A-posteriori error estimates for pressure-correction schemes

A-posteriori Fehlerabschätzungen für Druckkorrektur Verfahren

Der Naturwissenschaftlichen Fakultät
der Friedrich-Alexander-Universität
Erlangen-Nürnberg

zur

Erlangung der Doktorgrades Dr. rer. nat.

vorgelegt von

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FAKULTÄT

Als Dissertation genehmigt von der Naturwissen-
schaftlichen Fakultät der Friedrich-Alexander-Universität
Erlangen-Nürnberg

Tag der mündlichen Prüfung: 12.08.2016

Vorsitzender des Promotionsorgans: Prof. Dr. Jörn Wilms

Gutachter: Prof. Dr. Eberhard Bänsch

Prof. Dr. Charalambos Makridakis

Die Zeit promoviert jeden

Abstract

In this thesis, a-posteriori error estimates for time discretizations of the incompressible time dependent Stokes equations by pressure-correction methods are presented. Pressure-correction methods are splitting schemes, which decouple the velocity and the pressure. As a result the Stokes equations reduce to much easier schemes, which can be solved by cost-efficient algorithm. In a first step, a-posteriori estimates for the stationary Stokes system, discretized by the two-step backward differential formula method (BDF2), are presented. This allows to compare the a-posteriori estimators of the discretized Stokes system with the estimators of the pressure-correction scheme.

In the second part of the thesis rigorous proofs of global upper bounds for the incremental pressure correction scheme discretized by backward Euler scheme as well as for the two-step backward differential formula method (BDF2) in rotational form are presented. Moreover, rate optimality of the estimators are shown for velocity (in case of backward Euler and BDF2 in rotational form) and pressure (in case of Euler). Computational experiments confirm the theoretical results.

Key words: A-posteriori error analysis, projection methods, fractional step methods, semi-discrete Navier–Stokes equations, two-step backward differentiation formula reconstruction, backward Euler

Acknowledgments

This thesis originates from my work on pressure-correction schemes with the focus on a-posteriori error estimates.

First of all, I want to express my heartfelt thanks to my supervisor Prof. Dr. Eberhard Bänsch. He showed me the fascinating field of the analysis and the numerics of the Navier–Stokes equations and he gave me the opportunity to work collaboratively with him on pressure-projection schemes for solving these equations. His continuous support, great knowledge, and policy of an always open door guided me through all difficulties during my work. Moreover, I would like to take the opportunity to express my gratitude to Prof. Dr. Thomas Richter and Prof. Dr. Wolfgang Borchers, who kindly agreed to act as examiners in the defense of the thesis. I would also like to thank all current and past members of the AM3 group: Christopher and Steffen Basting, Melanie Bittl, Kathrin Bäuml, Dr. Michael Fried, Dr. Nico Neuß, Dr. Cornelia Schneider, Stefanie Reuter and Stephan Weller. Their encouragement and help in all kinds of numerical and mathematical difficulties, the always pleasant atmosphere and good mood kept me motivated during my years of work. Special thanks to my colleague Stefanie Reuter and my friend Theodor Wolfrum for proofreading the manuscript. At AM3 I have not only found greatly appreciated colleagues but also many friends. Thank you for the nice time.

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1 Introduction

The scope of this thesis is the development of a-posteriori error estimators for projection schemes with a first or second order time discretization. In this chapter the interest in projection schemes for solving the Navier–Stokes equations and the need of a-posteriori error estimator are motivated. Furthermore the aim of this work as well as an outline of the thesis are given.

1.1 Motivation

The *Navier–Stokes equations* are one of the most important equations in fluid mechanics. They describe the behavior of flows in fluids. The Navier–Stokes equations are a system of nonlinear partial differential equations of second order. In the first half of the nineteenth century *Claude Louis Marie Henri Navier* and *George Gabriel Stokes* discovered independently from each other the momentum equation for Newtonian fluids such as water, honey or air.

The numerical approximation of the Navier–Stokes equations is still a challenging topic and there is a wide range of numerical methods. In this thesis *projection methods* are considered. Instead of solving a saddle point problem in each time step, velocity and pressure are decoupled and one has to solve a transport equation for the velocity and a Poisson equation for the pressure. These two problems are well-known and there exist a large number of efficient algorithms.

The history of projection schemes dates back to the work of Chorin and Temam [8,9,37]. Since then there has been an abundance of articles dealing with extending and improving the original scheme. In particular, a serious drawback of the original Chorin scheme and its first improvements is a non-physical behavior of the pressure, where a pressure error boundary layer results in a reduced rate of convergence, see for instance [31,34].

It is impossible to give a rough overview of even the most relevant contributions to projection methods. However, a short overview will be given in Chapter 4. For a more detailed overview I would recommend the excellent overview paper of Guermond, Mineev and Shen in [15] and the references cited there.

Since realistic applications from engineering are highly complex and therefore result in expensive simulations, one needs efficient numerical algorithms with low computational cost on the one hand and a certain accuracy on the other hand. A large class of appropriate methods are so-called *adaptive methods*, which have become very popular and powerful tools for certain classes of partial differential equations (PDE), at least in the context of finite elements. Adaptive algorithms are naturally related to error control.

One distinguishes between adaptive methods in time and space. In context of space discretization, the concept for a adaptive finite element strategy is the following: use a spatial indicator to pick the elements of a triangulation with a large contribution of the error and refine them in smaller elements. For the second group, consisting of adaptive methods in time, one uses a time indicator in order to adjust the time step size. Such indicators are typically a-posteriori error estimators, which should have the following properties: first the a-posteriori error estimator is computable by only using given data, the discretization and already calculated discrete quantities. Second the cost of calculating the estimator should be comparable to the cost of assembling the stiffness matrix. And third the estimator should be close to the error between exact and discrete solution itself, i.e. the error can be bounded by the estimator.

While for elliptic problems the theory and practice of adaptive methods has reached sound maturity, error control for time dependent problems is still a challenging area. In the last years there were contributions by [2, 3, 26–28, 40] concerning parabolic problems. For time dependent problems involving elliptic constraints like the incompressible Navier-Stokes equations, there is even less theory available.

In this article, among further topics, a-posteriori estimates for first order schemes as well as for second order schemes in so called *rotational form* are reviewed. The vast majority of work on projection schemes in the literature deals with the semi-discrete case, i.e. schemes still continuous in space. Besides additional technical difficulties the reason for this is the focus on the *splitting error* introduced by projection schemes and the variety of possible space discretizations.

In [24, 25] the standard non-incremental Chorin-Temam scheme is considered. There, global upper bounds as well as local lower bounds between the error and the estimators are given. However, these estimates are not completely a-posteriori. Full a-posteriori estimators are derived in [6] for the non-incremental Chorin scheme with backward Euler time discretization. *Goal oriented error estimators* for the incremental pressure correction scheme are presented in [32]. Note that non-incremental schemes are limited to $\tau^{1/2}$ order of convergence, no matter what kind of time discretization is used in the viscous sub-step, cf. Chapter 4.

1.2 Aim

The aim of this study is to develop a-posteriori error estimates for time discretization by backward Euler for the standard incremental scheme and two-step backward differential formula method (BDF2) for the rotational pressure-correction method.

The estimators are based on the methodology developed in [2, 3, 28] and then further used in [5, 11, 12, 22]. The key point is the definition of an auxiliary function, called *reconstruction*, i.e. an interpolant in time (piecewise linear for the first order case and piecewise quadratic in case of the BDF scheme in rotational form) and a reconstruction

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of the discrete numerical solution to the continuous space, cf. Chapter 2.5.

Further the optimality of the developed estimators will be proven. In this context optimality of the estimators is understood in the sense that they asymptotically behave like known a-priori estimates. For the first order scheme optimality of the estimators for velocity and pressure can be shown. For the second order BDF2 scheme in rotational form proofs for the optimality of the velocity estimators are given. If, however, the estimator for the pressure is of optimal order remains open.

Notice the limitations of the analysis regarding a fully practical devise for computations:

- As usual with residual based a-posteriori estimators the error is bounded by the estimator up to a certain constant that is difficult to quantify.
- Some of the a-posteriori estimators involve the evaluation of *negative norms* like in [1, 28] for parabolic problems. From a computational point of view this is completely prohibitive, since this evaluation may be computationally more expensive than solving the scheme.
- Since the present analysis is rather technical, further difficulties are avoided by assuming rather high regularity of the problem. It is well known, that regularity for the time dependent Stokes problem for $t \searrow t_0 = 0$ is an issue itself, see Chapter 2.2. Alternatively one could already run the flow problem for some time before one starts the computational scheme. Notice that the papers cited in [15] dealing with second order convergence all assume sufficient regularity of the exact solution.
- Since in this thesis the focus is on the splitting error introduced by projection schemes, the projection methods are analyzed in the semi-discrete case, still continuous in space. Analysis for the fully discrete case will be published in a forthcoming paper and is motivated in the outlook.

Nevertheless, to the best of my knowledge, this is the first rigorous a-posteriori error analysis related to the *incremental* pressure correction scheme. Moreover, I do not know of any a-posteriori analysis for higher order splitting schemes in rotational form.

In the following the analysis is restricted to the case of constant time steps. Of course, a-posteriori estimators are supposed to be used in an adaptive strategy with adaptive time step control. However, the generalization to non constant time steps is my current work and will be introduced in Chapter 7.2.

1.3 Outline

The rest of this thesis is organized as follows: In Chapter 2, notation and some results for the Helmholtz and Stokes projection are introduced. Further the used techniques for deriving a-posteriori estimates in this thesis are presented. In Chapter 3 the instationary

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Stokes system is considered. Using BDF2 as time discretization, a-priori and a-posteriori estimates are developed for the fully discretized scheme. Before presenting the main work of the thesis, a short overview of projection schemes is given in Chapter 4. The main results, the a-posteriori estimates, are stated in Chapter 5 for the standard incremental pressure-correction scheme discretized by implicit Euler and in Chapter 6 for the scheme in rotational form discretized by BDF2. Finally in Chapter 7 the thesis is summarized and an outlook on current and future work is given.

2 Notation and Preliminaries

2.1 Function spaces

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be an open, connected and bounded domain with a sufficiently smooth boundary $\Gamma := \partial\Omega$. Consider the incompressible Navier–Stokes equations on a finite time interval $]0, T[$: find the velocity u and the pressure p fulfilling

$$\begin{aligned} \partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p &= f && \text{in } \Omega \times]0, T[, \\ \operatorname{div} u &= 0 && \text{in } \Omega \times]0, T[, \\ u &= 0 && \text{on } \Gamma \times]0, T[, \\ u(t = 0) &= u_0 && \text{in } \Omega \end{aligned}$$

for a (sufficiently regular) right hand side f . The boundary condition for the velocity is set to zero for the sake of simplicity, and u_0 is the given initial vector field.

In the following the nonlinear term is dropped, as the focus of our analysis is the error introduced by time discretization and splitting. Thus, consider the approximation of the time-dependent Stokes equations (setting $\nu = 1$ for convenience):

$$\begin{aligned} \partial_t u - \Delta u + \nabla p &= f && \text{in } \Omega \times]0, T[, \\ \operatorname{div} u &= 0 && \text{in } \Omega \times]0, T[, \\ u &= 0 && \text{on } \Gamma \times]0, T[, \\ u(t = 0) &= u_0 && \text{in } \Omega. \end{aligned} \tag{2.1}$$

In the following it is assumed that the unique solution (u, p) to the above system is sufficiently smooth in time and space, cf. Chapter 2.2.

Denote by $H^m(\Omega)$ ($m = 0, 1, \dots$) the standard Sobolev spaces with norms $\|\cdot\|_m$. The norm and inner product of $L^2(\Omega) = H^0(\Omega)$ is denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ respectively. To handle the homogeneous Dirichlet boundary condition introduce $H_0^1(\Omega) := \{v \in H^1(\Omega) : v|_\Gamma = 0\}$. Thanks to the Poincaré inequality, for $v \in H_0^1(\Omega)$, the norms $\|\nabla v\|$ and $\|v\|_1$ are equivalent. For a function space V its dual space is denoted by V^* . The dual space of $H_0^1(\Omega)$ is set by $H^{-1} := H_0^1(\Omega)^*$ with norm

$$\|u\|_{-1} := \sup_{0 \neq v \in H_0^1(\Omega)} \frac{\langle u, v \rangle}{\|\nabla v\|}$$

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To handle the velocity and the pressure in the Navier–Stokes equations introduce the spaces

$$X := H_0^1(\Omega)^d, \quad M := H^1(\Omega) \cap L_0^2(\Omega), \quad Y := L^2(\Omega)^d.$$

Further define the divergence free spaces

$$J := \{v \in X : \operatorname{div} v = 0\}, \quad H := \{v \in Y : \operatorname{div} v = 0, v \cdot n|_\Gamma = 0\}.$$

The *Helmholtz projection* $\mathcal{H} : Y \rightarrow H$ is defined by

$$\langle \mathcal{H}u, v \rangle = \langle u, v \rangle, \quad \forall u \in Y, v \in H,$$

and $\mathcal{S} : X \rightarrow J$ denotes the *Stokes projection*, defined by

$$\langle \nabla \mathcal{S}u, \nabla v \rangle = \langle \nabla u, \nabla v \rangle, \quad \forall u \in X, v \in J.$$

Finally, write $a \lesssim b$ for two functions or quantities a, b , whenever there is a generic constant C , such that $a \leq Cb$.

The next two lemmata bound the error of the Helmholtz projection $\mathcal{H}$ and Stokes projection $\mathcal{S}$ in different norms.

Lemma 2.1.

Suppose $u \in X$, $s = 0, 1$. Then the Helmholtz projection $\mathcal{H}$ satisfies

$$\|(I - \mathcal{H})u\|_s \lesssim \|\operatorname{div} u\|_{H^{1-s}(\Omega)^*}. \quad (2.2)$$

Proof : The difference $v = (I - \mathcal{H})u$ fulfills

$$-v + \nabla q = 0, \quad \operatorname{div} v = g, \quad v \cdot n|_\Gamma = 0$$

for some $q \in M$ and $g = \operatorname{div} u$. Applying the div-operator to the first equation one gets $-\Delta q = -g$. Stability and regularity for the Laplace operator imply

$$\|q\|_{s+1} \lesssim \|g\|_{H^{1-s}(\Omega)^*}, \quad s = 0, 1.$$

Thus one concludes

$$\|v\|_s \lesssim \|q\|_{s+1} \lesssim \|g\|_{H^{1-s}(\Omega)^*}.$$

□

Lemma 2.2.

Suppose $u \in X \cap H^{-s+1}(\Omega)^d$, $s \in \{-1, 0, 1\}$. Then the Stokes projection $\mathcal{S}$ satisfies

$$\|(I - \mathcal{S})u\|_s \lesssim \|\operatorname{div} u\|_{H^{1-s}(\Omega)^*}. \quad (2.3)$$

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Proof : Let $v = (\mathcal{S} - I)u$ and $g = \operatorname{div} u$. Then the definition of the Stokes projection implies

$$-\Delta v + \nabla q = 0, \quad \operatorname{div} v = g$$

for some $q \in M$. The estimate for $s = 1$ follows directly from the variational formulation in combination with the inf-sup condition. For $s \in \{-1, 0\}$ and $w \in H^{-s}$ the solution (z, r)

$$-\Delta z + \nabla r = w, \quad \operatorname{div} z = 0, \quad \int_{\Omega} r(x) dx = 0$$

fulfills

$$\|z\|_{-s+2} + \|r\|_{-s+1} \lesssim \|w\|_{-s}, \quad (2.4)$$

because of the regularity of the Stokes system. Then:

$$\begin{aligned} \|g\|_{H^{1-s}(\Omega)^*} &= \sup_{\tilde{r} \in H^{1-s}(\Omega)} \frac{|\langle g, \tilde{r} \rangle|}{\|\tilde{r}\|_{1-s}} = \sup_{\tilde{r} \in H^{1-s}(\Omega)} \frac{|\langle \operatorname{div} v, \tilde{r} \rangle|}{\|\tilde{r}\|_{1-s}} = \sup_{\tilde{r} \in H^{1-s}(\Omega)} \frac{|\langle v, \nabla \tilde{r} \rangle|}{\|\tilde{r}\|_{1-s}} \\ &\geq \frac{|\langle v, \nabla r \rangle|}{\|r\|_{1-s}} \gtrsim \frac{|\langle v, w + \Delta z \rangle|}{\|w\|_{-s}} \\ &= \frac{|\langle v, w \rangle + \langle \Delta v, z \rangle|}{\|w\|_{-s}} = \frac{|\langle v, w \rangle + \langle \nabla q, z \rangle|}{\|w\|_{-s}} \\ &= \frac{|\langle v, w \rangle - \langle q, \operatorname{div} z \rangle|}{\|w\|_{-s}} = \frac{|\langle v, w \rangle|}{\|w\|_{-s}}. \end{aligned}$$

Since $w \in H^{-s}$ is arbitrary, the proof is complete. $\square$

Be aware that Lemma 2.1 and 2.2 are essential tools for the a-posteriori analysis of projection schemes. They bound the error resulting of neglecting the solenoidal condition.

The *Stokes Operator* $\mathcal{A} : J \cap H^2(\Omega)^d \rightarrow H$ is defined as

$$\mathcal{A}u := -\mathcal{H}\Delta u, \quad \forall u \in D(\mathcal{A}) = J \cap H^2(\Omega)^d.$$

$\mathcal{A}$ is unbounded, positive self-adjoint and closed in H with domain $D(\mathcal{A})$. Its inverse $\mathcal{A}^{-1}$ is compact in H . Note that $\mathcal{A}^{-1}$ can be extended to a bounded operator

$$\mathcal{A}^{-1} : J^* \rightarrow J.$$

Lemma 2.3.

Let $v \in X$, then the following norm equivalence holds

$$\|v\|_{J^*}^2 \leq \langle v, \mathcal{A}^{-1}v \rangle = \|\nabla \mathcal{A}^{-1}v\|^2 \leq \|v\|_{J^*}^2. \quad (2.5)$$

Proof : For $v \in X$ set $w = \mathcal{A}^{-1}v$. Then $(w, r) \in J \times M$ is the solution of

$$-\Delta w + \nabla r = v, \quad \operatorname{div} w = 0, \quad w|_{\Gamma} = 0.$$

It follows

$$\begin{aligned}\|v\|_{J^*}^2 &= \sup_{x \in J} \frac{|\langle v, x \rangle|^2}{\|\nabla x\|^2} = \sup_{x \in J} \frac{|\langle -\Delta w + \nabla r, x \rangle|^2}{\|\nabla x\|^2} \\ &= \sup_{x \in J} \frac{|\langle \nabla w, \nabla x \rangle|^2}{\|\nabla x\|^2} \leq \|\nabla w\|^2 = \langle w, -\Delta w \rangle,\end{aligned}$$

which proves the first inequality and the identity in Eq. (2.5). The second inequality is easily obtained by regularity of the Stokes system. $\square$

For any Banach space V , $L^p((t_1, t_2); V)$ is the space of Lebesgue measurable X -valued functions $u : (t_1, t_2) \rightarrow V$ with

$$\begin{aligned}\|u\|_{L^p((t_1, t_2); V)} &:= \left(\int_{t_1}^{t_2} \|u(t)\|_V^p dt \right)^{1/p}, & 1 \leq p < \infty, \\ \|u\|_{L^\infty((t_1, t_2); V)} &:= \operatorname{ess\,sup}_{t_1 < t < t_2} \|u(t)\|_V, & p = \infty,\end{aligned}$$

and $\|\cdot\|_{L^p(V)} := \|\cdot\|_{L^p((0, T); V)}$.

2.2 Regularity of the solution

Throughout this thesis sufficient regularity is assumed for the solution (u, p) of the Navier–Stokes equations and the instationary Stokes system Eq. (2.1), respectively, particularly that $\|u(t)\|_m$ and $\|p(t)\|_{H^m \cap L_0^2}$ remain bounded as $t \searrow 0$. However, this required regularity of the solution cannot be realistically assumed; it holds only in the presence of nonlocal compatibility conditions of various orders for the data u_0 and f at time $t = 0$, cf. [18, 19].

In particular, to show that the terms $\|u(t)\|_3$ or $\|\nabla \partial_t u(t)\|$ remain bounded as $t \searrow 0$, which is required for standard error estimates, there must necessarily exist a solution $p_0 \in H^1(\Omega) \cap L_0^2$ of the overdetermined Neumann problem

$$\begin{aligned}-\Delta p_0 &= \operatorname{div}(u_0 \cdot \nabla u_0 - f(0)) && \text{in } \Omega, \\ \nabla p_0 &= \Delta u_0 + f(0) - u_0 \cdot \nabla u_0 && \text{on } \Gamma,\end{aligned}\tag{2.6}$$

for the initial pressure. Eq. (2.6) follows directly as $t \searrow 0$ by applying the divergence operator div and the trace operator $\gamma_0 : H^1(\Omega) \rightarrow H^{1/2}(\Gamma)$ to the Navier–Stokes equation. Eq. (2.6) amounts to a *compatibility condition* to be satisfied by f and u_0 , which is virtually unverifiable for given data due to its nonlocal character. Moreover, even in simple test cases it is rarely satisfied.

In [18], however, it is shown under the assumption that Ω is of class C^2 and the prescribed data f and u_0 satisfy

$$u_0 \in J \cap H^2(\Omega), \quad f \in L^\infty(0, T; L^2), \quad \text{and} \quad \partial_t f(t) \in L^\infty(0, T; L^2), \tag{2.7}$$

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that the solution of the Neumann problem

$$\begin{aligned} -\Delta p_0 &= \operatorname{div}(u_0 \cdot \nabla u_0 - f(0)) && \text{in } \Omega, \\ \nabla p_0 \cdot n &= (\Delta u_0 + f(0) - u_0 \cdot \nabla u_0) \cdot n && \text{on } \Gamma, \end{aligned} \quad (2.8)$$

determines the initial pressure p_0 fulfilling

$$\begin{aligned} \|\nabla(p(t) - p_0)\| &\rightarrow 0 && \text{as } t \searrow 0, \\ \partial_t u(t) &\rightarrow -\nabla p_0 + \Delta u_0 + f(0) - u_0 \cdot \nabla u_0 && \text{as } t \searrow 0. \end{aligned}$$

In this theses, however, higher regularity is needed, as for instance the a-priori estimates for the instationary Stokes System in Theorem 3.1. Moreover, the cited a-priori estimates for the presented pressure projection schemes in Chapter 4 require among others the fourth order time derivative of the velocity to be bounded as $t \searrow 0$. This induces higher order compatibility conditions, meaning compatibility conditions on higher order time derivatives of the given data f and u_0 . Due to the lack of such compatibility conditions, it can be shown that estimates for the solution (u, p) of the form

$$\|u(t)\|_m + \|p(t)\|_{H^{m-1} \cap L_0^2} \lesssim t^{1-m/2}, \quad \text{as } t \searrow 0,$$

remain valid and require only the smoothness of the data and local compatibility conditions, like $u_0 \in J$. Let $(U_h(t), P_h(t))$ be the semi-discrete solution to the Navier–Stokes equations. In [18, 19] Heywood and Rannacher show under appropriate assumptions error estimates of order m for $m \in \{2, 3, 4, 5\}$ of the form

$$\|(u - U_h)(t)\| \leq h^m E_1(t), \quad \|(p - P_h)(t)\| \leq h^{m-1} E_2(t),$$

where the error constants $E_1(t)$, $E_2(t)$ are independent of h , but become singular as $t \searrow 0$, if $m \geq 3$.

Regarding pressure projection schemes it is common practice in literature to assume sufficient regularity for proving a-priori estimates, cf. [15, 17, 33, 35, 36]. In [30], however, Prohl weights the error by the term $\rho(t) := \min\{t, 1\}$. This allows a-priori error estimates without requiring nonlocal compatibility conditions for the data u_0 and f to be satisfied.

2.3 Discrete spaces and operators

Let $(\mathcal{T}_h)_{h>0}$ be a family of regular, quasi-uniform triangulations of Ω of mesh size h , $0 < h < 1$, and X_h as well as M_h mixed finite element approximation of X and M , respectively. Assume that the spatial discretization $X_h \times M_h$ is of order $k \geq 2$. This means that for each $v \in X \cap H^{k+1}(\Omega)^d$ and $q \in M \cap H^k(\Omega)$ there exist approximations $\mathcal{I}_h v \in X_h$ and $\mathcal{J}_h q \in M_h$ such that

$$\|\nabla(v - \mathcal{I}_h v)\| \leq Ch^k \|v\|_{k+1}, \quad \|q - \mathcal{J}_h q\|_{L_0^2(\Omega)} \leq Ch^k \|q\|_k. \quad (2.9)$$

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Denote by $-\Delta_h : X_h \rightarrow X_h$ the linear continuous operator, such that

$$(-\Delta_h u_h, v_h) = (\nabla u_h, \nabla v_h) \quad \forall u_h \in X_h, v_h \in X_h.$$

Let us also introduce the linear operator $B_h : X_h \rightarrow M_h$ and its transpose $B_h^\top : M_h \rightarrow X_h$, such that

$$(B_h v_h, q_h) = (v_h, B_h^\top q_h) = -(\operatorname{div} v_h, q_h) \quad \forall v_h \in X_h, q_h \in M_h.$$

Assume in the sequel that B_h is surjective; that is to say, the mixed finite element approximation $X_h \times M_h$ satisfies the inf-sup condition:

$$\exists c > 0, \quad \inf_{q_h \in M_h} \sup_{v_h \in X_h} \frac{(B_h v_h, q_h)}{\|v_h\|_1 \|q_h\|_0} \geq c. \quad (2.10)$$

The space of discrete divergence-free vector fields is denoted by

$$J_h := \{v_h \in X_h : \langle \operatorname{div} v_h, q_h \rangle = 0 \ \forall q_h \in M_h\}.$$

$\mathcal{S}_h : X \times M \rightarrow J_h$ is defined as the *discrete Stokes projector* and reads

$$\langle \nabla \mathcal{S}_h(u, p), \nabla v_h \rangle = \langle \nabla u, \nabla v \rangle - \langle p, \operatorname{div} v_h \rangle, \quad \forall (u, p) \in X \times M, v_h \in J_h.$$

In the following denote $\mathcal{S}_h u := \mathcal{S}_h(u, p)$ for the sake of readability. Note, in the case of a discrete pair $(U_h, P_h) \in X_h \times M_h$ it holds

$$\langle \nabla \mathcal{S}_h U_h, \nabla v_h \rangle = \langle \nabla U_h, \nabla v_h \rangle, \quad \forall v_h \in J_h.$$

In [19] Heywood and Rannacher prove the following approximation estimates for $\mathcal{S}_h$:

Lemma 2.4.

Let (u, p) sufficiently regular in space and time. Then it holds

$$\|u - \mathcal{S}_h u\| + h \|\nabla(u - \mathcal{S}_h u)\| \lesssim h^{k+1} \left(\|u\|_{k+1} + \|p\|_{H^k \setminus \mathbb{R}} \right), \quad (2.11)$$

$$\|\partial_t(u - \mathcal{S}_h u)\| + h \|\nabla \partial_t(u - \mathcal{S}_h u)\| \lesssim h^{k+1} \left(\|u_t\|_{k+1} + \|p_t\|_{H^k \setminus \mathbb{R}} \right). \quad (2.12)$$

In addition to the velocity estimates one also needs an estimate for the associated pressure.

Lemma 2.5.

For given $u \in H^{k+1}(\Omega)^d \cap X$ and $p \in H^k(\Omega) \cap M$ let $(\mathcal{S}_h u, \bar{P}_h) \in X_h \times M_h$ be the solution of the mixed problem

$$\begin{aligned} \langle \nabla \mathcal{S}_h u, \nabla v_h \rangle + \langle \nabla \bar{P}_h, v_h \rangle &= \langle \nabla u, \nabla v_h \rangle + \langle \nabla p, v_h \rangle, & \forall v_h \in X_h, \\ \langle \operatorname{div} \mathcal{S}_h u, q_h \rangle &= 0, & \forall q_h \in M_h. \end{aligned} \quad (2.13)$$

Then it holds

$$\|\bar{P}_h - p\|_{L^2 \setminus \mathbb{R}} \leq h^k \left(\|u\|_{k+1} + \|p\|_{H^k \setminus \mathbb{R}} \right).$$

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Proof : The inf-sup stability (2.10) and the interpolation operator $\mathcal{J}_h$ deduce

$$\|\bar{P}_h - p\|_{L^2 \setminus \mathbb{R}} \lesssim \|\nabla(I - S_h)u\| + \|(I - \mathcal{J}_h)p\|.$$

Then the use of Eq. (2.9) and Lemma 2.4 complete the proof. $\square$

Let Y_h , a finite dimensional subspace of $L^2(\Omega)^d$ associated with the norm of $L^2(\Omega)$ with $X_h \subset Y_h$. Further introduce another discrete version of the divergence operator $C_h : Y_h \rightarrow M_h$ and its transpose $C_h^\top : M_h \rightarrow Y_h$ by

$$\langle C_h v_h, q_h \rangle = \langle v_h, C_h^\top q_h \rangle = \langle v_h, \nabla q_h \rangle \quad \forall v_h \in Y_h, q_h \in M_h.$$

The relation between B_h and C_h is clarified in the following proposition (cf. [16, Proposition 3.1])

Proposition 2.6.

C_h is an extension of B_h and $\pi_h^\top C_h^\top = B_h^\top$; in other words one has the following commutative diagrams:

$$\begin{array}{ccc} X_h & \xrightarrow{B_h} & M_h \\ \pi_h \downarrow & \nearrow C_h & \\ Y_h & & \end{array} \qquad \begin{array}{ccc} X_h & \xleftarrow{B_h^\top} & M_h \\ \pi_h^\top \uparrow & \nwarrow C_h^\top & \\ Y_h & & \end{array}$$

where π_h is the continuous injection of X_h into Y_h and $\pi_h^\top$ the L^2 projection of Y_h onto X_h .

Note that as a consequence of the assumed surjectivity of B_h also C_h is surjective. By defining the kernel of C_h , namely $H_h := \ker C_h$, one can formulate a discrete counterpart of the *Helmholtz decomposition* $Y = H \oplus \nabla(H^1(\Omega))$, (cf. [16, Corollary 3.2]),

$$Y_h = H_h \oplus C_h^\top(M_h).$$

$C_h^\top$ is characterized in the following Lemma (cf. [16, Lemma 3.4])

Lemma 2.7.

$C_h^\top$ is the restriction of ∇ to M_h .

Thus, one is able to define the *discrete Helmholtz projector* $\mathcal{H}_h : Y \rightarrow H_h$

$$\langle \mathcal{H}_h u, v_h \rangle = \langle u, v_h \rangle, \quad \forall u \in Y, v_h \in H_h$$

and as discrete counterpart to $\mathcal{A}$, the *discrete Stokes operator* $\mathcal{A}_h : X_h \rightarrow H_h$ by

$$\mathcal{A}_h u_h := \mathcal{H}_h \Delta_h u_h, \quad \forall u_h \in X_h.$$

Remark 2.8.

The introduction of the space Y_h and the discrete version of the divergence operator C_h are necessary for the analysis of projection methods in a spatially discrete form, which are indicated in the outline.

2.4 Time discretization

For the time discretization let $N \in \mathbb{N}$, $N \geq 2$ and denote by $\tau := T/N$ the constant time step. Let $t_n := n\tau$ for $n = 0, \dots, N$ be a uniform partition of $[0, T]$, and $I_n := (t_{n-1}, t_n]$. For a sequence $v_0, v_1, \dots$ set

$$\bar{\partial}v^n := \frac{v^n - v^{n-1}}{\tau}, \quad \bar{\partial}^2 := \bar{\partial}(\bar{\partial}v^n) = \frac{v^n - 2v^{n-1} + v^{n-2}}{\tau^2},$$

and recursively

$$\bar{\partial}^i v^n := \bar{\partial}(\bar{\partial}^{i-1} v^n), \quad i \leq n.$$

Further define the *BDF2 backward differences operator* by

$$\bar{\partial}^B v^n := \frac{\tau}{2} \bar{\partial}^2 v^n + \bar{\partial}v^n = \frac{1}{2\tau} (3v^n - 4v^{n-1} + v^{n-2}).$$

The next lemma shows that $\bar{\partial}^B$ is a second order approximation:

Lemma 2.9.

For a given time dependent $v \in C^3(V)$ it holds

$$\|(\bar{\partial}^B - \partial_t)v|_{t=t_n}\|_V \leq \tau^2 \int_{t_{n-2}}^{t_n} \|\partial_{ttt}u(t)\|_V dt. \quad (2.14)$$

Proof : Let $v \in C^3(V)$. Then the lemma is easily proved by Taylor expansion:

$$\begin{aligned} v^{n-1} &= v^n - \tau \partial_t v^n + \tau^2 \partial_{tt} v^n - \int_{t_{n-1}}^{t_n} \frac{(t_{n-1} - t)^2}{2!} \partial_{ttt} v(t) dt, \\ v^{n-2} &= v^n - 2\tau \partial_t v^n + 4\tau^2 \partial_{tt} v^n - \int_{t_{n-2}}^{t_n} \frac{(t_{n-2} - t)^2}{2!} \partial_{ttt} v(t) dt. \end{aligned}$$

Thus, one obtains

$$(\bar{\partial}^B - \partial_t)v^n = \frac{1}{\tau} \left(\frac{3}{2}v^n - 2v^{n-1} + \frac{1}{2}v^{n-2} - \tau \partial_t v^n \right) \leq C\tau^2 \sup_{t \in (t_{n-2}, t_n)} \|\partial_{ttt}v(t)\|_V.$$

□

The discrete counterpart of $L^p(V)$ are defined as follows: for a sequence $v^{i_1}, \dots, v^{i_m} \in V$, $i_1, \dots, i_m \in \mathbb{N}$, $i_1 \leq \dots \leq i_m$, and a given $\tau > 0$ let

$$\begin{aligned} \|u\|_{\ell^p(i_1, i_m; V)} &:= \left(\sum_{n=i_1}^{i_m} \tau \|u^n\|_V^p \right)^{1/p}, & 1 \leq p < \infty, \\ \|u\|_{\ell^\infty(i_1, i_m; V)} &:= \max_{i_1 \leq n \leq i_m} \|u^n\|_V, & p = \infty, \end{aligned}$$

and set $\|u\|_{\ell^p(V)} := \|u\|_{\ell^p(1, N; V)}$.

The following identities are easily proved by using properties of the inner product:

$$2\langle 3a^n - 4a^{n-1} + a^{n-2}, a^n \rangle = \|a^n\|^2 - \|a^{n-1}\|^2 + \tau^4 \|\bar{\partial}^2 a^n\|^2 + \|2a^n - a^{n-1}\|^2 - \|2a^{n-1} - a^{n-2}\|^2, \quad (2.15)$$

$$2\langle a - b, c - b \rangle = \|a - b\|^2 + \|c - b\|^2 - \|a - c\|^2, \quad (2.16)$$

$$2\langle a - b, a \rangle = \|a\|^2 - \|b\|^2 + \|a - b\|^2. \quad (2.17)$$

2.5 A-posteriori techniques

In the *a-priori analysis*, error estimates are established by inserting the exact solution into the discretized equation. To this end the exact solution has to be projected onto the discrete space. Take, for example, the heat equation

$$\partial_t u(t) - \Delta u = 0, \quad u|_\Gamma = 0, \quad u(0) = u_0, \quad (2.18)$$

where the right hand side is set to zero to simplify the demonstration. The equation discretized in space and time (BDF2) reads:

Calculate $U_h^n \in X_h$ for $n \geq 2$, such that

$$\bar{\partial}^B U_h^n - \Delta_h U_h^n = 0. \quad (2.19)$$

Subtract Eq. (2.18) from Eq. (2.19) to derive the error equation

$$\langle \bar{\partial}^B (U_h^n - u(t_n)), v_h \rangle + \langle \nabla (U_h^n - u(t_n)), \nabla v_h \rangle = 0, \quad \forall v_h \in X_h. \quad (2.20)$$

Now, one would like to set $v_h = e^n := U_h^n - u(t_n)$ but $u(t_n) \notin X_h$. Hence, the error is split by an adequate discrete size:

$$e^n = \theta_h^n + \eta^n := U_h^n - \mathcal{R}_h u(t_n) + \mathcal{R}_h u(t_n) - u(t_n),$$

where $\mathcal{R}_h : X \rightarrow X_h$ is the *Ritz projection* defined by

$$\langle \nabla \mathcal{R}_h u, \nabla v_h \rangle = \langle \nabla u, \nabla v_h \rangle, \quad \forall v_h \in X_h.$$

Using the Ritz projection the error equation (2.20) can be reformulated to

$$\langle \bar{\partial}^B \theta_h^n, v_h \rangle + \langle \nabla \theta_h^n, \nabla v_h \rangle = \langle -\bar{\partial}^B \eta_n, v_h \rangle, \quad \forall v_h \in X_h.$$

Due to the fact that $\theta_h^n \in X_h$ one is able to set $v_h = \theta_h^n$ and to bound the error θ_h^n by the interpolation error η_n which can be controlled by standard interpolation estimates.

In contrast, the *a-posteriori analysis* requires a different procedure: to derive error estimates one has to insert the discrete solution into the exact equation. This approach is presented in the next Chapters.

2.5.1 Reconstruction in time

To compare the discrete solution U_h , that is only available at discrete points in time to the continuous solution $u(t)$, an approximation $U_h(t)$ of $u(t) \forall t \in]0, T[$ is needed. Therefore one introduces a piecewise linear function, interpolating a sequence $\{v^n\}_{n=0,\dots,N}$ at the nodes:

$$v(t) := v^n + (t - t_n)\bar{\partial}v^n, \quad t \in I_n, \quad n = 1, \dots, N. \quad (2.21)$$

Note that for a discrete quantity like U_h the functions $U_h(t)$ are piecewise linear, whereas U_h^n are piecewise constant in time.

Even though the interpolant $U_h(t)$, defined in (2.21), is a second order approximation, its residual, i.e. the error between the continuous system (2.1) and the approximated solution, is only of first order, cf. [1] for parabolic problems.

To obtain optimal order error estimates, follow the approach of Akrivis and Chatzipantelidis in [1] and modify the interpolant $v(t)$ using the piecewise quadratic *three-point reconstruction* $\hat{v}(t)$, interpolating a sequence $\{v^n\}_{n=0,\dots,N}$ in t_{n-2}, t_{n-1}, t_n :

$$\hat{v}(t) := v(t) + \frac{1}{2}(t - t_n)(t - t_{n-1})\bar{\partial}^2v^n, \quad t \in I_n, \quad n = 2, \dots, N. \quad (2.22)$$

Its time derivative is given by

$$\partial_t \hat{v}(t) = \bar{\partial}^B v^n + (t - t_n)\bar{\partial}^2v^n, \quad t \in I_n, \quad n = 2, \dots, N. \quad (2.23)$$

This three-point reconstruction was also proposed by Lozinski, Picasso and Prachittham in [27] for the Crank–Nicolson scheme.

Figure 2.1: Linear interpolant ($\circ$) vs. three-point reconstruction ($\square$).

In Figure 2.1 an example is given for the linear interpolant and the three-point reconstruction.

Additionally the analysis of the projection schemes needs an approximation of the difference quotient $\bar{\partial}v^n$. For $t \in I_n, n = 2, \dots, N$ define

$$\bar{\partial}v(t) := \bar{\partial}v^n + (t - t_n)\bar{\partial}^2v^n. \quad (2.24)$$

Note for a discrete quantity the difference between the time derivative of the linear interpolant $\partial_t v(t)$ (cf. Eq. (2.21)) and the linear interpolant of the difference quotient $\bar{\partial} v(t)$.

2.5.2 Reconstruction in space: elliptic reconstruction

In this Chapter the idea of the elliptic reconstruction introduced by Makridakis and Nochetto in [29] is presented, see also Figure 2.2. Let $u \in X$ be the exact solution and $U_h(t) \in X_h$ the time dependent finite element approximation of u . The objective is to calculate an intermediate object V between u and U_h fulfilling the property that U_h is an *elliptic* approximation of V . For this *elliptic* error $U_h - V$ the Galerkin orthogonality holds and a-posteriori bounds are available by using standard approaches.

Note that V is only a theoretical object and is not computable. However, the *time dependent* error $V - u$ satisfies the original equation with computable data. V is called *reconstruction*.

Figure 2.2: Elliptic reconstruction.

Using the above example of the heat equation the elliptic reconstruction $V \in X$ is defined by

$$V^n := (-\Delta)^{-1} U_h^n,$$

i.e. $V^n \in X$ is the solution of

$$-\Delta V^n = -\Delta_h U_h^n, \quad V^n|_{\Gamma} = 0.$$

In case of the heat equation $V(t)$ fulfills

$$\begin{aligned} \partial_t \hat{V}(t) \underbrace{-\Delta V(t)}_{=-\Delta_h U_h(t)} &= \partial_t(\hat{V} - \hat{U}_h)(t) + \partial_t \hat{U}_h(t) - \Delta_h U_h(t) \\ &= \partial_t(\hat{V} - \hat{U}_h)(t) + \underbrace{\bar{\partial}^B U_h^n - \Delta_h U_h^n}_{=0} + (t - t_n) \bar{\partial} \underbrace{(\bar{\partial} U_h^n - \Delta_h U_h^n)}_{=-\frac{\tau}{2} \bar{\partial}^2 U_h^n} \\ &= \partial_t(\hat{V} - \hat{U}_h)(t) - \frac{\tau}{2} (t - t_n) \bar{\partial}^3 U_h^n. \end{aligned}$$

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So the equation satisfied by the error error $V - u$ reads

$$\partial_t(\hat{V} - u)(t) - \Delta(V - u) = \underbrace{\partial_t(\hat{V} - \hat{U}_h)(t)}_{\text{elliptic estimator}} \underbrace{- \frac{\tau}{2}(t - t_n)\bar{\partial}^3 U_h^n}_{\text{temporal estimator}}.$$

and its residual is composed of computable quantities.

Note that the *elliptic reconstruction* is the continuous analogue of the *Ritz projection*.

Thus, in summary it can be said, that the error splitting by the reconstruction V

$$U(t) - u(t) = [U(t) - V(t)] + [V(t) - u(t)]$$

reduces the problem to that of an elliptic estimator and the estimate of the time discretization.

3 The instationary Stokes system

In this Chapter a-posteriori error estimates for the instationary Stokes system (Eq. (2.1)) are presented. On the one hand these results are interesting in themselves, on the other hand these estimates will be used as reference for adequate estimates for pressure projection schemes in Chapter 6.

The discretization of Eq. (2.1) in time by BDF2 and in space reads as follows: Let U_h^0 and U_h^1 be given. For $n \geq 2$ calculate $(U_h^n, P_h^n) \in X_h \times M_h$ by

$$\begin{aligned} \bar{\partial}^B U_h^n - \triangle_h U_h^n + B_h^\top P_h^n &= f_h^n, \\ B_h U_h^n &= 0, \end{aligned} \quad (3.1)$$

where $f_h^n := \pi_h f(t_n)$.

To initialize the scheme the values U_h^0 and U_h^1 are needed. To this end let $(\mathcal{S}_h u_0, \bar{P}_h^0) \in J_h \times M_h$ be the solution of the Stokes system in Lemma 2.5. Then set $U_h^0 := \mathcal{S}_h u_0$, $P_h^0 := \bar{P}_h^0$ and calculate $(U_h^1, P_h^1) \in X_h \times M_h$ from the trapezoidal scheme:

$$\bar{\partial} U_h^1 - \triangle_h U_h^{1/2} + \nabla P_h^{1/2} = f^{1/2}, \quad U_h^1|_\Gamma = 0, \quad (3.2)$$

where $U_h^{1/2} := \frac{1}{2}(U_h^1 + U_h^0)$.

3.1 A-priori estimates for the fully discrete scheme

First a-priori estimates for the fully discrete schemes are deduced to validate the second order convergence of the time discretization in Eq. (3.1).

Theorem 3.1 (A-priori estimates for the instationary Stokes system).

$$\|U_h - u\|_{\ell^\infty(L^2)}^2 \lesssim h^{2(k+1)}(A^0 + A^1) + \tau^4 \|\partial_t^3 u\|_{L^1(H^{-1})}^2, \quad (3.3)$$

$$\|\nabla(U_h - u)\|_{\ell^2(L^2)}^2 \lesssim h^{2k}(A^0 + h^2 A^1) + \tau^4 \|\partial_t^3 u\|_{L^1(H^{-1})}^2, \quad (3.4)$$

$$\|\nabla(U_h - u)\|_{\ell^\infty(L^2)}^2 + \|P_h - p\|_{\ell^2(L^2)}^2 \lesssim h^{2k}(A^0 + h^2 A^1) + \tau^4 \|\partial_t^3 u\|_{L^1(L^2)}^2, \quad (3.5)$$

where $A^i := \|\partial_t^i u\|_{L^\infty(H^{k+1})}^2 + \|\partial_t^i p\|_{L^\infty(H^k \setminus \mathbb{R})}^2$, $i = 0, 1$.

Proof : Set $\theta_h^n := U_h^n - \mathcal{S}_h u_n$. Subtract Eq. (2.1) from Eq. (3.1) and combine this difference with the definition of the discrete Stokes projector in Eq. (2.13) to get the

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error equation

$$\bar{\partial}^B \theta_h^n - \triangle_h \theta_h^n + B_h^\top (P_h^n - \bar{P}_h^n) = \pi_h (\bar{\partial}^B (I - \mathcal{S}_h) u_n + (\partial_t - \bar{\partial}^B) u_n), \quad n \geq 2. \quad (3.6)$$

Test this equation with $4\tau\theta_h^n \in J_h$. With the identity (2.15) the first term reduces to

$$\begin{aligned} 4\tau \langle \bar{\partial}^B \theta_h^n, \theta_h^n \rangle &= 2 \langle 3\theta_h^n - 4\theta_h^{n-1} + \theta_h^{n-2}, \theta_h^n \rangle \\ &= \|\theta_h^n\|^2 - \|\theta_h^{n-1}\|^2 + \tau^4 \|\bar{\partial}^2 \theta_h^n\|^2 + \|2\theta_h^n - \theta_h^{n-1}\|^2 - \|2\theta_h^{n-1} - \theta_h^{n-2}\|^2. \end{aligned}$$

Due to $\theta_h^n \in J_h$ the pressure terms drop out and one obtains the error inequality

$$\begin{aligned} \|\theta_h^n\|^2 - \|\theta_h^{n-1}\|^2 + \tau^4 \|\bar{\partial}^2 \theta_h^n\|^2 + \|2\theta_h^n - \theta_h^{n-1}\|^2 - \|2\theta_h^{n-1} - \theta_h^{n-2}\|^2 + \tau \|\nabla \theta_h^n\|^2 \\ \lesssim \tau \|\bar{\partial}^B (I - \mathcal{S}_h) u_n\|_{J_h^*}^2 + \tau \|(\partial_t - \bar{\partial}^B) u_n\|_{J_h^*}^2. \end{aligned}$$

Summing up from $n = 2$ to N yields

$$\begin{aligned} \max_{1 \leq n \leq N} \|\theta_h^n\|^2 + \sum_{n=2}^N \tau \|\nabla \theta_h^n\|^2 \\ \lesssim \sum_{n=2}^N \tau \|\bar{\partial}^B (I - \mathcal{S}_h) u_n\|_{J_h^*}^2 + \sum_{n=2}^N \tau \|(\partial_t - \bar{\partial}^B) u_n\|_{J_h^*}^2 + \|\theta_h^1\|^2 + \|\theta_h^1 - \theta_h^0\|^2. \end{aligned}$$

For the first term on the right hand side one gets by using Lemma 2.4

$$\sum_{n=2}^N \tau \|\bar{\partial}^B (I - \mathcal{S}_h) u_n\|_{J_h^*}^2 \lesssim \sum_{n=1}^N \tau \|(I - \mathcal{S}_h) \bar{\partial} u_n\|^2 \lesssim h^{2(k+1)} A^1. \quad (3.7)$$

Use Eq. (2.14) and the fact that $\theta_h^0 = 0$ to obtain

$$\max_{1 \leq n \leq N} \|\theta_h^n\|^2 + \sum_{n=2}^N \tau \|\nabla \theta_h^n\|^2 \lesssim \|\theta_h^1\|^2 + h^{2(k+1)} A^1 + \tau^4 \|\partial_t^3 u\|_{L^1(H^{-1})}^2. \quad (3.8)$$

One gets the velocity estimate Eq. (3.3) by using the initial estimate Eq. (3.10) below, Eq. (3.8), Lemma 2.4 and the triangle inequality:

$$\max_{1 \leq n \leq N} \|U_h^n - u_n\| \lesssim \max_{1 \leq n \leq N} \|\theta_h^n\| + \max_{1 \leq n \leq N} \|(I - \mathcal{S}_h) u_n\|.$$

In the same way Eq. (3.4) is proven.

In order to prove the pressure estimate Eq. (3.5) the term $\|\bar{\partial}^B U_h^n - \partial_t u\|$ has to be bounded. To this end test the error equation (3.6) by $4\tau \bar{\partial}^B \theta_h^n$ and use the identity (2.15) to get

$$\begin{aligned} \tau \|\bar{\partial}^B \theta_h^n\|^2 + \|\nabla \theta_h^n\|^2 - \|\nabla \theta_h^{n-1}\|^2 + \tau^4 \|\nabla \bar{\partial} \theta_h^n\|^2 + \|\nabla (2\theta_h^n - \theta_h^{n-1})\|^2 - \|\nabla (2\theta_h^{n-1} - \theta_h^{n-2})\|^2 \\ \lesssim \tau \|\bar{\partial}^B (I - \mathcal{S}_h) u_n\|^2 + \tau \|(\partial_t - \bar{\partial}^B) u_n\|^2. \end{aligned}$$

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Summing up from $n = 2$ to N and using that $\theta_h^0 = 0$ one obtains

$$\begin{aligned} \sum_{n=2}^N \tau \|\bar{\partial}^B \theta_h^n\|^2 + \max_{1 \leq n \leq N} \|\nabla \theta_h^n\|^2 \\ \lesssim \|\nabla \theta_h^1\|^2 + \sum_{n=2}^N \tau \|(I - \mathcal{S}_h) \bar{\partial}^B u_n\|^2 + \sum_{n=2}^N \tau \|(\partial_t - \bar{\partial}^B) u_n\|^2. \end{aligned} \quad (3.9)$$

Next the pressure estimate Eq. (3.5) will be proven. Using the inf-sup condition (2.10) one obtains

$$\begin{aligned} \|P_h^n - p_n\| &\leq \|P_h^n - \bar{P}_h^n\| + \|\bar{P}_h^n - p_n\| \lesssim \sup_{v_h \in X_h} \frac{\langle P_h^n - \bar{P}_h^n, \operatorname{div} v_h \rangle}{\|\nabla v_h\|} + \|\bar{P}_h^n - p_n\| \\ &= \sup_{v_h \in X_h} \frac{\langle \bar{\partial}^B U_h^n - \partial_t u_n, v_n \rangle + \langle \nabla \theta_h^n, \nabla v_h \rangle}{\|\nabla v_h\|} + \|\bar{P}_h^n - p_n\| \\ &\leq \|\bar{\partial}^B \theta_h^n\| + \|(I - \mathcal{S}_h) \bar{\partial}^B u_n\|_{-1} + \|(\bar{\partial}^B - \partial_t) u_n\|_{-1} + \|\nabla \theta_h^n\| + \|\bar{P}_h^n - p_n\|. \end{aligned}$$

Square this inequality, multiply by τ and use Eq. (3.9) to get

$$\begin{aligned} \sum_{n=1}^N \tau \|P_h^n - p_n\|^2 &\lesssim \|\nabla \theta_h^1\|^2 + \tau \|P_h^1 - p_1\|^2 \\ &\quad + \sum_{n=2}^N \tau \left(\|(I - \mathcal{S}_h) \bar{\partial}^B u_n\|^2 + \|(\partial_t - \bar{\partial}^B) u_n\|^2 + \|\nabla \theta_h^n\|^2 + \|\bar{P}_h^n - p_n\|^2 \right). \end{aligned}$$

Finally, using the initial estimate Eqs. (3.10)–(3.13) below, the velocity estimate Eq. (3.8) and Lemma Eq. 2.5 the pressure estimate in Eq. (3.5) is proven. The $\ell^\infty(L^2)$ estimate for the gradients of the velocity is easily derived by Eq. (3.9) and Lemma 2.4. $\square$

It remains to bound the initial errors.

Lemma 3.2.

Assume the exact solution of the Stokes system (2.1) is sufficiently regular. Let $(\mathcal{S}_h u_0, \bar{P}_h^0)$ the solution of the Stokes system in Lemma 2.5 and set $(U_h^0, P_h^0) = (\mathcal{S}_h u_0, \bar{P}_h^0)$. Calculate $U_h^1 \in X_h$ and $P_h^1 \in M_h$ by Eq. (3.2). Then it holds the semi-discrete estimate

$$\begin{aligned} \|U_h^1 - \mathcal{S}_h u_1\|^2 + \tau \|\nabla(U_h^1 - \mathcal{S}_h^1 u_1)\|^2 + \tau^2 \|P_h^1 - \bar{P}_h^1\|^2 \\ \lesssim \tau^2 h^{2(k+1)} \|\partial_t u\|_{L^\infty((0,t_1);H^{k+1})}^2 + \tau^6 \|\partial_t^3 u\|_{L^\infty((0,t_1);L^2)}^2 \end{aligned} \quad (3.10)$$

and the fully discrete estimates

$$\|U_h^1 - u_1\|^2 \lesssim h^{2(k+1)} \cdot A^0 + \tau^6 \|\partial_t^3 u\|_{L^\infty((0,t_1);L^2)}^2, \quad (3.11)$$

$$\|\nabla(U_h^1 - u_1)\|^2 \lesssim h^{2k} \cdot A^0 + \tau h^{2(k+1)} \cdot A^1 + \tau^5 \|\partial_t^3 u\|_{L^\infty((0,t_1);L^2)}^2, \quad (3.12)$$

$$\|P_h^1 - p_1\|^2 \lesssim h^{2k} \cdot A^0 + h^{2(k+1)} \cdot A^1 + \tau^4 \|\partial_t^3 u\|_{L^\infty((0,t_1);L^2)}^2, \quad (3.13)$$

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hold, where

$$A^i := \|\partial_t^i u\|_{L^\infty((0,t_1);H^{k+1})}^2 + \|\partial_t^i p\|_{L^\infty((0,t_1);H^k \setminus \mathbb{R})}^2.$$

Proof : Set $\theta_h^i := U_h^i - \mathcal{S}_h u_i$. Use the Stokes projector defined in Lemma 2.5 and note that $\theta_h^0 = 0$ to deduce the error equation

$$\frac{1}{\tau}\theta_h^1 - \frac{1}{2}\Delta_h \theta_h^1 + \frac{1}{2}B_h^\top(P_h^1 - \bar{P}_h^1) = (I - \mathcal{S}_h)\bar{\partial}u_1 + (\partial_t u_{1/2} - \bar{\partial}u_1).$$

Test this equation by $\tau\theta_h^1 \in J_h$ to obtain

$$\begin{aligned} \|\theta_h^1\|^2 + \tau\|\nabla\theta_h^1\|^2 &\lesssim \tau^2\|(I - \mathcal{S}_h)u\bar{\partial}u_1\|^2 + \tau^2\|(\partial_t u_{1/2} - \bar{\partial}u_1)\|^2 \\ &\lesssim \tau^2 h^{2(k+1)}\|\partial_t u\|_{L^\infty((0,t_1);H^{k+1})}^2 + \tau^6\|\partial_t^3 u\|_{L^\infty((0,t_1);L^2)}^2, \end{aligned}$$

where Lemma 2.4 and Taylor expansion have been used.

Note that for θ_h^1 one obtains the sharper estimate

$$\|\theta_h^1\|^2 \lesssim h^{2(k+1)}\|u\|_{L^\infty((0,t_1);H^{k+1})}^2 + \tau^6\|\partial_t^3 u\|_{L^\infty((0,t_1);L^2)}^2.$$

The pressure estimate is obtained by the inf-sup condition: use Lemma 2.4 and Lemma 2.5 to conclude the proof. $\square$

3.2 A-posteriori estimates for the fully discrete scheme

Now a-posteriori estimates for Eq. (3.1) are presented by using the techniques introduced in Chapter 2.5. To this end define the *Stokes Reconstruction* V^n , cf. [23], as

$$V^n := \mathcal{A}^{-1}(-\Delta_h U_h^n + B_h^\top P_h^n),$$

i.e. $(V^n, R^n) \in X \times M$ is the solution of

$$-\Delta V^n + \nabla R^n = -\Delta_h U_h^n + B_h^\top P_h^n, \quad \operatorname{div} V^n = 0, \quad V|_\Gamma = 0. \quad (3.14)$$

Assume that the spatial estimators of the stationary Stokes problem are given:

Assumption 3.3.

For $1 \leq n \leq N$ let $(V^n, R^n) \in X \times M$ be the solution of Eq. (3.14) and (U_h^n, P_h^n) be the solution of Eqs. (3.2) and (3.1). Assume that there exist a-posteriori estimator functions $\eta_u^n = \eta_u^n((U_h^n, P_h^n); Y)$, $\eta_p^n = \eta_p^n((U_h^n, P_h^n); M)$, depending on (U_h^n, P_h^n) and the corresponding norm, such that

$$\begin{aligned} \|U_h^n - V^n\|_Y &\leq \eta_u^n((U_h^n, P_h^n); Y), \quad Y \in \{H^{-1}, L^2, H_0^1\}, \\ \|P_h^n - R^n\| &\leq \eta_p^n((U_h^n, P_h^n); M). \end{aligned} \quad (3.15)$$

Additionally for $n = 0$ there exists an estimator function $\eta^0 = \eta^0(Y)$ depending on the initial values (u_0, p_0) and the corresponding norm, such that

$$\|(I - \mathcal{S}_h)u_0\|_Y \leq \eta^0(Y).$$

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Assumption 3.3 is proven for common conform discretizations such as the Taylor–Hoods elements [7], the mini–elements [39] or the nonconforming Crouzeix–Raviart elements [10] and nonconforming P_1/P_0 elements [20]. In [4] a comparison of several a-posteriori error estimates for the stationary Stokes problem is given.

Next the error estimators are defined.

Definition 3.4 (Error estimators).

Let us introduce the *time error estimators*

$$\begin{aligned}\mathcal{E}_1 &:= \sum_{n=2}^N \tau \|\bar{\partial}^2 U_h^n\|^2, & \mathcal{E}_2 &:= \sum_{n=3}^N \tau \|\bar{\partial}^3 U_h^n\|_{J^*}^2, & \mathcal{E}_3 &:= \max_{2 \leq n \leq N} \|\bar{\partial}^2 U_h^n\|^2, \\ \mathcal{E}_6 &:= \sum_{n=3}^N \tau \|\bar{\partial}^3 U_h^n\|^2, & \mathcal{E}_7 &:= \max_{2 \leq n \leq N} \|\nabla \bar{\partial}^2 U_h^n\|^2,\end{aligned}$$

the *Stokes reconstruction error estimators*

$$\begin{aligned}\mathcal{E}_4 &:= \tau (\|\bar{\partial}^2 f_h^2 - \frac{3}{2\tau} \Psi_1\|_{J^*}^2 + \|\bar{\partial}^2 f_h^3 - \frac{3}{2} \bar{\partial}^3 U_h^3 + \frac{1}{2\tau} \Psi_1\|_{J^*}^2) + \sum_{n=4}^N \tau \|\bar{\partial}^2 (f_h^n - \bar{\partial}^B U_h^n)\|_{J^*}^2, \\ \mathcal{E}_5 &:= \tau (\|\bar{\partial}^2 f_h^2 - \frac{3}{2\tau} \Psi_1\|^2 + \|\bar{\partial}^2 f_h^3 - \frac{3}{2} \bar{\partial}^3 U_h^3 + \frac{1}{2\tau} \Psi_1\|^2) + \sum_{n=4}^N \tau \|\bar{\partial}^2 (f_h^n - \bar{\partial}^B U_h^n)\|^2,\end{aligned}$$

the *space error estimators*

$$\begin{aligned}\Lambda_1 &:= \sum_{n=0}^N \tau \eta_u^n((U_h^n, P_h^n); L^2), & \Lambda_2 &:= \max_{0 \leq n \leq N} \eta_u^n((U_h^n, P_h^n); L^2), \\ \Lambda_3 &:= \sum_{n=1}^N \tau \eta_u^n((\bar{\partial} U_h^n, \bar{\partial} P_h^n); J^*), & \Lambda_4 &:= \sum_{n=1}^N \tau \eta_u^n((U_h^n, P_h^n); H_0^1), \\ \Lambda_5 &:= \max_{0 \leq n \leq N} \eta_u^n((U_h^n, P_h^n); H_0^1), & \Lambda_6 &:= \sum_{n=1}^N \tau \eta_u^n((\bar{\partial} U_h^n, \bar{\partial} P_h^n); L^2), \\ \Lambda_7 &:= \sum_{n=0}^N \tau \eta_p^n((U_h^n, P_h^n); M),\end{aligned}$$

and the *data approximation error estimators*

$$\begin{aligned}\mathcal{F}_1 &:= \int_0^{t_1} \|f_h^{1/2} + (t - t_{1/2}) \bar{\partial} f_h^{1/2} - f(t)\|_{J^*}^2 dt + \sum_{n=2}^N \int_{t_{n-1}}^{t_n} \|f_h^n + (t - t_n) \bar{\partial} f_h^n - f(t)\|_{J^*}^2 dt, \\ \mathcal{F}_2 &:= \int_0^{t_1} \|f_h^{1/2} + (t - t_{1/2}) \bar{\partial} f_h^{1/2} - f(t)\|^2 dt + \sum_{n=2}^N \int_{t_{n-1}}^{t_n} \|f_h^n + (t - t_n) \bar{\partial} f_h^n - f(t)\|^2 dt,\end{aligned}$$

where $\Psi_1 := \bar{\partial}^2 U_h^2 - \triangle_h \bar{\partial} U_h^1 + B_h^\top \bar{\partial} P_h^1 - \bar{\partial} f^1$.

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In the following the a-posteriori error estimates are stated in different norms. The proofs are outlined to Chapter 3.3.

Theorem 3.5 ($L^2(L^2)$ a-posteriori error estimates for the velocity).

Let U_h be the solution of Eqs. (3.1)-(3.2) and u the exact solution of Eq. (2.1). Then it holds

$$\|U_h - u\|_{L^2(L^2)}^2 \lesssim \tau^4(\mathcal{E}_1 + \mathcal{E}_2) + \Lambda_1 + \mathcal{F}_1 + \|(I - \mathcal{S}_h)u_0\|_{J^*}^2 + \tau^3\|\Psi_1\|_{J^*}^2, \quad (3.16)$$

where the estimators and $\mathcal{F}_1$ are defined in Definition 3.4.

An estimate in a stronger norm is stated in the following theorem.

Theorem 3.6 ($L^\infty(L^2)$ and $L^2(H^1)$ a-posteriori error estimates for the velocity).

Let U_h be the solution of Eqs. (3.1)-(3.2) and u be the exact solution of Eq. (2.1). Then it holds

$$\|U_h - u\|_{L^\infty(L^2)}^2 \lesssim \eta^0(L^2) + \tau^3\|\Psi_1\|_{J^*}^2 + \tau^4(\mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4) + \Lambda_2 + \Lambda_3 + \mathcal{F}_1, \quad (3.17)$$

$$\|\nabla(U_h - u)\|_{L^2(L^2)}^2 \lesssim \eta^0(L^2) + \tau^3\|\Psi_1\|_{J^*}^2 + \tau^4(\mathcal{E}_2 + \mathcal{E}_4) + \Lambda_3 + \Lambda_4 + \mathcal{F}_1, \quad (3.18)$$

where the estimators and $\mathcal{F}_1$ are defined in Definition 3.4.

In the next theorem the pressure error estimate and the gradient error in the stronger $L^\infty(L^2)$ norm are given.

Theorem 3.7 ($L^2(L^2)$ a-posteriori error estimates for the pressure).

Let (U_h, P_h) be the solution of Eqs. (3.1)-(3.2) and (u, p) the exact solution of Eq. (2.1). Then it holds

$$\|\nabla(U_h - u)\|_{L^\infty(L^2)}^2 \lesssim \eta^0(H_0^1) + \tau^3\|\Psi_1\|^2 + \tau^4(\mathcal{E}_5 + \mathcal{E}_6 + \mathcal{E}_7) + \Lambda_5 + \Lambda_6 + \mathcal{F}_2, \quad (3.19)$$

$$\|P_h - p\|_{L^2(L^2)}^2 \lesssim \eta^0(H_0^1) + \tau^3\|\Psi_1\|^2 + \tau^4(\mathcal{E}_5 + \mathcal{E}_6) + \Lambda_6 + \Lambda_7 + \mathcal{F}_2, \quad (3.20)$$

where the estimators and $\mathcal{F}_2$ are defined in Definition 3.4.

Note that the estimates in Theorem 3.5–3.7 are optimal in time meaning that the time error estimators $\mathcal{E}_1 - \mathcal{E}_6$ have the same asymptotic behavior as the a-priori estimates in Theorem 3.1:

$$\|U_h - u\|_{L^\infty(L^2)}^2 + \|\nabla(U_h - u)\|_{L^\infty(L^2)}^2 + \|P_h - p\|_{L^2(L^2)}^2 \lesssim \tau^4 + \mathcal{O}(h^k)$$

In [1] a-posteriori estimates for general semi-discrete parabolic problems discretized in time by BDF2 were established. If one considers the heat equation (2.18), for example, one gets by [1]:

$$\|U - u\|_{L^2(L^2)}^2 \lesssim \tau^4 \left(\sum_{n=2}^N \tau \|\bar{\partial}^2 U^n\|^2 + \sum_{n=3}^N \tau \|\bar{\partial}^3 U^n\|_{-1}^2 \right).$$

This corresponds to the estimators $\mathcal{E}_1$ and $\mathcal{E}_2$ in Theorem 3.5.

3.3 Proofs of Theorem 3.5, 3.6 and 3.7

3.3.1 Auxiliary statements

Before the theorems can be proved estimates for the initial errors and bounds for the Stokes reconstruction have to be shown.

First the error equation is derived by using the three-point reconstruction $\hat{U}_h(t)$ for $U_h(t)$, as illustrated for the heat equation in Chapter 2.5

Lemma 3.8.

Let $n \geq 3$ and $t \in I_n$. Then it holds in H^{-1}

$$\begin{aligned} \partial_t \hat{U}_h(t) - \Delta U_h(t) + \nabla P_h(t) \\ = -\Delta(U_h - V)(t) + \nabla(P_h - R)(t) - \frac{\tau}{2}(t - t_n)\bar{\partial}^3 U_h^n + f_h(t), \end{aligned} \quad (3.21)$$

where $f_h(t) := f_h^n + (t - t_n)\bar{\partial} f_h^n$.

Proof : Using the definition of $\hat{U}_h(t)$ in Eq. (2.22) the following identity for $n \geq 3$ holds

$$\begin{aligned} \partial_t \hat{U}_h(t) - \Delta U_h(t) + \nabla P_h(t) &= \bar{\partial}^B U_h^n + (t - t_n)\bar{\partial}^2 U_h^n - \Delta U_h(t) - \Delta_h U_h(t) \\ &\quad + \nabla P_h(t) + B_h^\top P_h(t) + \underbrace{\Delta_h U_h(t) - B_h^\top P_h(t)}_{=\Delta V(t) - \nabla R(t)} \\ &= \underbrace{\bar{\partial}^B U_h^n - \Delta_h U_h^n + B_h^\top P_h^n}_{=f_h^n} + (t - t_n)\bar{\partial} \underbrace{(-\Delta_h U_h^n + B_h^\top P_h^n + \bar{\partial} U_h^n)}_{f_h^n - \frac{\tau}{2}\bar{\partial}^2 U_h^n} \\ &\quad - \Delta(U_h - V)(t) + \nabla(P_h - R)(t) \\ &= -\Delta(U_h - V)(t) + \nabla(P_h - R)(t) + f_h(t) - \frac{\tau}{2}(t - t_n)\bar{\partial}^3 U_h^n. \end{aligned}$$

□

Now upper bounds of the initial errors are proven.

Lemma 3.9 (Initial estimates).

Let (u, p) be the exact solution of Eq. (2.1) and $(U_h^0, P_h^0) := (\mathcal{S}_h u_0, \bar{P}_h^0)$ the solution of Eq. (2.13). Calculate (U_h^1, P_h^1) by Eq. (3.2) and (U_h^2, P_h^2) by Eq. (3.1). Then

$$\begin{aligned} \|\hat{U}_h - u\|_{L^\infty((0, t_2); J^*)}^2 + \|U_h - u\|_{L^2((0, t_2); L^2)}^2 &\lesssim \eta^0(H^{-1}) + \tau^5 \|\bar{\partial}^2 U_h^2\|^2 + \tau^3 \|\Psi_1\|_{J^*}^2 \\ &\quad + \sum_{n=0}^2 \tau \|U_h^n - V^n\|^2 + \int_0^{t_1} \|(f_h^{1/2} - f)(t)\|_{J^*}^2 + \int_{t_1}^{t_2} \|(f_h - f)(t)\|_{J^*}^2, \end{aligned} \quad (3.22)$$

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$$\begin{aligned} \|U_h - u\|_{L^\infty((0,t_2);L^2)}^2 &\lesssim \eta^0(L^2) + \tau^3 \|\Psi_1\|_{J^*}^2 + \tau^5 \|\nabla \bar{\partial}^2 V^2\|^2 + \tau^4 \|\bar{\partial}^2 U_h^2\|^2 \\ &\quad + \max_{0 \leq n \leq 2} \|U_h^n - V^n\|^2 + \sum_{n=1}^2 \tau \|\bar{\partial}(U_h^n - V^n)\|_{J^*}^2 \\ &\quad + \int_0^{t_1} \|(f_h^{1/2} - f)(t)\|_{J^*}^2 + \int_{t_1}^{t_2} \|(f_h - f)(t)\|_{J^*}^2, \end{aligned} \quad (3.23)$$

$$\begin{aligned} \|\nabla(U_h - u)\|_{L^2((0,t_2);L^2)}^2 &\lesssim \eta^0(L^2) + \tau^3 \|\Psi_1\|_{J^*}^2 + \tau^5 \|\nabla \bar{\partial}^2 V^2\|^2 + \sum_{n=0}^2 \tau \|\nabla(U_h^n - V^n)\|^2 \\ &\quad + \sum_{n=1}^2 \tau \|\bar{\partial}(U_h^n - V^n)\|_{J^*}^2 + \int_0^{t_1} \|(f_h^{1/2} - f)(t)\|_{J^*}^2 + \int_{t_1}^{t_2} \|(f_h - f)(t)\|_{J^*}^2, \end{aligned} \quad (3.24)$$

$$\begin{aligned} \|\nabla(U_h - u)\|_{L^\infty((0,t_2);L^2)}^2 &\lesssim \eta^0(H_0^1) + \tau^3 \|\Psi_1\|^2 + \tau^4 \|\nabla \bar{\partial}^2 U_h^2\|^2 + \tau^5 \|\Delta \bar{\partial}^2 V^2\|^2 \\ &\quad + \max_{0 \leq n \leq 2} \|\nabla(U_h^n - V^n)\|^2 + \sum_{n=1}^2 \tau \|\bar{\partial}(U_h^n - V^n)\|^2 \\ &\quad + \int_0^{t_1} \|(f_h^{1/2} - f)(t)\|^2 + \int_{t_1}^{t_2} \|(f_h - f)(t)\|^2, \end{aligned} \quad (3.25)$$

$$\begin{aligned} \|P_h - p\|_{L^2((0,t_2);L^2)}^2 &\lesssim \eta^0(H_0^1) + \tau^3 \|\Psi_1\|^2 + \tau^5 \|\Delta \bar{\partial}^2 V^2\|^2 + \sum_{n=0}^2 \tau \|P_h^n - R^n\|^2 \\ &\quad + \sum_{n=1}^2 \tau \|\bar{\partial}(U_h^n - V^n)\|^2 + \int_0^{t_1} \|(f_h^{1/2} - f)(t)\|^2 + \int_{t_1}^{t_2} \|(f_h - f)(t)\|^2, \end{aligned} \quad (3.26)$$

where η^0 is defined in Assumption 3.3 and $\Psi_1 := \bar{\partial}^2 U_h^2 - \Delta_h \bar{\partial} U_h^1 + B_h^\top \bar{\partial} P_h^1 - \bar{\partial} f_h^1$.

Proof : Let $t \in I_1 = (0, t_1]$. Define the three-point reconstruction

$$\hat{U}_h(t) = U_h(t) + \frac{1}{2}(t - t_0)(t - t_1)\bar{\partial}^2 U_h^2, \quad \partial_t \hat{U}_h(t) = \bar{\partial} U_h^1 + (t - t^{1/2})\bar{\partial}^2 U_h^2,$$

where the identity $t - t^{1/2} = \frac{\tau}{2} + (t - t_1)$ was used. $\hat{U}_h(t)$ fulfills in H^{-1}

$$\begin{aligned} &\partial_t \hat{U}_h(t) - \Delta U_h(t) + \nabla P_h(t) \\ &= \bar{\partial} U_h^1 + (t - t^{1/2})\bar{\partial}^2 U_h^2 - \Delta(U_h - V)(t) + \nabla(P_h - R)(t) - \underbrace{\Delta V(t) + \nabla R(t)}_{-\Delta_h U_h(t) + B_h^\top P_h(t)} \\ &= \Delta U_h^{1/2} - B_h^\top P_h^{1/2} + f_h^{1/2} + (t - t^{1/2})\bar{\partial}^2 U_h^2 - \Delta(U_h - V)(t) \\ &\quad + \nabla(P_h - R)(t) - \Delta_h U_h(t) + B_h^\top P_h(t) \\ &= (t - t_{1/2})(\bar{\partial}^2 U_h^2 - \Delta_h \bar{\partial} U_h^1 + B_h^\top \bar{\partial} P_h^1 - \bar{\partial} f^1) + f_h^{1/2} + (t - t_{1/2})\bar{\partial} f_h^1 \\ &\quad - \Delta(U_h - V)(t) + \nabla(P_h - R)(t). \end{aligned}$$

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Subtract Eq. (2.1) from this equation to deduce the error equation for $t \in I_1$

$$\begin{aligned} \partial_t(\hat{U}_h - u)(t) - \Delta(U_h - u)(t) + \nabla(P_h - p)(t) \\ = (t - t_{1/2})\Psi_1 + (f_h^{1/2} - f)(t) - \Delta(U_h - V)(t) + \nabla(P_h - R)(t), \end{aligned} \quad (3.27)$$

where $\Psi_1 := \bar{\partial}^2 U_h^2 - \Delta_h \bar{\partial} U_h^1 + B_h^\top \bar{\partial} P_h^1 - \bar{\partial} f_h^1$ and $f_h^{1/2}(t) := f_h^{1/2} + (t - t_{1/2})\bar{\partial} f_h^1$.

Test Eq. (3.27) by $(\hat{U}_h - u)(t)$ and integrate from $t = 0$ to t_1 to obtain

$$\begin{aligned} \sup_{t \in [0, t_1]} \|(\hat{U}_h - u)(t)\|_{J^*}^2 + \int_0^{t_1} (\|U_h - u(t)\|^2 + \|(\hat{U}_h - u)(t)\|^2) dt \\ \lesssim \|(\hat{U}_h - u)(0)\|_{J^*}^2 + \tau^5 \|\bar{\partial}^2 U_h^2\|^2 + \sum_{n=0}^1 \tau \|U_h^n - V^n\|^2 \\ + \int_0^{t_1} \|(f_h^{1/2} - f)(t)\|_{J^*}^2 + \tau^3 \|\Psi_1\|_{J^*}^2, \end{aligned} \quad (3.28)$$

where $\|(\hat{U}_h - u)(0)\|_{J^*}^2 = \|U_h^0 - u_0\|_{J^*}^2 = \|(\mathcal{S}_h - I)u_0\|_{J^*}^2 \leq \eta^0(H^{-1})$.

Now rearrange Eq. (3.27) to get for $t \in I_1$

$$\begin{aligned} \partial_t(\hat{V} - u)(t) - \Delta(V - u)(t) + \nabla(R - p)(t) \\ = \partial_t(\hat{V} - \hat{U}_h)(t) + (t - t_{1/2})\Psi_1 + (F^{1/2} - f)(t). \end{aligned} \quad (3.29)$$

Similarly to the proof of Theorem 3.6 test Eq. (3.29) by $(V - u)(t) \in J$ and integrate from $t = 0$ to t_1 to obtain

$$\begin{aligned} \max_{t \in [0, t_1]} \|(\hat{V} - u)(t)\|^2 + \int_0^{t_1} (\|\nabla(\hat{V} - u)(t)\|^2 + \|\nabla(V - u)(t)\|^2) dt \\ \lesssim \|(\hat{V} - u)(0)\|^2 + \int_0^{t_1} (\|\partial_t(\hat{V} - \hat{U}_h)(t)\|_{J^*}^2 + \tau^2 \|\Psi_1\|_{J^*}^2) dt \\ + \int_0^{t_1} (\|\nabla(V - \hat{V})(t)\|^2 + \|(f_h^{1/2} - f)(t)\|_{J^*}^2) dt. \end{aligned} \quad (3.30)$$

Using the triangle inequality yields

$$\begin{aligned} \max_{t \in [0, t_1]} \|(\hat{U}_h - u)(t)\|^2 &\leq \max_{t \in [0, t_1]} \|(\hat{U}_h - U)(t)\|^2 + \max_{t \in [0, t_1]} \|(\hat{V} - \hat{U}_h)(t)\|^2 + \max_{t \in [0, t_1]} \|(\hat{V} - u)(t)\|^2 \\ &\lesssim \tau^4 \|\bar{\partial}^2 U_h^2\|^2 + \max_{0 \leq n \leq 2} \|V^n - U_h^n\|^2 + \|(I - \mathcal{S}_h)u_0\|^2 + \tau^3 \|\Psi_1\|_{J^*}^2 \\ &\quad + \sum_{n=1}^2 \tau \|\bar{\partial}(V^n - U_h^n)\|_{J^*}^2 + \tau^5 \|\nabla \bar{\partial}^2 V^2\|^2 + \int_0^{t_1} \|(f_h^{1/2} - f)(t)\|_{J^*}^2 dt. \end{aligned}$$

On a similar way the $L^2(H^1)$ estimate is obtained by splitting the error into

$$\int_0^{t_1} \|\nabla(U_h - u)(t)\|^2 dt \leq \int_0^{t_1} \|\nabla(U_h - V)(t)\|^2 dt + \int_0^{t_1} \|\nabla(V - u)(t)\|^2 dt.$$

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Let now $t \in I_2 = (t_1, t_2]$. The three-point reconstruction $\hat{U}_h(t)$, as defined in Eq. (2.22), fulfills in H^{-1}

$$\begin{aligned} \partial_t \hat{U}_h(t) - \Delta U_h(t) + \nabla P_h(t) \\ = -\Delta(U_h - V)(t) + \nabla(P_h - R)(t) - \frac{(t - t_2)}{2} \Psi_1 + f_h^2 + (t - t_2) \bar{\partial} f_h^2. \end{aligned} \quad (3.31)$$

The same procedure as above proves Eqs. (3.22)–(3.23).

For estimating Eqs. (3.25)–(3.26) proceed exactly like in the proof of Theorem 3.7 using the error equations (3.29) and (3.31), respectively. $\square$

Finally the Stokes reconstruction has to be bounded by computable data.

Lemma 3.10.

The Stokes reconstruction defined in Eq. (3.14) fulfills

$$\|\nabla \bar{\partial}^2 V\|_{\ell(2,N;L^2)}^2 \lesssim \mathcal{E}_4, \quad \|\Delta \bar{\partial} V\|_{\ell(2,N;L^2)}^2 \lesssim \mathcal{E}_5, \quad (3.32)$$

where $\mathcal{E}_4, \mathcal{E}_5$ are defined in Definition 3.4.

PROOF : It follows directly from Eq. (3.14) that

$$\|\nabla \bar{\partial}^2 V^n\| \leq \|\bar{\partial}^2(-\Delta_h U_h^n + B_h^\top P_h^n)\|_{J^*}.$$

For $n \geq 4$ this is equivalent to

$$\|\nabla \bar{\partial}^2 V^n\| \leq \|\bar{\partial}^2(f_h^n - \bar{\partial}^B U_h^n)\|_{J^*}.$$

In the case $n = 2, 3$ the initial equation (4.3) has to be used. As in Lemma 3.9 define

$$\Psi_1 := \bar{\partial}^2 U_h^2 - \Delta_h \bar{\partial} U_h^1 + B_h^\top \bar{\partial} P_h^1 - \bar{\partial} f_h^1.$$

It follows

$$\begin{aligned} \bar{\partial}^2(-\Delta_h U_h^3 + B_h^\top P_h^3) &= \frac{1}{\tau^2} \left(\underbrace{-\Delta_h U_h^3 + B_h^\top P_h^3}_{=f_h^3 - \frac{\tau}{2} \bar{\partial}^2 U_h^3 - \bar{\partial} U_h^3} + \underbrace{2\Delta_h U_h^2 - 2B_h^\top P_h^2}_{=\tau \bar{\partial}^2 U_h^2 + 2\bar{\partial} U_h^2 - 2f_h^2} - \Delta_h U_h^1 + B_h^\top P_h^1 \right) \\ &= \frac{1}{\tau^2} \left(\tau^2 \bar{\partial}^2 f_h^3 - \frac{\tau^2}{2} \bar{\partial}^3 U_h^3 + \frac{\tau}{2} \Psi_1 - \bar{\partial} U_h^3 + 2\bar{\partial} U_h^2 - \underbrace{\Delta_h U_h^{1/2} + B_h^\top P_h^{1/2} - f_h^{1/2}}_{-\bar{\partial} U_h^1} \right), \\ &= \bar{\partial}^2 f_h^3 - \frac{3}{2} \bar{\partial}^3 U_h^3 + \frac{1}{2\tau} \Psi_1, \\ \bar{\partial}^2(-\Delta_h U_h^2 + B_h^\top P_h^2) &= \frac{1}{\tau^2} \left(\underbrace{-\Delta_h U_h^2 + B_h^\top P_h^2}_{=f_h^2 - \frac{\tau}{2} \bar{\partial}^2 U_h^2 - \bar{\partial} U_h^2} + \underbrace{2\Delta_h U_h^1 - 2B_h^\top P_h^1 - \Delta_h U_h^0 + B_h^\top P_h^0}_{=\tau(\bar{\partial}^2 U_h^2 - \Psi_1 - \bar{\partial} f_h^1) + \Delta_h U_h^1 - B_h^\top P_h^1} \right) \\ &= \frac{1}{\tau^2} \left(\tau^2 \bar{\partial}^2 f_h^2 + \tau \bar{\partial}^2 U_h^2 - \bar{\partial} U_h^2 - \tau \Psi_1 - \underbrace{\frac{\tau}{2} \bar{\partial}^2 U_h^2 + \Delta_h U_h^1 - B_h^\top P_h^1 + f_h^1}_{=-\frac{\tau}{2} \Psi_1 + \Delta_h U_h^{1/2} - B_h^\top P_h^{1/2} + f_h^{1/2} = -\frac{\tau}{2} \Psi_1 + \bar{\partial} U_h^1} \right) \\ &= \bar{\partial}^2 f_h^2 - \frac{3}{2\tau} \Psi_1. \end{aligned}$$

□

3.3.2 Proof of Theorem 3.5

Proof : (**Theorem 3.5**) Subtracting Eq. (2.1) from Eq. (3.21) and testing with $v \in J$ yields for $n \geq 3$

$$\begin{aligned} & \underbrace{\langle \partial_t(\hat{U}_h - u)(t), v \rangle}_{(A)} + \underbrace{\langle \nabla(U_h - u)(t), \nabla v \rangle}_{(B)} \\ &= \underbrace{\langle -\Delta(U_h - V)(t) \rangle}_{(C)} + \underbrace{\langle f_h(t) - f(t), v \rangle - \frac{\tau}{2}(t - t_n \langle \bar{\partial}^3 U_h^n, v \rangle)}_{(D)} \end{aligned}$$

Set $v = w(t) := \mathcal{A}^{-1}(\hat{U}_h - u)(t)$, i.e. $(w(t), r(t))$ is solution of

$$-\Delta w(t) + \nabla r(t) = (\hat{U}_h - u)(t), \quad \operatorname{div} w(t) = 0, \quad w(t)|_\Gamma = 0.$$

Using the stability for $\nabla r(t)$ and identity (2.16) one obtains

$$\begin{aligned} (A) &= \frac{1}{2} \frac{d}{dt} \|\nabla w(t)\|^2, \\ (B) &= \langle (U_h - u)(t), -\Delta w(t) \rangle = \langle (U_h - u)(t), (\hat{U}_h - u)(t) \rangle - \langle (U_h - u)(t), \nabla r(t) \rangle \\ &= \frac{1}{2} \|(U_h - u)(t)\|^2 + \frac{1}{2} \|(\hat{U}_h - u)(t)\|^2 - \frac{1}{2} \|(\hat{U}_h - U_h)(t)\|^2 - \langle (U_h - V)(t), \nabla r(t) \rangle \\ &\geq \frac{1}{2} \|(U_h - u)(t)\|^2 + \frac{1}{2} \|(\hat{U}_h - u)(t)\|^2 - \frac{1}{2} \|(\hat{U}_h - U_h)(t)\|^2 - \|(U_h - V)(t)\| \|(\hat{U}_h - u)(t)\|, \\ (C) &= \langle U_h(t) - V(t), -\Delta w(t) \rangle = \langle (U_h - V)(t), -\nabla r(t) + (\hat{U}_h - u)(t) \rangle \\ &\leq C_\varepsilon \|(U_h - V)(t)\|^2 + \varepsilon \|(\hat{U}_h - u)(t)\|, \\ (D) &\lesssim (\|(F - f)(t)\|_{J^*} + \tau^2 \|\bar{\partial}^3 U_h^n\|_{J^*}) \|\nabla w(t)\|. \end{aligned}$$

Collect these results, integrate from $t = t_2$ to T and use the Gronwall Lemma and Youngs inequality to get

$$\begin{aligned} & \sup_{t \in [t_2, T]} \|\nabla w(t)\|^2 + \int_{t_2}^T (\|(U_h - u)(t)\|^2 + \|(\hat{U}_h - u)(t)\|^2) dt \\ &\lesssim \|\nabla w(t_2)\|^2 + \int_{t_2}^T \left(\|(\hat{U}_h - U_h)(t)\|^2 + \|(U_h - V)(t)\|^2 + \|(F - f)(t)\|_{J^*}^2 + \tau^4 \|\bar{\partial}^3 U_h^n\|_{J^*}^2 \right) dt \end{aligned}$$

The second and third term are handled as follows

$$\int_{t_2}^T \|(\hat{U}_h - U_h)(t)\|^2 dt \lesssim \tau^4 \sum_{n=3}^N \tau \|\bar{\partial}^2 U_h^n\|^2, \quad \int_{t_2}^T \|(U_h - V)(t)\|^2 dt \leq \sum_{n=2}^N \tau \|U_h^n - V^n\|^2.$$

This estimate combined with the initial estimate in Eq. (3.22) and Assumption 3.3 proves the theorem. □

3.3.3 Proof of Theorem 3.6

Proof : **(Theorem 3.6)** Rearrange terms in Eq. (3.21) to obtain the identity for the Stokes reconstruction

$$\partial_t \hat{V}(t) - \Delta V(t) + \nabla R(t) = \partial_t(\hat{V} - \hat{U}_h)(t) + F(t) + \frac{\tau}{2}(t - t_n)\bar{\partial}^3 U_h^n.$$

Subtract Eq. (2.1) from the above identity to get the error equation for $t \in I_n$, $n \geq 3$,

$$\begin{aligned} \partial_t(\hat{V} - u)(t) - \Delta(V - u)(t) + \nabla(R - p)(t) \\ = \partial_t(\hat{V} - \hat{U}_h)(t) + (F - f)(t) + \frac{\tau}{2}(t - t_n)\bar{\partial}^3 U_h^n. \end{aligned} \quad (3.33)$$

Test Eq. (3.33) by $(\hat{V} - u)(t) \in J$ and obtain

$$\begin{aligned} \frac{d}{dt} \|(\hat{V} - u)(t)\|^2 + \|\nabla(\hat{V} - u)(t)\|^2 + \|\nabla(V - u)(t)\|^2 \\ \lesssim \|\partial_t(\hat{V} - \hat{U}_h)(t)\|_{J^*}^2 + \|\nabla(V - \hat{V})(t)\|^2 + \tau^4 \|\bar{\partial}^3 U_h^n\|_{J^*}^2 + \|(F - f)(t)\|_{J^*}^2, \end{aligned}$$

where identity (2.16) has been used. This expression, integrated from $t = t_2$ to T , results in

$$\begin{aligned} \max_{t \in [t_2, T]} \|(\hat{V} - u)(t)\|^2 + \int_{t_2}^T (\|\nabla(\hat{V} - u)(t)\|^2 + \|\nabla(V - u)(t)\|^2) dt \\ \lesssim \int_{t_2}^T (\|\partial_t(\hat{V} - \hat{U}_h)(t)\|_{J^*}^2 + \|\nabla(V - \hat{V})(t)\|^2 + \|(F - f)(t)\|_{J^*}^2) dt \\ + \|(\hat{V} - u)(t_2)\|^2 + \tau^4 \sum_{n=3}^N \tau \|\bar{\partial}^3 U_h^n\|_{J^*}^2. \end{aligned} \quad (3.34)$$

For the first and second term on the right hand side one gets

$$\int_{t_2}^T \|\partial_t(\hat{V} - \hat{U}_h)(t)\|_{J^*}^2 \lesssim \sum_{n=2}^N \tau \|\bar{\partial}(V^n - U_h^n)\|_{J^*}^2, \quad \int_{t_2}^T \|\nabla(V - \hat{V})(t)\|^2 \lesssim \tau^4 \sum_{n=3}^N \tau \|\nabla \bar{\partial}^2 V^n\|^2.$$

Use the triangle inequality to get

$$\begin{aligned} \max_{t \in [0, T]} \|(\hat{U}_h - u)(t)\|^2 &\lesssim \max_{t \in [0, t_2]} \|(\hat{U}_h - u)(t)\|^2 + \max_{t \in [t_2, T]} \|(\hat{U}_h - \hat{U}_h)(t)\|^2 \\ &+ \max_{t \in [t_2, T]} \|(\hat{V} - \hat{U}_h)(t)\|^2 + \max_{t \in [t_2, T]} \|(\hat{V} - u)(t)\|^2 \\ &\lesssim \max_{t \in [0, t_2]} \|(\hat{U}_h - u)(t)\|^2 + \tau^4 \max_{3 \leq n \leq N} \|\bar{\partial}^2 U_h^n\|^2 + \max_{1 \leq n \leq N} \|U_h^n - V^n\|^2 \\ &+ \sum_{n=2}^N \tau \|\bar{\partial}(V^n - U_h^n)\|_{J^*}^2 + \tau^4 \sum_{n=3}^N \tau \|\nabla \bar{\partial}^2 V^n\|^2 + \tau^4 \sum_{n=3}^N \tau \|\bar{\partial}^3 U_h^n\|_{J^*}^2 + \mathcal{F}_1. \end{aligned}$$

This combined with Assumption 3.3, Eq. (3.11) and Lemma 3.10 proves Eq. (3.17).

3 The instationary Stokes system

In a similar way Eq. (3.18) is bounded

$$\int_0^T \|\nabla(U_h - u)(t)\|^2 \leq \int_0^{t_2} \|\nabla(U_h - u)(t)\|^2 + \int_{t_2}^T \|\nabla(U_h - V)(t)\|^2 + \int_{t_2}^T \|\nabla(V - u)(t)\|^2.$$

Once again use Assumption 3.3, Eq. (3.12) and Lemma 3.10 to complete the proof. $\square$

3.3.4 Proof of Theorem 3.7

Proof : Test Eq. (3.33) with $\partial_t(\hat{V} - u)(t) \in J$ to obtain for $t \in I_n$, $n \geq 3$

$$\|\partial_t(\hat{V} - u)(t)\|^2 + \langle -\Delta(V - u), \partial_t(\hat{V} - u) \rangle \lesssim \|\partial_t(\hat{V} - \hat{U}_h)(t)\|^2 + \|(f_h - f)(t)\|^2 + \tau^4 \|\bar{\partial}^3 U_h\|^2,$$

where the second term on the left hand side is handled by

$$\langle -\Delta(V - u), \partial_t(\hat{V} - u) \rangle = \langle \nabla(\hat{V} - u)(t), \nabla \partial_t(\hat{V} - u) \rangle + \langle -\Delta(V - \hat{V}), \partial_t(\hat{V} - u) \rangle.$$

Combine these equations and integrate from $t = t_2$ to T

$$\begin{aligned} & \int_{t_2}^T \|\partial_t(\hat{V} - u)(t)\|^2 + \max_{t \in [t_2, T]} \|\nabla(\hat{V} - u)\|^2 \lesssim \|\nabla(\hat{V} - u)(t_2)\|^2 \\ & + \sum_{n=2}^N \tau \|\bar{\partial}(V^n - U_h^n)\|^2 + \tau^4 \sum_{n=3}^N \tau (\|\bar{\partial}^3 U_h^n\|^2 + \|\Delta \bar{\partial}^2 V^n\|^2) + \int_{t_2}^T \|(f_h - f)(t)\|^2. \end{aligned} \quad (3.35)$$

To prove Eq. (3.19) use the triangle inequality

$$\begin{aligned} \max_{t \in [0, T]} \|\nabla(U_h - u)(t)\|^2 & \leq \max_{t \in [0, t_2]} \|\nabla(U_h - u)(t)\|^2 + \max_{t \in [t_2, T]} \|\nabla(U_h - \hat{U}_h)(t)\|^2 \\ & + \max_{t \in [t_2, T]} \|\nabla(\hat{U}_h - \hat{V})(t)\|^2 + \max_{t \in [t_2, T]} \|\nabla(\hat{V} - u)(t)\|^2 \\ & \leq \tau^4 \max_{3 \leq n \leq N} \|\nabla \bar{\partial}^2 U_h^n\|^2 + \max_{1 \leq n \leq N} \|\nabla(U_h^n - V^n)\|^2 \\ & + \max_{t \in [0, t_2]} \|\nabla(U_h - u)(t)\|^2 + \max_{t \in [t_2, T]} \|\nabla(\hat{V} - u)(t)\|^2 \end{aligned}$$

and the initial estimate Eq. (3.25) and Eq. (3.33).

It remains to prove the pressure estimate Eq. (3.20). The inf-sup condition and Eq. (3.33) lead to

$$\begin{aligned} \|(R - p)(t)\| & \lesssim \sup_{\substack{v \in X \\ \|\nabla v\|=1}} \langle \nabla(R - p)(t), v \rangle \\ & = \sup_{\substack{v \in X \\ \|\nabla v\|=1}} \langle -\partial_t(\hat{V} - u) + \Delta(V - u) + \partial_t(\hat{V} - \hat{U}_h) + (F - f) + (t - t_n) \frac{\tau}{2} \bar{\partial}^3 U_h^n, v \rangle \\ & \leq \|\partial_t(\hat{V} - u)\|_{-1} + \|\nabla(V - u)\| + \|\partial_t(\hat{V} - \hat{U}_h)\|_{-1} + \|F - f\|_{-1} + \tau^2 \|\bar{\partial}^3 U_h^n\|_{-1} \\ & \leq \|\partial_t(\hat{V} - u)\| + \|\nabla(V - u)\| + \|\partial_t(\hat{V} - \hat{U}_h)\| + \|(F - f)(t)\| + \tau^2 \|\bar{\partial}^3 U_h^n\| \end{aligned}$$

3 The instationary Stokes system

Square this equation, integrate from $t = t_2$ to T and use Eqs. (3.34) and (3.35) to obtain

$$\begin{aligned} \int_{t_2}^T \|(R - p)(t)\|^2 &\lesssim \|\nabla(\hat{V} - u)(t)\|^2 + \sum_{n=2}^N \tau \|\bar{\partial}(U_h^n - V^n)\|^2 \\ &+ \tau^4 \sum_{n=3}^N \tau (\|\nabla \bar{\partial}^2 V^n\|^2 + \|\Delta \bar{\partial}^2 V^n\|^2 + \|\bar{\partial}^3 U_h^n\|^2) + \int_{t_2}^T \|(F - f)\|^2. \end{aligned} \quad (3.36)$$

Now split the pressure error

$$\int_0^T \|(P_h - p)(t)\|^2 \leq \int_0^{t_2} \|(P_h - p)(t)\|^2 + \int_{t_2}^T \|(P_h - R)(t)\|^2 + \int_{t_2}^T \|(R - p)(t)\|^2,$$

and use the initial estimates Eq. (3.26) and Eq. (3.36) to complete the proof. $\square$

4 A short overview of projection methods

Projection methods are effective procedures to solve the time-dependent incompressible Navier–Stokes equations. The key advantage of the projection methods is to decouple the velocity from the pressure.

The main idea of projection methods is based on the *Helmholtz decomposition* of any vector field into a divergence-free (solenoidal) part and an irrotational part:

$$u = u_{\text{sol}} + u_{\text{irrot}} = u_{\text{sol}} + \nabla\phi, \quad (4.1)$$

i.e. the space $Y = L^2(\Omega)^d$ can be written as direct sum of the divergence-free space H and ∇M :

$$Y = H + \nabla M.$$

Apply the divergence operator to Eq. (4.1) and multiply Eq. (4.1) with the normal n to obtain

$$-\Delta\phi = -\operatorname{div} u, \text{ in } \Omega, \quad \nabla\phi \cdot n = u \cdot n \text{ on } \Gamma.$$

This Poisson equation with Neumann boundary condition the scalar function ϕ can be solved for a known u and then the solenoidal part of u can be extracted using the relation

$$u_{\text{sol}} = u - \nabla\phi.$$

This idea is the essence of projection methods for solving incompressible Navier–Stokes equation.

In the following *pressure-correction schemes* are presented. These are time-dependent techniques made up of two sub-steps for each time step: in the first sub-step the pressure is treated explicitly or ignored and a provisional velocity is computed, which is corrected in the second step by projecting onto the space H .

For a general overview of projection methods the reader is referred to the paper of Shen, Guermond and Mineev [15].

4.1 Teman's and Chorin's projection method

In the late sixties Chorin [8] and Temam [37] proposed the simplest *pressure-correction* scheme. By using the implicit Euler time discretization the algorithm reads as follows:

Algorithm 4.1 (Teman's and Chorin's projection method).

Set $U^0 := u_0$, then for $n \geq 1$ compute in the first sub-step, accounting for viscous effects, the provisional velocity $\tilde{U}^n$ by

$$\frac{\tilde{U}^n - U^{n-1}}{\tau} - \Delta \tilde{U}^n = f(t_n), \quad \tilde{U}^n|_{\Gamma} = 0,$$

and in the second one accounting for incompressibility calculate (U^n, P^n) by solving

$$\frac{U^n - \tilde{U}^n}{\tau} + \nabla P^n = 0, \quad \operatorname{div} U^n = 0, \quad U^n \cdot n|_{\Gamma} = 0.$$

To realize the second sub-step apply the divergence operator div to get the following Poisson problem with Neumann boundary condition

$$-\Delta P^n = -\frac{1}{\tau} \operatorname{div} \tilde{U}^n, \quad \nabla P^n \cdot n|_{\Gamma} = 0,$$

where the boundary condition follows by testing the second sub-step with the normal n on the boundary Γ . Then U^n is updated by

$$U^n = \tilde{U}^n - \tau \nabla P^n.$$

The second sub-step is usually called the *projection step*, because $U^n = \mathcal{H}\tilde{U}^n$:

$$\langle U^n, v \rangle = \langle \tilde{U}^n, v \rangle - \tau \langle \nabla P^n, v \rangle = \langle \tilde{U}^n, v \rangle + \tau \underbrace{\langle P^n, \operatorname{div} v \rangle}_{=0} = \langle \tilde{U}^n, v \rangle \quad \forall v \in H, \quad U^n \in H.$$

The scheme fulfills the following a-priori estimates which are proven, for instance, in [30, 31, 33].

Theorem 4.2 (A-priori estimates for Teman's and Chorin's scheme).

Assume the exact solution (u, p) of Eq. (2.1) is sufficiently smooth. Then the solution $(\tilde{U}^n, U^n, P^n)$ of the Teman and Chorin scheme satisfies the error estimates

$$\begin{aligned} \|\tilde{U} - u\|_{\ell^\infty(L^2)} + \|U - u\|_{\ell^\infty(L^2)} &\leq C(u, p, T) \tau, \\ \|P - p\|_{\ell^\infty(L^2)} + \|\tilde{U} - u\|_{\ell^\infty(H^1)} &\leq C(u, p, T) \tau^{1/2}. \end{aligned}$$

The pressure error and the velocity error in H^1 suffer from a suboptimal convergence rate of $1/2$. In [31] Rannacher explains that this is caused by the artificial boundary condition for the pressure

$$\nabla P^n \cdot n|_{\Gamma} = 0$$

and shows that this condition induces a numerical boundary layer for the pressure, cf. Figure 4.1. In [31] Rannacher show that the Teman and Chorin scheme can be interpreted as a *semi-explicit pressure stabilization method* of the instationary Stokes equation and investigates the following model system (cf. [30])

$$\begin{aligned} \partial_t u_\epsilon - \Delta u_\epsilon + \nabla p_\epsilon &= f, \\ \operatorname{div} u_\epsilon - \epsilon \Delta p_\epsilon &= 0, \quad \nabla p_\epsilon \cdot n|_{\Gamma} = 0, \end{aligned}$$

Figure 4.1: Pressure error plot for the Temam and Chorin scheme in [31].

which satisfies the a-priori estimates

$$\begin{aligned} \|u - u_\epsilon\|_{L^\infty(L^2)} &\lesssim \epsilon, \\ \|u - u_\epsilon\|_{L^\infty(H^1)} + \|p - p_\epsilon\|_{L^\infty(L^2)} &\lesssim \sqrt{\epsilon}. \end{aligned}$$

This implies that the decoupling of velocity from pressure causes an irreducible *splitting error* for the velocity of order $\epsilon = \tau$. As a consequence, the convergence rate of a higher-order time discretization of order $k \geq 2$ is always limited by the splitting error of order 1. Hence, the use of a higher-order time discretization does not improve the convergence rate.

In summary, it can be stated that there exist two main drawbacks of the Temam and Chorin scheme. The artificial boundary condition for the pressure inducing a numerical boundary layer on one hand and on the other the splitting error of order $\mathcal{O}(\tau)$ that limits the scheme to use a higher-order time discretization.

4.2 The standard incremental pressure-correction scheme

The restriction of a first order time discretization was first overcome by Goda in [13]. He had the idea to add the old value of the pressure gradient in the first sub-step and update the second sub-step accordingly afterwards. This improves the splitting error to order 2 and the convergence rate can be improved by the use of a second order time discretization. Subsequently the *Backward Difference Formula of second order* (BDF2) is used although other time discretizations are possible. The BDF2 discretization has the advantage of being A-stable but requires the calculation of additional initial values, which will be discussed next.

Initialization of the scheme

In [17] Guermond and Shen recommend to calculate one backward Euler step to initialize the scheme.

Algorithm 4.3 (Initialization of the scheme by backward Euler).

Set $U^0 := u_0$ and $P^0 := p(0)$, where $p(0)$ is solution of

$$-\Delta p(0) = -\operatorname{div} f(0), \quad \nabla p(0) \cdot n|_{\Gamma} = (f(0) + \Delta u_0) \cdot n|_{\Gamma}. \quad (4.2)$$

Calculate $(\tilde{U}^1, U^1, P^1)$ by the the following first-order pressure correction scheme:

$$\begin{aligned} \frac{\tilde{U}^1 - U^0}{\tau} - \Delta \tilde{U}^1 + \nabla P^0 &= f(t_1), \quad \tilde{U}^1|_{\Gamma} = 0, \\ \frac{U^1 - \tilde{U}^1}{\tau} + \nabla(P^1 - P^0) &= 0, \quad \operatorname{div} U^1 = 0, \quad U^1 \cdot n|_{\Gamma} = 0. \end{aligned}$$

In [17] it is shown that $(\tilde{U}^1, U^1, P^1)$ satisfy the following estimate

$$\|\tilde{U}^1 - u(t_1)\| + \sqrt{\tau} \|\tilde{U}^1 - u(t_1)\|_1 + \tau \|P^1 - p(t_1)\| \lesssim \tau^2,$$

which is sufficient to prove the a-priori estimates for the standard scheme as well as for the rotational scheme (cf. Theorem 4.7 and Theorem 4.9).

As opposed to this, in [1] Akrivis and Chatzipantelidis demonstrated that one gets suboptimal convergence results for the a-posteriori analysis, if one uses Algorithm 4.3 to calculate the initial values. Due to this drawback they recommend to use a higher order discretization and propose the trapezoidal method:

Algorithm 4.4 (Initialization of the scheme using the trapezoidal rule).

Set $U^0 := u_0$ and $P^0 := p(0)$, where $p(0)$ is solution of Eq. (4.2). Calculate $(\tilde{U}^1, U^1, P^1)$ by the following pressure correction scheme, using the trapezoidal rule:

$$\frac{\tilde{U}^1 - U^0}{\tau} - \Delta \tilde{U}^{1/2} + \nabla P^0 = f^{1/2}, \quad \tilde{U}^1|_{\Gamma} = 0, \quad (4.3)$$

$$\frac{U^1 - \tilde{U}^1}{\tau} + \nabla(P^1 - P^0) = 0, \quad \operatorname{div} U^1 = 0, \quad U^1 \cdot n|_{\Gamma} = 0, \quad (4.4)$$

where $\tilde{U}^{1/2} = \frac{\tilde{U}^1 + U^0}{2}$ and $f^{1/2} = \frac{f(t_1) + f(0)}{2}$.

The standard incremental pressure-correction scheme

The standard incremental pressure-correction scheme reads as follows:

Algorithm 4.5 (The standard incremental pressure-correction scheme).

For given $(\tilde{U}^1, U^1, P^1)$ calculate for $n \geq 2$ in a first sub-step the provisional velocity $\tilde{U}^n$ by

$$\frac{3\tilde{U}^n - 4U^{n-1} + U^{n-2}}{2\tau} - \Delta \tilde{U}^n + \nabla P^{n-1} = f(t_n), \quad \tilde{U}^n|_{\Gamma} = 0,$$

and (U^n, P^n) by

$$\frac{3U^n - 3\tilde{U}^n}{2\tau} + \nabla(P^n - P^{n-1}) = 0, \quad \operatorname{div} U^n = 0, \quad U^n \cdot n|_\Gamma = 0.$$

This scheme was made popular by Van Kan in [21] using the Crank-Nicolson discretization instead of BDF2. That's why it is also known as *Van Kan's scheme*.

The convergence behavior was studied by Shen in [36] for the Crank-Nicolson time stepping and by Guermond in [14] for the fully discrete case using BDF2 as time discretization:

Theorem 4.6 (A-priori estimate for the standard pressure-correction scheme).

Assume the exact solution is sufficiently regular. Then the solution of Algorithm 4.5 with initial values given by Algorithm 4.3 or 4.4 satisfies

$$\begin{aligned} \|\tilde{U} - u\|_{\ell^2(L^2)} + \|U - u\|_{\ell^2(L^2)} &\lesssim \tau^2, \\ \|\tilde{U} - u\|_{\ell^\infty(H^1)} + \|P - p\|_{\ell^\infty(L^2)} &\lesssim \tau. \end{aligned}$$

Algorithm 4.5 can be reformulated by eliminating U from the algorithm. Insert the second sub-step

$$U^i = \tilde{U}^i - \frac{2\tau}{3} \nabla(P^i - P^{i-1})$$

for $i = n-1, n-2$, in the first equation of Algorithm 4.5 and apply the divergence operator to the second step to obtain

$$\begin{aligned} \bar{\partial}^B \tilde{U}^n - \Delta \tilde{U}^n + \nabla(P^n + \frac{4}{3}P^{n-1} - \frac{5}{3}P^{n-2} + \frac{1}{3}P^{n-3}) &= f, \quad \tilde{U}^n|_\Gamma = 0, \\ \operatorname{div} \tilde{U}^n - \tau \frac{2}{3} \Delta(P^n - P^{n-1}) &= 0, \quad \nabla P^n \cdot n|_\Gamma = \nabla P^{n-1} \cdot n|_\Gamma. \end{aligned}$$

This allows to interpret Algorithm 4.5 as a second-order time discretization of the perturbed system (cf. [36])

$$\begin{aligned} \partial_t u_\epsilon - \Delta u_\epsilon + \nabla p_\epsilon &= f, \quad u_\epsilon|_\Gamma = 0, \\ \operatorname{div} u_\epsilon - \epsilon^2 \Delta \partial_t p_\epsilon &= 0, \quad \nabla \partial_t p_\epsilon \cdot n|_\Gamma = 0, \end{aligned}$$

In [30] Prohl shows the following a-priori estimates for this perturbed system :

Theorem 4.7.

Let the exact solution (u, p) and f of Eq. (2.1) be sufficiently regular, then

$$\|u_\epsilon - u\|_{L^2(L^2)} + \sqrt{\epsilon} \|u_\epsilon - u\|_{L^\infty(L^2)} + \epsilon \|u_\epsilon - u\|_{L^\infty(H^1)} + \epsilon \|p_\epsilon - p\|_{L^\infty(L^2)} \lesssim \epsilon^2.$$

Setting $\epsilon = \tau$ shows that the decoupling of velocity and pressure in Algorithm 4.5 generates a *splitting error* of order $\mathcal{O}(\tau^2)$ in the $L^2(L^2)$ norm. But there is still a non-physical Neumann boundary condition for the pressure

$$\nabla P^n \cdot n|_\Gamma = \nabla P^{n-1} \cdot n|_\Gamma = \dots \nabla P^0 \cdot n|_\Gamma. \quad (4.5)$$

This numerical boundary layer limits the accuracy of the scheme for the pressure and velocity gradient.

4.3 The rotational incremental pressure-correction scheme

Although the standard incremental pressure-correction scheme Algorithm 4.5 is of second order it is still unsatisfactory in view of the non-physical Neumann boundary condition for the pressure (4.5). To improve this, in [38] Timmermans, Mineev and Van De Vosse had the idea to slightly modify the second sub-step. By only adding the divergence of the provisional velocity $\tilde{U}^n$ in the pressure update one obtains a consistent pressure boundary condition.

Algorithm 4.8 (The rotational incremental pressure-correction scheme).

For given $(\tilde{U}^1, U^1, P^1)$ calculate for $n \geq 2$ in a first sub-step the provisional velocity $\tilde{U}^n$ by

$$\frac{3\tilde{U}^n - 4U^{n-1} + U^{n-2}}{2\tau} - \Delta\tilde{U}^n + \nabla P^{n-1} = f(t_n), \quad \tilde{U}^n|_{\Gamma} = 0,$$

and in the second sub-step (U^n, P^n) by

$$\frac{3U^n - 3\tilde{U}^n}{2\tau} + \nabla(P^n - P^{n-1} + \operatorname{div}\tilde{U}^n) = 0, \quad \operatorname{div}U^n = 0, \quad U^n \cdot n|_{\Gamma} = 0.$$

Before answering the question, why the modified Algorithm 4.8 performs better than the standard scheme of Algorithm 4.5, the improvement of the pressure boundary layer is demonstrated on a numerical example. Let

$$\begin{aligned} u(x, y, t) &= \pi \sin(t) \left(\sin(2\pi y) \sin^2(\pi x), -\sin(2\pi x) \sin^2(\pi y) \right)^{\top}, \\ p(x, y, t) &= \sin(t) \cos(\pi x) \sin(\pi y). \end{aligned}$$

be the solution of Eq. (2.1) on $\Omega =]-0.5, 0.5[^2$ with a respective right hand side f . In Figure 4.2 the pressure error is plotted at end time $T = 1$ with time step $\tau = 0.01$ on the left side for the standard scheme (Algorithm 4.5) and on the right side for the rotational from (Algorithm 4.8). It is clearly recognizable that the boundary layer for the rotational scheme is reduced. To understand the improvement analytically, take the sum of the two sub-steps of Algorithm 4.8 to get

$$\bar{\partial}^B U^n - \Delta\tilde{U}^n + \nabla \operatorname{div}\tilde{U}^n + \nabla P^n = f(t_n), \quad \operatorname{div}U^n = 0, \quad U^n \cdot n|_{\Gamma} = 0. \quad (4.6)$$

From vector analysis the following relation between the Laplacian and rotation operator is known:

$$-\Delta v + \nabla \operatorname{div}v = \nabla \times \nabla \times v.$$

Apply the rotation operator twice to the second sub-step of Algorithm 4.8 and use the fact that $\nabla \times \nabla = 0$ to get

$$\nabla \times \nabla \times \tilde{U}^n = \nabla \times \nabla \times U^n.$$

With this relation and the fact that $\operatorname{div}U^n = 0$, one yields

$$-\Delta\tilde{U}^n + \nabla \operatorname{div}\tilde{U}^n = \nabla \times \nabla \times \tilde{U}^n = \nabla \times \nabla \times U^n = -\Delta U^n + \nabla \operatorname{div}U^n = -\Delta U^n,$$

Figure 4.2: Pressure error at end time $T = 1$ with time step size $\tau = 0.01$. Standard scheme (left) and rotation form (right) with BDF2.

and Eq. (4.6) changes into

$$\bar{\partial}^B U^n - \Delta U^n + \nabla P^n = f(t_n), \quad \operatorname{div} U^n = 0, \quad U^n \cdot n|_{\Gamma} = 0. \quad (4.7)$$

In view of the exact equation (2.1), Eq. (4.7) is, up to the inexact tangential boundary condition for the velocity U^n , a consistent discretization. Further, by multiplying Eq. (4.7) with the normal vector on the boundary one gets a consistent pressure boundary condition:

$$\nabla P^n \cdot n|_{\Gamma} = (f(t_n) + \Delta U^n) \cdot n|_{\Gamma},$$

since the exact pressure fulfills $\nabla p(t) \cdot n|_{\Gamma} = (f(t) + \Delta u(t)) \cdot n|_{\Gamma}$.

The better convergence behavior is also stated in the a-priori analysis, which Guermond and Shen proved in [17]:

Theorem 4.9 (A-priori estimate for the rotational pressure-correction scheme).

Assume the exact solution is sufficiently regular. Then the solution of Algorithm 4.8 with initial values given by Algorithm 4.3 or 4.4 satisfies

$$\begin{aligned} \|\tilde{U} - u\|_{\ell^2(L^2)} + \|U - u\|_{\ell^2(L^2)} &\lesssim \tau^2, \\ \|\tilde{U} - u\|_{\ell^2(H^1)} + \|U - u\|_{\ell^2(H^1)} + \|P - p\|_{\ell^2(L^2)} &\lesssim \tau^{3/2}. \end{aligned}$$

Concerning the estimate for the standard Algorithm 4.5 in Theorem 4.6 the rotational scheme predicts a better convergence rate for the velocity gradients and the pressure error of order $\mathcal{O}(\tau^{3/2})$ instead of $\mathcal{O}(\tau)$.

4.4 Implementation of pressure-correction schemes

In the pressure-correction schemes two velocities arise. The provisional velocity $\tilde{U}$ which has the correct boundary conditions but is not solenoidal and the projected velocity

4 A short overview of projection methods

U . U is divergence-free but has a wrong boundary condition. Regarding the a-priori estimates (Theorem 4.6 and 4.9) one surprisingly realizes that U and $\tilde{U}$ yield the same convergence behavior; therefore from the analytically point of view there is no objective reason to prefer one to the other.

Focusing on the implementational point of view there are, indeed, two arguments to the benefit of using $\tilde{U}$. First notice that U is not in general an element of the finite element space: it follows from the second sub-step that U behave like ∇P ; thus by using $\mathcal{C}^0(\bar{\Omega})$ finite elements for the pressure, for example, the velocity U is discontinuous at the interface between elements. Hence, one cannot calculate U with standard finite elements. Second, by simple algebraic manipulations the velocity U can be removed completely from the algorithm, for example, Algorithm 4.8. Reposition of the second sub-step yields

$$U^i = \tilde{U}^i - \frac{2\tau}{3}\nabla\Phi^i, \quad i \geq 2,$$

where

$$\Phi^i := P^i - P^{i-1} + \operatorname{div} \tilde{U}^i.$$

Hence, replace U^{n-1} and U^{n-2} in the first sub-step of Algorithm 4.8 to obtain

$$\bar{\partial}^B \tilde{U}^n - \Delta \tilde{U}^n + \nabla(P^{n-1} + \frac{4}{3}\Phi^{n-1} - \frac{1}{3}\Phi^{n-2}) = f^n, \quad \tilde{U}^n|_{\Gamma} = 0,$$

calculate Φ^n by the Poisson problem

$$-\Delta\Phi^n = -\frac{3}{2\tau}\operatorname{div} \tilde{U}^n, \quad \nabla\Phi \cdot n|_{\Gamma} = 0,$$

and set

$$P^n = \Phi^n + P^{n-1} - \operatorname{div} \tilde{U}^n.$$

5 The standard incremental pressure-correction scheme discretized by implicit Euler

In this Chapter a-posteriori error estimates for the standard incremental pressure-correction scheme discretized by implicit Euler are developed. Although using implicit Euler as time discretization is not that interesting for practical purposes, the analysis for the implicit Euler scheme helps to comprehend the procedure and strategy of proving a-posteriori estimates in general. This comprehension is essential to prove more technical estimates of higher order time discretization.

The standard incremental pressure correction scheme of (formally) first order using backward Euler to approximate Eq. (2.1) reads as follows.

Algorithm 5.1 (The standard scheme discretized by implicit Euler).

Let $U^0 = u_0$ and $P^0 := p(0)$, where $p(0)$ is calculated by Eq. (4.2). Then for $n \geq 1$ perform the two following steps. In the first sub-step the provisional velocity $\tilde{U}^n$ is sought, such that

$$\frac{\tilde{U}^n - U^{n-1}}{\tau} - \Delta \tilde{U}^n + \nabla P^{n-1} = f^n, \quad \tilde{U}^n|_{\Gamma} = 0. \quad (5.1)$$

In the second sub-step find (U^n, P^n) fulfilling

$$\frac{U^n - \tilde{U}^n}{\tau} + \nabla(P^n - P^{n-1}) = 0, \quad \operatorname{div} U^n = 0, \quad U^n \cdot n|_{\Gamma} = 0. \quad (5.2)$$

In view of the implementation U can be easily removed from Algorithm 5.1 (cf. the discussion in Chapter 4.4). Replace U^{n-1} in Eq. (5.1) by the second sub-step (5.2) and set $\tilde{U}^0 = U^0$ to obtain

$$\bar{\partial} \tilde{U}^n - \Delta \tilde{U}^n + \nabla \gamma^n = f^n, \quad \tilde{U}^n|_{\Gamma} = 0, \quad \text{where } \gamma^n := \begin{cases} 2P^{n-1} - P^{n-2} & n \geq 2, \\ P^0 & n = 1. \end{cases} \quad (5.3)$$

Note, that the second step implements the projection $U^n = \mathcal{H} \tilde{U}^n$ and that it is equivalent to solve the Poisson equation

$$-\Delta \Phi^n = -\frac{1}{\tau} \operatorname{div} \tilde{U}^n, \quad \nabla \Phi^n \cdot n|_{\Gamma} = 0, \quad (5.4)$$

and the updates

$$\begin{aligned} U^n &= \tilde{U}^n - \tau \nabla \Phi^n, \\ P^n &= P^{n-1} + \Phi^n. \end{aligned} \quad (5.5)$$

For convenience set additionally $\tilde{U}^0 := u_0$, $P^{-1} := P^0$ and $\tilde{U}^{-1}$ by the equation

$$\frac{\tilde{U}^0 - \tilde{U}^{-1}}{\tau} - \Delta \tilde{U}^0 + \nabla P^0 = f^0, \quad \text{i.e.} \quad \tilde{U}^{-1} = \tau(-\Delta \tilde{U}^0 + \nabla P^0 - f^0) + \tilde{U}^0. \quad (5.6)$$

In [33, Theorem 2] Shen proved the following a-priori estimate

Theorem 5.2 (A-priori estimates for the standard pressure-correction scheme).

Suppose the solution (u, p) and the right hand side f of Eq. (2.1) is sufficiently regular. Then the following a-priori estimate holds

$$\|U - u\|_{\ell^\infty(L^2)}^2 + \|\nabla(\tilde{U} - u)\|_{\ell^2(L^2)}^2 + \|P - p\|_{\ell^2(L^2)}^2 \lesssim \tau^2.$$

This estimate is proven for the nonlinear Navier–Stokes equations but holds obviously also for Eq. (2.1). Note that the proof of the pressure error estimate in [33] is incorrect and Shen corrected the proof in [35].

5.1 A-posteriori estimates

Now the a-posteriori estimates are presented in natural energy norms. Further, an estimate for the velocity error is provided in the $L^2(L^2(\Omega))$ norm, which consists of easier computable error estimators. Moreover, optimality of the estimators for velocity and pressure are shown. Here, optimality is understood in the sense that the a-posteriori estimators are asymptotically of the same order as the corresponding a-priori error estimates. The respective proofs are outlined in Chapter 5.3.

The following error estimators are defined.

Definition 5.3 (Error estimators).

Let us introduce the **time error estimators**

$$\hat{e}_1 := \|\nabla \bar{\partial} \tilde{U}\|_{\ell^2(L^2)}^2, \quad \hat{e}_5 := \|\bar{\partial}^2 \tilde{U}\|_{\ell^2(2,N;L^2)}^2, \quad \hat{e}_6 := \|\bar{\partial} \tilde{U}\|_{\ell^2(L^2)}^2,$$

the **splitting error estimators**

$$\begin{aligned} \hat{e}_2 &:= \tau^2 \|\nabla \bar{\partial} P\|_{\ell^\infty(L^2)}^2, & \hat{e}_3 &:= \tau^2 \|\nabla \bar{\partial}^2 P\|_{\ell^2(2,N;L^2)}^2, \\ \hat{e}_4 &:= \frac{1}{\tau^2} \|\operatorname{div} \tilde{U}\|_{\ell^2(L^2)}^2, & \hat{e}_7 &:= \tau^2 \|\nabla \bar{\partial} P\|_{\ell^2(L^2)}^2, \end{aligned}$$

and the **data approximation error estimators**

$$\begin{aligned} \hat{\mathcal{F}}_1 &:= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|f^n - f(t)\|_{J^*}^2 dt, & \hat{\mathcal{F}}_2 &:= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|f^n + (t - t_n) \bar{\partial} f^n - f(t)\|_{-1}^2 dt, \\ \hat{\mathcal{F}}_3 &:= \sum_{n=2}^N \int_{t_{n-1}}^{t_n} \|\bar{\partial} f^n - \partial_t f(t)\|_{J^*}^2 dt. \end{aligned}$$

where $\tilde{U}^{-1}$ is defined in Eq. (5.6) and $P^{-1} := P^0$.

Theorem 5.4 (A-posteriori error estimates for the standard scheme).

Let the solution (u, p) of Eq. (2.1) be sufficiently smooth in time and space and let $U^0 = \tilde{U}^0 = u_0 \in J$. Then the semi-discrete solutions $(\tilde{U}^n, U^n, P^n)$ of Eqs. (5.1)–(5.2) satisfy the a-posteriori estimates

$$\|u - \tilde{U}\|_{L^\infty(L^2)}^2 + \|u - U\|_{L^\infty(L^2)}^2 \lesssim \tau^2(\hat{\epsilon}_1 + \hat{\epsilon}_2 + \hat{\epsilon}_3) + \hat{\mathcal{F}}_1, \quad (5.7)$$

$$\|\nabla(u - \tilde{U})\|_{L^2(L^2)}^2 + \|\nabla(u - U)\|_{L^2(L^2)}^2 \lesssim \tau^2(\hat{\epsilon}_1 + \hat{\epsilon}_3 + \hat{\epsilon}_4) + \hat{\mathcal{F}}_1, \quad (5.8)$$

$$\|P - p\|_{L^2(L^2)}^2 \lesssim \tau^2(\hat{\epsilon}_1 + \hat{\epsilon}_3 + \hat{\epsilon}_4 + \hat{\epsilon}_5) + \hat{\mathcal{F}}_2 + \hat{\mathcal{F}}_3, \quad (5.9)$$

where the estimators are defined in Definition 5.3.

For a weaker norm in time one can show the following theorem:

Theorem 5.5 (L2–L2 a-posteriori error estimates for the velocity).

Under the assumptions of Theorem 5.4 the semi-discrete solutions $(\tilde{U}^n, U^n)$ of Eq. (5.1)–(5.2) satisfy the a-posteriori estimates

$$\|\tilde{U} - u\|_{L^2(L^2)}^2 + \|U - u\|_{L^2(L^2)}^2 \lesssim \tau^2(\hat{\epsilon}_6 + \hat{\epsilon}_7) + \hat{\mathcal{F}}_1, \quad (5.10)$$

where the estimators are defined in Definition 5.3.

By comparing the velocity error estimates in Theorem 5.4 and 5.5 one realizes that the stronger L^∞ norm in time requires more complicated estimators, which computation can be more difficult.

Clearly, the $\hat{\mathcal{F}}$ terms in Definition 5.3 are of first order with respect to the time step size. The next theorem, which is proved in Chapter 5.3 shows that the above a-posteriori estimates for the other terms are also optimal in the sense that they yield first order convergence, which is the same order as the a-priori error estimates.

Theorem 5.6 (Stability for the estimates for Euler).

Suppose u_0, p_0 and f are sufficiently regular. Then the above estimators are stable, i.e.

$$\hat{\epsilon}_1 + \hat{\epsilon}_2 + \hat{\epsilon}_3 + \hat{\epsilon}_4 + \hat{\epsilon}_5 + \hat{\epsilon}_6 + \hat{\epsilon}_7 \leq C(u_0, p_0, f), \quad (5.11)$$

where C is independent of τ .

5.2 Computational results for the standard incremental Euler scheme

In this Chapter computational examples are presented to verify the analysis numerically. Similar to [17] the following example on $\Omega =]-0.5, 0.5[^2$ is chosen

$$\begin{aligned} u(t, x) &= \pi \sin(t + 0.2) [\sin(2\pi x_2) \sin(\pi x_1)^2, -\sin(2\pi x_1) \sin(\pi x_2)^2]^\top, \\ p(t, x) &= \sin(t + 0.2) \cos(\pi x_1) \sin(\pi x_2) \end{aligned}$$

with a right hand side f , such that the above pair (u, p) is a solution to (2.1) for $\nu = 0.1$. The time interval is $]0, 0.5[$. A finite element $\mathcal{P}_2\text{--}\mathcal{P}_1$ Taylor–Hood discretization is used to compute the numerical solutions. In order to get close to the semi-discrete case, a fine grid consisting of $2 \times 66,049$ degrees of freedom for the velocity and 16,641 degrees of freedom for the pressure is used. Consequently the error due to space discretization is very small.

With the estimators of Definition 5.3

$$\begin{aligned}\hat{\epsilon}_1 &:= \|\nabla \bar{\partial} \tilde{U}\|_{\ell^2(L^2)}^2, & \hat{\epsilon}_2 &:= \tau^2 \|\nabla \bar{\partial} P\|_{\ell^\infty(L^2)}^2, & \hat{\epsilon}_3 &:= \tau^2 \|\nabla \bar{\partial}^2 P\|_{\ell^2(L^2)}^2, \\ \hat{\epsilon}_4 &:= \frac{1}{\tau^2} \|\operatorname{div} \tilde{U}\|_{\ell^2(L^2)}^2, & \hat{\epsilon}_5 &:= \|\bar{\partial}^2 \tilde{U}\|_{\ell^2(L^2)}^2, & \hat{\epsilon}_6 &:= \|\bar{\partial} \tilde{U}\|_{\ell^2(L^2)}^2, \\ \hat{\epsilon}_7 &:= \tau^2 \|\nabla \bar{\partial} P\|_{\ell^2(L^2)}^2,\end{aligned}$$

define

$$\begin{aligned}\Theta_{Euler,1}^2 &:= \tau^2 (\hat{\epsilon}_1 + \hat{\epsilon}_2 + \hat{\epsilon}_3), & \Theta_{Euler,2}^2 &:= \tau^2 (\hat{\epsilon}_1 + \hat{\epsilon}_3 + \hat{\epsilon}_4), \\ \Theta_{Euler,3}^2 &:= \tau^2 (\hat{\epsilon}_1 + \hat{\epsilon}_3 + \hat{\epsilon}_4 + \hat{\epsilon}_5), & \Theta_{Euler,4}^2 &:= \tau^2 (\hat{\epsilon}_6 + \hat{\epsilon}_7).\end{aligned}$$

With these notations Theorems 5.4 and 5.5 read as follows:

$$\begin{aligned}\|u - \tilde{U}\|_{L^\infty(L^2)}^2 &\lesssim \Theta_{Euler,1}^2 + \hat{\mathcal{F}}_1, \\ \|\nabla(u - \tilde{U})\|_{L^2(L^2)}^2 &\lesssim \Theta_{Euler,2}^2 + \hat{\mathcal{F}}_1, \\ \|P - p\|_{L^2(L^2)}^2 &\lesssim \Theta_{Euler,3}^2 + \hat{\mathcal{F}}_2 + \hat{\mathcal{F}}_3, \\ \|\tilde{U} - u\|_{L^2(L^2)}^2 &\lesssim \Theta_{Euler,4}^2 + \hat{\mathcal{F}}_1,\end{aligned}$$

where $\hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2, \hat{\mathcal{F}}_3$ are defined in Definition 5.3.

For simplicity only the Θ_{Euler} -terms are presented, since the other terms are just data approximation terms. As it can be seen in Fig. 5.1 the estimators are of the same order as the true error except for the $L^2(V)$ norm, for which the true error is slightly superconvergent in this example. The stagnation of this error for smaller time step sizes is caused by the space discretization error that eventually becomes dominant for very small values of τ .

5.3 Proofs of Theorems 5.4, 5.5 and 5.6

Before proving the theorems, some lemmata have to be provided.

5.3.1 Auxiliary statements

The first lemma provides an upper bound of the error between the time derivative of the exact solution $\partial_t u$ and the interpolant of the difference quotient $\bar{\partial} \tilde{U}(t)$ defined in Eq.(2.24). This estimate is required for proving the pressure error in Theorem 5.4.

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Figure 5.1: Euler: errors and (scaled) estimators versus time step size; $\|u - \tilde{U}\|_{L^\infty(L^2)}$ and $\Theta_{Euler,1}$; $\|\nabla(u - \tilde{U})\|_{L^2(L^2)}^2$ and $\Theta_{Euler,2}$; $\|P - p\|_{L^2(L^2)}^2$ and $\Theta_{Euler,3}$; $\|u - \tilde{U}\|_{L^2(L^2)}$ and $\Theta_{Euler,4}$ (from top left to bottom right); triangles have slope 1.

Lemma 5.7.

Let the solution (u, p) of Eq. (2.1) be sufficiently smooth in time and space. Then the semi-discrete solutions $(\tilde{U}^n, P^n)$ of Eqs. (5.1)–(5.2) satisfy the estimate

$$\|\bar{\partial}\tilde{U} - \partial_t u\|_{L^2(L^2)}^2 \lesssim \tau^2 \|\nabla \bar{\partial}^2 \tilde{U}\|_{\ell^2(L^2)}^2 + \tau^4 \|\nabla \bar{\partial}^2 P\|_{\ell^2(L^2)}^2 + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|\bar{\partial} f^n - \partial_t f(t)\|_{J^*}^2,$$

where U^{-1} is defined in Eq. (5.6) and $P^{-1} = P^0$.

Proof : Apply the $\bar{\partial}$ operator to Eq. (5.3) to obtain for $n \geq 1$

$$\bar{\partial}^2 \tilde{U}^n - \Delta \bar{\partial} \tilde{U}^n + \nabla \bar{\partial} \gamma^n = \bar{\partial} f^n,$$

where $\gamma^0 := 0$ and $\tilde{U}^{-1}$ is defined in Eq. (5.6). With the definition of the interpolant of the difference quotient in Eq. (2.24) this yields to

$$\partial_t \bar{\partial} \tilde{U}(t) - \Delta \bar{\partial} \tilde{U}(t) + \nabla \bar{\partial} \gamma^n = \bar{\partial} f^n - (t - t_n) \Delta \bar{\partial}^2 \tilde{U}^n.$$

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From this equation subtract the time derivative of Eq. (2.1) and set $\zeta(t) := \bar{\partial}\tilde{U}(t) - \partial_t u(t)$ for convenience to get the error equation for $v \in J$:

$$\langle \partial_t \zeta(t), v \rangle + \langle -\Delta \zeta(t), v \rangle = (t - t_n) \langle -\Delta \bar{\partial}^2 \tilde{U}^n, v \rangle + \langle \bar{\partial} f^n - \partial_t f(t), v \rangle. \quad (5.12)$$

Now let $(v(t), r(t))$ be the solution of the Stokes system

$$-\Delta v(t) + \nabla r(t) = \zeta(t), \quad \operatorname{div} v(t) = 0, \quad v(t)|_\Gamma = 0,$$

i.e. $v(t) = \mathcal{A}^{-1} \zeta(t)$. Then

$$\|v(t)\|_s + \|r(t)\|_{s-1} \lesssim \|\zeta(t)\|_{s-2}, \quad s = 1, 2 \quad (5.13)$$

holds due to regularity of the Stokes system. Take the above v as test function in Eq. (5.12). The first term on the left hand side in Eq. (5.12) can be rewritten as:

$$\begin{aligned} \langle \partial_t \zeta(t), v(t) \rangle &= \langle \partial_t (-\Delta v(t) + \nabla r(t)), v(t) \rangle \\ &= \frac{1}{2} \frac{d}{dt} \|\nabla v(t)\|^2 - \underbrace{\langle r(t), \operatorname{div} v(t) \rangle}_{=0} = \frac{1}{2} \frac{d}{dt} \|\nabla v(t)\|^2. \end{aligned}$$

Integrating by parts the second term in Eq. (5.12) reads

$$\begin{aligned} -\langle \Delta \zeta(t), v(t) \rangle &= \langle \zeta(t), -\Delta v(t) \rangle = \|\zeta(t)\|^2 - \langle \nabla r(t), \zeta(t) \rangle \\ &= \|\zeta(t)\|^2 - \langle \nabla r(t), d\tilde{U}(t) - dU(t) \rangle, \end{aligned}$$

where the fact $\operatorname{div} u(t) = 0$ and $\operatorname{div} U(t) = 0$ has been used. To bound the last term in the above estimate the second sub-step of the scheme Eq. (5.2) is used:

$$\tilde{U}^n - U^n = \tau^2 \nabla \bar{\partial} P^n.$$

From this one infers

$$\begin{aligned} \langle d\tilde{U}(t) - dU(t), \nabla r(t) \rangle &= \langle \bar{\partial}(\tilde{U}^n - U^n), \nabla r(t) \rangle + (t - t_n) \langle \bar{\partial}^2(\tilde{U}^n - U^n), \nabla r(t) \rangle \\ &= \tau^2 \langle \bar{\partial}^2 \nabla P^n, \nabla r(t) \rangle + (t - t_n) \tau^2 \langle \bar{\partial}^3 \nabla P^n, \nabla r(t) \rangle \\ &\lesssim \tau^2 \|\zeta(t)\| \left(\|\nabla \bar{\partial}^2 P^{n-1}\| + \|\nabla \bar{\partial}^2 P^n\| \right), \end{aligned}$$

where the regularity estimate Eq. (5.13) has been used for $r(t)$ with $s = 2$.

The terms on the right hand side in Eq. (5.12) are estimated as follows:

$$\begin{aligned} (t - t_n) \langle -\Delta \bar{\partial}^2 \tilde{U}^n, v(t) \rangle &= (t - t_n) \langle \nabla \bar{\partial}^2 \tilde{U}^n, \nabla v(t) \rangle \leq \tau \|\nabla \bar{\partial}^2 \tilde{U}^n\| \cdot \|\nabla v(t)\|, \\ \langle \bar{\partial} f^n - \partial_t f(t), v \rangle &\leq \|\bar{\partial} f^n - \partial_t f(t)\|_{J^*} \|\nabla v(t)\|. \end{aligned}$$

Putting everything together finally yields for $t \in I_n$

$$\begin{aligned} &\frac{d}{dt} \|\nabla v(t)\|^2 + \|\zeta(t)\|^2 \\ &\lesssim \left(\tau \|\nabla \bar{\partial}^2 \tilde{U}^{n+1}\| + \|\bar{\partial} f^{n+1} - \partial_t f(t)\|_{J^*} \right) \|\nabla v(t)\| + \tau^4 \left(\|\nabla \bar{\partial}^2 P^{n+1}\|^2 + \|\nabla \bar{\partial}^2 P^n\|^2 \right). \end{aligned}$$

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Integrating the above equation from $t = 0$ to T and applying Gronwall's Lemma yields

$$\begin{aligned} & \max_{t \in [0, T]} \|\nabla v(t)\|^2 + \int_0^T \|\zeta(t)\|^2 dt \\ & \lesssim \|\nabla v(0)\|^2 + \tau^2 \sum_{n=1}^N \tau \|\nabla \bar{\partial}^2 \tilde{U}^n\|^2 + \tau^4 \sum_{n=1}^N \tau \|\nabla \bar{\partial}^2 P^n\|^2 + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|\bar{\partial} f^n - \partial_t f(t)\|_{J^*}^2 dt. \end{aligned}$$

Finally, with the norm equivalence (2.5), one concludes

$$\|\nabla v(0)\| \lesssim \|\zeta(0)\|_{J^*} = \|\bar{\partial} \tilde{U}^0 - u_t(0)\|_{J^*} = \|\Delta(\tilde{U}^0 - u(0)) + \nabla(P^0 - p(0))\|_{J^*} = 0.$$

This proves the lemma. $\square$

In the next lemma bounds of the divergence of the provisional velocity in negative norms are established.

Lemma 5.8.

Under the assumptions of Lemma 5.7 it holds

$$\|\operatorname{div} \tilde{U}\|_{L^\infty(H^1(\Omega)^*)}^2 \lesssim \tau^4 \|\nabla \bar{\partial} P\|_{\ell^\infty(L^2)}^2, \quad (5.14)$$

$$\|\operatorname{div} \partial_t \tilde{U}\|_{L^2(H^2(\Omega)^*)}^2 \lesssim \tau^4 \|\nabla \bar{\partial}^2 P\|_{\ell^2(L^2)}^2, \quad (5.15)$$

where U^{-1} is defined in Eq. (5.6) and $P^{-1} := P^0$.

Proof : Eq. (5.14) follows from $U(t) \cdot n|_\Gamma = 0$ and by Eq. (5.2):

$$\begin{aligned} \|\operatorname{div} \tilde{U}(t)\|_{L^\infty(H^1(\Omega)^*)}^2 & \leq \max_{1 \leq n \leq N} \|\operatorname{div} \tilde{U}^n\|_{H^1(\Omega)^*}^2 = \max_{1 \leq n \leq N} \sup_{q \in H^1(\Omega)} \frac{|\langle \operatorname{div}(\tilde{U}^n - U^n), q \rangle|^2}{\|q\|_1^2} \\ & = \max_{1 \leq n \leq N} \sup_{q \in H^1(\Omega)} \frac{|\langle \tilde{U}^n - U^n, \nabla q \rangle|^2}{\|q\|_1^2} \leq \tau^4 \max_{1 \leq n \leq N} \|\nabla \bar{\partial} P^n\|^2. \end{aligned}$$

The second equation Eq. (5.15) is obtained by

$$\begin{aligned} \int_0^T \|\operatorname{div} \partial_t \tilde{U}\|_{H^2(\Omega)^*}^2 dt & = \sum_{n=1}^N \tau \|\operatorname{div} \bar{\partial} \tilde{U}^n\|_{H^2(\Omega)^*}^2 = \sum_{n=1}^N \tau \sup_{q \in H^2} \frac{|\langle \operatorname{div} \bar{\partial}(\tilde{U}^n - U^n), q \rangle|^2}{\|q\|_2^2} \\ & = \sum_{n=1}^N \tau \sup_{q \in H^2} \frac{|\langle \bar{\partial}(\tilde{U}^n - U^n), \nabla q \rangle|^2}{\|q\|_2^2} \\ & \leq \sum_{n=1}^N \tau \|\bar{\partial}(\tilde{U}^n - U^n)\|^2 \leq \sum_{n=1}^N \tau^5 \|\nabla \bar{\partial}^2 P^n\|^2, \end{aligned}$$

where in the last step Eq. (5.2) has been used. $\square$

Remark 5.9.

It is interesting to note that Eq. (5.15) can be replaced by a sharper estimate. Instead of estimating

$$\|(I - \mathcal{S})\partial_t \tilde{U}\|_{L^2(J^*)}^2 \lesssim \|\operatorname{div} \partial_t \tilde{U}\|_{L^2(H^2(\Omega)^*)}^2$$

as the proof of Theorem 5.4, the corresponding error term can be bounded by values on the boundary only:

$$\|(I - \mathcal{S})\partial_t \tilde{U}\|_{L^2(J^*)}^2 \lesssim \tau^4 \|\bar{\partial}^2 P\|_{\ell^2(H^{-1/2}(\Gamma))}^2.$$

5.3.2 Proof of Theorem 5.4

Now Theorem 5.4 will be proved. Since the divergence of $\tilde{U}$ is not zero, the derived error equation cannot be tested with the error $\tilde{U}(t) - u(t)$ itself in order to neglect any pressure terms. Therefore the error is subdivided into a divergence-free part $\mathcal{S}\tilde{U} - u$ and $\mathcal{S}\tilde{U} - \tilde{U}$ which is controlled by Lemma 2.2.

Proof : (**Theorem 5.4**) Using the definition in Eq. (2.21) for the interpolant $\tilde{U}(t)$ one verifies that for an arbitrary function $v \in J$, $t \in I_n$ the following is true

$$\begin{aligned} & \langle \partial_t \mathcal{S}\tilde{U}(t), v \rangle + \langle \nabla \mathcal{S}\tilde{U}(t), \nabla v \rangle \\ &= \langle \partial_t (\mathcal{S}\tilde{U} - \tilde{U})(t), v \rangle + \langle \partial_t \tilde{U}(t), v \rangle + \langle \nabla \tilde{U}^n, \nabla v \rangle + \langle \nabla \tilde{U}(t) - \nabla \tilde{U}^n, \nabla v \rangle \\ &= \langle \partial_t (\mathcal{S}\tilde{U} - \tilde{U})(t), v \rangle + \langle f^n, v \rangle + \langle \nabla \tilde{U}(t) - \nabla \tilde{U}^n, \nabla v \rangle. \end{aligned}$$

Thereof subtract Eq. (2.1) to arrive at the error equation for all $v \in J$ and $t \in I_n$

$$\begin{aligned} & \langle \partial_t (\mathcal{S}\tilde{U} - u)(t), v \rangle + \langle \nabla (\mathcal{S}\tilde{U} - u)(t), \nabla v \rangle \\ &= \langle \partial_t (\mathcal{S}\tilde{U} - \tilde{U})(t), v \rangle + (t - t_n) \langle \nabla \bar{\partial} \tilde{U}^n, \nabla v \rangle + \langle f^n - f(t), v \rangle. \end{aligned} \quad (5.16)$$

Choose $v = (\mathcal{S}\tilde{U} - u)(t) \in J$ in Eq. (5.16) to obtain for $t \in I_n$

$$\begin{aligned} & \frac{d}{dt} \|(\mathcal{S}\tilde{U} - u)(t)\|^2 + \|\nabla (\mathcal{S}\tilde{U} - u)(t)\|^2 \\ & \lesssim \|(\mathcal{S} - I)\partial_t \tilde{U}(t)\|_{J^*}^2 + \tau^2 \|\nabla \bar{\partial} \tilde{U}^n\|^2 + \|f^n - f(t)\|_{J^*}^2. \end{aligned} \quad (5.17)$$

Integrating Eq. (5.17) from $t = 0$ to T yields

$$\begin{aligned} & \max_{t \in [0, T]} \|(\mathcal{S}\tilde{U} - u)(t)\|^2 + \int_0^T \|\nabla (\mathcal{S}\tilde{U} - u)(t)\|^2 dt \\ & \lesssim \|(\mathcal{S}\tilde{U} - u)(0)\|^2 + \int_0^T \|(\mathcal{S} - I)\partial_t \tilde{U}(t)\|_{J^*}^2 dt \\ & \quad + \tau^2 \sum_{n=1}^N \tau \|\nabla \bar{\partial} \tilde{U}^n\|^2 dt + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|f^n - f(t)\|_{J^*}^2 dt \\ & \lesssim \tau^4 \sum_{n=1}^N \tau \|\nabla \bar{\partial}^2 P^n\|^2 + \tau^2 \sum_{n=1}^N \tau \|\nabla \bar{\partial} \tilde{U}^n\|^2 dt + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|f^n - f(t)\|_{J^*}^2 dt, \end{aligned}$$

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where Lemma 2.2 with $s = -1$, Lemma 5.8, and the requirements of the initial data $\tilde{U}^0 = u_0 \in J$ were used. Observing that

$$\tilde{U} - u = (\mathcal{S}\tilde{U} - u) + (\tilde{U} - \mathcal{S}\tilde{U}) = (\mathcal{S}\tilde{U} - u) + (I - \mathcal{S})\tilde{U},$$

using Lemma 2.2 with $s = 0$ and Lemma 5.8 proves (5.7). Likewise one proves Eq. (5.8) using the estimate

$$\int_0^T \|\nabla(I - \mathcal{S})\tilde{U}(t)\|^2 dt \lesssim \int_0^T \|\operatorname{div} \tilde{U}(t)\|^2 dt \lesssim \sum_{n=1}^n \tau \|\operatorname{div} \tilde{U}^n\|^2.$$

The estimates for $\|U - u\|$ and $\|\nabla(U - u)\|$ in Eqs. (5.7) and (5.8) are derived by writing

$$U - u = (\tilde{U} - u) + (U - \tilde{U})$$

and using the triangle inequality, Lemma 2.1, and Lemma 5.8.

To prove the pressure estimate write $\nabla(p - P)(t)$ according to definition Eq. (2.21), the sum of Eqs. (5.1) and (5.2), and the continuous equation Eq. (2.1) as

$$\nabla(p - P)(t) = (\bar{\partial}U - \partial_t u)(t) - \Delta(\tilde{U} - u)(t) - f^n + f(t) - (t - t_n)\bar{\partial}f^n.$$

Note, that $\bar{\partial}\tilde{U}(t)$ is the linear interpolant of the difference quotient as Eq. (2.24). Next, due to the inf-sup condition, for $t \in I_n$,

$$\begin{aligned} \|(p - P)(t)\| &\lesssim \sup_{\substack{v \in V \\ \|v\|=1}} |\langle (p - P)(t), \operatorname{div} v \rangle| = \sup_{\substack{v \in V \\ \|v\|=1}} |\langle \nabla(p - P)(t), v \rangle| \\ &= \sup_{\substack{v \in V \\ \|v\|=1}} \left| \langle (\bar{\partial}U - \partial_t u)(t), v \rangle + \langle \nabla(\tilde{U} - u)(t), \nabla v \rangle - \langle f^n - f(t), v \rangle - (t - t_n) \langle \bar{\partial}f^n, v \rangle \right| \\ &\lesssim \|(\bar{\partial}U - \partial_t u)(t)\|_{-1} + \|\nabla(\tilde{U} - u)(t)\| + \|f^n + (t - t_n)\bar{\partial}f^n - f(t)\|_{-1} \\ &\lesssim \|\mathcal{H}(\bar{\partial}\tilde{U} - \partial_t u)(t)\| + \|\nabla(\tilde{U} - u)(t)\| + \|f^n + (t - t_n)\bar{\partial}f^n - f(t)\|_{-1} \\ &\lesssim \|(\bar{\partial}\tilde{U} - \partial_t u)(t)\| + \|\nabla(\tilde{U} - u)(t)\| + \|f^n + (t - t_n)\bar{\partial}f^n - f(t)\|_{-1}, \end{aligned}$$

holds, where the stability of the Helmholtz operator $\mathcal{H}$ was used. Squaring and integrating from $t = 0$ to T , using Lemma 5.7, and the gradient estimate for the velocity error proves the pressure estimate in (5.8), which completes the proof. $\square$

5.3.3 Proof of Theorem 5.5

Proof : (**Theorem 5.5**) Subtracting the continuous equation (2.1) from Eq. (5.3) yields for $v \in J$

$$\langle \bar{\partial}\tilde{U}^n - \partial_t u(t), v \rangle + \langle -\Delta(\tilde{U} - u)(t), v \rangle = (t - t_n) \langle -\Delta\bar{\partial}\tilde{U}^n, v \rangle + \langle f^n - f(t), v \rangle.$$

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With the notation $\tilde{e}(t) := \tilde{U}(t) - u(t)$ (and $e(t) := U(t) - u(t)$) the above error equation reads

$$\langle \partial_t \tilde{e}(t), v \rangle + \langle -\Delta \tilde{e}(t), v \rangle = (t - t_n) \langle -\Delta \bar{\partial} \tilde{U}^{n+1}, v \rangle + \langle f^n - f(t), v \rangle. \quad (5.18)$$

For $t \in I_n$ let $(w(t), r(t))$ be the solution of

$$-\Delta w(t) + \nabla r(t) = \tilde{e}(t), \quad \operatorname{div} w(t) = 0, \quad w(t)|_\Gamma = 0,$$

hence $w = \mathcal{A}^{-1} \tilde{e}(t)$. The following regularity estimate holds

$$\|w\|_s + \|r\|_{s-1} \lesssim \|\tilde{e}\|_{s-2}, \quad s = 1, 2. \quad (5.19)$$

Choose $v := w(t)$ in Eq. (5.18). The first term on the left hand side gives

$$\langle \partial_t \tilde{e}(t), w(t) \rangle = \langle \partial_t (-\Delta w(t) + \nabla r(t)), w(t) \rangle = \langle \partial_t \nabla w(t), \nabla w(t) \rangle = \frac{1}{2} \frac{d}{dt} \|\nabla w(t)\|^2.$$

For the second term, integration by parts yields

$$\begin{aligned} \langle -\Delta \tilde{e}(t), w(t) \rangle &= \langle \tilde{e}(t), -\Delta w(t) \rangle = \langle \tilde{e}(t), \tilde{e}(t) - \nabla r(t) \rangle \\ &= \|\tilde{e}(t)\|^2 - \langle \tilde{e}(t) - e(t), \nabla r(t) \rangle. \end{aligned}$$

The last term on the right hand side of the above equation is estimated by

$$\langle \tilde{e}(t) - e(t), \nabla r(t) \rangle \leq \|\tilde{U}(t) - U(t)\| \|\nabla r(t)\| \lesssim \|\tilde{U}(t) - U(t)\| \|\tilde{e}(t)\|,$$

where the regularity estimate Eq. (5.19) has been used. The first term on the right hand side in Eq. (5.18) yields

$$\begin{aligned} (t - t_n) \langle -\Delta \bar{\partial} \tilde{U}^n, w(t) \rangle &= (t - t_n) \langle \bar{\partial} \tilde{U}^n, -\Delta w(t) \rangle \\ &= (t - t_n) \langle \bar{\partial} \tilde{U}^n, \tilde{e}(t) - \nabla r(t) \rangle \\ &\lesssim \tau \|\bar{\partial} \tilde{U}^n\| \|\tilde{e}(t)\|. \end{aligned}$$

Collecting the above estimates, applying Young's inequality and integrating from $t = 0$ to T results in

$$\begin{aligned} &\max_{t \in [0, T]} \|\nabla w(t)\|^2 + \int_0^T \|\tilde{e}(t)\|^2 \\ &\lesssim \|\nabla w(0)\|^2 + \tau^2 \sum_{n=1}^N \tau \|\bar{\partial} \tilde{U}^n\|^2 + \int_0^T \|\tilde{U}(t) - U(t)\|^2 dt + \left(\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|f^n - f(t)\|_{J^*} \right)^2. \end{aligned}$$

For the third term on the right hand side, the second sub-step Eq. (5.2) of the scheme deduces

$$\int_0^T \|\tilde{U}(t) - U(t)\|^2 dt \lesssim \sum_{n=1}^N \tau \|\tilde{U}^n - U^n\|^2 \leq \tau^4 \sum_{n=1}^N \tau \|\nabla \bar{\partial} P^n\|^2.$$

The estimate for $\|U - u\|_{L^2(L^2)}$ is obtained by again writing

$$U - u = (\tilde{U} - u) + (U - \tilde{U}) = \tilde{e} + (U - \tilde{U}).$$

This, together with the fact $w(0) = 0$ proves the theorem. $\square$

5.3.4 Proof of optimality Theorem 5.6

Next the optimality of our estimators $\hat{e}_1, \dots, \hat{e}_7$ of Definition 5.3 for an adequate choice of the initial data is shown.

First a stability result is deduced.

Lemma 5.10.

The standard incremental Euler scheme (5.1) and (5.2) fulfills the following stability estimate for all $i \geq 0$:

$$\begin{aligned} \|\bar{\partial}^i U\|_{\ell^\infty(i,N;L^2)}^2 + \tau^2 \|\nabla \bar{\partial}^i P\|_{\ell^\infty(i,N;L^2)}^2 + \frac{1}{\tau} \|\bar{\partial}^i (\tilde{U} - U)\|_{\ell^2(i+1,N;L^2)}^2 + \tau \|\bar{\partial}^{i+1} U\|_{\ell^2(i+1,N;L^2)}^2 \\ + \|\nabla \bar{\partial}^i \tilde{U}\|_{\ell^2(i+1,N;L^2)}^2 \lesssim \|\bar{\partial}^i U^i\|^2 + \tau^2 \|\nabla \bar{\partial}^i P^i\|^2 + \|\bar{\partial}^i f\|_{\ell^2(i,N;H^{-1})}^2. \end{aligned}$$

Proof : Apply the backward difference operator $\bar{\partial}^i$ to Eqs. (5.1) and (5.2) to obtain for $n \geq i + 1$

$$\frac{\bar{\partial}^i (\tilde{U}^n - U^{n-1})}{\tau} - \Delta \bar{\partial}^i \tilde{U}^n + \nabla \bar{\partial}^i P^{n-1} = \bar{\partial}^i f^n, \quad (5.20)$$

$$\frac{1}{\tau} \bar{\partial}^i (U^n - \tilde{U}^n) + \nabla \bar{\partial}^i (P^n - P^{n-1}) = 0. \quad (5.21)$$

Next, multiply Eq. (5.20) by $2\tau \bar{\partial}^i \tilde{U}^{n+1}$. Using the relation Eq. (2.17), the first term on the left hand side can be treated as

$$\begin{aligned} 2\langle \bar{\partial}^i (\tilde{U}^n - U^{n-1}), \bar{\partial}^i \tilde{U}^n \rangle &= \|\bar{\partial}^i \tilde{U}^n\|^2 - \|\bar{\partial}^i U^{n-1}\|^2 + \|\bar{\partial}^i (\tilde{U}^n - U^{n-1})\|^2 \\ &= \|\bar{\partial}^i \tilde{U}^n\|^2 - \|\bar{\partial}^i U^n\|^2 + \|\bar{\partial}^i (\tilde{U}^n - U^n)\|^2 + \|\bar{\partial}^i (U^n - U^{n-1})\|^2, \end{aligned}$$

where the last step follows from Eq. (5.21), i.e. $\bar{\partial}^i (\tilde{U}^n - U^n) \perp \bar{\partial}^i (U^n - U^{n-1}) \in H$. Hence

$$\begin{aligned} \|\bar{\partial}^i \tilde{U}^n\|^2 + \|\bar{\partial}^i (\tilde{U}^n - U^n)\|^2 - \|\bar{\partial}^i U^{n-1}\|^2 + \tau^2 \|\bar{\partial}^{i+1} U^n\|^2 \\ + 2\tau \|\nabla \bar{\partial}^i \tilde{U}^n\|^2 + 2\tau \langle \nabla \bar{\partial}^i P^{n-1}, \bar{\partial}^i \tilde{U}^n \rangle = 2\tau \langle \bar{\partial}^i f^n, \bar{\partial}^i \tilde{U}^n \rangle. \end{aligned} \quad (5.22)$$

Now, rearrange Eq. (5.21) to get

$$\bar{\partial}^i U^n + \tau \nabla \bar{\partial}^i P^n = \bar{\partial}^i \tilde{U}^n + \tau \nabla \bar{\partial}^i P^{n-1}$$

and squaring this yields

$$\|\bar{\partial}^i U^n\|^2 + \tau^2 \|\nabla \bar{\partial}^i P^n\|^2 = \|\bar{\partial}^i \tilde{U}^n\|^2 + 2\tau \langle \bar{\partial}^i \tilde{U}^n, \nabla \bar{\partial}^i P^{n-1} \rangle + \tau^2 \|\nabla \bar{\partial}^i P^{n-1}\|^2. \quad (5.23)$$

By adding Eqs. (5.22) and (5.23) one obtains

$$\begin{aligned} \|\bar{\partial}^i U^n\|^2 - \|\bar{\partial}^i U^{n-1}\|^2 + \|\bar{\partial}^i (\tilde{U}^n - U^n)\|^2 + \tau^2 \|\bar{\partial}^{i+1} U^n\|^2 \\ + \tau \|\nabla \bar{\partial}^i \tilde{U}^n\|^2 + \tau^2 \|\nabla \bar{\partial}^i P^n\|^2 - \tau^2 \|\nabla \bar{\partial}^i P^{n-1}\|^2 \lesssim \tau \|\bar{\partial}^i f^n\|_{-1}^2. \end{aligned}$$

Summing up from $n = i + 1$ to N proves the lemma. $\square$

It is the aim to show the stability of the $\hat{e}$ -terms defined in Definition 5.3, i.e.

$$\hat{e}_1 + \hat{e}_2 + \hat{e}_3 + \hat{e}_4 + \hat{e}_5 + \hat{e}_6 + \hat{e}_7 \leq C,$$

where C is a constant only depending on data.

As an easy consequence of Lemma 5.10 the following corollary holds:

Corollary 5.11.

The error estimators $\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_5$ and $\hat{e}_6$ of Definition 5.3 are bounded by

$$\hat{e}_1 + \hat{e}_2 + \hat{e}_6 + \hat{e}_7 \lesssim \|\bar{\partial}\tilde{U}^1\|^2 + \tau^2\|\nabla\bar{\partial}P^1\|^2 + \|\bar{\partial}f\|_{\ell^2(H^{-1})}^2, \quad (5.24)$$

$$\hat{e}_3 + \hat{e}_5 \lesssim \|\bar{\partial}^2\tilde{U}^2\|^2 + \tau^2\|\nabla\bar{\partial}^2P^2\|^2 + \|\nabla\bar{\partial}P^1\|^2 + \|\bar{\partial}^2f\|_{\ell^2(2,N;H^{-1})}^2. \quad (5.25)$$

Proof : The estimates follow directly from Lemma 5.10 combined with the identities

$$\begin{aligned} \bar{\partial}U^1 &= \frac{U^1 - u_0}{\tau} = \frac{\tilde{U}^1 - u_0}{\tau} - \nabla(P^1 - p_0) = \bar{\partial}\tilde{U}^1 - \tau\nabla\bar{\partial}P^1, \\ \bar{\partial}^2U^2 &= \frac{\bar{\partial}U^2 - \bar{\partial}U^1}{\tau} = \frac{\bar{\partial}\tilde{U}^2 - \tau^2\nabla\bar{\partial}^2P^2 - \bar{\partial}\tilde{U}^1 + \tau\nabla\bar{\partial}P^1}{\tau} = \bar{\partial}^2\tilde{U}^2 - \tau\nabla\bar{\partial}^2P^2 + \nabla\bar{\partial}P^1. \end{aligned}$$

$\square$

Adopting the procedure in [1] the initial values $\bar{\partial}^i\tilde{U}^i$ and $\bar{\partial}^iP^i$, $i = 1, 2$, on the right hand sides of Eqs. (5.24), (5.25) can be bounded by a-priori estimates of the initial errors:

Lemma 5.12.

Suppose $\tilde{U}^0 = U^0 = u_0$ and $P^0 = p_0$. Then the following a-priori estimate holds for $i = 1, 2$:

$$\|\tilde{e}^i\|^2 + \tau\|\nabla\tilde{e}^i\|^2 + \tau^2\|\Delta\tilde{e}^i\|^2 + \tau^4\|\nabla\bar{\partial}P^i\|^2 \lesssim \tau^4(\|\partial_{tt}u\|_{L^\infty((0,t_i);L^2)}^2 + \|\nabla\partial_t p\|_{L^\infty((0,t_i);L^2)}^2).$$

Proof : Use Eqs. (5.1) and (2.1) to get the error equation

$$\tilde{e}^1 - \tau\Delta\tilde{e}^1 = \tau R^1 + \tau^2\nabla\bar{\partial}p_1,$$

where $R^i := (\partial_t - \bar{\partial})u(t_i)$. From standard PDE theory one derives

$$\|\tilde{e}^1\|^2 + \tau\|\nabla\tilde{e}^1\|^2 + \tau^2\|\Delta\tilde{e}^1\|^2 \lesssim \tau^4(\|\partial_{tt}u\|_{L^\infty((0,t_1);L^2)}^2 + \|\nabla\bar{\partial}p_1\|^2).$$

Eq. (5.2) and the stability of the Helmholtz operator yields

$$\begin{aligned} \|\nabla\bar{\partial}P^1\| &\lesssim \frac{1}{\tau^2}(\|\tilde{e}^1\| + \|e^1\|) \lesssim \frac{1}{\tau^2}\|\tilde{e}^1\| \lesssim \|\partial_{tt}u\|_{L^\infty((0,t_1);L^2)} + \|\nabla\partial_t p\|_{L^\infty((0,t_1);L^2)}, \\ \|\nabla(P^1 - p_1)\| &\lesssim \frac{1}{\tau}\|\tilde{e}^1\| + \|\nabla(p_1 - p_0)\| \lesssim \tau(\|\partial_{tt}u\|_{L^\infty((0,t_1);L^2)} + \|\nabla\partial_t p\|_{L^\infty((0,t_1);L^2)}). \end{aligned}$$

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For the second time step the error equation reads

$$\tilde{e}^2 - \tau \Delta \tilde{e}^2 = \tau R^2 + \tau^2 \nabla \bar{\partial} p_2 + \tau \nabla (p_1 - P^1) + \tau^2 \nabla \bar{\partial} P^1.$$

With the same arguments as above one obtains

$$\|\tilde{e}^2\|^2 + \tau \|\nabla \tilde{e}^2\|^2 + \tau^2 \|\Delta \tilde{e}^2\|^2 \lesssim \tau^4 (\|\partial_{tt} u\|_{L^\infty((0,t_2);L^2)}^2 + \|\nabla \partial_t p\|_{L^\infty((0,t_2);L^2)}^2)$$

and

$$\|\nabla \bar{\partial} P^2\| \lesssim \frac{1}{\tau^2} \|\tilde{e}^2\| \lesssim \|\partial_{tt} u\|_{L^\infty((0,t_2);L^2)} + \|\nabla \partial_t p\|_{L^\infty((0,t_2);L^2)}.$$

□

To prove optimality of the estimator $\hat{e}_4$ of Definition 5.3 the a-priori estimate of the gradient error $\|\nabla(\tilde{U} - u)\|$ in Theorem 5.2 is used:

$$\hat{e}_4 = \frac{1}{\tau^2} \|\operatorname{div} \tilde{U}\|_{\ell^2(L^2)}^2 = \frac{1}{\tau^2} \|\operatorname{div}(\tilde{U} - u)\|_{\ell^2(L^2)}^2 \lesssim \frac{1}{\tau^2} \|\nabla(\tilde{U} - u)\|_{\ell^2(L^2)}^2 \lesssim C.$$

Collecting all the above estimates and observing that

$$\bar{\partial} \tilde{U}^1 = (\tilde{U}^1 - u_1)/\tau + (u_1 - u_0)/\tau = \tilde{e}^1/\tau + \bar{\partial} u_1,$$

and likewise for $\bar{\partial}^2 \tilde{U}^2$ proves the Theorem 5.6.

6 The rotational pressure-correction scheme discretized by BDF2

In this chapter the main results are presented: A-posteriori error estimates for the rotational pressure-correction scheme discretized by the two-step backward differential formula method (BDF2), which are stated in Theorems 6.2 and 6.3. By use of the trapezoidal scheme for calculating the initial values an initial estimate is given in Theorem 6.4. The a-posteriori estimates combined with the initial estimate are claimed in Corollary 6.5. To show optimality of the velocity estimator, Theorem 6.6 gives a stability result.

Recall Algorithm 4.8: Using the BDF2 formula to approximate the time derivative, the pressure-correction scheme in rotational form for (2.1) reads as follows. Let U^0 , U^1 and P^1 be given. Then for $n \geq 0$ in the first sub-step accounting for viscous diffusion, a velocity $\tilde{U}^n$ is sought, such that

$$\begin{aligned} \frac{3\tilde{U}^n - 4U^{n-1} + U^{n-2}}{2\tau} - \Delta \tilde{U}^n + \nabla P^{n-1} &= f^n, \\ \tilde{U}^n|_{\Gamma} &= 0, \end{aligned} \quad (6.1)$$

and in the second step calculate (U^n, P^n) , such that

$$\begin{aligned} \frac{3U^n - 3\tilde{U}^n}{2\tau} + \nabla(P^n - P^{n-1} + \operatorname{div} \tilde{U}^n) &= 0, \\ \operatorname{div} U^n &= 0, \\ U^n \cdot n|_{\Gamma} &= 0. \end{aligned} \quad (6.2)$$

Set $\Phi^n := P^n - P^{n-1} + \operatorname{div} \tilde{U}^n$, Eq. (6.2) is equivalent to

$$\begin{aligned} -\Delta \Phi^n &= -\frac{3}{2\tau} \operatorname{div} \tilde{U}^n, \quad \nabla \Phi^n \cdot n|_{\Gamma} = 0, \\ U^n &= \tilde{U}^n - \frac{2}{3}\tau \nabla \Phi^n, \\ P^n &= P^{n-1} + \Phi^n - \operatorname{div} \tilde{U}^n. \end{aligned} \quad (6.3)$$

As demonstrated in Chapter 4.4, Eqs. (6.1) and (6.2) can be reformulated to

$$\bar{\partial}^B \tilde{U}^n - \Delta \tilde{U}^n + \nabla(P^{n-1} + \frac{4}{3}\Phi^{n-1} - \frac{1}{3}\Phi^{n-2}) = f^n, \quad \tilde{U}^n|_{\Gamma} = 0. \quad (6.4)$$

6.1 A-posteriori estimates

Now the error estimators and the appropriate estimates are presented. The proofs of the theorems are given in Chapter 6.3.

Definition 6.1 (Error estimators).

For arbitrary initial values U^0, U^1 and P^1 introduce the **time error estimators**

$$\begin{aligned}\epsilon_1 &:= \|\nabla \bar{\partial}^2 \tilde{U}\|_{\ell^2(3,N;L^2)}^2, & \epsilon_2 &:= \|\bar{\partial}^3 \tilde{U}\|_{\ell^2(3,N;J^*)}^2, & \epsilon_3 &:= \|\bar{\partial}^2 \tilde{U}\|_{\ell^2(3,N;L^2)}^2, \\ \epsilon_7 &:= \|\bar{\partial}^3 \tilde{U}\|_{\ell^2(4,N;L^2)}^2, & \epsilon_8 &:= \|\bar{\partial}^4 \tilde{U}\|_{\ell^2(4,N;J^*)}^2, & \epsilon_9 &:= \|\bar{\partial}^2 \tilde{U}\|_{\ell^\infty(3,N;L^2)}^2,\end{aligned}$$

the **splitting error estimators**

$$\begin{aligned}\epsilon_4 &:= \frac{1}{\tau^2} \|\nabla \Phi\|_{\ell^\infty(2,N;L^2)}^2, & \epsilon_5 &:= \frac{1}{\tau^2} \|\nabla \Phi\|_{\ell^2(2,N;L^2)}^2, & \epsilon_6 &:= \frac{1}{\tau} \|\nabla \bar{\partial} \Phi\|_{\ell^2(3,N;L^2)}^2, \\ \epsilon_{10} &:= \frac{1}{\tau^3} \|\operatorname{div} \tilde{U}^n\|_{\ell^2(3,N;L^2)}^2,\end{aligned}$$

the **data approximation errors**

$$\begin{aligned}\mathcal{F}_1 &:= \|F - f\|_{L^2((t_2,T);J^*)}^2, & \mathcal{F}_2 &:= \|F - f\|_{L^2((t_2,T);H^{-1})}^2, \\ \mathcal{F}_3 &:= \sum_{n=4}^N \int_{t_{n-1}}^{t_n} \|\bar{\partial}_B f^n + (t - t_n) \bar{\partial}^2 f^n - \partial_t f(t)\|_{J^*}^2 dt,\end{aligned}$$

and the **initial approximation errors**

$$\begin{aligned}\eta_U^0 &:= \|\tilde{U} - u\|_{L^\infty((0,t_2);L^2)}^2 + \|\tilde{U} - U\|_{L^\infty((0,t_2);L^2)}^2 + \frac{1}{\tau} \|\tilde{U}^1 - U^1\|^2 + \tau \|\bar{\partial}(\tilde{U}^1 - U^1)\|^2, \\ \eta_{\nabla U}^0 &:= \|\nabla(\tilde{U} - u)\|_{L^2((0,t_2);L^2)}^2 + \|\nabla(\tilde{U} - U)\|_{L^2((0,t_2);L^2)}^2 + \frac{1}{\tau} \|\tilde{U}^1 - U^1\|^2 + \tau \|\bar{\partial}(\tilde{U}^1 - U^1)\|^2, \\ \eta_P^0 &:= \|P - p\|_{L^2((0,t_3);L^2)}^2 + \frac{1}{\tau} \|\tilde{U}^1 - U^1\|^2 + \tau \|\bar{\partial}(\tilde{U}^2 - U^2)\|^2.\end{aligned}$$

If one uses Algorithm 4.4 for calculating the initial values U^0, U^1 and P^1 , one can also include the initial approximation errors η_U^0 and $\eta_{\nabla U}^0$ in the estimators. With the same notation the redefined estimators read:

$$\begin{aligned}\epsilon_1 &:= \|\nabla \bar{\partial}^2 \tilde{U}\|_{\ell^2(2,N;L^2)}^2, & \epsilon_2 &:= \|\bar{\partial}^3 \tilde{U}\|_{\ell^2(3,N;J^*)}^2, \\ \epsilon_{2,0} &:= \frac{1}{\tau} \|\bar{\partial}^2 \tilde{U}^2 - \bar{\partial}(\Delta \tilde{U}^1 + f^1)\|_{J^*}^2, & \epsilon_3 &:= \|\bar{\partial}^2 \tilde{U}\|_{\ell^2(2,N;L^2)}^2, \\ \epsilon_4 &:= \frac{1}{\tau^2} \|\nabla \Phi\|_{\ell^\infty(L^2)}^2, & \epsilon_5 &:= \frac{1}{\tau^2} \|\nabla \Phi\|_{\ell^2(L^2)}^2, \\ \epsilon_6 &:= \frac{1}{\tau} \|\nabla \bar{\partial} \Phi\|_{\ell^2(L^2)}^2, & \epsilon_9 &:= \|\bar{\partial}^2 \tilde{U}\|_{\ell^\infty(2,N;L^2)}^2, \\ \epsilon_{10} &:= \frac{1}{\tau^3} \|\operatorname{div} \tilde{U}\|_{\ell^2(L^2)}^2,\end{aligned} \tag{6.5}$$

where $\Phi^0 = 0$, and the data approximation error is

$$\mathcal{F} := \int_0^{t_1} \|f^{1/2} + (t - t^{1/2})\bar{\partial}f^1 - f(t)\|_{J_*}^2 + \sum_{n=2}^N \int_{t_{n-1}}^{t_n} \|f^n + (t - t_n)\bar{\partial}f^n - f(t)\|_{J_*}^2 \quad (6.6)$$

Theorem 6.2 (A-posteriori error estimates for the rotational scheme).

Suppose the solution (u, p) to System (2.1) is smooth enough in time and space. Then the semi-discrete solutions $(\tilde{U}^n, U^n, P^n)$ of Eqs. (6.1) and (6.2) satisfy the estimates

$$\|\tilde{U} - u\|_{L^\infty(L^2)}^2 + \|U - u\|_{L^\infty(L^2)}^2 \lesssim \eta_U^0 + \tau^4(\epsilon_1 + \epsilon_2 + \epsilon_4 + \epsilon_9) + \tau^3\epsilon_6 + \mathcal{F}_1, \quad (6.7)$$

$$\|\nabla(\tilde{U} - u)\|_{L^2(L^2)}^2 + \|\nabla(U - u)\|_{L^2(L^2)}^2 \lesssim \eta_{\nabla U}^0 + \tau^4(\epsilon_1 + \epsilon_2) + \tau^3(\epsilon_6 + \epsilon_{10}) + \mathcal{F}_1, \quad (6.8)$$

$$\|P - p\|_{L^2(L^2)}^2 \lesssim \eta_P^0 + \tau^4(\epsilon_1 + \epsilon_7 + \epsilon_8) + \tau^3(\epsilon_6 + \epsilon_{10}) + \mathcal{F}_2 + \mathcal{F}_3, \quad (6.9)$$

where the estimators are denoted in Definition 6.1.

As done the Euler scheme, dual techniques enable the derivation of an improved estimate for the velocity in a weaker norm.

Theorem 6.3 (L2–L2 a-posteriori error estimates for the velocity).

Under the assumptions of Theorem 6.2 it holds

$$\|\tilde{U} - u\|_{L^2(L^2)}^2 + \|U - u\|_{L^2(L^2)}^2 \lesssim \eta_U^1 + \tau^4(\epsilon_2 + \epsilon_3 + \epsilon_5) + \mathcal{F}_1, \quad (6.10)$$

where the estimators are denoted in Definition 6.1.

If one uses the trapezoidal scheme, defined in Eqs. (4.3)–(4.4), to calculate the initial values, the initial errors η_U^0 , $\eta_{\nabla U}^0$ and η_U^1 can be estimated as follows:

Theorem 6.4 (Initial estimates).

Set $\tilde{U}^0 = U^0 = u_0$ and $P^0 = p_0$. Let $(\tilde{U}^1, U^1, P^1)$ be calculated by Eqs. (4.3)–(4.4). Then the initial errors η_U^0 , $\eta_{\nabla U}^0$ and η_U^1 of Theorem 6.2 and Theorem 6.3 are bounded as follows:

$$\begin{aligned} \eta_U^0 &\lesssim \tau^4 \|\bar{\partial}^2 \tilde{U}^2\|^2 + \tau^3 \|\bar{\partial}^2 \tilde{U}^2 - \bar{\partial}(\Delta \tilde{U}^1 + f^1)\|_{J_*}^2 \\ &\quad + \sum_{i=1}^2 \left(F_i + \tau^2 \|\nabla \Phi^i\|^2 + \tau^3 \|\nabla \bar{\partial} \Phi^i\|^2 \right), \end{aligned} \quad (6.11)$$

$$\eta_{\nabla U}^0 \lesssim \tau^3 \|\bar{\partial}^2 \tilde{U}^2 - \bar{\partial}(\Delta \tilde{U}^1 + f^1)\|_{J_*}^2 + \sum_{i=1}^2 \left(F_i + \tau \|\operatorname{div} \tilde{U}^i\|^2 + \tau^3 \|\nabla \bar{\partial} \Phi^i\|^2 \right), \quad (6.12)$$

$$\eta_U^1 \lesssim \tau^3 \|\bar{\partial}^2 \tilde{U}^2 - \bar{\partial}(\Delta \tilde{U}^1 + f^1)\|_{J_*}^2 + \tau^5 \|\bar{\partial}^2 \tilde{U}^2\|^2 + \sum_{i=1}^2 \left(F_i + \tau^3 \|\nabla \Phi^i\|^2 \right), \quad (6.13)$$

where $\Phi^0 = 0$ and

$$F_1 := \int_0^{t_1} \|f^{1/2} + (t - t^{1/2})\bar{\partial}f^1 - f(t)\|_{J_*}^2 dt, \quad F_2 := \int_{t_1}^{t_2} \|f^2 + (t - t_2)\bar{\partial}f^2 - f(t)\|_{J_*}^2 dt.$$

Combining Theorems 6.2, 6.3 and 6.4 one arrives at the following a-posteriori estimates.

Corollary 6.5.

Under the assumptions of Theorems 6.2, 6.3 and 6.4 the semi-discrete solutions $(\tilde{U}^n, U^n)$ satisfy the a-posteriori estimates

$$\begin{aligned} \|\tilde{U} - u\|_{L^\infty(L^2)}^2 + \|U - u\|_{L^\infty(L^2)}^2 &\lesssim \tau^4(\epsilon_1 + \epsilon_2 + \epsilon_{2,0} + \epsilon_4 + \epsilon_9) + \tau^3\epsilon_6 + \mathcal{F}, \\ \|\nabla(\tilde{U} - u)\|_{L^2(L^2)}^2 + \|\nabla(U - u)\|_{L^2(L^2)}^2 &\lesssim \tau^4(\epsilon_1 + \epsilon_2 + \epsilon_{2,0}) + \tau^3(\epsilon_6 + \epsilon_{10}) + \mathcal{F}, \\ \|\tilde{U} - u\|_{L^2(L^2)}^2 + \|U - u\|_{L^2(L^2)}^2 &\lesssim \tau^4(\epsilon_2 + \epsilon_{2,0}\epsilon_3 + \epsilon_5) + \mathcal{F}, \end{aligned}$$

where the estimators are denoted in Definition 6.1. Note that the initial values are included in the estimators, cf. Eqs. (6.5)–(6.6).

Clearly, $\mathcal{F}$ is of second order with respect to the time step size. The next theorem shows that also the remaining ϵ -terms are yielding optimal order estimates for the velocity.

Theorem 6.6 (Stability for the estimates for BDF2).

Suppose the assumptions and conditions of Corollary 6.5 hold. Then, by using the trapezoidal rule for calculating the initial values, the estimators $\epsilon_1, \dots, \epsilon_6, \epsilon_9, \epsilon_{10}$ defined in (6.5) for the velocity are stable, i.e.

$$\epsilon_1 + \epsilon_2 + \epsilon_{2,0} + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_9 + \epsilon_{10} \leq C(u, p, f),$$

where C is independent of τ .

Note that the above theorem yields the same asymptotic estimates as the a-priori estimates in Theorem 4.9:

$$\begin{aligned} \|\tilde{U} - u\|_{L^2(L^2)}^2 + \|U - u\|_{L^2(L^2)}^2 &\lesssim \tau^4, \\ \|\tilde{U} - u\|_{L^\infty(L^2)}^2 + \|U - u\|_{L^\infty(L^2)}^2 + \|\nabla(\tilde{U} - u)\|_{L^2(L^2)}^2 + \|\nabla(U - u)\|_{L^2(L^2)}^2 &\lesssim \tau^3. \end{aligned}$$

6.2 Computational results for the rotational pressure-correction scheme

To verify the analysis of the above Theorems, the same example as for the standard incremental Euler scheme in Chapter 5.2 is used.

$$\begin{aligned} u(t, x) &= \pi \sin(t + 0.2) [\sin(2\pi x_2) \sin(\pi x_1)^2, -\sin(2\pi x_1) \sin(\pi x_2)^2]^\top, \\ p(t, x) &= \sin(t + 0.2) \cos(\pi x_1) \sin(\pi x_2) \end{aligned}$$

with a right hand side f , such that the above pair (u, p) is a solution to (2.1) for $\nu = 0.1$ on $\Omega =]-0.5, 0.5[^2$.

Recall the a-posteriori estimators of Definition 6.1:

$$\begin{aligned}
 \epsilon_1 &:= \|\nabla \bar{\partial}^2 \tilde{U}\|_{\ell^2(2,N;L^2)}^2, & \epsilon_2 &:= \|\bar{\partial}^3 \tilde{U}\|_{\ell^2(3,N;J^*)}^2, & \epsilon_3 &:= \|\bar{\partial}^2 \tilde{U}\|_{\ell^2(2,N;L^2)}^2, \\
 \epsilon_4 &:= \frac{1}{\tau^2} \|\nabla \Phi\|_{\ell^\infty(L^2)}^2, & \epsilon_5 &:= \frac{1}{\tau^2} \|\nabla \Phi\|_{\ell^2(L^2)}^2, & \epsilon_6 &:= \frac{1}{\tau} \|\nabla \bar{\partial} \Phi\|_{\ell^2(L^2)}^2, \\
 \epsilon_7 &:= \|\bar{\partial}^3 \tilde{U}\|_{\ell^2(4,N;L^2)}^2, & \epsilon_8 &:= \|\bar{\partial}^4 \tilde{U}\|_{\ell^2(4,N;J^*)}^2, & \epsilon_9 &:= \|\bar{\partial}^2 \tilde{U}\|_{\ell^\infty(2,N;L^2)}^2, \\
 \epsilon_{10} &= \frac{1}{\tau^3} \|\operatorname{div} \tilde{U}\|_{\ell^2(L^2)}^2, & \epsilon_{2,0} &:= \frac{1}{\tau} \|\bar{\partial}^2 \tilde{U}^2 - \bar{\partial}(\Delta \tilde{U}^1 + f^1)\|_{J^*}^2
 \end{aligned}$$

with $\Phi^0 = 0$. Define

$$\begin{aligned}
 \Theta_{BDF,1}^2 &:= \tau^4(\epsilon_1 + \epsilon_2 + \epsilon_4 + \epsilon_9) + \tau^3 \epsilon_6, & \Theta_{BDF,2}^2 &:= \tau^4(\epsilon_1 + \epsilon_2) + \tau^3(\epsilon_6 + \epsilon_{10}), \\
 \Theta_{BDF,3}^2 &:= \tau^4(\epsilon_1 + \epsilon_7 + \epsilon_8) + \tau^3(\epsilon_6 + \epsilon_{10}), & \Theta_{BDF,4}^2 &:= \tau^4(\epsilon_2 + \epsilon_3 + \epsilon_5).
 \end{aligned}$$

Thus Corollary 6.5 can be formulated as

$$\begin{aligned}
 \|\tilde{U} - u\|_{L^\infty(L^2)}^2 &\lesssim \Theta_{BDF,1}^2 + \tau^4 \epsilon_{2,0} + \mathcal{F}, \\
 \|\nabla(\tilde{U} - u)\|_{L^2(L^2)}^2 &\lesssim \Theta_{BDF,2}^2 + \tau^4 \epsilon_{2,0} + \mathcal{F}, \\
 \|\tilde{U} - u\|_{L^2(L^2)}^2 &\lesssim \Theta_{BDF,4}^2 + \tau^4 \epsilon_{2,0} + \mathcal{F},
 \end{aligned}$$

and the pressure estimates Eq. (6.9) can be rewritten as

$$\|P - p\|_{L^2(L^2)}^2 \lesssim \eta_P^0 + \Theta_{BDF,3}^2 + \mathcal{F}_2 + \mathcal{F}_3.$$

As in Chapter 5.2, only the Θ_{BDF} -terms are presented, since the other terms are only data approximation terms. Fig. 6.1 shows that the estimators for the $L^\infty(L^2)$ and $L^2(V)$ -norms are of the expected order of $\tau^{3/2}$. In the present example, the true error in the $L^\infty(L^2)$ -norm is slightly super-convergent with rate τ^2 . Again, the stagnation of the error in the $L^2(V)$ -norm for smaller time step sizes is caused by the space discretization error that eventually becomes dominant for very small values of τ .

The $L^2(L^2)$ pressure error decays as $\tau^{1.5}$, as expected and predicted by a-priori theory. The estimator, however, is slightly suboptimal with a rate of approximately $\tau^{1.3}$.

The behavior of the estimator for the $L^2(L^2)$ -norm is in perfect agreement with the theoretical result as well as with the correct error.

Remark 6.7.

Note that in order to get an optimal convergence rate for term

$$\epsilon_2 = \sum_{n=3}^N \tau \|\bar{\partial}^3 \tilde{U}^n\|_{J^*}^2,$$

6 The rotational pressure-correction scheme discretized by BDF2

Figure 6.1: BDF2: errors and (scaled) estimators versus time step size; $\|u - \tilde{U}\|_{L^\infty(L^2)}$ and $\Theta_{BDF,1}$; $\|\nabla(u - \tilde{U})\|_{L^2(L^2)}^2$ and $\Theta_{BDF,2}$; $\|P - p\|_{L^2(L^2)}^2$ and $\Theta_{BDF,3}$; $\|u - \tilde{U}\|_{L^2(L^2)}$ and $\Theta_{BDF,4}$ (from top left to bottom right); triangles have slope 1.5 except for the bottom right picture, where the slope is 2.0.

it is mandatory to use the J^* -norm. The reason for using this negative norm is to cancel a gradient term in deriving Eq. (6.35) in Lemma 6.12 that otherwise could not be controlled with sufficient order. Note that Lemma 6.12 in turn is instrumental to finally prove the stability Theorem 6.6, see Chapter 6.3.5 below.

Fig. 6.2 reports on the results using the $L^2(\Omega)$, V^* , and J^* -norm, for the ϵ_2 term, confirming the necessity to use the J^* -norm. The ϵ_2 term has a rate of τ^2 in the J^* -norm and $\tau^{3/2}$, both in the V^* norm and in the L^2 -norm. On the space-continuous level computing the J^* (H^{-1})-norm requires solving a Stokes (Poisson) problem, see Eq. (2.5). In the fully discrete case, this norm was approximated using the solution of a discretized Stokes (Poisson) problem. Note that this is completely prohibitive in practice.

Figure 6.2: Estimator term $\tau^2 \left(\sum_n \tau \|\partial^3 \tilde{U}^n\|_X^2 \right)^{1/2}$ for $X = L^2(\Omega)^d$, $X = V^*$ and $X = J^*$; triangle has slope 2.0.

6.3 Proofs of Theorems

Before proving the theorems some lemmata have to be provided.

6.3.1 Auxiliary statements

In the first lemma bounds are given for divergence of the three-point reconstruction of the provisional velocity in negative norms.

Lemma 6.8.

Assume $\tilde{U}^0 = U^0 = u_0$ and $(\tilde{U}^1, U^1, P^1)$ be given. Then the solutions $\tilde{U}^n$ and U^n of Eqs. (6.1)–(6.2) fulfill

$$\begin{aligned} \|\operatorname{div} \hat{\tilde{U}}\|_{L^\infty(t_2, T; H^1(\Omega)^*)}^2 &\lesssim \max \left\{ \|U^1 - \tilde{U}^1\|^2, \tau^2 \|\nabla \Phi\|_{\ell^\infty(2, N; L^2)}^2 \right\}, \\ \|\tilde{U} - U\|_{L^2(t_2, T; L^2)}^2 &\lesssim \tau \|\tilde{U}^1 - U^1\|^2 + \tau^2 \|\nabla \Phi\|_{\ell^2(L^2)}^2, \\ \|\operatorname{div} \hat{\tilde{U}}\|_{L^2(t_2, T; L^2)}^2 &\lesssim \|\operatorname{div} \tilde{U}\|_{\ell^2(L^2)}^2, \\ \|\operatorname{div} \partial_t \hat{\tilde{U}}\|_{L^2(t_2, T; H^2(\Omega)^*)}^2 &\lesssim \frac{1}{\tau} \|\tilde{U}^1 - U^1\|^2 + \tau^2 \|\nabla \bar{\partial} \Phi\|_{\ell^2(2, N; L^2)}^2. \end{aligned}$$

Proof : The first and fourth terms are estimated as follows

$$\begin{aligned}
 \|\operatorname{div} \hat{\tilde{U}}\|_{L^\infty(t_2, T; H^1(\Omega)^*)}^2 &\lesssim \max_{1 \leq n \leq N} \|\operatorname{div}(\tilde{U}^n - U^n)\|_{H^1(\Omega)^*}^2 \lesssim \max_{1 \leq n \leq N} \|\tilde{U}^n - U^n\|^2, \\
 \|\operatorname{div} \partial_t \hat{\tilde{U}}\|_{L^2(t_2, T; H^2(\Omega)^*)}^2 &\lesssim \sum_{n=1}^N \tau \|\operatorname{div} \bar{\partial} \tilde{U}^n\|_{H^2(\Omega)^*}^2 = \sum_{n=1}^N \tau \sup_{q \in H^2(\Omega)} \frac{|\langle \operatorname{div} \bar{\partial} \tilde{U}^n, q \rangle|^2}{\|q\|_2^2} \\
 &= \sum_{n=1}^N \tau \sup_{q \in H^2(\Omega)} \frac{|\langle \operatorname{div} \bar{\partial}(\tilde{U}^n - U^n), q \rangle|^2}{\|q\|_2^2} \\
 &= \sum_{n=1}^N \tau \sup_{q \in H^2(\Omega)} \frac{|\langle \bar{\partial}(\tilde{U}^n - U^n), \nabla q \rangle|^2}{\|q\|_2^2} \\
 &\leq \frac{1}{\tau} \|\tilde{U}^1 - U^1\|^2 + \sum_{n=2}^N \tau \|\bar{\partial}(\tilde{U}^n - U^n)\|^2.
 \end{aligned}$$

The rest of the proof follows directly from Eq. (6.3). $\square$

In the next Lemma a bound of the time derivative of the error is given. The proof of lemma is quite technical and requires to introduce an interpolant of the BDF2 term:

$$d\tilde{U}(t) := \bar{\partial}^B \tilde{U}^n + (t - t_n) \bar{\partial}^2 \tilde{U}^n + \frac{1}{2} (t - t_n) (\tau + (t - t_{n-1})) \bar{\partial}^3 \tilde{U}^n \quad (6.14)$$

for all $t \in I_n$. $dU(t)$ is defined correspondingly. It is easily verified that $d\tilde{U}(t)$ is continuous. Its time derivative is given by

$$\partial_t d\tilde{U}(t) = \bar{\partial}^2 \tilde{U}^n + \tau \bar{\partial}^3 \tilde{U}^n + (t - t_n) \bar{\partial}^3 \tilde{U}^n. \quad (6.15)$$

Lemma 6.9.

Under the assumptions of Lemma 6.8 the interpolant $d\tilde{U}(t)$, defined in (6.14), fulfills

$$\begin{aligned}
 \|d\tilde{U}(t) - \partial_t u(t)\|_{L^2(t_3, T; L^2)}^2 &\lesssim \frac{1}{\tau} \|\tilde{U}^1 - U^1\|^2 + \tau \|\bar{\partial}(\tilde{U}^2 - U^2)\|^2 + \|\nabla(\tilde{U}^3 - u(t_3))\|^2 \\
 &+ \tau^4 \left(\|\bar{\partial}^4 \tilde{U}\|_{\ell^2(4, N; J^*)}^2 + \|\nabla \bar{\partial}^3 \tilde{U}\|_{\ell^2(4, N; L^2)}^2 \right) + \tau^2 \|\nabla \bar{\partial} \Phi\|_{\ell^2(3, N; L^2)}^2 \\
 &+ \sum_{n=4}^N \int_{t_{n-1}}^{t_n} \|\bar{\partial}^B f^n + (t - t_n) \bar{\partial}^2 f^n - \partial_t f\|_{J^*}^2 dt.
 \end{aligned}$$

Proof : Applying $\bar{\partial}^B = \frac{\tau}{2} \bar{\partial}^2 + \bar{\partial}$ to Eq. (6.4), yields for $n \geq 4$:

$$\frac{\tau^2}{4} \langle \bar{\partial}^4 \tilde{U}^n, v \rangle + \tau \langle \bar{\partial}^3 \tilde{U}^n, v \rangle + \langle \bar{\partial}^2 \tilde{U}^n, v \rangle - \langle \Delta \bar{\partial}^B \tilde{U}^n, v \rangle = \bar{\partial}^B f^n \quad \forall v \in J.$$

Rearrange this equation and combine it with Eqs. (6.14) and (6.15) to obtain

$$\begin{aligned}
 & \langle \partial_t d\tilde{U}(t) - \Delta d\tilde{U}(t), v \rangle \\
 &= -\frac{\tau^2}{4} \langle \bar{\partial}^4 \tilde{U}^n, v \rangle + (t - t_n) \langle \bar{\partial}^2 (\bar{\partial} \tilde{U}^n - \Delta \tilde{U}^n), v \rangle \\
 &\quad - \frac{1}{2} (t - t_n) (\tau + (t - t_{n-1})) \langle \Delta \bar{\partial}^3 \tilde{U}^n, v \rangle + \langle \bar{\partial}^B f^n, v \rangle \\
 &= \left(-\frac{\tau^2}{4} - \frac{\tau}{2} (t - t_n) \right) \langle \bar{\partial}^4 \tilde{U}^n, v \rangle - \frac{1}{2} (t - t_n) (\tau + (t - t_{n-1})) \langle \Delta \bar{\partial}^3 \tilde{U}^n, v \rangle \\
 &\quad + \langle \bar{\partial}^B f^n + (t - t_n) \bar{\partial}^2 f^n, v \rangle.
 \end{aligned}$$

Set $\tilde{\omega}(t) := d\tilde{U}(t) - u_t(t)$, $\omega(t) := dU(t) - u_t(t)$, $t \in I_n$, $n \geq 4$, and derive the error equation by subtracting the above equation from the time derivative of (2.1):

$$\langle \partial_t \tilde{w}(t), v \rangle - \langle \Delta \tilde{w}(t), v \rangle = \langle T_1, v \rangle + \langle \Delta T_2, v \rangle + \langle \bar{\partial}^B f^n + (t - t_n) \bar{\partial}^2 f^n - f_t(t), v \rangle, \quad (6.16)$$

for $v \in J$, $t \in I_n$, where

$$T_1 := \left(-\frac{\tau^2}{4} + \frac{\tau}{2} (t - t_n) \right) \bar{\partial}^4 \tilde{U}^n, \quad T_2 := -\frac{1}{2} (t - t_n) (\tau + (t - t_{n-1})) \bar{\partial}^3 \tilde{U}^n.$$

Now, following the line of Lemma 5.7, let $(v(t), r(t))$ be the solution of the Stokes system

$$-\Delta v(t) + \nabla r(t) = \tilde{\omega}(t), \quad \operatorname{div} v(t) = 0, \quad v(t)|_\Gamma = 0,$$

i.e. $v(t) = \mathcal{A}^{-1} \tilde{\omega}(t)$. Regularity of the Stokes system gives

$$\|v(t)\|_s + \|r(t)\|_{s-1} \lesssim \|\tilde{\omega}(t)\|_{s-2}, \quad s = 1, 2. \quad (6.17)$$

Take the above v as test function in Eq. (6.16). Then for the first term on the left hand side one obtains

$$\langle \partial_t \tilde{\omega}(t), v(t) \rangle = \langle \partial_t (-\Delta v(t) + \nabla r(t)), v(t) \rangle = \frac{1}{2} \frac{d}{dt} \|\nabla v(t)\|^2.$$

For the second term on the left hand side integrating by parts yields

$$\begin{aligned}
 \langle -\Delta \tilde{\omega}(t), v(t) \rangle &= \langle \tilde{\omega}(t), -\Delta v(t) \rangle = \|\tilde{\omega}(t)\|^2 - \langle \nabla r(t), \tilde{\omega}(t) \rangle \\
 &\geq \|\tilde{\omega}(t)\|^2 - \|\nabla r(t)\| \|\tilde{\omega}(t) - \omega(t)\| \\
 &\geq \|\tilde{\omega}(t)\|^2 - C \|\tilde{\omega}(t)\| \|\tilde{\omega}(t) - \omega(t)\| \\
 &\geq \frac{1}{2} \|\tilde{\omega}(t)\|^2 - C \|d\tilde{U}(t) - dU(t)\|^2,
 \end{aligned}$$

where the regularity estimate Eq. (6.17) for $r(t)$ with $s = 2$ and Young's inequality were used.

The terms of the right hand side in Eq. (6.16) are treated as follows:

$$\begin{aligned}\langle T_1, v(t) \rangle &\lesssim \tau^2 \|\bar{\partial}^4 \tilde{U}^n\|_{J^*} \|\nabla v(t)\|, \\ \langle \Delta T_2, v(t) \rangle &\lesssim \|\nabla T_2\| \|\nabla v(t)\| \lesssim \tau^2 \|\nabla \bar{\partial}^3 \tilde{U}^n\| \|\nabla v(t)\|, \\ \langle \bar{\partial}^B f^n + (t - t_n) \bar{\partial}^2 f^n - f_t(t), v \rangle &\lesssim \|\bar{\partial}^B f^n + (t - t_n) \bar{\partial}^2 f^n - f_t(t)\|_{J^*} \|\nabla v(t)\|.\end{aligned}$$

Collecting the above estimates and integrating from $t = t_3$ to T , yields

$$\begin{aligned}\max_{t \in [t_3, T]} \|\nabla v(t)\|^2 + \int_{t_3}^T \|\tilde{\omega}(t)\|^2 dt &\lesssim \|\nabla v(t_3)\|^2 + \int_{t_3}^T \|d\tilde{U}(t) - dU(t)\|^2 \\ &\quad + \sum_{n=4}^N \int_{t_{n-1}}^{t_n} \|\nabla v(t)\| \left(\tau^2 \|\bar{\partial}^4 \tilde{U}^n\|_{J^*} + \tau^2 \|\nabla \bar{\partial}^3 \tilde{U}^n\| \right. \\ &\quad \left. + \|\bar{\partial}^B f^n + (t - t_n) \bar{\partial}^2 f^n - \partial_t f\|_{J^*} \right). \quad (6.18)\end{aligned}$$

With Eq. (2.5) one gets

$$\|\nabla v(t_3)\| \lesssim \|d\tilde{U}(t_3) - u_t(t_3)\|_{J^*} = \|\bar{\partial}^B \tilde{U}^3 - u_t(t_3)\|_{J^*} = \|\Delta(\tilde{U}^3 - u(t_3))\|_{J^*} \lesssim \|\nabla(\tilde{U}^3 - u(t_3))\|.$$

For the second term on the right hand side of Eq. (6.18), using Eq. (6.2), one gets

$$\begin{aligned}\int_{t_3}^T \|d\tilde{U}(t) - dU(t)\|^2 &\lesssim \sum_{n=1}^N \tau \|\bar{\partial}(\tilde{U}^n - U^n)\|^2 \\ &= \frac{1}{\tau} \|\tilde{U}^1 - U^1\|^2 + \tau \|\bar{\partial}(\tilde{U}^2 - U^2)\|^2 + \tau^2 \sum_{n=3}^N \tau \|\nabla \bar{\partial} \Phi^n\|^2.\end{aligned}$$

Applying Young's inequality and then the Gronwall Lemma to Eq. (6.18) proves the lemma. $\square$

In the following lemma Eq. (6.4) is reformulated to a continuous equation in time, which is necessary to obtain an error equation. Using the three-point reconstruction (2.22) and the linear interpolant (2.21) one gets

Lemma 6.10.

For $n \geq 3$, $t \in I_n$ and $v \in J$ it holds

$$\langle \partial_t \hat{\tilde{U}}(t), v \rangle + \langle \nabla \tilde{U}(t), \nabla v \rangle = \langle F(t) - \varphi(t), v \rangle, \quad (6.19)$$

where $F(t) = f^n + (t - t_n) \bar{\partial} f^n$ and $\varphi(t) = (t - t_n) \frac{\tau}{2} \bar{\partial}^3 \tilde{U}^n$.

Proof : Using the definition of the reconstructions (2.21) and (2.23) one deduces from Eq. (6.4) for $v \in J$, $n \geq 3$, $t \in I_n$:

$$\begin{aligned}\langle \partial_t \hat{\tilde{U}}(t), v \rangle + \langle \nabla \tilde{U}(t), \nabla v \rangle &= \langle \bar{\partial}^B \tilde{U}^n, v \rangle + \langle \nabla \tilde{U}^n, \nabla v \rangle + (t - t_n) (\langle \bar{\partial}^2 \tilde{U}^n, v \rangle + \langle \nabla \bar{\partial} \tilde{U}^n, \nabla v \rangle) \\ &= f^n + (t - t_n) \left(\langle \bar{\partial}^2 \tilde{U}^n, v \rangle + \langle \nabla \bar{\partial} \tilde{U}^n, \nabla v \rangle \right) \\ &= f^n + (t - t_n) \bar{\partial} f^n - (t - t_n) \frac{\tau}{2} \langle \bar{\partial}^3 \tilde{U}^n, v \rangle \\ &=: \langle F(t) - \varphi(t), v \rangle.\end{aligned}$$

□

6.3.2 Proof of Theorem 6.2

Theorem 6.2 is proved following the procedure of the Euler case in Theorem 5.4. To get optimality of the error estimators the three-point reconstruction (2.22) instead of the linear interpolant (2.21) has to be used.

Proof : (**Theorem 6.2**) Subtracting Eq. (2.1) from Eq. (6.19) and using the Stokes projection yields the error equation for $t \in I_n$, $n \geq 3$ and for all $v \in J$:

$$\langle \partial_t(\mathcal{S}\hat{U} - u)(t), v \rangle + \langle \nabla(\mathcal{S}\tilde{U} - u)(t), \nabla v \rangle = \langle (\mathcal{S} - I)\partial_t\hat{U}(t), v \rangle + \langle F(t) - f(t) - \varphi(t), v \rangle.$$

Set $v = (\mathcal{S}\hat{U} - u)(t)$:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|(\mathcal{S}\hat{U} - u)(t)\|^2 + \langle \nabla(\mathcal{S}\tilde{U} - u)(t), \nabla(\mathcal{S}\hat{U} - u)(t) \rangle \\ \leq \left(\|F(t) - f(t) - \varphi(t)\|_{J^*} + \|(\mathcal{S} - I)\partial_t\hat{U}(t)\|_{J^*} \right) \|\nabla(\mathcal{S}\hat{U} - u)(t)\|. \end{aligned}$$

Applying the identity (2.16) to the second term on the left hand side yields

$$\begin{aligned} \frac{d}{dt} \|(\mathcal{S}\hat{U} - u)(t)\|^2 + \|\nabla(\mathcal{S}\tilde{U} - u)(t)\|^2 + \frac{1}{2} \|\nabla(\mathcal{S}\hat{U} - u)(t)\|^2 \\ \leq \|\nabla\mathcal{S}(\hat{U} - \tilde{U})\|^2 + 4\|F(t) - f(t) - \varphi(t)\|_{J^*}^2 + 4\|(\mathcal{S} - I)\partial_t\hat{U}(t)\|_{J^*}^2. \end{aligned} \quad (6.20)$$

Integrating Eq. (6.20) from $t = t_2$ to T and using the stability of $\mathcal{S}$ implies

$$\begin{aligned} \max_{t \in [t_2, T]} \|(\mathcal{S}\hat{U} - u)(t)\|^2 + \int_{t_2}^T \left(\|\nabla(\mathcal{S}\tilde{U} - u)(t)\|^2 + \|\nabla(\mathcal{S}\hat{U} - u)(t)\|^2 \right) \\ \lesssim \|(\mathcal{S}\hat{U} - u)(t_2)\|^2 + \int_{t_2}^T \left(\|\nabla(\hat{U} - \tilde{U})\|^2 \right. \\ \left. + \|F(t) - f(t) - \varphi(t)\|_{J^*}^2 + \|(\mathcal{S} - I)\partial_t\hat{U}(t)\|_{J^*}^2 \right). \end{aligned} \quad (6.21)$$

Using the triangle inequality, the fact that $\hat{U}(t_2) = \tilde{U}(t_2)$, Lemma 2.2, and Eq. (6.3) the first term on the right hand side can be bounded by

$$\begin{aligned} \|(\mathcal{S}\hat{U} - u)(t_2)\| &\lesssim \|(\tilde{U} - u)(t_2)\| + \|(\mathcal{S}\tilde{U} - \tilde{U})(t_2)\| \\ &\leq \|(\tilde{U} - u)(t_2)\| + \|\operatorname{div} \tilde{U}(t_2)\|_{H^1(\Omega)^*} \\ &= \|(\tilde{U} - u)(t_2)\| + \|\operatorname{div}(\tilde{U}(t_2) - U(t_2))\|_{H^1(\Omega)^*} \\ &\leq \|(\tilde{U} - u)(t_2)\| + \|\tilde{U}^2 - U^2\| \\ &\lesssim \|(\tilde{U} - u)(t_2)\| + \tau^2 \|\nabla \varphi^2\|. \end{aligned}$$

The second term is estimated using the definition of the reconstruction Eq. (2.22):

$$\|\nabla(\hat{\tilde{U}} - \tilde{U})(t)\| = \left\| \frac{1}{2}(t - t_n)(t - t_{n-1})\nabla\bar{\partial}^2\tilde{U}^n \right\| \leq \frac{1}{2}\tau^2\|\nabla\bar{\partial}^2\tilde{U}^n\|.$$

The estimate for $\|F(t) - f(t) - \varphi(t)\|_{J^*}^2$ in Eq. (6.21) follows directly from its definition in Eq. (6.19). It remains to estimate $\|(\mathcal{S} - I)\partial_t\hat{\tilde{U}}(t)\|_{J^*}^2$. By Lemma 2.2 one has

$$\|(\mathcal{S} - I)\partial_t\hat{\tilde{U}}(t)\|_{J^*}^2 \lesssim \|\operatorname{div}\partial_t\hat{\tilde{U}}(t)\|_{H^2(\Omega)^*}^2,$$

which can be further estimated by Lemma 6.8.

Putting everything together, the error $\|\tilde{U} - u\|$ in Eq. (6.7) can be bounded by using the triangle inequality

$$\|(\tilde{U} - u)(t)\| \leq \|\tilde{U}(t) - \mathcal{S}\hat{\tilde{U}}(t)\| + \|\mathcal{S}\hat{\tilde{U}}(t) - u(t)\|$$

and then Lemma 2.2 applied to (6.21):

$$\|\tilde{U}(t) - \mathcal{S}\hat{\tilde{U}}(t)\| \leq \|(I - \mathcal{S})\hat{\tilde{U}}\| + \frac{1}{2}\|(t - t_n)(t - t_{n-1})\bar{\partial}^2\tilde{U}^n\| \leq \|\operatorname{div}\hat{\tilde{U}}\|_{H^1(\Omega)^*} + \frac{\tau^2}{2}\|\bar{\partial}^2\tilde{U}^n\|.$$

Finally, apply Lemma 6.8 to bound the divergence term in the above estimate.

The estimate for $\|U - u\|$ is derived from Lemma 2.1 and again using the triangle inequality:

$$U - u = (U - \tilde{U}) + (\tilde{U} - u) = (\mathcal{H} - I)\tilde{U} + (\tilde{U} - u).$$

In a similar way, again using Lemma 2.2 and Lemma 2.1 Eq. (6.8) can be proved.

For proving the pressure estimate (6.9) use the linear interpolant of the pressure

$$P(t) = P^n + (t - t_n)\bar{\partial}P^n \quad \text{for } t \in I_n.$$

Note that Eq. (4.7) implies

$$\nabla P(t) = -\bar{\partial}^B U^n + \triangle U^n + (t - t_n) \left(-\bar{\partial}\bar{\partial}^B U^n + \triangle\bar{\partial}U^n \right) + F(t).$$

From the inf-sup condition, using the interpolants $U(t)$ and $dU(t)$, one gets

$$\begin{aligned} \|P(t) - p(t)\| &\lesssim \sup_{v \in V} \frac{\langle \nabla P(t) - \nabla p(t), v \rangle}{\|\nabla v\|} \\ &\lesssim \left\| \bar{\partial}^B U^n + (t - t_n) \left(\bar{\partial}\bar{\partial}^B U^n \right) - \partial_t u(t) \right\|_{-1} + \|\nabla(U - u)(t)\| + \|F(t) - f(t)\|_{-1} \\ &\lesssim \|dU(t) - \partial_t u(t)\|_{-1} + \tau^2 \|\bar{\partial}^3 \tilde{U}^n\|_{-1} + \|\nabla(U - u)(t)\| + \|F(t) - f(t)\|_{-1} \\ &\lesssim \|\mathcal{H}(d\tilde{U}(t) - \partial_t u(t))\| + \tau^2 \|\bar{\partial}^3 \tilde{U}^n\|_{-1} + \|\nabla(U - u)(t)\| + \|F(t) - f(t)\|_{-1} \\ &\lesssim \|d\tilde{U}(t) - \partial_t u(t)\| + \tau^2 \|\bar{\partial}^3 \tilde{U}^n\|_{-1} + \|\nabla(U - u)(t)\| + \|F(t) - f(t)\|_{-1}, \end{aligned}$$

where the fact $\bar{\partial}\bar{\partial}^B = \bar{\partial}^2 + \frac{\tau}{2}\bar{\partial}^3$, and the stability of the Helmholtz operator were used. Squaring the above inequality, integrating from $t = t_2$ to T , using Lemma 6.9 and Eq. (6.8) proves the pressure estimate. $\square$

6.3.3 Proof of Theorem 6.3

PROOF : (Theorem 6.3) Denote $\tilde{e} := \tilde{U} - u$ and $\hat{e} := \hat{U} - u$. Subtract Eq. (2.1) from Eq. (6.19) and set $v = \mathcal{A}^{-1}\hat{e}(t)$ to get for $t \in I_n$, $n \geq 3$,

$$\langle \partial_t \hat{e}(t), \mathcal{A}^{-1}\hat{e}(t) \rangle + \langle \nabla \tilde{e}(t), \nabla \mathcal{A}^{-1}\hat{e}(t) \rangle \leq \|F(t) - f(t) - \varphi(t)\|_{J^*} \|\nabla \mathcal{A}^{-1}\hat{e}(t)\|. \quad (6.22)$$

First consider the term $\langle \partial_t \hat{e}, \mathcal{A}^{-1}\hat{e} \rangle$. Let $(w(t), r(t)) \in V \times W$ be the solution of

$$-\Delta w(t) + \nabla r(t) = \hat{e}(t), \quad \operatorname{div} w(t) = 0, \quad (6.23)$$

hence $w(t) = \mathcal{A}^{-1}\hat{e}(t)$. The following regularity estimate holds:

$$\|w(t)\|_s + \|r(t)\|_{s-1} \lesssim \|\hat{e}(t)\|_{s-2}, \quad s = 1, 2.$$

Taking the time derivative of Eq. (6.23) one gets $w_t(t) = \mathcal{A}^{-1}\partial_t \hat{e}(t)$. Thus

$$\langle \partial_t \hat{e}, \mathcal{A}^{-1}\hat{e} \rangle = \langle -\Delta w_t + \nabla r_t, w \rangle = \langle \nabla w_t, \nabla w \rangle = \frac{1}{2} \frac{d}{dt} \|\nabla w(t)\|^2 = \frac{1}{2} \frac{d}{dt} \|\nabla \mathcal{A}^{-1}\hat{e}\|^2.$$

Using the definition of $\hat{U}$ the second term is split as

$$\langle \nabla \tilde{e}, \nabla \mathcal{A}^{-1}\hat{e} \rangle = \langle \nabla \tilde{e}, \nabla \mathcal{A}^{-1}\tilde{e} \rangle + \frac{1}{2}(t - t_n)(t - t_{n-1}) \langle \nabla \tilde{e}, \nabla \mathcal{A}^{-1}\bar{\partial}^2 \tilde{U}^n \rangle.$$

Solving again Eq. (6.23), but now with right hand side $\tilde{e}(t)$ instead of $\hat{e}(t)$, and denoting the corresponding dual solution by $(w(t), r(t))$, leads to

$$\begin{aligned} \langle \nabla \tilde{e}(t), \nabla \mathcal{A}^{-1}\tilde{e}(t) \rangle &= \langle \tilde{e}, -\Delta \mathcal{A}^{-1}\tilde{e} \rangle = \langle \tilde{e}, -\Delta w \rangle \\ &= \langle \tilde{e}, \tilde{e} - \nabla r \rangle = \|\tilde{e}\|^2 - \langle \nabla r, \tilde{e} - e \rangle \\ &\geq \|\tilde{e}\|^2 - \|\nabla r\| \|\tilde{e} - e\| \\ &\gtrsim \|\tilde{e}\|^2 - \|\tilde{e}\| \|\tilde{e} - e\| \\ &\geq (1 - \varepsilon) \|\tilde{e}(t)\|^2 - C_\varepsilon \|(\tilde{e} - e)(t)\|^2 \end{aligned}$$

for $\varepsilon \in (0, 1)$. Last, define the right hand side in Eq. (6.23) as

$$\xi := \frac{1}{2}(t - t_n)(t - t_{n-1}) \bar{\partial}^2 \tilde{U}^n,$$

denote the corresponding dual solution again by (w, r) , and apply a similar computation as above. This gives

$$-\langle \nabla \tilde{e}, \nabla \mathcal{A}^{-1}\xi \rangle \leq \varepsilon \|\tilde{e}\|^2 + C_\varepsilon \|\xi\|^2.$$

Putting everything together and choosing ε sufficiently small, modifies Eq. (6.22) to

$$\frac{1}{2} \frac{d}{dt} \|\nabla \mathcal{A}^{-1}\hat{e}(t)\|^2 + \|\tilde{e}\|^2 \lesssim \|\xi\|^2 + \|\tilde{e} - e\|^2 + \|F(t) - f(t) - \varphi\|_{J^*} \|\nabla \mathcal{A}^{-1}\hat{e}(t)\|.$$

Integrating from $t = t_2$ to T and applying Gronwall's Lemma gives

$$\begin{aligned} \max_{t \in [t_2, T]} \|\hat{e}\|_{J^*}^2 + \int_{t_2}^T \|\tilde{e}(t)\|^2 dt \\ \lesssim \|\hat{e}(t_2)\|_{J^*}^2 + \int_{t_2}^T (\|\xi\|^2 + \|\tilde{e} - e\|^2 + \|F(t) - f(t) - \varphi\|_{J^*}^2) dt, \end{aligned}$$

where relation (2.5) was used. The first term of the right hand side results in

$$\|\hat{e}(t_2)\|_{J^*} = \|\tilde{U}^2 - u(t_2)\|_{J^*}.$$

For the second and third terms one obtains

$$\begin{aligned} \int_{t_2}^T (\|\xi\|^2 + \|\tilde{e} - e\|^2) dt &\leq \tau^4 \sum_{n=3}^N \tau \|\bar{\partial}^2 \tilde{U}^n\|^2 + \sum_{n=2}^N \tau \|\tilde{U}^n - U^n\|^2 \\ &= \tau^4 \sum_{n=3}^N \tau \|\bar{\partial}^2 \tilde{U}^n\|^2 + \tau^2 \sum_{n=2}^N \tau \|\nabla \Phi^n\|^2. \end{aligned}$$

To complete the proof the term $\|F(t) - f(t) - \varphi\|_{J^*}^2$ is directly estimated from its definition in Eq. (6.19). $\square$

6.3.4 Proof of Theorem 6.4

The proof of the initial estimates is quite similar to the proofs of Theorem 6.2 and 6.3, but one has to introduce a modified three-point reconstruction.

Proof : (**Theorem 6.4**) It is

$$\begin{aligned} \eta_U^0 &:= \underbrace{\|\tilde{e}\|_{L^\infty((0, t_2); L^2)}^2}_{A_1} + \underbrace{\|\tilde{U} - U\|_{L^\infty((0, t_2); L^2)}^2}_{A_2} + \underbrace{\frac{1}{\tau} \|\tilde{U}^1 - U^1\|^2}_{A_3}, \\ \eta_{\nabla U}^0 &:= \underbrace{\|\nabla \tilde{e}\|_{L^2((0, t_2); L^2)}^2}_{A_4} + \underbrace{\|\nabla(\tilde{U} - U)\|_{L^2((0, t_2); L^2)}^2}_{A_5} + A_3, \\ \eta_U^1 &:= \underbrace{\|\tilde{U} - u\|_{L^2((0, t_2); L^2)}^2}_{A_6} + \underbrace{\|\tilde{U} - U\|_{L^2((0, t_2); L^2)}^2}_{A_7}. \end{aligned}$$

A_3 one gets immediately by Eq. (4.4):

$$A_3 = \tau \|\nabla \Phi^1\|^2 = \tau^3 \|\nabla \bar{\partial} \Phi^1\|^2.$$

A_2 and A_7 fulfill

$$\begin{aligned} A_7 &\leq 2\tau A_2 \lesssim \tau \max_{i \in \{0, 1, 2\}} \|\tilde{U}^i - U^i\| \\ &\leq \tau (\|\tilde{U}^1 - U^1\|^2 + \|\tilde{U}^2 - U^2\|^2) \\ &= \tau^3 (\|\nabla \Phi^1\|^2 + \|\nabla \Phi^2\|^2). \end{aligned}$$

To bound A_1 let first $t \in (0, t_1]$. Define

$$\hat{\tilde{U}}(t) := \tilde{U}(t) + \frac{1}{2}(t - t_0)(t - t_1)\bar{\partial}^2\tilde{U}^2, \quad t \in I_1,$$

and use Eq. (4.3) and Eq. (2.1) to get

$$\begin{aligned} & \langle \partial_t(\mathcal{S}\hat{\tilde{U}} - u)(t), v \rangle + \langle \nabla(\mathcal{S}\tilde{U} - u)(t), \nabla v \rangle \\ &= \langle \partial_t(\mathcal{S} - I)\tilde{U}(t), v \rangle - (t - t^{1/2})\langle \Delta\bar{\partial}\tilde{U}^1 + \bar{\partial}f^1 - \bar{\partial}^2\tilde{U}^2, v \rangle \\ & \quad + \langle f^1 + (t - t^{1/2})\bar{\partial}f^1 - f(t), v \rangle, \end{aligned} \quad (6.24)$$

for $v \in J$.

Following the lines of Theorem 6.2 one obtains

$$\begin{aligned} & \max_{t \in [0, t_1]} \|(\mathcal{S}\hat{\tilde{U}} - u)(t)\|^2 + \int_0^{t_1} \|\nabla(\mathcal{S}\tilde{U} - u)(t)\|^2 \\ & \lesssim \tau \|\operatorname{div} \bar{\partial}\tilde{U}^1\|_{H^1(\Omega)^*}^2 + \tau^3 \|\bar{\partial}^2\tilde{U}^2 - \bar{\partial}(\Delta\tilde{U}^1 + f^1)\|_{J^*}^2 + F_1 \\ & \lesssim \tau \|\nabla\Phi^1\|^2 + \tau^3 \|\bar{\partial}^2\tilde{U}^2 - \bar{\partial}(\Delta\tilde{U}^1 + f^1)\|_{J^*}^2 + F_1. \end{aligned} \quad (6.25)$$

Now consider $t \in (t_1, t_2]$.

$$\begin{aligned} & \langle \partial_t\hat{\tilde{U}}(t), v \rangle + \langle \nabla\tilde{U}(t), \nabla v \rangle = \langle f^2, v \rangle + (t - t_2)(\langle \bar{\partial}^2\tilde{U}^2, v \rangle + \langle \nabla\bar{\partial}\tilde{U}^2, \nabla v \rangle), \\ &= \langle f^2 + (t - t_2)\bar{\partial}f^2, v \rangle + \frac{t - t_2}{\tau} \langle f^1 - f^{1/2}, v \rangle + \frac{t - t_2}{\tau} \langle \bar{\partial}\tilde{U}^2 - \Delta\tilde{U}^2 - f^2, v \rangle \\ & \quad - \frac{t - t_2}{\tau} \underbrace{\langle \bar{\partial}\tilde{U}^1 - \Delta\tilde{U}^{1/2} - f^{1/2}, v \rangle}_{=0 \ \forall \ v \in J} + \frac{t - t_2}{\tau} \langle \Delta(\tilde{U}^1 - \tilde{U}^{1/2}), v \rangle \\ &= \langle f^2 + (t - t_2)\bar{\partial}f^2, v \rangle - \frac{t - t_2}{2} \langle \bar{\partial}^2\tilde{U}^2 - \bar{\partial}(\Delta\tilde{U}^1 + f^1), v \rangle, \end{aligned} \quad (6.26)$$

for $v \in J$.

Replace Eq. (6.19) by Eq. (6.26) in the proof of Theorem 6.2 to obtain

$$\begin{aligned} & \max_{t \in [t_1, t_2]} \|(\mathcal{S}\hat{\tilde{U}} - u)(t)\|^2 + \int_{t_1}^{t_2} \|\nabla(\mathcal{S}\tilde{U} - u)(t)\|^2 \\ & \lesssim \sum_{i=1}^2 \tau \|\operatorname{div} \bar{\partial}\tilde{U}^i\|_{H^1(\Omega)^*}^2 + \tau^3 \|\bar{\partial}^2\tilde{U}^2 - \bar{\partial}(\Delta\tilde{U}^1 + f^1)\|_{J^*}^2 + \|\mathcal{S}\tilde{U}^1 - u^1\|^2 + F_2 \\ & \lesssim \sum_{i=1}^2 \left(\tau^3 \|\nabla\bar{\partial}\Phi^i\|^2 + F_i \right) + \tau^3 \|\bar{\partial}^2\tilde{U}^2 - \bar{\partial}(\Delta\tilde{U}^1 + f^1)\|_{J^*}^2. \end{aligned} \quad (6.27)$$

The triangle inequality and Lemma 2.2 yield

$$\begin{aligned} A_1 & \leq \max_{t \in [0, t_2]} \|(S - I)\tilde{U}\|^2 + \max_{t \in [0, t_2]} \|\mathcal{S}(\tilde{U} - \hat{\tilde{U}})(t)\|^2 + \max_{t \in [0, t_2]} \|(\mathcal{S}\hat{\tilde{U}} - u)(t)\|^2 \\ & \lesssim \tau^2 (\|\nabla\Phi^1\|^2 + \|\nabla\Phi^2\|^2) + \tau^4 \|\bar{\partial}^2\tilde{U}^2\|^2 \\ & \quad + \max_{t \in [0, t_1]} \|(\mathcal{S}\tilde{U} - u)(t)\|^2 + \max_{t \in [t_1, t_2]} \|(\mathcal{S}\hat{\tilde{U}} - u)(t)\|^2. \end{aligned}$$

Equations (6.25)–(6.27) give the final bound for A_1 .

By the same arguments A_4 can be bounded as:

$$\begin{aligned} A_4 &\leq \int_0^{t_2} \|\nabla(\mathcal{S} - I)\tilde{U}(t)\|^2 + \int_0^{t_1} \|\nabla(\mathcal{S}\tilde{U} - u)(t)\|^2 + \int_{t_1}^{t_2} \|\nabla(\mathcal{S}\tilde{U} - u)(t)\|^2 \\ &\leq \tau(\|\operatorname{div} \tilde{U}^1\|^2 + \|\operatorname{div} \tilde{U}^2\|^2) + \int_0^{t_1} \|\nabla(\mathcal{S}\tilde{U} - u)(t)\|^2 + \int_{t_1}^{t_2} \|\nabla(\mathcal{S}\tilde{U} - u)(t)\|^2. \end{aligned}$$

Lemma 2.1 gives:

$$A_5 \leq \tau(\|\nabla(U^1 - \tilde{U}^1)\|^2 + \|\nabla(U^2 - \tilde{U}^2)\|^2) \lesssim \tau(\|\operatorname{div} \tilde{U}^1\|^2 + \|\operatorname{div} \tilde{U}^2\|^2).$$

It remains to bound A_6 . For $t \in [0, t_1]$ one obtains with the techniques of Theorem 6.3 using Eq. (6.24)

$$\max_{t \in [0, t_1]} \|\tilde{e}(t)\|_{J^*}^2 + \int_0^{t_1} \|\tilde{e}(t)\|^2 \lesssim \tau^3 \|\bar{\partial}^2 \tilde{U}^2 - \bar{\partial}(\Delta \tilde{U}^1 + f^1)\|_{J^*}^2 + \tau^3 \|\nabla \Phi^1\|^2 + F_1.$$

In the interval $[t_1, t_2]$, use Eq. (6.26) in the proof of Theorem 6.3 to get

$$\begin{aligned} \max_{t \in [t_1, t_2]} \|\tilde{e}(t)\|_{J^*}^2 + \int_{t_1}^{t_2} \|\tilde{e}(t)\|^2 \\ \lesssim \|\tilde{e}(t_1)\|_{J^*}^2 + \tau^3 (\|\nabla \Phi^2\|^2 + \|\bar{\partial}^2 \tilde{U}^2 - \bar{\partial}(\Delta \tilde{U}^1 + f^1)\|_{J^*}^2) + \tau^5 \|\bar{\partial}^2 \tilde{U}^2\|^2 + F_2. \end{aligned}$$

Collecting the results above proves the theorem. $\square$

6.3.5 Proof of optimality Theorem 6.6

In this Chapter it is shown that the estimates for the velocity in Theorem 6.2 and Theorem 6.3 are optimal if one uses Algorithm 4.4 to calculate the initial values.

First two stability estimates are stated.

Lemma 6.11 (Stability).

Let $\tilde{e}^n := \tilde{U}^n - u_n$, $e^n := U^n - u_n$. The scheme defined in Eq. (6.1)–(6.2) fulfills the following stability results for all $i \geq 1$:

$$\begin{aligned} &\|\bar{\partial}^i e\|_{\ell^\infty(i+2, N; L^2)}^2 + \tau^3 \|\bar{\partial}^{i+2} e\|_{\ell^2(i+2, N; L^2)}^2 + \|\nabla \bar{\partial}^i \tilde{e}\|_{\ell^2(i+2, N; L^2)}^2 + \frac{1}{\tau} \|\bar{\partial}^i (\tilde{U} - U)\|_{\ell^2(i+2, N; L^2)}^2 \\ &+ \frac{1}{\tau} \|\operatorname{div} \bar{\partial}^{i-1} \tilde{U}\|_{\ell^\infty(i+2, N; L^2)}^2 + \|\nabla (\bar{\partial}^{i-1} \Phi - \tau \bar{\partial}^i p)\|_{\ell^\infty(i+2, N; L^2)}^2 \\ &\lesssim \|\bar{\partial}^i e^{i+1}\|^2 + \|\bar{\partial}^i (2e^{i+1} - e^i)\|^2 + \frac{2}{\tau} \|\operatorname{div} \bar{\partial}^{i-1} \tilde{U}^{i+1}\|^2 \\ &\quad + \tau^2 \left(\|\partial_t^{(2+i)} u\|_{L^\infty(V^*)}^2 + \|\nabla \partial_t^{(i+1)} p\|_{L^\infty(L^2)}^2 \right) \\ &\quad + \|\nabla (\bar{\partial}^{i-1} \Phi^{i+1} - \tau \bar{\partial}^i p^{i+1})\|^2, \quad (6.28) \end{aligned}$$

$$\begin{aligned}
 & \|\bar{\partial}^i \tilde{e}\|_{\ell^\infty(i+2,N;J^*)}^2 + \tau^3 \|\bar{\partial}^{i+2} \tilde{e}\|_{\ell^2(i+2,N;J^*)}^2 + \|\bar{\partial}^i \tilde{e}\|_{\ell^2(i+2,N;L^2)}^2 \\
 & \lesssim \|\bar{\partial}^i \tilde{e}^{i+1}\|_{J^*}^2 + \|\bar{\partial}^i (2\tilde{e}^{i+1} - \tilde{e}^i)\|_{J^*}^2 + \tau(\|\bar{\partial}^i e^{i+1}\| + \|\bar{\partial}^i (2e^{i+1} - e^i)\|^2) \\
 & \quad + \tau^3(\|\partial_t^{(2+i)} u\|_{L^\infty(V^*)}^2 + \|\nabla \partial_t^{(1+i)} p\|_{L^\infty(L^2)}^2) + \tau^4 \|\partial_t^{(3+i)} u\|_{L^\infty(J^*)}^2 \\
 & \quad + \|\operatorname{div} \bar{\partial}^{i-1} \tilde{U}^{i+1}\| + \tau \|\nabla(\bar{\partial}^{i-1} \Phi^{i+1} - \tau \bar{\partial}^i p^{i+1})\|^2. \quad (6.29)
 \end{aligned}$$

Proof : Subtract Eq. (2.1) from Eq. 6.1 and apply the time difference operator $\bar{\partial}^i$ to obtain for $n \geq i+2$, $i \geq 1$, $v \in V$,

$$\begin{aligned}
 & \frac{1}{2\tau} \langle \bar{\partial}^i (3\tilde{e}^n - 4e^{n-1} + e^{n-2}), v \rangle \\
 & \quad + \langle \nabla \bar{\partial}^i \tilde{e}^n, \nabla v \rangle + \langle \nabla \bar{\partial}^i (P^{n-1} - p_n), v \rangle = \langle \bar{\partial}^i R^n, v \rangle, \quad (6.30)
 \end{aligned}$$

where $R^n := \partial_t u_n - \bar{\partial}^B u_n$. Set $v = 4\tau \bar{\partial}^i \tilde{e}^n$. The first term on the left hand side can be treated as follows:

$$\begin{aligned}
 2\langle \bar{\partial}^i (3\tilde{e}^n - 4e^{n-1} + e^{n-2}), \bar{\partial}^i \tilde{e}^n \rangle &= 6\langle \bar{\partial}^i (\tilde{e}^n - e^n), \bar{\partial}^i \tilde{e}^n \rangle &=: I_1 \\
 &+ 2\langle \bar{\partial}^i (3e^n - 4e^{n-1} + e^{n-2}), \bar{\partial}^i (\tilde{e}^n - e^n) \rangle &=: I_2 \\
 &+ 2\langle \bar{\partial}^i (3e^n - 4e^{n-1} + e^{n-2}), \bar{\partial}^i e^n \rangle &=: I_3
 \end{aligned}$$

Use the identities (2.17) and (2.15) to obtain

$$\begin{aligned}
 I_1 &= 3\|\bar{\partial}^i \tilde{e}^n\|^2 - 3\|\bar{\partial}^i e^n\|^2 + 3\|\bar{\partial}^i (\tilde{e}^n - e^n)\|^2, \\
 I_3 &= \|\bar{\partial}^i e^n\|^2 - \|\bar{\partial}^i e^{n-1}\|^2 + \tau^4 \|\bar{\partial}^{i+2} e^n\|^2 + \|\bar{\partial}^i (2e^n - e^{n-1})\|^2 - \|\bar{\partial}^i (2e^{n-1} - e^{n-2})\|^2.
 \end{aligned}$$

Because of Eq. (6.2) it holds

$$I_2 = 2\langle \bar{\partial}^i (3e^n - 4e^{n-1} + e^{n-2}), \nabla \bar{\partial}^i \Phi^n \rangle = 0, \quad \text{since } \operatorname{div} e^n = 0.$$

Collecting all these identities yields

$$\begin{aligned}
 & 3\|\bar{\partial}^i \tilde{e}^n\|^2 - 3\|\bar{\partial}^i e^n\|^2 + 3\|\bar{\partial}^i (\tilde{e}^n - e^n)\|^2 + \|\bar{\partial}^i e^n\|^2 - \|\bar{\partial}^i e^{n-1}\|^2 \\
 & \quad + \tau^4 \|\bar{\partial}^{i+2} e^n\|^2 + \|\bar{\partial}^i (2e^n - e^{n-1})\|^2 - \|\bar{\partial}^i (2e^{n-1} - e^{n-2})\|^2 \\
 & \quad + 4\tau \|\nabla \bar{\partial}^i \tilde{e}^n\|^2 + 4\tau \langle \nabla \bar{\partial}^i (P^{n-1} - p^n), \bar{\partial}^i \tilde{e}^n \rangle = 4\tau \langle \bar{\partial}^i R^n, \bar{\partial}^i \tilde{e}^n \rangle. \quad (6.31)
 \end{aligned}$$

Now, applying $\bar{\partial}^i$ to Eq. (6.2) and rearranging terms gives

$$\begin{aligned}
 \frac{3}{2\tau} \bar{\partial}^i e^n + \nabla \underbrace{(\bar{\partial}^i P^n + \frac{1}{\tau} \operatorname{div} \bar{\partial}^{i-1} \tilde{U}^n - \bar{\partial}^i p^n)}_{=\frac{1}{\tau} \bar{\partial}^{i-1} \Phi^n} &= \frac{3}{2\tau} \bar{\partial}^i \tilde{e}^n + \nabla \underbrace{(\bar{\partial}^i P^{n-1} + \frac{1}{\tau} \operatorname{div} \bar{\partial}^{i-1} \tilde{U}^{n-1} - \bar{\partial}^i p^n)}_{=\frac{1}{\tau} \bar{\partial}^{i-1} \Phi^{n-1}}.
 \end{aligned}$$

By squaring, integrating over Ω , and multiplying by $\frac{4}{3}\tau^2$ one obtains

$$\begin{aligned} 3\|\bar{\partial}^i e^n\|^2 + \frac{4}{3}\|\nabla(\bar{\partial}^{i-1}\Phi^n - \tau\bar{\partial}^i p^n)\|^2 \\ = 3\|\bar{\partial}^i \tilde{e}^n\|^2 + \frac{4}{3}\|\nabla(\bar{\partial}^{i-1}\Phi^{n-1} - \tau\bar{\partial}^i p^n)\|^2 \\ + 4\tau\langle \bar{\partial}^i \tilde{e}^n, \nabla\bar{\partial}^i(P^{n-1} - p^n) \rangle - 4\langle \operatorname{div} \bar{\partial}^i \tilde{U}^n, \operatorname{div} \bar{\partial}^{i-1} \tilde{U}^{n-1} \rangle. \end{aligned}$$

The second term on the right hand side results in

$$\begin{aligned} \frac{4}{3}\|\nabla(\bar{\partial}^{i-1}\Phi^{n-1} - \tau\bar{\partial}^i p^n)\|^2 &= \frac{4}{3}\left(\|\nabla(\bar{\partial}^{i-1}\Phi^{n-1} - \tau\bar{\partial}^i p^{n-1})\| + \tau^2\|\nabla\bar{\partial}^{i+1}p^n\|\right)^2 \\ &= \frac{4}{3}\|\nabla(\bar{\partial}^{i-1}\Phi^{n-1} - \tau\bar{\partial}^i p^{n-1})\|^2 \\ &\quad + \frac{8\tau^2}{3}\|\nabla(\bar{\partial}^{i-1}\Phi^{n-1} - \tau\bar{\partial}^i p^{n-1})\|\|\nabla\bar{\partial}^{i+1}p^n\| + \frac{4}{3}\tau^4\|\nabla\bar{\partial}^{i+1}p^n\|^2 \\ &\leq \frac{4}{3}(1+\tau)\|\nabla(\bar{\partial}^{i-1}\Phi^{n-1} - \tau\bar{\partial}^i p^{n-1})\|^2 + C\tau^3\|\nabla\bar{\partial}^{i+1}p^n\|^2. \end{aligned}$$

For the last term on the right hand side it holds

$$\begin{aligned} -4\langle \operatorname{div} \bar{\partial}^i \tilde{U}^n, \operatorname{div} \bar{\partial}^{i-1} \tilde{U}^{n-1} \rangle &= \frac{4}{\tau}\langle \operatorname{div} \bar{\partial}^{i-1} \tilde{U}^{n-1} - \operatorname{div} \bar{\partial}^{i-1} \tilde{U}^n, \operatorname{div} \bar{\partial}^{i-1} \tilde{U}^{n-1} \rangle \\ &= \frac{2}{\tau}\left(\|\operatorname{div} \bar{\partial}^{i-1} \tilde{U}^{n-1}\|^2 - \|\operatorname{div} \bar{\partial}^{i-1} \tilde{U}^n\|^2 + \tau^2\|\operatorname{div} \bar{\partial}^i \tilde{U}^n\|^2\right). \end{aligned}$$

These estimates combined with Eq. (6.31) give

$$\begin{aligned} \|\bar{\partial}^i e^n\|^2 - \|\bar{\partial}^i e^{n-1}\|^2 + \tau^4\|\bar{\partial}^{i+2}e^n\|^2 + 3\|\bar{\partial}^i(\tilde{U}^n - U^n)\|^2 \\ + \frac{2}{\tau}\left(\|\operatorname{div} \bar{\partial}^{i-1} \tilde{U}^n\|^2 - \|\operatorname{div} \bar{\partial}^{i-1} \tilde{U}^{n-1}\|^2\right) \\ + \|\bar{\partial}^i(2e^n - e^{n-1})\|^2 - \|\bar{\partial}^i(2e^{n-1} - e^{n-2})\|^2 + 4\tau\|\nabla\bar{\partial}^i \tilde{e}^n\|^2 \\ + \frac{4}{3}\|\nabla(\bar{\partial}^{i-1}\Phi^n - \tau\bar{\partial}^i p^n)\|^2 - \frac{4}{3}(1+\tau)\|\nabla(\bar{\partial}^{i-1}\Phi^{n-1} - \tau\bar{\partial}^i p^{n-1})\|^2 \\ = 4\tau\langle \bar{\partial}^i R^n, \bar{\partial}^i \tilde{e}^n \rangle + 2\tau\|\operatorname{div} \bar{\partial}^i \tilde{U}^n\|^2 + C\tau^3\|\nabla\bar{\partial}^{i+1}p^n\|^2 \\ \leq C(\tau\|\bar{\partial}^i R^n\|_{-1}^2 + \tau^3\|\nabla\bar{\partial}^{i+1}p^n\|^2) + 3\tau\|\nabla\bar{\partial}^i \tilde{U}^n\|^2. \end{aligned}$$

By summing up from $n = i + 2$ to N one obtains

$$\begin{aligned} \max_{i+2 \leq n \leq N} \|\bar{\partial}^i e^n\|^2 + \tau^3 \sum_{n=i+2}^N \tau\|\bar{\partial}^{i+2}e^n\|^2 + \sum_{n=i+2}^N \tau\|\nabla\bar{\partial}^i \tilde{e}^n\|^2 + \frac{1}{\tau} \sum_{n=i+2}^N \tau\|\bar{\partial}^i(\tilde{U}^n - U^n)\|^2 \\ + \frac{1}{\tau} \max_{i+2 \leq n \leq N} \|\operatorname{div} \bar{\partial}^{i-1} \tilde{U}^n\|^2 + \max_{i+2 \leq n \leq N} \|\bar{\partial}^i(2e^n - e^{n-1})\|^2 \\ + \max_{i+2 \leq n \leq N} \|\nabla(\bar{\partial}^{i-1}\Phi^n - \tau\bar{\partial}^i p^n)\|^2 \\ \leq \|\bar{\partial}^i e^{i+1}\|^2 + \|\bar{\partial}^i(2e^{i+1} - e^i)\|^2 + \frac{2}{\tau}\|\operatorname{div} \bar{\partial}^{i-1} \tilde{U}^{i+1}\|^2 \\ + \|\nabla(\bar{\partial}^{i-1}\Phi^{i+1} - \tau\bar{\partial}^i p^{i+1})\|^2 + \tau^4\|\bar{\partial}^i \partial_t^{(3)} u\|_{L^\infty(V^*)}^2 + \tau^2\|\nabla \partial_t^{(i+1)} p\|_{L^\infty(L^2)}^2, \end{aligned}$$

which proves Eq. (6.28).

In order to prove Eq. (6.29) subtract Eq. (2.1) from Eq. (6.4), and apply ∂^i to obtain for $v \in J$, $i \geq 0$, $n \geq i + 2$

$$\frac{1}{2\tau} \langle \bar{\partial}^i (3\tilde{e}^n - 4\tilde{e}^{n-1} + \tilde{e}^{n-2}), v \rangle + \langle -\Delta \bar{\partial}^i \tilde{e}^n, v \rangle = \langle \bar{\partial}^i R^n, v \rangle.$$

Set $v = 4\tau \mathcal{A}^{-1} \bar{\partial}^i \tilde{e}^n$. Using identity (2.15) and the norm equivalence Eq. (2.5) the first term on the left hand side can be written as

$$\begin{aligned} & 2 \langle \bar{\partial}^i (3\tilde{e}^n - 4\tilde{e}^{n-1} + \tilde{e}^{n-2}), \mathcal{A}^{-1} \bar{\partial}^i \tilde{e}^n \rangle \\ & \gtrsim \|\bar{\partial}^i \tilde{e}^n\|_{J^*}^2 - \|\bar{\partial}^i \tilde{e}^{n-1}\|_{J^*}^2 + \|\bar{\partial}^i (2\tilde{e}^n - \tilde{e}^{n-1})\|_{J^*}^2 - \|\bar{\partial}^i (2\tilde{e}^{n-1} - \tilde{e}^{n-2})\|_{J^*}^2 + \tau^4 \|\bar{\partial}^{i+2} \tilde{e}^n\|_{J^*}^2. \end{aligned}$$

The second term on the left hand side can be bounded from below as

$$\begin{aligned} 4\tau \langle -\Delta \bar{\partial}^i \tilde{e}^n, \mathcal{A}^{-1} \bar{\partial}^i \tilde{e}^n \rangle &= 4\tau \langle \bar{\partial}^i \tilde{e}^n, -\Delta \mathcal{A}^{-1} \bar{\partial}^i \tilde{e}^n \rangle \\ &= 4\tau \|\bar{\partial}^i \tilde{e}^n\|^2 - 4\tau \langle \bar{\partial}^i (\tilde{U}^n - U^n), \nabla r \rangle \\ &\geq \tau \|\bar{\partial}^i \tilde{e}^n\|^2 - C\tau \|\bar{\partial}^i (\tilde{U}^n - U^n)\|^2, \end{aligned}$$

where r is a pressure function fulfilling

$$\|\nabla r\| \lesssim \|\bar{\partial}^i \tilde{e}^n\|.$$

Collecting these estimates results in

$$\begin{aligned} & \|\bar{\partial}^i \tilde{e}^n\|_{J^*}^2 - \|\bar{\partial}^i \tilde{e}^{n-1}\|_{J^*}^2 + \|\bar{\partial}^i (2\tilde{e}^n - \tilde{e}^{n-1})\|_{J^*}^2 \\ & \quad - \|\bar{\partial}^i (2\tilde{e}^{n-1} - \tilde{e}^{n-2})\|_{J^*}^2 + \tau^4 \|\bar{\partial}^{i+2} \tilde{e}^n\|_{J^*}^2 + \tau \|\bar{\partial}^i \tilde{e}^n\|^2 \\ & \lesssim \tau \|\bar{\partial}^i R^n\|_{J^*} \|\bar{\partial}^i \tilde{e}^n\|_{J^*} + \tau \|\bar{\partial}^i (\tilde{U}^n - U^n)\|^2. \end{aligned}$$

Summing up from $n = i + 2$ to N one arrives at

$$\begin{aligned} & \max_{i+2 \leq n \leq N} (\|\bar{\partial}^i \tilde{e}^n\|_{J^*}^2 + \|\bar{\partial}^i (2\tilde{e}^n - \tilde{e}^{n-1})\|_{J^*}^2) + \tau^3 \sum_{n=i+2}^N \tau \|\bar{\partial}^{i+2} \tilde{e}^n\|_{J^*}^2 + \sum_{n=i+2}^N \tau \|\bar{\partial}^i \tilde{e}^n\|^2 \\ & \lesssim \max_{i+2 \leq n \leq N} \|\bar{\partial}^i \tilde{e}^n\|_{J^*} \sum_{n=i+2}^N \tau \|\bar{\partial}^i R^n\|_{J^*} \\ & \quad + \sum_{n=i+2}^N \tau \|\bar{\partial}^i (\tilde{U}^n - U^n)\|^2 + \|\bar{\partial}^i \tilde{e}^{i+1}\|_{J^*}^2 + \|\bar{\partial}^i (2\tilde{e}^{i+1} - \tilde{e}^i)\|_{J^*}^2. \end{aligned}$$

Now use Eq. (6.28) to bound the second term on the right hand side by

$$\begin{aligned} & \sum_{n=i+2}^N \tau \|\bar{\partial}^i (\tilde{U}^n - U^n)\|^2 \lesssim \tau \|\bar{\partial}^i e^{i+1}\|^2 + \tau \|\bar{\partial}^i (2e^{i+1} - e^i)\|^2 + \|\operatorname{div} \bar{\partial}^{i-1} \tilde{U}^{i+1}\|^2 \\ & \quad + \tau \|\nabla (\bar{\partial}^{i-1} \Phi^{i+1} - \tau \bar{\partial}^i p^{i+1})\|^2 + \tau^3 (\|\partial_t^{(2+i)} u\|_{L^\infty(V^*)}^2 + \|\nabla \partial_t^{(i+1)} p\|_{L^\infty(L^2)}^2). \end{aligned}$$

This finally proves Eq. (6.29). $\square$

Lemma 6.11 shows, that the sum of the difference quotients is bounded by initial values. To bound these, initial a-priori estimates are presented in the following Lemma.

Lemma 6.12.

Suppose $\tilde{U}^0 = U^0 = u_0$, $P^0 = p_0$ and $(\tilde{U}^1, U^1, P^1)$ are calculated by the trapezoidal method in Eqs. (4.3)–(4.4). Then it holds

$$\|\tilde{U}^1 - u_1\| + \tau^{\frac{1}{2}} \|\nabla(\tilde{U}^1 - u_1)\| + \tau \|\Delta(\tilde{U}^1 - u_1)\| + \tau \|\nabla\Phi^1\| \lesssim \tau^2 Y_1, \quad (6.32)$$

$$\|\tilde{U}^2 - u_2\| + \tau^{\frac{1}{2}} \|\nabla(\tilde{U}^2 - u_2)\| + \tau \|\Delta(\tilde{U}^2 - u_2)\| + \tau \|\nabla\Phi^2\| \lesssim \tau^2 (Y_1 + Y_2), \quad (6.33)$$

$$\|\tilde{U}^3 - u_3\| + \tau^{\frac{1}{2}} \|\nabla(\tilde{U}^3 - u_3)\| + \tau \|\Delta(\tilde{U}^3 - u_3)\| + \tau \|\nabla\Phi^3\| \lesssim \tau^2 \sum_{i=1}^3 Y_i, \quad (6.34)$$

$$\|\bar{\partial}(\tilde{U}^1 - u_1)\|_{J^*} + \|\bar{\partial}(\tilde{U}^2 - u_2)\|_{J^*} \lesssim \tau^{\frac{3}{2}} (Y_1 + Y_2), \quad (6.35)$$

where

$$Y_1 := \|\nabla\bar{\partial}p_1\| + \tau^{\frac{1}{2}} (\|\partial_t^{(3)}u\|_{L^\infty((0,t_1);V^*)} + \|\partial_{tt}p\|_{L^\infty((0,t_1);L^2)} + \|\partial_{tt}f\|_{L^\infty((0,t_1);V^*)}),$$

$$Y_2 := \|\nabla\bar{\partial}p_2\| + \tau^{\frac{1}{2}} \|\partial_t^{(3)}u\|_{L^\infty((t_0,t_2);V^*)}, \quad Y_3 := \|\nabla\bar{\partial}p_3\| + \tau^{\frac{1}{2}} \|\partial_t^{(3)}u\|_{L^\infty((t_1,t_3);V^*)}.$$

Proof : Let R^1 be the consistency error of the first step,

$$R^1 := \bar{\partial}u^1 - \Delta u^{1/2} + \nabla p^{1/2} - f^{1/2},$$

fulfilling

$$\|R^1\|_{-1} \lesssim \tau^2 \left(\|\partial_t^{(3)}u\|_{L^\infty((0,t_1),V^*)} + \|\partial_{tt}p\|_{L^\infty((0,t_1),L^2)} + \|\partial_{tt}f\|_{L^\infty((0,t_1),V^*)} \right). \quad (6.36)$$

The error equation corresponding to Eq. (4.3) is

$$\tilde{e}^1 - \frac{\tau}{2} \Delta \tilde{e}^1 = \frac{\tau^2}{2} \nabla \bar{\partial} p^1 - \tau R^1.$$

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$$\|\nabla\Phi^1\| \leq \frac{\|e^1\| + \|\tilde{e}^1\|}{\tau} \leq 2 \frac{\|\tilde{e}^1\|}{\tau}$$

prove immediately Eq. (6.32). Let $R^2 := \partial_t u_2 - \bar{\partial}^B u_2$. This gives

$$3\tilde{e}^2 - 2\tau \Delta \tilde{e}^2 = 4e^1 - 2\tau \Phi^1 + 2\tau^2 (\nabla \bar{\partial} p^2 + \nabla \bar{\partial} p^1) - 2\tau R^2.$$

The same procedure as above yields Eq. (6.33)–(6.34). Eq. (6.35) is obtained by

$$\begin{aligned} \|\bar{\partial}\tilde{e}^1\|_{J^*} &= \sup_{0 \neq v \in J} \frac{\langle \bar{\partial}\tilde{e}^1, v \rangle}{\|\nabla v\|} = \sup_{0 \neq v \in J} \frac{\langle \Delta\tilde{e}^1 - 2R^1, v \rangle}{2\|\nabla v\|} \lesssim \|\nabla\tilde{e}^1\| + \|R^1\|_{-1}, \\ \|\bar{\partial}\tilde{e}^2\|_{J^*} &= \sup_{0 \neq v \in J} \frac{\langle \bar{\partial}\tilde{e}^2, v \rangle}{\|\nabla v\|} = \frac{1}{3} \sup_{0 \neq v \in J} \frac{\langle 2\Delta\tilde{e}^2 + \bar{\partial}\tilde{e}^1 - 2R^2, v \rangle}{2\|\nabla v\|} \\ &\lesssim \|\bar{\partial}\tilde{e}^1\|_{J^*} + \|\nabla\tilde{e}^2\| + \|R^2\|_{-1}. \end{aligned}$$

□

Recall Definition 6.1 of the respective a-posteriori estimator using the the trapezoidal method in Eqs. (4.3)–(4.4):

$$\begin{aligned}\epsilon_1 &:= \|\nabla \bar{\partial}^2 \tilde{U}\|_{\ell^2(2,N;L^2)}^2, & \epsilon_2 &:= \|\bar{\partial}^3 \tilde{U}\|_{\ell^2(3,N;J^*)}^2, & \epsilon_{2,0} &:= \frac{1}{\tau} \|\bar{\partial}^2 \tilde{U}^2 - \bar{\partial}(\Delta \tilde{U}^1 + f^1)\|_{J^*}^2, \\ \epsilon_3 &:= \|\bar{\partial}^2 \tilde{U}\|_{\ell^2(2,N;L^2)}^2, & \epsilon_4 &:= \frac{1}{\tau^2} \|\nabla \Phi\|_{\ell^\infty(L^2)}^2, & \epsilon_5 &:= \frac{1}{\tau^2} \|\nabla \Phi\|_{\ell^2(L^2)}^2, \\ \epsilon_6 &:= \frac{1}{\tau} \|\nabla \bar{\partial} \Phi\|_{\ell^2(L^2)}^2, & \epsilon_9 &:= \|\bar{\partial}^2 \tilde{U}\|_{\ell^\infty(2,N;L^2)}^2, & \epsilon_{10} &:= \frac{1}{\tau^3} \|\operatorname{div} \tilde{U}\|_{\ell^2(L^2)}^2.\end{aligned}$$

Now Theorem 6.6 can be proved under sufficiently regularity assumptions, i.e. $\partial_{ttt}f \in L^\infty(V^*)$, $u_{ttt} \in L^\infty(V^*)$, $u_{tt} \in L^\infty(H_0^1)$, $p_t \in L^\infty(H^1)$, $p_{tt} \in L^\infty(H^1)$ and $\partial_t^{(4)}u \in L^\infty(J^*)$.

Proof : (**Theorem 6.6**) Using Lemma 6.11 and the fact that $\tilde{U}^n = \tilde{e}^n + u^n$ the estimators are bounded from above by

$$\begin{aligned}\epsilon_1 &\lesssim \tau \|\nabla \bar{\partial}^2 \tilde{e}^2\|^2 + \tau \|\nabla \bar{\partial}^2 \tilde{e}^3\|^2 + \sum_{n=4}^N \tau \|\nabla \bar{\partial}^2 \tilde{e}^n\|^2 + \sum_{n=2}^N \tau \|\nabla \bar{\partial}^2 u(t_n)\|^2 \\ &\lesssim \|\partial_{tt}u\|_{L^\infty(H_0^1)}^2 + \|\nabla \partial_{tt}p\|_{L^\infty(L^2)}^2 + \|\partial_t^{(3)}u\|_{L^\infty(V^*)}^2 + \tau \|\nabla \bar{\partial}^2 \tilde{e}^2\|^2 \\ &\quad + \tau \|\nabla \bar{\partial}^2 \tilde{e}^3\|^2 + \|\bar{\partial}^2 \tilde{e}^2\|^2 + \|\bar{\partial}^2 \tilde{e}^3\|^2 + \frac{1}{\tau} \|\operatorname{div} \bar{\partial} \tilde{U}^3\|^2 + \|\nabla \bar{\partial} \Phi^3\|^2,\end{aligned}$$

$$\begin{aligned}\epsilon_{2,0} &= \frac{1}{\tau^3} \|2(f^2 + \Delta \tilde{U}^2) - 2\bar{\partial} \tilde{U}^2 - (f^1 + \Delta \tilde{U}^1) + (f^0 + \Delta u_0)\|_{J^*}^2 \\ &= \frac{1}{\tau^3} \|2(f^2 + \Delta u_2) + 2\Delta \tilde{e}^2 - 2\bar{\partial} \tilde{e}^2 - 2\bar{\partial} u_2 - (f_1 + \Delta u_1) - \Delta \tilde{e}^1 + (f_0 - \Delta u_0)\|_{J^*}^2 \\ &= \frac{1}{\tau^3} \|(\partial_t(2u_2 - u_1 + u_0) - 2\bar{\partial} u_2) + 2\Delta \tilde{e}^2 - \Delta \tilde{e}^1 + 2\bar{\partial} \tilde{e}^2\|_{J^*}^2 \\ &\lesssim \tau \|\partial_t^{(3)}u\|_{J^*}^2 + \frac{1}{\tau^3} \left(\|\nabla \tilde{e}^2\|^2 + \|\nabla \tilde{e}^1\|^2 + \|\bar{\partial} \tilde{e}^2\|^2 \right),\end{aligned}$$

$$\begin{aligned}\epsilon_2 &\lesssim \sum_{n=3}^N \tau \|\bar{\partial}^3 \tilde{e}^n\|_{J^*}^2 + \sum_{n=3}^N \|\bar{\partial}^3 u(t_n)\|_{J^*}^2 \\ &\lesssim \|\partial_t^{(3)}u\|_{L^\infty(V^*)}^2 + \|\nabla \partial_{tt}p\|_{L^\infty(L^2)}^2 + \tau \|\partial_t^{(4)}u\|_{L^\infty(J^*)}^2 \\ &\quad + \frac{1}{\tau^3} (\|\bar{\partial} \tilde{e}^1\|_{J^*}^2 + \|\bar{\partial} \tilde{e}^2\|_{J^*}^2 + \|\operatorname{div} \tilde{U}^2\|^2) + \frac{1}{\tau^2} (\|\bar{\partial} \tilde{e}^1\|^2 + \|\bar{\partial} \tilde{e}^2\|^2 + \|\nabla \Phi^2\|^2),\end{aligned}$$

$$\begin{aligned}
 \epsilon_3 + \epsilon_9 &\lesssim \max_{2 \leq n \leq N} \|\bar{\partial}^2 e^n\|^2 + \tau^2 \max_{2 \leq n \leq N} \|\nabla \bar{\partial}^2 \Phi^n\|^2 + \max_{2 \leq n \leq N} \|\bar{\partial}^2 u(t_n)\|^2 \\
 &\lesssim \|\bar{\partial}^2 e^2\|^2 + \|\bar{\partial}^2 e^3\|^2 + \max_{4 \leq n \leq N} \|\bar{\partial}^2 e^n\|^2 + \|\nabla \bar{\partial} \Phi^1\|^2 + \|\nabla \bar{\partial} \Phi^2\|^2 + \|\nabla \bar{\partial} \Phi^3\|^2 \\
 &\quad + \max_{4 \leq n \leq N} \|\nabla(\bar{\partial} \Phi^n - \tau \bar{\partial}^2 p^n)\|^2 + \max_{3 \leq n \leq N} \|\nabla \bar{\partial} p(t_n)\|^2 + \|\partial_{tt} u\|_{L^\infty(L^2)}^2 \\
 &\lesssim \|\partial_{tt} u\|_{L^\infty(L^2)}^2 + \|\nabla \partial_t p\|_{L^\infty(L^2)}^2 + \|\nabla \partial_{tt} p\|_{L^\infty(L^2)}^2 + \|\partial_t^{(3)} u\|_{L^\infty(V^*)}^2 \\
 &\quad + \|\bar{\partial}^2 \tilde{e}^2\|^2 + \|\bar{\partial}^2 \tilde{e}^3\|^2 + \frac{1}{\tau} \|\operatorname{div} \bar{\partial} \tilde{U}^3\|^2 + \frac{1}{\tau^2} \|\nabla \Phi^1\|^2 + \|\nabla \bar{\partial} \Phi^2\|^2 + \|\nabla \bar{\partial} \Phi^3\|^2, \\
 \\
 \epsilon_4 + \epsilon_5 &\lesssim \frac{1}{\tau^2} (\|\nabla \Phi^1\|^2 + \|\nabla \Phi^2\|^2) + \frac{1}{\tau^2} \max_{3 \leq n \leq N} \|\nabla(\Phi^n - \tau \bar{\partial} p^n)\|^2 + \frac{1}{\tau^2} \max_{3 \leq n \leq N} \|\nabla \tau \bar{\partial} p^n\|^2 \\
 &\lesssim \|\nabla \partial_t p\|_{L^\infty(L^2)}^2 + \|\partial_t^{(3)} u\|_{L^2(V^*)}^2 + \|\nabla \partial_{tt} p\|_{L^2(L^2)}^2 \\
 &\quad + \frac{1}{\tau^2} (\|\bar{\partial} \tilde{e}^1\|^2 + \|\bar{\partial} \tilde{e}^2\|^2 + \|\nabla \Phi^1\|^2 + \|\nabla \Phi^2\|^2) + \frac{1}{\tau^3} \|\operatorname{div} \tilde{U}^2\|^2, \\
 \\
 \epsilon_6 &= \frac{1}{\tau^2} \|\nabla \Phi^1\|^2 + \|\nabla \bar{\partial} \Phi^2\|^2 + \frac{1}{\tau^3} \sum_{n=3}^N \tau \|\bar{\partial}(\tilde{U}^n - U^n)\|^2 \\
 &\lesssim \|\partial_t^{(3)} u\|_{L^\infty(V^*)}^2 + \|\nabla \partial_{tt} p\|_{L^\infty(L^2)}^2 \\
 &\quad + \frac{1}{\tau^2} (\|\bar{\partial} \tilde{e}^1\|^2 + \|\bar{\partial} \tilde{e}^2\|^2 + \|\nabla \Phi^1\|^2 + \|\nabla \Phi^2\|^2) + \frac{1}{\tau^3} \|\operatorname{div} \tilde{U}^2\|^2, \\
 \\
 \epsilon_{10} &\lesssim \frac{1}{\tau^2} \|\operatorname{div} \tilde{U}^1\|^2 + \frac{1}{\tau^2} \|\operatorname{div} \tilde{U}^2\|^2 + \frac{1}{\tau^3} \max_{3 \leq n \leq N} \|\operatorname{div} \tilde{U}^n\|^2 \\
 &\lesssim \frac{1}{\tau^2} (\|\nabla \tilde{e}^1\|^2 + \|\bar{\partial} \tilde{e}^2\|^2 + \|\bar{\partial} \tilde{e}^1\|^2 + \|\nabla \Phi^2\|^2) + \frac{1}{\tau^3} \|\nabla \tilde{e}^2\|^2 \\
 &\quad + \|\nabla \bar{\partial} p(t_2)\|^2 + \|\partial_t^{(3)} u\|_{L^\infty(V^*)}^2 + \|\nabla \partial_t^{(2)} p\|_{L^\infty(J^*)}^2
 \end{aligned}$$

These estimates combined with Lemma 6.12 prove the theorem. $\square$

7 Conclusion and Outlook

7.1 Conclusion

In this thesis a-posteriori error estimates for the standard incremental pressure-correction scheme discretized by backward Euler, as well as for the rotational pressure-correction scheme discretized by the two-step backward differential formula method (BDF2) were presented.

Before analyzing the pressure-correction schemes, first, a-posteriori estimates for the fully discrete instationary Stokes system discretized in time by BDF2 were developed. Considering only the time estimators of Definition 3.4 the most critical one is the third order difference quotient

$$\mathcal{E}_2 = \sum_{n=3}^N \tau \|\bar{\partial}^3 U_h^n\|_{J^*}^2,$$

in the negative J^* norm. Calculating this norm is prohibitive in practice since one has to solve a Stokes problem, cf. the discussion in Chapter 6.2. This critical estimator $\mathcal{E}_2$ is characteristic for BDF2 time discretizations and also occurs in parabolic problems, cf. [1].

In Chapter 4 different pressure-projection schemes were introduced and the advantages and disadvantages were discussed. I advise the reader to use the *rotational pressure-correction scheme* because in contrast to the others it has the following advantages:

- second order splitting error, which makes it worthwhile to use a second order time discretization,
- a consistent pressure boundary condition.

Before the results for the rotational pressure-correction scheme were presented, in Chapter 5 the standard pressure-correction scheme discretized by implicit Euler were considered. This simpler scheme is not that interesting for practical use but helps to understand, how to use the analytical tools, in order to apply them to more complex algorithms, like the rotational pressure-correction scheme discretized by BDF2.

As expected the a-posteriori estimates for the Euler scheme has convergence rate $\mathcal{O}(\tau)$, which was stated in Theorem 5.4 and 5.5. The stability of the a-posteriori estimator was proven in Theorem 5.6.

The main results of this thesis were outlined in Chapter 6: the a-posteriori estimates for the rotational pressure-correction scheme discretized by BDF2. To the best of my

7 Conclusion and Outlook

knowledge, it is the first a-posteriori analysis for a second order time discretization for pressure-correction schemes. In Theorems 6.2–6.4 the a-posteriori estimates for the velocity and the pressure were presented and in Theorem 6.6 it was shown that the velocity results are optimal, i.e. that the estimates have the same convergence rate as the appropriate a-priori estimates. For the pressure estimate the optimality cannot be claimed because I was not able to bound the estimator

$$\epsilon_8 = \|\bar{\partial}^4 \tilde{U}\|_{\ell^2(4,N;J^*)}^2$$

of Definition 6.1. This time estimator of fourth order difference quotients results from the decoupling of the velocity and the pressure and does not appear in the estimate of the instationary Stokes system in Theorem 3.7.

Comparing the a-posteriori estimates of the velocity in Corollar 6.5 with the time dependent Stokes system (cf. Chapter 3) the following observations are possible:

- The $L^2(L^2)$ estimate for the projection scheme has the same convergence behavior as the respective estimate for the Stokes system, but needs the additional estimator ϵ_5 , which results from the splitting.
- However, in the $L^\infty(L^2)$ and $L^2(H^1)$ norm the splitting estimators ϵ_6 and ϵ_{10} prevent the scheme to be fully second order and reduce the convergence rate to 1.5.

7.2 Outlook

In the following current and future projects are presented. While in this thesis a-posteriori estimates for rotational schemes for the semi-discrete case with constant time step were presented, the main task is to develop a-posteriori estimator for the fully discrete scheme with adaptive meshes and variable time steps.

7.2.1 Fully discrete a-posteriori estimates for pressure correction scheme for a fixed mesh

The fully discrete version of Algorithm 4.8 for a fixed mesh reads:

Algorithm 7.1 (The fully discrete rotational scheme for a fixed mesh).

For given $(\tilde{U}_h^1, U_h^1, P_h^1)$ calculate for $n \geq 2$ in a first sub-step the provisional velocity $\tilde{U}_h^n$ by

$$\frac{3\tilde{U}^n - 4i_h^\top U_h^{n-1} + i_h^\top U_h^{n-2}}{2\tau} - \triangle_h \tilde{U}^n + B_h^\top P_h^{n-1} = f_h^n,$$

and in the second sub-step (U_h^n, P_h^n) by

$$\frac{3}{2\tau}(U_h^n - i_h \tilde{U}^n) + C_h^\top \Phi_h^n = 0, \quad C_h \tilde{U}^n = 0,$$

where

$$\Phi_h^n = P_h^n - P_h^{n-1} + C_h \tilde{U}^n.$$

7 Conclusion and Outlook

As in the semi-discrete case the projection step is equivalent to the following Poisson problem: calculate $\Phi_h^n \in M_h$, such that

$$(\nabla \Phi_h^n, \nabla q_h) = -\frac{(\operatorname{div} \tilde{U}^n, q_h)}{\tau}, \quad \forall q_h \in M_h,$$

and set

$$U_h^n = \tilde{U}^n - \tau \nabla \Phi_h^n.$$

In the implementation one can eliminate U_h to obtain for $n \geq 2$

$$\bar{\partial}_B \tilde{U}^n - \Delta_h \tilde{U}^n + B_h^\top \gamma_h^{n-1} = \tilde{f}_h^n,$$

where

$$\gamma_h^n = \begin{cases} P_h^n + \frac{4}{3}\Phi_h^n - \frac{1}{3}\Phi_h^{n-1} & n \geq 3 \\ P_h^2 + \frac{4}{3}\Phi_h^2 & n = 2, \\ P_h^1 & n = 1 \end{cases}, \quad \tilde{f}_h^n = \begin{cases} f_h^n & n \geq 4 \\ f_h^3 + \frac{1}{2\tau}(\tilde{U}_h^1 - i_h^\top U_h^1) & n = 3, \\ f_h^2 - \frac{2}{\tau}(\tilde{U}_h^1 - i_h^\top U_h^1) + \frac{1}{2\tau}(\tilde{U}_h^0 - i_h^\top U_h^0) & n = 2 \end{cases}.$$

The analysis of the fully-discrete scheme is more or less straightforward but more technical. The results will be published in a forthcoming paper.

7.2.2 A-posteriori estimates with adaptive meshes and variable time step

In view of applications the main task is to derive an adaptive strategy for Algorithm 7.1 by using changing meshes and variable time steps.

Therefore define for arbitrary points in time $t_0 < \dots < t_N$ the variable time step $\tau_n := t_n - t_{n-1}$ and the time step ratio $\omega_n := \frac{\tau_n}{\tau_{n-1}}$. The difference quotients are defined recursively in the following way

$$\bar{\partial} U^n := \frac{U^n - U^{n-1}}{\tau_n}, \quad \bar{\partial}^i U^n := \frac{\bar{\partial}^{i-1} U^n - \bar{\partial}^{i-1} U^{n-1}}{\tau_n}, \quad n \geq i.$$

Then the *BDF2 operator for variable time steps* is defined by

$$\bar{\partial}^B U^n = \left(1 + \frac{\omega_n}{1 + \omega_n}\right) \bar{\partial} U^n - \frac{\omega_n}{1 + \omega_n} \bar{\partial} U^{n-1} = \frac{\omega_n}{1 + \omega_n} \tau_n \bar{\partial}^2 U^n + \bar{\partial} U^n.$$

Note that $\bar{\partial}^B$ does not commute with $\bar{\partial}$ for variable time steps.

In order to understand the behavior of BDF2 for adaptive meshes and variable time steps one has to simplify the problem and consider the heat equation (cf. Chapter 2.5). For each time interval $t \in I_n := (t_{n-1}, t_n]$ define a FEM space $X^n := X_{h_n} \subset H_0^1(\Omega)^d$ and denote by $\Pi_n := \Pi_{h_n}$ the L^2 -projection onto X^n . Note that in general $\bar{\partial} U^n \notin X^n$. Therefore define

$$\bar{\partial}_k^i U^n := \Pi_k \bar{\partial}^i U^n \in X^k, \quad \bar{\partial}_k^B U^n := \Pi_k \bar{\partial}^B U^n \in X^k.$$

The discretized scheme reads:

7 Conclusion and Outlook

Algorithm 7.2 (Variable fully discretized heat equation).

Assume U^0, U^1 are given. Then calculate $U^n \in X^n$ for $n \geq 2$ by

$$\bar{\partial}_n^B U^n - \Delta_n U^n = \Pi_n f(t_n).$$

Up to my knowledge there is no a-posteriori error analysis for parabolic problems discretized in time by BDF2 with variable time step and adaptive meshes. The first step will be to develop a-posteriori estimators for Algorithm 7.2.

In the second step I will apply these techniques for the heat equation to the instationary Stokes system.

The last task will be to develop error estimates for the fully discrete rotational pressure-correction scheme for adaptive meshes and variable time steps. Changing the mesh at arbitrary time requires to project or interpolate the discrete values from the old onto the new mesh. Concerning projection schemes one must additionally analyze after which sub-step this interpolation or projection should be done to minimize the error due to the mesh change.

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