***A priori* error analysis for finite element approximations of the Stokes problem on dynamic meshes**

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In this article we study finite element approximations of the time-dependent Stokes system on dynamically changing meshes. Applying the backward Euler method for time discretization we use the discrete Helmholtz or Stokes projection to evaluate the solution at time t_{n-1} on the new spatial mesh at time t_n . The theoretical results consist of *a priori* error estimates that show a dependence on the time step size not better than $\mathcal{O}(1/\Delta t)$. These surprisingly pessimistic upper bounds are complemented by numerical examples giving evidence for a negative convergence rate, at least for a large range of time step sizes, and in this sense backing our theory. These observations imply that using adaptive meshes for incompressible flow problems is delicate and requires further investigation.

Keywords: Stokes equations; dynamic finite element meshes; adaptivity.

1. Introduction

Many applications in fluid dynamics involve localized and time-dependent phenomena such as sharp moving fronts, interior or boundary layers, shocks and complicated structures, where important physical phenomena take place. Examples can be found in technical and environmental engineering or in the life sciences. The reliable and efficient numerical approximation of solutions to such problems is still a challenging task. To resolve the dynamics of complex nonstationary flows with high accuracy and efficiency, it is necessary to use dynamic meshes (that is, meshes changing in time) for the discretization in space; cf., for example, Hoffmann & Johnson (2003) and Bänsch & Schmidt (2003). Flow problems with free boundaries have also enjoyed increasing interest in industrial applications; cf., for example, Bänsch (2001), Cuvelier (1985) and Tobiska & Matthies (2002). In the numerical simulation of flows with free boundaries the computational domain, as well as the spatial mesh, change in each time step and have to be recomputed.

In general, dynamically changing meshes require a projection of the discrete solution at the old time level onto a different spatial mesh at the new time step. Too-frequent mesh changes, especially when the mesh topology is also changed, may yield serious convergence problems. This was shown in Dupont (1982) for even the simple case of the heat equation in one dimension: changing the mesh in each time step by $h/2$ to the left, and then in the next time step back to the right, resulted in a dissipation term that

FIG. 1. von Karman vortex shedding. Line integral convolution visualization of the velocity for a certain time instant.

FIG. 2. Pressure versus time steps at a monitor point slightly behind the obstacle.

led to a completely wrong solution if $h^4/\Delta t \rightarrow \infty$. The reason for this is that the frequent and (for small time steps) rapid mesh changes introduce an artificial dissipation strong enough to damp the numerical solution to 0, whereas the exact solution is nonvanishing.

The situation is even more delicate when an elliptic constraint is present. More specifically, in computing incompressible flows, the elliptic constraint of the solenoidal condition has to be taken into account. Numerical experiments have shown that depending on the specific projection this can cause additional and even stronger inaccuracies; cf., for example, [Besier & Wollner \(2011\)](#) (in that paper, corresponding numerical results stimulate a theoretical investigation of using different projections). Note that this effect is different from the error mechanism for Dupont's example.

As an illustration, we computed a von Karman vortex shedding as depicted in Fig. 1. The flow is coming from the left given by a parabolic inflow profile, on the right outlet a do-nothing outflow condition is prescribed. The obstacle that gives rise to oscillatory flow behaviour for a medium Reynolds number is of a rectangular shape in order not to modify the geometry during mesh changes. The typical flow pattern is caught by probing the flow at a monitor point slightly behind the obstacle in Fig. 2. Figure 3 shows computations with a fixed fine mesh and for meshes that are randomly coarsened every 16 time steps (with simple interpolation of the old data onto the new mesh). Clearly, the pressure reacts in a very sensitive and instantaneous way to mesh changes, whereas the velocity behaves more reasonably for a long time. Thus, we deduce that the pressure is much more affected by mesh changes than the velocity, which is a phenomenon not present in, for instance, the heat equation.

FIG. 3. y component of the velocity (left), pressure (middle) and zoom into pressure plot (right), with and without mesh changes, at a monitor point slightly behind the obstacle.

In this paper we analyse theoretically and numerically finite element discretizations of nonstationary, incompressible flow problems on dynamic meshes. We will demonstrate that meshes varying in time can result in a loss of the order of convergence and even lead to a negative order of convergence of the discrete solution with respect to the time step size.

Despite its relevance, there are only a few articles dealing with the effect of mesh changes for parabolic problems. Let us mention the work of Dupont (1982), Eriksson & Johnson (1991, 1995) and Yang (2000). In the pioneering work of Dupont (1982), for a one-dimensional parabolic model problem, it is shown that changing the mesh in a particular way in each time step leads to convergence of the numerical solution to a wrong answer. In Eriksson & Johnson (1991, 1995) an *a priori* error analysis based on duality arguments is given for finite element approximations of a class of parabolic problems. Optimal-order error estimates are established under restrictions on the mesh changes, which have to satisfy either that the finite element spaces are embedded, $\mathcal{T}_n \subset \mathcal{T}_{n-1}$, or that the restriction $h_n^2 \leq C \Delta t_n$ on the discretization step sizes is satisfied, where h_n and Δt_n denote the spatial and temporal step size, respectively, and C is a constant. In Yang (2000) an error estimate for finite element approximations of a parabolic equation is established. There, the error is split into three parts, the temporal discretization error, the spatial discretization error and a projection error related to the number of mesh changes. However, no further estimate or convergence rate is given for the projection error. A naive estimate of this part leads to an upper bound depending reciprocally on the time step size.

Even less, however, is known for time-dependent problems with an elliptic constraint. As a prototype problem, in this paper we present *a priori* error estimates for finite element approximations with stable pairs of spaces (cf. Brezzi & Fortin, 1982) on dynamic meshes for the Stokes problem

$$\begin{aligned} \partial_t u - \Delta u + \nabla p &= f, \quad \nabla \cdot u = 0 \text{ on } (0, T) \times \Omega, \\ u &= 0 \text{ on } (0, T) \times \partial\Omega, \quad u|_{t=0} = u_0 \text{ on } \Omega. \end{aligned} \quad (1.1)$$

Here, $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ is a bounded polyhedral domain, u denotes the velocity field, p is the pressure, f is a prescribed external force and u_0 denotes the prescribed initial velocity. Restricting our analysis to the linear Stokes problem modelling creeping flow, instead of considering the Navier–Stokes equations, is only done for the sake of simplicity. The effects of mesh changes are also observed for the Stokes system. The *a priori* error analysis of finite element approximations on dynamic meshes is of fundamental importance for ensuring their convergence as well as understanding and evaluating the asymptotic behaviour of discretizations. Studies of complex dynamic meshes, in particular those where the mesh topology is changing and the meshes are not obtained by refining or coarsening, are rare in

the literature; cf., for example, [Hoffmann & Johnson \(2003\)](#) and [Dupont \(1982\)](#). In our error analysis we follow the line of [Yang \(2000\)](#). We bound the approximation error by contributions that are due to the temporal discretization, the spatial discretization and the mesh changes. This separation of the error contributions allows us to study each source of inaccuracy separately. In contrast to [Yang \(2000\)](#), we derive convergence rates for the error caused by mesh changes. In our numerical experiments (see Section 5) we illustrate the sharpness of the given error estimates.

As mentioned above, in the numerical approximation of incompressible fluid flow the velocity approximation U_{n-1} at time t_{n-1} that has been computed on the mesh $\mathcal{T}_{n-1}$ has to be evaluated on the new spatial mesh $\mathcal{T}_n$ at time t_n . While U_{n-1} satisfies a *discrete* solenoidal constraint with respect to the test space corresponding to mesh $\mathcal{T}_{n-1}$, this condition no longer holds on the new spatial mesh at time t_n , since the pair of finite element function spaces for velocity and pressure are strongly connected to the underlying mesh.

To get an idea of why this may lead to problems, let us consider an analogue of this situation for the semiimplicit Stokes equation, discretized in time by the backward Euler method,

$$\frac{U_n - U_{n-1}}{\Delta t} - \Delta U_n + \nabla P_n = 0.$$

Taking the divergence of this equation leads to

$$\operatorname{div} \frac{U_n - U_{n-1}}{\Delta t} - \Delta \operatorname{div} U_n + \Delta P_n = 0.$$

In the case that U_n is divergence free, we get

$$\Delta P_n = \frac{\operatorname{div} U_{n-1}}{\Delta t}.$$

Now, if U_{n-1} is not divergence free, then the Poisson pressure equation has an artificial right-hand side, scaled by $1/\Delta t$.

Thus, in the spatial discrete case, the loss of the *discrete* divergence-free constraints leads to inaccuracies, in particular, in the pressure approximation. This problem was observed in [Bänsch & Schmidt \(2003\)](#), where the authors propose a generalized Stokes projection of the old solution onto the new grid in order to cure spurious pressure peaks. In a recent work, [Besier & Wollner \(2011\)](#) analyse this problem in more detail. In particular, [Besier & Wollner \(2011\)](#) study the convergence behaviour of the solution for decreasing time step sizes under the rather strong assumption of only two mesh changes. The observation in [Besier & Wollner \(2011\)](#) is that the standard L^2 projection of the velocity field at the old time onto the new spatial mesh $\mathcal{T}_{n-1}$ leads to diverging pressure approximations and, thereby, is not suitable for computations on dynamic meshes. Instead, the authors suggest and study the application of a divergence-free L^2 and a divergence-free H_0^1 projection.

In the present article we will show that neither the divergence-free L^2 nor the divergence-free H_0^1 projection are capable of handling dynamic finite element meshes. In particular, we give upper bounds for the error, which even suggest *divergence* of the discrete velocity and pressure, if the mesh is changed in each time step, as it might be necessary in applications of practical interest. This is done for two different test cases in Section 5.

At the current stage of our research we do not know how to eliminate the inaccuracies resulting from too-frequent mesh changes in time and how to project the velocity field onto the new spatial mesh in an optimal way without accumulating errors or unphysical oscillations in the variables close to the time of the mesh change. The intention of this work is to point out this source of trouble, to

understand the phenomenon and to stimulate interest in analysing and solving the problem that is of practical relevance for all flow simulations on adaptive finite element meshes or with free boundaries and changing meshes. Moreover, to the best of our knowledge, an *a priori* error analysis for the finite element approximation of the Stokes problem on dynamically changing meshes has not been published before and may be of common interest.

Even though the Navier–Stokes equations are not explicitly studied here, similar observations will be made for Navier–Stokes flows. This was also pointed out in [Besier & Wollner \(2011\)](#).

The rest of this paper is organized as follows. In Section 2 we introduce the finite element methods on dynamic meshes that are used here. In particular, two different methods for projecting the velocity at the old time onto the new spatial mesh are described. In Section 3 an *a priori* error analysis is given for the schemes introduced in Section 2. Error estimates including convergence rates for Stokes approximations on dynamic finite element meshes are proved. In Section 4 the projection error is studied in the case that only mesh refinements and coarsenings are permitted. In Section 5 the results of our numerical experiments are presented. They nicely illustrate the theoretical results and demonstrate the difficulties of accurate Stokes approximations on rapidly changing finite element meshes.

2. Approximation schemes

Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$, be a bounded Lipschitz domain. For the sake of simplicity our notation does not differ between vector- and scalar-valued functions. As usual, $L^2(\Omega)$ denotes the space of functions defined and quadratically summable in Ω and $\|\cdot\|_0$ its norm. We denote the inner product in L^2 by $\langle \cdot, \cdot \rangle$. By $H^m(\Omega) = W^{m,2}(\Omega)$, with $m \in \mathbb{N}_0$, we denote standard Sobolev spaces, equipped with the norm $\|u\|_m := (\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|^2)^{1/2}$ and $\|u\| := \|u\|_0$. $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in $H^1(\Omega)$, where $C_0^\infty(\Omega)$ is the space of all infinitely differentiable functions with compact support. Furthermore, let $J(\Omega)$ be the space of divergence-free functions given by

$$J(\Omega) := \{\varphi \in H_0^1(\Omega) \mid \operatorname{div} \varphi = 0\}.$$

$H^m(\Omega)/\mathbb{R}$ denotes the quotient space of equivalence classes of elements of $H^m(\Omega)$ differing by constants. The norm in $H^m(\Omega)/\mathbb{R}$ is defined by $\|p\|_{H^m/\mathbb{R}} = \inf_{c \in \mathbb{R}} \|p + c\|_m$. For $m = 0$ we set $L^2 \setminus \mathbb{R} := H^0 \setminus \mathbb{R}$. $L^p(0, T; X)$, for any Banach space X , is the space of measurable X -valued functions $u : (0, T) \rightarrow X$ such that

$$\begin{aligned} \|u\|_{L^p(0, T; X)} &:= \left(\int_0^T \|u(t)\|_X^p dt \right)^{1/p}, \quad 1 \leq p < \infty, \\ \|u\|_{L^\infty(0, T; X)} &:= \operatorname{ess\,sup}_{0 < t < T} \|u(t)\|_X, \quad p = \infty. \end{aligned}$$

The norm in $L^p(0, T; X)$ is denoted by $\|\cdot\|_{L^p(X)} := \|\cdot\|_{L^p(0, T; X)}$, $1 \leq p \leq \infty$. We will use C as a generic constant, which may have different values in different occurrences.

With the above notation, the Stokes system (1.1) in weak form reads as follows: find $u \in L^2(0, T; H_0^1(\Omega))$, $p \in L^2(0, T; L^2(\Omega)/\mathbb{R})$ with $\partial_t u \in L^2(0, T; L^2(\Omega))$ such that

$$\begin{aligned} \langle \partial_t u, v \rangle + \langle \nabla u, \nabla v \rangle - \langle p, \operatorname{div} v \rangle &= \langle f, v \rangle \quad \forall v \in H_0^1(\Omega), \\ \langle q, \operatorname{div} u \rangle &= 0 \quad \forall q \in L^2(\Omega), \end{aligned} \tag{2.1}$$

for almost all $t \in (0, T)$, where $f \in L^2(0, T; L^2(\Omega))$.

The divergence-free weak formulation used hereafter with the pressure variable eliminated is given as follows: find $u \in L^2(0, T; J(\Omega))$ with $\partial_t u \in L^2(0, T; L^2(\Omega))$ such that

$$\langle \partial_t u, v \rangle + \langle \nabla u, \nabla v \rangle = \langle f, v \rangle \quad \forall v \in J(\Omega), \quad (2.2)$$

for almost all $t \in (0, T)$.

Clearly, (2.1) implies (2.2). Conversely, (2.2) implies the existence of a function p satisfying (2.1) and p can be recovered by

$$\langle p, \operatorname{div} v \rangle = \langle \partial_t u, v \rangle + \langle \nabla u, \nabla v \rangle - \langle f, v \rangle \quad \forall v \in H_0^1(\Omega) \quad \text{and almost all } t \in (0, T). \quad (2.3)$$

Next, we introduce the discrete schemes that are studied in this work. For the temporal discretization of the Stokes system (1.1) we consider a mesh $\{t_n\}$, $n = 0, \dots, N$, with $t_0 = 0$ and $t_N = T$, for the time variable t and define $\Delta t_n = t_n - t_{n-1}$. For $n = 0, 1, \dots, N$, let $\mathcal{T}_n = \{K\}$ be a quasi-uniform conforming triangulation of mesh size h_n , $0 < h_n < 1$, of the domain Ω at time t_n . The maximal mesh size is denoted by $h := \max_n h_n$. The finite element spaces to be used for approximating the velocity space $H_0^1(\Omega)$ and the pressure space $L^2(\Omega)$ at time t_n are of the form

$$\begin{aligned} V_n &= \{v_n \in H_0^1(\Omega) \mid v_n|_K \in \mathbb{P}(K) \quad \forall K \in \mathcal{T}_n\}, \\ Q_n &= \{q_n \in L^2(\Omega) \mid q_n|_K \in \mathbb{Q}(K) \quad \forall K \in \mathcal{T}_n\}, \end{aligned}$$

where $\mathbb{P}(K)$ and $\mathbb{Q}(K)$ are spaces of polynomials on K and $n = 0, 1, \dots, N$.

We assume that the discrete spaces satisfy the following approximation properties: for each $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$ and $p \in H^k(\Omega)$ and for $n = 0, 1, \dots, N$, there exist approximations $i_n u \in V_n$ and $j_n p \in Q_n$ such that

$$\begin{aligned} \|u - i_n u\| + h_n \|\nabla(u - i_n u)\| &\leq C h_n^{k+1} \|u\|_{k+1}, \\ \|p - j_n p\|_{L^2(\Omega)/\mathbb{R}} &\leq C h_n^k \|p\|_{H^k(\Omega)/\mathbb{R}}. \end{aligned} \quad (2.4)$$

The finite element space that is used for approximating the set $J(\Omega)$ of all divergence-free velocity fields at time t_n is defined by

$$J_n := \{u_n \in V_n \mid \langle q_n, \operatorname{div} u_n \rangle = 0 \quad \forall q_n \in Q_n\}. \quad (2.5)$$

We note that the finite element spaces V_n , Q_n and J_n no matching list are allowed to change in time. To ensure that the discrete pressure $p_n \in Q_n$ depends continuously on the discrete velocity $u_n \in J_n$ uniformly in h_n for $n = 0, 1, \dots, N$, we assume that the inf-sup condition holds.

For each $p_n \in Q_n$ there exists a nontrivial function $\varphi_n \in V_n$ such that

$$|\langle p_n, \operatorname{div} \varphi_n \rangle| \geq C \|\nabla \varphi_n\| \|p_n\|_{L^2/\mathbb{R}}. \quad (2.6)$$

For the solution $u \in J(\Omega)$ and $p \in L^2/\mathbb{R}$ of (1.1) we define approximations $S_n u \in J_n$ satisfying

$$\langle \nabla S_n u, \nabla v_n \rangle = \langle \nabla u, \nabla v_n \rangle - \langle p, \operatorname{div} v_n \rangle \quad \forall v_n \in J_n. \quad (2.7)$$

We remark that $S_n u$ is vaguely defined since S_n depends on the pressure as well. But for reasons of simplicity we follow the notation in Heywood & Rannacher (1982, Section 4).

Furthermore, we assume that the following error estimates are satisfied for $n = 0, 1, \dots, N$:

$$\begin{aligned} \|(I - S_n)u\| + h_n \|\nabla((I - S_n)u)\| &\leq Ch_n^{k+1} (\|u\|_{k+1} + \|p\|_{H^k/\mathbb{R}}), \\ \|(I - S_n)\partial_t u\| + h_n \|\nabla((I - S_n)\partial_t u)\| &\leq Ch_n^{k+1} (\|\partial_t u\|_{k+1} + \|\partial_t p\|_{H^k/\mathbb{R}}). \end{aligned} \quad (2.8)$$

Supposing that the solution of the steady form of the Stokes system is sufficiently regular, these estimates are verified, for example, in [Heywood & Rannacher \(1988\)](#).

Next, we define the L^2 or Helmholtz projection $\Pi_n : H_0^1(\Omega) \mapsto J_n$ by

$$\langle \Pi_n u, v_n \rangle = \langle u, v_n \rangle \quad \forall v_n \in J_n, \quad (2.9)$$

and likewise the H^1 or Stokes projection $\Lambda_n : H_0^1(\Omega) \mapsto J_n$ by

$$\langle \nabla \Lambda_n u, \nabla v_n \rangle = \langle \nabla u, \nabla v_n \rangle \quad \forall v_n \in J_n. \quad (2.10)$$

REMARK 2.1 An equivalent formulation of the above definitions in terms of the mixed finite element pair (V_n, Q_n) reads as follows: for $U \in V$ find a pair $(\hat{U}, \hat{P}) \in V_n \times Q_n$ such that

$$\begin{aligned} \langle \hat{U} - U, v_n \rangle + \langle \hat{P}, \operatorname{div} v_n \rangle &= 0 \quad \forall v_n \in V_n, \\ \langle q_n, \operatorname{div} \hat{U} \rangle &= 0 \quad \forall q_n \in Q_n \end{aligned} \quad (2.11)$$

for the Helmholtz projection and $q = 1$ for the Stokes projection, respectively.

$$\begin{aligned} \langle \nabla(\hat{U} - U), \nabla v_n \rangle + \langle \hat{P}, \operatorname{div} v_n \rangle &= 0 \quad \forall v_n \in V_n, \\ \langle q_n, \operatorname{div} \hat{U} \rangle &= 0 \quad \forall q_n \in Q_n. \end{aligned} \quad (2.12)$$

Here, we use the backward Euler method for discretizing the divergence-free weak form (2.2) of the Stokes system (1.1) in time. Then, we consider the following fully discrete scheme for solving (1.1).

ALGORITHM 2.2 Suppose that U_0 is an initial approximation of $u(\cdot, 0)$. For $n = 1, \dots, N$, first compute the projection $\hat{U}_{n-1} \in J_n$ by a grid projection, either $\hat{U}_{n-1} = \Pi_n U_{n-1}$ or $\hat{U}_{n-1} = \Lambda_n U_{n-1}$, and then compute $U_n \in J_n$ by

$$\left\langle \frac{U_n - \hat{U}_{n-1}}{\Delta t_n}, v_n \right\rangle + \langle \nabla U_n, \nabla v_n \rangle = \langle f_n, v_n \rangle \quad \forall v_n \in J_n, \quad (2.13)$$

as well as $P_n \in Q_n$ by

$$\langle P_n, \operatorname{div} v_n \rangle = \left\langle \frac{U_n - \hat{U}_{n-1}}{\Delta t_n}, v_n \right\rangle + \langle \nabla U_n, \nabla v_n \rangle - \langle f_n, v_n \rangle \quad \forall v_n \in V_n, \quad (2.14)$$

where $f_n = f(t_n)$.

3. Error estimates on dynamic meshes

A lot of work has been done on the numerical analysis of finite element approximations of the Stokes or even Navier–Stokes equations; cf., for example, [Heywood & Rannacher \(1982, 1990\)](#), [Bause \(2005\)](#),

Girault & Raviart (1987) and Temam (1983). Either energy techniques or functional analytical methods like semigroup theory have been applied. In particular, convergence rates have been proved. However, in all of these analyses, static finite element meshes have been assumed. As far as we know, error estimates for finite element approximations of the Stokes system on dynamic meshes have not been established yet. In this section we provide error estimates for the Stokes system on arbitrarily changing meshes. These estimates yield *upper* bounds. Assuming that the estimates are not too pessimistic they might give a better understanding of the effects of the divergence of finite element approximations on dynamic meshes in the case of decreasing time step sizes. In our numerical computations we have observed such cases of divergence. Examples illustrating this effect are given in Section 5.

For the sake of completeness and for comparison we shall recall some error estimates that were proved in He (2008) for finite element Navier–Stokes approximations on meshes not changing in time. If the Navier–Stokes equations are discretized in time by the backward Euler method and in space by a stable pair of finite element spaces for velocity and pressure of polynomial order k and $k - 1$, respectively, fulfilling (2.4), and if the mesh does not change in time, then for $n = 1, \dots, N$ it holds

$$\|U_n - u(t_n)\| \leq C(\Delta t + h^{k+1}) \quad \text{and} \quad \|P_n - p(t_n)\| \leq C(\Delta t + h^k). \quad (3.1)$$

3.1 *A priori estimates on dynamic meshes by the Helmholtz projection*

First, we consider Algorithm 2.2 equipped with the Helmholtz projection (2.9) of the velocity field U_{n-1} on the new spatial mesh $\mathcal{T}_n$. For this scheme the following error estimates can be established.

THEOREM 3.1 Suppose that u and p are sufficiently regular solutions of (2.2) and (2.3), respectively. For $n = 1, \dots, N$ let U_n and P_n be the solution of Algorithm 2.2 with grid projection (2.9). Then, the error estimates

$$\max_{1 \leq n \leq m} \|U_n - u_n\|^2 \leq C \left(E_0 + \max_{1 \leq n \leq m} \|(I - S_n)u_n\|^2 \right), \quad (3.2)$$

$$\sum_{n=1}^m \Delta t_n \|\nabla(U_n - u_n)\|^2 \leq C \left(E_0 + \sum_{n=1}^m \Delta t_n \|\nabla(I - S_n)u_n\|^2 \right), \quad (3.3)$$

$$\sum_{n=1}^m \Delta t_n \|P_n - p_n\|^2 \leq C \left(E_1 + \sum_{n=1}^m \Delta t_n \|\nabla(I - S_n)u_n\|^2 \right) \quad (3.4)$$

are satisfied for $1 \leq m \leq N$, where for $i = 0, 1$,

$$\begin{aligned} E_i = & \|U_0 - u_0\|^2 + \left[\sum_{n=1}^m \int_{t_{n-1}}^{t_n} \|(I - S_n)\partial_t u\| \, dt \right]^2 \\ & + \left[\sum_{n=1}^m \Delta t_n \int_{t_{n-1}}^{t_n} \|\partial_t^2 u\| \, dt \right]^2 + \left[\sum_{n=1}^m h_n^{-i} \|(S_n - S_{n-1})u_{n-1}\| \right]^2. \end{aligned}$$

Here, S_n denotes the projection defined by (2.7) and $(u_n, p_n) := (u(t_n), p(t_n))$.

Proof. We put

$$\begin{aligned}\theta_n &= U_n - S_n u_n, & \hat{\theta}_{n-1} &= \hat{U}_{n-1} - S_n u_{n-1}, \\ \eta_n &= u_n - S_n u_n, & \hat{\eta}_{n-1} &= u_{n-1} - S_n u_{n-1},\end{aligned}$$

and define the time discretization error

$$\rho_n := \frac{u_n - u_{n-1}}{\Delta t_n} - \partial_t u_n.$$

Subtracting (2.13) from (2.1) yields

$$\left\langle \frac{U_n - \hat{U}_{n-1}}{\Delta t_n} - \frac{u_n - u_{n-1}}{\Delta t_n}, v_n \right\rangle + \langle \nabla(U_n - u_n), \nabla v_n \rangle + \langle p_n, \operatorname{div} v_n \rangle = -\langle \rho_n, v_n \rangle,$$

where

$$\|\rho_n\| = \left\| \frac{u_n - u_{n-1}}{\Delta t_n} - \partial_t u_n \right\| \leq \int_{t_{n-1}}^{t_n} \|\partial_t^2 u\| \, dt. \quad (3.5)$$

Using (2.7), it follows that

$$\left\langle \frac{\theta_n - \hat{\theta}_{n-1}}{\Delta t_n}, v_n \right\rangle + \langle \nabla \theta_n, \nabla v_n \rangle = \left\langle \frac{\eta_n - \hat{\eta}_{n-1}}{\Delta t_n} - \rho_n, v_n \right\rangle \quad \forall v_n \in J_n. \quad (3.6)$$

The Helmholtz projection (2.9) satisfies

$$\langle \hat{\theta}_{n-1} - \theta_{n-1}, v_n \rangle = \langle \hat{\eta}_{n-1} - \eta_{n-1}, v_n \rangle \quad \forall v_n \in J_n. \quad (3.7)$$

Choosing $v_n = \theta_n$ in (3.6), with (3.7) and the identity

$$\langle \theta_n - \theta_{n-1}, \theta_n \rangle = \frac{1}{2}(\|\theta_n\|^2 - \|\theta_{n-1}\|^2 + \|\theta_n - \theta_{n-1}\|^2),$$

we obtain the error equation

$$\frac{1}{2}\|\theta_n\|^2 - \frac{1}{2}\|\theta_{n-1}\|^2 + \Delta t_n \|\nabla \theta_n\|^2 \leq \|\theta_n\|(\|\hat{\eta}_{n-1} - \eta_{n-1}\| + \|\eta_n - \hat{\eta}_{n-1}\| + \Delta t_n \|\rho_n\|). \quad (3.8)$$

Summing (3.8) from $n = 1$ to m with $1 \leq m \leq N$, we obtain

$$\begin{aligned}& \max_{1 \leq n \leq m} \|\theta_n\|^2 - \|\theta_0\|^2 + \sum_{n=1}^m \Delta t_n \|\nabla \theta_n\|^2 \\& \leq 2 \sum_{n=1}^m \|\theta_n\|(\|\hat{\eta}_{n-1} - \eta_{n-1}\| + \|\eta_n - \hat{\eta}_{n-1}\| + \Delta t_n \|\rho_n\|) \\& \leq \frac{1}{2} \max_{1 \leq n \leq m} \|\theta_n\|^2 + C \left(\left[\sum_{n=1}^m \|\hat{\eta}_{n-1} - \eta_{n-1}\| \right]^2 + \left[\sum_{n=1}^m \|\eta_n - \hat{\eta}_{n-1}\| \right]^2 + \left[\sum_{n=1}^m \Delta t_n \|\rho_n\| \right]^2 \right).\end{aligned} \quad (3.9)$$

For the interpolation error it holds

$$\left[\sum_{n=1}^m \|\eta_n - \hat{\eta}_{n-1}\| \right]^2 \leq C \left[\sum_{n=1}^m \int_{t_{n-1}}^{t_n} \|(I - S_n) \partial_t u\| dt \right]^2.$$

Together with the identity $\|\hat{\eta}_{n-1} - \eta_{n-1}\| = \|(S_n - S_{n-1})u_{n-1}\|$ this proves assertions (3.2) and (3.3).

It remains to show (3.4). The inf-sup condition (2.6) yields

$$\begin{aligned} \|P_n - p_n\| &\leq \|p_n - j_n p_n\| + C \sup_{v_n \in V_n} \frac{\langle P_n - j_n p_n, \operatorname{div} v_n \rangle}{\|\nabla v_n\|} \\ &\leq (1 + C) \|p_n - j_n p_n\| + C \sup_{v_n \in V_n} \frac{\langle P_n - p_n, \operatorname{div} v_n \rangle}{\|\nabla v_n\|}, \end{aligned} \quad (3.10)$$

where $j_n p_n$ is the approximation of p_n defined in (2.4). Subtracting (2.14) from (2.3), we get

$$\langle P_n - p_n, \operatorname{div} v_n \rangle = \left\langle \frac{U_n - u_n - (\hat{U}_{n-1} - u_{n-1})}{\Delta t_n}, v_n \right\rangle + \langle \nabla(U_n - u_n), \nabla v_n \rangle + \langle \rho_n, v_n \rangle. \quad (3.11)$$

Substituting (3.11) into (3.10), we obtain

$$\|P_n - p_n\| \leq C \left(\|p_n - j_n p_n\| + \frac{\|\theta_n - \hat{\theta}_{n-1}\|_{V_n^*}}{\Delta t_n} + \frac{\|\eta_n - \hat{\eta}_{n-1}\|_{V_n^*}}{\Delta t_n} + \|\nabla(U_n - u_n)\| + \|\rho_n\| \right), \quad (3.12)$$

where $\|\cdot\|_{V_n^*}$ is the discrete dual norm

$$\|w_n\|_{V_n^*} := \sup_{v_n \in V_n, v_n \neq 0} \frac{\langle w_n, v_n \rangle}{\|\nabla v_n\|}.$$

Squaring (3.12), multiplying by Δt_n and summing from $n = 1$ to m , it follows

$$\begin{aligned} \sum_{n=1}^m \Delta t_n \|P_n - p_n\|^2 &\leq C \left(\sum_{n=1}^m \frac{\|\theta_n - \hat{\theta}_{n-1}\|_{V_n^*}^2}{\Delta t_n} + \sum_{n=1}^m \frac{\|\eta_n - \hat{\eta}_{n-1}\|_{V_n^*}^2}{\Delta t_n} \right. \\ &\quad \left. + \sum_{n=1}^m \Delta t_n \|p_n - j_n p_n\|^2 + \sum_{n=1}^m \Delta t_n \|\nabla(U_n - u_n)\|^2 + \sum_{n=1}^m \Delta t_n \|\rho_n\|^2 \right). \end{aligned} \quad (3.13)$$

Note that

$$\frac{\|\eta_n - \hat{\eta}_{n-1}\|_{V_n^*}^2}{\Delta t_n} = \frac{1}{\Delta t_n} \left\| (1 - S_n) \int_{t_{n-1}}^{t_n} \partial_t u(t) dt \right\|_{V_n^*}^2 \leq C \Delta t_n \|(1 - S_n) \partial_t u(t)\|^2. \quad (3.14)$$

Owing to (2.14), (3.3), (3.5) and (3.14), it only remains to estimate the first term on the right-hand side of (3.13).

Testing (3.6) by $v_n = \theta_n - \hat{\theta}_{n-1}$ yields, using the inverse estimate,

$$\frac{1}{\Delta t_n} \|\theta_n - \hat{\theta}_{n-1}\|^2 \leq \left(\frac{1}{\Delta t_n} \|\eta_n - \hat{\eta}_{n-1}\| + \|\rho_n\| \right) \|\theta_n - \hat{\theta}_{n-1}\| + Ch_n^{-1} \|\nabla \theta_n\| \|\theta_n - \hat{\theta}_{n-1}\|.$$

By applying Young's inequality and summing we obtain

$$\sum_{n=1}^m \frac{\|\theta_n - \hat{\theta}_{n-1}\|^2}{\Delta t_n} \leq C \left(\sum_{n=1}^m \Delta t_n \frac{\|\eta_n - \hat{\eta}_{n-1}\|^2}{\Delta t_n} + \sum_{n=1}^m \Delta t_n \|\rho_n\|^2 + \sum_{n=1}^m h_n^{-1} \Delta t_n \|\nabla \theta_n\|^2 \right). \quad (3.15)$$

Combining this inequality with (3.5), (3.14) and (3.9) proves the assertion (3.4). $\square$

Inequalities (3.2–3.4) show that the discretization error of the velocity and pressure variable are governed by four quantities reflecting different sources of errors: the first of the contributions to the right-hand side terms in (3.2–3.4) measures the error of the approximation of the initial value u_0 , the second and third term of the right-hand sides describe the error of the discretization in space and time, respectively, and the last part describes the projection error that is caused by the mesh changes.

The projection error is bounded from above by

$$\begin{aligned} \left[\sum_{n=1}^m \|(S_n - S_{n-1})u_{n-1}\| \right]^2 &= \left[\sum_{n=1}^m \|(S_n - I)u_{n-1} - (S_{n-1} - I)u_{n-1}\| \right]^2 \\ &\leq \left[\sum_{n=1}^m \delta_{n-1} (\|(S_n - I)u_{n-1}\| + \|(S_{n-1} - I)u_{n-1}\|) \right]^2, \end{aligned} \quad (3.16)$$

where $\delta_n = 0$ if $\mathcal{T}_n = \mathcal{T}_{n-1}$ and $\delta_n = 1$ otherwise. Applying (2.8), we obtain from (3.16) that

$$\sum_{n=1}^m \|(S_n - S_{n-1})u_{n-1}\| \leq \sum_{n=1}^m \delta_{n-1} (h_n^{k+1} + h_{n-1}^{k+1}) \leq C \cdot M h^{k+1} \leq C \frac{h^{k+1}}{\Delta t}, \quad (3.17)$$

where $M \leq N$ denotes the number of mesh changes, $h = \max_{0 \leq n \leq N} h_n$ and $\Delta t = \min_{1 \leq n \leq N} \Delta t_n$. In (3.17), each mesh change leads to an additional error part depending reciprocally on the time step size Δt . Of course, we do not know if this estimate is really sharp. It might be pessimistic. The sharpness of (3.17) will be studied further in Section 5 by numerical experiments.

Together, Theorem 3.1 and inequality (3.17) yield the following asymptotic error estimates for fully discrete Stokes approximations on dynamic finite element meshes if the Helmholtz projection (2.9) of the velocity field U_{n-1} onto the new spatial mesh $\mathcal{T}_n$ is used.

COROLLARY 3.2 Under the assumptions of Theorem 3.1 it holds

$$\begin{aligned} \max_{1 \leq n \leq m} \|U_n - u_n\|^2 &= \mathcal{O} \left(\|U_0 - u_0\|^2 + \Delta t^2 + h^{2(k+1)} + \frac{h^{2(k+1)}}{\Delta t^2} \right), \\ \sum_{n=1}^m \Delta t \|\nabla(U_n - u_n)\|^2 &= \mathcal{O} \left(\|U_0 - u_0\|^2 + \Delta t^2 + h^{2k} + \frac{h^{2(k+1)}}{\Delta t^2} \right), \\ \sum_{n=1}^m \Delta t \|P_n - p_n\|^2 &= \mathcal{O} \left(\|U_0 - u_0\|^2 + \Delta t^2 + h^{2k} + \frac{h^{2k}}{\Delta t^2} \right), \end{aligned} \quad (3.18)$$

where $h = \max_{1 \leq n \leq m} h_n$.

REMARK 3.3 Note that the projection error (3.16) also occurs in the case of the heat equation, where the Stokes operator is replaced by the elliptic projection operator; cf. Yang (2000). However, because of the saddle point structure of the Stokes system, is more delicate.

3.2 A priori estimates on dynamic meshes by the Stokes projection

Next, we study Algorithm 2.2 equipped with the Stokes projection (2.10) of the velocity field U_{n-1} onto the new spatial mesh $\mathcal{T}_n$.

THEOREM 3.4 Suppose that u and p are sufficiently regular solutions of (2.2) and (2.3). For $n = 1, \dots, N$ let U_n and P_n be solutions of Algorithm 2.2 with grid projection (2.10). Then, the error estimates

$$\sum_{n=1}^m \Delta t_n \|U_n - u_n\|^2 \leq C \left(H + \sum_{n=1}^m \Delta t_n \|(I - S_n)u_n\|^2 \right), \quad (3.19)$$

$$\max_{1 \leq n \leq m} \|\nabla(U_n - u_n)\|^2 \leq C \left(H + \max_{1 \leq n \leq m} \|\nabla(I - S_n)u_n\|^2 \right), \quad (3.20)$$

$$\sum_{n=1}^m \Delta t_n \|P_n - p_n\|^2 \leq C \left(H + \sum_{n=1}^m \Delta t_n \|(I - S_n)u_n\|^2 \right) \quad (3.21)$$

are satisfied for $1 \leq m \leq N$, where

$$\begin{aligned} H &= \|\nabla(U_0 - u_0)\|^2 + \sum_{n=1}^m \frac{1}{\Delta t_n} \left[\int_{t_{n-1}}^{t_n} \|(I - S_n)\partial_t u\| \, dt \right]^2 \\ &\quad + \sum_{n=1}^m \Delta t_n \left[\int_{t_{n-1}}^{t_n} \|\partial_t^2 u\| \, dt \right]^2 + \sum_{n=1}^m \frac{\|\nabla(S_n - S_{n-1})u_{n-1}\|^2}{\Delta t_n}. \end{aligned}$$

Here, S_n denotes the projection defined by (2.7) and $(u_n, p_n) := (u(t_n), p(t_n))$.

Proof. First, we prove the assertion (3.20). Similarly to the proof of Theorem 3.1 we put

$$\begin{aligned} \theta_n &= U_n - S_n u_n, & \hat{\theta}_{n-1} &= \hat{U}_{n-1} - S_n u_{n-1}, \\ \eta_n &= u_n - S_n u_n, & \hat{\eta}_{n-1} &= u_{n-1} - S_n u_{n-1}. \end{aligned}$$

Now we choose $v_n = \theta_n - \hat{\theta}_{n-1}$ in the error equation (3.6) and obtain by Young's inequality,

$$\frac{\|\theta_n - \hat{\theta}_{n-1}\|^2}{\Delta t_n} + \|\nabla \theta_n\|^2 - \|\nabla \hat{\theta}_{n-1}\|^2 \leq 2\Delta t_n \|\rho_n\|^2 + 2\frac{\|\eta_n - \hat{\eta}_{n-1}\|^2}{\Delta t_n}. \quad (3.22)$$

Next, we note that the Stokes projection (2.10) implies

$$\langle \nabla \hat{\theta}_{n-1} - \nabla \theta_{n-1}, \nabla v_n \rangle = \langle \nabla \hat{\eta}_{n-1} - \nabla \eta_{n-1}, \nabla v_n \rangle \quad \forall v_n \in J_n.$$

Choosing $v_n = \hat{\theta}_{n-1}$ we obtain, by Young's inequality,

$$\|\nabla \hat{\theta}_{n-1}\|^2 - \|\nabla \theta_{n-1}\|^2 \leq 2\sqrt{2}\|\nabla \hat{\eta}_{n-1} - \nabla \eta_{n-1}\| \|\nabla \theta_{n-1}\| + 4\|\nabla \hat{\eta}_{n-1} - \nabla \eta_{n-1}\|^2. \quad (3.23)$$

Inserting (3.23) into (3.22) and summing the resulting inequality from $n = 1$ to m gives

$$\begin{aligned} & \sum_{n=1}^m \frac{\|\theta_n - \hat{\theta}_{n-1}\|^2}{\Delta t_n} + \max_{1 \leq n \leq m} \|\nabla \theta_n\|^2 \\ & \leq \|\nabla \theta_0\|^2 + 2\sqrt{2} \max_{0 \leq n \leq m} \|\nabla \theta_n\| \sum_{n=1}^m \|\nabla \hat{\eta}_{n-1} - \nabla \eta_{n-1}\| \\ & \quad + C \sum_{n=1}^m \left(\Delta t_n \|\rho_n\|^2 + \frac{\|\eta_n - \hat{\eta}_{n-1}\|^2}{\Delta t_n} + \|\nabla \hat{\eta}_{n-1} - \nabla \eta_{n-1}\|^2 \right) \\ & \leq \|\nabla \theta_0\|^2 + \frac{1}{2} \max_{0 \leq n \leq m} \|\nabla \theta_n\|^2 + C \left(\left[\sum_{n=1}^m \|\nabla \hat{\eta}_{n-1} - \nabla \eta_{n-1}\| \right]^2 \right. \\ & \quad \left. + \sum_{n=1}^m \left(\Delta t_n \|\rho_n\|^2 + \frac{\|\eta_n - \hat{\eta}_{n-1}\|^2}{\Delta t_n} + \|\nabla \hat{\eta}_{n-1} - \nabla \eta_{n-1}\|^2 \right) \right). \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} & \sum_{n=1}^m \frac{\|\theta_n - \hat{\theta}_{n-1}\|^2}{\Delta t_n} + \max_{1 \leq n \leq m} \|\nabla \theta_n\|^2 \\ & \leq C \left(\|\nabla \theta_0\|^2 + \sum_{n=1}^m \Delta t_n \|\rho_n\|^2 + \sum_{n=1}^m \frac{\|\eta_n - \hat{\eta}_{n-1}\|^2}{\Delta t_n} + \sum_{n=1}^m \frac{\|\nabla \hat{\eta}_{n-1} - \nabla \eta_{n-1}\|^2}{\Delta t_n} \right). \quad (3.24) \end{aligned}$$

Finally, Eq. (3.20) is proved by applying the triangle inequality to $\|\nabla(U_n - u_n)\|$ and using the estimate (3.24).

To show the error estimate (3.19), we multiply (3.6) by Δt_n^2 and choose $v_n = \theta_n$. Then, we obtain that

$$\frac{\Delta t_n}{2} (\|\theta_n\|^2 - \|\hat{\theta}_{n-1}\|^2 + \|\theta_n - \hat{\theta}_{n-1}\|^2) + \Delta t_n^2 \|\nabla \theta_n\|^2 \leq \Delta t_n^2 \|\rho_n\| \|\theta_n\| + \Delta t_n \|\eta_n - \hat{\eta}_{n-1}\| \|\theta_n\|. \quad (3.25)$$

Next, in (3.25) we shall bound the term $\|\hat{\theta}_{n-1}\|$ from above. By the Poincaré inequality together with inequality (3.23) we get

$$\|\hat{\theta}_{n-1}\|^2 \leq C \|\nabla \hat{\theta}_{n-1}\|^2 \leq C (\|\nabla \theta_{n-1}\|^2 + \|\nabla \hat{\eta}_{n-1} - \nabla \eta_{n-1}\|^2). \quad (3.26)$$

Combining (3.25) and (3.26) and summing the resulting relation from $n = 1$ to m yields

$$\begin{aligned} \sum_{n=1}^m \Delta t_n \|\theta_n\|^2 &\leq C \left(\sum_{n=1}^m \Delta t_n^3 \|\rho_n\|^2 + \sum_{n=1}^m \Delta t_n \|\eta_n - \hat{\eta}_{n-1}\|^2 \right. \\ &\quad \left. + \sum_{n=1}^m \Delta t_n \|\nabla \theta_{n-1}\|^2 + \sum_{n=1}^m \Delta t_n \|\nabla \hat{\eta}_{n-1} - \nabla \eta_{n-1}\|^2 \right), \end{aligned} \quad (3.27)$$

where the third term on the right-hand side of (3.27) can be further estimated by means of

$$\sum_{n=1}^m \Delta t_n \|\nabla \theta_{n-1}\|^2 \leq \max_{1 \leq n \leq m} \|\nabla \theta_{n-1}\|^2 \sum_{n=1}^m \Delta t_n \leq T \max_{0 \leq n \leq m} \|\nabla \theta_n\|^2.$$

Combined with (3.24), this proves the assertion (3.19).

The proof of the error estimate (3.21) for the pressure variable directly follows from inequalities (3.13) and (3.24). $\square$

Similarly to the case of the Helmholtz projection of U_{n-1} on $\mathcal{T}_n$ considered above, the upper bounds given in Theorem 3.4 for the Stokes projection are governed by four quantities again, which measure the different sources of errors. Estimating the error caused by the mesh changes as in (3.16) and (3.17), it holds that

$$\sum_{n=1}^m \|\nabla (S_n - S_{n-1}) u_{n-1}\| \leq C \frac{h^k}{\Delta t}, \quad (3.28)$$

where $h = \max_n h_n$ and $\Delta t = \min_n \Delta t_n$. Of course, the upper bound in (3.28) might be pessimistic again.

Together, Theorem 3.4 and inequality (3.28) yield the following asymptotic error estimates for fully discrete Stokes approximations on dynamic finite element meshes if the Stokes projection (2.10) of the velocity field U_{n-1} onto the new spatial mesh $\mathcal{T}_n$ is used and constant time steps Δt_n are assumed.

COROLLARY 3.5 Under the assumptions of Theorem 3.4 it holds that

$$\begin{aligned} \sum_{n=1}^m \Delta t \|U_n - u_n\|^2 &= \mathcal{O} \left(\|\nabla (U_0 - u_0)\|^2 + \Delta t^2 + h^{2(k+1)} + \frac{h^{2k}}{\Delta t^2} \right), \\ \max_{1 \leq n \leq m} \|\nabla (U_n - u_n)\|^2 &= \mathcal{O} \left(\|\nabla (U_0 - u_0)\|^2 + \Delta t^2 + h^{2k} + \frac{h^{2k}}{\Delta t^2} \right), \\ \sum_{n=1}^m \Delta t \|P_n - p_n\|^2 &= \mathcal{O} \left(\|\nabla (U_0 - u_0)\|^2 + \Delta t^2 + h^{2k} + \frac{h^{2k}}{\Delta t^2} \right), \end{aligned} \quad (3.29)$$

where $h = \max_{1 \leq n \leq m} h_n$.

We shall now compare the error estimates given in Theorem 3.1 for the Helmholtz projection (2.9) of the velocity field U_{n-1} onto the new spatial mesh $\mathcal{T}_n$ with the error estimates given in Theorem 3.4 for the Stokes projection (2.10) of U_{n-1} . For this we introduce the discrete norms

$$\begin{aligned}\|v\|_{\ell^2(L^2)} &:= \|v\|_{\ell^2(0,T;L^2(\Omega))} := \left(\sum_{1 \leq n \leq m} \Delta t_n \|v(\cdot, t_n)\|_{L^2(\Omega)}^2 \right)^{1/2}, \\ \|v\|_{\ell^\infty(L^2)} &:= \|v\|_{\ell^\infty(0,T;L^2(\Omega))} := \max_{1 \leq n \leq m} \|v(\cdot, t_n)\|_{L^2(\Omega)}.\end{aligned}$$

According to (3.17) and (3.28), an estimate of the contributions to the error bounds in Theorems 3.1 and 3.4, respectively, that is, due to the projection of U_{n-1} on $\mathcal{T}_n$ is given by

$$\sum_{n=1}^m |(S_n - S_{n-1})u_{n-1}|_q \leq Ch^{k+1-q}, \quad (3.30)$$

where $q=1$ for the Stokes projection (2.10) and $q=0$ for the Helmholtz projection (2.9) with $h = \max_{1 \leq n \leq N} h_n$. Of course, the estimate (3.30) might be pessimistic. This will be studied further in Section 5. Combined with (3.30), Theorems 3.1 and 3.4 imply the following convergence rates for fully discrete Stokes approximations on finite element meshes varying in time: for the velocity field it holds that

$$\begin{aligned}\text{Helmholtz projection: } \|U - u\|_{\ell^\infty(L^2)} &= \mathcal{O} \left(\|U_0 - u_0\| + \Delta t + h^{k+1} + \frac{h^{k+1}}{\Delta t} \right), \\ \text{Stokes projection: } \|U - u\|_{\ell^2(L^2)} &= \mathcal{O} \left(\|\nabla(U_0 - u_0)\| + \Delta t + h^{k+1} + \frac{h^k}{\Delta t} \right),\end{aligned}$$

and for the pressure variable we have that

$$\begin{aligned}\text{Helmholtz projection: } \|P - p\|_{\ell^2(L^2)} &= \mathcal{O} \left(\|U_0 - u_0\| + \Delta t + h^k + \frac{h^k}{\Delta t} \right), \\ \text{Stokes projection: } \|P - p\|_{\ell^2(L^2)} &= \mathcal{O} \left(\|\nabla(U_0 - u_0)\| + \Delta t + h^k + \frac{h^k}{\Delta t} \right).\end{aligned}$$

First, we note that our upper error bounds for Stokes approximations on dynamic finite element meshes depend reciprocally on the time step size which implies a restriction of the mesh size h with respect to the magnitude of the temporal step size Δt in order to ensure convergence for vanishing mesh sizes in time and space. On static meshes the error estimates given in (3.1) yield convergence without such a smallness condition for h . Furthermore, we observe that in the approximation of the velocity field, one order of convergence gets lost with respect to the discretization in space if the Stokes projection instead of the Helmholtz projection of U_{n-1} onto the new spatial mesh $\mathcal{T}_n$ is used. Moreover, the error estimate holds in the weaker $\|\cdot\|_{\ell^2(L^2)}$ norm. We note that the reciprocal dependence on Δt arises only in the part of the upper bound originating from mesh changes. Our numerical calculations in Section 5 will show that both variants of projecting U_{n-1} onto the new spatial mesh $\mathcal{T}_n$ are still unsatisfactory and can lead to divergence of the fully discrete unknowns if completely uncontrolled mesh changes are performed.

4. Projection error for mesh refining and coarsening

In this section we study the error behaviour of finite element Stokes approximations in the case that only mesh refinements and coarsenings are permitted. In contrast to Section 3 we thus no longer allow arbitrary mesh changes. By restricting ourselves to refinement and coarsening steps we mimic mesh changes that could be based on adaptivity. This is the most popular application of dynamic mesh changes in practice. We will prove error bounds that no longer depend reciprocally on the time step size and hold uniformly with respect to the time step size, similarly to the case of static finite element meshes.

In (3.17) and (3.30), the worst case estimate

$$\sum_{n=1}^m |(S_n - S_{n-1})u_{n-1}|_q \leq C \frac{h^{k+1-q}}{\Delta t}$$

was given for the error parts of Stokes approximations that are due to the changes, where $q = 0$ if the Helmholtz projection (2.9) is used and $q = 1$ if the Stokes projection (2.10) is used. If we restrict the mesh changes to pure refinement steps, then the following sharper estimate of the error parts (cf. Theorems 3.1 and 3.4) reflecting the mesh changes can be established.

THEOREM 4.1 Suppose that u and p are sufficiently regular solutions of (2.2) and (2.3). Let h_0 denote the finite element mesh size of the initial grid $\mathcal{T}_0$. Suppose that the initial grid is at most Q times, $Q \leq m$, uniformly refined with mesh sizes $h_i = \frac{1}{2}h_{i-1}$, $1 \leq i \leq Q$. Then, for the error part that is due to the mesh changes (cf. Theorems 3.1 and 3.4), it holds that

$$\sum_{n=1}^m |(S_n - S_{n-1})u_{n-1}|_q \leq Ch_0^{k+1-q} (\|u\|_{L^\infty([0,T];H^{k+1-q})} + \|p\|_{L^\infty([0,T];H^{k-q}\setminus\mathbb{R})}), \quad (4.1)$$

where $q = 0$ for the Helmholtz projection and $q = 1$ for the Stokes projection. The constant C does not depend on the number of refinements.

Proof. Using (2.8), (3.16) and (3.17), we obtain that

$$\sum_{n=1}^m |(S_n - S_{n-1})u_{n-1}|_q \leq C \sum_{i=1}^Q (h_i^{k+1-q} - h_{i-1}^{k+1-q}) A_{i-1},$$

where $A_i = \|u(t_i)\|_{k+1-q} + \|p(t_i)\|_{H^{k-q}\setminus\mathbb{R}}$ and $i \in I = \{n = 1, \dots, m \mid \mathcal{T}_n \neq \mathcal{T}_{n-1}\}$. For regular mesh refinements with $h_i = \frac{1}{2}h_{i-1}$ and, therefore, $h_i = h_0/2^{-i}$ it follows that

$$\begin{aligned} \sum_{n=1}^m |(S_n - S_{n-1})u_{n-1}|_q &\leq Ch_0^{k+1-q} \|A\|_{L^\infty([0,T])} \sum_{i=1}^Q 2^{-i(k+1-q)} + 2^{-(i-1)(k+1-q)} \\ &\leq Ch_0^{k+1-q} \|A\|_{L^\infty([0,T])} \sum_{i=0}^Q 2^{-i(k+1-q)}. \end{aligned}$$

Since the sum on the right-hand side is bounded from above independently of Q , assertion (4.1) is proved. $\square$

REMARK 4.2 If pure mesh coarsenings are permitted, we get analogously that

$$\sum_{n=1}^m |(S_n - S_{n-1})u_{n-1}|_q \leq Ch_Q^{k+1-q} (\|u\|_{L^\infty([0,T];H^{k+1-q})} + \|p\|_{L^\infty([0,T];H^{k-q}(\mathbb{R})}), \quad (4.2)$$

where h_Q denotes the finite element mesh size of the coarsest grid.

Together, inequalities (4.1) and (4.2) indicate that mesh changes based on a refining and coarsening strategy of adaptivity should not lead to diverging approximations due to the performed mesh changes. This conclusion is in accordance with the observations and results that are presented in [Besier & Wollner \(2011\)](#).

REMARK 4.3 [Karakashian & Makridakis \(2004\)](#) study space-time continuous Galerkin methods with mesh modification in time for semilinear second-order hyperbolic equations. They show that mesh modifications lead to additional terms, known as ‘jump terms’, in the *a priori* error analysis if no further restrictions on the mesh changes are made. Precisely, the jump terms are the elliptic projection of the solution and its time derivative at times t_n where mesh changes occur. These terms, albeit small, might cause the divergence of the numerical method. Thus, the structure of the *a priori* error estimate that is given in [Karakashian & Makridakis \(2004\)](#) for hyperbolic problems is in good agreement with our error estimates for Stokes approximations given Theorems 3.1 and 3.4. Karakashian and Makridakis further show that the jump terms can be eliminated under the ‘reasonable’ assumption that all meshes contain a reference mesh $\mathcal{T}_h$. In such cases, convergence rates of optimal order can be guaranteed. Finally, they prove that an increased accuracy is obtained for the jump terms if the meshes differ only in regions of small area. In the two-dimensional case this region has to be an area of size $\mathcal{O}(h^2)$. Under this assumption about the meshes, optimal convergence rates are established again for the second-order hyperbolic model problem. Such restrictions on the mesh changes are not studied here because our interest is to show theoretically and numerically that arbitrary mesh changes can lead to divergence of Stokes approximations, even for divergence-free projections onto the modified meshes, and to stimulate interest in analysing this phenomenon.

5. Numerical studies

In this section we study numerically the behaviour of finite element approximations of the nonstationary Stokes equations on meshes varying in time. We show that changing the mesh in an ‘uncontrolled way’ between two time steps can cause a negative order of convergence of the discrete solution. Moreover, the numerical calculations illustrate the *a priori* convergence rates that were proved in Section 3 and characterize to what extent they are sharp. Numerical calculations are presented for both the Helmholtz and the Stokes projection.

In [Besier & Wollner \(2011\)](#) it is shown that a small number of mesh changes (in their case one mesh change) could be tolerated without significant loss in accuracy if the Helmholtz or the Stokes projection of the velocity approximation is used instead of its L^2 projection for evaluating the discrete velocity on the new spatial mesh. Here, we will show that this observation does not hold true if the mesh varies in each time step as might arise in complex applications with sharp moving fronts or free boundaries. In Fig. 4 we demonstrate that the pressure error of a single mesh change will be smoothed, but if we change the meshes several times, the error blows up.

EXAMPLE 5.1 In our first example we study the convergence rates of the finite element approximation of the nonstationary Stokes problem. We consider problem (1.1) with $\Omega = (0, 1)^2$

FIG. 4. Plots of time versus pressure error $\|P_n - p_n\|$ for Example 5.1 with time step $\Delta t = 0.0001$. On the left-hand side two mesh changes, on the right-hand side ten mesh changes between triangulations with grid parameter $h = 0.2$; cf. Fig. 5.

FIG. 5. Iterating triangulation in $(0, 1)^2$ with grid parameter $h = 0.1$.

on the time interval $[0, 0.01]$. The analytical solution is given by $u(x_1, x_2) = (\frac{1}{2} \sin(4\pi x_1) \cos(2\pi x_2), \sin(2\pi x_2)(2 \sin^2(2\pi x_1) - 1))^T$ and $p(x_1, x_2) = \frac{5}{2} \cos(4\pi x_1) \cos(2\pi x_2)$ with a corresponding right-hand-side term f . On the boundary $\partial\Omega$ we impose the Dirichlet data given by the analytical solution. The numerical solution is calculated by NAVIER (Bänsch, 1998) using $\mathcal{P}_2/\mathcal{P}_1$ -Taylor–Hood elements. Since it is impossible to computationally realize arbitrary mesh changes without committing *variational crimes* we use meshes that are switched every time step between the triangulations shown in Fig. 5.

In Table 1 we summarize the calculated errors of velocity, velocity gradient and pressure for the Helmholtz projection for grid parameter $h = 0.05$. We observe that under uniform refinement of the temporal discretization the errors increase as claimed in Theorem 3.1 and Corollary 3.2. The (divergence) rates given there are slightly more pessimistic compared with the numerical results that give a rate between $1/\sqrt{\Delta t}$ and $1/\Delta t$.

The results based on using the Stokes projection are summarized in Table 2 for grid parameter $h = 0.05$. Similarly to the case of the Helmholtz projection we observe divergence of the solution for decreasing time step sizes, as predicted by Theorem 3.4 and Corollary 3.5. Again, the numerical results are slightly less pessimistic, yielding rates between $1/\sqrt{\Delta t}$ and $1/\Delta t$.

To compare the Helmholtz with the Stokes projection in Fig. 6, the velocity error $\|U - u\|_{\ell^\infty(L^2)}$ and the pressure error $\|P - P\|_{\ell^2(L^2)}$ are plotted for the grid parameter $h = 0.05$. We realize that the Stokes projection is more sensitive concerning the predicted convergence rate than the Helmholtz projection.

TABLE 1 *Time step, error of velocity, velocity gradient and pressure with respective experimental order of convergence for the Helmholtz projection with grid parameter $h = 0.05$*

Δt	$\ U - u\ _{\ell^\infty(L^2)}$			$\ U - u\ _{\ell^2(L^2)}$			$\ \nabla(U - u)\ _{\ell^\infty(L^2)}$			$\ \nabla(U - u)\ _{\ell^2(L^2)}$			$\ P - p\ _{\ell^2(L^2)}$		
2.00e-3	3.13e-4	—		2.90e-5	—		4.00e-2	—		3.73e-3	—		1.63e-2	—	
1.00e-3	3.16e-4	-0.01		2.93e-5	-0.01		3.94e-2	0.02		3.73e-3	0.00		1.61e-2	0.02	
5.00e-4	3.17e-4	-0.00		2.93e-5	-0.00		3.88e-2	0.03		3.67e-3	0.02		1.59e-2	0.02	
2.50e-4	3.21e-4	-0.02		3.00e-5	-0.03		3.88e-2	-0.00		3.67e-3	-0.00		1.58e-2	0.01	
1.25e-4	3.53e-4	-0.14		3.28e-5	-0.13		4.18e-2	-0.11		3.88e-3	-0.08		1.52e-2	0.05	
6.25e-5	4.49e-4	-0.35		4.03e-5	-0.30		5.05e-2	-0.27		4.48e-3	-0.21		1.50e-2	0.03	
3.13e-5	6.56e-4	-0.55		5.63e-5	-0.48		6.41e-2	-0.34		5.53e-3	-0.30		1.52e-2	-0.03	
1.56e-5	1.07e-3	-0.71		8.83e-5	-0.65		7.86e-2	-0.29		7.04e-3	-0.35		1.54e-2	-0.02	
7.81e-6	1.89e-3	-0.82		1.52e-4	-0.78		9.75e-2	-0.31		9.08e-3	-0.37		1.67e-2	-0.11	
3.91e-6	3.50e-3	-0.89		2.77e-4	-0.87		1.28e-1	-0.39		1.18e-2	-0.38		1.93e-2	-0.21	
1.95e-6	6.67e-3	-0.93		5.23e-4	-0.91		1.78e-1	-0.47		1.60e-2	-0.43		2.46e-2	-0.35	
9.77e-7	1.28e-2	-0.94		9.98e-4	-0.93		2.73e-1	-0.62		2.35e-2	-0.56		3.42e-2	-0.48	
4.88e-7	2.43e-2	-0.93		1.90e-3	-0.93		4.60e-1	-0.75		3.81e-2	-0.70		5.04e-2	-0.56	

TABLE 2 *Time step, error of velocity, velocity gradient and pressure with respective experimental order of convergence for the Stokes projection with grid parameter $h = 0.05$*

Δt	$\ U - u\ _{\ell^\infty(L^2)}$			$\ U - u\ _{\ell^2(L^2)}$			$\ \nabla(U - u)\ _{\ell^\infty(L^2)}$			$\ \nabla(U - u)\ _{\ell^2(L^2)}$			$\ P - p\ _{\ell^2(L^2)}$		
2.00e-3	6.37e-4	—		4.91e-5	—		4.08e-2	—		3.79e-3	—		1.63e-2	—	
1.00e-3	1.19e-3	-0.90		8.77e-5	-0.84		4.35e-2	-0.09		3.96e-3	-0.06		1.61e-2	0.02	
5.00e-4	2.33e-3	-0.97		1.71e-4	-0.96		5.32e-2	-0.29		4.55e-3	-0.20		1.61e-2	-0.00	
2.50e-4	4.60e-3	-0.98		3.39e-4	-0.99		8.20e-2	-0.62		6.52e-3	-0.52		1.62e-2	-0.02	
1.25e-4	9.06e-3	-0.98		6.72e-4	-0.99		1.49e-1	-0.86		1.14e-2	-0.81		1.75e-2	-0.11	
6.25e-5	1.77e-2	-0.96		1.32e-3	-0.97		2.83e-1	-0.93		2.17e-2	-0.92		2.23e-2	-0.35	
3.13e-5	3.37e-2	-0.93		2.54e-3	-0.95		5.34e-1	-0.92		4.12e-2	-0.93		3.47e-2	-0.64	
1.56e-5	6.18e-2	-0.88		4.74e-3	-0.90		9.72e-1	-0.86		7.63e-2	-0.89		5.87e-2	-0.76	
7.81e-6	1.06e-1	-0.78		8.40e-3	-0.82		1.65e+0	-0.77		1.34e-1	-0.81		9.69e-2	-0.72	
3.91e-6	1.67e-1	-0.65		1.37e-2	-0.70		2.55e+0	-0.62		2.14e-1	-0.68		1.47e-1	-0.60	
1.95e-6	2.34e-1	-0.49		2.00e-2	-0.55		3.50e+0	-0.46		3.08e-1	-0.52		1.98e-1	-0.43	
9.77e-7	2.94e-1	-0.33		2.63e-2	-0.39		4.30e+0	-0.30		3.95e-1	-0.36		2.38e-1	-0.27	
4.88e-7	3.35e-1	-0.19		3.13e-2	-0.25		4.85e+0	-0.17		4.61e-1	-0.22		2.64e-1	-0.15	

Furthermore, we observe a smoothing of the error for $\Delta t \rightarrow 0$ for the Stokes projection. This seems not completely surprising, since we do not expect the discrete solution actually to tend to infinity. Note that also in the classical example in Dupont (1982), the error tends to a fixed value (the numerical solution there just decayed to 0) when $\Delta t \rightarrow 0$ and that our analysis provides just an *upper* bound for the error. The main point, however, is that for a large range of time step sizes the error increases and the negative order of convergence is close to the predicted one, at least for the velocity.

Thus, we may conclude that neither the Helmholtz nor the Stokes projection can cure the effect of too-frequent mesh changes.

In Fig. 7 we visualize profiles of the calculated solution at different times to illustrate how the solution deteriorates in time for a small time step size.

FIG. 6. Plots of velocity $\|U - u\|_{\ell^\infty(L^2)}$ and pressure error $\|P - p\|_{\ell^2(L^2)}$ versus time step size for grid parameter $h = 0.05$.

EXAMPLE 5.2 In our second numerical experiment we study the model problem introduced in [Besier & Wollner \(2011, Section 3.2\)](#). Here, the exact solution of the Stokes problem on $\Omega = (0, 1)^2$ is given by

$$u(x_1, x_2) = \begin{pmatrix} \cos^2\left(\frac{\pi x_1}{2}\right) \cos\left(\frac{\pi x_2}{2}\right) \sin\left(\frac{\pi x_2}{2}\right) \\ -\cos\left(\frac{\pi x_1}{2}\right) \sin\left(\frac{\pi x_1}{2}\right) \cos^2\left(\frac{\pi x_2}{2}\right) \end{pmatrix},$$

$$p(x_1, x_2) = \cos\left(\frac{\pi x_1}{2}\right) \sin\left(\frac{\pi x_1}{2}\right) \cos\left(\frac{\pi x_2}{2}\right) \sin\left(\frac{\pi x_2}{2}\right).$$

FIG. 7. Pressure (top) and velocity magnitude (bottom) for the Stokes projection with $\Delta t = 3.91\text{e-}6$ and $h = 0.05$ at times $t = 0, 6.25\text{e-}4, 3.00\text{e-}3$ and $9.00\text{e-}3$. The figures show the deterioration of the solution in time for a small time step size.

TABLE 3 Time step, error of velocity and pressure with experimental order of convergence for Helmholtz and Stokes projection with grid parameter $h = 0.05$

Δt	Helmholtz projection				Stokes projection			
	$\ U - u\ _{L^\infty(L^2)}$		$\ P - p\ _{\ell^2(L^2)}$		$\ U - u\ _{\ell^\infty(L^2)}$		$\ P - p\ _{\ell^2(L^2)}$	
2.00e-3	2.79e-7	—	6.54e-6	—	3.16e-7	—	6.53e-6	—
1.00e-3	2.75e-7	0.02	7.13e-6	-0.13	4.19e-7	-0.41	7.19e-6	-0.14
5.00e-4	2.75e-7	-0.00	7.30e-6	-0.03	6.91e-7	-0.72	7.73e-6	-0.10
2.50e-4	2.92e-7	-0.09	7.90e-6	-0.11	1.33e-6	-0.95	9.66e-6	-0.32
1.25e-4	3.59e-7	-0.30	9.62e-6	-0.28	2.73e-6	-1.04	1.49e-5	-0.63
6.25e-5	5.25e-7	-0.55	1.33e-5	-0.46	5.75e-6	-1.07	2.62e-5	-0.81
3.13e-5	8.10e-7	-0.63	1.90e-5	-0.52	1.23e-5	-1.10	4.69e-5	-0.84
1.56e-5	1.26e-6	-0.63	2.64e-5	-0.48	2.64e-5	-1.10	8.18e-5	-0.80
7.81e-6	2.03e-6	-0.69	3.58e-5	-0.44	5.56e-5	-1.08	1.36e-4	-0.73
3.91e-6	3.50e-6	-0.78	4.87e-5	-0.44	1.13e-4	-1.02	2.12e-4	-0.64
1.95e-6	6.25e-6	-0.84	6.74e-5	-0.47	2.17e-4	-0.94	3.09e-4	-0.54

Again, in each time step, we switch the triangulation between the meshes shown in Fig. 5. The errors $\|U - u\|_{\ell^\infty(L^2)}$ and $\|P - p\|_{\ell^2(L^2)}$ for the Helmholtz and Stokes projection are given in Table 3 for grid parameter $h = 0.05$ and visualized in Fig. 8.

In Besier & Wollner (2011, Section 5.1), the authors analyse the behaviour of the pressure approximation for a decreasing sequence of time steps and a spatial mesh coarsened or refined in each time step. The Helmholtz and the Stokes projection are used for evaluating the velocity approximations on the new meshes. Their conclusion is that the divergence of the pressure variable can be avoided by using either projection. In contrast, our observation is that if the grid is changed in each time step in an ‘uncontrolled’ way, neither the Helmholtz nor the Stokes projection is capable of preventing a loss in the rate of convergence, or even the divergence, of the discrete approximations. Thus, this topic needs further investigation.

Fig. 8. Plots of velocity $\|U - u\|_{\ell^\infty(L^2)}$ and pressure error $\|P - p\|_{\ell^2(L^2)}$ versus time step size for grid parameter $h = 0.05$.

REMARK 5.3 It is worthwhile mentioning that for the numerical simulations it was mandatory to compute the term on the right-hand side arising in Algorithm 2.2 given by $\langle \hat{U}_{n-1}/\Delta t_n, v_n \rangle$ exactly, that is, without performing *variational crimes* using numerical quadrature. If using numerical quadrature, the divergence of the solution would be much more pronounced and could even lead to blow-up in time of the error. This fact is somehow reminiscent of stability issues for the Lagrange–Galerkin method with nonexact integration; see Morton *et al.* (1988). For practical reasons we were therefore limited to mesh changes like those used in our experiments and we could not perform simulations for really arbitrary changes.

6. Conclusion

In this article we studied finite element approximations of the time-dependent Stokes system on dynamically changing meshes. Applying the most robust backward Euler method for time discretization we

introduced the discrete Helmholtz or Stokes projection to evaluate the solution at time t_n on the new mesh at time t_{n+1} . We proved the *a priori* estimates

$$\begin{aligned}\text{Helmholtz projection: } \|U - u\|_{\ell^\infty(L^2)} &= \mathcal{O} \left(\|U_0 - u_0\| + \Delta t + h^{k+1} + \frac{h^{k+1}}{\Delta t} \right), \\ \text{Stokes projection: } \|U - u\|_{\ell^2(L^2)} &= \mathcal{O} \left(\|\nabla(U_0 - u_0)\| + \Delta t + h^{k+1} + \frac{h^k}{\Delta t} \right),\end{aligned}$$

for the velocity error and

$$\begin{aligned}\text{Helmholtz projection: } \|P - p\|_{\ell^2(L^2)} &= \mathcal{O} \left(\|U_0 - u_0\| + \Delta t + h^k + \frac{h^k}{\Delta t} \right), \\ \text{Stokes projection: } \|P - p\|_{\ell^2(L^2)} &= \mathcal{O} \left(\|\nabla(U_0 - u_0)\| + \Delta t + h^k + \frac{h^k}{\Delta t} \right),\end{aligned}$$

for the pressure error. These surprisingly pessimistic upper bounds are illustrated by numerical examples giving evidence that the convergence rate is negative at least for a large range of time step sizes. These observations imply that using adaptive meshes for flow problems is delicate and needs further mathematical investigation.

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