k \, dx
\]

The boundary integrals vanish due to the same reasons as before (imposed boundary conditions of \( \text{div}_h \mathbf{m}^k \) and \( \text{curl}_h \mathbf{m}^k \)). In the two-dimensional case, the computations are completely analogous, i.e. the curl-related term is treated as follows,

\[
- \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \text{curl} \text{curl}_h \mathbf{m}^k \cdot (\mathbf{h}_a)_h^k \, dx = - \frac{\tau \sigma \mu_0 \beta}{2} \int_{\Omega} \text{curl} \text{curl}_h \mathbf{m}^k \cdot (\mathbf{h}_a)_h^k \, dx
\]

The terms from (4.77) or (4.78) in case of \( d = 2 \) will be treated with Young’s inequality and absorption later on. The proof will be continued in the case of the choice (4.18a) which obviously is more difficult to handle (comparing (4.79) with (4.77)) than the alternative (4.18b).

Using that the trilinear form \((\mathbf{a}, \mathbf{b}, \mathbf{c}) \mapsto (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\) is alternating (see (A.26) for the two-dimensional case), i.e.

\[
(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = -(\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a},
\]

combined with \( \mathbf{m}^k \times \mathbf{m}^k = \mathbf{0} \), the third term on the right hand side of (4.65) is simplified. Collecting the information which has been achieved since (4.65), one obtains the identity

\[
\frac{\alpha_3 \mu_0}{2} \int_{\Omega} |\mathbf{m}^k|^2 \, dx - \frac{\alpha_3 \mu_0}{2} \int_{\Omega} |\mathbf{m}^{k-1}|^2 \, dx + \frac{\alpha_3 \mu_0}{2} \int_{\Omega} |\mathbf{m}^k - \mathbf{m}^{k-1}|^2 \, dx
\]

\[
+ \frac{\mu_0 \alpha_1}{2} \int_{\Omega} |\nabla R^k|^2 \, dx - \frac{\mu_0 \alpha_1}{2} \int_{\Omega} |\nabla R^{k-1}|^2 \, dx + \frac{\mu_0 \alpha_1}{2} \int_{\Omega} |\nabla R^k - \nabla R^{k-1}|^2 \, dx
\]

\[
+ \frac{\tau \mu_0 \alpha_1}{\tau_{\text{rel}}} \int_{\Omega} |\nabla R^k|^2 \, dx + \frac{\tau \mu_0 \alpha_3}{\tau_{\text{rel}}} \int_{\Omega} |\mathbf{m}^k|^2 \, dx + \frac{\tau \mu_0 \alpha_3}{\tau_{\text{rel}}} \int_{\Omega} \chi^{-1}_{r,h} |\mathbf{h}^k|^2 \, dx
\]

\[
+ \tau \sigma \mu_0 \alpha_3 \int_{\Omega} |\text{div}_h \mathbf{m}^k|^2 \, dx + \tau \sigma \mu_0 \alpha_1 \int_{\Omega} |\text{div}_h \mathbf{h}^k|^2 \, dx + \tau \sigma \mu_0 \alpha_3 \int_{\Omega} |\text{curl}_h \mathbf{m}^k|^2 \, dx
\]

\[
= \tau \mu_0 \alpha_1 \int_{\Omega} (\mathbf{h}_a)_h^k \cdot (\mathbf{h}_a)_h^{k-1} \cdot \nabla R^k \, dx + \frac{\mu_0 \beta}{2} \int_{\Omega} (\mathbf{m}^k - \mathbf{m}^{k-1}) \cdot (\mathbf{h}_a)_h^k \, dx
\]

\[
+ \frac{\beta \tau \mu_0}{2 \tau_{\text{rel}}} \int_{\Omega} \mathbf{m}^k \cdot (\mathbf{h}_a)_h^k \, dx + \frac{\tau \mu_0 \alpha_3}{\tau_{\text{rel}}} \int_{\Omega} \chi^{-1}_{r,h} \mathbf{h}^k \cdot \mathbf{m}^k \, dx
\]
Adding (4.82) to the results (4.58) and (4.64) from before gives
\[
\rho_0 \int_{\Omega} |\mathbf{u}^k|^2 \, dx - \rho_0 \int_{\Omega} |\mathbf{u}^{k-1}|^2 \, dx + \rho_0 \int_{\Omega} |\mathbf{u}^{k} - \mathbf{u}^{k-1}|^2 \, dx
\]
\[+ D \int_{\Omega} g_{s,h}^k \, dx - D \int_{\Omega} g_{s,h}^{k-1} \, dx
\]
\[+ \frac{\alpha_3 \mu_0}{2} \int_{\Omega'} |\mathbf{m}^k|^2 \, dx - \frac{\alpha_3 \mu_0}{2} \int_{\Omega'} |\mathbf{m}^{k-1}|^2 \, dx + \frac{\alpha_3 \mu_0}{2} \int_{\Omega'} |\mathbf{m}^k - \mathbf{m}^{k-1}|^2 \, dx
\]
\[+ \mu_0 \alpha_3 \int_{\Omega} |\nabla \mathbf{h}^k|^2 \, dx + \mu_0 \alpha_3 \int_{\Omega'} |\mathbf{m}^k - \mathbf{m}^{k-1}|^2 \, dx
\]
\[+ \frac{\mu_0 \alpha_3}{2} \int_{\Omega'} \nabla |\mathbf{h}^k|^2 \, dx + \frac{\mu_0 \alpha_3}{2} \int_{\Omega'} \nabla |\mathbf{m}^k - \mathbf{m}^{k-1}|^2 \, dx
\]
\[+ \tau \int_{\Omega'} 2\eta |\mathbf{Dw}^k|^2 \, dx + \tau \int_{\Omega'} \mathcal{I}_{h,1} \left( \frac{f_{h}^{(s),k-1} |\mathbf{v}_{part}^k|^2}{K} \right) \, dx
\]
\[\quad + \tau \int_{\Omega'} \mathcal{I}_{h,2} |\mathbf{m}^k|^2 \, dx + \tau \mu_0 \alpha_3 \int_{\Omega'} |\mathbf{h}^k| |\mathbf{Dw}^k| \, dx
\]
\[\quad + \tau \mu_0 \alpha_3 \int_{\Omega'} |\nabla \mathbf{h}^k|^2 \, dx + \tau \mu_0 \alpha_3 \int_{\Omega'} |\mathbf{m}^k|^2 \, dx
\]
\[\leq \tau \mu_0 \alpha_3 \int_{\Omega'} \left( \frac{(\mathbf{h})^{k-1} - (\mathbf{h})^{k-1}}{\tau} \right) \cdot \nabla \mathbf{R}^k \, dx + \frac{\mu_0 \beta}{2} \int_{\Omega} (\mathbf{m}^k - \mathbf{m}^{k-1}) \cdot (\mathbf{h})^{k-1}_h \, dx
\]
\[+ \frac{\beta \tau \mu_0}{2} \int_{\Omega} \mathbf{m}^k \cdot (\mathbf{h})^{k-1}_h \, dx + \frac{\beta \tau \mu_0}{2} \int_{\Omega} \mathbf{m}^k \cdot \mathbf{m}^{k-1}_h \, dx
\]
\[+ \frac{\beta \tau \mu_0}{2} \int_{\Omega} |\nabla \mathbf{h}^k|^2 \, dx - \frac{\beta \tau \mu_0}{2} \int_{\Omega} \mathbf{m}^k \cdot (\mathbf{h})^{k-1}_h \, dx
\]
\[+ \frac{\beta \tau \mu_0}{2} \int_{\Omega} |\mathbf{Dw}^k| |\mathbf{m}^k| \, dx + \alpha_1 \tau \mu_0 \int_{\Omega'} |\mathbf{h}^k| |\mathbf{Dw}^k| \, dx
\]
\[+ \frac{\beta \tau \mu_0}{2} \int_{\Omega'} |\mathbf{m}^k| |\mathbf{Dw}^k| \, dx.
\]


\[ -\frac{\tau\mu_0\beta}{2} \int_{\Omega} \mathbf{m}^{k-1} \cdot \left( \frac{(\mathbf{h}_h^k)^k - (\mathbf{h}_h^{k-1})^k}{\tau h} \right) \, d\mathbf{x} \leq \frac{\tau\mu_0|\beta|}{4} \int_{\Omega} |\mathbf{m}^{k-1}|^2 \, d\mathbf{x} + \frac{\tau\mu_0|\beta|}{4} \int_{\Omega} \left| \frac{(\mathbf{h}_h^k)^k - (\mathbf{h}_h^{k-1})^k}{\tau h} \right|^2 \, d\mathbf{x} \]

\[ \frac{\beta\tau\mu_0}{2\tau_{rel}} \int_{\Omega} \mathbf{m}^k \cdot (\mathbf{h}_h)^k \, d\mathbf{x} \leq \frac{\tau\mu_0\alpha_3}{4\tau_{rel}} \int_{\Omega} |\mathbf{m}^k|^2 \, d\mathbf{x} + \frac{\tau\mu_0\beta^2}{8\alpha_3\tau_{rel}} \int_{\Omega} |(\mathbf{h}_h)^k|^2 \, d\mathbf{x}, \]

\[ \frac{\tau\mu_0\alpha_1}{\tau_{rel}} \int_{\Omega'} (\mathbf{h}_h)^k \cdot \nabla R^k \, d\mathbf{x} \leq \frac{\tau\mu_0\alpha_1}{4\tau_{rel}} \int_{\Omega'} |\nabla R^k|^2 \, d\mathbf{x} + \frac{\tau\mu_0\alpha_1}{\tau_{rel}} \int_{\Omega'} |(\mathbf{h}_h)^k|^2 \, d\mathbf{x}, \]

\[ \frac{\beta\tau\sigma\mu_0}{2} \int_{\Omega} \text{div}_h \mathbf{m}^k \cdot \text{div}_h (\mathbf{h}_h)^k \, d\mathbf{x} \leq \frac{\tau\sigma\mu_0\alpha_3}{2} \int_{\Omega} |\text{div}_h \mathbf{m}^k|^2 \, d\mathbf{x} + \frac{\tau\sigma\mu_0\beta^2}{8\alpha_3} \int_{\Omega} |\text{div}_h (\mathbf{h}_h)^k|^2 \, d\mathbf{x}, \]

\[ \alpha_1\tau\mu_0 \int_{\Omega'} \text{div}_h \mathbf{h}^k \cdot \text{div}_h (\mathbf{h}_h)^k \, d\mathbf{x} \leq \frac{\alpha_1\tau\sigma\mu_0}{2} \int_{\Omega'} |\text{div}_h \mathbf{h}^k|^2 \, d\mathbf{x} + \frac{\alpha_1\tau\sigma\mu_0}{2} \int_{\Omega'} |\text{div}_h (\mathbf{h}_h)^k|^2 \, d\mathbf{x}, \]

\[ \frac{\beta\tau\sigma\mu_0}{2} \int_{\Omega} \text{curl}_h \mathbf{m}^k \cdot \text{curl}_h (\mathbf{h}_h)^k \, d\mathbf{x} \leq \frac{\tau\sigma\mu_0\alpha_3}{2M_0} \int_{\Omega} |\text{curl}_h \mathbf{m}^k|^2 \, d\mathbf{x} + \frac{M_0\tau\sigma\mu_0\beta^2}{8\alpha_3} \int_{\Omega} |\text{curl}_h (\mathbf{h}_h)^k|^2 \, d\mathbf{x} \]

\[ \leq M_0 \int_{\Omega} \mathcal{I}_{h,1}(|\text{curl}_h \mathbf{m}^k|^2) \, d\mathbf{x} \]

In the last estimate, norm-equivalence between $\| \cdot \|_{L^2(\Omega)}$ and $\| \cdot \|_{h,\Omega}$ has been used and $M_0 > 1$ denotes the constant – independent of the mesh size – in the respective equivalence estimate, see e.g. **Lemma 4.2**.

The remaining terms are

\[ \frac{\tau\mu_0\alpha_3}{\tau_{rel}} \int_{\Omega} \lambda_{r,h}^{k-1} \mathbf{h}^k \cdot \mathbf{m}^k \, d\mathbf{x} \leq \frac{\tau\mu_0\alpha_3}{4\tau_{rel}} \int_{\Omega} |\mathbf{m}^k|^2 \, d\mathbf{x} + \frac{\tau\mu_0\alpha_3}{\tau_{rel}} \int_{\Omega} |\lambda_{r,h}^{k-1} \mathbf{h}^k|^2 \, d\mathbf{x} \]

\[ \text{and} \]

\[ -\frac{\tau\mu_0\beta}{2\tau_{rel}} \int_{\Omega} \lambda_{r,h}^{k-1} \mathbf{h}^k \cdot (\mathbf{h}_h)^k \, d\mathbf{x} \leq \frac{\tau\mu_0|\beta|}{4\tau_{rel}} \int_{\Omega} |(\mathbf{h}_h)^k|^2 \, d\mathbf{x} + \frac{\tau\mu_0|\beta|}{4\tau_{rel}} \int_{\Omega} |\lambda_{r,h}^{k-1} \mathbf{h}^k|^2 \, d\mathbf{x}. \]

Note that the $|\mathbf{m}^k|^2$-term can be absorbed now. The norms $\| \cdot \|_{L^2(\Omega)}$ and $\| \cdot \|_{h,\Omega}$ are equivalent independently from the mesh size $h > 0$, see **Lemma 4.2**. Therefore,

\[ \int_{\Omega} |\lambda_{r,h}^{k-1} \mathbf{h}^k|^2 \, d\mathbf{x} \leq M_0 \int_{\Omega} \hat{\mathcal{I}}_{h,1} \left( |\lambda_{r,h}^{k-1} (\mathbf{h}^k)|^2 \right) \, d\mathbf{x} \leq M_0 \int_{\Omega} \hat{\mathcal{I}}_{h,1} \left( |\mathbf{h}^k|^2 \right) \, d\mathbf{x} \]

\[ \leq M_0^2 \int_{\Omega} |\mathbf{h}^k|^2 \, d\mathbf{x}. \]

\[ \leq M_0 \int_{\Omega} \hat{\mathcal{I}}_{h,1} \left( |\lambda_{r,h}^{k-1} (\mathbf{h}^k)|^2 \right) \, d\mathbf{x} \leq M_0 \int_{\Omega} \hat{\mathcal{I}}_{h,1} \left( |\mathbf{h}^k|^2 \right) \, d\mathbf{x} \]

\[ \leq M_0^2 \int_{\Omega} |\mathbf{h}^k|^2 \, d\mathbf{x}. \]

(4.84)
Next, it will be used that $\|c^{k-1}\|_{L^1(\Omega)}$ is bounded by the entropy $\|s_{k,h}^{k-1}\|_{L^1(\Omega)}$, for instance
\[
\int_\Omega g_{s,h}^{k-1} \, dx \geq K_3 \int_\Omega |c^{k-1}| \, dx - K_4 |\Omega| \tag{4.85}
\]
uniformly in $h, s, k - 1$ for some constants $K_3, K_4 > 0$. The proof is as follows.

It is known that $g_s(c^{k-1}) + 1 \geq 0$, hence, exploiting norm equivalence of $\| \cdot \|_{L^2(\Omega)}$ and $\| \cdot \|_{h,\Omega}$ again, see Lemma 4.2,
\[
\int_\Omega g_{s,h}^{k-1} \, dx = \int_\Omega \mathcal{T}_{h,1}((\sqrt{g_s(c^{k-1})} + 1)^2 - 1) \, dx \geq \frac{1}{M_0} \int_\Omega |\sqrt{g_s(c^{k-1})} + 1|^2 \, dx - |\Omega|,
\]
where $M_0 > 1$. Then,
\[
\int_\Omega g_{s,h}^{k-1} \, dx \geq \frac{1}{M_0} \left( \int_\Omega (g_s(c^{k-1}) + 1) \, dx - |\Omega| \right) = \frac{1}{M_0} \int_\Omega g_s(c^{k-1}) \, dx.
\]
Apply Lemma 3.22 to obtain the result. This also yields the estimate
\[
\int_\Omega |K_1 + K_2| \sqrt{|c^{k-1}|^2} \, dx \leq K_5 \int_\Omega g_{s,h}^{k-1} \, dx + K_6 |\Omega|
\]
for some constants $K_5, K_6 > 0$ which are independent of the discretization parameters $\tau, h > 0$.

Putting everything together (starting with (4.83)) one obtains
\[
\frac{\rho_0}{2} \int_\Omega |u^k|^2 \, dx - \frac{\rho_0}{2} \int_\Omega |u^{k-1}|^2 \, dx + \frac{\rho_0}{2} \int_\Omega |u^k - u^{k-1}|^2 \, dx
\]
\[
+ D \int_\Omega g_{s,h}^k \, dx - D \int_\Omega g_{s,h}^{k-1} \, dx
\]
\[
+ \frac{\mu_0 \beta}{2} \int_\Omega (m^k \cdot (h_a)^k_h - m^{k-1} \cdot (h_a)^{k-1}_h) \, dx - \frac{\tau \mu_0 |\beta|}{4} \int_\Omega |m^{k-1}|^2 \, dx
\]
\[
+ \frac{\alpha_3 \mu_0}{2} \int_\Omega |m^k|^2 \, dx - \frac{\alpha_3 \mu_0}{2} \int_\Omega |m^{k-1}|^2 \, dx + \frac{\alpha_3 \mu_0}{2} \int_\Omega |m^k - m^{k-1}|^2 \, dx
\]
\[
+ \frac{\mu_0 \alpha_1}{2} \int_\Omega |\nabla R|^2 \, dx - \frac{\mu_0 \alpha_1}{2} \int_\Omega |\nabla R^{k-1}|^2 \, dx + \frac{\mu_0 \alpha_1}{2} \int_\Omega |\nabla R - \nabla R^{k-1}|^2 \, dx
\]
\[
+ \tau \int_\Omega 2\eta |\text{Diag} u^k|^2 \, dx + \tau \int_\Omega \mathcal{T}_{h,1} (f_{h,1}^{(s),k-1} |V_{\text{part}}|^2 \frac{K}{K}) \, dx
\]
\[
+ \frac{\tau \sigma \mu_0 \alpha_3}{2} \int_\Omega |\text{Diag} m^k|^2 \, dx + \frac{\tau \sigma \mu_0 \alpha_1}{2} \int_\Omega |\text{Diag} h^k|^2 \, dx
\]
\[
+ \frac{\tau \sigma \mu_0 \alpha_3}{2} \int_\Omega \mathcal{T}_{h,1}(|\text{curl} h^k|^2) \, dx + \frac{\tau \mu_0 \alpha_3}{2 \tau_{rel}} \int_\Omega |m^k|^2 \, dx
\]
\[
+ \frac{\tau \mu_0 \alpha_1}{2 \tau_{rel}} \int_\Omega |\nabla R|^2 \, dx + \frac{\tau \mu_0 \alpha_1}{\tau_{rel}} \int_\Omega \chi_{h,h}^{k-1,k} |h^k|^2 \, dx
\]
\[
\leq \tau \mu_0 \alpha_1 \tau_{rel} \int_\Omega \left( \frac{|(h_a)^k_h - (h_a)^{k-1}_h|^2}{\tau} \right) \, dx + \frac{\tau \mu_0 |\beta|}{4} \int_\Omega \left( \frac{|(h_a)^k_h - (h_a)^{k-1}_h|^2}{\tau} \right) \, dx
\]
\[
+ \frac{\tau \mu_0 \alpha_1}{\tau_{rel}} \int_\Omega \left( |(h_a)^k_h|^2 + \frac{\tau \mu_0 |\beta|}{\tau_{rel}} \right) \int_\Omega \left( \frac{|(h_a)^k_h - (h_a)^{k-1}_h|^2}{\tau} \right) \, dx
\]
\[
+ \frac{\tau \mu_0 \alpha_1}{\tau_{rel}} \int_\Omega \left( |(h_a)^k_h|^2 + \frac{\tau \mu_0 |\beta|}{\tau_{rel}} \right) \int_\Omega \left( \frac{|(h_a)^k_h - (h_a)^{k-1}_h|^2}{\tau} \right) \, dx
\]
\[
+ \frac{\tau \mu_0 \alpha_1}{4 \alpha_3 \tau_{rel}} \int_\Omega \left( |(h_a)^k_h|^2 + \frac{\tau \mu_0 |\beta|}{\tau_{rel}} \right) \int_\Omega \left( \frac{|(h_a)^k_h - (h_a)^{k-1}_h|^2}{\tau} \right) \, dx
\]
\[
+ \frac{\tau \mu_0 \alpha_1}{\tau_{rel}} \int_\Omega \left( |(h_a)^k_h|^2 + \frac{\tau \mu_0 |\beta|}{\tau_{rel}} \right) \int_\Omega \left( \frac{|(h_a)^k_h - (h_a)^{k-1}_h|^2}{\tau} \right) \, dx
\]
\[
+ \frac{\tau \mu_0 \alpha_1}{8 \alpha_3} \int_\Omega \left( |\text{Diag} (h_a)^k_h|^2 \right) \, dx + \alpha \tau \mu_0 \frac{2}{\tau_{rel}} \int_\Omega \left( |\text{Diag} (h_a)^k_h|^2 \right) \, dx
\]
\[
+ \frac{M_0 \tau \sigma \mu_0 \alpha_1}{8 \alpha_3} \int_\Omega |\text{curl} (h_a)^k_h|^2 \, dx.
\tag{4.86}
In order to conclude, a discrete version of Gronwall’s inequality has to be used, see [72, p. 120]. For its application, consider the estimate

\[
\frac{\mu_0\beta}{2} \int_{\Omega} m^k \cdot (h_n)_h^k \, dx - \frac{\mu_0\beta}{2} \int_{\Omega} m^{k-1} \cdot (h_n)_h^{k-1} \, dx \\
+ \frac{\alpha_3 \mu_0}{2} \int_{\Omega} |m^k|^2 \, dx - \frac{\alpha_3 \mu_0}{2} \int_{\Omega} |m^{k-1}|^2 \, dx + D \int_{\Omega} g_{s,h}^k \, dx - D \int_{\Omega} g_{s,h}^{k-1} \, dx \\
+ \frac{\rho_0}{2} \int_{\Omega} |u^k|^2 \, dx - \frac{\rho_0}{2} \int_{\Omega} |u^{k-1}|^2 \, dx + \frac{\mu_0\alpha_1}{2} \int_{\Omega'} |\nabla R_k|^2 \, dx - \frac{\mu_0\alpha_1}{2} \int_{\Omega'} |\nabla R^{k-1}|^2 \, dx \\
\leq \tau K_7 D \int_{\Omega} g_{s,h}^{k-1} \, dx + \frac{\tau \mu_0\beta}{4} \int_{\Omega} |m^{k-1}|^2 \, dx + \tau (K_8 |\Omega| + C^{k,k-1})
\]

(4.87)

for some \(K_7, K_8 > 0\) (independent from the discretization parameters \(\tau, h > 0\)) which follows easily from the one above, where

\[
C^{k,k-1} = \mu_0\alpha_1 \tau_{rel} \int_{\Omega} \left| \frac{(h_n)_h^k - (h_n)_h^{k-1}}{\tau} \right|^2 \, dx + \frac{\mu_0\beta}{4} \int_{\Omega} \left| \frac{(h_n)_h^k - (h_n)_h^{k-1}}{\tau} \right|^2 \, dx \\
+ \frac{\mu_0\beta^2}{4\alpha_3 \tau_{rel}} \int_{\Omega} |(h_n)_h^k|^2 \, dx + \frac{\mu_0\alpha_1}{\tau_{rel}} \int_{\Omega'} |(h_n)_h^k|^2 \, dx + \frac{\mu_0\beta}{4\tau_{rel}} \int_{\Omega'} |(h_n)_h^k|^2 \, dx \\
+ \frac{\sigma_\mu \mu_0^2}{8\alpha_3} \int_{\Omega} |\text{div}(h_n)_h^k|^2 \, dx + \frac{\alpha_1 \sigma_\mu \mu_0}{2} \int_{\Omega'} |\text{div}(h_n)_h^k|^2 \, dx + \frac{M_0 \sigma_\mu \mu_0^2}{8\alpha_3} \int_{\Omega} |\text{curl}(h_n)_h^k|^2 \, dx
\]

(4.88)

for some constants \(K_9, K_{10} > 0\) which are independent of the discretization parameters \(\tau, h > 0\).

In total, after summing over all time steps \(k \in \{1, ..., N\}, 1 \leq N \leq n_T\), one obtains from (4.87) the estimate

\[
\frac{\mu_0\beta}{2} \int_{\Omega} m^N \cdot (h_n)_h^N \, dx + \frac{\alpha_3 \mu_0}{2} \int_{\Omega} |m^N|^2 \, dx + D \int_{\Omega} g_{s,h}^N \, dx \\
+ \frac{\rho_0}{2} \int_{\Omega} |u^N|^2 \, dx + \frac{\mu_0\alpha_1}{2} \int_{\Omega'} |\nabla R^N|^2 \, dx \\
\leq \int_{\Omega} m^0 \cdot (h_n)_h^0 \, dx + \frac{\alpha_3 \mu_0}{2} \int_{\Omega} |m^0|^2 \, dx + D \int_{\Omega} g_{s,h}^0 \, dx \\
+ \frac{\rho_0}{2} \int_{\Omega} |u^0|^2 \, dx + \frac{\mu_0\alpha_1}{2} \int_{\Omega'} |\nabla R^0|^2 \, dx \\
+ \tau K_{11} \sum_{l=1}^{N} \left( D \int_{\Omega} g_{s,h}^{l-1} \, dx + \frac{\alpha_3 \mu_0}{4} \int_{\Omega} |m^{l-1}|^2 \, dx \right) + \tau \sum_{l=1}^{N} K_8 |\Omega| + \tau \sum_{l=1}^{N} C^{l,l-1}_{h_{a-1}}
\]

(4.89)

for some \(K_{11} > 0\) which was chosen in such a way that the coefficient in front of the \(|m^{l-1}|^2\)-term in parentheses could be adapted. It easily follows that \(\tau \sum_{l=1}^{N} K_8 |\Omega| \leq \tau n_T |\Omega| \leq TK_8 |\Omega|\) is bounded.

It turns out that \(\tau \sum_{l=1}^{N} C^{l,l-1}_{h_{a-1}}\) is also bounded as can be seen by the following considerations.

Notice that due to non-negativity of the individual constants \(C^{l,l-1}_{h_{a-1}}\) one can without loss of generality assume \(N = n_T\). First, examine the case of choice (4.18b) for the discretization of \(h_n\). The terms related to \(\text{div}(h_n)_h^0\) or \(\text{curl}(h_n)_h^0\) on the right-hand side do not exist in this case (cf. (4.79) and (4.76)). Hence, the first term on the right-hand side of (4.88) is bounded by \(\|h_n(t, \cdot)\|_{L^2(\Omega)^d}^2\) because of the \(L^2\)-stability of the \(L^2\)-projection. Summing up over all time steps and multiplying by \(\tau\) yields an approximation of \(\|h_n\|_{L^2(I; L^2(\Omega)^d)}^2\). The latter is immediately
We will now continue to bound the right-hand side of (4.89). The initial data \( C \) with estimates of the type (4.90). Consequently, the boundedness follows, i.e. there exists a given in (N3).

For the right-hand side of (4.88), use the fundamental theorem of calculus, i.e. after exploiting the \( L^2 \)-stability of the projection one is left with

\[
\tau \int_{\Omega} \frac{\| \mathbf{h}_a(t^{k}, \cdot) - \mathbf{h}_a(t^{k-1}, \cdot) \|^2}{\tau} \, dx = \frac{1}{\tau} \int_{\Omega} \left[ \int_{\nu-1}^{t^k} \partial_t \mathbf{h}_a \, dt \right]^2 \, dx \leq \int_{\nu-1}^{t^k} \left| \partial_t \mathbf{h}_a \right|^2 \, dx \, dt, \tag{4.90}
\]

where Hölder’s inequality and Fubini’s theorem have been used in the last step. Summing up over all time steps gives the \( L^2(\Omega; L^2(\Omega)^d) \)-norm of \( \partial_t \mathbf{h}_a \) which is finite due to the regularity of \( \partial_t \mathbf{h}_a \) given in [N3].

In the case of discretization (4.18a), the \( H^2 \)-regularity in space will be used to exploit standard interpolation error estimates. The error (at time \( t^k \)) in the \( H^1 \)-norm (and consequently the \( L^2 \)-norms of \( \text{div} \) and \( \text{curl} \)) is bounded by the \( H^2 \)-norm of \( \mathbf{h}_a(t^{k}, \cdot) \), see [16, Theorem 3.1.6]. Hence, the first term on the right-hand side of (4.88) is uniformly bounded by

\[
\| (\mathbf{h}_a)_{\tau}^{k} \|^2_{H(\text{div,curl})(\Omega)} \leq 2 \| (\mathbf{h}_a)_{\tau}^{k} - \mathbf{h}_a(t^{k}, \cdot) \|_{H^1(\Omega)^d}^2 + 2 \| \mathbf{h}_a(t^{k}, \cdot) \|_{L^2(\Omega)^d}^2 \leq C \| \mathbf{h}_a(t^{k}, \cdot) \|_{L^2(\Omega)^d}^2.
\]

Summing over all time steps and multiplying by \( \tau \) again gives a uniformly bounded approximation of the time-integral as in the previous case. For the difference quotient in time (second term on the right-hand side of (4.88)), use

\[
\tau \sum_{l=1}^{n} C_{h_a}^{l-1} \leq C_{h_a},
\]

where \( C_{h_a} \) only depends on \( \mathbf{h}_a \) but not on the discretization parameters \( \tau, h > 0 \).

We will now continue to bound the right-hand side of (4.89). The initial data \( \mathbf{m}^0 \) is \( L^2 \)-integrable. For the \( (\mathbf{h}_a)_{\tau}^{l} \)-related term use the same techniques as in the proof of boundedness in regard to \( C_{h_a}^{l-1} \), which yields boundedness in \( L^2(\Omega)^d \). Therefore, by Young’s inequality the first term on the right-hand side of (4.89) is bounded. The term \( g_{\tau}^{0} \) has at most quadratic growth in \( \mathbf{c}^0 \) and therefore is in \( L^1(\Omega) \) as \( c^0 \) is in \( L^2(\Omega) \). Hence, the second term on the right-hand side of (4.89) is bounded. The first term on the left-hand side of (4.89) will be put to the right-hand side and estimated via Young’s inequality, i.e.

\[
-\frac{\mu_0 \beta}{2} \int_{\Omega} \mathbf{m}^N \cdot (\mathbf{h}_a)_{\tau} \, dx \leq \frac{\mu_0 \alpha_3}{4} \int_{\Omega} |\mathbf{m}^N|^2 \, dx + \frac{\mu_0 \beta^2}{4 \alpha_3} \int_{\Omega} |(\mathbf{h}_a)_{\tau}^N|^2 \, dx. \tag{4.91}
\]

The term \( \mathbf{h}_a^N \) is bounded in \( L^2(\Omega)^d \) uniformly in \( 1 \leq N \leq n_T \) due to similar arguments as before. For instance, the \( L^2 \)-stability of the \( L^2 \)-projection in case of discretization (4.18a) or the estimates of the interpolation error in case of (4.18b) yield boundedness by \( H(\text{div,curl})(\Omega) \)-norms of \( \mathbf{h}_a(t^{N}, \cdot) \) uniformly in \( 1 \leq N \leq n_T \) and \( H_2(\Omega)^d \), respectively. Then, the result follows from the continuous embedding \( H^1(\Omega; L^2(\Omega)^d) \hookrightarrow C([0, T]; L^2(\Omega)^d) \) or \( H^1(\Omega; H^2(\Omega)^d) \hookrightarrow C([0, T]; H^2(\Omega)^d) \), respectively.
By absorption and skipping non-negative terms on the left-hand side of (4.89), one ends up with
\[
\frac{\alpha_3 \mu_0}{4} \int_{\Omega} |m|^2 \, dx + D \int_{\Omega} g_{s,h}^N \, dx \\
\quad \leq \tau K_{11} \sum_{l=1}^{N} \left( \int_{\Omega} g_{s,h}^{l-1} \, dx + \frac{\alpha_3 \mu_0}{4} \int_{\Omega} |m|^{l-1}|^2 \, dx \right) + C_{h_a,|\Omega|,0,T},
\]  
(4.92)
where $C_{h_a,|\Omega|,0,T} > 0$ only depends on $h_a, |\Omega|$, initial data and ending time $T$. Using a discrete version of Gronwall’s lemma [72, p. 120], the estimate
\[
\frac{\alpha_3 \mu_0}{4} \int_{\Omega} |m|^2 \, dx + D \int_{\Omega} g_{s,h}^N \, dx \leq C_{h_a,|\Omega|,0,T} e^{K_{11}T} \quad \forall N \in \{1, \ldots, n_T\}
\]  
(4.93)
follows. Here, $K_{11}$ depends on $r > 0$, see (N2 4.2), but not the discretization parameters $h, \tau > 0$. Notice that the version of Gronwall’s lemma which we reference above requires the left-hand side to non-negative but $g_{s,h}^N \geq -1$ is only bounded from below. However, it is straightforward to extend the result to this case by changing the constant on the right-hand side by another contribution only depending on $|\Omega|$.

Reinserting (4.93) into (4.86) and summing up over all time steps $k \in \{1, \ldots, N\}, 1 \leq N \leq n_T$, yields
\[
\frac{\rho_0}{2} \int_{\Omega} |u|^2 \, dx + \frac{\rho_0}{2} \sum_{i=1}^{N} \int_{\Omega} |u^i - u^{i-1}|^2 \, dx + D \int_{\Omega} g_{s,h}^N \, dx \\
+ \frac{\mu_0 \beta}{2} \int_{\Omega} (m^N \cdot (h_a)_h^N - m^0 \cdot (h_a)_h^0) \, dx \\
+ \frac{\alpha_3 \mu_0}{2} \int_{\Omega} |m|^2 \, dx + \frac{\alpha_3 \mu_0}{2} \sum_{l=1}^{N} \int_{\Omega} |m^l - m^{l-1}|^2 \, dx \\
+ \frac{\mu_0 \alpha_3}{2} \int_{\Omega'} |\nabla R|^2 \, dx + \frac{\mu_0 \alpha_1}{2} \sum_{l=1}^{N} \int_{\Omega'} |\nabla R^l - \nabla R^{l-1}|^2 \, dx \\
+ \tau \sum_{l=1}^{N} \int_{\Omega} 2\eta |Du^l|^2 \, dx + \tau \sum_{l=1}^{N} \int_{\Omega} I_{h,1} \left( f_h^{(e),l-1} \frac{|v_{\text{part}}|^2}{K} \right) \, dx \\
+ \frac{\tau \sigma \mu_0 \alpha_3}{2} \sum_{l=1}^{N} \int_{\Omega} |\text{div}_h m^l|^2 \, dx + \frac{\tau \sigma \mu_0 \alpha_1}{2} \sum_{l=1}^{N} \int_{\Omega'} |\text{div}_h h^l|^2 \, dx \\
+ \frac{\tau \mu_0 \alpha_3}{2 \tau_{rel}} \sum_{l=1}^{N} \int_{\Omega} \left| I_{h,1} (|\text{curl}_h m^l|)^2 \right| \, dx + \frac{\tau \mu_0 \alpha_3}{2 \tau_{rel}} \sum_{l=1}^{N} \int_{\Omega} |m^l|^2 \, dx \\
+ \frac{\tau \mu_0 \alpha_1}{2 \tau_{rel}} \sum_{l=1}^{N} \int_{\Omega} |\nabla R^l|^2 \, dx + \frac{\tau \mu_0 \alpha_1}{2 \tau_{rel}} \sum_{l=1}^{N} \int_{\Omega} \chi_{\tau,h}^{l-1} |h^l|^2 \, dx \leq \frac{\rho_0}{2} \int_{\Omega} |u^0|^2 \, dx + D \int_{\Omega} g_{s,h}^0 \, dx + \frac{\alpha_3 \mu_0}{2} \int_{\Omega} |m^0|^2 \, dx + \frac{\mu_0 \alpha_1}{2} \int_{\Omega} |\nabla R^0|^2 \, dx \\
+ T \left( \frac{\mu_0 \beta}{\tau_{rel}} + \frac{T \mu_0 \beta}{4 \tau_{rel}} \right) M_0^2 K_0 |\Omega| + \left( \frac{\mu_0 \beta}{\tau_{rel}} + \frac{\mu_0 \beta}{4 \tau_{rel}} \right) \frac{M_0^2 K_0}{D} C_{h_a,|\Omega|,0,T} e^{K_{11}T} + C_{h_a} \\
+ T \frac{\alpha_3}{\beta} C_{h_a,|\Omega|,0,T} e^{K_{11}T}.
\]
Here, the last term on the right-hand side comes from the term $-\frac{\tau \mu_0 \beta}{2 \tau_{rel}} \int_{\Omega} |m^{k-1}|^2 \, dx$ on the left-hand side of (4.86) in combination with (4.93), which obviously vanishes if $\beta = 0$. Hence, while above estimate is only valid for $\beta \neq 0$, a simpler estimate is valid for $\beta = 0$. 
The fourth term on the left-hand side above will be dealt with Young’s inequality. The terms related to initial time are obviously bounded. The $|\mathbf{m}|^{2}$ term (see for instance (4.91)) can be absorbed and $(h_{0})_{N}^{2}$ is uniformly bounded (for $1 \leq N \leq n_{T}$) in $L^{2}(\Omega)^{d}$ due to $H^{1}(I; H^{2}(\Omega)^{d}) \hookrightarrow C([0, T]; H^{2}(\Omega)^{d})$. This concludes the proof.

**Theorem 4.6.** Let $(u^{k}, p^{k}, c^{k}, R^{k})_{k=0,\ldots,n_{T}}$ be a discrete solution to the scheme (4.54) and let

$$f_{h}^{(s),k-1} := I_{h,1} \left( \frac{\max(s, c^{k-1})^{2-m}}{\tau(1-c^{k-1})} \right)$$

(4.94)

for all $k = 1, \ldots, n_{T}$. Then

$$\frac{1}{2} \int_{\Omega} \rho_{h}^{k}|u^{k}|^{2} \, dx + \frac{1}{2} \sum_{l=1}^{k} \int_{\Omega} \rho_{h}^{l-1}|u^{l} - u^{l-1}|^{2} \, dx + 2\tau \eta \sum_{l=1}^{k} \int_{\Omega} |D\mathbf{u}|^{2} \, dx$$

$$+ D \int_{\Omega} I_{h,1}(g_{s}(c^{k}) + \Theta[g_{s}(1 - c^{k})]) \, dx + \frac{\tau \mu_{0}}{2} \sum_{l=1}^{k} \int_{\Omega} (1 + \mathbf{I}_{h} \chi_{s,h}^{k})|\mathbf{h}^{k}|^{2} \, dx + \frac{\mu_{0}}{2} \sum_{l=1}^{k} \int_{\Omega} (1 + \mathbf{I}_{h} \chi_{s,h}^{l-1})|\mathbf{h}^{l} - \mathbf{h}^{l-1}|^{2} \, dx$$

$$+ \tau \mu_{0} \sum_{l=1}^{k} \int_{\Omega} \chi_{s,h}^{k} |\mathbf{h}^{l}|^{2} \, dx$$

$$\leq \frac{1}{2} \int_{\Omega} \rho_{h}^{k}|u^{k}|^{2} \, dx + D \int_{\Omega} I_{h,1}(g_{s}(c^{k}) + \Theta[g_{s}(1 - c^{k})]) \, dx + \frac{\mu_{0}}{2} \int_{\Omega} (1 + \mathbf{I}_{h} \chi_{s,h}^{0})|\mathbf{h}^{0}|^{2} \, dx$$

$$+ \frac{\mu_{0}}{2} \sum_{l=1}^{k} \int_{\Omega} |(\mathbf{h}_{a})_{h} - (\mathbf{h}_{a})^{l-1}_{h}|^{2} \, dx + \tau \mu_{0} \sum_{l=1}^{k} \int_{\Omega} |(\mathbf{h}_{a})^{l-1}_{h}|^{2} \, dx. \tag{4.95}$$

**Proof:** Choose $\mathbf{v} := \tau \mathbf{u}^{k}$ in (4.54a). From the first two terms on the left-hand side of (4.54a) one gets

$$\frac{1}{2} \int_{\Omega} \rho_{h}^{k}|u^{k}|^{2} \, dx + \frac{1}{2} \int_{\Omega} \rho_{h}^{k-1}|u^{k-1}|^{2} \, dx - \frac{1}{2} \int_{\Omega} \rho_{h}^{k}|u^{k-1} \cdot u^{k}| \, dx - \frac{1}{2} \int_{\Omega} \rho_{h}^{k-1}|u^{k-1} \cdot u^{k-1}| \, dx$$

$$+ \frac{1}{2} \int_{\Omega} \rho_{h}^{k} u^{k-1} \cdot u^{k} \, dx - \frac{1}{2} \int_{\Omega} \rho_{h}^{k-1} u^{k-1} \cdot u^{k} \, dx$$

$$= \frac{1}{2} \int_{\Omega} \rho_{h}^{k}|u^{k}|^{2} \, dx + \frac{1}{2} \int_{\Omega} \rho_{h}^{k-1}|u^{k-1}|^{2} \, dx - \int_{\Omega} \rho_{h}^{k-1} u^{k-1} \cdot u^{k} \, dx$$

$$= \frac{1}{2} \int_{\Omega} \rho_{h}^{k}|u^{k}|^{2} \, dx + \frac{1}{2} \int_{\Omega} \rho_{h}^{k-1}|u^{k} - u^{k-1}|^{2} \, dx - \frac{1}{2} \int_{\Omega} |\rho_{h}^{k-1}| u^{k-1}|^{2} \, dx$$

where

$$u^{k-1} \cdot u^{k} = \frac{1}{2}(|u^{k} - u^{k-1}|^{2} - |u^{k-1}|^{2} - |u^{k-1}|^{2})$$

was used. This leads to

$$\frac{1}{2} \int_{\Omega} \rho_{h}^{k}|u^{k}|^{2} \, dx + \frac{1}{2} \int_{\Omega} \rho_{h}^{k-1}|u^{k} - u^{k-1}|^{2} \, dx - \frac{1}{2} \int_{\Omega} \rho_{h}^{k-1}|u^{k-1}|^{2} \, dx$$

$$+ 2\tau \eta \int_{\Omega} |D\mathbf{u}|^{2} \, dx - \tau \int_{\Omega} p_{h}^{k} \, div \mathbf{u}^{k} \, dx$$

$$\tag{4.94}$$

$$= -\tau D \int_{\Omega} c^{k-1} \nabla I_{h,1}(g_{s}(c^{k}) - \Theta[g_{s}(1 - c^{k})]) \cdot \mathbf{u}^{k} \, dx$$

$$+ \frac{\tau \mu_{0}}{2} \int_{\Omega} c^{k-1} \mathbf{u}^{k} \cdot \nabla I_{h,1} \left( \frac{\delta \chi_{s,h}^{k}}{\delta c}(c^{k})^{2} \mathbf{h}_{1,h}^{2} \right) \, dx. \tag{4.96}$$
In (4.54e) we choose \( \psi := \tau D \mathcal{I}_{h,1}(g_s'(c^k) - \Theta [g_s'(1 - c^k)]) - \frac{\mu_0}{2} \mathcal{I}_{h,1} \left( \frac{\delta \chi_{s,h}^{k-1}}{\delta c} (c^k) (|h^k|^2)_{1,h} \right) \), see (4.39) for the definition of the switch \( \Theta \). First, observe that convexity of \( g_s \) (and \( g_s(1 - c) \)) combined with [Lemma 4.1] gives

\[
D \int \mathcal{I}_{h,1}(g_s(c^k) + \Theta [g_s(1 - c^k)]) \, dx - D \int \mathcal{I}_{h,1}(g_s(c^{k-1}) + \Theta [g_s(1 - c^{k-1})]) \, dx \\
\leq D \int \mathcal{I}_{h,1}((c^k - c^{k-1})(g_s'(c^k) + \frac{\delta}{\delta c} \Theta [g_s(1 - c^k)])) \, dx \\
= D \int \mathcal{I}_{h,1}((c^k - c^{k-1}) \mathcal{I}_{h,1}(g_s'(c^k) - \Theta [g_s'(1 - c^k)])) \, dx,
\]

where in the case of 'model W' the term \( \Theta [g_s'(1 - c^k)] = 1 \) in the last term above should have vanished. But we can achieve consistency in notation by keeping this term without changing the result because of conservation of mass,

\[
D \int \mathcal{I}_{h,1}(c^k - c^{k-1}) \, dx \overset{\psi=1 \text{ in (4.54c)}}{=} 0.
\]

This allows to continue without a distinction of cases.

The magnetic contribution of the test function to the time difference term from (4.54c) gives rise to the term

\[
\int \mathcal{I}_{h,1} \left( (c^k - c^{k-1}) \mathcal{I}_{h,1} \left( \frac{\delta \chi_{s,h}^{k-1}}{\delta c} (c^k) (|h^k|^2)_{1,h} \right) \right) \, dx \\
= \int_{[c^{k-1} \neq c^k]} \mathcal{I}_{h,1} \left( (c^k - c^{k-1}) \frac{\delta \chi_{s,h}^{k-1}}{\delta c} (c^k) (|h^k|^2)_{1,h} \right) \, dx \\
\overset{4.49}{=} \int_{[c^{k-1} \neq c^k]} \mathcal{I}_{h,1} \left( (c^k - c^{k-1}) \chi_s(c^k) - \chi_s(c^{k-1}) \right) (|h^k|^2)_{1,h} \, dx \\
= \int_{\Omega} \mathcal{I}_{h,1} \left( \mathcal{I}_{h,1}(\chi_s(c^k) - \chi_s(c^{k-1})) (|h^k|^2)_{1,h} \right) \, dx \\
\overset{\varepsilon \mathcal{P}_1(\Omega) = c_h}{=} \int_{\Omega} (\chi_{s,h}^k - \chi_{s,h}^{k-1}) (|h^k|^2) \, dx. \quad (4.97)
\]

In total, testing (4.54c) by \( \psi \) as presented above yields

\[
D \int \mathcal{I}_{h,1}(g_s(c^k) + \Theta [g_s(1 - c^k)]) \, dx - D \int \mathcal{I}_{h,1}(g_s(c^{k-1}) + \Theta [g_s(1 - c^{k-1})]) \, dx \\
- \frac{\mu_0}{2} \int (\chi_{s,h}^k - \chi_{s,h}^{k-1}) |h^k|^2 \, dx \\
\leq \tau D \int c^{k-1} u^k \cdot \nabla \mathcal{I}_{h,1}(g_s'(c^k) - \Theta [g_s'(1 - c^k)]) \, dx \\
- \frac{\tau \mu_0}{2} \int c^{k-1} u^k \cdot \nabla \mathcal{I}_{h,1} \left( \frac{\delta \chi_{s,h}^{k-1}}{\delta c} (c^k) (|h^k|^2)_{1,h} \right) \, dx \\
+ \tau D \int \hat{\mathcal{I}}_{h,1} \left( c^{k-1} v_{part}^k \cdot \nabla \mathcal{I}_{h,1}(g_s'(c^k) - \Theta [g_s'(1 - c^k)]) \right) \, dx \\
- \frac{\tau \mu_0}{2} \int \hat{\mathcal{I}}_{h,1} \left( c^{k-1} v_{part}^k \cdot \nabla \mathcal{I}_{h,1} \left( \frac{\delta \chi_{s,h}^{k-1}}{\delta c} (c^k) (|h^k|^2)_{1,h} \right) \right) \, dx. \quad (4.98)
\]
Choose $\theta := \frac{\tau}{K} \mathcal{I}_{h,1}^d \left( f^{(s),k-1}_h \varphi^{k}_\text{part} \right)$ in (4.54e) to obtain

$$
\frac{\tau}{K} \int_{\Omega} \mathcal{I}_{h,1} \left( f^{(s),k-1}_h |\varphi^{k}_\text{part}|^2 \right) \, dx
= -\tau D \int_{\Omega} \tilde{I}_{h,1} \left( c^{k-1}_s \nabla \mathcal{I}_{h,1} \left( g'_s(c^k) - \Theta[g'_s(1 - c^k)] \right) \cdot \varphi^{k}_\text{part} \right) \, dx
+ \frac{\tau \mu_0}{2} \int_{\Omega} \tilde{I}_{h,1} \left( c^{k-1}_s \varphi^{k}_\text{part} \cdot \nabla \mathcal{I}_{h,1} \left( \frac{\delta \chi_{s,h}^{-1}}{\delta c}((h_k^2)_1,h) \right) \right) \, dx. \tag{4.99}
$$

Summing up (4.96), (4.98) and (4.99) yields

$$
\frac{1}{2} \int_{\Omega} \rho_h^k |u^k|^2 \, dx + \frac{1}{2} \int_{\Omega} \rho_h^{k-1} |u^k - u^{k-1}|^2 \, dx + 2\tau \eta \int_{\Omega} |D u^k|^2 \, dx
+ D \int_{\Omega} \mathcal{I}_{h,1} \left( g_s(c^k) + \Theta(1 - c^k) \right) \, dx + \frac{\tau}{K} \int_{\Omega} \mathcal{I}_{h,1} \left( f^{(s),k-1}_h |\varphi^{k}_\text{part}|^2 \right) \, dx
- \frac{\mu_0}{2} \int_{\Omega} (\chi_{s,h} - \chi_{s,h}^{-1}) |h_k|^2 \, dx
= \frac{1}{2} \int_{\Omega} \rho_h^{k-1} |u^{k-1}|^2 \, dx + D \int_{\Omega} \mathcal{I}_{h,1} \left( g_s(c^{k-1}) + \Theta(1 - c^{k-1}) \right) \, dx. \tag{4.100}
$$

The magnetostatic problem (4.54e) will be tested by $S := \mu_0 R^k$ and its analogous version in time step $k - 1$ with $S := -\mu_0 R^k$. Summing up both results, on the left-hand side one gets

$$
\mu_0 \int_{\Omega'} (h^k - h^{k-1}) \cdot h^k \, dx + \mu_0 \int_{\Omega} (\chi_{s,h} h^k - \chi_{s,h}^{-1} h^{k-1}) \cdot h^k \, dx
= \frac{\mu_0}{2} \int_{\Omega'} |h^k|^2 \, dx + \frac{\mu_0}{2} \int_{\Omega'} |h^k - h^{k-1}|^2 \, dx - \frac{\mu_0}{2} \int_{\Omega'} |h^{k-1}|^2 \, dx
+ \mu_0 \int_{\Omega} (\chi_{s,h} |h^k|^2 + \chi_{s,h}^{-1} (h^k - h^{k-1}) \cdot h^k - \chi_{s,h}^{-1} |h^{k-1}|^2) \, dx, \tag{4.101}
$$

where

$$(4.102)
I = \chi_{s,h} |h^k|^2 + \frac{1}{2} \chi_{s,h}^{-1} |h^k|^2 + \frac{1}{2} \chi_{s,h} |h^k - h^{k-1}|^2 - \frac{1}{2} \chi_{s,h}^{-1} |h^{k-1}|^2 - \chi_{s,h}^{-1} |h^k|^2
= \frac{1}{2} \chi_{s,h} |h^k|^2 + \frac{1}{2} \chi_{s,h}^{-1} |h^k - h^{k-1}|^2 - \frac{1}{2} \chi_{s,h}^{-1} |h^{k-1}|^2 + \frac{1}{2} (\chi_{s,h} - \chi_{s,h}^{-1}) |h^k|^2.
$$

By means of (4.102), testing (4.54e) as mentioned before yields

$$
\frac{\mu_0}{2} \int_{\Omega'} |h^k|^2 \, dx + \frac{\mu_0}{2} \int_{\Omega'} |h^k - h^{k-1}|^2 \, dx - \frac{\mu_0}{2} \int_{\Omega'} |h^{k-1}|^2 \, dx
+ \frac{\mu_0}{2} \int_{\Omega} \chi_{s,h} |h^k|^2 \, dx + \frac{\mu_0}{2} \int_{\Omega} \chi_{s,h}^{-1} |h^k - h^{k-1}|^2 \, dx - \frac{\mu_0}{2} \int_{\Omega} \chi_{s,h}^{-1} |h^{k-1}|^2 \, dx
+ \frac{\mu_0}{2} \int_{\Omega} (\chi_{s,h} - \chi_{s,h}^{-1}) |h^k|^2 \, dx
= \tau \mu_0 \int_{\Omega'} (h^k_h - (h^k)_h) \frac{1}{\tau} \cdot h^k \, dx. \tag{4.103}
$$

Test (4.54e) another time, choosing $S := \tau \mu_0 R^k$, and get

$$
\tau \mu_0 \int_{\Omega'} |h^k|^2 \, dx + \tau \mu_0 \int_{\Omega} \chi_{s,h} |h^k|^2 \, dx = \tau \mu_0 \int_{\Omega'} (h^k_h)_h \cdot h^k \, dx. \tag{4.104}
$$
Combining (4.100), (4.103) and (4.104), application of Young’s inequality yields

\[
\frac{1}{2} \int_{\Omega} \rho_h |u^k|^2 \, dx + \frac{1}{2} \int_{\Omega} \rho_h^{-1} |u^k - u^{k-1}|^2 \, dx + 2\tau \eta \int_{\Omega} |Du^k|^2 \, dx
\]

\[
+ D \int_{\Omega} I_{h,1}(g_s(c^k) + \Theta[g_s(1 - c^k)]) \, dx + \frac{\tau}{K} \int_{\Omega} I_{h,1} \left( \|f^s \|^2 \right) \, dx
\]

\[
+ \frac{\mu_0}{2} \int_{\Omega'} (1 + \mathbf{1}_\Omega \chi_{s,h}^k) |h|^2 \, dx + \frac{\mu_0}{2} \int_{\Omega'} (1 + \mathbf{1}_\Omega \chi_{s,h}^{k-1}) |h^k - h^{k-1}|^2 \, dx
\]

\[
+ \tau \mu_0 \int_{\Omega} \chi_{s,h}^k |h^k|^2 \, dx
\]

\[
\leq \frac{1}{2} \int_{\Omega} \rho_h^{-1} |u^{k-1}|^2 \, dx + D \int_{\Omega} I_{h,1}(g_s(c^{k-1}) + \Theta[g_s(1 - c^{k-1})]) \, dx
\]

\[
+ \frac{\tau \mu_0}{2} \int_{\Omega'} \left( (h_a)_h^k - (h_a)_h^{k-1} \right)^2 \, dx + \frac{\mu_0}{2} \int_{\Omega'} (1 + \mathbf{1}_\Omega \chi_{s,h}^{k-1}) |h^{k-1}|^2 \, dx + \frac{\tau \mu_0}{2} \int_{\Omega'} |(h_a)_h^k|^2 \, dx.
\]

From this, the claim follows easily by summation over all time steps.

By the same arguments as in the proof of Theorem 4.5, see for instance (4.90), the \( h_a \)-related terms in (4.95) are uniformly bounded for \( h, \tau > 0 \).

### 4.2 Existence of discrete solutions

This section is about the proof of existence of discrete solutions via Schaefer’s fixed point theorem. Arbitrary data from the time step \( k - 1 \), \( k \in \mathbb{N} \), is assumed to be given (not necessarily a solution) and the existence of a solution in time step \( k \) will be proven for arbitrarily fixed \( \tau, h > 0 \).

This way, for arbitrary initial data the solution in time step 1 exists and by induction it exists for all time steps. Hence,

- fix a time step \( k \in \{1, ..., n_T\} \) arbitrarily

and

- let \( u^{k-1} \in \mathcal{U}_h, c^{k-1} \in \mathcal{C}_h \) and \( m^{k-1} \in \mathcal{M}_h \) be given.

Also, \( (h_a)_h^k \in \mathcal{D}_1(\overline{\Omega})^d \) is given, too, as \( h_a \) is external given data. This section will be written for the three-dimensional case, but wherever necessary, remarks about the case \( d = 2 \) will be made.

The usual approach to eliminate the pressure and conclude its existence afterwards will be pursued. For this, consider the finite element space

\[
\mathcal{U}_h := \left\{ u \in \mathcal{U}_h \middle| \int_{\Omega} \text{div} \, u_q \, dx = 0 \ \forall q \in \mathcal{P}_h \right\}
\]

(4.105)

which allows to eliminate (4.37b) or (4.54b), respectively, and the pressure term in (4.37a) or (4.54a), respectively.

#### 4.2.1 Model GW

The discretization of "model GW", see (4.37), will be discussed first. Consider new auxiliary variables \( (\hat{u}, \hat{c}, \hat{R}, \hat{m}) \in \mathcal{U}_h \times \mathcal{C}_h \times \mathcal{R}_h \times \mathcal{M}_h \) to be given. With those at hand, the numerical scheme
will be linearized. With a slight misuse of notation, let \( v_{\text{part}} : C_h \times R_h \times \mathcal{M}_h \rightarrow C_h \) be defined pointwise by

\[
\int_\Omega \mathcal{I}_{h,1}(v_{\text{part}}(c^k, R^k, m^k) \cdot \theta) \, dx = -KD \int_\Omega \hat{\mathcal{I}}_{h,1}(\frac{f_h}{((c^k)^{k-1})_{s,h}} \nabla g_{s,h}^k \cdot \theta) \, dx \\
\quad + K\mu_0 b_h^m \left( I_{h,1}^d \left( \frac{f_h}{((c^k)^{k-1})_{s,h}} \right), (\alpha_1 \nabla R^k |_{\Omega} + \frac{\beta}{2}(h_a)_h^k |_{\Omega} - \alpha_3 m^k), m^k \right). \tag{4.106}
\]

The well-posedness of \( v_{\text{part}} \) is obvious in finite dimensions. With that, the variable \( v_{\text{part}}^k \) is not part of the scheme anymore as it can be seen as function which depends on the other unknowns.

The following abbreviations will be used,

\[
\hat{h} := \nabla \hat{R} , \quad \hat{g}_{s,h} := \mathcal{I}_{h,1}(g_{s}(\hat{c})), \\
\hat{v}_{\text{part}} := v_{\text{part}}((\hat{c}, \hat{R}, \hat{m}), \\
\hat{\chi}_{r, h} := \mathcal{I}_{h,1}(\chi_r((\hat{c}^{k-1}, \hat{h})).
\]

Moreover, let

\[
\text{div}_h \hat{m} \in \mathcal{P}_2^{\text{zero}}(\Omega), \quad \text{curl}_h \hat{m} \in \mathcal{P}_1(\Omega)^d \cap H_0(\text{curl})(\Omega)
\]

and \( \text{div}_h \hat{h} \in \mathcal{P}_2(\Omega) \cap H_0^1(\Omega \setminus \partial \Omega) \)

be defined as in (4.32), (4.31) (or (4.34) if \( d = 2 \)) or (4.33), respectively, where \( m^k \) is replaced with \( \hat{m} \) and \( h^k \) is replaced with \( \hat{h} \).

The linearized scheme is as follows. Find functions \((u^k, c^k, R^k, m^k)\) in \( \Omega_h \times C_h \times R_h \times \mathcal{M}_h \) such that for all test functions \((v, \psi, S, n) \in \Omega_h \times C_h \times R_h \times \mathcal{M}_h \) the equations

\[
\rho_0 \int_\Omega \left( \frac{(u^k - u^{k-1})}{\tau} \right) \cdot v \, dx + 2\eta \int_\Omega D u^k : D v \, dx \\
\quad + \frac{\rho_0}{2} \int_\Omega (u^{k-1} \cdot \nabla) \hat{u} \cdot v \, dx - \frac{\rho_0}{2} \int_\Omega (u^{k-1} \cdot \nabla) v \cdot \hat{u} \, dx \\
\quad = -D \int_\Omega c^{k-1} \nabla \hat{g}_{s,h} \cdot v \, dx + \mu_0 b_h^m (v, (\alpha_1 \hat{h} |_{\Omega} + \frac{\beta}{2}(h_a)_h^k |_{\Omega} - \alpha_3 \hat{m}), \hat{m}) \\
\quad + \frac{\mu_0}{2} \int_\Omega (\hat{m} \times (\alpha_1 \hat{h} + \frac{\beta}{2}(h_a)_h^k)) \cdot \text{curl} v \, dx,
\]

\[
\int_\Omega \mathcal{I}_{h,1} \left( \frac{(c^{k-1} - c^{k-2})}{\tau} \right) \psi \, dx = \int_\Omega e^{k-1} \hat{u} \cdot \nabla \psi \, dx + \int_\Omega \hat{I}_{h,1}((c^{k-1} v_{\text{part}} \cdot \nabla \psi) \, dx, \tag{4.107b}
\]

\[
\int_{\Omega'} \nabla R^k \cdot \nabla S \, dx = \int_{\Omega'} (h_a)_h^k \cdot \nabla S \, dx - \int_\Omega \hat{m} \cdot \nabla S \, dx, \tag{4.107c}
\]

\[
\int_{\Omega'} (m^k - m^{k-1}) \cdot n \, dx = b_h^m (\hat{u}, n, \hat{m}) - b_h^m (v_{\text{part}}, n, \hat{m}) \\
\quad = \frac{1}{2} \int_\Omega \text{curl} \hat{u} \times \hat{m} \cdot n \, dx - \frac{1}{\tau_{\text{rel}}} \int_\Omega (\hat{m} - \hat{\chi}_{r,h} \hat{h}) \cdot n \, dx \\
\quad - \sigma \int_\Omega \text{curl}_h \hat{m} \cdot n \, dx + \sigma \int_\Omega \nabla \text{div}_h \hat{m} \cdot n \, dx \tag{4.107d}
\]

hold. In the two-dimensional setting, the \( \text{curl}_h \text{curl}_h \) operator will be replaced with the operator \( \text{Curl}_h \text{Curl}_h \). Also, see the other notational remarks in the proximity of (4.38).
4.2 Existence of discrete solutions

Via a fixed point theorem we will prove existence of \((\hat{u}, \hat{c}, \hat{R}, \hat{m}) \in \mathcal{U}_h \times \mathcal{C}_h \times \mathcal{R}_h \times \mathcal{M}_h\) such that the solution of (4.107) coincides with these auxiliary variables, i.e. it is a solution to (4.37). Above linearized system is an uncoupled linear system, hence unique solvability can be deduced easily. Let

\[
\mathcal{L}^{GW} : \mathcal{U}_h \times \mathcal{C}_h \times \mathcal{R}_h \times \mathcal{M}_h \to \mathcal{U}_h \times \mathcal{C}_h \times \mathcal{R}_h \times \mathcal{M}_h, \\
(\hat{u}, \hat{c}, \hat{R}, \hat{m}) \mapsto (\hat{u}^k, \hat{c}^k, \hat{R}^k, \hat{m}^k)
\]

be the well-defined solution operator of (4.107). The following results are concerned with the requirements for Schaefer’s fixed point theorem.

**Lemma 4.7.** The operator \(\mathcal{L}^{GW}\) from (4.108) is continuous and compact.

**Proof:** Consider sequences \(\hat{u}_n \to \hat{u}\) in \(\mathcal{U}_h\), \(\hat{c}_n \to \hat{c}\) in \(\mathcal{C}_h\), \(\hat{R}_n \to \hat{R}\) in \(\mathcal{R}_h\), \(\hat{m}_n \to \hat{m}\) in \(\mathcal{M}_h\), and set

\[
(\hat{u}_n^k, \hat{c}_n^k, \hat{R}_n^k, \hat{m}_n^k) := \mathcal{L}^{GW}(\hat{u}_n, \hat{c}_n, \hat{R}_n, \hat{m}_n).
\]

For given \(\hat{u} \in \mathcal{U}_h\), \(\hat{c} \in \mathcal{C}_h\) and \(\hat{R} \in \mathcal{R}_h\), \(\hat{m} \in \mathcal{M}_h\) let

\[
(\hat{u}^k, \hat{c}^k, \hat{R}^k, \hat{m}^k) := \mathcal{L}^{GW}(\hat{u}, \hat{c}, \hat{R}, \hat{m}).
\]

As the finite element spaces are finite dimensional, the choice of norms is arbitrary and will be specified as needed in the following computations. Recall that \(b_h^m\) (cf. (4.30)) is a continuous trilinear form, hence it is bounded in the sense

\[
|b_h^m(a, b, c)| \leq C\|a\|_{L^\infty(\Omega)} d \text{ or } L^2(\Omega)^d \|b\|_{L^2(\Omega)^d} \|c\|_{L^2(\Omega)^d}.
\]

Also, the estimate

\[
\|I_{h,1}(\cdot)\|_{L^\infty(\Omega)} \leq \|\cdot\|_{L^\infty(\Omega)}
\]

will be used multiple times. The latter is true as the maximum and minimum function values of a piecewise linear finite element function \(I_{h,1}(f)\) are attained at one of the nodes of the mesh and the nodal values of \(f\) will not be changed by \(I_{h,1}\). Hence, the range of \(f\) includes at least all minimum and all maximum function values of \(I_{h,1}(f)\) which proves (4.110). From there it is easy to prove the vector-valued analogue

\[
\|I_{h,1}^d(\cdot)\|_{L^\infty(\Omega)^d} = \sup_{x \in \Omega} \sqrt{(I_{h,1}^d(\cdot)(x))^2 + \ldots + (I_{h,1}^d(\cdot)(x))^2}
\]

\[
\leq \sup_{x \in \Omega} \sum_{i=1}^d ||I_{h,1}^d(\cdot)(x)|| = \sum_{i=1}^d \|I_{h,1}^d(\cdot)(x)||_{L^\infty(\Omega)}
\]

\[
\leq \sum_{i=1}^d \|\cdot\|_{L^\infty(\Omega)} \leq d \|\cdot\|_{L^\infty(\Omega)^d}.
\]

Before examining the continuity of the variables \((\hat{u}^k, \hat{c}^k, \hat{R}^k, \hat{m}^k)\), a few auxiliary results need to be proven.

i) \(g'_s(\hat{c}_n) \to g'_s(\hat{c})\),

ii) \(v_{\text{part}}(\hat{c}_n, \hat{R}_n, \hat{m}_n) \to v_{\text{part}}(\hat{c}, \hat{R}, \hat{m})\),

iii) \(\chi_r(c^{k-1}, \nabla \hat{R}_n) \to \chi_r(c^{k-1}, \nabla \hat{R})\).

Ad i): From (4.20), global Lipschitz-continuity of \(g_s'(\cdot)\) (with Lipschitz-constant \(\frac{1}{s}\)) is evident. Hence, e.g.

\[
\|g_s'(\hat{c}_n) - g_s'(\hat{c})\|_{L^2(\Omega)} \leq \frac{1}{s}\|\hat{c}_n - \hat{c}\|_{L^2(\Omega)} \to 0,
\]

which concludes the proof of this claim.
Ad [iv] Now consider
\[
\int_{\Omega} I_{h,1}((v_{\text{part}}(\hat{c}_n, \hat{R}_n, \hat{m}_n) - v_{\text{part}}(\hat{c}, \hat{R}, \hat{m})) \cdot \theta) \, dx
\]

\[\begin{align*}
\text{[4.106]} & \quad -KD \int_{\Omega} I_{h,1} \left( \frac{j_{p,h}^{(s),k-1}}{(c^{k-1})_{s,h}} \nabla (g'_s(\hat{c}_n) - g'_s(\hat{c})) \cdot \theta \right) \, dx \\
+ \quad & \quad K \mu_0 b_h^{m} \left( I_{h,1}^d \left( \frac{j_{p,h}^{(s),k-1}}{(c^{k-1})_{s,h}} \theta \right), (\alpha_1(\nabla \hat{R}_n - \nabla \hat{R})|_{\Gamma} - \alpha_3(\hat{m}_n - \hat{m})), \hat{m}_n \right) \\
+ \quad & \quad K \mu_0 b_h^{m} \left( I_{h,1}^d \left( \frac{j_{p,h}^{(s),k-1}}{(c^{k-1})_{s,h}} \theta \right), (\alpha_1 \nabla \hat{R}|_{\Gamma} + \frac{\gamma}{2} (\hat{h}_n)^k_{h} \nabla \hat{R})|_{\Gamma} - \alpha_3 \hat{m}), (\hat{m}_n - \hat{m}) \right)
\end{align*}\]

\[\begin{align*}
\text{[4.110]} & \quad \leq \|g'_s(\hat{c}_n) - g'_s(\hat{c})\|_{W^{1,\infty}(\Omega)} \\
+ \quad & \quad K \mu_0 C \left\| I_{h,1}^d \left( \frac{j_{p,h}^{(s),k-1}}{(c^{k-1})_{s,h}} \theta \right) \right\|_{L^{\infty}(\Omega)^d} \|\alpha_1(\nabla \hat{R}_n - \nabla \hat{R}) - \alpha_3(\hat{m}_n - \hat{m})\|_{L^2(\Omega)^d} \|\hat{m}_n\|_{L^2(\Omega)^d} \\
+ \quad & \quad K \mu_0 C \left\| I_{h,1}^d \left( \frac{j_{p,h}^{(s),k-1}}{(c^{k-1})_{s,h}} \theta \right) \right\|_{L^{\infty}(\Omega)^d} \|\alpha_1 \nabla \hat{R} + \frac{\gamma}{2} (\hat{h}_n)^k_{h} - \alpha_3 \hat{m}\|_{L^2(\Omega)^d} \|\hat{m}_n - \hat{m}\|_{L^2(\Omega)^d} \\
\leq & \quad C' \left( \int_{\Omega} I_{h,1}(\|\theta\|^2) \, dx \right)^{1/2} \left( \|g'_s(\hat{c}_n) - g'_s(\hat{c})\|_{W^{1,\infty}(\Omega)} \right) \\
+ \quad & \quad (\|\hat{R}_n - \hat{R}\|_{H^1(\Omega)} + \|\hat{m}_n - \hat{m}\|_{L^2(\Omega)^d}) \|\hat{m}_n\|_{L^2(\Omega)^d} \\
+ \quad & \quad (\|\nabla \hat{R}\|_{L^2(\Omega)^d} + \|\hat{h}_n\|_{H^1(\Omega)} + \|\hat{m}\|_{L^2(\Omega)^d}) \|\hat{m}_n - \hat{m}\|_{L^2(\Omega)^d} \\
\rightarrow & \quad 0 \quad \text{due to [3]} \\
\rightarrow & \quad 0 \quad \text{bounded due to convergence} \\
\rightarrow & \quad 0 \\
\end{align*}\]

where \(C'\) depends on \(c^{k-1}\) and \(|\Omega|\). Choose \(\theta := (v_{\text{part}}(\hat{c}_n, \hat{R}_n, \hat{m}_n) - v_{\text{part}}(\hat{c}, \hat{R}, \hat{m}))\) and divide by \(\|v_{\text{part}}(\hat{c}_n, \hat{R}_n, \hat{m}_n) - v_{\text{part}}(\hat{c}, \hat{R}, \hat{m})\|_{h,\Omega,d}\) to obtain the result.

Ad [iii] By norm-equivalence one can assume \(\hat{R}_n \to \hat{R}\) in \(W^{1,\infty}(\Omega')\). Hence, \(|\nabla \hat{R}_n(x) - \nabla \hat{R}(x)| \leq C\) for some \(C > 0\) uniformly for \(n \in \mathbb{N}\) and \(x \in \Omega').\) Now, it is sufficient to consider \(\chi_c(c^{k-1}, \cdot)\) on the closed ball (a compact domain) \(W := \overline{B_{\|\nabla \hat{R}\|_{L^\infty(\Omega)^d} + C} \subset \mathbb{R}^d}\). Notice that \(\nabla \hat{R}(x), \nabla \hat{R}_n(x) \in W\) for all \(n \in \mathbb{N}\) and \(x \in \Omega'\). Therefore, as \(\chi_c(c^{k-1}, \cdot)\) is uniformly continuous on \(W\), for any \(\varepsilon > 0\) there exists \(0 < \delta < C\) such that for sufficiently large \(n \in \mathbb{N}\) with \(|\nabla \hat{R}_n - \nabla \hat{R}|_{L^\infty(\Omega)^d} < \delta\) the function values satisfy the estimate

\[\|\chi_c(c^{k-1}, \nabla \hat{R}_n) - \chi_c(c^{k-1}, \nabla \hat{R})\|_{L^\infty(\Omega)} < \varepsilon.\]

Hence, as \(\varepsilon\) was arbitrary and \(\hat{R}_n \to \hat{R}\) the convergence follows.

With those auxiliary results, the continuity of the unknowns \((u_k, c_k, R_k, m_k)\) is straightforward. Continuity with respect of the variable \(R^k\) will be considered first. Subtracting both respective
equations (4.107c) and choosing $S := \nabla R_n^k - \nabla R_k$ yields

$$
\int_{\Omega} (\nabla R_n^k - \nabla R_k) \cdot (\nabla R_n^k - \nabla R_k) \, dx = - \int_{\Omega} (\hat{m}_n - \hat{m}) \cdot (\nabla R_n^k - \nabla R_k) \, dx \\
\leq \frac{1}{2} \| \hat{m}_n - \hat{m} \|_{L^2(\Omega)^d}^2 + \frac{1}{2} \| \nabla R_n^k - \nabla R_k \|_{L^2(\Omega)^d}^2
$$

and consequently $\| \nabla R_n^k - \nabla R_k \|_{L^2(\Omega)^d} \to 0$ via absorption.

Analogously, continuity of the variable $c^k$ follows by exploiting norm-equivalence and the fact that $c^{k-1} \in L^\infty(\Omega)$ is fixed as well as the continuity of $v_{\text{part}}$ (cf. auxiliary result [ii]), i.e.

$$
\int_{\Omega} I_{h,1}((c_n^k - c^k)(c_n^k - c^k)) \, dx \leq \tau \int_{\Omega} c^{k-1}(\hat{u}_n - \hat{u}) \cdot \nabla (c_n^k - c^k) \, dx \\
+ \tau \int_{\Omega} \hat{I}_{h,1}(c^{k-1}(v_{\text{part}}(\hat{c}_n, \hat{R}_n, \hat{m}_n) - v_{\text{part}}(\hat{c}, \hat{R}, \hat{m})) \cdot \nabla (c_n^k - c^k) \, dx \\
\leq C_\delta \left( \| \hat{u}_n - \hat{u} \|_{L^2(\Omega)^d}^2 + \| v_{\text{part}}(\hat{c}_n, \hat{R}_n, \hat{m}_n) - v_{\text{part}}(\hat{c}, \hat{R}, \hat{m}) \|_{L^2(\Omega)^d}^2 \right) \\
\leq \frac{C_\delta}{\tau} \left( \| \hat{u}_n - \hat{u} \|_{L^2(\Omega)^d}^2 + \| v_{\text{part}}(\hat{c}_n, \hat{R}_n, \hat{m}_n) - v_{\text{part}}(\hat{c}, \hat{R}, \hat{m}) \|_{L^2(\Omega)^d}^2 \right) \\
\leq \Delta \| c^k - c^{k-1} \|_{H^1(\Omega)}^2 + \frac{\sigma}{\tau} \int_\Omega I_{h,1}((c_n^k - c^k)^2) \, dx
$$

for any $\delta > 0$, where $C$ depends on given data $c^{k-1}$ and $\delta > 0$ only. Absorption yields the result.

Multilinear terms can be handled by a splitting like $a_n b_n - ab = a_n (b_n - b) + (a_n - a) b$, where $b$ is fixed and $a_n$ bounded due to convergence $a_n \to a$. Hence, concerning the variable $u^k$, one gets

$$
\frac{\rho_0}{2} \| u_n^k - u^k \|_{L^2(\Omega)^d}^2 + 2\eta \| Du_n^k - Du^k \|_{L^2(\Omega)^d \times d} \\
\frac{4.107a}{2} = \frac{\rho_0}{2} \int_{\Omega} (u^{k-1} \cdot \nabla) (u_n^k - \hat{u}) \cdot (u_n^k - u^k) \, dx \\
+ \frac{\rho_0}{2} \int_{\Omega} (u^{k-1} \cdot \nabla) (u_n^k - u^k) \cdot (\hat{u}_n - \hat{u}) \, dx \\
- D \int_{\Omega} c^{k-1} \nabla I_{h,1}(g_n'(\hat{c}_n) - g_n'(\hat{c})) \cdot (u_n^k - u^k) \, dx \\
+ \mu_0 b_n^m (u_n^k - u^k, \alpha_1 (\nabla \hat{R}_n - \nabla \hat{R}) |_{\Gamma_T} - \alpha_3 (\hat{m}_n - \hat{m}), \hat{m}_n) \\
+ \mu_0 b_n^m (u_n^k - u^k, \alpha_1 \nabla \hat{R} |_{\Gamma_T} - \alpha_3 \hat{m}_n - \hat{m}) \\
+ \frac{\mu_0}{2} \int_{\Omega} (\hat{m}_n - \hat{m}) \times (\alpha_1 (\nabla \hat{R}_n - \nabla \hat{R})) \cdot \text{curl}(u_n^k - u^k) \, dx \\
+ \frac{\mu_0}{2} \int_{\Omega} (\hat{m} \times (\alpha_1 (\nabla \hat{R}_n - \nabla \hat{R}))) \cdot \text{curl}(u_n^k - u^k) \, dx \\
\leq C \left( \| \hat{u}_n - \hat{u} \|_{H^1(\Omega)^d} \right) + u_n^k - u^k \|_{L^2(\Omega)^d} + \| u_n^k - u^k \|_{H^1(\Omega)^d} \| \hat{u}_n - \hat{u} \|_{L^2(\Omega)^d} \\
\leq \frac{C_\delta}{\tau} \left( \| \hat{u}_n - \hat{u} \|_{L^2(\Omega)^d}^2 + \| u_n^k - u^k \|_{H^1(\Omega)^d}^2 \right) \\
\leq \frac{C_\delta}{\tau} \left( \| \hat{u}_n - \hat{u} \|_{L^2(\Omega)^d}^2 + \| u_n^k - u^k \|_{L^2(\Omega)^d}^2 \right)
$$
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+bh^n u_n - u^k ||L^2(Ω)^d|| + ||\hat{u}_n - \hat{u}||_{H^1(Ω)^d} \leq C' ||u_n^k - u^k||_{L^2(Ω)^d},

where C' depends on given data – particularly u^{-1}, c^{-1}, m^{-1} and (h_a)_n^k. The terms in the big parentheses go to zero and thereby it is easy to prove that the left-hand side vanishes in the limit – just omit the second term on the left-hand side and divide the inequality by ||u_n^k - u^k||_{L^2(Ω)^d}.

In case of the variable m^k one infers from (4.107d) and the analogous procedure as in case of u^k the estimate

\frac{1}{d} ||m_n^k - m^k||_{L^2(Ω)^d} \leq \frac{1}{d} \left( b_h^m(\hat{u}_n - \hat{u}, m_n^k - m^k, \hat{m}) + b_h^m(\hat{u}_n, m_n^k - m^k, \hat{m}_n - \hat{m}) + b_h^m(\hat{u}_n, m_n^k - m^k, \hat{m}_n - \hat{m}) + \frac{1}{2} \int_Ω \text{curl}(\hat{u}_n - \hat{u}) \times \hat{m} \cdot (m_n^k - m^k) \, dx + \frac{1}{2} \int_Ω \text{curl}(\hat{u}_n \times (\hat{m}_n - \hat{m})) \cdot (m_n^k - m^k) \, dx - \frac{1}{d_{rel}} \int_Ω (\hat{m}_n - \hat{m}) \cdot (m_n^k - m^k) \, dx + \frac{1}{d_{rel}} \int_Ω (\chi_r(e^{-k,1}, \nabla \hat{R}_n) - \chi_r(e^{-k,1}, \nabla \hat{R})) \nabla \hat{R} \cdot (m_n^k - m^k) \, dx + \frac{1}{d_{rel}} \int_Ω \chi_r(e^{-k,1}, \nabla \hat{R}_n) (\nabla \hat{R}_n - \nabla \hat{R}) \cdot (m_n^k - m^k) \, dx - \sigma \int_Ω \text{curl}(\text{curl}_{h}(\hat{m}_n - \text{curl}_{h} \hat{m}) \cdot (m_n^k - m^k) \, dx + \sigma \int_Ω \nabla(\text{div}_{h} \hat{m}_n - \text{div}_{h} \hat{m}) \cdot (m_n^k - m^k) \, dx \leq C' ||m_n^k - m^k||_{L^2(Ω)^d} \left( ||\hat{u}_n - \hat{u}||_{L^2(Ω)^d} + ||\hat{m}_n - \hat{m}||_{L^2(Ω)^d} + ||\hat{u}_n - \hat{u}||_{H^1(Ω)^d} + ||\hat{m}_n - \hat{m}||_{L^2(Ω)^d} + ||\hat{m}_n - \hat{m}||_{L^2(Ω)^d} + \nabla(\chi_r(e^{-k,1}, \nabla \hat{R}_n) - \chi_r(e^{-k,1}, \nabla \hat{R})) \nabla \hat{R} \cdot (m_n^k - m^k) \right),
where $C' > 0$ depends on the boundedness constant of $b^h_\ell$ (cf. (4.109)), the given variables $\hat{R}$ and $\hat{m}$ as well as the boundedness constants of the converging sequences $(\bar{u}_n)_n \in \mathbb{N}$, $(v_{\text{part}}(\hat{c}_n, \hat{R}_n, \hat{m}_n))_n \in \mathbb{N}$ (cf. [ii]) and $(\chi_{(\hat{x}_r^{k-1}, \nabla \hat{R}_n^2)})_n \in \mathbb{N}$ (cf. [iii]). It suffices to examine the terms with curl$_h$ or div$_h$.

Norm equivalence allows to examine

$$\int_\Omega \mathcal{I}_{h,1}(|\text{curl}_h \hat{m}_n - \text{curl}_h \hat{m}|^2) \, dx$$

instead of $\|\text{curl}_h \hat{m}_n - \text{curl}_h \hat{m}\|_{H^1(\Omega)^d}$ if $d=2$

From (4.31) or (4.34) combined with norm-equivalence in finite dimensions one obtains

$$\int_\Omega \mathcal{I}_{h,1}(|\text{curl}_h \hat{m}_n - \text{curl}_h \hat{m}|^2) \, dx = \int_\Omega (\hat{m}_n - \hat{m}) \cdot \text{curl}(\text{curl}_h \hat{m}_n - \text{curl}_h \hat{m}) \, dx$$

and from there curl$_h \hat{m}_n \to$ curl$_h \hat{m}$ is straightforward. The div$_h$-related term is treated analogously.

Hence, $\hat{m}^k$ depends continuously on $(\hat{u}, \hat{c}, \hat{R}, \hat{m})$, which concludes continuity.

The compactness is proven as follows. Due to finite dimensionality it suffices to proof boundedness of solutions if the operator’s arguments are bounded. As the solution solves a linear system, boundedness in terms of given data and arguments of $\mathcal{L}^{GW}$ is obvious.
where $\text{div}_h(h_a)^k_h$ and $\text{curl}_h(h_a)^k_h$ are defined via duality, i.e.

$$
\int_{\Omega'} \text{div}_h(h_a)^k_h \varphi_0 \, dx = - \int_{\Omega'} (h_a)^k_h \cdot \nabla \varphi_0 \, dx \quad \forall \varphi_0 \in \mathcal{P}_2^{\text{zero}}(\Omega'),
$$

$$
\int_{\Omega'} I_{h,1}(\text{curl}_h(h_a)^k_h \cdot \Phi_0) \, dx = - \int_{\Omega'} (h_a)^k_h \cdot \text{curl} \Phi_0 \, dx \quad \forall \Phi_0 \in \mathcal{P}_1(\Omega')^d \cap H_{00}(\text{curl})(\Omega),
$$
or in the two-dimensional case

$$
\int_{\Omega'} I_{h,1}(\text{curl}_h(h_a)^k_h \cdot \varphi_0) \, dx = - \int_{\Omega'} (h_a)^k_h \cdot \text{Curl} \varphi_0 \, dx \quad \forall \varphi_0 \in \mathcal{P}_2^{\text{zero}}(\Omega')
$$

instead of the the second line above.

**Proof:** Replace $(u^k, e^k, R^k, m^k)$ according to (4.113) with $\frac{1}{\lambda} (\tilde{u}, \tilde{e}, \tilde{R}, \tilde{m})$ for an arbitrary $\lambda \in (0, 1]$ and choose the following test functions in (4.107),

- $v := \tau \tilde{u}$,
- $\psi := \tau D\tilde{g}_{s,h}$,
- $S := \mu_0 \alpha_1 \left( \frac{1}{\lambda} + \frac{\tau_{\text{rel}}}{\lambda} \right) \tilde{R} - \tau \sigma \mu_0 \alpha_1 \text{div} \tilde{h}$,
- $n := \alpha_3 \mu_0 \tilde{m} - \frac{\tau_{\text{rel}}}{2} (h_a)^k_h|_{\Omega} - \mu_0 \alpha_1 \tilde{h}|_{\Omega},$

as well as $\theta := \frac{\tau}{\pi} I_{h,1} \left( f_h^{(s),k-1} \varphi_{\text{part}} \right)$ in (4.106), where

$$
f_h^{(s),k-1} := I_{h,1}((c^{k-1})^2 - m)
$$
is the same term as in (4.56). The computations are quite similar to those from the proof of Theorem 4.5. From (4.107a) one gets

$$
\rho_0 \int_{\Omega} \left( \frac{\tilde{u}}{\lambda} - u^{k-1} \right) \cdot \tilde{u} \, dx + \frac{2\tau \eta}{\lambda} \int_{\Omega} |D\tilde{u}|^2 \, dx
$$

$$
= - \tau D \int_{\Omega} e^{k-1} \nabla \tilde{g}_{s,h} \cdot \tilde{u} \, dx + \tau \mu_0 \beta \tilde{m} \left( \alpha_1 \tilde{h} \right|_{\partial \Omega} + \frac{\beta}{2} (h_a)^k_h|_{\partial \Omega} - \alpha_3 \tilde{m} \right), 
$$

$$
+ \frac{\tau \mu_0}{2} \int_{\Omega} \tilde{m} \times (\alpha_1 \tilde{h} + \frac{\beta}{2} (h_a)^k_h) \cdot \text{curl} \tilde{u} \, dx,
$$

from (4.107b)

$$
D \int_{\Omega} I_{h,1} \left( \frac{\tilde{e}}{\lambda} - e^{k-1} \tilde{g}_{s,h} \right) \, dx = \tau D \int_{\Omega} e^{k-1} \tilde{u} \cdot \nabla \tilde{g}_{s,h} \, dx + \tau D \int_{\Omega} \tilde{I}_{h,1}(c^{k-1}) \varphi_{\text{part}} \cdot \nabla \tilde{g}_{s,h} \, dx,
$$

from (4.107c) with the help of (4.33), (4.32) one gets

$$
\frac{\mu_0 \alpha_1}{\lambda} \left( \frac{1}{\lambda} + \frac{\tau_{\text{rel}}}{\lambda} \right) \int_{\Omega'} |\tilde{h}|^2 \, dx + \frac{\tau \sigma \mu_0 \alpha_1}{\lambda} \int_{\Omega'} |\text{div}_h \tilde{h}|^2 \, dx
$$

$$
= \mu_0 \alpha_1 \left( \frac{1}{\lambda} + \frac{\tau_{\text{rel}}}{\lambda} \right) \int_{\Omega'} (h_a)^k_h \cdot \tilde{h} \, dx - \mu_0 \alpha_1 \left( \frac{1}{\lambda} + \frac{\tau_{\text{rel}}}{\lambda} \right) \int_{\Omega} \tilde{m} \cdot \tilde{h} \, dx
$$

$$
+ \tau \sigma \mu_0 \alpha_1 \int_{\Omega'} \text{div}_h(h_a)^k_h \text{div}_h \tilde{h} \, dx - \tau \sigma \mu_0 \alpha_1 \int_{\Omega} \text{div}_h \tilde{m} \text{div}_h \tilde{h} \, dx,
$$

and from (4.107d) with the help of (4.31), (4.32) and using

$$
(\cdot) \times \tilde{m} \cdot \tilde{m} = 0, \quad \text{curl} \tilde{u} \times a \cdot b = a \times b \cdot \text{curl} \tilde{u}
$$
one gets
\[
\mu_0 \alpha_3 \int_{\Omega} \left( \mathbf{m} - \mathbf{m}^{k-1} \right) \cdot \mathbf{m} \, dx - \frac{\mu_0 \beta}{2} \int_{\Omega} \left( \mathbf{m} - \mathbf{m}^{k-1} \right) \cdot (\mathbf{h}_a)^k_h \, dx \\
- \mu_0 \alpha_1 \int_{\Omega} \left( \mathbf{m} - \mathbf{m}^{k-1} \right) \cdot \hat{\mathbf{h}} \, dx + \tau \mu_0 b_h^m \left( \mathbf{v}_{\text{part}}, \left( \alpha_1 \hat{\mathbf{h}}^\alpha |_{\Gamma_T} + \frac{\beta}{2} (\mathbf{h}_a)^k |_{\Gamma_T} - \alpha_3 \hat{\mathbf{m}}, \hat{\mathbf{m}} \right) \right) \\
+ \tau \mu_0 b_h^m \left( \mathbf{v}_{\text{part}}, (\alpha_1 \hat{\mathbf{h}}^\alpha |_{\Gamma_T} + \frac{\beta}{2} (\mathbf{h}_a)^k |_{\Gamma_T} - \alpha_3 \hat{\mathbf{m}}, \hat{\mathbf{m}} \right) \right) \\
= -\frac{\tau \mu_0}{2} \int_{\Omega} \mathbf{m} \times (\alpha_1 \hat{\mathbf{h}} + \frac{\beta}{2} (\mathbf{h}_a)^k_h) \cdot \text{curl} \, \mathbf{u} \, dx \\
- \frac{\tau \mu_0 \alpha_3}{\tau_{\text{rel}}} \int_{\Omega} |\mathbf{m}|^2 \, dx + \frac{\tau \mu_0 \beta}{2 \tau_{\text{rel}}} \int_{\Omega} \mathbf{m} \cdot (\mathbf{h}_a)^k_h \, dx \\
+ \frac{\tau \mu_0 \alpha_3}{\tau_{\text{rel}}} \int_{\Omega} \hat{\chi}_{r,h} \hat{\mathbf{h}} \cdot \mathbf{m} \, dx - \frac{\tau \mu_0 \beta}{2 \tau_{\text{rel}}} \int_{\Omega} \hat{\chi}_{r,h} \hat{\mathbf{h}} \cdot (\mathbf{h}_a)^k_h \, dx - \frac{\tau \mu_0 \alpha_3}{\tau_{\text{rel}}} \int_{\Omega} \hat{\chi}_{r,h} |\hat{\mathbf{h}}|^2 \, dx \\
- \tau \mu_0 \alpha_3 \int_{\Omega} \mathcal{I}_{h,1}(|\text{curl}_h \mathbf{m}|^2) \, dx + \frac{\tau \mu_0 \beta}{2} \int_{\Omega} \mathcal{I}_{h,1}(\text{curl}_h \mathbf{m} \cdot \text{curl}_h (\mathbf{h}_a)^k_h) \, dx \\
+ \tau \mu_0 \alpha_3 \int_{\Omega} \text{curl}_h \mathbf{m} \cdot \hat{\mathbf{h}} \, dx - \tau \mu_0 \alpha_3 \int_{\Omega} \text{div}_h |\mathbf{m}|^2 \, dx \\
+ \frac{\tau \mu_0 \alpha_3}{2} \int_{\Omega} \text{div}_h \mathbf{m} \cdot \text{div}_h (\mathbf{h}_a)^k_h \, dx + \tau \mu_0 \alpha_3 \int_{\Omega} \text{div}_h \mathbf{m} \cdot \text{div}_h \hat{\mathbf{h}} \, dx.
\]

It can easily be seen within the lines of the proof of (4.5) that \( \int_{\Omega} \text{curl}_h \mathbf{m} \cdot \hat{\mathbf{h}} \, dx = 0 \), see (4.74). The key to the proof is integration by parts and the fact that \( \hat{\mathbf{h}} \times \mathbf{v} = \nabla \hat{\mathbf{R}} \times \mathbf{v} \) does not jump across element boundaries. Finally, from (4.100) we obtain
\[
\frac{\tau}{K} \int_{\Omega} \mathcal{I}_{h,1}(f_h^{(s,k-1)} \hat{\mathbf{v}}_{\text{part}}) \, dx = -\tau D \int_{\Omega} \hat{\mathbf{v}}_{h,1} \left( \chi^{k-1} \nabla \hat{\mathbf{g}}^s_{h} \cdot \hat{\mathbf{v}}_{\text{part}} \right) \, dx \\
+ \tau \mu_0 b_h^m \left( \mathbf{v}_{\text{part}}, (\alpha_1 \hat{\mathbf{h}}^k |_{\Gamma_T} + \frac{\beta}{2} (\mathbf{h}_a)^k |_{\Gamma_T} - \alpha_3 \mathbf{m}^k, \mathbf{m}^k) \right).
\]

Summing up the results yields
\[
\rho_0 \int_{\Omega} \left( \frac{\hat{\mathbf{u}}}{\lambda} - \mathbf{u}^{k-1} \right) \cdot \mathbf{u} \, dx + D \int_{\Omega} \mathcal{I}_{h,1} \left( \frac{\hat{\mathbf{u}}}{\lambda} - \chi^{k-1} \hat{\mathbf{g}}^s_{h} \right) \, dx + \alpha_3 \mu_0 \int_{\Omega} \left( \frac{\mathbf{m}}{\lambda} - \mathbf{m}^{k-1} \right) \cdot \mathbf{m} \, dx \\
+ \frac{2 \tau \eta}{\lambda} \int_{\Omega} |\mathbf{D}\mathbf{u}|^2 \, dx + \frac{\tau}{K} \int_{\Omega} \mathcal{I}_{h,1}(f_h^{(s,k-1)} |\mathbf{v}_{\text{part}}|^2) \, dx + \frac{\tau \mu_0 \alpha_3}{\tau_{\text{rel}}} \int_{\Omega} |\hat{\mathbf{m}}|^2 \, dx \\
+ \mu_0 \alpha_1 \left( \frac{\hat{\mathbf{u}}}{\lambda} + \frac{\tau}{\tau_{\text{rel}}} \right) \int_{\Omega} |\mathbf{h}|^2 \, dx + \frac{\tau \mu_0 \alpha_1}{\tau_{\text{rel}}} \int_{\Omega} \hat{\chi}_{r,h} |\hat{\mathbf{h}}|^2 \, dx + \frac{\tau \sigma \mu_0 \alpha_1}{\lambda} \int_{\Omega} |\text{div}_h \hat{\mathbf{h}}|^2 \, dx \\
+ \tau \alpha_3 \mu_0 \alpha_3 \int_{\Omega} \mathcal{I}_{h,1}(|\text{curl}_h \mathbf{m}|^2) \, dx + \tau \mu_0 \alpha_1 \int_{\Omega} |\text{div}_h \mathbf{m}|^2 \, dx \\
= \frac{\tau \mu_0 \alpha_1}{\tau_{\text{rel}}} \int_{\Omega} (\mathbf{h}_a)^k_h \cdot \hat{\mathbf{h}} \, dx + \frac{\mu_0 \alpha_1}{\lambda} \int_{\Omega'} (\mathbf{h}_a)^k_h \cdot \hat{\mathbf{h}} \, dx + \tau \sigma \mu_0 \alpha_1 \int_{\Omega'} \text{div}_h (\mathbf{h}_a)^k_h \, dx \\
+ \frac{\tau \alpha_3 \mu_0 \alpha_3}{\tau_{\text{rel}}} \int_{\Omega} \hat{\chi}_{r,h} \mathbf{m} \cdot \hat{\mathbf{h}} \, dx - \alpha_1 \mu_0 \int_{\Omega} \mathbf{m}^{k-1} \cdot \hat{\mathbf{h}} \, dx \\
+ \frac{\tau \alpha_3 \mu_0 \alpha_3}{\tau_{\text{rel}}} \int_{\Omega} \hat{\chi}_{r,h} \mathbf{m} \cdot \hat{\mathbf{h}} \, dx - \alpha_1 \mu_0 \int_{\Omega} \mathbf{m}^{k-1} \cdot \hat{\mathbf{h}} \, dx \\
+ \frac{\mu_0 \beta}{2} \int_{\Omega} \left( \frac{\mathbf{m}}{\lambda} - \mathbf{m}^{k-1} \right) \cdot (\mathbf{h}_a)^k_h \, dx + \frac{\tau \beta \mu_0}{2 \tau_{\text{rel}}} \int_{\Omega} \mathbf{m} \cdot (\mathbf{h}_a)^k_h \, dx \\
- \frac{\tau \beta \mu_0}{2 \tau_{\text{rel}}} \int_{\Omega} \hat{\chi}_{r,h} \mathbf{m} \cdot (\mathbf{h}_a)^k_h \, dx + \frac{\tau \beta \mu_0}{2} \int_{\Omega} \mathcal{I}_{h,1}(\text{curl}_h \mathbf{m} \cdot \text{curl}_h (\mathbf{h}_a)^k_h) \, dx \\
+ \frac{\tau \beta \mu_0 \alpha_3}{2} \int_{\Omega} \text{div}_h \mathbf{m} \cdot \text{div}_h (\mathbf{h}_a)^k_h \, dx. \tag{4.114}
\]

For terms of the type \((\frac{a}{\lambda} - b)a\) one can proceed by means of
\[
\left(\frac{a}{\lambda} - b\right)a = (a - b)a + (\frac{a}{\lambda} - 1)a^2 = (\frac{a}{\lambda} - \frac{1}{2})a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a - b)^2.
\]
Moreover,
\[ I_{h,1} \left( \left( \frac{\hat{c}}{\lambda} - c^{k-1} \right) \hat{g'}_{s,h} \right) = I_{h,1} \left( \left( \frac{\hat{c}}{\lambda} - c^{k-1} \right) \hat{g'}_{s,h} \right) =: I + II. \]

In the same way as in the proof of [Theorem 4.5] the convexity of \( g_s \) (combined with [Lemma 4.1]) will be exploited, i.e.
\[
I \geq g_{s,h}(\hat{c}) - g_{s,h}(c^{k-1}), \\
II \geq (\frac{1}{\lambda} - 1)(g_{s,h}(\hat{c}) - g_{s,h}(0)).
\]

For convenience, above computation is valid as interpolation operators as part of the argument of other interpolation operators are redundant and can be omitted. Therefore, \( g_s \) is directly accessible. The terms on the right-hand side of (4.114) are treated by Young’s inequality as follows.

\[
\frac{\tau \mu_0 \alpha_1}{\tau_{rel}} \int_{\Omega'} (h_a)^k \cdot \hat{h} \, dx 
\leq \frac{\tau \mu_0 \alpha_1}{2 \tau_{rel}} \int_{\Omega'} |(h_a)^k|^2 \, dx + \frac{\tau \mu_0 \alpha_1}{2 \tau_{rel}} \int_{\Omega'} |\hat{h}|^2 \, dx,
\]

\[
\frac{\mu_0 \alpha_1}{\lambda} \int_{\Omega'} (h_a)^k \cdot \hat{h} \, dx
\leq \mu_0 \alpha_1 \int_{\Omega'} |(h_a)^k|^2 \, dx + \frac{\mu_0 \alpha_1}{4 \lambda^2} \int_{\Omega'} |\hat{h}|^2 \, dx,
\]

\[
\tau \sigma \mu_0 \alpha_1 \int_{\Omega'} \text{div}_h (h_a)^k \cdot \text{div}_h \hat{h} \, dx
\leq \frac{\lambda \tau \sigma \mu_0 \alpha_1}{2} \int_{\Omega'} |\text{div}_h (h_a)^k|^2 \, dx + \frac{\tau \sigma \mu_0 \alpha_1}{2 \lambda} \int_{\Omega'} |\text{div}_h \hat{h}|^2 \, dx,
\]

\[
\frac{\tau \alpha_3 \mu_0}{\tau_{rel}} \int_{\Omega} \hat{x}_r \cdot \hat{h} \, dx
\leq \frac{\tau \mu_0 \alpha_3}{\tau_{rel}} \int_{\Omega} |\hat{x}_r|^2 \, dx + \frac{\tau \alpha_3 \mu_0}{4 \tau_{rel}} \int_{\Omega} |\hat{m}|^2 \, dx,
\]

\[
\frac{\alpha_1 \mu_0}{2} \int_{\Omega} \hat{m}^{k-1} \cdot \hat{h} \, dx 
\leq \alpha_1 \mu_0 \lambda^2 \int_{\Omega} |\hat{m}^{k-1}|^2 \, dx + \frac{\alpha_1 \mu_0}{4 \lambda^2} \int_{\Omega} |\hat{h}|^2 \, dx,
\]

\[
\frac{\mu_0 \beta}{2} \int_{\Omega} \hat{m} \cdot (h_a)^k \, dx
\leq \mu_0 \alpha_3 \int_{\Omega} |\hat{m}|^2 \, dx + \frac{\mu_0 \beta}{4 \lambda \alpha_3} \int_{\Omega} |(h_a)^k|^2 \, dx,
\]

\[
\frac{\mu_0 \beta}{2} \int_{\Omega} \hat{m}^{k-1} \cdot (h_a)^k \, dx
\leq \frac{\mu_0 |\beta|}{4} \int_{\Omega} |\hat{m}^{k-1}|^2 \, dx + \frac{\mu_0 |\beta|}{4} \int_{\Omega} |(h_a)^k|^2 \, dx,
\]

\[
\frac{\tau \beta \mu_0}{2 \tau_{rel}} \int_{\Omega} \hat{m} \cdot (h_a)^k \, dx 
\leq \frac{\tau \mu_0 \alpha_3}{4 \tau_{rel}} \int_{\Omega} |\hat{m}|^2 \, dx + \frac{\tau \beta \mu_0}{4 \tau_{rel} \alpha_3} \int_{\Omega} |(h_a)^k|^2 \, dx.
\]
- \frac{\tau \beta \mu_0}{2 \tau_{\text{rel}}} \int_\Omega \hat{x}_{r,h} \cdot (\hat{h}_h)^k \, dx \\
\leq \frac{\tau |\beta| \mu_0}{4 \tau_{\text{rel}}} \int_\Omega |\hat{x}_{r,h}|^2 \, dx + \frac{\tau |\beta| \mu_0}{4 \tau_{\text{rel}}} \int_\Omega |(\hat{h}_h)^k|^2 \, dx,

\frac{\tau \beta \mu_0 \sigma}{2} \int_\Omega I_{h,1}(\text{curl}_h \hat{m} \cdot \text{curl}_h (\hat{h}_h)^k) \, dx \\
\leq \frac{\tau \mu_0 \sigma \alpha_3}{2} \int_\Omega I_{h,1}(|\text{curl}_h \hat{m}|^2) \, dx + \frac{\tau \beta^2 \mu_0 \sigma}{8 \alpha_3} \int_\Omega I_{h,1}(|\text{curl}_h (\hat{h}_h)^k|^2) \, dx,

\frac{\tau \beta \mu_0 \sigma}{2} \int_\Omega \text{div}_h \hat{m} \cdot \text{div}_h (\hat{h}_h)^k \, dx \\
\leq \frac{\tau \mu_0 \sigma \alpha_3}{2} \int_\Omega |\text{div}_h \hat{m}|^2 \, dx + \frac{\tau \beta^2 \mu_0 \sigma}{8 \alpha_3} \int_\Omega |\text{div}_h (\hat{h}_h)^k|^2 \, dx.

Recall that the terms \text{div}_h (\hat{h}_h)^k and \text{curl}_h (\hat{h}_h)^k depend on given data and therefore are fixed. Combining all results so far yields

\rho_0 (\chi - \frac{1}{2}) \int_\Omega |\hat{u}|^2 \, dx + \frac{D}{\chi} \int_\Omega g_{s,h} (\hat{c}) \, dx + \alpha_3 \mu_0 (\frac{3}{4} - \frac{1}{2}) \int_\Omega |\hat{m}|^2 \, dx \\
+ \frac{\rho_0}{2} \int_\Omega |\hat{u} - u^{k-1}|^2 \, dx + \frac{\alpha_3 \mu_0}{2} \int_\Omega |\hat{m} - m^{k-1}|^2 \, dx \\
+ \frac{2 \tau \eta}{\lambda} \int_\Omega |D\hat{u}|^2 \, dx + \frac{\tau}{\lambda} \int_\Omega I_{h,1}(f^{(s),k-1}_h \cdot \hat{v}_{\text{part}}^2) \, dx + \frac{\tau \alpha_3 \mu_0}{2 \tau_{\text{rel}}} \int_\Omega |\hat{m}|^2 \, dx \\
+ \frac{\mu_0 \alpha_1}{2 \lambda} (\chi + \frac{\tau_{\text{rel}}}{\tau_0}) \int_\Omega |\hat{h}|^2 \, dx + \frac{\tau \alpha_1 \mu_0}{\tau_{\text{rel}}} \int_\Omega |\hat{x}_{r,h}|^2 \, dx + \frac{\tau \sigma \mu_0 \alpha_1}{2 \lambda} \int_\Omega |\text{div}_h \hat{h}|^2 \, dx \\
+ \frac{\tau \alpha_3 \mu_0 \sigma}{2} \int_\Omega I_{h,1}(|\text{curl}_h \hat{m}|^2) \, dx + \frac{\tau \alpha_3 \mu_0 \sigma}{2} \int_\Omega |\text{div}_h \hat{m}|^2 \, dx \\
\leq \rho_0 \int_\Omega |u^{k-1}|^2 \, dx + \left( \frac{\alpha_3 \mu_0}{2} + \alpha_1 \mu_0 \lambda^2 + \frac{\mu_0 |\beta|}{4} \right) \int_\Omega |m^{k-1}|^2 \, dx \\
+ D \int_\Omega g_{s,h}(0) \, dx \\
+ \left( \frac{\tau \alpha_3 \mu_0}{\tau_{\text{rel}}} + \frac{\tau |\beta| \mu_0}{4 \tau_{\text{rel}}} \right) \left( 2 K_1^2 |\Omega| + 2 K_2^2 \int_\Omega |\epsilon^{k-1}| \, dx \right) \\
+ \left( \mu_0 \alpha_1 + \frac{\tau \mu_0 \alpha_1}{2 \tau_{\text{rel}}} \right) \int_\Omega |(\hat{h}_h)^k|^2 \, dx + \frac{\tau \sigma \mu_0 \alpha_1 \lambda}{2} \int_\Omega |\text{div}_h (\hat{h}_h)^k|^2 \, dx \\
+ \left( \frac{\mu_0 \beta^2}{4 \alpha_3 \lambda} + \frac{\mu_0 |\beta|}{4} + \frac{\tau \beta^2 \mu_0}{4 \tau_{\text{rel}} \alpha_3} + \frac{\tau |\beta| \mu_0}{4 \tau_{\text{rel}}} \right) \int_\Omega |(\hat{h}_h)^k|^2 \, dx \\
+ \frac{\tau \beta^2 \mu_0 \sigma}{8 \alpha_3} \int_\Omega I_{h,1}(|\text{curl}_h (\hat{h}_h)^k|^2) \, dx + \frac{\tau \beta^2 \mu_0 \sigma}{8 \alpha_3} \int_\Omega |\text{div}_h (\hat{h}_h)^k|^2 \, dx.

In order to get a uniform estimate with respect to \lambda \in (0,1], we will multiply above inequality by \lambda. After straightforward estimates related to \lambda one obtains

\rho_0 \int_\Omega |\hat{u}|^2 \, dx + D \int_\Omega g_{s,h}(\hat{c}) \, dx + \frac{\alpha_3 \mu_0}{4} \int_\Omega |\hat{m}|^2 \, dx + 2 \tau \eta \int_\Omega |D\hat{u}|^2 \, dx \\
+ \frac{\mu_0 \alpha_1}{2} (1 + \frac{\tau_{\text{rel}}}{\tau_0}) \int_\Omega |\hat{h}|^2 \, dx + \frac{\tau \sigma \mu_0 \alpha_1}{2} \int_\Omega |\text{div}_h \hat{h}|^2 \, dx \\
\leq \rho_0 \int_\Omega |u^{k-1}|^2 \, dx + \left( \frac{\alpha_3 \mu_0}{2} + \alpha_1 \mu_0 + \frac{\mu_0 |\beta|}{4} \right) \int_\Omega |m^{k-1}|^2 \, dx

+ \lambda D \int_\Omega g^{k-1}_{s,h} \, dx + D(1 - \lambda) \int_\Omega g_{s,h}(0) \, dx \\
+ \left( \frac{\tau \alpha \mu_0}{\tau_{\text{rel}}} + \frac{\tau |\beta| \mu_0}{4 \tau_{\text{rel}}} \right) \int_\Omega |(h_{s,h})_k^h| \, dx + \frac{\tau_\sigma \mu_0 \alpha_1}{2} \int_\Omega' |\nabla h_{s,h}(0)|^2 \, dx \\
+ \left( \frac{\mu_0 \beta^2}{4 \alpha_3} + \frac{\mu_0 |\beta|}{4 \tau_{\text{rel}} \alpha_3} + \frac{\tau |\beta| \mu_0}{4 \tau_{\text{rel}}} \right) \int_\Omega |(h_{s,h})_k^h|^2 \, dx \\
+ \frac{\tau \beta^2 \mu_0 \sigma}{8 \alpha_3} \int_\Omega \mathcal{I}_{h_{s,h}}(|\nabla h_{s,h}(0)|^2) \, dx + \frac{\tau \beta^2 \mu_0 \sigma}{8 \alpha_3} \int_\Omega |\nabla h_{s,h}(0)|^2 \, dx.

Note that the term $g_s$ is bounded from below by $-1$. Hence,

$$\lambda D \int_\Omega g^{k-1}_{s,h} \, dx = \lambda D \int_\Omega (g^{k-1}_{s,h} + 1) \, dx - \lambda D |\Omega| \leq D \int_\Omega (g^{k-1}_{s,h} + 1) \, dx \quad (4.115)$$

and analogously

$$(1 - \lambda) D \int_\Omega g_{s,h}(0) \, dx \leq D \int_\Omega (g_{s,h}(0) + 1) \, dx. \quad (4.116)$$

This implies the claim.

**Corollary 4.9.** The set $V_{GW}$ from (4.112) is bounded.

**Proof:** From Lemma 4.8 the existence of a constant $C > 0$ can be deduced such that

$$\|\hat{u}\|_{L^2(\Omega)^d}^2 + \int_\Omega g_{s,h}(\hat{c}) \, dx + \|\hat{m}\|_{L^2(\Omega)^d}^2 + \|\hat{h}\|_{L^2(\Omega)^d}^2 \leq C. \quad (4.117)$$

The constant does depend on model parameters, given data from time step $k - 1$, external data and $\tau, h > 0$ (which is permissible for the sole purpose of proving existence of discrete solutions).

The second term on the left hand side of (4.117) can be used to bound the $L^1$-norm of $\hat{c}$ as can be seen from Lemma 3.22. In finite dimensions the choice of norms does not matter, which is why the proof is complete.

**Theorem 4.10.** The numerical scheme (4.37) admits at least one discrete solution for all time steps $k \in \{1, ..., n_T\}$.

**Proof:** It is sufficient to only consider a fixed time step $k$. Combining the results from Lemma 4.7 and Corollary 4.9, Schaefer’s fixed point theorem is applicable to the operator $L_{GW}$ from (4.108). Hence, a discrete solution exists.

**4.2.2 Model W and model B**

The existence of discrete solutions for "model W" or "model B", respectively, is proven similarly to the case of "model GW". One proceeds as in the proofs of Lemma 4.7 and Lemma 4.8. Then by means of Schaefer’s fixed point theorem the existence of discrete solutions will be evident.
Consider auxiliary variables \((\hat{\mathbf{u}}, \hat{c}, \hat{R}) \in \mathcal{U}_h \times \mathcal{C}_h \times \mathcal{R}_h\) to be given. Redefine \(\mathbf{v}_{\text{part}} := \mathbf{v}_{\text{part}}(\hat{c}, \hat{R})\), where \(\mathbf{v}_{\text{part}} : (c^k, R^k) \rightarrow v_{\text{part}}^k\) is defined pointwise by

\[
\int \mathcal{I}_{h,1}(\mathbf{v}_{\text{part}}(c^k, R^k) \cdot \mathbf{v}) \, \mathrm{d}x = -KD \int \mathcal{I}_{h,1} \left( \Theta[1 - c^{k-1}] \nabla \mathbf{I}_{h,1}(g_s'(c^k) - \Theta[g'_s(1 - c^k)]) \cdot \mathbf{v} \right) \, \mathrm{d}x + \frac{K \mu_0}{2} \int \mathcal{I}_{h,1} \left( \Theta[1 - c^{k-1}] \varphi h_{\text{part}} \cdot \nabla \mathbf{I}_{h,1} \left( \frac{\delta k_{\text{h}}}{\delta c} (c^k)(|\nabla R^k|^2)_{1,h} \right) \right) \, \mathrm{d}x
\]

for all \((\mathbf{v}, \psi, S) \in \mathcal{U}_h \times \mathcal{C}_h \times \mathcal{R}_h\). Well-posedness of \(\mathbf{L}^{W|B}\) can be deduced easily as the system is uncoupled and linear.

**Lemma 4.11.** The operator \(\mathbf{L}^{W|B}\) from (4.119) is continuous and compact.

**Proof:** Consider sequences \(\mathbf{u}_{n} \rightarrow \mathbf{u}, \hat{c}_{n} \rightarrow \hat{c}\) in \(\mathcal{C}_h, \hat{R}_{n} \rightarrow \hat{R}\) in \(\mathcal{R}_h\) and set

\[
(u^k, c^k, R^k) := \mathbf{L}^{W|B}(\mathbf{u}_n, \hat{c}_n, \hat{R}_n).
\]

For given \(\mathbf{u} \in \mathcal{U}_h, \hat{c} \in \mathcal{C}_h\) and \(\hat{R} \in \mathcal{R}_h\) let

\[
(u^k, c^k, R^k) := \mathbf{L}^{W|B}(\mathbf{u}, \hat{c}, \hat{R}).
\]

The following auxiliary results will be proven first.

i) \(\Theta[g'_s(1 - \hat{c}_n)] \rightarrow \Theta[g'_s(1 - \hat{c})]\),

ii) \(\frac{\delta k_{\text{h}}}{\delta c} (\hat{c}_n) \rightarrow \frac{\delta k_{\text{h}}}{\delta c} (\hat{c})\),

iii) \(\mathbf{v}_{\text{part}}(\hat{c}_n, \hat{R}_n, \hat{m}_n) \rightarrow \mathbf{v}_{\text{part}}(\hat{c}, \hat{R}, \hat{m})\).

Remember, the choice of norms is arbitrary due to the finite dimensional setting.
Ad [ii]: In case of 'model W' there is nothing to prove, see (4.39). In case of 'model B' Lipschitz-continuity of $g'_s$, as in the proof of [iii] in Lemma 4.7 yields the result.

Ad [iii]: A simple splitting of the type $a_nb_n - ab = a_n(b_n - b) + (a_n - a)b$ leads to the terms

$$
\frac{\delta x_{k-1}}{\delta c} (\hat{e}_n) (|\nabla \hat{R}_n|^2)_{1,h} - \frac{\delta x_{k-1}}{\delta c} (\hat{e})(|\nabla \hat{R}|^2)_{1,h}
$$

$$
= (\frac{\delta x_{k-1}}{\delta c} (\hat{e}_n) - \frac{\delta x_{k-1}}{\delta c} (\hat{e})) (|\nabla \hat{R}_n|^2)_{1,h} + \frac{\delta x_{k-1}}{\delta c} (\hat{e})(|\nabla \hat{R}_n|^2 - |\nabla \hat{R}|^2)_{1,h}.
$$

Consider the $L^1$-norms of the terms on the right-hand side, i.e.

$$
\|\frac{\delta x_{k-1}}{\delta c} (\hat{e}_n) - \frac{\delta x_{k-1}}{\delta c} (\hat{e}) (|\nabla \hat{R}_n|^2)_{1,h}\|_{L^1(\Omega)} \leq \|\frac{\delta x_{k-1}}{\delta c} (\hat{e}_n) - \frac{\delta x_{k-1}}{\delta c} (\hat{e})\|_{L^2(\Omega)} \|(|\nabla \hat{R}_n|^2)_{1,h}\|_{L^2(\Omega)} \leq \| |\nabla \hat{R}_n|^2\|_{L^2(\Omega)}
$$

(L^2-stability of $(\cdot)_{1,h}$, see (4.45))

and analogously

$$
\|\frac{\delta x_{k-1}}{\delta c} (\hat{e}_n) - \frac{\delta x_{k-1}}{\delta c} (\hat{e}) (|\nabla \hat{R}_n|^2 - |\nabla \hat{R}|^2)_{1,h}\|_{L^1(\Omega)} \leq \|\frac{\delta x_{k-1}}{\delta c} (\hat{e}_n) - \frac{\delta x_{k-1}}{\delta c} (\hat{e})\|_{L^2(\Omega)} \|(|\nabla \hat{R}_n - \nabla \hat{R}|)_{1,h}\|_{L^2(\Omega)} + \||\nabla \hat{R}_n - \nabla \hat{R}|_{L^2(\Omega)}\|_{L^2(\Omega)} \leq \||\nabla \hat{R}_n - \nabla \hat{R}|_{L^2(\Omega)}\|_{L^2(\Omega)_{d}} \|\nabla \hat{R}_n + \nabla \hat{R}|_{L^2(\Omega)}\|_{L^2(\Omega)_{d}}
$$

bounded due to convergence of $R_n$.

Therefore, it is sufficient to prove

$$
\frac{\delta x_{k-1}}{\delta c} (\hat{e}_n) \to \frac{\delta x_{k-1}}{\delta c} (\hat{e})
$$

in order to conclude the $L^1$-convergence $\frac{\delta x_{k-1}}{\delta c} (\hat{e}_n) (|\nabla \hat{R}_n|^2)_{1,h} \to \frac{\delta x_{k-1}}{\delta c} (\hat{e})(|\nabla \hat{R}|^2)_{1,h}$. The latter can be achieved as follows. First, one can drop the interpolation operator in the definition of $\frac{\delta x_{k-1}}{\delta c}$, see (4.49), by exploiting norm-equivalence (Lemma 4.2). Then we exploit the Lipschitz-continuity (see (4.51)) of $\frac{\delta x_{k-1}}{\delta c}$, where the Lipschitz constant is $\frac{3}{4s}$, see Section A.5. Then

$$
\|\frac{\delta x_{k-1}}{\delta c} (\hat{e}_n) - \frac{\delta x_{k-1}}{\delta c} (\hat{e})\|_{L^2(\Omega)} \leq M_0 \|\hat{e}_n - \hat{e}\|_{L^2(\Omega)} \to 0
$$

which proves the claim.

Ad [iii] From (4.118) one obtains, by introducing the abbreviations

$$
C_1(c^{k-1}) := KD\Theta[1 - c^{k-1},r]f^{(s),k-1}_{h}((c^{k-1})_{s},h),
$$

$$
C_2(c^{k-1}) := K\mu_0\Theta[1 - c^{k-1},r]c^{-1}f^{(s),k-1}_{h}((c^{k-1})_{s},h),
$$

the identity

$$
\int_{\Omega} \mathcal{I}_{h,1}((\nu_{part}(\hat{e}_n, \hat{R}_n) - \nu_{part}(\hat{e}, \hat{R})) \cdot \theta) \, dx
$$
4.2 Existence of discrete solutions

\[ = -\int_{\Omega} \tilde{I}_{h,1}(C(c^{k-1})\nabla I_{h,1}((g'_s(\hat{c}_n) - g'_s(\hat{c})) - (\Theta[g'_s(1 - \hat{c}_n)] - \Theta[g'_s(1 - \hat{c}))]) \cdot \theta) \, dx \]

\[ + \int_{\Omega} \tilde{I}_{h,1}(C_2(c^{k-1})\theta \cdot \nabla I_{h,1} (\frac{\delta x^{k-1}}{\delta c}(\hat{c})) (|\nabla \hat{R}|^2)_{1,h} - \frac{\delta x^{k-1}}{\delta c}(\hat{c})(|\nabla R|^2)_{1,h} ) \, dx \]

\[ \leq \|C_1(c^{k-1})\|_{L^\infty(\Omega)} \left( \|\nabla I_{h,1}((g'_s(\hat{c}_n) - g'_s(\hat{c})) - (\Theta[g'_s(1 - \hat{c}_n)] - \Theta[g'_s(1 - \hat{c}))])\|_{L^\infty(\Omega)^d} \right) \|\theta\|_{L^\infty(\Omega)^d} 
\]

\[ + \|C_2(c^{k-1})\|_{L^\infty(\Omega)} \left( \|\nabla I_{h,1} (\frac{\delta x^{k-1}}{\delta c}(\hat{c})) (|\nabla \hat{R}|^2)_{1,h} - \frac{\delta x^{k-1}}{\delta c}(\hat{c})(|\nabla R|^2)_{1,h} )\|_{L^\infty(\Omega)^d} \right) \|\theta\|_{L^\infty(\Omega)^d} 
\]

\[ \leq C'(\int_{\Omega} |\theta|^2 \, dx)^{\frac{1}{2}} \left( \|g'_s(\hat{c}_n) - g'_s(\hat{c})\|_{L^2(\Omega)} + \|\Theta[g'_s(1 - \hat{c}_n)] - \Theta[g'_s(1 - \hat{c}))\|_{L^2(\Omega)} \right) 
\]

\[ + \|\frac{\delta x^{k-1}}{\delta c}(\hat{c}) (|\nabla \hat{R}|^2)_{1,h} - \frac{\delta x^{k-1}}{\delta c}(\hat{c})(|\nabla R|^2)_{1,h} \|_{L^2(\Omega)} \right) \]

where \( C' > 0 \) depends on given data \( c^{k-1} \). Choosing \( \theta := (v_{\text{part}}(\hat{c}_n, \hat{R}_n) - v_{\text{part}}(\hat{c}, \hat{R})) \) and dividing by \( \|v_{\text{part}}(\hat{c}_n, \hat{R}_n) - v_{\text{part}}(\hat{c}, \hat{R})\|_{h,\Omega,d} \) yields the result.

Now, the main part of the proof proceeds. Starting with \( u^k \), one subtracts the respective equations for \( u^k \), which are satisfied by \( u^k \) and \( u^k_n \) and exploits norm-equivalence to obtain the estimate

\[ \int_{\Omega} I_{h,1}(\rho_{\text{reg}}(\hat{c}_n)) - I_{h,1}(\rho_{\text{reg}}(\hat{c})) \cdot u^k \cdot v \, dx + 2\eta \int_{\Omega} (Dv_n - Du^k) : Dv \, dx \]

\[ = -\int_{\Omega} I_{h,1}(\rho_{\text{reg}}(\hat{c}_n)) - I_{h,1}(\rho_{\text{reg}}(\hat{c})) \cdot u^k \cdot v \, dx \]

\[ + \int_{\Omega} I_{h,1}(\rho_{\text{reg}}(\hat{c}_n)) - I_{h,1}(\rho_{\text{reg}}(\hat{c})) \cdot u^{k-1} \cdot v \, dx \]

\[ - \frac{1}{2} \int_{\Omega} \rho_h^k(u^{k-1} \cdot \nabla)(\hat{u}_n - \hat{u}) \cdot v \, dx + \frac{1}{2} \int_{\Omega} \rho_h^k(u^{k-1} \cdot \nabla) v \cdot (\hat{u}_n - \hat{u}) \, dx \]

\[ - \frac{1}{2} \int_{\Omega} (\Theta'[g^{k-1}](v_{\text{part}}(\hat{c}_n, \hat{R}_n) - v_{\text{part}}(\hat{c}, \hat{R})) \cdot \nabla) v \cdot (\hat{u}_n - \hat{u}) \, dx \]

\[ - \frac{1}{2} \int_{\Omega} (\Theta'[g^{k-1}]v_{\text{part}}(\hat{c}_n, \hat{R}_n) \cdot \nabla)(\hat{u}_n - \hat{u}) \cdot v \, dx \]

\[ + \frac{1}{2} \int_{\Omega} (\Theta'[g^{k-1}](v_{\text{part}}(\hat{c}_n, \hat{R}_n) - v_{\text{part}}(\hat{c}, \hat{R})) \cdot \nabla) v \cdot (\hat{u}_n - \hat{u}) \, dx \]

\[ + \frac{1}{2} \int_{\Omega} (\Theta'[g^{k-1}]v_{\text{part}}(\hat{c}_n, \hat{R}_n) \cdot \nabla) v \cdot (\hat{u}_n - \hat{u}) \, dx \]

\[ - D \int_{\Omega} c^{k-1} \nabla I_{h,1}((g'_s(\hat{c}_n) - g'_s(\hat{c})) - (\Theta[g'_s(1 - \hat{c}_n)] - \Theta[g'_s(1 - \hat{c}))]) \cdot v \, dx \]

\[ + \frac{\mu_0}{2} \int_{\Omega} c^{k-1} \cdot \nabla I_{h,1} (\frac{\delta x^{k-1}}{\delta c}(\hat{c})) (|\nabla \hat{R}|^2)_{1,h} - \frac{\delta x^{k-1}}{\delta c}(\hat{c})(|\nabla R|^2)_{1,h} \, dx \]
Therefore, \( C' > 0 \) only depends on fixed quantities \((\tau) \) is fixed as well in this proof. Choose \( \mathbf{v} := (\mathbf{u}_n^k - \mathbf{u}_n^k) \) above, divide by \( \|u_n^k - u_n^k\|_{L^2(\Omega)^d} \) and notice \( \frac{\delta_{\mathbf{c}_1}(\rho_{\text{reg}}(\mathbf{c}_n)) + \delta_{\mathbf{c}_1}(\rho_{\text{reg}}(\mathbf{c}_n))}{2} \geq \min(\rho_1, \rho_2) > 0 \) in order to obtain the result.

From (4.120b) one obtains via norm-equivalence

\[
\int_{\Omega} I_{k,1}((c_n^k - c^k)^{\mathbf{v}}) \, \text{d}x = \tau \int_{\Omega} c_n^{k-1}(\mathbf{u}_n^k - \mathbf{u}) \cdot \nabla \psi \, \text{d}x
\]

\[
+ \int_{\Omega} I_{k,1}(c_n^k - c^k)(\mathbf{v}_{\text{part}}(\mathbf{c}_n^k, \hat{R}_n) - \mathbf{v}_{\text{part}}(\mathbf{c}_n, \hat{R})) \cdot \nabla \psi \, \text{d}x
\]

\[
\leq C' \left( \int_{\Omega} I_{k,1}(\mathbf{v})^2 \, \text{d}x \right)^{\frac{1}{2}} \left( \|u_n^k - u_n^k\|_{L^2(\Omega)^d} + \|\mathbf{v}_{\text{part}}(\mathbf{c}_n^k, \hat{R}_n) - \mathbf{v}_{\text{part}}(\mathbf{c}_n, \hat{R})\|_{L^2(\Omega)^d} \right),
\]

from which the result follows easily after choosing \( \psi := (c_n^k - c^k) \).

From (4.120c) one obtains

\[
\int_{\Omega} (\nabla R_n^k - \nabla R^k) \cdot \nabla S \, \text{d}x + \int_{\Omega} I_{k,1}(\chi_s(\mathbf{c}_n^k))(\nabla R_n^k - \nabla R^k) \cdot \nabla S \, \text{d}x
\]

\[
= - \int_{\Omega} I_{k,1}(\chi_s(\mathbf{c}_n^k) - \chi_s(\mathbf{c}_n^k)) \nabla R_n^k \cdot \nabla S \, \text{d}x
\]

\[
\leq C' \|\nabla S\|_{L^2(\Omega)^d} \left( \|\chi_s(\mathbf{c}_n^k) - \chi_s(\mathbf{c}_n^k)\|_{L^2(\Omega)} \right),
\]

where \( C' > 0 \) depends on \( R^k \) which does not depend on \( n \in \mathbb{N} \). Choosing \( S := (R_n^k - R^k) \) and using Lipschitz continuity of \( \chi_s \), see (4.50), and \( \omega > -1 \) (cf. below (4.46)) the result follows.

Therefore, \( L^{W/B} \) is continuous. Due to finite dimensionality compactness follows from boundedness of solutions if the operator’s arguments are bounded. As the solution solves a linear system, boundedness in terms of given data and arguments of \( L^{W/B} \) is obvious. \( \square \)

For applicability of Schaefer’s fixed point theorem the set

\[
V^{W/B} := \{ \mathbf{y} \in \mathcal{U}_h \times \mathcal{C}_h \times \mathcal{R}_h | \mathbf{y} = \lambda L^{W/B} \mathbf{y} \text{ for any } \lambda \in (0,1) \}
\]

(4.121)
is supposed to be bounded. This corresponds to the replacement of
\[
(u^k, c^k, R^k) \quad \text{with} \quad \frac{1}{\lambda}(\hat{u}, \hat{c}, \hat{R})
\] in (4.120).

**Lemma 4.12.** The set \(V^{W|B}\) from (4.121) is bounded.

**Proof:** Boundedness of functions that satisfy (4.120) when \((u^k, c^k, R^k)\) is replaced with \(\frac{1}{\lambda}(\hat{u}, \hat{c}, \hat{R})\) (independently of \(\lambda \in (0, 1]\)) has to be proven. Choose test functions

- \(v := \tau \hat{u}\),
- \(\psi := \tau D I_{h,1}(g'_s(\hat{\hat{c}}) - \Theta[g'_s(1 - \hat{\hat{c}})]) - \frac{\mu_0}{2} I_{h,1} \left( \frac{\delta x_{h,-1}^k}{\delta c}(\hat{\hat{c}})(|\hat{\hat{u}}|^2)_{1,h} \right),\)

in (4.120) as well as \(\theta := \frac{\tau}{K} f_h^{(s),k-1} \hat{\nu}_{\text{part}}\) in (4.118), where

\[
f_h^{(s),k-1} := I_{h,1} \left( \frac{(c^{k-1})_{1,m}}{\Theta[1 - (c^{k-1})^m]} \right)
\]
is defined as in (4.94). The computations are very similar as in the proof of Theorem 4.6. Summing up the tested equations, one obtains

\[
\int \frac{\rho_h + \rho_h^{k-1}}{2} (\hat{u} - u^{k-1}) \hat{u} \, dx + \frac{1}{2} \int A_h (\hat{u} - \hat{u}^{k-1}) \cdot \hat{u} \, dx + \frac{2\tau \eta}{\lambda} \int \Omega \hat{D} \hat{u}^2 \, dx
+ D \int_{\Omega} I_{h,1} \left( \left( \frac{\hat{c}}{\hat{x}} - c^{k-1} \right) I_{h,1}(g'_s(\hat{\hat{c}}) - \Theta[g'_s(1 - \hat{\hat{c}})]) \right) \, dx
+ \frac{\tau}{K} \int_{\Omega} I_{h,1} \left( f_h^{(s),k-1} |\hat{\nu}_{\text{part}}|^2 \right) \, dx
- \frac{\mu_0}{2} \int_{\Omega} I_{h,1} \left( \left( \frac{\hat{c}}{\hat{x}} - c^{k-1} \right) I_{h,1} \left( \frac{\delta x_{h,1}^k}{\delta c}(\hat{\hat{c}})(|\hat{\hat{u}}|^2)_{1,h} \right) \right) \, dx
= 0.
\]

Recalling that inner interpolation operators within the argument of an outer interpolation operator are redundant, the last term above yields

\[
\begin{align*}
- \frac{\mu_0}{2} & \int_{\Omega} I_{h,1} \left( \left( \frac{\hat{c}}{\hat{x}} - c^{k-1} \right) I_{h,1} \left( \frac{\delta x_{h,1}^k}{\delta c}(\hat{\hat{c}})(|\hat{\hat{u}}|^2)_{1,h} \right) \right) \, dx \\
= & - \frac{\mu_0}{2} \int_{[\hat{c} \neq c^{k-1}]} \underbrace{I_{h,1} \left( \frac{\hat{c}}{\hat{x}} - c^{k-1} \right) I_{h,1} \left( \frac{\delta x_{h,1}^k}{\delta c}(\hat{\hat{c}})(|\hat{\hat{u}}|^2)_{1,h} \right)} \, dx \\
= & \frac{\mu_0}{2} \int_{\Omega} I_{h,1} \left( \hat{x}_{s,h} - \chi_{s,h}(\hat{\hat{c}}) \right) |\hat{\hat{u}}|^2 \, dx - \frac{1}{\lambda} \frac{\mu_0}{2} \int_{\Omega} I_{h,1} \left( \frac{\delta x_{h,1}^k}{\delta c}(\hat{\hat{c}})(|\hat{\hat{u}}|^2)_{1,h} \right) \, dx \\
= & - \frac{\mu_0}{2} \int_{\Omega} \hat{x}_{s,h} |\hat{\hat{u}}|^2 \, dx + \frac{\mu_0}{2} \int_{\Omega} \chi_{s,h} |\hat{\hat{u}}|^2 \, dx \\
= & - \frac{1}{\lambda} \frac{\mu_0}{2} \int_{\Omega} I_{h,1} \left( \frac{\delta x_{h,1}^k}{\delta c}(\hat{\hat{c}})(|\hat{\hat{u}}|^2)_{1,h} \right) \, dx.
\end{align*}
\]

(4.123)
The bad terms (first and third term of the right-hand side of \((4.123)\)) can be estimated from below by

\[
- \frac{\mu_0}{2\lambda} \int_{\Omega} \hat{\chi}_{s,h} |\hat{h}|^2 \, dx = - \left( \frac{1}{\lambda} - 1 \right) \frac{\mu_0}{2} \int_{\Omega} \hat{\chi}_{s,h} |\hat{h}|^2 \, dx - \frac{\mu_0}{2} \int_{\Omega} \hat{\chi}_{s,h} |\hat{h}|^2 \, dx
\]

\[
\geq \mathcal{I}_{h,1}(\chi_0 \hat{c})
\]

\[
\geq \mathcal{I}_{h,1} \left( \frac{\delta \chi_{s,h}^k}{\delta c}(\hat{c}) \right)
\]

\[
\leq - \left( \frac{1}{\lambda} - 1 \right) \frac{\mu_0}{2} \int_{\Omega} \mathcal{I}_{h,1} \left( \frac{\delta \chi_{s,h}^k}{\delta c}(\hat{c}) \right) |\hat{h}|^2 \, dx - \frac{\mu_0}{2} \int_{\Omega} \hat{\chi}_{s,h} |\hat{h}|^2 \, dx,
\]

where it was used that \(\chi_{s,\hat{c}}^{(k)} - \chi_{s,\hat{c}}^{(k-1)} \hat{c} \leq (\max_{z \in \mathbb{R}} \chi_s'(z)) \hat{c} \leq \chi_0 \hat{c} \leq \chi_\hat{c}(\hat{c}) \) (cf. \((4.50)\)).

Choosing \(S := \mu_0 \hat{R} \) in \((4.120c)\) one gets

\[
\frac{\mu_0}{\lambda} \int_{\Omega'} |\hat{h}|^2 \, dx + \frac{\mu_0}{\lambda} \int_{\Omega} \hat{\chi}_{s,h} |\hat{h}|^2 \, dx = \mu_0 \int_{\Omega'} (h_a)_h^k \cdot \hat{h} \, dx \leq \frac{\mu_0}{2\lambda} \int_{\Omega'} |\hat{h}|^2 \, dx + \frac{\mu_0}{2} \int_{\Omega'} |(h_a)_h^k|^2 \, dx.
\]

Altogether, one arrives at

\[
\int_{\Omega} \left( \frac{\rho_h + \rho_h^{k-1}}{2} (\frac{\hat{u}}{\chi} - u^{k-1}) \hat{u} \, dx + \frac{1}{2} \int_{\Omega} (\zeta_h - \rho_h^{k-1}) u^{k-1} \cdot \hat{u} \, dx + \frac{2\eta}{\lambda} \int_{\Omega} |\nabla \hat{u}|^2 \, dx
\]

\[
+ D \int_{\mathcal{I}_{h,1}} (\frac{\chi}{\lambda} - c^{k-1}) \mathcal{I}_{h,1}(\zeta_h (\hat{c}) - \Theta[g_s'(1 - \hat{c})]) \, dx + \frac{\tau}{\kappa} \int_{\mathcal{I}_{h,1}} |(h_{\hat{c}}^{(s),k-1}|\nabla \psi_{\text{part}}|^2) \, dx
\]

\[
+ \frac{\mu_0}{2} \int_{\Omega} \hat{\chi}_{s,h} |\hat{h}|^2 \, dx + \frac{\mu_0}{2} \int_{\Omega} \hat{\chi}_{s,h} |\hat{h}|^2 \, dx + \frac{\mu_0}{2} \int_{\Omega'} |\hat{h}|^2 \, dx
\]

\[
\leq \frac{\mu_0}{2} \int_{\Omega'} |(h_a)_h^k|^2 \, dx.
\]

(4.124)

Continue by means of

\[
(\frac{1}{\lambda} - b) a = (a - b)a + (\frac{1}{\lambda} - 1)a^2 = (\frac{1}{\lambda} - \frac{1}{2})a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a - b)^2.
\]

This yields

\[
\int_{\Omega} \frac{\rho_h + \rho_h^{k-1}}{2} (\frac{\hat{u}}{\chi} - u^{k-1}) \hat{u} \, dx
\]

\[
= (\frac{1}{\lambda} - \frac{1}{2}) \int_{\Omega} \frac{\rho_h + \rho_h^{k-1}}{2} |\hat{u}|^2 \, dx - \frac{1}{2} \int_{\Omega} \frac{\rho_h + \rho_h^{k-1}}{2} |u^{k-1}|^2 \, dx + \frac{1}{2} \int_{\Omega} \frac{\rho_h + \rho_h^{k-1}}{2} |\hat{u} - u^{k-1}|^2 \, dx.
\]

And by \(ab = -\frac{1}{2}(a - b)^2 + \frac{1}{2}a^2 + \frac{1}{2}b^2\) one gets

\[
\frac{1}{2} \int_{\Omega} (\zeta_h - \rho_h^{k-1}) u^{k-1} \cdot \hat{u} \, dx
\]

\[
= - \frac{1}{2} \int_{\Omega} \frac{\rho_h - \rho_h^{k-1}}{2} |\hat{u} - u^{k-1}|^2 \, dx + \frac{1}{2} \int_{\Omega} \frac{\rho_h - \rho_h^{k-1}}{2} |\hat{u}|^2 \, dx + \frac{1}{2} \int_{\Omega} \frac{\rho_h - \rho_h^{k-1}}{2} |u^{k-1}|^2 \, dx.
\]

Adding the last two identities above, one arrives at

\[
\int_{\Omega} \frac{\rho_h + \rho_h^{k-1}}{2} (\frac{\hat{u}}{\chi} - u^{k-1}) \hat{u} \, dx + \frac{1}{2} \int_{\Omega} (\zeta_h - \rho_h^{k-1}) u^{k-1} \cdot \hat{u} \, dx
\]

\[
= \frac{1}{2} \int_{\Omega} \frac{\rho_h}{2} |\hat{u}|^2 \, dx + (\frac{1}{\lambda} - 1) \int_{\Omega} \frac{\rho_h^{k-1}}{2} |\hat{u}|^2 \, dx - \frac{1}{2} \int_{\Omega} \rho_h^{k-1} |u^{k-1}|^2 \, dx + \frac{1}{2} \int_{\Omega} \rho_h^{k-1} |\hat{u} - u^{k-1}|^2 \, dx.
\]
Moreover,
\[ I_{h,1} \left( \left( \frac{\xi}{\lambda} - c^{k-1} \right) I_{h,1}(g_s(\hat{c}) - \Theta[g_s(1 - \hat{c})]) \right) = I_{h,1} \left( \left( \frac{\xi}{\lambda} - c^{k-1} \right) I_{h,1}(g_s(\hat{c}) - \Theta[g_s(1 - \hat{c})]) \right) + (\frac{1}{\lambda} - 1) I_{h,1}((\hat{c} - 0)I_{h,1}(g_s(\hat{c}) - \Theta[g_s(1 - \hat{c})])) =: I + II. \]

Convexity of \( g_s \) (together with Lemma 4.1) is exploited, i.e.
\[ I \geq I_{h,1}(g_s(\hat{c}) + \Theta[g_s(1 - \hat{c})]) - I_{h,1}(g_s(c^{k-1}) + \Theta[g_s(1 - c^{k-1})]), \]
\[ II \geq (\frac{1}{\lambda} - 1) I_{h,1}(g_s(\hat{c}) + \Theta[g_s(1 - \hat{c})]) - I_{h,1}(g_s(0) + \Theta[g_s(1)]). \]

Note that in case of \( \Theta \equiv 1 \) convexity can be exploited, too, and inner interpolation operators are redundant from the perspective of the outer most interpolation operator. One computes
\[ I + II = \frac{1}{\lambda} I_{h,1}(g_s(\hat{c}) + \Theta[g_s(1 - \hat{c})]) - I_{h,1}(g_s(c^{k-1}) + \Theta[g_s(1 - c^{k-1})]) - (\frac{1}{\lambda} - 1) I_{h,1}(g_s(0) + \Theta[g_s(1)]). \]

Inserting the recent results into (4.112) and multiplying by \( \lambda \) gives
\[ \int_{\Omega} \frac{\xi}{\lambda} |\hat{u}|^2 \, dx + (1 - \lambda) \int_{\Omega} \frac{g_h^{k-1}}{2} |\hat{u}|^2 \, dx + \lambda \int_{\Omega} \frac{g_h^{k-1}}{2} |\hat{u} - u^{k-1}|^2 \, dx \]
\[ + 2\tau \int_{\Omega} |\hat{D} \hat{u}|^2 \, dx + D \int_{\Omega} I_{h,1}(g_s(\hat{c}) + \Theta[g_s(1 - \hat{c})]) \, dx \]
\[ + \frac{\tau \lambda}{K} \int_{\Omega} I_{h,1}(\lambda_{k-1}\hat{\psi}_{\text{part}}^2) \, dx + \frac{\mu_0 \lambda}{2} \int_{\Omega} \chi_{s,h}^{k-1} |\hat{h}|^2 \, dx \]
\[ + \frac{\mu_0}{2} \int_{\Omega} \hat{h}_{s,h} |\hat{h}|^2 \, dx + \frac{\mu_0}{2} \int_{\Omega'} |\hat{h}|^2 \, dx \]
\[ \leq \frac{\mu_0 \lambda}{2} \int_{\Omega'} |(h_{s,h})^{k-1}|^2 \, dx + \lambda \int_{\Omega} \frac{g_h^{k-1}}{2} |u^{k-1}|^2 \, dx + \lambda D \int_{\Omega} I_{h,1}(g_s(c^{k-1}) + \Theta[g_s(1 - c^{k-1})]) \, dx \]
\[ + (1 - \lambda) D \int_{\Omega} I_{h,1}(g_s(0) + \Theta[g_s(1)]) \, dx. \]

Straightforward estimates related to \( \lambda \) give
\[ \int_{\Omega} \frac{\xi}{\lambda} |\hat{u}|^2 \, dx + 2\tau \int_{\Omega} |\hat{D} \hat{u}|^2 \, dx + D \int_{\Omega} I_{h,1}(g_s(\hat{c}) + \Theta[g_s(1 - \hat{c})]) \, dx \]
\[ + \frac{\mu_0}{2} \int_{\Omega} \hat{h}_{s,h} |\hat{h}|^2 \, dx + \frac{\mu_0}{2} \int_{\Omega'} |\hat{h}|^2 \, dx \]
\[ \leq \frac{\mu_0}{2} \int_{\Omega'} |(h_{s,h})^{k-1}|^2 \, dx + \int_{\Omega} \frac{g_h^{k-1}}{2} |u^{k-1}|^2 \, dx + D \int_{\Omega} I_{h,1}(g_s(0) + \Theta[g_s(1)]) \, dx + 2D |\Omega| \]
\[ + D \int_{\Omega} I_{h,1}(g_s(c^{k-1}) + \Theta[g_s(1 - c^{k-1})]) \, dx + 2D |\Omega|, \quad (4.125) \]

where analogous arguments as in (4.115) and (4.116) have been used. From there on, one can proceed similar as in the proof of Corollary 4.9, i.e. the term \( \Theta[g_s(1 - \hat{c})] \) is bounded from below and the term \( g_s(\hat{c}) \) suffices to bound the \( L^1 \)-norm of \( \hat{c} \). Hence, the functions \( (\hat{u}, \hat{c}, \hat{R}) \) are bounded independently of \( \lambda \).

**Theorem 4.13.** The numerical scheme (4.54) admits at least one discrete solution for all time steps \( k \in \{1, ..., n_T\} \).

**Proof:** It is sufficient to only consider a fixed time step \( k \). Combining the results from Lemma 4.11 and Lemma 4.12, Schaefer’s fixed point theorem is applicable to the operator \( \mathcal{L}^{W|B} \) from (4.119). Hence, a discrete solution exists.
4.3 Practical aspects

This section goes slightly into detail of some aspects that have not been discussed yet but are helpful or even mandatory for practical simulations. These considerations are tailored to make

- the implementation straightforward
- and the computational cost reasonably small.

The implementation of the numerical schemes from Section 4.1 was done in the framework of the inhouse code 'EconDrop3D' which was mainly developed by G. Grün, F. Klingbeil and S. Metzger [44,50] in the context of the priority program "SPP1506" of the 'Deutsche Forschungsgemeinschaft' (DFG) as a successor to the entropy consistent solver for lubrication-type equations 'EConLub2D' [11, Becker]. Further contributions to the software are due to H. Grillmeier [36], O. Sieber and ourselves – our main contribution is briefly presented in Section 4.3.1. For details about the program’s basic structure and abilities the reader shall be referred to [11,44].

First, let us motivate why special care is needed to perform robust simulations on basis of the models (Section 2) and discretizations (Section 4.1) of this thesis. In order to resolve the support of the magnetic particle density and the effect of the particle-induced stray field sufficiently well, high spatial resolution (locally in space) is needed. In case of "model GW", discontinuous finite elements appear in the discretization of the magnetization equation (4.37f) for stability reasons, which come along with a lot of degrees of freedom. In contrast, only conforming finite elements are used for "model W" and "model B", because here the magnetization equation has been replaced with a simple algebraic expression. On the other hand – the magnetic force being the main driving force – the particle evolution equation is convection-dominant which generally requires a certain smallness of time increments (see e.g. the Courant–Friedrichs–Lewy (CFL) condition [45]). Hence, the computational costs will be rather large, especially in case of "model GW". For this reason, this thesis will only include proof-of-concept simulations (see Section 5) in two space dimensions, i.e.

$$d = 2.$$  

**Remark 4.14.** The first simulations on basis of "model GW" were done in [39]. There – combined with a heuristic time adaptivity approach – a mixed scheme has been proposed where the convection-dominant particle evolution is discretized via finite volumes combined with Engquist-Osher flux and minmod limiter [45]. The transition between finite element solution and finite volume solution is done with the help of dual cells. The details are as follows.

**Given a nodal point** \( \mathbf{x}_j \in \Omega \), **consider the dual cell**

$$V_j := \{ \mathbf{x} \in \Omega \mid |\mathbf{x}_j - \mathbf{x}| < |\mathbf{x}_l - \mathbf{x}| \ \forall l = 1, ..., \dim \mathcal{P}_1(\Omega), l \neq j \} \ \forall j = 1, ..., \dim \mathcal{P}_1(\Omega).$$

A cell-wise constant finite-volume function \( c^{\text{FV}} : \Omega \rightarrow \mathbb{R} \), which is determined by its cell values \( c^{\text{FV}} := c^{\text{FV}}(\mathbf{x}_j) \equiv c^{\text{FV}}|_{V_j}, j = 1, ..., \dim \mathcal{P}_1(\Omega) \), is defined as solution of the finite volume discretization of (4.37c),

$$\frac{|V_j|}{\tau} (c^{\text{FV}}_j - c^{k-1}(\mathbf{x}_j)) + \sum_{n \in N_j} F_{jn}(U^k + V_{\text{part}}^k, c^{k-1}(\mathbf{x}_j), c^{k-1}(\mathbf{x}_n)) = 0 \quad (4.126)$$

for all \( j = 1, ..., \dim \mathcal{P}_1(\Omega) \), where \( N_j \) is the index set of those neighboring dual cells, which share a common face with the cell \( V_j \) and the term \( F_{jm}(U^k + V_{\text{part}}^k, \cdot, \cdot) \) denotes the Engquist-Osher flux combined with the minmod limiter [45]. From here, the finite element function \( c^k \) is defined by piecewise linear nodal interpolation (on the original triangulation) of the data points \( (\mathbf{x}_j, c^{\text{FV}}_j), j = 1, ..., \dim \mathcal{P}_1(\Omega) \).
However, the simulations in Section 5 will mainly be carried out with the original finite element scheme in order to be consistent with the numerical analysis of [39] and of this thesis. Moreover, a much simpler heuristic time adaptivity scheme will be used. A short comparison of the original finite element scheme and a mixed finite volume scheme based on numerical quality and computational costs is provided in Section 5.2.

4.3.1 Aspects of implementation

For conciseness, it will be assumed that the reader is familiar with basic techniques and vocabulary used in context of finite element methods. For simplicity, assume the space dimension to be $d = 2$. Our main contribution to the development of "EconDrop3D" is the implementation of non-conforming finite elements, most importantly the piecewise linear elements $\mathcal{D}_1(\Omega)$, which are used (among others) for the simulations shown in Section 5. One major difference to conforming elements is the type of basis functions. The discontinuity allows to use basis functions with small support which thereby can be orthonormalized easily. Orthonormal basis functions simplify for instance the computation and inversion of mass matrices as they turn out to be diagonal.

Basis functions of $\mathcal{D}_1(\Omega)$. In order determine an affine linear function on a triangle $\mathcal{K}$ one can use Lagrangian interpolation with the help of nodal values. Consider the reference triangle $\hat{\mathcal{K}}$ defined as in Figure 4.1. The usual local basis functions are given on $\hat{\mathcal{K}}$ by

\[\varphi_0(x, y) := x, \quad \varphi_1(x, y) := y, \quad \varphi_2(x, y) := 1 - x - y\] (4.127)

and by affine linear transformation the respective local basis functions of arbitrary triangles $\mathcal{K} \in \mathcal{T}_h(\Omega)$ can be constructed.

In the conforming case, i.e. piecewise continuous linear elements, all those local basis functions which are equal to 1 at a specific vertex will be combined into one function to obtain continuous nodal basis functions $\phi_i$, $i = 1, \ldots, \dim \mathcal{P}_1(\Omega)$. In the discontinuous case, those local basis functions

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.1}
\caption{Reference triangle.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.2}
\caption{Nodal basis functions of continuous (top row) and discontinuous (bottom row) piecewise linear functions in the one-dimensional case.}
\end{figure}
could already be used as global basis functions $\psi_i, i = 1, \ldots, \dim D_1(\Omega)$. Figure 4.2 illustrates this contrast in the one-dimensional case.

There is a significant advantage to the usage of basis functions with support in only one triangle. Those can be orthonormalized very easily. The local basis functions are available for various polynomial degrees and are hierarchical in the sense that e.g. the basis for the quadratic case includes the basis functions of the linear case and the linear case includes the basis function of the piecewise constant case, see e.g. [31]. For piecewise linear elements the orthonormal basis functions on the reference triangle $\hat{K}$ are given by

$$
\varphi_0^\perp(x, y) := \sqrt{2},
\varphi_1^\perp(x, y) := 2 - 6x,
\varphi_2^\perp(x, y) := 2\sqrt{3}(1 - x - 2y).
$$

For this thesis, the orthonormalized basis functions were used to implement the discontinuous piecewise linear finite elements. In some instances, software developers might like to think about (element-wise) nodal basis functions as they are quite prominent and easy to understand visually. Hence, conversion routines between the two different types of degrees of freedom have been included in the code. In detail, the conversion matrix from degrees of freedom of orthonormal basis functions to degrees of freedom of nodal basis functions is given by

$$
\begin{pmatrix}
\sqrt{2} & -4 & 0 \\
\sqrt{2} & 2 & -2\sqrt{3} \\
\sqrt{2} & 2 & 2\sqrt{3}
\end{pmatrix}.
$$

The latter is just a list of the function values of orthonormal basis functions $\varphi_i^\perp(x, y), i = 1, \ldots, 3$, on the reference triangle $\hat{K}$ at its corner points $(x, y) = (1, 0), (0, 1)$ or $(0, 0)$, respectively. Evidently this matrix is invertible.

The global degrees of freedom of $D_1(\Omega)$ are numbered triangle-wise, i.e. for $1 \leq i \leq M(\Omega)$ (recall that $M(\Omega)$ is the number of triangles (4.6)) the degrees of freedom $\sigma_{3i-2}, \ldots, \sigma_{3i}$ belong to the 3 basis functions of triangle $K_i$. Explicit representations of those degrees of freedom can be obtained via combining nodal function evaluation with the inverse of the conversion matrix in (4.128). Those degrees of freedom are associated to the mesh elements but are independent of the mesh nodes which are used to keep records of the degrees of freedom in the conforming case. For triangulations in the two-dimensional case, the dimension of discontinuous piecewise linear elements is

$$\dim D_1(\Omega) = 3M(\Omega),$$

which typically is a lot larger than $\dim P_1(\Omega)$ or even $\dim P_2(\Omega)$ where vertices and edge midpoints of triangles are shared by multiple triangles. As an example, in the simulation [GW1] of 'model GW' in Section 5 at time $t = 1.5$ with its current adaptive mesh,

$$\dim P_2(\Omega) = 75817,$n\dim D_1(\Omega) = 112764,$

according to the output of our inhouse software. The amount of degrees of freedom of the piecewise linear elements is negligible and the piecewise quadratic elements of $R$ are only scalar-valued, while $m$ and $u$ are vector-valued. In total, the magnetization used up approximately 44% of all degrees of freedom and the magnetic variables ($m$ and $R$, while ignoring the auxiliary quantities $\nabla_h m$ and $\nabla_h R$) used up approximately 59%. However, the memory usage of our simulations was of no concern on our typical office machines.
Assembly. Discontinuous Galerkin methods have gained quite much popularity and attention. On the first glance, it seems crucial for such methods to have easy access to information of the surface of simplices, i.e. edges in the two-dimensional setting, due to jumps or averages on interior faces (cf. (4.30)). However, it is common practice to compute the jumps and averages on surfaces based on information per volume – if possible. Advantages of a volume-based assembly are the simplification of

- the implementation, as volume-based routines play a fundamental role in finite element methods and are readily available,
- mesh partitioning which can be used for parallelization (see e.g. domain decomposition methods).

In case of 'model GW', discontinuous elements have been used to discretize the magnetization equation, see (4.37f), such that in various parts of the discrete scheme the trilinear form $b_h$ appeared which deals with jumps and mean values along interior edges of the triangulation, see (4.30). For convenience, the definition

$$b_h : \mathcal{P}_2(\Omega)^d \times \mathcal{M}_h \times \mathcal{M}_h \rightarrow \mathbb{R},$$

$$b_h(u, h, m) := \sum_{\xi \in T(\Omega)} \int_{\xi} (u \cdot \nabla)h \cdot m \, dx - \int_{\mathcal{F}_{int}} [h] \cdot \{m\}(u \cdot \nu_{\text{int}}) \, d\sigma$$

will be repeated.

Next, let us discuss how the integral over the interior faces (last term above) can be rewritten in terms of triangle-wise information. Actually, the above discretization comes from integration by parts triangle. Assume formally that $u, h, m$ are sufficiently smooth, for instance let $\nabla m$ exist globally. Further assume $\text{div} \ u = 0$ and $u|_{\partial \Omega} = 0$. Hence, the term $- \int_{\Omega} (u \cdot \nabla)m \cdot h \, dx$ is well-defined and serves as a starting point to obtain a discretization of

$$\int_{\Omega} (m \cdot \nabla)h \cdot u \, dx = \int_{\Omega} (u \cdot \nabla)h \cdot m \, dx$$

which represents the Kelvin force tested by $u$. Compute via integration by parts

$$\int_{\Omega} (u \cdot \nabla)m \cdot h \, dx = \sum_{\xi \in T_h(\Omega)} \int_{\xi} (u \cdot \nabla)h \cdot m \, dx - \sum_{\xi \in T_h(\Omega)} \int_{\partial \xi} (u \cdot \nu)(m \cdot h) \, d\sigma. \quad (4.129)$$

Due to zero boundary conditions of $u$ the only relevant boundary parts are the interior edges. Let $E$ be the common edge of two triangles $\xi^+$ and $\xi^-$ with outer unit normals $\nu^+$ and $\nu^- = -\nu^+$. Without loss of generality let $\nu_{\text{int}}$ from (4.11) be chosen to coincide with $\nu^+$ on $E$. Let quantities with upper index "+" denote the restriction onto $\xi^+$, e.g. $h^+ = h|_{\xi^+}$ and let the upper index "-" denote the analogous restriction onto $\xi^-$. Then $h^+|_{E}$ denotes the trace of $h$ from the view of $\xi^+$ which in general might differ from the the trace of $h^-$ for e.g. discontinuous $h$ (so far $h \in D_1(\Omega)^d$ is sufficient for the terms in (4.129) to be well-defined). Assume $m$ to be continuous on the interior edge $E$. The boundary integrals that contribute to an integral over the edge $E$ are the two summands

$$\int_{\partial \xi^+} (u \cdot \nu)(m \cdot h) \, d\sigma \quad \text{and} \quad \int_{\partial \xi^-} (u \cdot \nu)(m \cdot h) \, d\sigma$$

which yield on $E = \partial \xi^+ \cap \partial \xi^-$ the sum of integrands

$$(u^+ \cdot \nu^+)|_{E}(m^+ \cdot h^+)|_{E} + (u^- \cdot \nu^-)|_{E}(m^- \cdot h^-)|_{E}$$

$$= (u \cdot \nu^+)|_{E}(m \cdot h^+)|_{E} - (u \cdot \nu^+)|_{E}(m \cdot h^-)|_{E} = (u \cdot \nu_{\text{int}})|_{E}(m \cdot [h])|_{E}.$$
When choosing \( \mathbf{u} \in \mathcal{U}_h, \ h \in \mathcal{M}_h \) and \( \mathbf{m} \in \mathcal{M}_h \), only one adaption must be made in order to obtain a well-defined expression. Using the mean value \( \{ \mathbf{m} \} \) on \( E \) as approximation in case of discontinuous \( \mathbf{m} \), one obtains the formula for \( b^m_h \) above (or see (4.30)). Indeed due to its origin from boundary integrals of triangles, the formula can be completely rewritten in terms of boundary integrals of triangles. Rewrite

\[
\int_E [\mathbf{h}] \cdot \{ \mathbf{m} \} (\mathbf{u} \cdot \nu) \, d\sigma = \int_E (\mathbf{h}^+ - \mathbf{h}^-) \cdot \frac{1}{2} \mathbf{m} (\mathbf{u} \cdot \nu) \, d\sigma
\]

\[
= \frac{1}{2} \int_E \mathbf{h}^+ \cdot (\mathbf{m}^+ + \mathbf{m}^-) (\mathbf{u} \cdot \nu^+) + \frac{1}{2} \int_E \mathbf{h}^- \cdot (\mathbf{m}^+ + \mathbf{m}^-) (\mathbf{u} \cdot \nu^-) \, d\sigma
\]

\[
= \frac{1}{2} \int_{\partial \mathcal{X}}^+ \mathbb{1}_E \mathbf{h}^+ \cdot (\mathbf{m}^+ + \mathbf{m}^-) (\mathbf{u} \cdot \nu^+) \, d\sigma + \frac{1}{2} \int_{\partial \mathcal{X}}^- \mathbb{1}_E \mathbf{h}^- \cdot (\mathbf{m}^+ + \mathbf{m}^-) (\mathbf{u} \cdot \nu^-) \, d\sigma,
\]

where \( \mathbb{1}_E \) denotes the characteristic function of \( E \). Each interior edge belongs to exactly two triangles. Let \( \mathbf{m}^\Omega(\mathcal{X}) : \partial \mathcal{X} \to \mathbb{R}^d \) denote the traces of \( \mathbf{m} \) in view of the neighbor triangles of \( \mathcal{X} \) on the common edges they have with \( \mathcal{X} \). Summing up over all edges and recalling \( \mathbf{u}|_{\partial \Omega} = 0 \) yields

\[
\int_{\mathcal{F}_{\text{int}}} [\mathbf{h}] \cdot \{ \mathbf{m} \} (\mathbf{u} \cdot \nu) \, d\sigma = \frac{1}{2} \sum_{\mathcal{X} \in \mathcal{K}(\Omega)} \int_{\partial \mathcal{X}} (\mathbf{h} \cdot (\mathbf{m}^\Omega(\mathcal{X})) (\mathbf{u} \cdot \nu) \, d\sigma.
\]

The data of neighbor triangles are easily accessible as they are already stored for the sake of local refinement strategies. Now, the volume-based assembly is feasible.

For simplification, the assembly of the trilinear form \( b^m_h \) is done with the help of the magnetic field \( \mathbf{h}_{|\mathbb{P}} \in \mathcal{D}_1(\mathbb{P}) = \mathcal{M}_h \) directly instead of using \( \nabla R_{|\mathbb{P}} \) which has different degrees of freedom and involves a gradient. The degrees of freedom of the magnetic field \( \mathbf{h} = \nabla R \in \mathcal{D}_1(\mathbb{P}) \) for some \( R \in \mathcal{R}_h \) are obtained by solving the variational equation

\[
\int_{\Omega'} \mathbf{h} \cdot \mathbf{n} \, d\mathbf{x} = \int_{\Omega'} \nabla R \cdot \mathbf{n} \, d\mathbf{x} \quad \forall \mathbf{n} \in \mathcal{D}_1(\mathbb{P}).
\]

Due to orthogonality of basis functions the mass matrix corresponding to the left-hand side above is diagonal and easily invertible. Hence, this approach is easy to implement and efficient.

Furthermore, the assembly of discretization terms is based on precomputed values, i.e. integral values of certain combinations of basis functions on a reference triangle. For example, the mass matrix \( M \) of continuous piecewise linear elements,

\[
(M_{ij})_{i,j=1,\ldots,\text{dim} \mathcal{P}_1(\mathbb{P})} := \int_{\Omega} \phi_i \phi_j \, d\mathbf{x},
\]

where \( \phi_1, \ldots, \phi_{\text{dim} \mathcal{P}_1(\mathbb{P})} \) are the corresponding nodal basis functions of \( \mathcal{P}_1(\mathbb{P}) \), is assembled elementwise with the help of linear transformation from the \( 3 \cdot 3 = 9 \) precomputed integrals

\[
\int_{\mathcal{K}} \varphi_i \varphi_j \, d\mathbf{x}, \quad i, j = 1, \ldots, 3,
\]

on the reference triangle \( \mathcal{K} \), where \( \varphi_1, \varphi_2, \varphi_3 \) are the nodal basis functions on \( \mathcal{K} \) as in (4.127). As generalization, in the vector-valued case the basis functions would be grouped into \( d \) sets where the \( i \text{th} \) group contains the respective scalar valued basis functions in their \( i \text{th} \) component and 0 in their other components. This way, the vector-valued case boils down to the scalar-valued case.
In the case where neighbor information is needed there are actually quite a lot additional local integral values needed in contrast to the conforming case. While the assembly of integrals that do not feature boundary-terms with jumps or mean values get along with their own local degrees of freedom, in the non-conforming case the local degrees of freedom of the neighboring triangles play a role, too. In Figure 4.3 the possible scenarios two triangles with common edge can go through are depicted and all of them need to be considered individually to obtain the correct analytical values needed for the assembly. For the ease of implementation, no case is neglected by symmetry arguments which are difficult to come by as the basis functions generally all have different traces on each edge. Also take into account that in some cases the neighboring triangle needs to be scaled in size – further complicating symmetry arguments. Hence, a tensor storing values of boundary integrals in the two-dimensional case, which depend on neighbor information, stores nine times as many values as in the case where neighbor triangles are irrelevant. The local tensors needed for the numerical scheme (4.37) of this thesis have been computed with the help of the computer algebra system 'Maple'.

Figure 4.3: The nine neighborhood configurations of triangles with common edge at the example of the reference triangle $\hat{K}$.

needs to be scaled in size – further complicating symmetry arguments. Hence, a tensor storing values of boundary integrals in the two-dimensional case, which depend on neighbor information, stores nine times as many values as in the case where neighbor triangles are irrelevant. The local tensors needed for the numerical scheme (4.37) of this thesis have been computed with the help of the computer algebra system 'Maple'.

4.3.2 Solving nonlinear equations

The nonlinear equations of the numerical schemes are solved via a fixed point iteration, i.e. they are being linearized and the solution of the linearized scheme is used as input for the next iteration until no significant change can be detected anymore.

In case of 'model GW' the scheme is linearized almost identically to (4.107). The only difference is that (4.107c) and (4.107d) remain coupled by replacing $\hat{m}$ with $m^k$ in both (4.107c) and the second term on the right-hand side of (4.131e), and by replacing $\hat{h}$ with $h^k = \nabla R^k$ in (4.131c). Moreover, the solution of (4.107a) is based on the original formulation in (4.37a), i.e. the weak incompressibility condition in $U_h$ is taken as individual equation and the pressure term from (4.37a) is restored. Hence, for a fixed time step $k \in \{1,...,n_T\}$ let

\[
\mathcal{L}^{GW}: U_h \times \mathcal{P}_h \times \mathcal{C}_h \times \mathcal{R}_h \times \mathcal{M}_h \rightarrow U_h \times \mathcal{P}_h \times \mathcal{C}_h \times \mathcal{R}_h \times \mathcal{M}_h,
\]

\[
(u^k, P^k, c^k, R^k, m^k),
\]
where \((u^k, P^k, c^k, R^k, m^k)\) is the solution of the linearized system of equations
\[
\rho_0 \int_\Omega \left( \frac{u^k-u^{k-1}}{\tau} \cdot \nu \, dx - \int_{\Omega'} P^k \, div \, v \, dx + 2\eta \int_\Omega D u^k : Dv \, dx \right. \\
+ \left. \frac{\rho_0}{2} \int_\Omega (u^{k-1} \cdot \nu) \hat{\nu} \cdot \nu \, dx - \frac{\rho_0}{2} \int_{\Omega'} (u^{k-1} \cdot \nu) \nu \cdot \hat{\nu} \, dx \right)
= -D \int_\Omega e^{k-1} \nabla \hat{q}_{\rho,h} \cdot v \, dx + \mu_0 b^m_h(v, (\alpha_1 \hat{h} |_{\Omega} + \frac{\beta}{2} (h|_{\Omega}^k |_{\Omega} - \alpha_3 \hat{m}), \hat{m}) \\
+ \frac{\mu_0}{2} \int_\Omega (\hat{m} \times (\alpha_1 \hat{h} + \frac{\beta}{2} (h|_{\Omega}^k))) \cdot curl \, v \, dx,
\]
\[
\int_\Omega \nabla \hat{u} \cdot \n abla \psi \, dx = \int_\Omega e^{k-1} \hat{u} \cdot \nabla \psi \, dx + \int_{\Omega'} \hat{I}_{h,1}(e^{k-1} \hat{v}_{\text{part}} \cdot \nabla \psi) \, dx,
\]
\[
\int_{\Omega'} \nabla R^k \cdot \nabla S \, dx = \int_{\Omega'} (h|_{\Omega})^k \cdot \nabla S \, dx - \int_\Omega m^k \cdot \nabla S \, dx,
\]
\[
\int_\Omega \frac{(m^k-m^{k-1})}{\tau} \cdot n \, dx - b^m_h(\hat{u}, \hat{m}, \hat{m}) - b^m_h(\hat{v}_{\text{part}}, \hat{n}, \hat{m}) \\
= \frac{1}{2} \int_\Omega (\nabla \hat{u} \times \hat{m}) \cdot n \, dx - \frac{1}{\tau_{rel}} \int_\Omega (m^k - \hat{x}_{r,h} \nabla R^k) \cdot n \, dx \\
- \sigma \int_\Omega \nabla \hat{u} \cdot \hat{m} \cdot n \, dx + \sigma \int_\Omega \nabla \hat{u} \cdot \hat{m} \cdot n \, dx
\]
for all \(v \in U_h, q \in P_h, \psi \in C_h, S \in R_h \) and \(n \in M_h\), where \(\hat{v}_{\text{part}}\) is defined as in Section 4.2.1. Note that in practical simulations, we do not use mean value free finite element functions, but use, for instance in case of the pressure \(P^k\), the space \(P_1(\Omega)\) and adjust the mean value afterwards. The same is done for the magnetic potential \(R^k\). There are practical reasons for the slight modification regarding magnetism in contrast to \(L^{GW}\) from (4.108). We experienced that in some situations higher precision at solving the magnetic subsystem might yield more robustness and faster convergence rate of the fixed point iteration. However, there is no particular reason to avoid the original linearized system. Regarding well-posedness, notice that the system is decoupled into subsystems. The Navier-Stokes system is treated in standard literature. The particle density equation is uniquely solvable due to invertibility of the mass matrix (also recall the well-posed definition of \(\hat{v}_{\text{part}}\) from (4.106)). The magnetic subsystem was considered in [51]. Note that this modified subsystem requires sufficiently small susceptibility \(\|\hat{x}_{r,h}\|_{L^\infty(\Omega)} < 4\) or, alternatively, sufficiently small time increments. The simulations presented in Section 5 all meet the former requirement of small susceptibilities. Based on this linearization, the fixed point scheme for "model GW" is as follows.

Let data from the previous time step, or initial data in the first time step (also recall the definition of an initial magnetic field (4.35)), \(u^{k-1} \in U_h, c^{k-1} \in C_h, m^{k-1} \in M_h, R^{k-1} \in R_h\) be given. Set
\[
u^k := u^{k-1}, \quad c^0 := c^{k-1}, \quad m^0 := m^{k-1}, \quad R^0 := R^{k-1}
\]
and compute the \(l\)th iteration, \(l \geq 1\), as solution of the linearized system
\[
(u^l, c^l, R^l, m^l) := \hat{L}^{GW}(u^l_{l-1}, c^l_{l-1}, R^l_{l-1}, m^l_{l-1})
\]
with the help of a direct solver. The first index \(l_0 \geq 1\) such that for some arbitrary tolerance \(\varepsilon > 0\) and norm \(\| \cdot \|_k\), which is possibly different for each time step \(k \in \{1, \ldots, n_T\}\), the estimate
\[
\|(u^k_{l_0}, c^k_{l_0}, R^k_{l_0}, m^k_{l_0}) - (u^k_{l_0-1}, c^k_{l_0-1}, R^k_{l_0-1}, m^k_{l_0-1})\|_k < \varepsilon
\]
holds, terminates the iteration and

$$(u^k, c^k, R^k, m^k) := (u^k_{t_0}, c^k_{t_0}, R^k_{t_0}, m^k_{t_0})$$

is taken as solution of the current time step.

For the stopping criterion of the fixed point iteration, each $l^2$-residuum of the individual quantities and additionally the $l^2$-residuum of $h^k = \nabla R^k$ are computed and divided by the respective dimension (number of degrees of freedom) of the underlying finite element space – turning them into relative residua. Then the $l^\infty$-norm is applied to the tuple of individual residua but the residuum of the magnetic potential is weighted by a factor of $10^4$ to improve accuracy of the magnetic field (which is the gradient of the potential). The tolerance is chosen to be $\varepsilon = 10^{-12}$ for the simulations in Section 5.

If the finite volume discretization (4.126) was used in exchange for (4.131c), the velocities $u^k$ and $v^k_{part}$ are to be replaced with $\hat{u}$ or $v_{part}$, respectively. The other discretizations (for "model W" and "model B") are treated analogously based on their original linearization from Section 4.2. The linearized systems were solved with the direct solver 'MUMPS' [2].

### 4.3.3 Adaptivity in space and time

For simplicity, consider the two-dimensional case. The inhouse-code features local mesh refinement by halving triangles. If the local refinement creates hanging nodes on some triangles, the latter will be refined as well. This will be repeated until there are no more hanging nodes. The mesh is regular in the sense that all triangles are similar to the reference triangle, i.e. they have a 90 degrees angle with catheti of equal length. A snapshot of a locally refined mesh is given in Figure 4.4. The mesh refinement strategy is motivated as follows.

- Generally, the simulation of convection-dominant equations requires some smallness condition on the time increments in relation to the mesh size, see the CFL-condition [45]. Yet, if one reduces the time increment too much, the scheme’s numerical diffusion increases which can be fixed to some extent with limiters (as used in the mixed finite elements/finite volumes scheme of [39]). The difficulty to find a suitable time increment for adaptive meshes lies in the different sizes of the triangles. During our research it turned out that the specific schemes of this thesis are more robust if one creates a mesh which is uniform in the proximity of the magnetic particles' support. Therefore, the transport of mass occurs on a uniformly refined portion of the mesh which makes it easier to find suitable time increments.

- An earlier work on ferrofluids, see [51], shows the significance of the stray field which is induced by the magnetic particles. Therefore it might be crucial to resolve the stray field also at the boundary of the magnetic particles' support. This can be achieved by refining a neighborhood around the support of the magnetic particles.

![Figure 4.4: Trimmed image of the adaptive triangular mesh (black lines). The gray colored region indicates $c \geq 0.001 \max_{\Omega} c$, where $c$ is the initial data from the simulations in Section 5.](image-url)
Large values and large changes of magnetization field and stray field should not occur far away from the magnetic particles’ support. Therefore, certain neighborhoods of the current support of $c^{k-1}$ will be refined only. More precisely, let $d_{i,j}$, $i, j = 1, \ldots, \dim \mathcal{P}_1(\Omega)$ formally be the pseudo-distances

$$d_{i,j} := \begin{cases} \|x_i - x_j\|_{\infty}, & \text{if } c^{k-1}(x_j) > \delta_{\text{rel}} \max_{l=1, \ldots, \dim \mathcal{P}_1(\Omega)} |c^{k-1}(x_l)| \text{ and } c^{k-1}(x_j) > \delta_{\text{abs}}, \\ \infty, & \text{else}, \end{cases}$$

between nodes $i$ and $j$ which are located at the points $x_i$ or $x_j$, respectively. Observe that $d_{i,j}$ is infinite if node $j$ is not contained in the 'discrete' support of the particle density function – defined by absolute and relative thresholds $\delta_{\text{abs}}$ and $\delta_{\text{rel}}$. Define

$$\hat{d}_i := \min_{j=1, \ldots, \dim \mathcal{P}_1(\Omega)} d_{i,j}.$$ 

Then a node $i$ gets marked for refinement if $\hat{d}_i < C_d$, where $C_d$ controls the distance of the neighborhood from the 'discrete' support of the particles. The values

$$\delta_{\text{abs}} = 5 \cdot 10^{-5}, \quad \delta_{\text{rel}} = 1 \cdot 10^{-3} \quad \text{and} \quad C_d = \frac{1}{\sqrt{2}} h$$

have been chosen for the simulations in Section 5 but they can be chosen arbitrarily in general.

The preceding adaptivity scheme refines a neighborhood of the support of the particle density. Additionally, the support itself (as determined by $\delta_{\text{abs}}$ and $\delta_{\text{rel}}$ above) will be refined as well as the elements with nodes on the boundary of the simulation domain.

The concept of time adaptivity is based on a simple heuristic approach which differs from the more complicated ansatz in [39]. The simple choice to be proposed below prevents chaotic changes of the time increments. Therefore the evolution of time increments will more likely be similar for all models and the comparability of simulation results is improved.

Let $\tau_{\max} > \tau_{\min} > 0$ be given parameters that define the range of possible time increments. The time adaptive scheme starts with a time increment $\tau_0 \in [\tau_{\min}, \tau_{\max}]$. The time increment will be recalculated in every step as follows.

If the fixed point iteration described in Section 4.3.2

- has terminated successfully within 10 iteration steps for 10 consecutive time steps, then multiply the time increment by 2.
- If it has not terminated after 10 iteration steps, then immediately multiply the time increment by 0.5 and repeat the current time step. If the new time increment becomes less than $\tau_{\min}$, the simulation will abort.
- Otherwise, keep the time increment unchanged and proceed.

The idea behind this scheme is that numerical instabilities in regards to convection-dominant problems often show oscillations which quickly lead to blow-ups during the nonlinear solver (fixed point iteration in this case) or at least to an increased number of iteration steps. Hence, the amount of iterations can be used to detect instabilities. However, this method is not guaranteed to work well in every case. We have encountered cases where oscillations smoothened out shortly after appearing without causing blow-ups. Suitable choices of the range $[\tau_{\min}, \tau_{\max}]$ affect the stability of the scheme positively.
It turned out that for a careful choice of parameters the approach above worked quite well for the simpler models "model W" and "model B", see Section 5.1.2. In case of "model GW", relatively small time increments were needed for the simulation at larger times, accompanied by a suitably small $\tau_{\text{max}}$, such that the computational overhead of above time adaptivity scheme was unnecessarily high in the beginning of the simulation. This could be prevented by manually changing the parameters $0 < \tau_{\text{min}} < \tau_{\text{max}}$ over the course of time.

Be aware that neither the time adaptivity scheme from [39] nor standard schemes based on the CFL-condition [45] are guaranteed to work well for the simulations in this thesis. The main difficulty of solving the evolution equation of the magnetic particle density (4.37c) is the fact that the convective velocity $v^k_{\text{part}}$ is an unknown quantity – non-constant in space and time – and nonlinearly coupled to other equations. To the best of our knowledge, there is no combination of time adaptivity scheme and nonlinear solver which has been proven to work well in every numerical experiment under such circumstances.

The parameters for the simulations in Section 5 will partly be specified here. The parameter $\tau_{\text{min}} := 10^{-7}$ prevents infinite loops and the starting time increment will be chosen identical to the maximum increment, i.e. $\tau_0 := \tau_{\text{max}}$. However, the choice of $\tau_{\text{min}} > 0$ depends on the situation and will be stated during each of the presentations of the simulation results.
5 Simulations - proof of concept

The field-induced transport of magnetic particles will be examined through a set of simulations as presented in this section. They serve as a proof of concept for the three models "model GW" [2.89], "model W" [2.100] and "model B" [2.110]. Moreover, those three models will be compared in a fixed setting which is similar to the scenario in [39]. For convenience, the upcoming simulations are listed here.

(GW1) Simulation of "model GW" with \( f_p(c) := c \), which induces linear diffusion. A rather homogeneous magnetic field will be considered in the sense that its field lines tend to be parallel – however the magnitude is inhomogeneous. The domain of the magnetic field has been chosen to be identical to the domain of the fluid, i.e. \( \Omega = \Omega' \).

(GW2) Simulation of "model GW" with \( f_p(c) := c^2 \) as it was used in Section 3.4 to show existence of weak solutions. The other details coincide with the setting in (GW1).

(W1) Simulation of "model W" in the same scenario as in (GW1).

(W1') The same scenario as in (W1) but the magnetic field is discretized on a larger domain \( \Omega' \supset \supset \Omega \) than the fluid domain. Thus, the two cases \( \Omega \subset \subset \Omega' \) and \( \Omega = \Omega' \) will be compared.

(W2) Simulation of "model W" in the same scenario as in (GW2).

(B1) Simulation of "model B" in the same scenario as in (GW1).

For convenience, the above labels are a combination of the model name and the exponent in \( f_p(c) = c^m \), \( m \in \{1, 2\} \). The apostrophe ‘ denotes the case of \( \Omega' \) being strictly larger than \( \Omega \).

After the comparison, additional simulations will follow including the following features which distinguish the individual scenarios from each other.

(A1) The analogous scenario as in (W1) will be examined but with two different inhomogeneous external magnetic fields.

(A2) The analogous scenario as in (W1) but the external magnetic field is induced by two sources on opposite sides of the domain.

The latter simulations were performed using "model W" in order to save computational costs.

In general, the simulation setup involves a magnetic field generated by a dipole to the right of the simulation domain. The initial distribution of the magnetic particles is a cluster in the middle of \( \Omega \). In order to emphasize the magnetic contribution to the simulation, gravitational forces are neglected. A sketch of the scenario is given in Figure 5.1. In the following, details on discretization parameters, modeling parameters and regularization parameters will be summarized.

![Figure 5.1: Sketch of simulation setup. Black color indicates large density values c, white means c = 0. Blue field lines indicate the applied field \( h_a \) which is induced by a dipole, symbolized by a green/red bar magnet.](image)
Decomposition of computation domain and time interval. The fluid domain is given by

\[ \Omega = (0,1)^2. \]

The domain of the magnetic field is chosen to be identical, i.e. \( \Omega' = \Omega \), unless otherwise stated. In case of \( \Omega \subset \subset \Omega' \) the larger domain is given by \((-0.3,1.3)^2\).

Recall the adaptivity in space and time from Section 4.3.3. The root triangulation of \( \Omega' \) consists of multiple squares – in a \( 10 \times 10 \) grid if \( \Omega = \Omega' \) or otherwise in a \( 16 \times 16 \) grid – which contain exactly two right-angled triangles each and will be refined at least 4 times. During the simulation the mesh can be refined locally up to 4 additional times (8 times in total), where one step of refinement corresponds to halving the respective triangles once. With those levels of refinement, the length of the catheti of the right-angled triangles is in the range \([0.00625,0.025]\) and the mesh size (given by the hypotenuse of the triangles) is in the range

\[ h \in [0.0088,0.0354]. \]

The time domain is \( I = [0,T] \), where \( T > 0 \) depends on the individual simulation and will be mentioned when needed. The time adaptivity algorithm – see also Section 4.3.3 – uses different maximum time increments for the different simulations and will be mentioned before presenting the respective simulation results. The respective choices for \( \tau_{\text{max}} \) are based on computational experience.

External magnetic field. The applied magnetic field will be modeled via superposition of dipole fields. Let \( n_a \in \mathbb{N} \) be the number of dipoles to be used. Consider the dipole potentials

\[ \phi_{a_i}(\mathbf{x}) := C_{a_i} \frac{\mathbf{d}_{a_i} \cdot (\mathbf{x}_{a_i} - \mathbf{x})}{\| \mathbf{x}_{a_i} - \mathbf{x} \|^2}, \quad \text{for } \mathbf{x} \neq \mathbf{x}_{a_i}, \quad i = 1, \ldots, n_a \]

for \( \mathbf{x} \in \Omega' \) with directors \( \mathbf{d}_{a_i} \) and locations at \( \mathbf{x}_{a_i} \in \mathbb{R}^2 \setminus \Omega' \). The intensity of a dipole is scaled by the coefficients \( C_{a_i} \geq 0, \ i = 1, \ldots, n_a \). As the functions \( \phi_{a_i} \) are harmonic on \( \mathbb{R}^2 \setminus \{ \mathbf{x}_{a_i} \} \), where \( \mathbf{x}_{a_i} \notin \Omega' \), the choice

\[ \mathbf{h}_a(\mathbf{x},t) = \alpha_{\text{intensity}}(t) \sum_{i=1}^{n_a} \nabla \phi_{a_i}(\mathbf{x}) \]

with the bounded evolution of the intensity

\[ \alpha_{\text{intensity}}(t) = \min(\max(0, \frac{t}{0.01}), 1) \]

is consistent with the requirements of Section 2. In the upcoming presentation of simulation results, multiple different magnetic fields will be used. The details will be specified before the presentation of the respective simulation results.

More data and regularization parameters. The generic parameters of the magnetic energy (2.59a) in case of ’model GW’ are chosen as

\[ (\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (1, 1, 1, 0.01). \]

Hence, the magnetic energy is given by

\[ E_{\text{mag}} = \frac{\mu_0}{2} \int_{\Omega'} |\mathbf{h}|^2 \, d\mathbf{x} + 0.01 \frac{\mu_0}{2} \int_{\Omega} |\mathbf{m}|^2 \, d\mathbf{x} \]
and the Kelvin force (of the continuous setting) reads \( μ_0 (m \cdot \nabla) h - 0.001 μ_0 (\nabla m)^T m \) with the second term being a gradient – thus it could be absorbed in the pressure gradient in the momentum equation. For the particle mobility function \( f_p \) take
\[
f_p(c) = c^m
\]
with \( m \in \{1, 2\} \). The specific choice of \( m \) will be listed per simulation. Choose the cut-off parameter
\[
s = 10^{-3}
\]
which affects \( g_s, (\cdot)_s, f^{(s), k-1}_{p,h}, f^{(s), k-1}_{p,h} \) (cf. "model GW") and \( \chi_s \) (cf. "model W"). Here, \( \omega = -0.5 \), i.e. \( s = (10^{-3}, -0.5) \) (cf. (4.46)). For simplicity, "model GW" will be simulated based on the linear susceptibility from (2.34) which will be cut off from above by a threshold parameter \( r = 10^4 \), resulting in
\[
\chi_r(c) = \min(\max(0,c)\chi_0,r)
\]
for some \( \chi_0 \geq 0 \) to be specified later. In case the susceptibility does not depend on the magnetic field \( h \) (see (2.32)), the estimates in (3.4) are not available. Nevertheless, the analytical and numerical results can be achieved for a susceptibility as above as well, as can e.g. be seen from the a priori estimate in [40] where the simpler condition of bounded susceptibility has been used – in contrast to the square root like growth from (3.4) or (N2 4.2), respectively.

**Initial data.** Initial data for the particle density is
\[
c^{\text{init}}(x) := \begin{cases} 
\frac{1}{2} \left( \cos \left( \frac{|x-y|^2}{r^2} \pi \right) + 1 \right) & \text{if } |x-y|^2 \leq 1, \\
0 & \text{else}, 
\end{cases}
\]
with \( y := (0.5, 0.5) \), modeling a particle distribution of circular shape with vanishing derivative at the boundary of its support. Initial data for velocity field and magnetization are taken to be zero, i.e.
\[
u^{\text{init}} := 0 \quad \text{and} \quad m^{\text{init}} := 0.
\]
The discrete initial data is obtained by Lagrangian interpolation, i.e.
\[
u^0 \equiv 0, \quad c^0 \equiv I_{h,1}(c^{\text{init}}), \quad m^0 \equiv 0.
\]

**Model parameters.** Table 5.1 gives an overview of the model parameters. They are partially inspired by a realistic scenario of magnetic nanoparticles being suspended in water. However, note that the initial density distribution and the external magnetic field are purely artificial. The actual diffusivity of coated particles (see (2.4) and [71]) with magnetic properties is difficult to determine. However, mobility is very high and diffusivity is very low. The choices in Table 5.1 take this fact into consideration while also facilitating practical simulations. The susceptibility at time \( t = 0 \) lies in the range \([0, \chi_0 \max_\Omega c^{\text{init}}] = [0, 1]\), which is realistic for commercial ferrofluids.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( \rho_0 )</th>
<th>( \eta )</th>
<th>( \chi_0 )</th>
<th>( \mu_0 )</th>
<th>( \sigma )</th>
<th>( \tau_{\text{rel}} )</th>
<th>( K )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values (in SI base units)</td>
<td>1000</td>
<td>1</td>
<td>1</td>
<td>( 4\pi \cdot 10^{-7} )</td>
<td>0.01</td>
<td>( 10^{-9} )</td>
<td>500</td>
<td>( 10^{-6} )</td>
</tr>
</tbody>
</table>

**Table 5.1:** Choice of model parameters used in the computations. The dimension of \( \chi_0 \) depends on the type of (possibly rescaled) particle density \( c \), such that \( \chi_0 c \) is dimensionless.
Visualization. The particle density and the magnetic quantities will be visualized via the open-source software "paraview" in the following manner.

i) The particle density \( c \) will be visualized by black color levels – the linear interpolation between pure black, indicating the current maximum, and pure white, indicating the absence of particles (values less than or equal to zero). Additionally, contour lines of \( c \) are presented in gray colors, where dark gray indicates less and light gray more than 50% of the current maximum. There are only two color levels except of a linear interpolation in a narrow range around the 50% mark. Blue lines depict the field lines of the total magnetic field \( h \).

Figure 5.2: Sample visualization of density \( c \) and total magnetic field \( h \).

ii) The particle density will be visualized by black color levels as in i). The stray field \( h_d \) is represented by arrows of normalized length to improve visibility of arrows corresponding to small magnitudes of \( h_d \). Here, brighter color means larger magnitude. Dark blue arrows correspond to very low magnitudes which are predominant outside the particle density’s support. The magnetization \( m \) is illustrated by red/green cylinders – mimicking bar magnets – whose sizes are scaled according to the magnetization’s magnitude.

Figure 5.3: Sample visualization of density \( c \), magnetization \( m \) and stray field \( h_d \).

Wherever needed, additional visualization techniques will be introduced.

5.1 Model comparison in fixed scenarios

In most simulations the fluid domain and the magnetic field domain coincide, i.e. \( \Omega = \Omega' = (0, 1)^2 \). A simulation with \( \Omega' = (-0.3, 1.3)^2 \) will only be presented using "model W". The maximum time increment of the time adaptivity scheme (cf. Section 4.3.3) is specified per simulation. The applied magnetic field is induced by a single dipole (5.1) determined via the parameters

\[
\begin{align*}
d_{a1} &= \begin{pmatrix} -1 \\ 0 \end{pmatrix}, & C_{a1} &= 100, & x_{a1} &= \begin{pmatrix} 3 \\ 0.5 \end{pmatrix}.
\end{align*}
\] (5.3)

As the dipole’s position is rather far away from the simulation domain, the field lines of the applied magnetic field are only slightly curved – looking similar to those of a homogeneous field. However, the magnitude of the applied magnetic field decreases with growing distance from the dipole. As magnetic monopoles do not exist, i.e. the attractive or repulsive forces acting on the different poles, respectively, might level out, this decrease is crucial for magnetic attraction. In contrast, electric monopoles do exist and will be attracted even in truly homogeneous electric fields.

Let us comment on the significance of the magnetic field’s direction. The magnetic force mostly depends on the change of the absolute value of the magnetic field (cf. e.g. [41]),

\[
\mathbf{m} \approx \chi_{\text{lin}}(c) \mathbf{h}, \quad \text{curl} \mathbf{h} = 0 \quad \Rightarrow \quad (\mathbf{m} \cdot \nabla)\mathbf{h} \approx \frac{\chi_{\text{lin}}(c)}{2} \nabla(|\mathbf{h}|^2).
\]

If the field’s direction was reversed, a magnetic particle would be magnetized in reverse, too. Consequently, the product \((\mathbf{m} \cdot \nabla)\mathbf{h}\) would remain unchanged. Therefore, flipping the direction of the dipole in (5.3) should not change the particle’s motion in the setting of this thesis. They would be attracted to the right the same way independent of the sign of \(d_{a1}\). However, note that this statement becomes false when the initial magnetization is not zero.
5.1 Model comparison in fixed scenarios

5.1.1 Presentation of "model GW"

As indicated at the beginning of Section 5, "model GW" is examined by the two simulations (GW1) (linear diffusion, \( f_p(c) = c \)) and (GW2) (nonlinear diffusion, \( f_p(c) = c^2 \)). The magnetic field given via (5.3) is a rather homogeneous field in the sense that its field lines spread only slightly.

**[GW1] Linear Diffusion.** The initial maximum time increment has been chosen to be \( \tau_{\text{max}} = 0.0016 \), but has been decreased manually to \( \tau_{\text{max}} = 0.0008 \) starting from \( t = 0.6411 \) and to \( \tau_{\text{max}} = 0.0002 \) from \( t = 1.1655 \) onward. The time adaptivity scheme rarely changed the time increment at all. The manual decrease of the maximum time increment has been chosen as late as possible while avoiding numerical instabilities.

Figure 5.4 illustrates the transport of the magnetic nanoparticles towards the right boundary of \( \Omega \). One can see in Figure 5.4b that the field lines of the total magnetic field (depicted in blue) are conspicuously curved in the beginning of the simulation in regions where a large magnitude of the particle induced stray field \( h_d \) is to be expected. In contrast, the sketch in Figure 5.1 only indicated the external field \( h_a \). In the course of time, this disturbance vanishes as the particles accumulate at the right boundary of \( \Omega \).

The snapshots in Figure 5.4 indicate the tendency of the contour lines (in gray scale) of \( c \) to be perpendicular to the streamlines of \( h \). However, in a finite distance to the upper and lower boundaries of \( \Omega \), this tendency at time \( t = 16 \) is still disturbed significantly – presumably the result of the peak formation at the upper and lower boundaries starting at approximately \( t = 2.1 \) and still being visible at \( t = 16 \) from the curved contour lines. Moreover, observe an initial spreading...
of the support of the particle density also to the left – moving away from the magnetic field’s source. In fact, this is mostly caused due to magnetic effects, see Figure 5.5, where we plotted

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5_5.png}
\caption{Black color levels depict $c$ at $t = 0.5$, black lines are contour lines to the levels 0.01, 0.1 and 0.2. Arrows illustrate the fluxes corresponding to the magnetic (blue) and the diffusive (red) effects. The right image shows a magnified view of the left image.}
\end{figure}

the diffusive and magnetic contributions to the particle flux separately, i.e. $c^{k-1}(v_{\text{part}})^{\text{diff}}$, where $(v_{\text{part}})^{\text{diff}}$ corresponds to the first term on the right-hand side of (4.37d), and $c^{k-1}(v_{\text{part}})^{\text{mag}}$, where $(v_{\text{part}})^{\text{mag}}$ corresponds to the second term on the right-hand side of (4.37d), i.e.

\begin{align*}
\int_{\Omega} I_{h,1}((v_{\text{part}})^{\text{diff}} \cdot \theta) \, dx &= -KD \int_{\Omega} I_{h,1} \left( \frac{f_{p,h}}{(c^{k-1})_{s,h}} \nabla g_{s,h} \cdot \theta \right) \, dx, \\
\int_{\Omega} I_{h,1}((v_{\text{part}})^{\text{mag}} \cdot \theta) \, dx &= K\mu_0 b_{h}^{m} \left( I_{h,1} \left( \frac{f_{p,h}}{(c^{k-1})_{s,h}} \right) \right) \left\{ (\alpha_{1} h_{k}^{s})_{|\Omega} + \frac{\partial}{2} (h_{k}^{s})_{|\Omega} - \alpha_{3} m_{k}, m_{k} \right\}.
\end{align*}

This effect seems to be plausible in view of the stationary distribution (see Figure 5.4o) at the end of the simulation where the slope of $c$ is smaller than the slope of the initial data.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5_6.png}
\caption{Model GW – linear diffusion (GW1). Particle density $c$, stray field $h_{d}$ and magnetization $m$ visualized according to Figure 5.3.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5_7.png}
\caption{Model GW – linear diffusion (GW1). Particle density $c$, stray field $h_{d}$ and magnetization $m$ visualized according to Figure 5.3.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5_8.png}
\caption{Model GW – linear diffusion (GW1). Particle density $c$, stray field $h_{d}$ and magnetization $m$ visualized according to Figure 5.3.}
\end{figure}
in the opposite direction of $\mathbf{h}_a$ inside the particle density’s support – which agrees with the notion “demagnetizing field” which is widely used in physics to describe this part of the particle-induced field. In the beginning, the stray field outside of the particle density’s support looks, as expected, similar to the field of a permanent magnet, i.e. the field lines leave the particles’ support to the left where $\mathbf{h}_a$ is pointing to and reenter on the right side. Once, the particles have reached the right boundary, these field lines cannot enter from the right anymore as the computational domain ends. Up to Figure 5.6e one can still see on the very left of the particle density’s support a change in direction of the stray field – marking the end of the inner demagnetizing and the outer straying part of $\mathbf{h}_d$.

In Figure 5.6f and Figure 5.6g it can be seen that the stray field emerging from the spikes at the top and bottom seemingly forces a change in the directional layout of the magnetic field such that in Figure 5.6h it seems like the direction of the stray field has been reversed in the sense that it enters the magnetic particles’ support on the left instead of leaving it. Notice how this directional change should not affect the evolution of the particle density because the change happens only outside of the particle density’s support. From $t = 10.5$ until $t = 16$ the magnetic field does not show notable changes – hence no snapshots have been presented.

\[\text{(GW2) Nonlinear diffusion.}\] The maximum time increment of the time adaptivity scheme was chosen to be $\tau_{\text{max}} = 0.0008$. The time increment dipped below the maximum value two times for a very short time at approximately $t = 2.8$ and $t = 10.3$. At $t = 16.7476$ (not part of the presented images) the time increment dropped permanently to 0.0004.

The results are depicted in Figure 5.7 and Figure 5.8 in the same manner as before in the case of linear diffusion. From [33], finite speed of propagation can be expected. Accordingly, one can see in the beginning of the simulation that the particles stay more concentrated than in the case of linear diffusion. Due to the degenerate mobility (see e.g. (3.168)) the diffusive and magnetic effects slow down drastically in the middle of $\Omega$ when the majority of the particles have already reached the right boundary of $\Omega$. This further contrasts the different speeds at which the right side and left side of the particle density’s support arrive at the right boundary of $\Omega$. 

![Figure 5.7: Model GW – nonlinear diffusion](image) Particle density $c$ and total magnetic field $\mathbf{h}$ visualized according to Figure 5.2.
No further differences to the case of linear diffusion were noticed in regards to the particle density. The support of the particle density spreads slightly to the left in the beginning of the simulation, too, but not as much as in the case of linear diffusion – where the mobility is not degenerate. The behavior of the total magnetic field $\mathbf{h}$ is qualitatively the same, i.e. in the beginning its streamlines are curved in areas where large magnitudes of the stray field $\mathbf{h}_d$ are expected and in the course of time the curves flatten out until they disappear.

The magnetic stray field behaves qualitatively like the stray field in the case of linear diffusion at much earlier times. For instance, the image at $t = 16$ in Figure 5.8 is qualitatively the same as the image at $t = 2.7$ in Figure 5.6. Later ending times $T$ will be considered for "model W" which needs way less computational resources (cf. Table 5.3 and Table 5.4 for details on the computation time).

5.1.2 Presentation of "model W"

The two simulations of Section 5.1.1 with the rather homogeneous external magnetic field from (5.3) will be repeated using "model W" (as indicated in (W1) and (W2)). In addition, simulation (W1') serves as a comparison of the cases $\Omega = \Omega'$ and $\Omega \subset \subset \Omega'$ in the linear diffusion case.

(W1) Linear diffusion. For time adaptivity, we have chosen $\tau_{\text{max}} = 0.001$. In the beginning, the time increment stays at the value of $\tau_{\text{max}} = 0.001$. From $t = 10$ to $t = 16$ the plot in Figure 5.9 shows a decreasing trend and many oscillations which indicate that the time increment already is close to the upper limit for stable simulations. Hence, the time increment was approximately 'optimal' in the sense of reducing computational overhead. We would like to highlight that the simple heuristic time adaptivity scheme proposed in Section 4.3.3 worked really well in contrast to the case of "model GW" (Section 5.1.1) and the overall size of time increments remains rather high – compared to "model GW" – for more than half of the simulation. In the end of the simulation, the time increment has similar size as in the corresponding simulation (GW1) of "model GW".
Figure 5.9: Model W – linear diffusion (W1). Adaptive time increments over time. Notice that the time adaptivity scheme from Section 4.3.3 introduces jumping between specific values. Oscillations between two successive values are rather mild and only appear to be very frequent (almost looking like rectangular areas) in above graph due to the tight scaling of the time axis.

Figure 5.10: Model W – linear diffusion (W1), $\Omega = \Omega'$. Particle density $c$ and total magnetic field $h$ visualized according to Figure 5.2. When the magnetic field domain and the fluid domain coincide ($\Omega = \Omega'$), the evolution of the particle density is consistent with the results of "model GW", see Figure 5.10. However, the

Figure 5.11: Model W – linear diffusion (W1), $\Omega = \Omega'$. Particle density $c$, stray field $h_d$ and magnetization $m$ visualized according to Figure 5.3. Results differ from the simulation of "model GW", see the second row of Figure 5.6.
stray field \( h_d \) behaves differently in the beginning. In the first half of the simulation (no images presented), the evolution coincides with the results of ‘model GW’, see the first row of Figure 5.6. Starting from Figure 5.11b it can be seen that the reversion of the stray field to the left of the density’s support happens as well, but the overall layout in Figure 5.11d differs from Figure 5.6h. This discrepancy does not disappear over time, i.e. it is still present at \( T = 16 \). The reason for the different result is unknown. Further insight is available in Section 5.1.4.

**[W1']** Linear diffusion, \( \Omega \subset \subset \Omega' \). The evolution of the time increment is similar as in the case \( \Omega = \Omega' \), but \( \tau_{\text{max}} = 0.0005 \) has been chosen to be half as large as in the case \( \Omega = \Omega' \). The latter choice was necessary to avoid numerical instabilities which might be linked to the stray field in the regions of the transition from \( \Omega \) to \( \Omega' \setminus \overline{\Omega} \), especially at the corners of the fluid domain \( \Omega \).

At first glance, the evolution of the particle density, see Figure 5.12, seems to be similar as in the case \( \Omega = \Omega' \) (for either ”model W” or ”model GW”). Yet it differs in some aspects from the

![Figure 5.12](image)

**Figure 5.12:** Model W – linear diffusion [(W1')](#). Particle density \( c \) and total magnetic field \( h \) visualized according to Figure 5.2.

![Figure 5.13](image)

**Figure 5.13:** Magnified view of the top right corner of Figure 5.12b.
previous simulations of the linear diffusion case. The magnetic particles seem to avoid the top right and bottom right corners of $\Omega$ and the formation of spikes on the top and bottom parts of $\partial \Omega$ is more pronounced. For convenience, a magnified view of the upper right corner is given in Figure 5.13. Apart from that, no differences could be noticed in regards to the particle density $c$.

Notice that the avoidance of corners most likely requires strong gradients in the stray field. This might be the reason why the simulation required smaller time increments to avoid instabilities. In combination with the lighter gray level in the corner, the contour lines clearly show the lower density of magnetic particles in the corner. Moreover, one can see a bend in the total magnetic field whenever it enters or leaves the fluid domain $\Omega$.

The stray field $h_d$ greatly differs from the respective results of both "model GW" and "model W" in case of $\Omega = \Omega'$. As can be seen in Figure 5.14 there is no change of direction of the stray field (cf. bottom row of images in Figure 5.6) and, overall, it seems more realistic. It acts demagnetizing inside the support of the density of magnetic particles – as expected – and spreads outwards on the left side of the density’s support mimicking a bar magnet of the same orientation as the external field. Moreover, the field lines outside of $\Omega$ circle back to the right side of the particles where they enter the support of the particle density again.

Figure 5.14: Model W – linear diffusion (W1'), $\Omega \subset \subset \Omega'$. Particle density $c$, stray field $h_d$ and magnetization $m$ visualized according to Figure 5.3.

(W2) Nonlinear diffusion. Simulation (GW2) will be repeated using "model W" and a larger ending time $T = 80$. We set $\tau_{\text{max}} = 0.0005$ from the start but manually increased it to $\tau_{\text{max}} = 0.002$ from $t = 40$ on to save computational costs. The time adaptivity scheme did not take effect, the increments stayed constant during both halves of the simulation.

Up to $T = 16$ no visible changes could be detected compared to the previous simulation (GW2) based on "model GW". Even at $t = 80$ there are positive density values in the middle of the domain ($c(0.5, 0.5) \approx 0.024$, about 4.9% of the current maximum) which are hardly visible in Figure 5.15 due to the very bright grayscale level. This explains why the point where the demagnetizing nature of $h_d$ (arrows pointing against the direction of $h_d$) changes into the (outwards) straying
5.1 Model comparison in fixed scenarios

Figure 5.15: Model W – nonlinear diffusion \((\Omega^2, \Omega')\). Particle density \(c\) and total magnetic field \(h\) visualized according to Figure 5.2.

nature of \(h_d\) (spreading leftwards) is so far on the left of the domain, see Figure 5.16d. A general change of direction as in case of linear diffusion, cf. Figure 5.6f-Figure 5.6h ("model GW") or Figure 5.11 ("model W"), is expected to happen here as well. The latter is due to the inwards pointing arrows emerging from the spikes at the top and bottom of the domain. However, due to limited computational resources, the simulation has not been continued any further. In the time range \([0, 16]\) the results of "model W" are consistent with the results of "model GW".

Figure 5.16: Model W – nonlinear diffusion \((\Omega^2, \Omega')\). Particle density \(c\), stray field \(h_d\) and magnetization \(m\) visualized according to Figure 5.3.

5.1.3 Presentation of "model B"

This section focuses on the contrast between "model W" and "model B", which manifests in the different convective velocities for the particle density \(c\), cf. (2.100d) for "model W" and (2.110d) for "model B". At points \((t, x) \in I \times \Omega\) where \(c(t, x) = 1\), this contrast is larger due to the factor \((1 - c)\) in front of the magnetic force in (2.110d). Concerning the nonlinear diffusion case, an additional – compared to the linear case – factor \(c(t, x)\) appears in the mobility (cf. (3.168)). The latter factor is of no significance at such points where \(c(t, x) = 1\). Therefore the nonlinear diffusion case will be omitted in this section.

The quantity \(c\) is interpreted as volume concentration and the initial distribution of \(c\) as described in (5.2) attains the value 1 in the middle, i.e. \(c_{\text{init}}(0.5, 0.5) = 1\), such that the volume concentration...
is high and even the maximum physical volume concentration $1$ is attained. This extreme scenario is tailored to compare the models from a theoretical point of view and is not motivated by practical interests. Therefore, the difference between "model W" and "model B" should be as noticeable as possible while still being as comparable as possible by keeping all model parameters and external data unchanged.

**{(B1)} Linear diffusion.** The maximum time increment has been chosen to be the same as the one used for "model W", i.e. $\tau_{\text{max}} = 0.001$, and the evolution of time increments coincides qualitatively with the the evolution in case of "model W", see Figure 5.9.

The beginning of the simulation shows how the evolution of the particle density (and the stray field) is slowed down a bit in contrast to "model W" in regions where the particle volume concentration $c$ is close to $1$, see Figure 5.17 (and Figure 5.19). The snapshots in Figure 5.18 representing graphs of particle densities for both models at selected times, clearly show how particles in dense regions tend to avoid moving. However, after the spreading of the particle density’s support – when the density values have become significantly smaller – the differences between "model W" and "model B" become small. The final configuration at the right boundary of $\Omega$ is qualitatively the same for both models. Notice how in this scenario only the magnitude of the magnetic force in the convective velocity $\mathbf{v}$ of "model B" changes compared to "model W". Moreover, the diffusion in the simulations of this thesis has been chosen very small, see Table 5.1 hence the magnetic force is dominating. Therefore, while the local transport speeds might differ according to the "additional mobility" $(1-c)$, it is not surprising that the stationary distribution of $c$ at the end of the simulation is very similar to the respective result for "model W" in simulation (W1).
5.1 Model comparison in fixed scenarios

Figure 5.18: Visualization of graphs of the particle density $c$ in "model W" (left in each subimage) and "model B" (right in each subimage) at selected times $t$ in case of linear diffusion and $\Omega = \Omega'$. View from the side in direction of the $x_2$-axis. Recall that this extreme scenario (initial concentration attains very high values) is of theoretical interest only.

Apart from the aforementioned slowdown, the evolution of the stray field is qualitatively the same as in case of $\Omega = \Omega'$ (using "model W") including the discrepancy to "model GW" after the field changes its direction, see Figure 5.19.

Figure 5.19: Model B – linear diffusion $h_d$. Particle density $c$, stray field $h_d$ and magnetization $m$ visualized according to Figure 5.3.

5.1.4 Model evaluation

In this section the agreements and discrepancies of the various simulation results will be summarized. Particularly, the observations on the field-induced transport of magnetic nanoparticles will be recalled and a comparative evaluation of the three models will be conducted.

Agreements. The evolution of the particle density $c$ and the total magnetic field $h$ seems to be consistent for the simulations that have been presented in Section 5.1 throughout all models except of "model B", whose particle density generally evolves at slower pace. However, the final
equilibrium state seems to be consistent again. In cases where the particle density is low, 'model W' and 'model B' behave very similar. The following general observations apply to all simulations.

- The attraction of magnetic particles to the right boundary of $\Omega$ is stronger on the right side of the particle density’s support (as it is closer to the external 'magnet') than on the left side thereof such that the particles in the middle of $\Omega$ seem to lag behind compared to those which already reached the right boundary, see e.g. Figure 5.4 at $t = 1.1$ and the following images.

- Looking at the green/red bars which visualize the magnetization, the magnitude of the magnetization correlates with the values of the particle density – see e.g. Figure 5.6. This is an indication that the magnetization field is strongly coupled to the particle density.

- The particle density evolve in such a way that initial disturbances (see e.g. Figure 5.4b) in the magnetic field lines, which are a consequence of the particle-induced stray field, vanish. In the final configuration at the right boundary of $\Omega$ there is a relation between contour lines of the particle density and streamlines of the total magnetic field – they tend to be approximately perpendicular. However, the external field in (5.3) leads to a final density distribution at the right boundary, which seemingly has been constrained from the top and bottom. This causes an accumulation of particles which violates the aforementioned orthogonality of contour lines and field lines.

- The particles slightly spread to the left, too, in the beginning of the simulations. This is mostly caused by magnetic effects, not diffusive effects which are very small (cf. Table 5.1 and Figure 5.5). In Figure 5.20, the shapes of the distributions of the magnetic particle density at $t = 0$ and $t = 0.7$ are visualized simultaneously via contour lines.

\[\text{Figure 5.20: Model GW – linear diffusion (GW1). Support of particle density at time } t = 0 \text{ (circle) and } t = 0.7 \text{ (around the circle) illustrated by contour lines at level } 0.005.\]

- The particles stay significantly more concentrated in the case of nonlinear diffusion $f_p(c) = c^2$. This is expected behavior for degenerate parabolic equations due to finite speed of propagation [33], see also (3.168). Additionally, low density values slow down the convective velocity and the particles in the middle region of the fluid domain lag behind a lot more than in the case of linear diffusion. For this, see particularly Figure 5.15 at $t = 80$ where a very light shade of gray is still visible in the middle region.

**Discrepancies.** The main differences relate to the stray field and the quality of the numerical results. As the results of the linear diffusion case show, the evolution of the stray field $h_d$ is different in each of the simulations $\{\text{GW1} \}'(\text{model GW', } \Omega = \Omega'), [\text{W1}]'(\text{model W', } \Omega = \Omega')$ and $[\text{W1}](\text{model W', } \Omega \subset \subset \Omega')$. The evolution of the stray field in case of 'model B' is very similar to the case of 'model W'. The individual stray fields of the aforementioned simulations are arranged side by side, among other things, in Figure 5.23.
The reason for the different results for the stray field $h_d$ might be

- numerical problems
- or discrepancies in the models.

In order to analyze this issue, consider simulation $W1$, i.e. 'model W' in case of linear diffusion and $\Omega = \Omega'$. If one draws a curve in Figure 5.11d, see Figure 5.21, separating arrows pointing to the right from arrows pointing to the left, one obtains a shape similar to the support of the particle density $c$ at earlier times, e.g. $t = 4.0$. This motivates to investigate the numerical quality of the particle density – focusing on values close to zero.

Starting with 'model W' in case of $\Omega = \Omega'$, let us have a closer look at the contour lines of the particle density for values close to zero. First, consider negative density values, see Figure 5.22. One can clearly see contour lines which have a similar shape as the particle density’s support at earlier times. The minimum density value at $t = 16$ is -0.01348, which amounts to approximately 2.647% of the current maximum value given by 0.5093. In comparison, the analogous values in case of 'model GW' are $-0.002361$ for the minimum density and 0.5088 for the maximum density value, resulting in a proportion of 0.4640%. The respective values for some selected simulations, based on the original finite element schemes as well as mixed finite volume schemes, are contained in Table 5.2.

However, the numerical artifacts of the contour lines are visible also for small but positive values of the particle density $c$, for instance at level 0.001. In Figure 5.23, snapshots from multiple previous...
simulations are presented next to each other, including the results of 'model B' \(B1\), 'model W' \(W1\) and 'model GW' \(GW1\) in case of \(\Omega = \Omega'\) as well as the results of 'model W' in case of \(\Omega \subset \subset \Omega'\).

**Figure 5.23:** Comparison of level line (blue) for \(c\) at level 0.001 at \(t = 16\). Additionally, the particle density and the stray field are visualized as in Figure 5.3. In case of \((W1')\) where \(\Omega \subset \subset \Omega'\), only the fluid domain \(\Omega\) is presented.

Despite the numerical issues, the magnetic force seems to be accurate. Since, after all, the absolute density values in this inaccurately computed area are low, implying the magnetization to have low magnitude. Thereby magnetic force \((\mathbf{m} \cdot \nabla)\mathbf{h}\), including possible errors, are expected to be small and the evolution of the particle density does not seem to be affected.

So far, there is a slight hint that the less regular (on a formal basis, see Section 2.4.3) 'model W' suffers from numerical problems which change the layout of the stray field. For further investigation it might be helpful to have a look at the simulation \((W1)\) but change \(\Omega'\) to be strictly larger than \(\Omega\) at time \(t = 15\), when the changed layout and the numerical artifacts are already present, and resume the simulation. If the changed layout remains, the aforementioned numerical problems are the reason for the discrepancy. This approach is viable as the magnetic field does not require initial data. Hence, one can use the simulation results for \(u\), \(c\) and \(m\) at some time \(t >> 0\) as initial data for a new simulation.

Let us comment about the case \(\Omega \subset \subset \Omega'\) in 'model GW' first. In order to save computational

**Figure 5.24:** Comparison of the layout of the stray field \(\mathbf{h}\) in case of linear diffusion, visualized according to Figure 5.3. The case of 'model GW' with \(\Omega = \Omega'\) (simulation \((GW1)\)) is on the left, the new simulation of 'model GW' with \(\Omega \subset \subset \Omega'\), where initial data equals simulation data at \(t = 15\) of the respective simulation in the case \(\Omega = \Omega'\), is in the middle. The right image shows the simulation \((W1')\) i.e. 'model W' with \(\Omega \subset \subset \Omega'\).
resources, the simulation has not been performed from the start $t = 0$, but the original simulation \{(GW1)\} has been resumed at time $t = 15$ – after the reversion of the stray field’s direction (see Figure 5.6f–Figure 5.6h) – with $\Omega \subset\subset \Omega'$. One can see in Figure 5.24 that the field lines of the stray field immediately change to the same layout as in case of 'model W', see Figure 5.14. The particle density is still accumulating in the corners instead of the top and bottom boundary of $\Omega$ in contrast to the case of 'model W' – see the differences in the black color levels in Figure 5.24 (or Figure 5.13). However, a redistribution happens shortly after and the particle density qualitatively coincides at $t = 16$ with the results from the case of simulation \{(W1')\} (no images presented).

The analogous data for 'model W' is depicted in Figure 5.25. Again, the stray field’s layout immediately changes and coincides with the layout from simulation \{(W1')\}. At the same time, the inaccuracies in the particle density are still present and remain qualitatively unchanged until the end of the simulation, where $T = 17.925$ is the final time. In the end, this does not allow to conclude whether the difference in the simulation results is caused by numerical artifacts. Choosing $\Omega \subset\subset \Omega'$ can improve the numerical quality compared to $\Omega = \Omega'$, cf. Figure 5.23, but at the same time changes the model itself.

Concerning whether to choose 'model GW' or whether to choose the magnetic domain $\Omega'$ strictly larger than $\Omega$, we have found the following.

- Both measures improve the numerical quality of $c$ for small values (cf. Figure 5.23).
- The most realistic simulations of the stray field $h_d$ are obtained by considering a strictly larger magnetic field domain independent of the choice of model equations.

## 5.2 Concluding remarks on the numerical schemes

Here, the numerical schemes from Section 4.1 will be compared based on indicators of numerical quality (see Table 5.2) and computational performance (see Table 5.4). Moreover, we comment on ways to reduce computational costs by considering lower order schemes (piecewise constant magnetization and magnetic field) and an alternative algorithm to solve the magnetic subsystem based on an elimination approach. For simplicity, only the case of linear diffusion ($f_p(c) = c$) will be considered. Moreover, the discretizations of 'model W' and 'model B' are very similar, which is why 'model B' will be excluded from the examination.
For a deeper numerical comparison, mixed finite volume schemes will be considered, too. The simulations (GW1) and (W1) will be repeated using the finite volume approach for the particle density equation (cf. (4.126)) and will be labeled (GW1)-FV when performing simulation (GW1) based on the mixed finite volume scheme, (W1)-FV when performing simulation (W1) based on the mixed finite volume scheme.

Unless otherwise stated, the time adaptivity scheme from Section 4.3.3 is used. For better comparability, the finite volume simulations from [39] with an involved heuristic time adaptivity scheme will not be part of the comparison of computational time. The results of the finite volume based simulations which use the time adaptivity scheme from Section 4.3.3 are consistent with the respective pure finite element simulations (no images presented). For "model GW" the maximum increment was $\tau_{\text{max}} = 0.001$ and the actual increment has remained constant at $\tau = 0.0005$ since $t = 0.817$. For "model W" the maximum increment was $\tau_{\text{max}} = 0.001$ and the evolution of the time increment was similar as in the finite element case (Figure 5.9) but never went below $\tau = 0.00025$. This way, the evolutions of the time increments show similar behavior as in the pure finite element case (cf. Section 5.1.1 and Section 5.1.2). Thereby the previous results are comparable with the mixed finite volume simulations (GW1)-FV and (W1)-FV.

On numerical quality. Based on Figure 5.23, the numerical quality of the particle density $c$ seems to be significantly better in case of "model GW". For convenience, no further improvements of numerical quality of the contour line, determined via the equation $c(16, x) = 0.001$, could be noted in the respective mixed finite volume simulation (GW1)-FV (no image presented). One might ask if the finite volume approach improves the numerical quality in simulations of "model W" with respect to the particle density (especially for density values close to zero) and the stray field. The corresponding simulation has been performed, but no significant changes have been detected, see Figure 5.26. In regards to the evolution of the particle density and the stray field, no deviations to the case of the pure finite element scheme could be noted.

Let us take a look at the minimum density values of simulations using "model GW" or "model W" (in the linear diffusion case). The violation of non-negativity is expected to become worse over time which is why the beginning of the simulations will be skipped for this examination. However, the minimum density values most likely do not evolve monotonously, i.e. they might oscillate. Therefore, the arithmetic mean of the density values will be presented in the time range $[10, 16]$, see Table 5.2. The minimum values are all negative and the largest minimum, which is found using "model GW" with the mixed finite volume approach and the time adaptivity of this thesis, is
5.2 Concluding remarks on the numerical schemes

Table 5.2: Time-averaged values of both the minimum of the magnetic particle density and its relation to the respective maximum density. Recall, \( t^k \) is the discrete time of the \( k^{th} \) time step, \( c^k \) is the discrete solution at time \( t^k \) and \( \text{avg} \) denotes the average. All discrete times within the range \([10, 16]\) are used to compute the averages.

<table>
<thead>
<tr>
<th>Simulation</th>
<th>( \text{avg}<em>{t^k \in [10, 16]} \left[ \min</em>{\Omega} c^k \right] )</th>
<th>( \text{avg}<em>{t^k \in [10, 16]} \left[ \frac{\min</em>{\Omega} c^k}{\max_{\Omega} c^k} \right] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(GW1)-FV with time adaptivity from [39]</td>
<td>-0.001961</td>
<td>0.3849%</td>
</tr>
<tr>
<td>(GW1)-FV</td>
<td>-9.702 \cdot 10^{-6}</td>
<td>1.908 \cdot 10^{-3} %</td>
</tr>
<tr>
<td>(W1)</td>
<td>-3.880 \cdot 10^{-8}</td>
<td>7.626 \cdot 10^{-6} %</td>
</tr>
<tr>
<td>(W1)-FV</td>
<td>-0.01121</td>
<td>2.201%</td>
</tr>
<tr>
<td>(W1')</td>
<td>-0.008675</td>
<td>1.703%</td>
</tr>
<tr>
<td></td>
<td>0.01119</td>
<td>2.178%</td>
</tr>
</tbody>
</table>

Table 5.3: Aforementioned indicators of the computational time.

<table>
<thead>
<tr>
<th>Simulation</th>
<th>(%Mag)</th>
<th>(#FPI)</th>
<th>( n_T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(GW1)</td>
<td>550.2%</td>
<td>2.306</td>
<td>76691</td>
</tr>
<tr>
<td>(GW1)-FV</td>
<td>517.8%</td>
<td>2.695</td>
<td>31188</td>
</tr>
<tr>
<td>(W1)</td>
<td>42.43%</td>
<td>3.446</td>
<td>27171</td>
</tr>
<tr>
<td>(W1)-FV</td>
<td>41.78%</td>
<td>3.699</td>
<td>20221</td>
</tr>
<tr>
<td>(W1')</td>
<td>49.83%</td>
<td>3.196</td>
<td>38410</td>
</tr>
</tbody>
</table>

On computational time. Another aspect of comparison is the computational time. The exact simulation time is not comparable as simulations have been run on multiple different machines under different external workloads. As point of reference the following indicators will be considered,

- \((%\text{Mag})\) the average percentage of computational time which is consumed by the magnetism in relation to the remaining (not the total) system,
- \((#\text{FPI})\) the average number of fixed point iterations which are needed to solve the nonlinear system,
- \((n_T)\) the total amount of time steps of the simulations.

As the time adaptivity scheme of [39] differs a lot, it is not included in the detailed comparison. The values are summarized in Table 5.3. From the values in Table 5.3 one can compute a characteristic number which allows for a rough comparison. Assuming the non-magnetic parts of the model to require fixed computational time \( \text{CT} \) for each fixed point iteration step, one can estimate the total computation time \( \text{CT}_{\text{total}} \) to be

\[
\text{CT}_{\text{total}} \approx (n_T)(\#\text{FPI})\left(1+(%\text{Mag})\right) \cdot \text{CT}.
\]

Although this thesis is not concerned with runtime optimization, the characteristic value \( \text{CT}_{\text{total}} \) and the indicators \(%\text{Mag} \), \( #\text{FPI} \) and \( n_T \) are useful for first insights on the performance evaluation of our particular models as we assume optimization with respect to convection dominance to affect all models equally. Optimization strategies which specifically focus on the magnetic
5.2 Concluding remarks on the numerical schemes

Simulation (W1)-FV < (W1) < (W1') < (GW1)-FV < (GW1)

CT\textsubscript{total} in CT/10\textsuperscript{5} 1.061 < 1.334 < 1.839 < 5.193 < 11.50

Table 5.4: Artificial characteristic number giving a hint on the total time of computation – the lower the better.

subsystem (having an impact especially on 'model GW") are briefly discussed at the end of this section. Table 5.4 shows that stabilization techniques with respect to convection dominance have a positive impact on the estimated computational time CT\textsubscript{total}. Moreover, establishing a mathematical foundation for reduced models like 'model W' which appear in the previous research line [Inhom-simpl] (see Section 1) is highly motivated from a practical perspective. See Figure 1.2 in order to see how the results of this thesis might be relevant for the latter.

Notice how the ratio of computation time for the magnetic equations in contrast to the remaining equations seems to be unreasonably high. Let us arrive at an alternative performance estimate by considering the degrees of freedom. The degrees of freedom at different selected times are presented in Table 5.5 in case of simulation (GW1). These results are representative for the other simulations (in case of linear diffusion), too, as the evolution of the particle density – which drives the local refinement – is similar throughout all of them. The degrees of freedom corresponding to the computation of the discrete divergence \( \text{div}_h (\cdot) \) (see (4.32)) are included in Table 5.5 as a separate system of equations needs to be solved. On the other hand, the computation time of the discrete curl \( \text{curl}_h (\cdot) \) (see (4.31)) is negligible as the lumped mass matrix is diagonal. At time \( t = 1 \) one would obtain for \( d = 2 \) the value 167.2% for the ratio of the degrees of freedom for the magnetic variables compared to the remaining quantities. Similar values arise at other times. This value is significantly lower than 500% as in Table 5.3. It seems that the software is struggling with the high dimensionality of the problem, i.e. the solving time per step does not scale linearly with the dimensionality of the input. However, reconfiguring or replacing parts of the algorithm might lead to improved results which are closer to an estimate based on the degrees of freedom. Under the assumption that the high percentage values (over 500%) in Table 5.3 could be replaced by approximately 170%, the resulting value of CT\textsubscript{total} in CT/10\textsuperscript{5} for 'model GW' would be 2.269 for the mixed finite volume scheme, (GW1)-FV, and 4.775 for the original finite element scheme, (GW1). Hence, the ranking from Table 5.4 would be unaffected, but the computational time would be cut in half.

Let us briefly comment on the evolution of time increments. They tend to get smaller in later time ranges of the simulation, see e.g. Figure 5.9. The numerical instabilities which have been observed during research are

- typical oscillations as expected from convection-dominant systems in the beginning of the simulation when the support of the particle density has not reached the right boundary yet (see Figure 5.27).
5.2 Concluding remarks on the numerical schemes

Figure 5.27: Visualization of numerical instabilities of the particle density as a graph \((x,c(t,x))\) for \(x \in \Omega\) at a fixed time \(t \in I\), view from the side in direction of the \(x_2\)-axis. Data comes from simulation \([GW1]\) – "model GW" with the original finite element scheme – where \(\tau_{\text{max}} = 0.005\). The results are representative for other models and schemes (including finite volume schemes), whenever the time increment is too large.

- overestimated density values at the right boundary accompanied by oscillations (Figure 5.28).

Figure 5.28: Graphs of the particle density \(c\) of deliberately unstable (red) simulations and previous stable simulations (black color levels) at the same time \(t\) are superimposed to each, where the \(x\)-axis corresponds to the \(x_1\)-component, the density-axis shows the density value \(c(x_1,x_2)\) and the remaining unlabeled axis corresponds to the \(x_2\)-component. Specifically, \([GW1]\) (left) was resumed from \(t = 10\) with \(\tau_{\text{max}} = 0.001\) and \([W1]\) (right) was resumed from \(t = 4\) with \(\tau_{\text{max}} = 0.004\) in order to cause instabilities. The number of fixed point iterations at the depicted times was 10 in each case, the maximum number before the time adaptivity is triggered (cf. Section 4.3.3). Note that it already might have been triggered earlier.

The magnetic force is strongest at the right boundary, leading to steep gradients and thus numerical challenges. As this is of no concern before the particles reached the right boundary, smaller time increments are especially required in the second half of the simulations.

As a tendency, the first type of instability mentioned above leads to failure of the simulation. The second type of instability occurs when most of the transport to the right boundary of \(\Omega\) has already been completed. Then, the high gradients of the overestimated particle density seemingly lead to strong magnetic forces (locally) which reduce the slope of the particle density – averting fatal instabilities. This might be an indication of the demagnetizing nature of the particle-induced field \(h_d\). Interestingly, way too large time increments facilitate the smoothing of instabilities. Nonetheless, a blow-up during the fixed point iteration is the most common outcome. Hence, especially in case of "model W" and suitable choices of \(\tau_{\text{max}}\) the instabilities caused immediate blow-ups which were handled by the time adaptivity scheme (cf. Section 4.3.3) very well.

On variations of the numerical schemes. Two different approaches to reduce computational time are considered numerically – the usage of lower order finite element spaces with respect only to the magnetic variables and the elimination of the magnetization equation in the magnetic
5.2 Concluding remarks on the numerical schemes

The examinations will all be carried out in the linear diffusion setting with $\Omega = \Omega'$. As the usage of finite volumes for the particle density (cf. (4.126)) already improves the numerical stability and thereby reduces the total amount of time steps (cf. Table 5.3 and Table 5.4) compared to the purely finite element based simulations [GW1] or [W1] respectively, it is most beneficial to apply the aforementioned techniques to the mixed finite volume schemes. Hence, the following tests do not include pure finite element schemes.

Looking at (4.17) and (C3 4.14) one might wonder why the polynomial degree of the finite element space $\mathcal{R}_h = \mathcal{P}_2^\text{mean}(\Omega')$ was chosen to be 2 instead of 1. It was already mentioned in Remark 4.3 how this raises stability concerns regarding the discretization of the Kelvin force (and similar terms) (4.30) in case of 'model GW' – cancellation effects seem likely. Such effects are not evident in case of the discretization of 'model W'. We tested 'model GW' and 'model W' using a lower order polynomial degree $\mathcal{R}_h = \mathcal{P}_1^\text{mean}(\Omega')$ for the discrete magnetic field potential and consequently a lower order polynomial degree $\mathcal{M}_h = D_0(\Omega')$ for the magnetization.

In case of 'model GW', the particle density $c$ showed numerical artifacts early on in the simulation – independent of our choices of time increments – which are most likely caused by inappropriately computed magnetic forces, see Figure 5.29 specifically in the upper right corner at $t = 5$ where one can see that the oscillations are linked to mesh elements. The overall transport of magnetic particles to the right side of $\Omega$ seems to be consistent with the previous results. However, these artifacts intensified over time and the simulation crashed. If a decrease of time increments solves this issue is unclear.

In case of 'model W', no numerical artifacts were noticed. However, in order to avoid numerical instabilities, a manual decrease of the maximum time increments to $\tau_{\text{max}} = 0.000125$ was necessary already from $t = 2.4435$ on. In comparison with the evolution of increments in Figure 5.9 for the same simulation with original polynomial order 2, this approach seems to be impractical. The simulation has been aborted. In the time range [0, 3.411] the average ratio [%Mag] of computational costs of magnetism compared to the other equations is 9.338%, the average amount [#FPI] of fixed point iterations is 3.270. The time increment $\tau = 0.001$ was sufficient for simulation [W1] until at least $t = 10$. Based on the current time increment $\tau = 0.000125$ the total amount of time steps will be estimated to be 8 times as much. Hence, the performance indicator $CT_{\text{total}}$ in $CT/10^5$ attains the estimated value 7.772, slightly better than 'model GW' (cf. Table 5.4). However, this is only a rough estimation. Much worse performance might be possible for later simulation times.

Altogether, it can be concluded that in the scenario of simulation [GW1] or [W1] respectively, lower order schemes are not practical.
Next, an approach to eliminate the magnetization equation in "model GW" will be considered. From the linearized scheme (4.131d, 4.131e) which is used for the fixed point iteration (solving the nonlinearly coupled system) one infers

\[
\left(1 + \frac{\tau}{\tau_{rel}}\right) \int_{\Omega} \mathbf{m}^k \cdot \mathbf{n} \, d\mathbf{x} = \frac{\tau}{\tau_{rel}} \int_{\Omega} \mathbf{r}_{rel} \cdot \mathbf{m} \, d\mathbf{x} + \int_{\Omega} \mathbf{m}^{k-1} \cdot \mathbf{n} \, d\mathbf{x} + \frac{\tau}{2} \int_{\Omega} (\text{curl} \, \mathbf{u} \times \mathbf{m}) \cdot \mathbf{n} \, d\mathbf{x} + \tau h_{rel}^{\text{m}} \mathbf{r}_{rel} \cdot \mathbf{n} \, d\mathbf{x} + \tau \sigma \int_{\Omega} \text{curl} \, \mathbf{m} \cdot \mathbf{n} \, d\mathbf{x} + \tau \sigma \int_{\Omega} \text{div} \, \mathbf{m} \cdot \mathbf{n} \, d\mathbf{x}
\]

for all \( \mathbf{n} \in \mathcal{M}_h \), where RHS collects all remaining terms depending on \( \mathbf{m}^{k-1}, \mathbf{m}, \mathbf{c}, \mathbf{r}, \mathbf{u}, \mathbf{v}_{\text{part}} \), which are all given data within the fixed point iteration, and

\[
\int_{\Omega'} \nabla R^k \cdot \nabla S \, d\mathbf{x} = \int_{\Omega'} (\mathbf{a}_h)^k \cdot \nabla S \, d\mathbf{x} - \int_{\Omega} \mathbf{m}^k \cdot \nabla S \, d\mathbf{x}
\]

for all \( S \in \mathcal{R}_h \). As \( \nabla S \) is a valid test function for (5.4), one easily arrives at

\[
\int_{\Omega'} \nabla R^k \cdot \nabla S \, d\mathbf{x} + \frac{\tau}{\tau_{rel} + \tau} \int_{\Omega} \mathbf{r}_{rel} \cdot \nabla R^k \cdot \nabla S \, d\mathbf{x} = \int_{\Omega'} (\mathbf{a}_h)^k \cdot \nabla S \, d\mathbf{x} - \text{RHS}(\nabla S)
\]

for all \( S \in \mathcal{R}_h \). After solving this equation which has significantly less degrees of freedom than the coupled system, the magnetization can simply be computed by inverting the mass matrix \( \mathbf{R} \). As \( \mathbf{R} \) is a valid test function for (5.4), one easily arrives at

\[
(5.4)
\]

\[
(5.5)
\]

\[
(5.6)
\]

The simulation \((GW1)\) (with finite volumes) was performed again until ending time \( T = 4 \) with the new solving technique. Concerning computational costs, the indicator \((\%\text{Mag})\) is 84.56\%, approximately twice as large as in case of "model W". The latter value is expected as a similar equation as in the scheme (4.131a) is solved and the discrete divergence of the magnetization is computed as well – based on the same finite element space of piecewise quadratic continuous functions. The indicator \((\#\text{FPI})\) is 3.642 and assuming – due to the similarity of the schemes – the total amount of time steps to be identical to the value in Table 5.3 \((GW1)\) in case of finite volumes, the performance indicator \(CT_{\text{total}}\) in \(CT/10^5\) is estimated to be 2.0963, a significant improvement. Nevertheless, there is a drawback. While the reduced system theoretically yields the same result, the average amount of fixed point iterations is higher – indicating less numerical accuracy. This might be problematic for simulations which are very sensitive to changes in the stray field. One example are the ferrofluid simulations from [51] which show the significance of the stray field using the example of the Rosensweig-instability (see e.g. Figure 1.1 in Section 1).

5.3 Study of different external magnetic fields

The goal of this section is to investigate how the simulation results change using different magnetic fields. For this, two types of fields will be considered,

- magnetic fields whose expected magnetic body force is of similar order/strength as in the previous simulations (Section 5.1 (5.3)) but with streamlines more curved, i.e. inhomogeneous fields in contrast to the rather homogeneous one of Section 5.1
- superpositions of the rather homogeneous dipole field from (5.3) with an analogous dipole field on the opposite side of \( \Omega \).

For simplicity, only "model W" in case of \( \Omega = \Omega' \) and linear diffusion will be considered.
The inhomogeneous fields are generated similarly as in (5.3), but the dipole is placed closer to the domain while reducing at the same time its maximum intensity. Let us motivate these choices.

Consider a dipole which is located at the origin \((0,0)\) pointing to the left, described by the potential
\[
\phi(x, y) := -\frac{x}{x^2 + y^2}, \quad \text{for } (x, y) \neq (0, 0).
\]

Along the one-dimensional \(x\)-axis the potential becomes \(-\frac{1}{x}\) and its derivative – the dipole field – becomes \(\frac{1}{x^2}\). Evidently, the field lines are more curved when closer to the origin due to the singularity of the dipole field at the origin, see e.g. Figure 5.30 (created with "GNU Octave"). Also the gradient of the field – which has the main bearing on the magnetic force – is a lot steeper.

Therefore, the intensity of the dipole potential will be decreased in a way that guarantees robust simulations. Due to the steeper gradient, it is impossible to have a magnetic force of the same order as before (cf. Section 5.1) globally in \(\Omega\). As a rule of thumb, the magnetic force will be comparatively stronger near the right boundary of \(\Omega\) but weaker in the middle of \(\Omega\) compared to the previous simulations. As a side-effect, the attraction of particles is expected to start very slowly which is why the final time of the corresponding simulations will has been chosen much higher than in Section 5.1.

Two different inhomogeneous fields will be examined, determined via (5.1) by the parameters
\[
d_{a1} = \left(\begin{array}{c} -1 \\ 0 \end{array}\right), \quad C_{a1} = 5, \quad x_{a1} = \left(\begin{array}{c} 1.75 \\ 0.5 \end{array}\right),
\]
\[
(5.7)
\]
or
\[
d_{a1} = \left(\begin{array}{c} -1 \\ 0 \end{array}\right), \quad C_{a1} = 0.5, \quad x_{a1} = \left(\begin{array}{c} 1.25 \\ 0.5 \end{array}\right).
\]
\[
(5.8)
\]
respectively. For comparison, in Section 5.1 the dipole’s position and intensity were \(x_{a1} = (3, 0.5)\) and \(C_{a1} = 100\). The dipoles above are successively placed closer to \(\Omega\) accompanied with lower intensity respectively.

Another type of external magnetic field to be considered in this section is a superposition of two dipoles which are located symmetrically to the center \((0.5, 0.5)\) of \(\Omega\). If one considers only a single dipole field, its direction does not matter in regards to the magnetic force \((\mathbf{m} \cdot \nabla)\mathbf{h}\), as the latter is unaffected if the sign of both \(\mathbf{h}\) and the aligned magnetization field \(\mathbf{m}\) changes. However, the magnetic force
\[
(\mathbf{m} \cdot \nabla)\mathbf{h} \approx (\chi_{\text{lin}}(c))\mathbf{h}_a \cdot \nabla)\mathbf{h}_a
\]
\[
(5.9)
\]
is nonlinear in $h_a$ and thereby it does not satisfy the superposition principle. Nevertheless, the particle density is expected to evolve symmetrically independent of the directions of the symmetrically placed dipoles. Let us briefly discuss the reason.

For the ease of presentation, consider equally intense dipoles described by $\phi_{a1}$, $\phi_{a2}$ and $\tilde{\phi}_{a2}$, where the respective positions $x_{a1}$, of dipole $\phi_{a1}$, and $x_{a2}$, of dipoles $\phi_{a2}$ and $\tilde{\phi}_{a2}$, are symmetric to the origin. Let the dipoles $\phi_{a1}$ and $\phi_{a2}$ be directed in the same direction and let the direction of $\tilde{\phi}_{a2}$ be the opposite, i.e.

$$d_{a2} = d_{a1} \quad \text{and} \quad \tilde{d}_{a2} = -d_{a1}.$$ 

The resulting dipole fields are denoted by

$$h_{a1} := \nabla \phi_{a1}, \quad h_{a2} := \nabla \phi_{a2} \quad \text{and} \quad \tilde{h}_{a2} := \nabla \tilde{\phi}_{a2}.$$ 

Assuming $m = \chi_{\text{lin}}(c)h$ and neglecting the stray field, it suffices to consider the terms $(h_a \cdot \nabla)h_a$ and $(\tilde{h}_a \cdot \nabla)\tilde{h}_a$, where

$$h_a := h_{a1} + h_{a2} \quad \text{and} \quad \tilde{h}_a := h_{a1} + \tilde{h}_{a2}$$

denote the respective external fields of the two superpositions. Based on (5.1), one easily computes

$$\phi_{a2} = -\phi_{a1}(\cdot) \Rightarrow h_{a2} = h_{a1}(\cdot)$$

$$\Rightarrow (h_a \cdot \nabla)h_a = ((h_{a1} + h_{a2}(\cdot)) \cdot \nabla)(h_{a1} + h_{a2}(\cdot))$$

and

$$\tilde{\phi}_{a2} = \phi_{a1}(\cdot) \Rightarrow \tilde{h}_{a2} = -h_{a1}(\cdot)$$

$$\Rightarrow (\tilde{h}_a \cdot \nabla)\tilde{h}_a = ((h_{a1} - h_{a2}(\cdot)) \cdot \nabla)(h_{a1} - h_{a2}(\cdot)).$$

In both cases, symmetry with respect to the origin is obvious.

Let two dipoles on opposite sides of $\Omega$ be determined via (5.1) by the parameters

$$d_{a1} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad d_{a2} = \mu d_{a1}, \quad C_{a1} = C_{a2} = 100, \quad x_{a1} = \begin{pmatrix} 3 \\ 0.5 \end{pmatrix}, \quad x_{a2} = \begin{pmatrix} -2 \\ 0.5 \end{pmatrix}, \quad (5.10)$$

where

$$\mu \in \{-1, 1\},$$

i.e. the dipole directions are chosen to be parallel ($\mu = 1$) or antiparallel ($\mu = -1$), respectively.

In addition to symmetry, it can be predicted that the support of the magnetic particles’ density will be compressed vertically in case of $\mu = 1$ and will be expanded vertically in case of $\mu = -1$. This can easily be seen by considering the sign of the applied dipole fields. Starting in the middle of $\Omega$, the field lines of both dipole fields are oriented horizontally. When moving away from the center vertically, the lines start to spread out vertically. Therefore, the sign of the dipole fields determines if the fields reinforce/compensate each other on horizontally or vertically. The resulting gradients in the magnetic field along the vertical axis $(0.5, y)$, $y \in [0, 1]$, lead to the aforementioned compression or expansion, see also Figure 5.31.
5.3 Study of different external magnetic fields

Figure 5.31: Sketch of the superposition \([\text{5.10}]\) of dipole fields and a hint on its impact on the magnetic force, based on the term \((\textbf{h}_a \cdot \nabla)\textbf{h}_a\) which is an approximation (cf. \((\text{5.9})\)) of the magnetic force up to rescaling by the susceptibility \(\chi_{\text{lin}}(c)\). Case \(\mu = 1\) is presented in the top row, case \(\mu = -1\) in the bottom row.

(A1) Inhomogeneous fields – part 1. First, the simulation which uses the field from \([\text{5.7}]\) will be presented. The maximum time increment is \(\tau_{\text{max}} = 0.001\). In the course of time, there were only three changes of the time increment, forming four time periods of qualitatively equal length, where the respective time increment either was identical to the maximum increment or its half.

Especially at the top right and bottom right corners of Figure 5.32 one can see how the contour lines (in gray color) of the particle density \(c\) are curved in a more pronounced manner than in the
earlier simulations of Section 5.1, approximately perpendicular to the magnetic field lines. No visible peaks have emerged from the right corners of the domain as in e.g. Figure 5.4 for \( t = 2.1 \). Also note that the spread of the density’s support to the left (in the beginning of the simulation) is slower than in Section 5.1 while the diffusion and mobility parameters are kept the same – further indicating that magnetic effects are the main cause for this effect (see also Figure 5.5).

As expected, the attraction of the particles to the right side starts very slowly (due to the low intensity of dipole field), also the magnetization in the middle of \( \Omega \) is rather small. Consequently, the size of the red/green cylinders in Figure 5.33 has been rescaled by a factor of 2 in contrast to the simulations from Section 5.1 to improve visibility. However, the magnetic force at the right boundary of \( \Omega \) is quite strong (due to a steep gradient in the magnetic field). This leads to a higher maximum density of 0.8944 (at \( t = 100 \)) in contrast to approximately 0.5 for the simulations of Section 5.1 (at \( t = 16 \)). At the same time, the area obtained by the particles’ support is much slimmer. This leads to significantly larger gradients of the particle density and to difficulties in numerical simulations which again motivate the choice of the reduced intensity in (5.7).

The stray field \( h_d \), see Figure 5.33, behaves similarly to the case of the rather homogeneous magnetic field from (5.3). At \( t = 30 \) it can be seen that the field lines close to the right corners begin pushing inwards as in e.g. Figure 5.6g but the change of direction that went along with it in Section 5.1 has not occurred yet at time \( t = 100 \). However, this change might still happen at a much later time.

\[ \text{(A1) Inhomogeneous fields – part 2.} \quad \text{Second, the results corresponding to the dipole field from (5.8) will be presented. The time increment remained constant at } \tau = 0.0005 \text{ during the whole simulation.} \]

Figure 5.33 and Figure 5.35 show the same features than the previous simulation but more pronounced. Here, the red/green cylinders depicting the magnetization have been rescaled by a factor of 6 in contrast to the simulations from Section 5.1 to improve visibility.
5.3 Study of different external magnetic fields

Figure 5.34: Model W – linear diffusion, \( \Omega = \Omega' \), (external) magnetic field as described by (5.8). Particle density \( c \) and total magnetic field \( h \) visualized according to Figure 5.2.

Figure 5.35: Model W – linear diffusion, \( \Omega = \Omega' \), (external) magnetic field as described by (5.8). Particle density \( c \), stray field \( h_d \) and magnetization \( m \) visualized according to Figure 5.3.

Even though at \( t = 30 \) the attraction to the right boundary is far from complete, the density values are very high (the maximum value is 1.052) and the particles accumulate only at a small portion of the boundary. However, it is beyond the scope of this thesis to pursue this scenario further due to the high computational costs. Recall that the particles are so densely packed at the right boundary of \( \Omega \) that there seem to be only two ways to deal with the corresponding steep gradients of the density – decreasing the time increment or weakening the magnetic field resulting in a later final time, the latter of which has been our choice. Moreover, notice how the density is larger than 1 and therefore one should think about using "model B" which formally is expected to bound the particle density \( c \) at its maximum physical value 1, if \( c \) has been chosen to model a volume concentration.

(A2) Multiple sources – part 1. Consider the case \( \mu = 1 \) first (cf. (5.10)) – where the dipole fields reinforce each other. For this simulation, the mixed finite volume scheme (cf. (4.126)) has been used to improve stability. Moreover, the time increments had to be adapted manually throughout the simulation to prevent instabilities. The increment started at \( \tau = 0.0005 \), was decreased to \( \tau = 0.00025 \) at \( t = 2.972 \) and to \( \tau = 0.000125 \) at \( t = 3.991 \).

As predicted earlier in Section 5.3, the support of the particle density is symmetric and it is thinner in vertical direction in the middle of \( \Omega \) than at the boundary, see Figure 5.36. Nevertheless, the support is not split up in two separate parts. The particle density seemingly reached its final distribution already shortly after \( t = 2.5 \). However, the numerical solution was found to oscillate slightly around the equilibrium, most likely due to numerical instabilities in regards to the steep gradients of \( c \) which are found at the top and bottom flanks in the horizontal center of the particle density’s support.

The stray field seemed to be stationary early on as well, see Figure 5.37. In contrast to the particle density, oscillations of the stray field around the supposed equilibrium could not be noted.
5.3 Study of different external magnetic fields

Figure 5.36: Model W – linear diffusion, $\Omega = \Omega'$, (external) magnetic field as described by (5.10), $\mu = 1$. Particle density $c$ and total magnetic field $h$ visualized according to Figure 5.2. Recall that the black color levels of $c$ are adapted for each image to the current maximum density value, i.e. mass is conserved despite the increasing size of the dark black areas.

Qualitatively, it is directed uniformly to the right – the opposite direction of the applied field $h_a$. The size of the red/green cylinders was rescaled by a factor of 0.5 compared to the earlier simulations in Section 5.1.

Figure 5.37: Model W – linear diffusion, $\Omega = \Omega'$, (external) magnetic field as described by (5.10), $\mu = 1$. Particle density $c$, stray field $h_d$ and magnetization $m$ visualized according to Figure 5.3. Recall that the black color levels of $c$ are adapted for each image to the current maximum density value, i.e. mass is conserved despite the increasing size of the dark black areas.
\textbf{(A2) Multiple sources – part 2.} Now, let us present the case $\mu = -1$ (cf. (5.10)) – where the dipole fields compensate each other. For this, the mixed finite volume scheme was used, too, and the maximum time increment was set to $\tau_{\text{max}} = 0.00025$ from the start. It was adapted manually to $0.001$ from $t = 3.8575$ on.

The predictions of symmetry and expansion of the support in vertical direction are met, see Figure 5.38. One can observe that the attraction to the left and right sides of $\Omega$ is stronger than

\begin{figure}[h!]
\centering
\includegraphics[width=\textwidth]{figure5_38}
\caption{Model W – linear diffusion, $\Omega = \Omega'$, (external) magnetic field as described by (5.10), $\mu = -1$. Particle density $c$ and total magnetic field $h$ visualized according to Figure 5.2.}
\end{figure}

\begin{figure}[h!]
\centering
\includegraphics[width=\textwidth]{figure5_39}
\caption{Model W – linear diffusion, $\Omega = \Omega'$, (external) magnetic field as described by (5.10), $\mu = -1$. Particle density $c$, stray field $h_d$ and magnetization $m$ visualized according to Figure 5.3.}
\end{figure}

the expansion towards the top and bottom boundary. Additionally, in the final distribution of the
density the center of $\Omega$ is 'numerically' empty, i.e. $c(16, (0.5, 0.5)) = 0.000199$. In contrast to the final distributions in Figure 5.32 and Figure 5.34 where accumulation of particles is predominant in the center of the left and right boundary segments, the particles in this case are mainly in the corners of $\Omega$. The stream lines of the total magnetic field seem to be slightly pulled to the center in the beginning of the simulation.

In Figure 5.39, the size of the green/red cylinders – depicting the magnetization – was rescaled by a factor of 2 in contrast to the visualizations in Section 5.1 (which is the reciprocal of the factor 0.5 used for Figure 5.37). The reason is the rather weak magnetization $m$ in the center of the simulation domain which is a consequence of compensation ($\mu = -1$) of the two dipole fields \((5.10)\). Later on, the magnetization is rather large in the corners of $\Omega$, see Figure 5.39h. The stray field $h_d$ in the beginning of the simulation enters the support of the particle density from the left and right and leaves it at the bottom and top – like the applied field $h_a$ does, see Figure 5.31d. This corresponds to the outer straying part of $h_d$, while on the inside of the density’s support the stray field is opposing the applied field as usual.

The simulation results of scenario (A2) clearly show the nonlinear nature of the magnetic force. A superposition principle does not even apply approximately. Comparing Figure 5.36 with Figure 5.38 or Figure 5.37 with Figure 5.39, respectively, the motion of the magnetic particles is greatly affected by the direction of a single dipole of the superposition.
Notation

The extensive notation used in this dissertation will be summarized here. In general, the following style guidelines apply.

- Bold face characters denote vector valued quantities in $\mathbb{R}^d$, where $d \in \{2, 3\}$. However, components of often used vector fields may be typeset with bold face characters, too, accompanied with lower indices.

- Plain characters are real valued scalars or elements of unspecified abstract vector spaces.

- A plain upper case character indicates that the main interest in this thesis is not the scalar itself but the gradient field it induces, e.g. the magnetic field $h = \nabla R$ is induced by the potential $R$. Other quantities that are related to $R$ may be written in the same style.

The explanations of notation are ordered in multiple groups. First, the continuous setting is explained, then the discrete setting. The following lists do not contain all symbols of this thesis. Additional notation may be introduced as needed during short passages or proofs. The latter will be explained in the respective parts of this thesis but will not be listed here. However, important notation or notation that will be reused much later after its first occurrence will be contained in the following.

### General notation of the continuous setting.

- $\Omega, \Omega'$ Spatial domains, where $\Omega \subset \Omega'$. See $\text{H1 3.1}$ for further assumptions.
- $T, I$ Final time $T > 0$ and time interval $I = (0, T)$.
- $d$ Spatial dimension, $d \in \{2, 3\}$. Will only be specified, if necessary.
- $\nu$ Outer unit normal vector.

### Continuous quantities and parameters.

- $u, p$ Velocity field and pressure, see (2.89a), (2.89b).
- $c$ Particle number density, see (2.89c).
- $h, R$ Magnetic field $h := \nabla R$ and its potential $R$, see (2.89f), (2.89g).
- $h_a$ Given external magnetic field, satisfying (2.13).
- $h_d$ This is the component of the total magnetic field $h = h_a + h_d$ (cf. (2.18)) which is induced by the magnetized particles – called stray field or demagnetizing field.
- $m, l_\Omega m$ Magnetization, see (2.89c), and its extension by 0 onto $\Omega$.
- $\mathbf{v}_{\text{part}}, J, k, T$ Convective velocity $\mathbf{v}_{\text{part}}$ of magnetic particle density $c$, see (2.89d), and the corresponding flux $J = c \mathbf{v}_{\text{part}}$, see (2.8). The former is identified via Onsager’s variational method in Section 2.3. The stress tensor $T$ and force density $k$ of the momentum equation are identified in Section 2.3 too.
- $u_{\text{init}}, c_{\text{init}}, m_{\text{init}}$ Initial data of the continuous models, see (2.91), (2.102) or (2.112).
- $\chi, \tilde{\chi}, \chi_{\text{lin}}, l_\Omega \chi_{\text{lin}}$ Susceptibility $\chi = \chi(c, h)$, see (2.32). The definition of $\chi(c, h)$ includes the factor $\tilde{\chi}(c)$ assumed to be subjected to a growth condition in the analytical part of this thesis, see (13 3.3). The linearized susceptibility is denoted by $\chi_{\text{lin}} = \chi_{\text{lin}}(c)$, see (2.34). The extension by 0 of $\chi_{\text{lin}}$ onto $\Omega'$ is denoted by $l_\Omega \chi_{\text{lin}}$.
- $f_p, f_p^n$ Nonlinear mobility function for the particle density, chosen as $f_p(c) = c^m$, $m \in [0, 2]$ in Section 3. Existence of distributional solutions established for $m = 2$. $(f_p^n)_{n \in \mathbb{N}}$ is an approximating sequence, see (3.157).
- $g_s^L, g_s$ Regularized entropic functions, see (3.73), (4.19). For convenience, let $0 < s < e < L$, where $e$ is Euler’s number.
Densities of the two-phase flow model, "model B", expressed in different ways, i.e. \( \hat{\rho}(\varphi) \) and \( \rho(c) \), see (2.46), (2.47). They are built by the respective fluids' pure densities \( \hat{\rho}_1, \hat{\rho}_2 \). In the single-phase case, the function is replaced by the constant \( \rho_0 \).

Within Section 2 \( \varphi \) is a phase field parameter \( \varphi = c - \varphi_1 \) describing the mixture of the two phases via volume densities of fluid particles \( \varphi_1 \) and magnetic particles \( c \).

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Within Section 2 \( \varphi \) is a phase field parameter \( \varphi = c - \varphi_1 \) describing the mixture of the two phases via volume densities of fluid particles \( \varphi_1 \) and magnetic particles \( c \).

Reguarlization parameter in the magnetization equation (2.89e).

Reguarlization parameter used in the particle density equation (3.140b) of the weak formulation.

Generic parameters defining the energy of the system (2.89), see (2.59a) or [39].

Mobility and diffusion parameters in the evolution equation of the particle density \( c \). Notice that the actual diffusion is given by their product \( KD \).

Dynamic viscosity of the carrier fluid.

Vacuum permeability/magnetic constant and relaxation time for rearrangement of magnetic spins in alignment with the magnetic field.

Members of a sequence of solutions either to the Galerkin scheme (3.93) or to (3.140) in the regularized case, see Definition 3.33 or Definition 3.54, respectively. There, \( h_n := \nabla R_n \).

Members of a sequence of convective velocities, defined by approximate solutions, see (3.128) on the discrete level and (3.159) on the level of solutions to the regularized model provided by Theorem 3.50.

Operators for the continuous setting.

\((\cdot)_t, \partial_t\) Differentiation with respect to time.

\(D\) Symmetric gradient of vector fields.

\(\text{div}, \nabla, \partial_{x_i}\) Divergence and gradient operators as well as partial derivative with respect to the \( i^{th}\) component of the spatial variable.

\(\text{curl}, \times, \text{Curl}\) The curl and \(\times\) operators defined in three dimensions as usual and defined according to Section A.4 in two dimensions. The operator \(\text{Curl}\) is another two-dimensional analogue of curl.

\((())_s, ((())^L), (((()))^k)\) Cut-off operator from below at \( s > 0 \), cf. (3.75) or cut-off from above at \( L > 0 \), cf. (3.158), or combination of both, respectively.

\(\Pi_{\mathcal{C}_n}, \Pi_{\mathcal{U}_n}, \Pi_{\mathcal{M}_n}\) \(L^2\)-orthogonal projection operators onto \( \mathcal{C}_n, \mathcal{U}_n, \mathcal{M}_n \), see (3.82), based on the basis representations (3.72), (3.67) and (3.33). Those are stable in the \(L^2\)-norm – due to orthogonality – and other norms, cf. (3.84).

\(\Pi_{\mathcal{R}_n}\) Projection operator, cf. (3.82), based on the basis (3.61). No stability result needed for this operator.

\(\Pi_{\mathcal{S}_n}\) \(L^2\)-orthogonal projection operator based on the basis (3.61), which is related to the basis of \(\mathcal{H}\), according to (3.64). See its definition in (3.81). Stability results are noted in (3.80).

Function spaces in the continuous setting and related spaces.

\(\mathcal{M}\) The space \(\mathcal{M}\) is used as starting point to find Galerkin ansatz space for the magnetization equation (3.140d), see (3.7).

\(\mathcal{H}, \mathcal{S}, \mathcal{S}^o\) Special subsets of \(\mathcal{M}\). The gradient fields in \(\mathcal{M}\) are denoted by \(\mathcal{H}\), cf. (3.17), while those gradient fields that have constant trace on \(\partial \Omega\) are denoted by \(\mathcal{S}\), cf. (3.18). The sum \(\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^o\) is direct and all spaces are equipped with the \(H^1(\Omega)^d\)-norm, see also Lemma 3.5.

\(\mathcal{V}\) Vector fields of \(\mathcal{M}\) that are no gradient fields. The complement \(\mathcal{V}\) is \(L^2\)-orthogonal to the gradient fields \(\mathcal{H}\) of \(\mathcal{M}\), see (3.20).
Basis functions and their auxiliary functions regarding the Galerkin approximation.

\( \mathcal{R} \)
Starting point to find Galerkin ansatz spaces for the magnetostatic equation (3.140c), see (3.8).

\( \mathcal{U} \)
This space serves as starting point to find Galerkin ansatz spaces for the Navier-Stokes equations (3.140a), see (3.14).

\( \mathcal{C} \)
This space serves as starting point to find Galerkin ansatz spaces for the particle density equation (3.140b), see (3.15).

\( \mathcal{U}_n, \mathcal{C}_n, \mathcal{R}_n, \mathcal{M}_n, \mathcal{K}_n \)
Finite dimensional subspaces of \( \mathcal{U}, \mathcal{C}, \mathcal{R}, \mathcal{M}, \mathcal{K} \) used for a Galerkin approximation argument, see (3.93) and Lemma 3.31.

\( L^p(I; L^q_{\text{loc}}(\Omega)) \)
Space of functions \( f \) that satisfy \( f \in L^p(I; L^q(\hat{\Omega})) \) for all \( \hat{\Omega} \subset \Omega \). In addition, whenever the notion \( L^q_{\text{loc}}^{s, \text{H}}(\cdot) \) inside a logical statement \( A \) occurs, it is an abbreviation of \( \forall \hat{\Omega} \subset \subset \Omega : A \) holds.

\( L^p(I; W^{k,q}_{\text{loc}}(\Omega)) \)
Analogously defined as \( L^p(I; L^q_{\text{loc}}(\Omega)) \).

\( W^{k,p}(I; X) \)
This is the space of functions in \( L^p(I; X) \) which have time derivatives up to order \( k \) which are in \( L^p(I; X) \) as well. This shall not be mistaken for a situation where the time derivative is in a dual space as it is defined in some literature.

\( L^p(\cdot; \cdot) \)
In a logical statement \( A \), the notion \( L^p(\cdot; \cdot) \) means that \( \forall r \in [1, p] : A \) holds.

\( H(\text{div})(\cdot), H(\text{div}_0)(\cdot) \)
Space of \( L^2(\cdot;\cdot)^d \)-functions with distributional divergence in \( L^2(\cdot;\cdot) \) in the former case. In the latter case, the divergence vanishes, additionally. Further information is available in Section A.1.

\( H(\text{curl})(\cdot), H(\text{curl}_0)(\cdot) \)
Analogously defined as the \( H(\text{div}) \)-spaces but with distributional curl instead of div.

\( H_{00}(\text{div})(\cdot) \)
Space of \( H(\text{div})(\cdot) \)-functions with vanishing distributional normal trace.

\( H_{00}(\text{curl})(\cdot) \)
Space of \( H(\text{curl})(\cdot) \)-functions with vanishing distributional tangential trace.

\( H(\text{div}, \text{curl})(\cdot) \)
Intersection \( H(\text{div})(\cdot) \cap H(\text{curl})(\cdot) \).

\( H(\text{div}_0, \text{curl}_0)(\cdot) \)
Intersection \( H(\text{div}_0)(\cdot) \cap H(\text{curl}_0)(\cdot) \).

\( H^2(\Omega) \)
Space of \( H^2 \)-functions with homogeneous Neumann boundary data, see (3.15).

Basis functions and their auxiliary functions regarding the Galerkin approximation.

\( s_i, s_i^+ \)
Basis functions of \( \mathcal{S} \) or \( \mathcal{S}^0 \), respectively, see (3.30).

\( h_i \)
Basis functions of \( \mathcal{H} \), combining \( s_j \) and \( s_j^+ \), cf. (3.31).

\( m_i \)
Basis functions of \( \mathcal{V} \).

\( \Psi_i^m \)
Basis functions of \( \mathcal{M} \) combining \( m_j \), \( h_j \), see (3.33).

\( \phi_i^\Omega \)
Potentials (on \( \Omega \)) of basis functions \( h_i \), mean value free on \( \Omega \), cf. (3.36).

\( \tilde{R}_i, R_i \)
Potentials – partly based on \( \phi_i^\Omega \) – that are used to construct a basis of \( \mathcal{R} \), see (3.44). Potentials \( \bar{R}_i \) are mean value free on \( \Omega' \), cf. (3.45).

\( \psi_i^{\text{dir}}, \tilde{\psi}_i \)
Eigenfunctions \( \psi_i^{\text{dir}} \) of the homogeneous Dirichlet-Laplace problem on \( \Omega \), cf. (3.43), and their constant extensions \( \tilde{\psi}_i \) onto \( \Omega \setminus \hat{\Omega} \), see (3.48). The latter functions were constructed for the proof of Lemma 3.17.

\( u_i^V \)
Eigenfunctions of the homogeneous Neumann-Laplace operator on \( V \). In this thesis, \( V = \Omega \) or \( V = \Omega' \).

\( \tilde{q}_i \)
Special functions constructed for the proof of Lemma 3.17, cf. (3.55).

\( p_i, q_i \)
Mean value free (cf. (3.48), (3.55)) variants of \( \tilde{p}_i, \tilde{q}_i \) used as test functions together with \( R_{2i+1} \) to prove that the set \( \{ \tilde{R}_i \}_{i \in \mathbb{N}} \) generates \( \mathcal{R} \).

\( \psi_i^R \)
Basis functions of \( \mathcal{R} \) combining the functions \( R_{2i} \) and re-orthogonalized functions based on \( R_{2i-1} \), see (3.62) and (3.63).

\( \Psi_i^u \)
Basis functions of \( \mathcal{U} \), see (3.67).

\( \Psi_i^C \)
Basis functions of \( L^2(\Omega) \) which are used for the particle density equation (3.140b), see (3.72).
General notation of the discrete finite element setting.

\( \tau, n_\tau, t^k \)

The time increment is denoted by \( \tau \) and in the time step \( k \) the time \( t^k := k \tau \) is reached. The number \( n_\tau \) denotes the last time step, i.e. \( t^{n_\tau} = T \). See (4.5).

\( \mathcal{T}_h(\Omega), \mathcal{T}_h(\Omega') \)

Triangulations of \( \Omega \) or \( \Omega' \), respectively.

\( \mathcal{K}, \mathcal{K}^+, \mathcal{K}^-, \mathcal{K}_i \)

Triangles.

\( P_k \)

Space of polynomials up to degree \( k \).

\( \mathcal{P}_k, \mathcal{P}_{k,\text{mean}}^0, \mathcal{P}_{k,\text{zero}}^0 \)

Space of continuous piecewise polynomial finite element functions of degree \( k \). Upper index 'mean' means that only functions with zero mean value are included. Upper index 'zero' denotes zero boundary values. See (4.7).

\( \mathcal{D}_k, \mathcal{D}_k^{\text{mean}} \)

Analogous to \( \mathcal{P}_k, \mathcal{P}_{k,\text{zero}}^0 \) but finite elements may be discontinuous, see (4.7).

\( \mathcal{U}_h, \mathcal{P}_h, \mathcal{C}_h, \mathcal{V}_h, \mathcal{R}_h, \mathcal{M}_h \)

Symbols for the finite element spaces used for the unknowns \( u, p, c, v_{\text{part}}, R, m \) of the models in Section 2. The specific choice of spaces can be seen in (4.17).

\( \mathcal{U}_h \)

Weakly divergence free finite element space which is used for the momentum equation – instead of \( \mathcal{U}_h \) which is used in the numerical schemes – in the section about discrete existence of solutions, Section 4.2.

\( \mathcal{F}_{\text{int}}, \nu_{\mathcal{F}_{\text{int}}} \)

Set of interior surfaces of the mesh of \( \Omega \) and the corresponding unit normal \( \nu_{\mathcal{F}_{\text{int}}} \) of which the direction is arbitrarily fixed, see (4.11) and above.

Discrete quantities.

\( u^k, p^k, c^k, v_{\text{part}}^k, P^k, m^k \)

Discrete solution of the numerical scheme in time step \( k \). The discrete quantities correspond to the respective continuous quantities \( u, p, c, v_{\text{part}}, R, m \).

\( u^0, c^0, m^0, R^0 \)

Discrete initial data, where \( R^0 \) is not explicitly given but computed as in (4.35) in case of ‘model GW’ or (4.52) for the other models.

\( (h_{\mathcal{U}})_h^k \)

Discrete external magnetic field at time \( t^k \), see (4.18).

\( g_{s,h}^k, g_{s}^k \)

Discrete versions of entropy and its derivative, see (4.19) and (4.20).

\( f_{\mathcal{p},h}^{(s),k-1}, f_{\mathcal{p},h}^{(s)} \)

Approximations of the particle mobility function \( f_\mathcal{p} \), see (4.23), (4.24).

\( \chi_{r}, \chi_{s} \)

Regularized susceptibilities (see (N2 4.2) or (4.46)) used for the discretization of ‘model GW’ or the other models, respectively. The parameter \( s = (s, \omega) \) controls the affected range and the start of the regularization from below.

\( \chi_{r,h}, \chi_{s,h}^k, \chi_{s,h}^k \)

Discrete versions of \( \chi_{r}, \chi_{s} \) (see (4.27) or (4.47)) and the extension of \( \chi_{s,h}^k \) by 0 onto \( \Omega' \).

\( b_{\mathcal{F}_{\text{int}}}^m \)

Trilinear form which is used to approximate Kelvin force and related coupling terms, see (4.30).

\( \text{curl}_h m^k, \text{div}_h m^k, \text{div}_h h^k \)

Discrete versions of \( \text{curl} m, \text{div} m, \text{div} h \) which are defined via duality as in (4.31) (see (4.34) if \( d = 2 \), (4.32), (4.33).

\( \Theta \)

This is a function used for switching between ‘model W’ and ‘model B’.

\( \rho^k_h \)

Discrete density of fluid particles, see (4.41).

\( c_h^{k-1,r} \)

Discrete version of cut-off (from above) particle density, see (4.42).

\( \delta^{k-1}_h, \delta^{k-1}_h \)

Projection onto space of continuous piecewise linear finite elements via the lumped \( L^2 \)-scalar product, see (4.44).

\( \delta 

\delta 

Discrete quantities – resembling \( u^k, c^k, R^k, h^k, m^k \) of the numerical scheme – which are used for the fixed point argument in Section 4.2.

\( \hat{\mathbf{v}}_{\text{part}}, \hat{\chi}_{r,h}, \hat{g}_{s,h}^k, \hat{g}_{s,h}, \hat{\chi}_{s,h}, \hat{\rho}_h \)

Some abbreviations, defined similarly as \( \hat{\mathbf{v}}_{\text{part}}^{k-1,h}, \hat{g}_{s,h}^k, g_{s,h}, \hat{\chi}_{s,h}, \hat{\chi}_{s,h}, \hat{\rho}_h \) but with arguments \( \hat{c}, \hat{R}, \hat{m}, \hat{h} \), see (4.106) and below as well as (4.118) and below.
Operators for the discrete setting.

\[ [], \{ \cdot \} \] Jump and average value, see (4.12).

\[ \mathcal{I}_{h,1}, \hat{\mathcal{I}}_{h,1}, I_{h,1}, \hat{I}_{h,1} \] Interpolation operator \( \mathcal{I}_{h,1} \) onto the space of linear discontinuous finite elements defined by elementwise Lagrangian interpolation (4.8). Applied to continuous functions, the result lies in the respective continuous finite element space and the symbol \( I_{h,1} \) is used for clarification. The operators with upper index \( d \) are vector-valued analogues.

\[ (\cdot)_{s,h} \] Discrete version of the cut-off operator (from below) which is used for the magnetic particle density, see (4.22).

\[ \mathcal{L}^{GW}, \mathcal{L}^{W|B} \] Solution operators used for the fixed point argument in Section 4.2 see (4.108), (4.119).
Appendix

A.1 The spaces $H(\text{div})(\Omega)$ and $H(\text{curl})(\Omega)$

Having a look on the magnetostatic equations

\[ \text{curl} \, h = 0 \quad \text{and} \quad \text{div} \, b = 0 \]

it is obvious that – analogously to the standard Sobolev spaces which admit a weak gradient – weak versions of the div and curl operators will be needed to construct Banach spaces that fit our setting. This gives raise to a class of function spaces which are defined similarly as the Sobolev spaces by means of distributional derivatives which can be identified with $L^2$-functions. Many literature is available on these spaces, see e.g. [18] or [24] for the use in discontinuous Galerkin methods. It is not the purpose of this thesis to give a thorough introduction about those spaces. Some details will be left out or adapted to the specific setting of this thesis. The interested reader is advised to read the aforementioned references.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Unless stated otherwise, the space dimension $d$ is either two or three, $d \in \{2, 3\}$. See Section A.4 for details on the two-dimensional curl-operator.

The distributional versions of the operators of interest can be defined (without introducing new notation) by

\[
\text{div} : L^2(\Omega)^d \to (H^1_0(\Omega))^', \\
(\text{div} f)(\varphi) := - \int_\Omega f \cdot \nabla \varphi \, dx \quad \forall \varphi \in H^1_0(\Omega)
\]

(A.1)

or

\[
\text{curl} : L^2(\Omega)^d \to (H^1_0(\Omega)^d)', \\
(\text{curl} f)(\Phi) := \int_\Omega f \cdot \text{curl} \, \Phi \, dx \quad \forall \Phi \in H^1_0(\Omega)^d,
\]

(A.2)

respectively.

**Definition A.1.** i) Space of functions with distributional $\text{div}$ in $L^2$,

\[
H(\text{div})(\Omega) := \left\{ f \in L^2(\Omega)^d \left| \exists g \in L^2(\Omega) : \int_\Omega g \varphi \, dx = (\text{div} f)(\varphi) \quad \forall \varphi \in H^1_0(\Omega) \right. \right\}.
\]

The $L^2$-function which is identified with the distributional derivative is also called $\text{div} \, f$ to simplify notation.

A scalar product on this space is given by

\[
(f, g)_{H(\text{div})(\Omega)} := (f, g)_{L^2(\Omega)^d} + (\text{div} f, \text{div} g)_{L^2(\Omega)}.
\]

(A.3)

ii) Space of functions with distributional $\text{curl}$ in $L^2$,

\[
H(\text{curl})(\Omega) := \left\{ f \in L^2(\Omega)^d \left| \exists g \in L^2(\Omega)^d : \int_\Omega g \cdot \Phi \, dx = (\text{curl} f)(\Phi) \quad \forall \Phi \in H^1_0(\Omega)^d \right. \right\}.
\]

The $L^2$-function which is identified with the distributional derivative is also called $\text{curl} \, f$ to simplify notation.
A scalar product on this space is given by
\[ \langle f, g \rangle_{H(\text{curl})(\Omega)} := \langle f, g \rangle_{L^2(\Omega)^d} + \langle \text{curl} f, \text{curl} g \rangle_{L^2(\Omega)^d}. \] (A.4)

iii) Space of functions with distributional div in \( L^2 \) and distributional curl in \( L^2 \),
\[ H(\text{div}, \text{curl})(\Omega) := H(\text{div})(\Omega) \cap H(\text{curl})(\Omega). \]

A scalar product of this space is given by
\[ \langle f, g \rangle_{H(\text{div})(\Omega)} := \langle f, g \rangle_{L^2(\Omega)^d} + \langle \text{div} f, \text{div} g \rangle_{L^2(\Omega)} + \langle \text{curl} f, \text{curl} g \rangle_{L^2(\Omega)^d}. \] (A.5)

**Remark A.2.** The space of \( H(\text{div}) \)-functions can be defined in any dimension \( d \in \mathbb{N} \). The space \( H(\text{curl})(\Omega) \) is defined in two and three dimensions, see Section A.4, as the curl-operator can be defined in the two-dimensional setting as well. In that case, in the definition of the scalar products (A.4), (A.5) the vector-valued \( L^2 \)-scalar products of the curl-terms has to be replaced with the scalar-valued product.

Another curl-related space in two dimensions is defined as follows,
\[ H(\text{curl})(\Omega) := \left\{ f \in L^2(\Omega) \mid \exists g \in L^2(\Omega)^2 : \int_{\Omega} g \cdot \Phi \, dx = \int_{\Omega} f \cdot \Phi \, dx \quad \forall \Phi \in H^1_0(\Omega)^2 \right\}. \] (A.6)

The spaces \( H(\text{div})(\Omega) \) and \( H(\text{curl})(\Omega) \) admit the definition of normal or tangential traces, respectively, in a distributional sense. The construction is based on integration by parts, i.e. see (A.29), (A.30) for three-dimensional and two-dimensional versions of the integration by parts formula for the curl-operator. By means of the inverse trace theorem [48], the traces can be defined as in [18] in the following distributional sense.

**Definition A.3.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded Lipschitz domain.

i) For \( f \in H(\text{div})(\Omega) \) the normal trace \( f \cdot \nu|_{\partial \Omega} \in (H^{1/2}(\partial \Omega))' \) is defined via
\[ \langle f \cdot \nu|_{\partial \Omega}, \varphi|_{\partial \Omega} \rangle_{(H^{1/2}(\partial \Omega))' \times H^{1/2}(\partial \Omega)} := \int_{\Omega} (\text{div} f \cdot \varphi + f \cdot \nabla \varphi) \, dx \quad \forall \varphi \in H^1(\Omega). \] (A.7)

ii) For \( f \in H(\text{curl})(\Omega) \) the tangential trace \( f \times \nu|_{\partial \Omega} \in (H^{1/2}(\partial \Omega)^d)' \) is defined by
\[ \langle f \times \nu|_{\partial \Omega}, \Phi|_{\partial \Omega} \rangle_{(H^{1/2}(\partial \Omega)^d)' \times H^{1/2}(\partial \Omega)^d} := \int_{\Omega} (f \cdot \text{curl} \Phi - \text{curl} f \cdot \Phi) \, dx \quad \forall \Phi \in H^1(\Omega)^d. \] (A.8)

In above definition, the normal traces are independent of the choice of \( \varphi \), which attains the trace \( \varphi|_{\partial \Omega} \). A specific choice can be found via the inverse trace theorem, i.e. there exists a linear and continuous right-inverse of the trace operator. The same applies to the tangential trace.

**Remark A.4.** The term \( f \times \nu \) is called "tangential trace" due to the following reason. Obviously, \( (f \times \nu) \cdot \nu = 0 \), hence the term \( f \times \nu \) is a tangential vector field. Let
\[ f^T := f - (f \cdot \nu)\nu \]
be the projection of \( f \) onto the tangential space. Then
\[ f \times \nu = (f^T + (f \cdot \nu)\nu) \times \nu = f^T \times \nu + 0 \]
\[ = f^T_1 \begin{pmatrix} 0 & -\nu_3 \\ -\nu_2 & 0 & \nu_1 \end{pmatrix} + f^T_2 \begin{pmatrix} \nu_3 \\ 0 \\ -\nu_1 \end{pmatrix} + f^T_3 \begin{pmatrix} -\nu_2 \\ \nu_1 \\ 0 \end{pmatrix} \]
is a transformation of the tangential projection of \( \mathbf{f} \). One can get the tangential components back by
\[
\begin{pmatrix}
0 & -\nu_3 & \nu_2 \\
\nu_3 & 0 & -\nu_1 \\
-\nu_2 & \nu_1 & 0
\end{pmatrix} (\mathbf{f} \times \nu) = \mathbf{f}_T^T \begin{pmatrix}
\nu_2^2 + \nu_3^2 \\
-\nu_1 \nu_2 \\
-\nu_1 \nu_3 \\
\nu_1^2 + \nu_3^2 \\
-\nu_2 \nu_3 \\
\nu_1 \nu_3
\end{pmatrix} + \mathbf{f}_T^T \begin{pmatrix}
-\nu_1 \nu_3 \\
\nu_1^2 + \nu_2^2 \\
\nu_2 \nu_3 \\
\nu_1^2 + \nu_3^2 \\
-\nu_2 \nu_3 \\
\nu_1 \nu_3
\end{pmatrix}.
\]

Moreover, \( (\mathbf{f} \times \nu) = \mathbf{0} \) if and only if \( \mathbf{f}_T = \mathbf{0} \).

Within this thesis the following subspaces will be considered.

**Definition A.5.** Let \( \Omega \) be a bounded Lipschitz-domain.

a) \( H(\Div_0)(\Omega) := \{ \mathbf{f} \in H(\Div)(\Omega) | \Div \mathbf{f} \equiv \mathbf{0} \} \).

b) \( H_{\sigma\sigma}(\Div)(\Omega) := \{ \mathbf{f} \in H(\Div)(\Omega) | \mathbf{f} \cdot \nu|_{\partial \Omega} \equiv \mathbf{0} \} \).

c) \( H(\Curl)(\Omega) := \{ \mathbf{f} \in H(\Curl)(\Omega) | \Curl \mathbf{f} \equiv \mathbf{0} \} \).

d) \( H_{\sigma\sigma}(\Curl)(\Omega) := \{ \mathbf{f} \in H(\Curl)(\Omega) | \mathbf{f} \times \nu|_{\partial \Omega} \equiv \mathbf{0} \} \).

e) \( H_{\sigma\sigma}(\Div)(\Omega) := H(\Div)(\Omega) \cap H_{\sigma\sigma}(\Div)(\Omega) \).

f) \( H_{\sigma\sigma}(\Curl)(\Omega) := H(\Curl)(\Omega) \cap H_{\sigma\sigma}(\Curl)(\Omega) \).

g) \( H(\Div,\Curl)(\Omega) := H(\Div)(\Omega) \cap H(\Curl)(\Omega) \).

h) \( H_{\sigma\sigma}(\Div,\Curl)(\Omega) := H_{\sigma\sigma}(\Div)(\Omega) \cap H_{\sigma\sigma}(\Curl)(\Omega) \).

i) \( H_{\sigma\sigma}(\Div,\Curl)(\Omega) := H_{\sigma\sigma}(\Div)(\Omega) \cap H(\Curl)(\Omega) \).

**Remark A.6.**

- It is well-known that the spaces in **Definition A.1** are Hilbert spaces and that the spaces in **Definition A.5** are closed subspaces of \( H(\Div)(\Omega) \) or \( H(\Curl)(\Omega) \) or \( H(\Div,\Curl)(\Omega) \), respectively.

- The spaces in **Definition A.5** which are involved with the curl-operator can be defined in two or three dimensions according to **Section A.4**. In the two-dimensional case the vector-valued zeros from above in [c] and [d] have to be replaced with scalar-valued zeros.

Having a look on magnetostatic equations again, they come together with transmission conditions. In the setting of this thesis, see **Section 2.1** the case of two domains arises,
\[
\Omega_1 := \Omega \text{ and } \Omega_2 := \Omega' \setminus \overline{\Omega}, \tag{A.9}
\]
where \( \Omega \subset \subset \Omega' \) and both are bounded Lipschitz domains.

Let the magnetic flux \( \mathbf{b} \) be smooth. Its normal component \( \mathbf{b} \cdot \nu \) is said to be continuous across \( \Gamma := \partial \Omega_1 \cap \partial \Omega_2 = \partial \Omega \), i.e. the values on \( \Gamma \) in view of either one of the volumes coincide,
\[
(\mathbf{b}|_{\Omega_1} \cdot \nu)|_\Gamma \equiv (\mathbf{b}|_{\Omega_2} \cdot \nu)|_\Gamma, \tag{A.10}
\]
where \( \nu \) is an outer unit normal of \( \Gamma \) which can be chosen arbitrarily (for each \( \mathbf{x} \in \Gamma \)). If one writes \( \mathbf{\nu}_1 \) and \( \mathbf{\nu}_2 \) for the outer unit normal vectors corresponding to \( \Omega_1 \) or \( \Omega_2 \), respectively, for a fixed \( \mathbf{x} \in \Gamma \) one obviously has \( \mathbf{\nu}_1 = -\mathbf{\nu}_2 \) and either \( \mathbf{\nu} = \mathbf{\nu}_1 \) or \( \mathbf{\nu} = \mathbf{\nu}_2 \). An expression, that
measures this continuity in \((A.10)\), and which does not depend on the choice of the normal, is given by

\[
[b] \cdot \nu|_\Gamma := \begin{cases} 
(b|_{\Omega_1} \cdot \nu)|_\Gamma - (b|_{\Omega_2} \cdot \nu)|_\Gamma & \text{in case of } \nu = \nu_1, \\
(b|_{\Omega_2} \cdot \nu)|_\Gamma - (b|_{\Omega_1} \cdot \nu)|_\Gamma & \text{in case of } \nu = \nu_2 
\end{cases}
\]

The expression \([b] \cdot \nu|_\Gamma\) is called 'normal jump' and is independent of the direction of \(\nu\) (inwards or outwards \(\Omega\)). In fact, as long as one restricts oneself to only normal or tangential jumps, only \(H(\text{div})\)-regular or \(H(\text{curl})\)-regular functions are needed to define above terms in a distributional sense. For instance, if \(b \in L^2(\Omega)^d\) satisfies \(b|_{\Omega_1} \in H(\text{div})(\Omega_1)\) and \(b|_{\Omega_2} \in H(\text{div})(\Omega_2)\) then \([b] \cdot \nu|_\Gamma \in (H^1(\Gamma))'\) could be defined via

\[
\langle [b] \cdot \nu|_\Gamma, \varphi|_\Gamma \rangle_{(H^1(\Gamma))' \times H^1(\Gamma)} := \langle b \cdot \nu_1|_{\partial\Omega_1}, \varphi|_{\partial\Omega_1} \rangle_{(H^1(\partial\Omega_1))' \times H^1(\partial\Omega_1)} + \langle b \cdot \nu_2|_{\partial\Omega_2}, \varphi|_{\partial\Omega_2} \rangle_{(H^1(\partial\Omega_2))' \times H^1(\partial\Omega_2)}
\]

for all \(\varphi \in H^1_0(\Omega')\).

### A.2 Banach space decompositions

In this subsection a few Helmholtz decompositions will be stated which will be utilized to show the similarities to the Banach space decomposition from [Section 3.1.1]. First, for convenience the probably best-known version of the Helmholtz decomposition will be stated. It is as follows,

\[
L^2(\Omega)^n = \nabla[H^1(\Omega)] \oplus L^2(\Omega)^n H_{00}(\text{div}_0)(\Omega),
\]

where \(\Omega \subset \mathbb{R}^n\) is a bounded Lipschitz-domain and \(n \in \mathbb{N}\) is arbitrary, see e.g. [18, equation (1.42)]. However, another type of decomposition is also quite popular and quite similar. See e.g. [18, equation (1.42)],

\[
L^2(\Omega)^n = \nabla[H^1_0(\Omega)] \oplus L^2(\Omega)^n H(\text{div}_0)(\Omega).
\]

Notice the trade-off between boundary information of the two summands. In the first version, the normal trace of the second term vanishes and in the second version the first term has a vanishing tangential trace. This can be seen easily from \(A.8\) via testing by gradient fields and integration by parts.

One can already see the connection of the Helmholtz decomposition to the \(H(\text{div})\)-regular functions. The connection to \(H(\text{curl})(\Omega)\) comes with another Helmholtz decomposition in case of \(n = 3\) and \(\Omega\) additionally being simply connected, see e.g. [18, pp. 216-235]. By combination of (1.42) (p. 216) with (1.79) and (1.78) (p. 235) from [18] and (in notation of [18]) \(\mathbb{1} = \{0\}\) for simply connected \(\Omega\) as well as \(\text{grad } \mathcal{H}^1(\Omega) = H(\text{div}_0, \text{curl}_0)(\Omega)\) for simply connected \(\Omega\) (by Poincaré’s lemma), one infers

\[
L^2(\Omega)^3 = \nabla[H^1_0(\Omega)] \oplus L^2(\Omega)^3 \text{curl}[H_{00}(\text{div}_0, \text{curl}_0)(\Omega)] \oplus L^2(\Omega)^3 H(\text{div}_0, \text{curl}_0)(\Omega).
\]

Here, the first term has vanishing tangential trace, the second term has vanishing normal trace (test by curl-fields in \(A.7\) and integrate by parts), and the last term has no boundary information at all. It is known that \(H_{00}(\text{div}_0, \text{curl}_0)(\Omega)\) (see [35] Remark 3.5 or [18]) and \(H_{00}(\text{div}_0, \text{curl}_0)(\Omega)\) (see [35] Remark 3.9) are trivial if \(\Omega\) is simply connected (and sufficiently regular). Otherwise little is known about those functions which is reflected in the fact that \(H(\text{div}, \text{curl})(\Omega)\) does neither embed continuously into \(H^1(\Omega)^3\) nor compactly into \(L^2(\Omega)^3\), see [6] for a counter example.
The decompositions (A.12) and (A.13) can easily be carried over to $H^k(\Omega)^n$ in case of sufficient boundary regularity, i.e. $\partial \Omega \in C^{k+1}$. The proof of higher regularity is a consequence of regularity theory of the Poisson problem. According to [18], the decomposition

$$H^k(\Omega)^n = \nabla[H^{k+1}(\Omega)] \oplus L^2(\Omega)^n \cap H^k(\Omega)^n$$

(A.15)

holds true. Similarly,

$$H^k(\Omega)^n = \nabla[H^1_0(\Omega)] \cap H^k(\Omega)^n \oplus L^2(\Omega)^n \cap H^k(\Omega)^n.$$  

(A.16)

As the decomposition from (A.16) with $H^k$-regularity is not explicitly included in [18], the proof will be included in this thesis. For this, a very useful lemma will be used.

**Lemma A.7.** Let $X, Y, Z$ be vector spaces and $T : X \to Y$, $S : Y \to Z$ linear operators. Then the following are equivalent:

i) $\ker(S \circ T) = \ker(T)$ and $\operatorname{Rng}(S \circ T) = \operatorname{Rng}(S)$.

ii) $Y = \operatorname{Rng}(T) \oplus \ker(S)$.

**Proof:** This is just a simpler version of [14, Lemma 2.2]. \hfill $\square$

**Lemma A.8.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{k,1}$-boundary, where $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$. The Helmholtz decomposition in (A.16) holds true algebraically and topologically, i.e.

$$H^k(\Omega)^d = \nabla[H^1_0(\Omega)] \cap H^k(\Omega)^d \oplus L^2(\Omega)^d \cap H^k(\Omega)^d$$

(A.17)

and there exists a constant $C > 0$ for all $H^k(\Omega)^d \ni f = \nabla u + w$, where $\nabla u \in \nabla[H^1_0(\Omega)] \cap H^k(\Omega)^d$ and $w \in H(\div) \cap H^k(\Omega)^d$ are determined uniquely, such that

$$\|f\|_{H^k(\Omega)^d} \leq \|\nabla u\|_{H^k(\Omega)^d} + \|w\|_{H^k(\Omega)^d} \leq C\|f\|_{H^k(\Omega)^d}. \quad (A.18)$$

**Proof:** In the Notation of Lemma A.7, define

$$X := H^1_0(\Omega) \cap H^{k+1}(\Omega), \quad Y := H^k(\Omega)^n, \quad Z := X',$$

$$X \xrightarrow{T} Y \xrightarrow{S} X',$$

$$T(u) := \nabla u, \quad S(f) := \left( \varphi \mapsto -\int_{\Omega} f \cdot \nabla \varphi \, dx \; \forall \varphi \in X \right).$$

The goal is to prove the conditions in i) of Lemma A.7. First, by using the definition of the operators, the kernel of $S \circ T$ is just given by

$$\ker(S \circ T) = \left\{ u \in X \left| \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = 0 \; \forall \varphi \in X \right. \right\} = \left\{ u \in X \left| \int_{\Omega} |\nabla u|^2 \, dx = 0 \right. \right\} = \ker(T).$$

The condition for the ranges leads to a well-known partial differential equation’s weak formulation – the Poisson problem with homogeneous Dirichlet data – which is given by

"Find $u \in X$ such that $\int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \int_{\Omega} f \cdot \nabla \varphi \, dx$ for all $\varphi \in X."  \quad (A.19)$$

and admits a unique solution in $H^1_0(\Omega)^n$. Classical regularity theory [34, Theorem 9.19] states that the solution, in fact, is in $H^{k+1}(\Omega)$. Therefore, $\operatorname{Rng}(S) \subset \operatorname{Rng}(S \circ T)$ and the converse implication is trivial.
It remains to characterize the kernel

\[ \ker(S) = \left\{ \Phi \in Y \left| \int_{\Omega} \Phi \cdot \nabla \varphi \, dx = 0 \quad \forall \varphi \in X \right. \right\}. \]

As all \( C^\infty_0 \)-functions are included in \( X \), the integral condition above is equivalent to vanishing distributional divergence, i.e. \( \ker(S) = H(\text{div}_0)(\Omega) \cap H^k(\Omega)^d \). This concludes the proof of the decomposition \( (A.17) \).

Concerned with the norm equivalence \( (A.18) \), take \( f \in H^k(\Omega)^d \) arbitrarily and let \( f = \nabla u + w \) according to the Helmholtz decomposition \( (A.17) \). Hence, \( u \) is a solution of \( (A.19) \). By regularity theory, \( u \) additionally satisfies the stability estimate

\[ \| \nabla u \|_{H^k(\Omega)^d} \lesssim \| u \|_{H^{k+1}(\Omega)} \leq C \| \text{div} f \|_{H^{k-1}(\Omega)} \lesssim \| f \|_{H^k(\Omega)^d}. \]

The other equivalence estimate follows from the triangle inequality. \( \square \)

With the help of the stability estimate in above remark one can prove equivalence of norms in Section 3.1.1. There, a Helmholtz-type decomposition \( M = S \oplus S^o \oplus V \) will be proven where the left-hand side is equipped with the \( H^3 \)-norm and the right-hand side naturally is equipped with the sum of the \( H^3 \)-norms of the individual summands. The equivalence of norms simplifies the mathematical analysis in Section 3.

The following argument is quite helpful in order to obtain more specific decompositions on basis of the already established ones. The decomposition from Section 3.1.1 is based on this result, see Lemma 3.5.

**Lemma A.9.** Consider the direct sum

\[ X = A \oplus B \]

and assume that for a subset \( Y \subset X \) one has \( A \subset Y \). Then one can conclude that

\[ X \cap Y = Y = A \oplus (B \cap Y). \]

**Proof:** Let \( y \in Y \subset X \). Then there are \( a \in A \) and \( b \in B \) such that \( y = a + b \). Now, as \( A \subset Y \), \( b = y - a \in Y \). Hence, \( b \in B \cap Y \). This implies \( Y \subset A \oplus (B \cap Y) \) and the converse is trivial. \( \square \)

As an example, Lemma A.9 will be applied to the Helmholtz decomposition in order to show a result in the spirit of Poincaré’s lemma about the existence of scalar potentials.

**Lemma A.10.** Let \( \Omega \subset \mathbb{R}^3 \) be a simply connected bounded domain with \( C^{1,1} \)-boundary. Then

\[ H(\text{curl}_0)(\Omega) = \nabla[H^1(\Omega)]. \]

**Proof:** By taking the intersection of both sides of \( (A.12) \) with \( H(\text{curl}_0)(\Omega) \) it follows that

\[ H(\text{curl}_0)(\Omega) = \left( \nabla[H^1(\Omega)] \oplus L^2(\Omega)^3 H_{n0}(\text{div}_0)(\Omega) \right) \cap H(\text{curl}_0)(\Omega) \]

\[ = \nabla[H^1(\Omega)] \oplus L^2(\Omega)^3 H_{n0}(\text{div}_0, \text{curl}_0)(\Omega). \]

By [35, Remark 3.5] the last summand is trivial and the claim follows. \( \square \)
A.3 Modeling aspects of magnetism

Coercivity of the magnetic energy of "model GW". Recall the definitions

\[ h_d = h - h_a \text{ (cf. } (2.18)) \text{ and } h = \nabla R \text{ (cf. } (2.16)). \]

The magnetic energy

\[ \mathcal{E}_{\text{mag}}^1 = \alpha_0 \frac{\mu_0}{2} \int_{\Omega'} (|h|^2 + h \cdot 1_{\Omega} m) \, dx - \alpha_1 \frac{\mu_0}{2} \int_{\Omega} h_d \cdot m \, dx \]

\[ - \alpha_2 \frac{\mu_0}{2} \int_{\Omega} h_a \cdot m \, dx + \alpha_3 \frac{\mu_0}{2} \int_{\Omega} |m|^2 \, dx \]

from (2.59a) is coercive with respect to \( m \) and \( h \). For this, consider the weak formulation (2.68) of the magnetostatic equations (2.89f), (2.89g), i.e.

\[ \int_{\Omega'} \nabla R \cdot \nabla s \, dx = \int_{\Omega'} h_a \cdot \nabla s \, dx - \int_{\Omega} m \cdot \nabla s \, dx \quad \forall S \in H_{\text{lin}}^1(\Omega'). \]

By setting \( S = h \) or \( S = h_d \) (see also (2.70a), (2.70b)) the identities

\[ \int_{\Omega'} (|h|^2 + h \cdot 1_{\Omega} m) \, dx = \int_{\Omega} h \cdot h_a \, dx, \]

and

\[ - \int_{\Omega} h_d \cdot m \, dx = \int_{\Omega'} |h_d|^2 \, dx = \int_{\Omega'} |h - h_a|^2 \, dx \]

follow. The reformulation of the energy into

\[ \mathcal{E}_{\text{mag}}^1 = \alpha_0 \frac{\mu_0}{2} \int_{\Omega'} h \cdot h_a \, dx + \alpha_1 \frac{\mu_0}{2} \int_{\Omega'} |h - h_a|^2 \, dx \]

\[ - \alpha_2 \frac{\mu_0}{2} \int_{\Omega} h_a \cdot m \, dx + \alpha_3 \frac{\mu_0}{2} \int_{\Omega} |m|^2 \, dx, \]  

is straightforward now. This representation features linear terms in \( h \) and \( m \) and quadratic quadratic terms in \( h \) and \( m \) with positive sign. Hence, coercivity can be observed.

About global (non-)existence of \( \nabla h \). Throughout this thesis – e.g. on page 16 below (2.15) or on page 21 below Remark 2.4 – it has been mentioned multiple times that one should consider \( h \) to have distributional divergence with \( L^2 \)-regularity only on \( \Omega \) and \( \Omega' \setminus \overline{\Omega} \) separately. On the contrary, in general \( \nabla h \) is not \( L^2 \)-regular globally on \( \Omega' \). The following considerations explain why \( h \in H(\nabla)(\Omega') \) may be unphysical from a modeling point of view.

The magnetic flux \( b = h + 1_{\Omega} m \) admits an \( L^2 \)-regular divergence globally on \( \Omega' \) due to the boundary condition \( [b] \cdot \nu|_{\partial \Omega} = 0 \), see (2.12a) and (2.14b). This can easily be deduced via integration by parts. If \( \nabla h \) exists globally on \( \Omega' \), implying \( [h] \cdot \nu|_{\partial \Omega} = 0 \) via integration by parts, then \( \nabla(1_{\Omega} m) = \nabla b - \nabla h \) exists globally, too. Hence, the jump condition \( [1_{\Omega} m] \cdot \nu|_{\partial \Omega} = 0 \) would follow which forces the doubtful boundary condition \( m \cdot \nu|_{\partial \Omega} = 0 \) onto the magnetization. On the other hand, within the modeling approach of this thesis, if \( m \cdot \nu|_{\partial \Omega} = 0 \) was true then \( [h] \cdot \nu|_{\partial \Omega} = 0 \), see (2.12a), which again leads to \( h \in H(\nabla)(\Omega') \). Therefore, those two properties,

a) \( h \in H(\nabla)(\Omega') \),

b) \( m \cdot \nu|_{\partial \Omega} = 0 \),
are equivalent within the framework of this thesis. Let us comment on each of these two conditions individually.

Ad a): Assume a) to hold. Then \( [h] \cdot \nu |_{\partial \Omega} = 0 \) combined with \( [h] \times \nu |_{\partial \Omega} = 0 \) (cf. (2.12b)) yields that the magnetic field would always be jump-free on \( \partial \Omega \) and therefore be "smooth" – despite the sudden change of magnetic properties – and the transmission condition for \( h \) is irrelevant. This contradicts basic knowledge taught in applied science classes of which the transmission conditions are clearly a part of.

If \( \Omega' = \Omega \), which automatically yields global existence of \( \text{div} \ h \) on \( \Omega' \), the aforementioned problem is hidden by that simplification. This raises more questions about applicability of such models. However, they are widely used in practice and the case \( \Omega' = \Omega \) is also covered in this thesis. For more information the reader is referred to e.g. [52] and the references therein, e.g. [12], [64].

Ad b): Condition b) itself seems unphysical due to the following considerations. Ferrofluidic suspensions have a rather low susceptibility as compared with solid magnetic materials (e.g. permanent bar magnets). Now, assume that the general direction and magnitude of the external magnetic field \( h_a \) is only slightly modified by the stray field \( h_d \) due to the low susceptibility. Moreover, let the magnetization \( m \) deviate from \( \chi(c,h)h \) only slightly and let \( \chi(c,h) \) be approximately constant. Such a setting is similar to the setting which was used in the numerical experiments of [51] (Rosensweig instability). There, an external field with approximately homogeneous direction (but decaying in magnitude) which is (approximately) perpendicular to a straight boundary was applied to a ferrofluid of (approximately) homogeneous density.

In this scenario, one could say that the magnetization \( m \) at the boundary is approximately directly proportional to the total magnetic field \( h \) which itself is approximately given by the external field \( h_a \). Normal components should not be disturbed too much by small deviations. Hence, a condition like \( m \cdot \nu |_{\partial \Omega} = 0 \) might be transferred to \( h_a \) (approximately) and thereby does not seem to be justified. However, as (2.13a) and (2.13b) hold true, the natural boundary conditions (2.38) do not seem to be contradictory and by Remark 2.4 the boundary conditions in (2.39) seem to be fine.

Despite the thought experiment above, which was based on weak stray fields, the stray field should not be neglected carelessly. In [51] the Rosensweig-instability was recaptured numerically, which relied on the presence of the stray field. If the magnetostatic equations (2.22) were to be dismissed and \( h \) were to be replaced with \( h_a \), the numerical experiment in [51] could not reproduce the famous surface instability.

A.4 Some definitions in two spatial dimensions

In this section, while adopting the notation of Dautray and Lions [18], curl-operators in the two-dimensional setting are defined, as well as the cross product. Some analogous properties will be discussed which are known from the three-dimensional case.

For \( a, b \in \mathbb{R}^2 \) the cross product is defined as in [39] by

\[
a \times b := a_1 b_2 - a_2 b_1 = -b \times a,
\]

and for a scalar \( c \in \mathbb{R} \), let

\[
c \times b := c \begin{pmatrix} -b_2 \\ b_1 \end{pmatrix} \quad \text{and} \quad b \times c := \begin{pmatrix} b_2 \\ -b_1 \end{pmatrix} c = -c \times b.
\]
Just as in the three-dimensional setting, where curl is formally defined by \( \text{curl} = \nabla \times \), the following curl-operators will be defined.

Let

\[
\text{curl} : C^1(\Omega)^2 \to C(\Omega),
\]

\[
\text{curl} v := \nabla \times v = \partial_{x_1} v_2 - \partial_{x_2} v_1
\]

and

\[
\text{Curl} : C^1(\Omega) \to C(\Omega)^2,
\]

\[
\text{Curl} v := \nabla \times v = \left( \begin{array}{c} \partial_{x_2} v \\ -\partial_{x_1} v \end{array} \right).
\]

**Remark A.11.** Treating a two-dimensional vector field \( u : \Omega \to \mathbb{R}^2 \) as a three-dimensional vector field \( \tilde{u} \) which is constant in \( x_3 \)-direction, e.g.

\[
\tilde{u}(x_1, x_2, x_3) := \begin{pmatrix} u_1(x_1, x_2) \\ u_2(x_1, x_2) \\ 0 \end{pmatrix},
\]

the curl of this flow field is readily computed to be

\[
\text{curl} \tilde{u} = \begin{pmatrix} 0 \\ 0 \\ \partial_{x_1} u_2 - \partial_{x_2} u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \text{curl} u \end{pmatrix}.
\]

This relation sheds light on the definition (A.23). Moreover, if \( \tilde{u} \) from above is a gradient field, \( \tilde{u} = \nabla \tilde{R} \), it is evident that \( \tilde{R} \) does not depend on \( x_3 \) and therefore \( u = \nabla R \) is a gradient field, too, where for instance \( R(x_1, x_2) = \tilde{R}(x_1, x_2, 0) \).

A similar relation sheds light on (A.24), by treating a scalar function \( u : \Omega \to \mathbb{R} \) as a three-dimensional vector field \( \tilde{u} \) which is constant in its first and second components, e.g.

\[
\tilde{u}(x_1, x_2, x_3) := \begin{pmatrix} 0 \\ 0 \\ u(x_1, x_2) \end{pmatrix}.
\]

Then, the curl is given by

\[
\text{curl} \tilde{u} = \begin{pmatrix} \partial_{x_2} u \\ -\partial_{x_1} u \\ 0 \end{pmatrix} = \begin{pmatrix} \text{Curl} u \\ 0 \end{pmatrix}.
\]

The composition of operators

\[
\text{curl} \nabla (\cdot) \text{ vanishes (A.25)}
\]

in two and three dimensions which is an easy consequence of their definitions.

**Remark A.12.** While the \( \times \) in a term of the type \( \text{curl} f \cdot \text{curl} g \) might seem irritating as the curl in two dimensions only yields a scalar field, this notation could be seen as scalar product of one-dimensional vector fields.

Moreover,

\[
(c \times b) \cdot a = \frac{\text{A.22}}{c(-b_2 a_1 + b_1 a_2) = -c(-a_2 b_1 + a_1 b_2)} = -(c \times a) \cdot b
\]

\[
= (b_1 a_2 - b_2 a_1) c \frac{\text{A.21}}{(b \times a) c} \]

such that, together with \( (c \times b) = -(b \times c) \) (cf. \( \text{A.22} \)) and \( b \times a = -a \times b \) (cf. \( \text{A.21} \)), the trilinear form on the left-hand side above is found to be alternating.
In three dimensions, the well known formula

\[-\Delta = \text{curl curl} \nabla \text{div}\]  \hspace{1cm} (A.27)

holds for sufficiently smooth vector fields. This identity is crucial for this thesis. The analogous formula in two dimensions reads

\[-\Delta = \text{Curl curl} \nabla \text{div}.\]  \hspace{1cm} (A.28)

Both identities can be verified easily by explicitly computing the derivatives while using the commutativity \(\partial_{x_i} \partial_{x_j} = \partial_{x_j} \partial_{x_i}, \ i,j \in \{1, \ldots, d\}\). The three-dimensional formula is well-known. The computation of the two-dimensional case will be included here,

\[
\begin{align*}
\text{(Curl curl} - \nabla \text{div}) \mathbf{a} &= \text{Curl}(\partial_{x_1} \mathbf{a}_2 - \partial_{x_2} \mathbf{a}_1) - \left( \frac{\partial_{x_1} \partial_{x_1} \mathbf{a}_1 + \partial_{x_1} \partial_{x_2} \mathbf{a}_2}{\partial_{x_2}} \right) \mathbf{a}
= \left( \partial_{x_2} (\partial_{x_1} \mathbf{a}_2 - \partial_{x_2} \mathbf{a}_1) \right) - \left( \partial_{x_1} \partial_{x_1} \mathbf{a}_1 + \partial_{x_1} \partial_{x_2} \mathbf{a}_2 \right)
= \left( \partial_{x_2} \partial_{x_1} \mathbf{a}_2 - \partial_{x_1} \partial_{x_2} \mathbf{a}_2 \right) - \Delta \mathbf{a} = -\Delta \mathbf{a}.
\end{align*}
\]

Remark A.13. Naturally, those definitions, properties and identities mentioned in this section can be extended to Sobolev-functions.

Another important identity is the integration by parts formula. For the curl-operator in three spatial dimensions the latter is given for smooth vector fields \(\mathbf{a}\) and \(\mathbf{b}\) by

\[
\int \Omega \text{curl} \mathbf{a} \cdot \mathbf{b} \, d\mathbf{x} = \int_{\partial \Omega} \mathbf{a} \cdot \mathbf{b} \, d\mathbf{\sigma} - \int_{\partial \Omega} \mathbf{a} \times \mathbf{\nu} \cdot \mathbf{b} \, d\mathbf{\sigma}.
\]  \hspace{1cm} (A.29)

If \(\mathbf{a}, \mathbf{b} \in H(\text{curl})(\Omega)\), above formula remains true where the boundary integral is meant in a distributional sense \(\langle \mathbf{a} \times \mathbf{\nu} | \partial \Omega, \mathbf{b} \rangle (H^\frac{1}{2}(\partial \Omega))^d \times H^\frac{1}{2}(\partial \Omega))\), see Definition A.3. This formula can easily be derived from Gauss's divergence theorem. The two-dimensional version of this formula is as follows. Let \(\mathbf{a}\) be a smooth vector field and \(\mathbf{b}\) be a smooth scalar function. Then,

\[
\int_{\Omega} \text{curl} \mathbf{a} \cdot \mathbf{b} \, d\mathbf{x} = \int_{\Omega} (\partial_{x_1} \mathbf{a}_2 \mathbf{b} - \partial_{x_2} \mathbf{a}_1 \mathbf{b}) \, d\mathbf{x}
= \int_{\Omega} (\mathbf{b} \partial_{x_2} \mathbf{a}_1 + \mathbf{a}_1 \partial_{x_2} \mathbf{b}) \, d\mathbf{x} + \int_{\partial \Omega} (\mathbf{a}_2 \mathbf{\nu}_1 \mathbf{b} - \mathbf{a}_1 \mathbf{\nu}_2 \mathbf{b}) \, d\mathbf{\sigma}
\]  \hspace{1cm} (A.30)

The formula remains true for \(\mathbf{a} \in H(\text{curl})(\Omega)\) and \(\mathbf{b} \in H(\text{Curl})(\Omega)\) (see its definition in (A.6)), where the boundary integral is meant in the distributional sense.

A.5 Supplementary statements

About \(L^{-1}_V\). For convenience, the definition of \(L^{-1}_V\) in Definition 3.10 will be repeated. On a bounded Lipschitz domain \(V \subset \mathbb{R}^d\) let \(L^{-1}_V : H^\frac{1}{2}(\partial V) \rightarrow H^1(V)\) be defined pointwise by the inhomogeneous Dirichlet-Laplace problem

\[
\begin{align*}
-\Delta L^{-1}_V f &= 0, \\
L^{-1}_V f |_{\partial V} &= f,
\end{align*}
\]
which is uniquely solvable in its weak form

\[
\begin{aligned}
&\text{for given } f \in H^\frac{1}{2}(\partial V) \text{ find } u \in H^1_0(V) \text{ such that } \\
&\forall \varphi_0 \in H^1_0(V) : \int_V \nabla u \cdot \nabla \varphi_0 \, dx = -\int_V \nabla (Ef) \cdot \nabla \varphi_0 \, dx,
\end{aligned}
\] (A.31)

where \( E : H^\frac{1}{2}(\partial V) \to H^1(V) \) is one of the continuous right-inverses of the trace operator and

\[
L_V^{-1} f := u + (Ef).
\] (A.32)

If \( \partial V \) is of class \( C^{k-1,1} \), \( k \in \mathbb{N} \), and \( (Ef) \in H^k(V) \), then \( u \in H^k(V) \), see [34] Theorem 9.19. Hence, \( u \) and \( L_V^{-1} f \) are \( H^1 \)-regular if \( \Delta (Ef) \) is \( H^2 \)-regular. The latter is true if \( f \in H^\frac{1}{2}(\partial V) \) by the inverse trace theorem (cf. [48]).

Let \( v \in \text{Rng}(L_V^{-1}) \). Then, by definition

\[
\int_V \nabla v \cdot \nabla \varphi_0 \, dx = 0 \quad \forall \varphi_0 \in H^1_0(V),
\]

which implies \( \nabla v \in H(\text{div}_0)(V) \). Due to \( \text{curl} \nabla = 0 \) (see (A.25) for the two-dimensional case) one obtains \( \nabla v \in H(\text{div}_0, \text{curl}_0)(V) \), which shows (3.41).

In the following, it will be examined why this operator is helpful to find extensions of suitable potentials in Section 3.1.2. If \( V \) was simply connected with \( C^{1,1} \)-boundary one can simply conclude equality \( \text{Rng}(\nabla L_V^{-1}) = H(\text{div}_0, \text{curl}_0)(V) \) by means of integration by parts. In detail, if there was an element \( f \in H(\text{div}_0, \text{curl}_0)(V) \) that is not in \( \text{Rng}(\nabla L_V^{-1}) \), then it could be assumed to be \( L^2 \)-orthogonal to that image, i.e.

\[
0 = \langle \nabla L_V^{-1} g, f \rangle_{L^2(V)} = \int_{\partial V} \frac{(L_V^{-1} g)|_{\partial V}}{\nu} f \cdot \nu \, d\sigma \quad \forall g \in H^\frac{1}{2}(\partial V).
\]

This implies the normal component on the boundary to vanish and, consequently, as the space \( H_{n0}(\text{div}_0, \text{curl}_0)(V) \) is trivial [35] Remark 3.5] for simply connected \( V \), equality can be concluded. However, in Figure 3.2 it can be seen that extensions in \( \nabla[H^1(V)] \cap H(\text{div}_0, \text{curl}_0)(V) \) need to be constructed, where \( V = \Omega \setminus \overline{\Omega} \) is not necessarily simply connected (i.e. it is certainly not in case of \( d = 2 \), where \( \Omega \) indicates a hole). It turns out that \( V \) does not need to be simply connected anymore if one restricts oneself to gradient fields. Let \( \nabla S \in \nabla[H^1(V)] \cap H(\text{div}_0, \text{curl}_0)(V) \) not be representable via \( \nabla L_V^{-1} \), then with the same argument as before \( \nabla S \cdot \nu|_{\partial V} = 0 \). Without loss of generality, choose \( S \in L^2_0(V) \) to be mean value free. Then \( S \) (weakly) solves the problem

\[
-\Delta S = 0, \\
\nabla S \cdot \nu|_{\partial V} = 0, \quad \text{with} \quad \int_V S \, dx = 0,
\]

of which \( S \equiv 0 \) is the unique solution. Hence, the operator indeed yields suitable extensions of potentials onto \( \Omega' \setminus \overline{\Omega} \) such that their respective gradient fields have vanishing div and curl, as required in the definition of \( \mathcal{R} \), (3.8).

**About \( g^L_s \).** The definition in (3.73) will be repeated,

\[
-1 \leq g^L_s(x) := \begin{cases} 
\frac{2}{3} + (\log s - 1)c - \frac{s}{2} & \text{for } c \leq s, \\
\frac{c}{2} \log c - c & \text{for } s < c < L, \\
\frac{c^2}{2^2} + (\log L - 1)c - \frac{L}{2} & \text{for } L \leq c.
\end{cases}
\]

Here, \( 0 < s < e < L \), where \( e \) is Euler’s number.
The following lines are about the lower bound of $g^L_s$. The minimum values of all three parts above will be determined by standard procedure. The second part $c \log c - c$ has its minimum at $c = 1$ which yields the value $-1$. The other two parts are of the same type which is why only one of them will be discussed. Consider

$$\frac{d}{dc} \left( \frac{c}{2s} + (\log s - 1)c - \frac{c}{2} \right) = \frac{c}{s} + (\log s - 1) = 0,$$

which yields $c = s - s \log s$ and thereby (second derivative is positive) the minimum value is given by $-\frac{1}{2}((1 - \log s)^2 + 1)$, which depends on $s$. Further examination is needed. In order to minimize with respect to $s > 0$, consider

$$\frac{d}{ds} \left( -\frac{s}{2}((1 - \log s)^2 + 1) \right) = -\frac{1}{2}((1 - \log s)^2 + 1) + (1 - \log s) = 0,$$

which yields $v := 1 - \log s \equiv 1$ by solving a quadratic equation and consequently $s = 1$. The minimum value then also is $-1$.

**About $\chi_s$.** Recall

$$\chi_s(c) := \begin{cases} 
\chi_0c & \text{if } c - \frac{\omega}{\chi_0} \geq s, \\
\chi_0 \left( -\frac{1}{16s} \left( c - \frac{\omega}{\chi_0} \right)^4 + \frac{3}{8s} \left( c - \frac{\omega}{\chi_0} \right)^2 \right) + \frac{1}{2} \left( c - \frac{\omega}{\chi_0} \right) + \frac{3s}{16} + \frac{\omega}{\chi_0} & \text{if } -s < c - \frac{\omega}{\chi_0} < s, \\
\omega & \text{if } c - \frac{\omega}{\chi_0} \leq -s.
\end{cases}$$

from (4.46). This paragraph sheds light on the properties (4.50) and (4.51), i.e.

i) $\omega \leq \chi_s \in C^2(\mathbb{R})$, ii) $\chi_0c \leq \chi_s(c)$,

iii) $0 \leq \chi'_s \leq \chi_0$, iv) $0 \leq \chi''_s \leq \frac{3}{16}$,

v) $\delta \chi^{k-1}_{s,c} \in C^{0,1}(\mathbb{R})$, vi) $0 \leq \delta \chi^{k-1}_{s,c} \leq \chi_0$.

First, checking for differentiability (see i)) is tedious but straightforward by considering the limit of the difference quotient $\frac{\chi_s(c+h) - \chi_s(c)}{h}$ for $0 \neq h \rightarrow 0$ at the points where $c - \frac{\omega}{\chi_0} \in \{-s, s\}$. Analogously, the same procedure for the first derivative (see below) shows that $\chi_s$ is differentiable twice.

The derivatives are given by

$$\chi'_s(c) = \begin{cases} 
\chi_0 & \text{if } c - \frac{\omega}{\chi_0} \geq s, \\
\chi_0 \left( -\frac{1}{16s} \left( c - \frac{\omega}{\chi_0} \right)^3 + \frac{3}{8s} \left( c - \frac{\omega}{\chi_0} \right)^2 + \frac{1}{2} \right) & \text{if } -s < c - \frac{\omega}{\chi_0} < s, \\
0 & \text{if } c - \frac{\omega}{\chi_0} \leq -s,
\end{cases}$$

$$\chi''_s(c) = \begin{cases} 
0 & \text{if } c - \frac{\omega}{\chi_0} \geq s, \\
\chi_0 \left( -\frac{3}{16s} \left( c - \frac{\omega}{\chi_0} \right)^2 + \frac{3}{4s} \right) & \text{if } -s < c - \frac{\omega}{\chi_0} < s, \\
0 & \text{if } c - \frac{\omega}{\chi_0} \leq -s.
\end{cases}$$

From the second derivative (strictly positive in the second case of its definition) and $\chi'_s(c) = 0$ one can conclude that the first derivative can only vanish for $c - \frac{\omega}{\chi_0} \leq -s$ which yields boundedness of $\chi_s$ from below by its minimum $\omega$. Therefore, i) is proven. Moreover, the derivative is monotonously increasing. From there it is straightforward to conclude ii) as $c \rightarrow \chi_0c$ has constant derivative, in contrast to monotonously increasing $\chi'_s$ (bounded by $\chi_0$), and coincides with $\chi_s$ for large arguments $c \geq \frac{\omega}{\chi_0} + s$. From monotonicity of $\chi'_s$, the lower bound 0 and upper bound $\chi_0$ of $\chi'_s$ follow immediately (cf. iii)) as well.
The maximum value of the second derivative is obviously attained at \( c = \frac{\omega}{\chi_0} \) which yields \( \frac{3}{4s} \) as upper bound (cf. iv)). The lower bound is 0, of course.

The difference quotient,

\[
\frac{\delta \chi_s^{k-1}}{\delta c}(c) := \begin{cases} 
\frac{\chi_s(c) - \chi_s(c^{k-1})}{c^{k-1} - c} & \text{if } c \neq c^{k-1}, \\
\chi'_s(c^{k-1}) & \text{if } c = c^{k-1},
\end{cases}
\]

of \( \chi_s \) from (4.48) has the following properties.

\[
\delta \chi_s^{k-1'}(z) = \begin{cases}
\chi'_s(z)(z - c^{k-1}) - \chi'_s(z) + \chi_s(c^{k-1}) & \text{Taylor} = -\frac{1}{2} \chi''_s(\xi_z) \text{ for a } \xi_z \text{ between } z \text{ and } c^{k-1},
\end{cases}
\]

Due to iv) (boundedness of second derivative) one can conclude that \( \frac{\delta \chi_s^{k-1}}{\delta c}(z) \) is (globally) Lipschitz continuous with Lipschitz-constant \( \frac{3}{4s} \). Moreover, the estimate \( 0 \leq \frac{\delta \chi_s^{k-1}}{\delta c} \leq \chi_0 \) follows from the mean value theorem for differentiable functions combined with iii) and hence the analogous estimate for \( \frac{\delta \chi_s^{k-1}}{\delta c} \) due to Lemma 4.1. This concludes the proof of the claims v) and vi).
References


