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DK Vu and P Steinmann

Chair of Applied Mechanics, Friedrich-Alexander-University of Erlangen-Nuremberg, Egerlandstrasse 5, 91058 Erlangen, Germany

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Abstract

In this work the formulation of spatial and material motion problems in nonlinear electro-elastostatics is considered using an energy approach, which takes into account the contribution of the free space surrounding a nonlinearly polarized body undergoing large deformation. The free space can have a huge impact on the electric field and on the deformation field inside a body made of materials with low electric permittivity such as the so-called electronic electroactive polymers (EEAPs). The contribution of the free space can be taken into account by using the electric flux and the Maxwell's traction acting on the boundary of the body. These two quantities can be expressed in terms of a stored energy density function. By using a stored energy density function for both the material body and the (finite or infinite) free space, the governing equations of both spatial and material motion problems are derived by considering the change of energy with respect to a change in the spatial or material configuration. In the spatial motion problem, well-known definitions for electric and mechanical quantities are derived. In the material motion problem, in addition to the derivation of configurational forces, this approach reveals the formulas for the part of energy that is released from the system material body, applied forces in response to a change in the material configuration, which are particularly useful in the study of defects such as crack propagation. The same approach can be used in the case of nonlinear electro-thermo-mechanical coupling and constitutes the direction for future works.

Keywords

configurational forces, free space, nonlinear coupling, nonlinear elasticity, nonlinear electricity, spatial and material motion

1. Introduction

The interaction between elastic bodies and electric fields is not a new subject of study in computational engineering. This is also true for the case of spatial and material motion problems in nonlinear electro-elastostatics. The spatial motion problem of nonlinear electro-elastostatics in particular and of nonlinear electro-elasticity in general has been the subject of many studies. Based on the basic laws of electricity and elasticity, the general equations and some analyses for the boundary-value problems of nonlinear electro-elasticity have been well established, see for example [1–6]. The material motion problem in nonlinear electro-elasticity has also received considerable attention [7–14]. Nevertheless, until recently the contribution of the free space in formulating the problem of nonlinear electro-elastostatics was often of minor interest. Under the application of an electric field,

Corresponding author:

DK Vu, Chair of Applied Mechanics, Friedrich-Alexander-University of Erlangen-Nuremberg, Egerlandstrasse 5, 91058 Erlangen, Germany.
Email: vu@itm.uni-erlangen.de

a polarizable body deforms because of electric forces exerted on electric dipoles that are arranged randomly inside the material. In finding the deformed state of the body under consideration (the spatial motion problem), the resulting electric field must be computed. In computing the resulting electric field, the free space surrounding the body should in principle be taken into account by using for example the formulas proposed in [15–17]. However, it is a well-known fact that for many materials the surrounding free space can be conveniently ignored because of its negligible contribution. This is particularly true if the electric permittivity of the material is much higher than that of free space. As consequence of this negligence the electric traction caused by the Maxwell's stress and the electric flux on the boundary of the body are often not considered in the spatial as well as in the material motion problems of nonlinear electro-elastostatics. Formulations of nonlinear electro-elastostatics neglecting the contribution of the free space can be found for example in [18–20]. With the discovery of new materials such as the so-called electronic electroactive polymers (EEAPs), which are used in developing artificial muscles [21–24], it has been observed that the above mentioned electric traction (Maxwell's traction) and electric flux may play an important role in determining spatial motion quantities such as the electric field vector, electric displacement vector, stress and deformation gradient tensors [25, 26]. The main reason here is that EEAPs have very low electric permittivity [27, 28] and therefore very weak polarization. Corresponding to this influence on spatial motion quantities, it is obvious that the free space is also important in computing material motion quantities such as configurational forces. In the present work, an energy approach is employed. In what follows, in order to pave the way for the derivation of the governing equations in nonlinear electro-elastostatics, we first use this energy approach in the spatial and material motion problems of nonlinear elastostatics. Next we use the same approach to extend the formulations of these two problems into nonlinear electro-elastostatics by taking into account both Maxwell's traction and the electric flux on the boundary of the material body. For more details on the material motion problem, its formulation and application, the readers are referred to, for example, [29–36].

2. Nonlinear elastostatics

In this section, the well-known governing equations of the spatial motion in nonlinear elastostatics, i.e. the balance equations of linear momentum and jump conditions, written in terms of the Cauchy stress tensor (in reference to the deformed configuration) or in terms of the Piola stress tensor (in reference to the undeformed configuration) are derived by considering the change of energy with respect to a variation of the deformed configuration. This approach is then used to derive the corresponding equations of the material motion problem by considering the change in energy with respect to a variation of the undeformed configuration.

2.1. Spatial motion problem in nonlinear elastostatics

Consider a body undergoing large deformation. The body is subjected to some external body and surface forces. We assume that the material behavior is elastic and can be inhomogeneous. Let us denote the undeformed configuration of the body by $\mathcal{B}_0$. The free space surrounding the body in which no matter is present is denoted by $\mathcal{S}_0$. Note that this free space can be finite or infinite. In the undeformed configuration $\mathcal{B}_0$ the position of a point P is denoted by $\mathbf{X}$ (material placement). Under the application of some given external body and surface forces (mechanical loads) the body deforms and its deformed configuration is denoted by $\mathcal{B}_t$. The free space surrounding $\mathcal{B}_t$ is denoted by $\mathcal{S}_t$. The position of the point P in $\mathcal{B}_t$ is denoted by $\mathbf{x}$ (spatial placement) and the map that maps P from the undeformed to the deformed configuration is called the spatial motion map and is denoted by $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\mathbf{X})$. The deformation at the point P is characterized by the spatial motion deformation gradient $\mathbf{F}$ defined as $\mathbf{F} := \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{X}}$.

The problem of finding the nonlinear motion map $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\mathbf{X})$ given the position $\mathbf{X}$ of each point P in $\mathcal{B}_0$ and the necessary boundary conditions is called the spatial motion problem. If the position $\mathbf{x}$ of each point P in $\mathcal{B}_t$ is given together with the necessary boundary conditions, then the problem of finding the position $\mathbf{X}$ of P in $\mathcal{B}_0$ or the material motion map $\mathbf{X} = \boldsymbol{\Phi}(\mathbf{x})$ is called the material motion problem. Corresponding to the inverse motion map $\boldsymbol{\Phi}$, the inverse of the spatial motion deformation gradient is called the material motion deformation

Figure 1. Undeformed ($\mathcal{B}_0$) and deformed ($\mathcal{B}_t$) configuration of a deformable body with the corresponding free spaces $\mathcal{S}_0$ and $\mathcal{S}_t$.

gradient and is defined as $\mathbf{f} := \frac{\partial \Phi}{\partial \mathbf{x}}$. For later use, we define two Cartesian coordinate systems with basis vectors $\{\mathbf{E}_I, I = 1, 2, 3\}$ in the undeformed configuration $\mathcal{B}_0$ and $\{\mathbf{e}_i, i = 1, 2, 3\}$ in the deformed configuration $\mathcal{B}_t$. By using these coordinate systems and the Einstein's summation convention, the position vectors $\mathbf{X}$ and $\mathbf{x}$ are expressed as $\mathbf{X} = X_I \mathbf{E}_I$ and $\mathbf{x} = x_i \mathbf{e}_i$. The spatial and material motion maps have the format $\boldsymbol{\varphi} = \varphi_i \mathbf{e}_i$ and $\boldsymbol{\Phi} = \Phi_I \mathbf{E}_I$. The spatial motion deformation gradient is presented by $\mathbf{F} = F_{iJ}(\mathbf{X}) \mathbf{e}_i \otimes \mathbf{E}_J$ with the coordinates $F_{iJ}(\mathbf{X}) = \frac{\partial \varphi_i(\mathbf{X})}{\partial X_J}$. For the material motion deformation gradient we have $\mathbf{f} = f_{ij}(\mathbf{x}) \mathbf{E}_I \otimes \mathbf{e}_j$ with the coordinates $f_{ij}(\mathbf{x}) = \frac{\partial \Phi_I(\mathbf{x})}{\partial x_j}$. In the free space outside the material body, we define an imaginary deformation field as that which is extended continuously from the material body into the free space. The continuity of this deformation field is assumed in such a way that across the boundary surface of the body there is no jump in the deformation gradient $\mathbf{F}$ and its inverse $\mathbf{f}$. Such an imaginary deformation field is helpful in formulating the governing equations of the spatial and material motion problems in terms of pull-back and push-forward quantities.

2.1.1. Governing equations in the deformed configuration. Let us consider first the spatial motion problem in reference to the deformed configuration $\mathcal{B}_t$. The quasi-static version of the balance equation of linear momentum and the jump condition in nonlinear elastostatics read

$$\begin{cases} \nabla_{\mathbf{x}} \cdot \boldsymbol{\sigma} + \mathbf{b}_t^{\text{mec}} = \mathbf{0} & \text{in } \mathcal{B}_t \\ \llbracket \boldsymbol{\sigma} \rrbracket \cdot \mathbf{n} = \mathbf{t}_t^{\text{mec}} & \text{on } \partial \mathcal{B}_t, \end{cases} \quad (1)$$

where $\boldsymbol{\sigma} = \sigma_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ is the Cauchy stress tensor, $\nabla_{\mathbf{x}} \cdot \boldsymbol{\sigma} = \frac{\partial \sigma_{ij}}{\partial x_j} \mathbf{e}_i$, $\mathbf{b}_t^{\text{mec}} = b_{ti}^{\text{mec}} \mathbf{e}_i$ is the mechanical body force, $\mathbf{n} = n_i \mathbf{e}_i$ is the unit outward pointing normal to the boundary $\partial \mathcal{B}_t$ and $\mathbf{t}_t^{\text{mec}} = t_{ti}^{\text{mec}} \mathbf{e}_i$ is the mechanical surface traction. The jump¹ of a quantity $(\bullet)$ at the boundary $\partial \mathcal{B}_t$ is defined as $\llbracket (\bullet) \rrbracket = (\bullet)_{\text{inside}} - (\bullet)_{\text{outside}}$, where $(\bullet)_{\text{inside}}$ is the value of the quantity under consideration inside the body and $(\bullet)_{\text{outside}}$ is the value outside the body. For the case of nonlinear elastostatics, the Cauchy stress tensor vanishes identically in the free space and the jump condition above becomes

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t}_t^{\text{mec}}, \quad (2)$$

which is also referred to as the boundary condition for the stress tensor $\boldsymbol{\sigma}$. By requiring that the Cauchy stress tensor be symmetric, the balance equation of angular momentum is satisfied automatically and is therefore not mentioned here explicitly.

2.1.2. Variational formulation in the deformed configuration. The above balance equation of linear momentum and boundary condition can be derived by using an energy approach, in which by keeping the material placement $\mathbf{X}$ unchanged ($\delta \mathbf{X} = \mathbf{0}$) and varying the spatial placement $\mathbf{x}$ an amount $\delta \mathbf{x} = \delta_X \boldsymbol{\varphi} \neq \mathbf{0}$, the change in the stored energy ($\delta_X \mathbf{w}_{\text{int}}^{\text{mec}}$) is equal to the change in the work done by the external forces ($\delta_X \mathbf{w}_{\text{ext}}^{\text{mec}}$)

$$\delta_X \mathbf{w}_{\text{int}}^{\text{mec}} - \delta_X \mathbf{w}_{\text{ext}}^{\text{mec}} = 0, \quad (3)$$

or, explicitly,

$$\delta_X \int_{\mathcal{B}_t} W_{tF} dv - \int_{\mathcal{B}_t} \mathbf{b}_t^{\text{mec}} \cdot \delta_X \boldsymbol{\varphi} dv - \int_{\partial \mathcal{B}_t} \mathbf{t}_t^{\text{mec}} \cdot \delta_X \boldsymbol{\varphi} ds = 0, \quad (4)$$

where W_{tF} is the stored energy density function² per unit volume of the deformed configuration $W_{tF} = W_{tF}(\mathbf{F}; \mathbf{X})$. Note that in addition to the deformation gradient $\mathbf{F}$, the dependency of the energy function W_{tF} on the material placement $\mathbf{X}$ of each point P is used to allow for material inhomogeneity. In the above equation the first term describes the change in the internal energy, the second and third terms are the change in the external energy due to the contribution of body and surface forces. This equation essentially states that with respect to the change $\delta \mathbf{x}$, the change in the internal energy being stored inside the material body must be equal to the change in the work done by the external forces. The localization of (4) leads to the following balance equation for a point inside the body

$$\nabla_x \cdot \left[W_{tF} \mathbf{i} + \frac{\partial W_{tF}}{\partial \mathbf{F}} \cdot \mathbf{F}^t \right] + \mathbf{b}_t^{\text{mec}} = \mathbf{0} \quad \text{in } \mathcal{B}_t \quad (5)$$

and the following boundary condition for a point on the boundary surface of the body

$$\left[W_{tF} \mathbf{i} + \frac{\partial W_{tF}}{\partial \mathbf{F}} \cdot \mathbf{F}^t \right] \cdot \mathbf{n} = \mathbf{t}_t^{\text{mec}} \quad \text{on } \partial \mathcal{B}_t, \quad (6)$$

where $\mathbf{i}$ is the second-order unit tensor $\mathbf{i} = \delta_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$. With hindsight gained by the balance equation of linear momentum in (1) and the boundary condition (2), the balance equation (5) and the boundary condition (6) motivate us to define the Cauchy stress tensor as

$$\boldsymbol{\sigma} := W_{tF} \mathbf{i} + \frac{\partial W_{tF}}{\partial \mathbf{F}} \cdot \mathbf{F}^t. \quad (7)$$

With these definitions we recover the traditional format (1) of the balance equation and jump condition by noting that no energy is stored in the free space outside the material body. Note that for the case of vanishing surface forces in which body forces can be derived from some potentials, an elegant approach can be used using the framework outlined in [37]. The case in which surface forces can be characterized by some surface potentials is not considered here because the spatial and material motion problems become more cumbersome.

2.1.3. Governing equations in the undeformed configuration. Let us consider the same spatial motion problem but in reference to the undeformed configuration $\mathcal{B}_0$. In reference to the undeformed configuration the governing equations of the problem are formulated with the help of the Piola stress tensor $\mathbf{P}$ defined as the pull-back of the Cauchy stress tensor $\boldsymbol{\sigma}$

$$\mathbf{P} := J \boldsymbol{\sigma} \cdot \mathbf{F}^{-t}, \quad (8)$$

where $J = \det \mathbf{F}$ and in index notation $\mathbf{P} = P_{iJ} \mathbf{e}_i \otimes \mathbf{E}_J$ with $P_{iJ} = J \sigma_{ik} F_{kJ}^{-t}$. In addition to the Piola stress tensor, in reference to the undeformed configuration the nominal mechanical body force is defined by

$$\mathbf{b}_0^{\text{mec}} = J \mathbf{b}_t^{\text{mec}} \quad (9)$$

and the nominal mechanical surface traction is

$$\mathbf{t}_0 = \zeta \mathbf{t}_t, \quad (10)$$

where $\zeta = J [\mathbf{N} \cdot \mathbf{C}^{-1} \cdot \mathbf{N}]^{1/2}$, $\mathbf{N} = N_I \mathbf{E}_I$ is the unit outward pointing normal to the boundary $\partial \mathcal{B}_0$ and $\mathbf{C}$ is the right Cauchy–Green tensor $\mathbf{C} = \mathbf{F}^t \cdot \mathbf{F}$. By using the above definitions for the stress, body force and traction, the governing equations presented in system (1) can be transformed into

$$\begin{cases} \nabla_X \cdot \mathbf{P} + \mathbf{b}_0^{\text{mec}} = \mathbf{0} & \text{in } \mathcal{B}_0 \\ \llbracket \mathbf{P} \rrbracket \cdot \mathbf{N} = \mathbf{t}_0^{\text{mec}} & \text{on } \partial \mathcal{B}_0, \end{cases} \quad (11)$$

which are the balance equation of linear momentum and jump condition in reference to the undeformed configuration.

2.1.4. Variational formulation in the undeformed configuration. The above system of governing equations can be derived by noting that under a variation $\delta_X \boldsymbol{\varphi}$ and in reference to the undeformed configuration, the equation (3) can be written as

$$\delta_X \int_{\mathcal{B}_0} W_{0F} dV - \int_{\mathcal{B}_0} \mathbf{b}_0^{\text{mec}} \cdot \delta_X \boldsymbol{\varphi} dV - \int_{\partial \mathcal{B}_0} \mathbf{t}_0^{\text{mec}} \cdot \delta_X \boldsymbol{\varphi} dS = 0, \quad (12)$$

where W_{0F} is the stored energy density function per unit volume of the undeformed configuration and is defined by its spatial counterpart as $W_{0F} = JW_{tF}$. By localizing the equation (12) we get the balance equation for a point inside the body

$$\nabla_X \cdot \frac{\partial W_{0F}}{\partial \mathbf{F}} + \mathbf{b}_0^{\text{mec}} = \mathbf{0} \quad \text{in } \mathcal{B}_0 \quad (13)$$

and the boundary condition for a point on the boundary surface of the body

$$\frac{\partial W_{0F}}{\partial \mathbf{F}} \cdot \mathbf{N} = \mathbf{t}_0^{\text{mec}} \quad \text{on } \partial \mathcal{B}_0. \quad (14)$$

The system (11) is then recovered by using the following definition of the Piola stress tensor

$$\mathbf{P} := \frac{\partial W_{0F}}{\partial \mathbf{F}} = J \left[W_{tF} \mathbf{i} + \frac{\partial W_{tF}}{\partial \mathbf{F}} \cdot \mathbf{F}^t \right] \cdot \mathbf{F}^{-t}, \quad (15)$$

which is fully compatible with the definition of the Cauchy stress tensor (7) and the pull-back formula (8).

Note that in this section, the governing equations of the spatial motion problem are derived by an energy approach in which the main argument is that under a change in the spatial placement of points (or particles), if the corresponding material placement is fixed, the change in the stored energy must be equal to the change in the work done by the applied forces. In the next section, we use this energy approach to derive the governing equations of the material motion problem both in reference to the undeformed and deformed configurations. In the material motion problem, the spatial placement $\mathbf{x}$ is considered as fixed and the corresponding material placement $\mathbf{X}$ is allowed to have some changes.

2.2. Material motion problem in nonlinear elastostatics

2.2.1. Formulation in the deformed configuration. The governing equations of the material motion problem can be derived by keeping the spatial placement $\mathbf{x}$ unchanged $\delta \mathbf{x} = \mathbf{0}$ and varying the material placement $\mathbf{X}$ by an amount $\delta \mathbf{X} = \delta_X \boldsymbol{\Phi} \neq \mathbf{0}$. Corresponding to the variation $\delta \mathbf{X}$, the new undeformed configuration is now $\mathcal{B}_0^\delta$. With this new undeformed configuration and with the application of external forces, the new deformed configuration should be $\mathcal{B}_t^\delta$. Obviously the energy (deformation energy and work done by external forces) related to the deformation process from $\mathcal{B}_0^\delta$ to $\mathcal{B}_t^\delta$ is different from that related to the deformation process from $\mathcal{B}_0$ to $\mathcal{B}_t$. In order to keep the deformed configuration unchanged, further additional energy (configurational energy) is needed in the form of work done by body and surface forces (configurational forces) to move $\mathcal{B}_t^\delta$ back to $\mathcal{B}_t$. This reasoning leads to the conclusion that under a change in the material placement of particles, if the corresponding spatial placement is fixed, then the change in the stored energy $\delta_x \mathbf{w}_{\text{int}}^{\text{mec}}$ is not equal to the change in the work done by the applied forces $\delta_x \mathbf{w}_{\text{ext}}^{\text{mec}}$ because some configurational energy $\delta_x \mathbf{w}_{\text{con}}^{\text{mec}}$ must be used to keep the spatial configuration of the body unchanged. This leads us then to the balance equation

$$\delta_x \mathbf{w}_{\text{int}}^{\text{mec}} - \delta_x \mathbf{w}_{\text{ext}}^{\text{mec}} - \delta_x \mathbf{w}_{\text{con}}^{\text{mec}} = 0, \quad (16)$$

which can be used to derive the governing equations of the material motion problem. Here, despite the fact that the change in the stored energy can be computed as

$$\delta_x \mathbf{w}_{\text{int}}^{\text{mec}} = \delta_x \int_{\mathcal{B}_t} W_{tF} dv = \int_{\mathcal{B}_t} \frac{\partial W_{tF}}{\partial \mathbf{f}} : \delta_x \mathbf{f} dv + \int_{\mathcal{B}_t} \frac{\partial W_{tF}}{\partial \boldsymbol{\Phi}} \cdot \delta_x \boldsymbol{\Phi} dv, \quad (17)$$

where $W_{tf} = W_{tf}(\mathbf{f}, \Phi)$ such that $W_{tf}|_{\mathbf{f}, \Phi} = W_{tF}|_{F, X}$, the change in the work done by the applied forces and the configurational energy cannot be computed explicitly. However, we can assume that the change in the work done by the applied forces can be presented by some volume and surface forces in the format

$$\delta_x \mathbf{w}_{\text{ext}}^{\text{mec}} = \int_{\mathcal{B}_t} \mathbf{B}_t^{\text{ext}} \cdot \delta_x \Phi \, dv + \int_{\partial \mathcal{B}_t} \mathbf{T}_t^{\text{ext}} \cdot \delta_x \Phi \, ds, \quad (18)$$

where $\mathbf{B}_t^{\text{ext}} = \mathbf{B}_t^{\text{ext}}(\mathbf{b}_t^{\text{mec}})$ is some material motion body force used in the computation of the change of energy due to the applied body force $\mathbf{b}_t^{\text{mec}}$ and $\mathbf{T}_t^{\text{ext}} = \mathbf{T}_t^{\text{ext}}(\mathbf{t}_t^{\text{mec}})$ is some material motion surface force used in the computation of the change of energy due to the applied surface force $\mathbf{t}_t^{\text{mec}}$. In the same way we assume that the configurational energy can be decomposed into a volume and a surface contribution as

$$\delta_x \mathbf{w}_{\text{con}}^{\text{mec}} = \int_{\mathcal{B}_t} \mathbf{B}_t^{\text{con}} \cdot \delta_x \Phi \, dv + \int_{\partial \mathcal{B}_t} \mathbf{T}_t^{\text{con}} \cdot \delta_x \Phi \, ds, \quad (19)$$

where $\mathbf{B}_t^{\text{con}}$ and $\mathbf{T}_t^{\text{con}}$ are some configurational body and surface forces. Note that for our system, the configurational energy must be non-positive

$$\delta_x \mathbf{w}_{\text{int}}^{\text{mec}} - \delta_x \mathbf{w}_{\text{ext}}^{\text{mec}} = \delta_x \mathbf{w}_{\text{con}}^{\text{mec}} \leq 0, \quad (20)$$

which essentially means that a change in the undeformed configuration is only possible when some energy is released from our system material body, applied forces, and used to alter the configuration of the body. The equality only happens in the case of zero dissipation or configurational equilibrium [37]. A very helpful example in demonstrating the use of the inequality (20) is the case of cracks, in which a crack can only propagate if this inequality is satisfied.

The localization of Equation (16) with the help of Equation (17), the assumptions (18) and (19) give us the balance equation

$$\nabla_x \cdot \frac{\partial W_{tf}}{\partial \mathbf{f}} - \frac{\partial W_{tf}}{\partial \Phi} + \mathbf{B}_t^{\text{ext}} + \mathbf{B}_t^{\text{con}} = \mathbf{0} \quad \text{in } \mathcal{B}_t \quad (21)$$

and the boundary condition

$$\frac{\partial W_{tf}}{\partial \mathbf{f}} \cdot \mathbf{n} - \mathbf{T}_t^{\text{ext}} - \mathbf{T}_t^{\text{con}} = \mathbf{0} \quad \text{on } \partial \mathcal{B}_t. \quad (22)$$

The above balance equation and boundary condition motivate the definition of the material motion Piola stress tensor

$$\mathbf{p} := \frac{\partial W_{tf}}{\partial \mathbf{f}}, \quad (23)$$

which in index notation can be written as $\mathbf{p} = p_{Ij} \mathbf{E}_I \otimes \mathbf{e}_j$ with the coordinates $p_{Ij} = \frac{\partial W_{tf}}{\partial f_{Ij}}$. By using this definition, the balance equation (21) becomes

$$\nabla_x \cdot \mathbf{p} - \frac{\partial W_{tf}}{\partial \Phi} + \mathbf{B}_t^{\text{ext}} + \mathbf{B}_t^{\text{con}} = \mathbf{0} \quad \text{in } \mathcal{B}_t. \quad (24)$$

By noting that the spatial motion Cauchy stress tensor $\boldsymbol{\sigma}$ can be computed in terms of the material motion deformation gradient $\mathbf{f}$ in the format

$$\boldsymbol{\sigma} = W_{tf} \mathbf{i} - \mathbf{f}^t \cdot \frac{\partial W_{tf}}{\partial \mathbf{f}}, \quad (25)$$

the material motion Piola stress tensor is related to $\boldsymbol{\sigma}$ by the relationship

$$\mathbf{p} = W_{tf} \mathbf{f}^{-t} - \mathbf{f}^{-t} \cdot \boldsymbol{\sigma}. \quad (26)$$

The use of this relationship in Equation (24) results in

$$\nabla_x \cdot \boldsymbol{\sigma} - \mathbf{f}^t \cdot \mathbf{B}_t^{\text{ext}} - \mathbf{f}^t \cdot \mathbf{B}_t^{\text{con}} = \mathbf{0} \quad \text{in } \mathcal{B}_t. \quad (27)$$

By comparing this equation with the balance equation of linear momentum of the spatial motion problem, i.e. the first equation of system (1), we arrive at the following relationship

$$\mathbf{B}_t^{\text{ext}} + \mathbf{B}_t^{\text{con}} = -\mathbf{f}^{-t} \cdot \mathbf{b}_t^{\text{mec}}. \quad (28.1-2)$$

By using this relationship, we cannot determine the material motion body force $\mathbf{B}_t^{\text{ext}}$ and the configurational body force $\mathbf{B}_t^{\text{con}}$ separately. However, the solution of the material motion problem in terms of $\mathbf{f}$ and $\mathbf{p}$ will not be affected if we define

$$\mathbf{B}_t^{\text{ext}} := -\mathbf{f}^{-t} \cdot \mathbf{b}_t^{\text{mec}} \quad \text{and} \quad \mathbf{B}_t^{\text{con}} := \mathbf{0}. \quad (29.1-2)$$

Also by using the definition of the material motion Piola stress tensor $\mathbf{p}$, the boundary condition (22) is now

$$\mathbf{p} \cdot \mathbf{n} = \mathbf{T}_t^{\text{ext}} + \mathbf{T}_t^{\text{con}} \quad \text{on } \partial \mathcal{B}_t \quad (30)$$

or in the format

$$W_{tf} \mathbf{f}^{-t} \cdot \mathbf{n} - \mathbf{f}^{-t} \cdot \mathbf{t}_t^{\text{mec}} = \mathbf{T}_t^{\text{ext}} + \mathbf{T}_t^{\text{con}} \quad \text{on } \partial \mathcal{B}_t. \quad (31)$$

As for the case of $\mathbf{B}_t^{\text{ext}}$ and $\mathbf{B}_t^{\text{con}}$ above, we cannot use this relationship to determine the material motion surface force $\mathbf{T}_t^{\text{ext}}$ and the configuration surface force $\mathbf{T}_t^{\text{con}}$ separately. However, if we define

$$\mathbf{T}_t^{\text{ext}} := -\mathbf{f}^{-t} \cdot \mathbf{t}_t^{\text{mec}} \quad \text{and} \quad \mathbf{T}_t^{\text{con}} := W_{tf} \mathbf{f}^{-t} \cdot \mathbf{n} \quad (32.1-2)$$

then the solution of the material motion problem in terms of $\mathbf{f}$ and $\mathbf{p}$ will not change.

By noting that in the free space surrounding the material body the material motion Piola stress tensor $\mathbf{p}$ vanishes, the above governing equations can be rewritten in the format

$$\begin{cases} \nabla_x \cdot \mathbf{p} + \mathbf{B}_t = \mathbf{0} & \text{in } \mathcal{B}_t \\ \llbracket \mathbf{p} \rrbracket \cdot \mathbf{n} = \mathbf{T}_t & \text{on } \partial \mathcal{B}_t, \end{cases} \quad (33)$$

where the corresponding body force is defined by $\mathbf{B}_t := -\frac{\partial W_{tf}}{\partial \boldsymbol{\Phi}} + \mathbf{B}_t^{\text{ext}} + \mathbf{B}_t^{\text{con}}$ and the surface force is $\mathbf{T}_t := \mathbf{T}_t^{\text{ext}} + \mathbf{T}_t^{\text{con}}$.

Remark. It should be noted that the definitions (29) and (32) can be employed to satisfy the balance equation (16), and therefore the governing equations (21) and (22), or equivalently the system (33), of the material motion problem, but for the computation of the configurational energy these definitions are normally not sufficient. In order to compute the configurational energy or the released energy $\delta_x \mathbf{w}_{\text{con}}^{\text{mec}}$, the value of the configurational forces $\mathbf{B}_t^{\text{con}}$ and $\mathbf{T}_t^{\text{con}}$ must be evaluated separately. The definitions (29) and (32) offer us only some assumed but not computed values of these forces and therefore can only be used to evaluate $\delta_x \mathbf{w}_{\text{con}}^{\text{mec}}$ under some certain conditions. An example of these conditions is the case in which there is no body force $\mathbf{b}_t^{\text{mec}} = \mathbf{0}$ and the surface force $\mathbf{t}_t^{\text{mec}}$ does not perform work with respect to a change in the undeformed configuration $\mathbf{t}_t^{\text{mec}} \cdot \delta_x \boldsymbol{\Phi} = 0$, for example in the case the body is loaded by some constant displacement (fixed grip condition). In this case the contribution of the material motion surface force $\mathbf{T}_t^{\text{ext}}$ to the change of energy vanishes, the definitions (32) apply and the configurational energy is

$$\delta_x \mathbf{w}_{\text{con}}^{\text{mec}} = \int_{\partial \mathcal{B}_t} \llbracket W_{tf} \rrbracket \mathbf{n} \cdot \mathbf{f}^{-1} \cdot \delta_x \boldsymbol{\Phi} \, ds, \quad (34)$$

in which it should be noted that the stored energy density function W_{tf} vanishes outside the material body.

2.2.2. *Formulation in the undeformed configuration.* In reference to the undeformed configuration $\mathcal{B}_0$, the change in the stored energy $\delta_x \mathbf{w}_{\text{int}}^{\text{mec}}$ can be computed by

$$\delta_x \mathbf{w}_{\text{int}}^{\text{mec}} = \delta_x \int_{\mathcal{B}_0} W_{0f} dV = \int_{\mathcal{B}_0} W_{0f} \delta_x dV + \int_{\mathcal{B}_0} \frac{\partial W_{0f}}{\partial \mathbf{f}} : \delta_x \mathbf{f} dV + \int_{\mathcal{B}_0} \frac{\partial W_{0f}}{\partial \Phi} \cdot \delta_x \Phi dV, \quad (35)$$

where W_{0f} is some stored energy density function per unit volume of the undeformed configuration defined as $W_{0f} = JW_{if}$. The change in the work done by the applied forces can be expressed in the format

$$\delta_x \mathbf{w}_{\text{ext}}^{\text{mec}} = \int_{\mathcal{B}_0} \mathbf{B}_0^{\text{ext}} \cdot \delta_x \Phi dV + \int_{\partial \mathcal{B}_0} \mathbf{T}_0^{\text{ext}} \cdot \delta_x \Phi dS, \quad (36)$$

where $\mathbf{B}_0^{\text{ext}} = \mathbf{B}_0^{\text{ext}}(\mathbf{b}_0^{\text{mec}})$ is some material motion body force used in the computation of the change of energy due to the applied body force $\mathbf{b}_0^{\text{mec}}$ and $\mathbf{T}_0^{\text{ext}} = \mathbf{T}_0^{\text{ext}}(\mathbf{t}_0^{\text{mec}})$ is some material motion surface force used in the computation of the change of energy due to the applied surface force $\mathbf{t}_0^{\text{mec}}$. The configurational energy is assumed to be decomposed into a volume and a surface contribution in the format

$$\delta_x \mathbf{w}_{\text{con}}^{\text{mec}} = \int_{\mathcal{B}_0} \mathbf{B}_0^{\text{con}} \cdot \delta_x \Phi dV + \int_{\partial \mathcal{B}_0} \mathbf{T}_0^{\text{con}} \cdot \delta_x \Phi dS, \quad (37)$$

where $\mathbf{B}_0^{\text{con}}$ and $\mathbf{T}_0^{\text{con}}$ are some configurational body and surface forces.

By using (35), (36) and (37) in Equation (16) and by localizing the result we arrive at the following balance equation for a point inside the material body

$$\nabla_X \cdot [W_{0f} \mathbf{I} + \frac{\partial W_{0f}}{\partial \mathbf{f}} \cdot \mathbf{f}^t] - \frac{\partial W_{0f}}{\partial \Phi} + \mathbf{B}_0^{\text{ext}} + \mathbf{B}_0^{\text{con}} = \mathbf{0}, \quad (38)$$

where $\mathbf{I} = \delta_{IJ} \mathbf{E}_I \otimes \mathbf{E}_J$, and the following boundary condition for a point on the boundary of the body

$$[W_{0f} \mathbf{I} + \frac{\partial W_{0f}}{\partial \mathbf{f}} \cdot \mathbf{f}^t] \cdot \mathbf{N} = \mathbf{T}_0^{\text{ext}} + \mathbf{T}_0^{\text{con}} \quad \text{on } \partial \mathcal{B}_0. \quad (39)$$

The balance equation and boundary condition above motivate the definition of the material motion Cauchy stress tensor Σ in the format

$$\Sigma := W_{0f} \mathbf{I} + \frac{\partial W_{0f}}{\partial \mathbf{f}} \cdot \mathbf{f}^t, \quad (40)$$

which in index notation can be written as $\Sigma = \Sigma_{IJ} \mathbf{E}_I \otimes \mathbf{E}_J$ with the coordinates $\Sigma_{IJ} = W_{0f} \delta_{IJ} + \frac{\partial W_{0f}}{\partial f_{Jk}} f_{Ik}$.

With this definition we obtain

$$\nabla_X \cdot \Sigma - \frac{\partial W_{0f}}{\partial \Phi} + \mathbf{B}_0^{\text{ext}} + \mathbf{B}_0^{\text{con}} = \mathbf{0}. \quad (41)$$

By expressing Σ in terms of the spatial motion deformation gradient $\mathbf{F}$ in the format

$$\Sigma := W_{0f} \mathbf{I} - \mathbf{F}^t \cdot \frac{\partial W_{0f}}{\partial \mathbf{F}}, \quad (42)$$

the relationship between Σ and the spatial motion Piola stress tensor $\mathbf{P}$ is

$$\Sigma := W_{0f} \mathbf{I} - \mathbf{F}^t \cdot \mathbf{P}. \quad (43)$$

The use of this relationship in Equation (41) gives us

$$\nabla_X \cdot \mathbf{P} - \mathbf{f}^t \cdot \mathbf{B}_0^{\text{ext}} - \mathbf{f}^t \cdot \mathbf{B}_0^{\text{con}} = \mathbf{0} \quad \text{in } \mathcal{B}_0. \quad (44)$$

By comparing this equation with the balance equation of linear momentum of the spatial motion problem written in terms of the Piola stress tensor, i.e. the first equation of system (11), we arrive at the following definitions for the material motion body force $\mathbf{B}_0^{\text{ext}}$ and for the configurational body force $\mathbf{B}_0^{\text{con}}$

$$\mathbf{B}_0^{\text{ext}} := -\mathbf{f}^{-t} \cdot \mathbf{b}_0^{\text{mec}} \quad \text{and} \quad \mathbf{B}_0^{\text{con}} := \mathbf{0}. \quad (45.1-2)$$

By using the definition of the material motion Cauchy stress tensor $\boldsymbol{\Sigma}$, the boundary condition (39) is now

$$\boldsymbol{\Sigma} \cdot \mathbf{N} = \mathbf{T}_0^{\text{ext}} + \mathbf{T}_0^{\text{con}} \quad \text{on } \partial\mathcal{B}_0 \quad (46)$$

or in the format

$$W_{0f} \mathbf{N} - \mathbf{f}^{-t} \cdot \mathbf{t}_0^{\text{mec}} = \mathbf{T}_0^{\text{ext}} + \mathbf{T}_0^{\text{con}} \quad \text{on } \partial\mathcal{B}_0, \quad (47)$$

which leads us to the definition of the material motion surface force $\mathbf{T}_0^{\text{ext}}$ and the configuration surface force $\mathbf{T}_0^{\text{con}}$ as

$$\mathbf{T}_0^{\text{ext}} := -\mathbf{f}^{-t} \cdot \mathbf{t}_0^{\text{mec}} \quad \text{and} \quad \mathbf{T}_0^{\text{con}} := W_{0f} \mathbf{N}. \quad (48.1-2)$$

Because in the free space surrounding the material body the material motion Cauchy stress tensor $\boldsymbol{\Sigma}$ vanishes, the above balance equation and boundary condition can be rewritten as

$$\begin{cases} \nabla_X \cdot \boldsymbol{\Sigma} + \mathbf{B}_0 = \mathbf{0} & \text{in } \mathcal{B}_0 \\ \llbracket \boldsymbol{\Sigma} \rrbracket \cdot \mathbf{N} = \mathbf{T}_0 & \text{on } \partial\mathcal{B}_0, \end{cases} \quad (49)$$

where $\mathbf{B}_0 := -\frac{\partial W_{0f}}{\partial \boldsymbol{\Phi}} + \mathbf{B}_0^{\text{ext}} + \mathbf{B}_0^{\text{con}}$ and $\mathbf{T}_0 := \mathbf{T}_0^{\text{ext}} + \mathbf{T}_0^{\text{con}}$.

Remark. Again it should be noted that the above definitions of the material motion forces and configurational forces satisfy the governing equations of the problem but can only be used to compute the released energy or the configurational energy under some certain conditions. In the case mentioned above where there is no body force $\mathbf{b}_0^{\text{mec}} = \mathbf{0}$ and the surface force $\mathbf{t}_0^{\text{mec}}$ does not perform work with respect to a change in the undeformed configuration $\mathbf{t}_0^{\text{mec}} \cdot \delta_x \boldsymbol{\Phi} = 0$, the configurational energy is computed by

$$\delta_x \mathbf{w}_{\text{con}}^{\text{mec}} = \int_{\partial\mathcal{B}_0} \llbracket W_{0f} \rrbracket \mathbf{N} \cdot \delta_x \boldsymbol{\Phi} \, dS, \quad (50)$$

where the stored energy density function W_{0f} vanishes outside the material body.

3. Nonlinear electro-elastostatics

In nonlinear electro-elastostatics, the spatial motion problem is defined as the problem of finding the deformation map $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X})$ under the application of mechanical and electrical loads. The inverse problem, the problem of finding the motion map $\mathbf{X} = \boldsymbol{\Phi}(\mathbf{x})$ given the spatial placement $\mathbf{x}$, the mechanical and electrical loads is defined as the material motion problem. In what follows, after recalling the governing equations of the spatial motion problem and the corresponding variational formulations, the governing equations of the material motion problem are derived by using the same energy approach presented above.

3.1. Spatial motion problem in nonlinear electro-elastostatics

3.1.1. Governing equations in the deformed configuration. Under the application of an external electric field, a polarizable body is subjected to electric body and surface forces. As these forces enter the governing equations of the spatial motion problem, the electric field inside and outside the material body must be determined. At every point inside the material body, by bypassing the Gauss's law for magnetism and the Ampere's law because of vanishing magnetic field, the electric field is determined by the Faraday's law of induction

$$\nabla_x \times \mathbf{e} = \mathbf{0} \quad (51)$$

and by the electric Gauss's law

$$\nabla_{\mathbf{x}} \cdot \mathbf{d} = 0, \quad (52)$$

where we assume that there is no free body charge inside the material body and the electric displacement $\mathbf{d}$ is computed as

$$\mathbf{d} = \epsilon_0 \mathbf{e} + \mathbb{p}, \quad (53)$$

where ϵ_0 is the electric permittivity of vacuum. In the above equation we assume that the electric polarization $\mathbb{p}$ is a fully nonlinear function of the electric field $\mathbf{e}$, the deformation gradient $\mathbf{F}$ and the material placement $\mathbf{X}$ in the format $\mathbb{p} = \mathbb{p}(\mathbf{e}, \mathbf{F}; \mathbf{X})$ and, as result, the electric displacement is also a fully nonlinear function of the electric field, the deformation gradient and the material placement $\mathbf{d} = \mathbf{d}(\mathbf{e}, \mathbf{F}; \mathbf{X})$. If we use the notation $(\bullet)^{\epsilon_0}$ to denote quantities at a point lying in S_t , i.e. outside the material body, then in the free space surrounding the body the above governing equations become

$$\nabla_{\mathbf{x}} \times \mathbf{e}^{\epsilon_0} = \mathbf{0}, \quad \nabla_{\mathbf{x}} \cdot \mathbf{d}^{\epsilon_0} = 0 \quad \text{with } \mathbf{d}^{\epsilon_0} := \epsilon_0 \mathbf{e}^{\epsilon_0}. \quad (54.1-3)$$

At the boundary of the body ∂B_t , the jump conditions for the electric field vector $\mathbf{e}$ and the electric displacement $\mathbf{d}$ are

$$[[\mathbf{e}]] \times \mathbf{n} = [\mathbf{e}^{\epsilon} - \mathbf{e}^{\epsilon_0}] \times \mathbf{n} = \mathbf{0} \quad \text{and} \quad [[\mathbf{d}]] \cdot \mathbf{n} = [\mathbf{d}^{\epsilon} - \mathbf{d}^{\epsilon_0}] \cdot \mathbf{n} = -\widehat{\varrho}_t^{\text{fre}}, \quad (55.1-2)$$

where the notation $(\bullet)^{\epsilon}$ is used to denote values inside the material body and $\widehat{\varrho}_t^{\text{fre}}$ is the free charge density per unit area of the deformed surface ∂B_t .

In order to solve the problem of electrostatics, a scalar electric potential ψ (called here spatial electric potential) is defined in such a way that the electric field vector can be computed by the spatial gradient of this potential

$$\mathbf{e} = -\nabla_{\mathbf{x}} \psi. \quad (56)$$

If we denote the electric body force that the electric field exerts on the matter by $\mathbf{b}_t^{\text{ele}}$, the balance equation of linear momentum is written as

$$\nabla_{\mathbf{x}} \cdot \boldsymbol{\sigma} + \mathbf{b}_t^{\text{mec}} + \mathbf{b}_t^{\text{ele}} = \mathbf{0}, \quad (57)$$

where the electric body force $\mathbf{b}_t^{\text{ele}}$ can be computed as a function of the electric field vector $\mathbf{e}$ and the electric polarization vector $\mathbb{p}$. This function in turn can be written as the divergence of a stress like tensor $\boldsymbol{\sigma}^{\text{ele}}$ (electric stress tensor) in the format

$$\mathbf{b}_t^{\text{ele}} = [\nabla_{\mathbf{x}} \mathbf{e}] \cdot \mathbb{p} = \nabla_{\mathbf{x}} \cdot \boldsymbol{\sigma}^{\text{ele}}, \quad (58)$$

where $\boldsymbol{\sigma}^{\text{ele}}$ is defined as³ $\boldsymbol{\sigma}^{\text{ele}} = \mathbf{e} \otimes \mathbf{d} - \frac{1}{2} \epsilon_0 [\mathbf{e} \cdot \mathbf{e}] \mathbf{i}$. Note that in the free space S_t surrounding the body of interest, the Cauchy stress tensor and mechanical body force vanish, the above-mentioned electric stress tensor $\boldsymbol{\sigma}^{\text{ele}}$ reduces to the Maxwell's stress tensor

$$\boldsymbol{\sigma}^{\text{max}} = \epsilon_0 \left[\mathbf{e} \otimes \mathbf{e} - \frac{1}{2} [\mathbf{e} \cdot \mathbf{e}] \mathbf{i} \right], \quad (59)$$

and the balance equation (57) becomes

$$\nabla_{\mathbf{x}} \cdot \boldsymbol{\sigma}^{\text{max}} = \mathbf{0}, \quad (60)$$

which is satisfied automatically.

For the case of linear polarization, in which we have $\mathbf{d} = \epsilon_m \mathbf{e}$ (where ϵ_m is the electric permittivity of the material body), the electric stress tensor $\boldsymbol{\sigma}^{\text{ele}}$ is symmetric and the balance equation of angular momentum requires that the Cauchy stress tensor be symmetric. However, for the case of nonlinear polarization, the electric field vector $\mathbf{e}$ and the electric displacement vector $\mathbf{d}$ are not parallel. In this case, the use of the balance equation of angular momentum leads to the requirement that not the Cauchy stress tensor $\boldsymbol{\sigma}$, but the combination of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{\text{ele}}$ be symmetric. This requirement motivates the use of a total stress tensor $\boldsymbol{\sigma}^{\text{tot}}$ defined as

$$\boldsymbol{\sigma}^{\text{tot}} = \boldsymbol{\sigma} + \boldsymbol{\sigma}^{\text{ele}}. \quad (61)$$

By using the total stress tensor σ^{tot} , the balance equation of linear momentum now has the same format as in the case of nonlinear elastostatics

$$\nabla_x \cdot \sigma^{\text{tot}} + b_t^{\text{mec}} = 0, \quad (62)$$

At the boundary of the body $\partial \mathcal{B}_t$, the total stress tensor must satisfy the jump condition

$$[[\sigma^{\text{tot}}]] \cdot n = t_t^{\text{mec}}, \quad (63)$$

where t_t^{mec} is the mechanical traction as defined before. It should be mentioned that this jump condition takes into account both the contribution of the Cauchy stress tensor σ and the contribution of the electric stress tensor σ^{ele} , which, as noted in [38], highlights the difficulty in separating the effect of the two stress tensors unambiguously using (currently available) mechanical means. For later use, we write the jump condition (63) in the format

$$\sigma^{\text{tot}, \epsilon} \cdot n = t_t^{\text{mec}} + t_t^{\text{max}}. \quad (64)$$

In the above equation, the spatial Maxwell's traction t_t^{max} is defined as $t_t^{\text{max}} = \sigma^{\text{max}} \cdot n$ and the relationship $\sigma^{\text{tot}} = \sigma^{\text{max}}$ applies for a point lying in the free space outside the material body. It is noteworthy that the above boundary condition differs from classical boundary conditions in the fact that while the mechanical traction t_t^{mec} is usually given in advance, the Maxwell's traction t_t^{max} is in many cases a quantity we still have to solve for.

3.1.2. Variational formulation in the deformed configuration. The above governing equations of the spatial motion problem in nonlinear electro-elastostatics can be derived by considering the change of energy when keeping the material placement unchanged $\delta X = 0$ and varying the spatial placement as well as the electric potential a small amount $\delta x = \delta_x \varphi \neq 0$, $\delta \psi = \delta_x \psi \neq 0$. As in the case of elastostatics, the change in the stored energy $\delta_X w_{\text{int}}^{\text{tot}}$ is equal to the change in the work done by the external forces $\delta_X w_{\text{ext}}^{\text{tot}}$

$$\delta_X w_{\text{int}}^{\text{tot}} - \delta_X w_{\text{ext}}^{\text{tot}} = 0, \quad (65)$$

where it should be noted that because of the presence of the electric field, the stored energy in nonlinear electro-elastostatics $w_{\text{int}}^{\text{tot}}$ has two contributions: a mechanical part and an electric part. This stored energy differs therefore from that of nonlinear elastostatics, i.e. $w_{\text{int}}^{\text{mec}}$, in the domain $\mathcal{B}_t \cup \mathcal{S}_t$ in which the energy is stored. For the problem of nonlinear electro-elastostatics, the energy is stored not only inside of material body $\mathcal{B}_t$ but also in the surrounding free space $\mathcal{S}_t$. Here the change in the stored energy $\delta_X w_{\text{int}}^{\text{tot}}$ can be computed as

$$\delta_X w_{\text{int}}^{\text{tot}} = \delta_X \int_{\mathcal{B}_t \cup \mathcal{S}_t} W_{tF}^{\text{tot}} \, dv, \quad (66)$$

where the stored energy density function per unit volume of the deformed configuration W_{tF}^{tot} is defined as a function of: (1) the gradient of the spatial motion map φ with respect to the material placement X , which is actually the deformation gradient defined in the previous section $\frac{\partial \varphi}{\partial X} := F$, (2) the gradient of the spatial electric potential ψ with respect to the material placement X , which turns out to be the pull-back version of the spatial electric field vector $\mathbb{E}$ in the undeformed configuration $\frac{\partial \psi}{\partial X} = F^t \cdot \mathbb{E} := \mathbb{E}$ ($\mathbb{E}$ is called the material electric field vector), and (3) the spatial placement X at each point inside the material body

$$W_{tF}^{\text{tot}} = W_{tF}^{\text{tot}}(F, \mathbb{E}; X). \quad (67)$$

For each point in the surrounding free space $\mathcal{S}_t$, the stored energy density function W_{tF}^{tot} reduces to $W_{tF}^{\text{tot}} = -\frac{1}{2} \mathbb{E}^{\epsilon_0} \cdot \mathbb{E}^{\epsilon_0} = -\frac{1}{2} \epsilon_0 \mathbb{E}^{\epsilon_0} \cdot \mathbb{E}^{\epsilon_0}$, or

$$W_{tF}^{\text{tot}} = -\frac{1}{2} \epsilon_0 \mathbb{E}^{\epsilon_0} \cdot C^{-1} \cdot \mathbb{E}^{\epsilon_0} := E_t \quad (68)$$

since no deformation energy is present in the free space $\mathcal{S}_t$ and the change (66) can be rewritten as

$$\delta_X \mathbf{w}_{\text{int}}^{\text{tot}} = \delta_X \int_{\mathcal{S}_t} E_t \, dv + \delta_X \int_{\mathcal{B}_t} W_{tF}^{\text{tot}} \, dv. \quad (69)$$

The change in the work done by the applied forces $\delta_X \mathbf{w}_{\text{ext}}^{\text{tot}}$ can be computed as the combination of the change in the works done by mechanical and electric loads

$$\delta_X \mathbf{w}_{\text{ext}}^{\text{tot}} = \int_{\mathcal{B}_t} \mathbf{b}_t^{\text{mec}} \cdot \delta_X \boldsymbol{\varphi} \, dv + \int_{\partial \mathcal{B}_t} \mathbf{t}_t^{\text{mec}} \cdot \delta_X \boldsymbol{\varphi} \, ds - \int_{\partial \mathcal{B}_t} \widehat{\varrho}_t^{\text{fre}} \delta \psi \, ds. \quad (70)$$

By using (69) and (70) in (65) and by requiring that Equation (65) be satisfied with respect to a variation of the spatial placement at fixed material placement, we get the following balance equation for a point inside the body

$$\nabla_x \cdot \left[W_{tF}^{\text{tot}} \mathbf{i} + \frac{\partial W_{tF}^{\text{tot}}}{\partial \mathbf{F}} \cdot \mathbf{F}^t \right] + \mathbf{b}_t^{\text{mec}} = \mathbf{0} \quad \text{in } \mathcal{B}_t \quad (71)$$

and the following jump condition for a point on the boundary of the body

$$\llbracket W_{tF}^{\text{tot}} \mathbf{i} + \frac{\partial W_{tF}^{\text{tot}}}{\partial \mathbf{F}} \cdot \mathbf{F}^t \rrbracket \cdot \mathbf{n} = \mathbf{t}_t^{\text{mec}} \quad \text{on } \partial \mathcal{B}_t, \quad (72)$$

where in the free space $\mathcal{S}_t$ we have $W_{tF}^{\text{tot}} \mathbf{i} + \frac{\partial W_{tF}^{\text{tot}}}{\partial \mathbf{F}} \cdot \mathbf{F}^t = \boldsymbol{\sigma}^{\text{max}}$. The comparison between the two systems (62)–(63) and (71)–(72) motivates us to define the total stress tensor $\boldsymbol{\sigma}^{\text{tot}}$ as

$$\boldsymbol{\sigma}^{\text{tot}} := \left[W_{tF}^{\text{tot}} \mathbf{i} + \frac{\partial W_{tF}^{\text{tot}}}{\partial \mathbf{F}} \cdot \mathbf{F}^t \right] \quad (73)$$

By requiring that Equation (65) be satisfied with respect to a variation of the electric potential ψ at fixed material placement, we get the following balance equation for a point inside the body

$$\nabla_x \cdot \left[-\frac{\partial W_{tF}^{\text{tot}}}{\partial \mathbb{E}} \right] = 0 \quad \text{in } \mathcal{B}_t \quad (74)$$

and the jump condition for a point on the boundary

$$\llbracket -\frac{\partial W_{tF}^{\text{tot}}}{\partial \mathbb{E}} \rrbracket \cdot \mathbf{n} = -\widehat{\varrho}_t^{\text{fre}} \quad \text{on } \partial \mathcal{B}_t. \quad (75)$$

By comparing with Equations (52) and (55.2), the balance equation (74) and the jump condition (75) motivate us to define the spatial electric displacement as

$$\mathbb{d} := -\frac{\partial W_{tF}^{\text{tot}}}{\partial \mathbb{E}}. \quad (76)$$

With the definitions (73) and (76) we recover the normal format of the governing equations in terms of the spatial total stress tensor $\boldsymbol{\sigma}^{\text{tot}}$ and in terms of the spatial electric displacement vector $\mathbb{d}$.

3.1.3. Governing equations in the undeformed configuration. In reference to the undeformed configuration, at every point inside the material body, the Faraday's law of induction is

$$\nabla_X \times \mathbb{E} = \mathbf{0}, \quad (77)$$

where the material version $\mathbb{E}$ of the electric field vector is defined by the pull-back of the spatial one $\mathbb{E} := \mathbf{F}^t \cdot \mathbf{e}$ as already mentioned above. The electric Gauss's law in $\mathcal{B}_0$ is

$$\nabla_X \cdot \mathbb{D} = 0, \quad (78)$$

where the material version of the electric displacement is defined by the pull-back formula

$$\mathbb{D} := J \mathbf{F}^{-1} \cdot \mathbf{d}. \quad (79)$$

By using the above definitions of the material electric field vector $\mathbb{E}$ and the material electric displacement vector $\mathbb{D}$, the relationship (53) can be rewritten in the format

$$\mathbb{D} = \epsilon_0 J \mathbf{C}^{-1} \cdot \mathbb{E} + J \mathbf{F}^{-1} \cdot \mathbb{P}, \quad (80)$$

which motivates the definition of the material version of the electric polarization $\mathbb{P}$ as

$$\mathbb{P} := J \mathbf{F}^{-1} \cdot \mathbb{P}. \quad (81)$$

In the free space outside the material body, the above governing equations become:

$$\nabla_X \times \mathbb{E}^{\epsilon_0} = \mathbf{0}, \quad \nabla_X \cdot \mathbb{D}^{\epsilon_0} = 0 \quad \text{with } \mathbb{D}^{\epsilon_0} := \epsilon_0 \mathbb{E}^{\epsilon_0}. \quad (82.1-3)$$

At the boundary of the body $\partial \mathcal{B}_0$, the jump condition for the electric field vector $\mathbb{E}$ and the electric displacement $\mathbb{D}$ are

$$[[\mathbb{E}]] \times \mathbf{N} = [\mathbb{E}^\epsilon - \mathbb{E}^{\epsilon_0}] \times \mathbf{N} = \mathbf{0} \quad \text{and} \quad [[\mathbb{D}]] \cdot \mathbf{N} = [\mathbb{D}^\epsilon - \mathbb{D}^{\epsilon_0}] \cdot \mathbf{N} = -\widehat{\varrho}_0^{\text{fre}}, \quad (83.1-2)$$

where $\widehat{\varrho}_0^{\text{fre}}$ is the free charge density per unit area of the undeformed surface $\partial \mathcal{B}_0$.

Corresponding to the spatial electric potential ψ the material electric potential is defined as $\Psi = \Psi(\mathbf{X})$ such that $\Psi(\mathbf{X}) = \psi(\mathbf{x}) \circ \boldsymbol{\varphi}(\mathbf{X})$. With this definition of the material electric potential Ψ , the material electric field vector $\mathbb{E}$ is computed by

$$\mathbb{E} = -\nabla_X \Psi. \quad (84)$$

The definitions of the material electric potential Ψ and the material electric field vector $\mathbb{E}$ comply with the pull-back formula $\mathbb{E} = \mathbf{F}^t \cdot \mathbf{e}$ since from the chain rule of derivatives we have $\nabla_X \Psi = \nabla_x \psi \cdot \mathbf{F}$.

In reference to the undeformed configuration $\mathcal{B}_0$, the balance equation of linear momentum in nonlinear electro-elastostatics is written in terms of the Piola stress tensor in the format

$$\nabla_X \cdot \mathbf{P} + \mathbf{b}_0^{\text{mec}} + \mathbf{b}_0^{\text{ele}} = \mathbf{0}, \quad (85)$$

where the electric body force $\mathbf{b}_0^{\text{ele}}$ is the counterpart of $\mathbf{b}_t^{\text{ele}}$ and can be computed by

$$\mathbf{b}_0^{\text{ele}} = J \mathbf{b}_t^{\text{ele}} = \nabla_X [\mathbf{F}^{-t} \cdot \mathbb{E}] \cdot \mathbb{P} = \nabla_X \cdot \mathbf{P}^{\text{ele}}. \quad (86)$$

In the above equation the electric stress tensor $\mathbf{P}^{\text{ele}}$ is defined as

$$\mathbf{P}^{\text{ele}} = \mathbf{F}^{-t} \cdot \mathbb{E} \otimes \mathbb{D} - \frac{1}{2} \epsilon_0 J [\mathbb{E} \cdot \mathbf{C}^{-1} \cdot \mathbb{E}] \mathbf{F}^{-t}. \quad (87)$$

In the free space surrounding the material body this stress reduces to the material version $\mathbf{P}^{\text{max}}$ of the Maxwell's stress tensor

$$\mathbf{P}^{\text{max}} = \epsilon_0 J \left[[\mathbf{F}^{-t} \cdot \mathbb{E}] \otimes [\mathbf{F}^{-t} \cdot \mathbb{E}] - \frac{1}{2} [\mathbb{E} \cdot \mathbf{C}^{-1} \cdot \mathbb{E}] \mathbf{I} \right] \cdot \mathbf{F}^{-t}, \quad (88)$$

in which, as assumed above, the deformation gradient $\mathbf{F}$ in the free space is defined as the extended version of that defined inside the material body.

Corresponding to the total stress tensor σ^{tot} in reference to the deformed configuration, the total stress tensor $\mathbf{P}^{\text{tot}}$ in reference to the undeformed configuration is defined as

$$\mathbf{P}^{\text{tot}} = \mathbf{P} + \mathbf{P}^{\text{ele}}, \quad (89)$$

which, by noting that $\mathbf{P}^{\text{ele}} = J\sigma^{\text{ele}} \cdot \mathbf{F}^{-t}$, can be shown to be the pull-back version of its counterpart σ^{tot} in the undeformed configuration

$$\mathbf{P}^{\text{tot}} = J\sigma^{\text{tot}} \cdot \mathbf{F}^{-t}. \quad (90)$$

By using the total stress tensor $\mathbf{P}^{\text{tot}}$, the balance equation (85) has the same format as in the case of nonlinear elastostatics

$$\nabla_X \cdot \mathbf{P}^{\text{tot}} + \mathbf{b}_0^{\text{mec}} = \mathbf{0}. \quad (91)$$

At the boundary of the body $\partial\mathcal{B}_0$, the total stress tensor $\mathbf{P}^{\text{tot}}$ must satisfy the jump condition

$$[[\mathbf{P}^{\text{tot}}]] \cdot \mathbf{N} = \mathbf{t}_0^{\text{mec}}, \quad (92)$$

which can be written explicitly in the format

$$\mathbf{P}^{\text{tot},\epsilon} \cdot \mathbf{N} = \mathbf{t}_0^{\text{mec}} + \mathbf{t}_0^{\text{max}}. \quad (93)$$

In the above equation, the material version $\mathbf{t}_0^{\text{max}}$ of the Maxwell's traction is defined as $\mathbf{t}_0^{\text{max}} = \mathbf{P}^{\text{max}} \cdot \mathbf{N}$ and in the free space we have $\mathbf{P}^{\text{tot}} = \mathbf{P}^{\text{max}}$.

3.1.4. Variational formulation in the undeformed configuration. The governing equations of the spatial motion problem and of the electric field inside or outside the material body above can be derived by considering the balance equation of energy changes (65) in reference to the undeformed configuration. Here the change in the stored energy $\delta_X \mathbf{w}_{\text{int}}^{\text{tot}}$ can be computed using the formula

$$\delta_X \mathbf{w}_{\text{int}}^{\text{tot}} = \delta_X \int_{\mathcal{B}_0 \cup \mathcal{S}_0} W_{0F}^{\text{tot}} dV, \quad (94)$$

where the stored energy density function per unit volume of the undeformed configuration W_{0F}^{tot} is defined as $W_{0F}^{\text{tot}} = JW_{iF}^{\text{tot}}$.

For a point lying in the free space $\mathcal{S}_0$, the stored energy density function W_{0F}^{tot} reduces to

$$W_{0F}^{\text{tot}} = -\frac{1}{2} \epsilon_0 J \mathbb{E}^{\epsilon_0} \cdot \mathbf{C}^{-1} \cdot \mathbb{E}^{\epsilon_0} := E_0 \quad (95)$$

and the change of the stored energy can be rewritten as

$$\delta_X \mathbf{w}_{\text{int}}^{\text{tot}} = \delta_X \int_{\mathcal{S}_0} E_0 dV + \delta_X \int_{\mathcal{B}_0} W_{0F}^{\text{tot}} dV. \quad (96)$$

The change in the work done by the applied forces $\delta_X \mathbf{w}_{\text{ext}}^{\text{tot}}$ can be computed by

$$\delta_X \mathbf{w}_{\text{ext}}^{\text{tot}} = \int_{\mathcal{B}_0} \mathbf{b}_0^{\text{mec}} \cdot \delta_X \boldsymbol{\varphi} dV + \int_{\partial\mathcal{B}_0} \mathbf{t}_0^{\text{mec}} \cdot \delta_X \boldsymbol{\varphi} dS - \int_{\partial\mathcal{B}_0} \widehat{\mathcal{Q}}_0^{\text{fre}} \delta\psi dS. \quad (97)$$

By using (96) and (97) in (65) and by requiring that Equation (65) be satisfied with respect to a variation of the spatial placement at fixed material placement, we obtain the following balance equation

$$\nabla_X \cdot \frac{\partial W_{0F}^{\text{tot}}}{\partial \mathbf{F}} + \mathbf{b}_0^{\text{mec}} = \mathbf{0} \quad \text{in } \mathcal{B}_0 \quad (98)$$

and jump condition

$$\llbracket \frac{\partial W_{0F}^{\text{tot}}}{\partial \mathbf{F}} \rrbracket \cdot \mathbf{N} = \mathbf{t}_0^{\text{mec}} \quad \text{on } \partial \mathcal{B}_0, \quad (99)$$

which motivate us to define the total stress tensor $\mathbf{P}^{\text{tot}}$ as

$$\mathbf{P}^{\text{tot}} := \frac{\partial W_{0F}^{\text{tot}}}{\partial \mathbf{F}}. \quad (100)$$

Note that the definitions of the spatial and material stresses $\boldsymbol{\sigma}^{\text{tot}}$ and $\mathbf{P}^{\text{tot}}$ in terms of the stored energy density functions W_{tF}^{tot} and W_{0F}^{tot} are compatible with the pull-back formula (90).

By using (96) and (97) in (65) and by requiring that Equation (65) be satisfied with respect to a variation of the electric potential ψ at fixed material placement, we obtain the balance equation

$$\nabla_X \cdot \left[-\frac{\partial W_{0F}^{\text{tot}}}{\partial \mathbb{E}} \right] = 0 \quad \text{in } \mathcal{B}_0 \quad (101)$$

and the jump condition

$$\llbracket -\frac{\partial W_{0F}^{\text{tot}}}{\partial \mathbb{E}} \rrbracket \cdot \mathbf{N} = -\widehat{\mathcal{Q}}_0^{\text{fre}} \quad \text{on } \partial \mathcal{B}_0, \quad (102)$$

which lead us to the definition of the material electric displacement as

$$\mathbb{D} := -\frac{\partial W_{0F}^{\text{tot}}}{\partial \mathbb{E}}. \quad (103)$$

Again it should be noted that, as in the case of stresses, the definitions of the spatial and material electric displacements $\mathfrak{d}$ and $\mathbb{D}$ in terms of the stored energy density functions W_{tF}^{tot} and W_{0F}^{tot} are fully compatible with the pull-back formula (79).

3.2. Material motion problem in nonlinear electro-elastostatics

3.2.1. Formulation in the deformed configuration. The governing equations of the material motion problem in nonlinear electro-elastostatics can be derived if we keep the spatial placement $\mathbf{x}$ unchanged ($\delta \mathbf{x} = \mathbf{0}$) and vary the material placement $\mathbf{X}$ and the material electric potential Ψ by an amount $\delta \mathbf{X} = \delta_X \boldsymbol{\Phi} \neq \mathbf{0}$ and $\delta \Psi = \delta_X \Psi \neq 0$, respectively. However, as it is different from the case of nonlinear elastostatics, the contribution of the electric field inside and outside the material body must be taken into account. By using an argument similar to that in the case of nonlinear elastostatics, we consider the following balance equation of energy changes:

$$\delta_X \mathbf{w}_{\text{int}}^{\text{tot}} - \delta_X \mathbf{w}_{\text{ext}}^{\text{tot}} - \delta_X \mathbf{w}_{\text{con}}^{\text{tot}} = 0. \quad (104)$$

The change of the stored energy in the above equation can be computed by

$$\delta_X \mathbf{w}_{\text{int}}^{\text{tot}} = \delta_X \int_{\mathcal{B}_t \cup \mathcal{S}_t} W_{tF}^{\text{tot}} \, dv, \quad (105)$$

where W_{tF}^{tot} is the stored energy density function per unit volume of the deformed configuration, which is defined as a function of: (1) the gradient of the material motion map $\boldsymbol{\Phi}$ with respect to the spatial placement $\mathbf{x}$ or actually the material motion deformation gradient $\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{x}} := \mathbf{f}$, (2) the gradient of the material electric potential Ψ with respect to the spatial placement $\mathbf{x}$, which is the spatial electric field vector $\mathfrak{e}$ in the deformed configuration $\frac{\partial \Psi}{\partial \mathbf{x}} = \mathbf{f}^t \cdot \frac{\partial \Psi}{\partial \mathbf{X}} =: \mathfrak{e}$, and (3) the material motion map $\boldsymbol{\Phi}$ at each point inside the material body

$$W_{tF}^{\text{tot}} = W_{tF}^{\text{tot}}(\mathbf{f}, \mathfrak{e}, \boldsymbol{\Phi}) \quad (106)$$

such that $W_{if}^{\text{tot}}|_{f,\mathbb{E},\Phi} = W_{if}^{\text{tot}}|_{F,\mathbb{E},X}$. This definition leads us to

$$\delta_x \mathbf{w}_{\text{int}}^{\text{tot}} = \int_{\mathcal{B}_t \cup \mathcal{S}_t} \frac{\partial W_{if}^{\text{tot}}}{\partial \mathbf{f}} : \delta_x \mathbf{f} \, dv + \int_{\mathcal{B}_t \cup \mathcal{S}_t} \frac{\partial W_{if}^{\text{tot}}}{\partial \Phi} \cdot \delta_x \Phi \, dv + \int_{\mathcal{B}_t \cup \mathcal{S}_t} \frac{\partial W_{if}^{\text{tot}}}{\partial \mathbb{E}} \cdot \delta \mathbb{E} \, dv. \quad (107)$$

The change in the work done by the external forces is assumed to be expressed by

$$\delta_x \mathbf{w}_{\text{ext}}^{\text{tot}} = \int_{\mathcal{B}_t} \mathbf{B}_t^{\text{tot,ext}} \cdot \delta_x \Phi \, dv + \int_{\partial \mathcal{B}_t} \mathbf{T}_t^{\text{tot,ext}} \cdot \delta_x \Phi \, ds - \int_{\partial \mathcal{B}_t} \widehat{\mathcal{Q}}_t^{\text{fre}} \delta \Psi \, ds, \quad (108)$$

where $\mathbf{B}_t^{\text{tot,ext}}$ and $\mathbf{T}_t^{\text{tot,ext}}$ are some material motion body and surface forces that can be computed as functions of $\mathbf{b}_t^{\text{mec}}$ and $\mathbf{t}_t^{\text{mec}}$ in the format $\mathbf{B}_t^{\text{tot,ext}} = \mathbf{B}_t^{\text{tot,ext}}(\mathbf{b}_t^{\text{mec}})$ and $\mathbf{T}_t^{\text{tot,ext}} = \mathbf{T}_t^{\text{tot,ext}}(\mathbf{t}_t^{\text{mec}})$. In addition, similar to the case of elastostatics, the configurational energy is assumed to be decomposed into a volume and a surface contribution as

$$\delta_x \mathbf{w}_{\text{con}}^{\text{tot}} = \int_{\mathcal{B}_t} \mathbf{B}_t^{\text{tot,con}} \cdot \delta_x \Phi \, dv + \int_{\partial \mathcal{B}_t} \mathbf{T}_t^{\text{tot,con}} \cdot \delta_x \Phi \, ds, \quad (109)$$

where $\mathbf{B}_t^{\text{tot,con}}$ and $\mathbf{T}_t^{\text{tot,con}}$ are some configurational body and surface forces.

By using (107), (108) and (109) in Equation (104) and by requiring that Equation (104) be satisfied with respect to a variation of the material placement at fixed spatial placement, we obtain the following balance equation for a point lying inside the material body

$$\nabla_x \cdot \frac{\partial W_{if}^{\text{tot}}}{\partial \mathbf{f}} - \frac{\partial W_{if}^{\text{tot}}}{\partial \Phi} + \mathbf{B}_t^{\text{tot,ext}} + \mathbf{B}_t^{\text{tot,con}} = \mathbf{0} \quad \text{in } \mathcal{B}_t, \quad (110)$$

and the jump condition for a point lying on the boundary of the body

$$\llbracket \frac{\partial W_{if}^{\text{tot}}}{\partial \mathbf{f}} \rrbracket \cdot \mathbf{n} = \mathbf{T}_t^{\text{tot,ext}} + \mathbf{T}_t^{\text{tot,con}} \quad \text{on } \partial \mathcal{B}_t. \quad (111)$$

The above balance equations and jump condition motivate us to define the total material motion Piola stress tensor in the format

$$\mathbf{p}^{\text{tot}} := \frac{\partial W_{if}^{\text{tot}}}{\partial \mathbf{f}}, \quad (112)$$

or in index notation $\mathbf{p}^{\text{tot}} = p_{ij} \mathbf{E}_I \otimes \mathbf{e}_j$ with the coordinates $p_{ij}^{\text{tot}} = \frac{\partial W_{if}^{\text{tot}}}{\partial f_{ij}}$. With this definition, the balance equation (110) is rewritten as

$$\nabla_x \cdot \mathbf{p}^{\text{tot}} - \frac{\partial W_{if}^{\text{tot}}}{\partial \Phi} + \mathbf{B}_t^{\text{tot,ext}} + \mathbf{B}_t^{\text{tot,con}} = \mathbf{0} \quad \text{in } \mathcal{B}_t. \quad (113)$$

Note that in the free space outside the material body, the stored energy density function W_{if}^{tot} reduces to E_t , i.e.

$$W_{if}^{\text{tot}} = -\frac{1}{2} \epsilon_o \mathbb{E}^{\epsilon_0} \cdot \mathbb{E}^{\epsilon_0} \quad \text{in } \mathcal{S}_t. \quad (114)$$

As consequence the total material motion Piola stress tensor vanishes identically

$$\mathbf{p}^{\text{tot}} = \mathbf{0} \quad \text{in } \mathcal{S}_t \quad (115)$$

and the balance equation (113) in the free space is satisfied automatically since $\frac{\partial W_{if}^{\text{tot}}}{\partial \Phi} = \mathbf{B}_t^{\text{tot,ext}} = \mathbf{B}_t^{\text{tot,con}} = \mathbf{0}$ in $\mathcal{S}_t$.

By expressing the total spatial motion Cauchy stress tensor in terms of the material motion deformation gradient $\mathbf{f}$ in the format

$$\boldsymbol{\sigma}^{\text{tot}} = W_{if}^{\text{tot}} \mathbf{i} - \mathbf{f}^t \cdot \frac{\partial W_{if}^{\text{tot}}}{\partial \mathbf{f}} + \mathbb{e} \otimes \mathbb{d}, \quad (116)$$

the total material motion Piola stress tensor $\mathbf{p}$ is related to $\boldsymbol{\sigma}^{\text{tot}}$ by the relationship

$$\mathbf{p}^{\text{tot}} = W_{if}^{\text{tot}} \mathbf{f}^{-t} - \mathbf{f}^{-t} \cdot \boldsymbol{\sigma}^{\text{tot}} + \mathbf{f}^{-t} \cdot \mathbb{e} \otimes \mathbb{d}. \quad (117)$$

The use of this relationship in Equation (113) results in

$$\nabla_x \cdot \boldsymbol{\sigma}^{\text{tot}} - \mathbf{f}^t \cdot \mathbf{B}_t^{\text{tot,ext}} - \mathbf{f}^t \cdot \mathbf{B}_t^{\text{tot,con}} = \mathbf{0} \quad \text{in } \mathcal{B}_t, \quad (118)$$

which, by comparing with the balance equation of linear momentum of the spatial motion problem, i.e. Equation (62), shows that the material motion body force $\mathbf{B}_t^{\text{tot,ext}}$ and the configurational body force $\mathbf{B}_t^{\text{tot,con}}$ can be defined as

$$\mathbf{B}_t^{\text{tot,ext}} := -\mathbf{f}^{-t} \cdot \mathbf{b}_t^{\text{mec}} \quad \text{and} \quad \mathbf{B}_t^{\text{tot,con}} := \mathbf{0}. \quad (119.1-2)$$

The material motion surface force $\mathbf{T}_t^{\text{tot,ext}}$ and the configurational surface force $\mathbf{T}_t^{\text{tot,con}}$ can be defined by considering the jump condition (111). By using the definition of the total material motion Piola stress tensor $\mathbf{p}^{\text{tot}}$, this jump condition is now

$$[\![\mathbf{p}^{\text{tot}}]\!] \cdot \mathbf{n} = \mathbf{T}_t^{\text{tot,ext}} + \mathbf{T}_t^{\text{tot,con}} \quad \text{on } \partial \mathcal{B}_t \quad (120)$$

or by using the relationship (117) and keeping in mind that the deformation gradient $\mathbf{f}$ is assumed to be continuous across the boundary of the material body

$$[\![W_{if}^{\text{tot}}]\!] \mathbf{f}^{-t} \cdot \mathbf{n} - \mathbf{f}^{-t} \cdot [\![\boldsymbol{\sigma}^{\text{tot}}]\!] \cdot \mathbf{n} + \mathbf{f}^{-t} \cdot [\![\mathbb{e} \otimes \mathbb{d}]\!] \cdot \mathbf{n} = \mathbf{T}_t^{\text{tot,ext}} + \mathbf{T}_t^{\text{tot,con}} \quad \text{on } \partial \mathcal{B}_t. \quad (121)$$

Because on the boundary of the body we have $[\![\boldsymbol{\sigma}^{\text{tot}}]\!] \cdot \mathbf{n} = \mathbf{t}_t^{\text{mec}}$ we arrive at the definition of the material motion surface force $\mathbf{T}_t^{\text{tot,ext}}$ and the configurational surface force $\mathbf{T}_t^{\text{tot,con}}$ as

$$\mathbf{T}_t^{\text{tot,ext}} := -\mathbf{f}^{-t} \cdot \mathbf{t}_t^{\text{mec}} \quad \text{and} \quad \mathbf{T}_t^{\text{tot,con}} := \mathbf{f}^{-t} \cdot [\![W_{if}^{\text{tot}}]\!] \mathbf{i} + \mathbb{e} \otimes \mathbb{d} \cdot \mathbf{n}. \quad (122.1-2)$$

By noting that $[\![W_{if}^{\text{tot}}]\!] \mathbf{i} + \mathbb{e} \otimes \mathbb{d} = W_{if}^{\text{tot},\epsilon} \mathbf{i} + \mathbb{e}^\epsilon \otimes \mathbb{d}^\epsilon + \frac{1}{2} \epsilon_0 \mathbb{e}^{\epsilon_0} \cdot \mathbb{e}^{\epsilon_0} - \mathbb{e}^{\epsilon_0} \otimes \mathbb{d}^{\epsilon_0} = W_{if}^{\text{tot},\epsilon} \mathbf{i} + \mathbb{e}^\epsilon \otimes \mathbb{d}^\epsilon - \boldsymbol{\sigma}^{\text{max}}$, we have the following formula for the configurational traction $\mathbf{T}_t^{\text{tot,con}}$

$$\mathbf{T}_t^{\text{tot,con}} := \mathbf{f}^{-t} \cdot \left[[W_{if}^{\text{tot},\epsilon} \mathbf{i} + \mathbb{e}^\epsilon \otimes \mathbb{d}^\epsilon] \cdot \mathbf{n} - \mathbf{t}_t^{\text{max}} \right], \quad (123)$$

which shows the contribution of the free space not only through its influence on different electric quantities but also through the Maxwell's traction $\mathbf{t}_t^{\text{max}}$.

With the above definitions, the governing equations of the material motion problem can be cast in the following format

$$\begin{cases} \nabla_x \cdot \mathbf{p}^{\text{tot}} + \mathbf{B}_t^{\text{tot}} = \mathbf{0} & \text{in } \mathcal{B}_t \\ [\![\mathbf{p}^{\text{tot}}]\!] \cdot \mathbf{n} = \mathbf{T}_t^{\text{tot}} & \text{on } \partial \mathcal{B}_t, \end{cases} \quad (124)$$

where the body force is defined by $\mathbf{B}_t^{\text{tot}} := -\frac{\partial W_{if}^{\text{tot}}}{\partial \Phi} + \mathbf{B}_t^{\text{tot,ext}} + \mathbf{B}_t^{\text{tot,con}}$ and the surface force is $\mathbf{T}_t^{\text{tot}} := \mathbf{T}_t^{\text{tot,ext}} + \mathbf{T}_t^{\text{tot,con}}$.

Remark. As is already noted in the case of nonlinear elastostatics, the definitions (119) and (122) are useful in satisfying the balance equation and jump condition of the material motion problem in nonlinear electroelastostatics, i.e. Equations (113) and (120) or system (124). However, they can only be used to compute the released energy or the configurational energy under some certain conditions. For the case in which there is no mechanical body force ($\mathbf{b}_t^{\text{mec}} = \mathbf{0}$) and the surface force $\mathbf{t}_t^{\text{mec}}$ does not perform work with respect to a change in the undeformed configuration, the configurational energy with respect to a variation $\delta_x \Phi$ can be computed by

$$\delta_x \mathbf{w}_{\text{con}}^{\text{tot}} = \int_{\partial \mathcal{B}_t} \left[\mathbf{f}^{-t} \cdot [W_{if}^{\text{tot},\epsilon} \mathbf{i} + \mathbb{e}^\epsilon \otimes \mathbb{d}^\epsilon] \cdot \mathbf{n} - \mathbf{t}_t^{\text{max}} \right] \cdot \delta_x \Phi \, ds. \quad (125)$$

3.2.2. Formulation in the undeformed configuration. In reference to the undeformed configuration $\mathcal{B}_0$, the material motion problem in nonlinear electro-elastostatics can also be characterized by the balance equation of energy changes (104), in which the change of the stored energy is computed by

$$\delta_x \mathbf{w}_{\text{int}}^{\text{tot}} = \delta_x \int_{\mathcal{B}_0} W_{0f}^{\text{tot}} dV, \quad (126)$$

where W_{0f}^{tot} is the stored energy density function per unit volume of the undeformed configuration $W_{0f}^{\text{tot}}(\mathbf{f}, \mathbb{E}, \Phi) = JW_{ff}^{\text{tot}}(\mathbf{f}, \mathbb{E}, \Phi)$ or, equivalently, $W_{0f}^{\text{tot}}|_{f, \mathbb{E}, \Phi} = W_{0f}^{\text{tot}}|_{F, \mathbb{E}, X}$. This definition leads us to

$$\begin{aligned} \delta_x \mathbf{w}_{\text{int}}^{\text{tot}} = & \int_{\mathcal{B}_0 \cup S_0} W_{0f}^{\text{tot}} \delta_x dV + \int_{\mathcal{B}_0 \cup S_0} \frac{\partial W_{0f}^{\text{tot}}}{\partial \mathbf{f}} : \delta_x \mathbf{f} dV \\ & + \int_{\mathcal{B}_0 \cup S_0} \frac{\partial W_{0f}^{\text{tot}}}{\partial \Phi} \cdot \delta_x \Phi dV + \int_{\mathcal{B}_0 \cup S_0} \frac{\partial W_{0f}^{\text{tot}}}{\partial \mathbb{E}} \cdot \delta \mathbb{E} dV. \end{aligned} \quad (127)$$

The change in the work done by the external forces is now computed as

$$\delta_x \mathbf{w}_{\text{ext}}^{\text{tot}} = \int_{\mathcal{B}_0} \mathbf{B}_0^{\text{tot}, \text{ext}} \cdot \delta_x \Phi dV + \int_{\partial \mathcal{B}_0} \mathbf{T}_0^{\text{tot}, \text{ext}} \cdot \delta_x \Phi dS - \int_{\partial \mathcal{B}_0} \widehat{\mathcal{Q}}_0^{\text{fre}} \delta \Psi dS, \quad (128)$$

where $\mathbf{B}_0^{\text{tot}, \text{ext}}$ and $\mathbf{T}_0^{\text{tot}, \text{ext}}$ are some material motion body and surface forces defined as functions of $\mathbf{b}_0^{\text{mec}}$ and $\mathbf{t}_0^{\text{mec}}$ in the format $\mathbf{B}_0^{\text{tot}, \text{ext}} = \mathbf{B}_0^{\text{tot}, \text{ext}}(\mathbf{b}_0^{\text{mec}})$ and $\mathbf{T}_0^{\text{tot}, \text{ext}} = \mathbf{T}_0^{\text{tot}, \text{ext}}(\mathbf{t}_0^{\text{mec}})$. The configurational energy is assumed to be calculated by

$$\delta_x \mathbf{w}_{\text{con}}^{\text{tot}} = \int_{\mathcal{B}_0} \mathbf{B}_0^{\text{tot}, \text{con}} \cdot \delta_x \Phi dV + \int_{\partial \mathcal{B}_0} \mathbf{T}_0^{\text{tot}, \text{con}} \cdot \delta_x \Phi dS, \quad (129)$$

where $\mathbf{B}_0^{\text{tot}, \text{con}}$ and $\mathbf{T}_0^{\text{tot}, \text{con}}$ are the counterparts of the configurational body and surface forces $\mathbf{B}_t^{\text{tot}, \text{con}}$ and $\mathbf{T}_t^{\text{tot}, \text{con}}$ in the undeformed configuration $\mathcal{B}_0$.

By using (127), (128) and (129) in Equation (104) and by requiring that Equation (104) be satisfied with respect to a variation of the material placement at fixed spatial placement, we get the balance equation

$$\nabla_X \cdot [W_{0f}^{\text{tot}} \mathbf{I} + \frac{\partial W_{0f}^{\text{tot}}}{\partial \mathbf{f}} \cdot \mathbf{f}^t] - \frac{\partial W_{0f}^{\text{tot}}}{\partial \Phi} + \mathbf{B}_0^{\text{tot}, \text{ext}} + \mathbf{B}_0^{\text{tot}, \text{con}} = \mathbf{0} \quad \text{in } \mathcal{B}_0 \quad (130)$$

and the jump condition

$$\llbracket W_{0f}^{\text{tot}} \mathbf{I} + \frac{\partial W_{0f}^{\text{tot}}}{\partial \mathbf{f}} \cdot \mathbf{f}^t \rrbracket \cdot \mathbf{N} = \mathbf{T}_0^{\text{tot}, \text{ext}} + \mathbf{T}_0^{\text{tot}, \text{con}} \quad \text{on } \partial \mathcal{B}_0, \quad (131)$$

which motivate us to define the total material motion Cauchy stress tensor as

$$\Sigma^{\text{tot}} := W_{0f}^{\text{tot}} \mathbf{I} + \frac{\partial W_{0f}^{\text{tot}}}{\partial \mathbf{f}} \cdot \mathbf{f}^t, \quad (132)$$

or in index notation $\Sigma^{\text{tot}} = \Sigma_{IJ}^{\text{tot}} \mathbf{E}_I \otimes \mathbf{E}_J$ with the coordinates $\Sigma_{IJ}^{\text{tot}} = W_{0f}^{\text{tot}} \delta_{IJ} + \frac{\partial W_{0f}^{\text{tot}}}{\partial f_{Ik}} f_{Jk}$. The balance equation (130) can therefore be rewritten as

$$\nabla_X \cdot \Sigma^{\text{tot}} - \frac{\partial W_{0f}^{\text{tot}}}{\partial \Phi} + \mathbf{B}_0^{\text{tot}, \text{ext}} + \mathbf{B}_0^{\text{tot}, \text{con}} = \mathbf{0} \quad \text{in } \mathcal{B}_0. \quad (133)$$

In the free space outside the material body, the stored energy density function W_{0f}^{tot} reduces to E_0

$$W_{0f}^{\text{tot}} = -\frac{1}{2}\epsilon_0 j^{-1} \mathbb{E}^{\epsilon_0} \cdot \mathbb{E}^{\epsilon_0} \quad \text{in } S_0, \quad (134)$$

where $j = \det \mathbf{f}$ and the total material motion Cauchy stress tensor, like its counterpart the total material Piola stress tensor, vanishes

$$\boldsymbol{\Sigma}^{\text{tot}} := \mathbf{0} \quad \text{in } S_0. \quad (135)$$

The balance equation (133) in the free space is satisfied automatically since $\frac{\partial W_{0f}^{\text{tot}}}{\partial \boldsymbol{\Phi}} = \mathbf{B}_0^{\text{tot,ext}} = \mathbf{B}_0^{\text{tot,con}} = \mathbf{0}$ in S_0 .

Note that the total material motion Cauchy stress tensor can also be expressed in terms of the spatial motion deformation gradient $\mathbf{F}$ as

$$\boldsymbol{\Sigma}^{\text{tot}} = W_{0f}^{\text{tot}} \mathbf{I} - \mathbf{F}^t \cdot \frac{\partial W_{0f}^{\text{tot}}}{\partial \mathbf{F}} + \mathbb{E} \otimes \mathbb{D}, \quad (136)$$

which leads us to the following relationship between $\boldsymbol{\Sigma}^{\text{tot}}$ and its counterpart, the total spatial motion Piola stress tensor $\mathbf{P}^{\text{tot}}$

$$\boldsymbol{\Sigma}^{\text{tot}} = W_{0f}^{\text{tot}} \mathbf{I} - \mathbf{F}^t \cdot \mathbf{P}^{\text{tot}} + \mathbb{E} \otimes \mathbb{D}. \quad (137)$$

By using this relationship in Equation (133) one obtains

$$\nabla_X \cdot \mathbf{P}^{\text{tot}} - \mathbf{f}^t \cdot \mathbf{B}_0^{\text{tot,ext}} - \mathbf{f}^t \cdot \mathbf{B}_0^{\text{tot,con}} = \mathbf{0} \quad \text{in } \mathcal{B}_0. \quad (138)$$

By comparing this equation with the balance equation of linear momentum of the spatial motion problem, i.e. Equation (91), we arrive at the following definitions for the material motion body force $\mathbf{B}_0^{\text{tot,ext}}$ and for the configurational body force $\mathbf{B}_0^{\text{tot,con}}$

$$\mathbf{B}_0^{\text{tot,ext}} := -\mathbf{f}^{-t} \cdot \mathbf{b}_0^{\text{mec}} \quad \text{and} \quad \mathbf{B}_0^{\text{tot,con}} := \mathbf{0}. \quad (139.1-2)$$

By using the definition of the material motion Cauchy stress tensor $\boldsymbol{\Sigma}$, the jump condition (131) is now

$$\llbracket \boldsymbol{\Sigma}^{\text{tot}} \rrbracket \cdot \mathbf{N} = \mathbf{T}_0^{\text{tot,ext}} + \mathbf{T}_0^{\text{tot,con}} \quad \text{on } \partial \mathcal{B}_0 \quad (140)$$

or in the format

$$\llbracket W_{0f}^{\text{tot}} \rrbracket \mathbf{N} - \mathbf{f}^{-t} \cdot \mathbf{t}_0^{\text{mec}} + \llbracket \mathbb{E} \otimes \mathbb{D} \rrbracket \cdot \mathbf{N} = \mathbf{T}_0^{\text{tot,ext}} + \mathbf{T}_0^{\text{tot,con}} \quad \text{on } \partial \mathcal{B}_0, \quad (141)$$

which motivates us to define the material motion surface force $\mathbf{T}_0^{\text{tot,ext}}$ and the configuration surface force $\mathbf{T}_0^{\text{tot,con}}$ as

$$\mathbf{T}_0^{\text{tot,ext}} := -\mathbf{f}^{-t} \cdot \mathbf{t}_0^{\text{mec}} \quad \text{and} \quad \mathbf{T}_0^{\text{tot,con}} := \llbracket W_{0f}^{\text{tot}} \mathbf{I} + \mathbb{E} \otimes \mathbb{D} \rrbracket \cdot \mathbf{N}. \quad (142.1-2)$$

By noting that in the free space surrounding the body $W_{0f}^{\text{tot}} \mathbf{I} + \mathbb{E} \otimes \mathbb{D} = -\frac{1}{2}\epsilon_0 j [\mathbb{E}^{\epsilon_0} \cdot \mathbf{C}^{-1} \cdot \mathbb{E}^{\epsilon_0}] \mathbf{I} + \mathbb{E}^{\epsilon_0} \otimes \mathbb{D}^{\epsilon_0} = \mathbf{F}^t \cdot \mathbf{P}^{\text{max}}$ we can rewrite the above formula for $\mathbf{T}_0^{\text{tot,con}}$ in the following format

$$\mathbf{T}_0^{\text{tot,con}} := \left[W_{0f}^{\text{tot},\epsilon} \mathbf{I} + \mathbb{E}^{\epsilon} \otimes \mathbb{D}^{\epsilon} \right] \cdot \mathbf{N} - \mathbf{f}^{-t} \cdot \mathbf{t}_0^{\text{max}}, \quad (143)$$

where the contribution of the free space can be seen through its influence on electric quantities such as $\mathbb{E}^{\epsilon}$, $\mathbb{D}^{\epsilon}$ and through the presence of the material version of the Maxwell's traction $\mathbf{t}_0^{\text{max}}$.

The above definitions lead us to the following system of governing equations

$$\begin{cases} \nabla_X \cdot \boldsymbol{\Sigma}^{\text{tot}} + \mathbf{B}_0^{\text{tot}} = \mathbf{0} & \text{in } \mathcal{B}_0 \\ \llbracket \boldsymbol{\Sigma}^{\text{tot}} \rrbracket \cdot \mathbf{N} = \mathbf{T}_0^{\text{tot}} & \text{on } \partial \mathcal{B}_0, \end{cases} \quad (144)$$

where $\mathbf{B}_0^{\text{tot}} := -\frac{\partial W_{0f}^{\text{tot}}}{\partial \Phi} + \mathbf{B}_0^{\text{tot,ext}} + \mathbf{B}_0^{\text{tot,con}}$ and $\mathbf{T}_0^{\text{tot}} := \mathbf{T}_0^{\text{tot,ext}} + \mathbf{T}_0^{\text{tot,con}}$.

Remark. When there is no mechanical body force ($\mathbf{b}_0^{\text{mec}} = \mathbf{0}$) and the surface force $\mathbf{t}_0^{\text{mec}}$ does not perform work with respect to a change in the undeformed configuration, the configurational energy respect to a variation $\delta_X \Phi$ is

$$\delta_X \mathbf{w}_{\text{con}}^{\text{tot}} = \int_{\partial \mathcal{B}_0} \left[\left[W_{0f}^{\text{tot},\epsilon} \mathbf{I} + \mathbb{E}^\epsilon \otimes \mathbb{D}^\epsilon \right] \cdot \mathbf{N} - \mathbf{f}^{-t} \cdot \mathbf{t}_0^{\text{max}} \right] \cdot \delta_X \Phi \, dS. \quad (145)$$

4. Conclusion

The formulation of spatial and material motion problems in nonlinear electro-elastostatics is considered in this work by using an energy approach and by paying careful attention to the contribution of the free space surrounding a nonlinearly polarized body undergoing large deformation. With the help of some stored energy density functions and an imaginary deformation field in the free space, the governing equations of the spatial and material motion problems are derived using the concept of total stresses. In the material motion problem, in addition to the derivation of configurational forces, this approach reveals the formulas for the part of energy that is released from the system material body, applied forces, in response to a change in the material configuration. In the case the contribution of the free space cannot be ignored, the formulas for the released energy presented here offer a useful tool in the study of defects, for example in the study of crack propagation. The same approach can be used in the case of nonlinear electro-thermo-mechanical coupling and constitutes the direction for future works.

Notes

1. For the sake of simplicity, we assume that there is no surface of discontinuity inside the material body. As consequence, all jump conditions for mechanical as well as electrical quantities are considered only at the boundary between the material body and the surrounding free space.
2. We assume that such stored energy density function exists.
3. The above definition of the electric stress tensor σ^{ele} is chosen for the sake of simplicity despite the fact that σ^{ele} is only defined up to an additive constant.

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Conflict of Interest

None declared.

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