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Towards an Entropic Analysis of Quantum Error Correction with Imperfections

Carlo Cafaro and Peter van Loock

*Optical Quantum Information Theory Group
Max-Planck Institute for the Science of Light, Germany
and Institute of Theoretical Physics, Universitat Erlangen-Nurnberg, Germany*

Abstract. The concept of entropy and the correct application of the Second Law of thermodynamics are essential in order to understand the reason why quantum error correction (QEC) is thermodynamically possible and no violation of the Second Law occurs during its execution.

We report here on our preliminary work aiming at an information-theoretic analysis extended to QEC in the presence of *imperfections*.

INTRODUCTION

Entropy plays a key-role in the foundations of quantum theory [1] whose statistical nature is evident when dealing with incomplete *information* gathered in quantum measurements. In general, measurements are performed to increase information about physical systems. This information, if appropriate, may in principle be used for a reduction of the thermodynamical entropy of such physical systems as we know from the thought construction of an intelligent being, the so-called Maxwell's demon. In his 1929 seminal paper [2], Szilard presented several examples (Szilard's engine) which show that additional information about a system yields a decrease in the entropy of the system. Szilard reaffirmed his belief in the *Second Law of thermodynamics* and that the measurement process, in some overall sense, requires energy dissipation.

Perhaps, there is no better arena than that provided by QEC to critically discuss about the links among entropy, information and thermodynamics. A QEC technique consists in encoding quantum information into a physical system in such a way that it can be saved from such effects. Furthermore, since the process of quantum measurement cannot perfectly discriminate among non-orthogonal states, the optimal strategy to encode information is to prepare the d -level quantum system in one out of d orthogonal states.

More specifically, the main motivation for our work was the will of gaining a better understanding of the following statement appeared in [3]: "Doing perfect error correction without perfect information gain is forbidden by the Second Law of thermodynamics via Landauer's principle. This is analogous to von Neumann's (1952) proof that being able to distinguish perfectly between two non-orthogonal states would lead directly to the violation of the Second Law of thermodynamics".

In Section II, we present an alternative proof that shows the square modulus of the overlap of non-orthogonal quantum states is the essential quantity that limits the effectiveness of discrimination between quantum states when no inconclusive measurement

outcome is permitted. In Section III, we discuss few entropic consequences of the non-orthogonality of quantum states in the analysis of a QEC cycle. Our final remarks appear in Section IV.

QUANTUM STATE DISCRIMINATION

The ability to determine the state of a quantum state is not only severely limited by thermodynamics, as pointed out earlier, but by quantum theory itself as well. In particular, even if they are known, non-orthogonal quantum states cannot be discriminated perfectly. The two most well-known optimum discrimination strategies are the "*optimum unambiguous error-free discrimination strategy*" and the "*optimum ambiguous discrimination with minimum error strategy*" [4]. In the former procedure, whenever a definitive answer is returned after a measurement on the state, the result should be unambiguous, at the expense of allowing inconclusive outcomes to occur. In the latter procedure, instead, one requires to have conclusive results only. This means that errors are unavoidable when the states are non-orthogonal. Based on the measurement outcome, in each single case then a guess has to be made as to what the state of the quantum system was. This procedure is known as *quantum hypothesis testing* [5]. The problem consists in finding the optimum measurement strategy that minimizes the probability of errors. In general, the explicit solution to a quantum hypothesis testing, which is an error-minimizing problem, is not trivial and analytical expressions have been derived only for a few special cases. For instance, the solution of the problem of distinguishing two pure non-orthogonal quantum states with minimum error is considered a pioneering work in quantum detection theory and was uncovered by Helstrom. The optimal value $\mathcal{P}_E \stackrel{\text{def}}{=} \min \mathcal{P}_{\text{err.}}$ of the probability of error $\mathcal{P}_{\text{err.}}$ obtained by Helstrom reads [4],

$$\mathcal{P}_E \stackrel{\text{def}}{=} 2^{-1} \left[1 - \left(1 - 4\eta_1\eta_2 |\langle \psi_1 | \psi_2 \rangle|^2 \right)^{\frac{1}{2}} \right], \quad (1)$$

where, in general, $\mathcal{P}_{\text{err.}}$ is defined as,

$$\mathcal{P}_{\text{err.}} \stackrel{\text{def}}{=} 1 - \mathcal{P}_{\text{corr.}} = 1 - \sum_{k=1}^N \eta_k \text{Tr}(\rho_k \Pi_k) \text{ with, } \sum_{k=1}^N \Pi_k = I_{D \times D}. \quad (2)$$

The quantity $\mathcal{P}_{\text{corr.}}$ is the probability to make a correct guess, D denotes the dimensionality of the physical space state, η_k are the *a priori* probabilities of occurrence of the quantum states, Π_k are the detection operators that characterize the measurement process and ρ_k are the density operators of the N states of a quantum system. As an illustrative example, consider the following two pure states $|\psi_1\rangle \stackrel{\text{def}}{=} 1/2|0\rangle + \sqrt{3}/2|1\rangle$ and $|\psi_2\rangle \stackrel{\text{def}}{=} \sqrt{3}/2|0\rangle + 1/2|1\rangle$ with equal a priori probabilities $\eta_1 = \eta_2 = 1/2$. It turns out that a convenient choice for the optimal von Neumann measurement operators is given by $\Pi_1 \stackrel{\text{def}}{=} |1\rangle\langle 1|$ and $\Pi_2 \stackrel{\text{def}}{=} |0\rangle\langle 0|$ (orthogonal detectors placed symmetrically around $|\psi_1\rangle$ and $|\psi_2\rangle$). In this case, $\mathcal{P}_E \stackrel{\text{def}}{=} \min \mathcal{P}_{\text{err.}} = 0.25$.

Discrimination of orthogonal states: an old proof

Before presenting our novel analysis, let us revisit the proof establishing that "*it is impossible to unambiguously distinguish non-orthogonal pure quantum states*". This assertion is proved by contradiction. We assume that non-orthogonal quantum states can be unambiguously distinguished and show that this leads to a contradiction. Consider two non-orthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$. Let $\mathcal{O}$ be an observable represented by the Hermitian operator $\hat{O}$ with eigenvalues λ_k and projection operators Π_k such that its measurement allows to unambiguously distinguish $|\psi_1\rangle$ and $|\psi_2\rangle$. This implies that eigenvalues λ_α and λ_β exist such that observation of λ_α (λ_β) unambiguously identifies $|\psi_1\rangle$ ($|\psi_2\rangle$) as the pre-measurement state. Formally, this means that the probability to observe λ_α (λ_β) when the pre-measurement state is $|\psi_1\rangle$ ($|\psi_2\rangle$) is one,

$$\langle \psi_1 | \Pi_\alpha | \psi_1 \rangle = 1 \text{ and } \langle \psi_2 | \Pi_\beta | \psi_2 \rangle = 1, \quad (3)$$

and thus the probability to observe λ_β (λ_α) when the pre-measurement state is $|\psi_1\rangle$ ($|\psi_2\rangle$) is zero,

$$\langle \psi_1 | \Pi_\beta | \psi_1 \rangle = 0 \text{ and } \langle \psi_2 | \Pi_\alpha | \psi_2 \rangle = 0. \quad (4)$$

Since $|\psi_1\rangle$ and $|\psi_2\rangle$ are assumed to be non-orthogonal, we can write

$$|\psi_2\rangle \stackrel{\text{def}}{=} c_1 |\psi_1\rangle + c_d |\psi_d\rangle, \quad (5)$$

where $|c_1|^2 + |c_d|^2 = 1$ and $|\psi_d\rangle$ is orthogonal to $|\psi_1\rangle$. Observe that $\langle \psi_1 | \Pi_\beta | \psi_1 \rangle = 0$ implies $\Pi_\beta |\psi_1\rangle = 0$ since

$$0 = \langle \psi_1 | \Pi_\beta | \psi_1 \rangle = \langle \psi_1 | \Pi_\beta \Pi_\beta | \psi_1 \rangle = \|\Pi_\beta |\psi_1\rangle\|^2, \quad (6)$$

and the only state with zero norm is the null state. Combining (5) with (6) allows us to explicitly evaluate $\langle \psi_2 | \Pi_\beta | \psi_2 \rangle$,

$$\langle \psi_2 | \Pi_\beta | \psi_2 \rangle = |c_d|^2 \langle \psi_d | \Pi_\beta | \psi_d \rangle. \quad (7)$$

Observe that,

$$1 = \langle \psi_d | \psi_d \rangle = \langle \psi_d | I | \psi_d \rangle = \sum_k \langle \psi_d | \Pi_k | \psi_d \rangle \geq \langle \psi_d | \Pi_\beta | \psi_d \rangle, \quad (8)$$

where the inequality appears since all terms in the sum are non-negative ($\langle \psi_d | \Pi_k | \psi_d \rangle$ is the probability that λ_k is the measurement outcome when the pre-measurement state is $|\psi_d\rangle$). Combining (7) and (8) gives,

$$1 \equiv \langle \psi_2 | \Pi_\beta | \psi_2 \rangle = |c_d|^2 \langle \psi_d | \Pi_\beta | \psi_d \rangle \leq |c_d|^2, \quad (9)$$

that is, $|c_d|^2 \geq 1$. However, recall that $|c_1|^2 + |c_d|^2 = 1$. Therefore, it must be $c_d = 1$ and $c_1 = 0$, that is $|\psi_2\rangle = |\psi_d\rangle$. Thus, for $|\psi_2\rangle$ to be unambiguously distinguishable from $|\psi_1\rangle$, the two pure quantum states must be orthogonal (in particular, any two orthogonal

entangled quantum states can be distinguished just as well using local operations and classical communication as they can globally [6]). However, we assumed that these states were non-orthogonal so that we arrived at a contradiction that proves the assertion.

We emphasize that the line of reasoning just presented exhibits two main features. First, no inconclusive outcome was allowed since the sum of the measurement operators add up to the unit operator. Second, no ambiguity (imperfect discrimination) was permitted as evident from Eqs. (3) and (4).

Discrimination of non-orthogonal states: a novel proof

In what follows, being within the ambiguous discrimination strategy framework, we "perturb" the old proof concerning the perfect discrimination of orthogonal-states in such a way to accommodate "imperfect/approximate" or, better yet, *ambiguous discrimination* of non-orthogonal states. We show, via a simple alternative route, that our reasoning is consistent with standard arguments that give the square modulus of the overlap of non-orthogonal quantum states as the essential quantity that limits the effectiveness of discrimination (with nonvanishing minimum error probability) between quantum states [7]. In particular, we check the compatibility of the main consequence of our analysis with the above-mentioned Helstrom's pioneering result in the limit of very small probability of error. Stated otherwise, we extend the above-reconsidered analysis preserving the first feature but relaxing the second one by introducing an "ambiguity-factor" δ which can be ultimately regarded as the generator of a non-vanishing probability of error within the scheme of optimum ambiguous discrimination. Specifically, our main working hypothesis is that Eqs. (3) and (4) assume the following new forms,

$$\langle \psi_1 | \Pi_\alpha | \psi_1 \rangle = 1 - \delta \text{ and } \langle \psi_2 | \Pi_\beta | \psi_2 \rangle = 1 - \delta, \quad (10)$$

and thus the probability to observe λ_β (λ_α) when the pre-measurement state is $|\psi_1\rangle$ ($|\psi_2\rangle$) assumes a non-vanishing value δ ,

$$\langle \psi_1 | \Pi_\beta | \psi_1 \rangle = \delta \text{ and } \langle \psi_2 | \Pi_\alpha | \psi_2 \rangle = \delta. \quad (11)$$

The non-orthogonality between $|\psi_1\rangle$ and $|\psi_2\rangle$ allows us to reconsider the decomposition given in Eq. (5). Inserting (5) into the second relation in (10), we get

$$1 - \delta \stackrel{\text{def}}{=} \langle \psi_2 | \Pi_\beta | \psi_2 \rangle = c_1^* \langle \psi_1 | \Pi_\beta | \psi_2 \rangle + c_1 \langle \psi_2 | \Pi_\beta | \psi_1 \rangle - |c_1|^2 \langle \psi_1 | \Pi_\beta | \psi_1 \rangle + |c_d|^2 \langle \psi_d | \Pi_\beta | \psi_d \rangle. \quad (12)$$

For the sake of clarity, we assume that the quantum states considered are *real*-valued and using Eqs. (10) and (11), Eq. (12) becomes

$$1 - \delta = 2c_1 \sqrt{\delta(1 - \delta)} - c_1^2 \delta + (1 - c_1^2) \langle \psi_d | \Pi_\beta | \psi_d \rangle. \quad (13)$$

However, Eq. (8) implies that $\langle \psi_d | \Pi_\beta | \psi_d \rangle \leq 1$ and, thus, we arrive at the following inequality constraint relating the "ambiguity-factor" δ and the overlap $c_1 \stackrel{\text{def}}{=} \langle \psi_1 | \psi_2 \rangle$,

$$(1 + \delta)c_1^2 - 2\sqrt{\delta(1 - \delta)}c_1 - \delta \leq 0. \quad (14)$$

Finally, in the limiting case of interest, $\delta \ll 1$, the inequality constraint (14) requires that

$$\delta \gtrsim \tilde{\eta}_{12} |\langle \psi_1 | \psi_2 \rangle|^2, \quad (15)$$

with $0 \leq \tilde{\eta}_{12} \stackrel{\text{def}}{=} (1 + \sqrt{2})^{-2} \leq 1$. Our analysis leads to the conclusion that $\delta_{\min.} \propto |\langle \psi_1 | \psi_2 \rangle|^2$, that is, it confirms that the square modulus of the overlap of non-orthogonal quantum states is the essential quantity that limits the effectiveness of discrimination between quantum states when no inconclusive measurement outcome is permitted [7]. We remark that in general the overlap $\mathcal{O}_{jk}$ between two arbitrary mixed quantum states ρ_j and ρ_k reads [8],

$$\mathcal{O}_{jk} \stackrel{\text{def}}{=} \text{Tr}(\rho_j \rho_k), \quad (16)$$

while for pure quantum states $\rho_i \stackrel{\text{def}}{=} |\psi_i\rangle \langle \psi_i|$ with $i = 1, 2$ (and $\rho_i^2 = \rho_i$), it turns out that $\text{Tr}(\rho_1 \rho_2) \stackrel{\text{def}}{=} |\langle \psi_1 | \psi_2 \rangle|^2$. For the sake of completeness, we also stress that there are cases where the effectiveness of discrimination between two non-orthogonal quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$ is limited by $|\langle \psi_1 | \psi_2 \rangle|$. This happens in a classification without errors where the modulus of the overlap sets the bound (the so-called Ivanovich-Dieks-Peres bound, [4]). In this alternative case, the discrimination procedure enables to infer with certainty whether the system was in the state $|\psi_1\rangle$ or $|\psi_2\rangle$ and leaves a minimum number of cases undecided. As an additional consistency check of our analysis, we notice that in analogy to the working condition $\delta \ll 1$, setting $\mathcal{P}_E \ll 1$, and re-arranging Helstrom's formula in (1) together with neglecting higher order infinitesimal terms in the Taylor-expansion of $|\langle \psi_1 | \psi_2 \rangle|^2$ in (1), we arrive at

$$\mathcal{P}_{\text{err.}} \geq \mathcal{P}_E \simeq \eta_1 \eta_2 |\langle \psi_1 | \psi_2 \rangle|^2. \quad (17)$$

Upon the reasonable identification of δ with $\mathcal{P}_{\text{err.}}$ (after all, "*ambiguity does cause errors*") and for a convenient choice of measurement operators, we find out that our positive numerical proportionality factor in (15) is less than unity and is in principle compatible with a suitable choice of a pair of *a priori* probabilities η_1 and η_2 in (17) (namely, $\eta_1 = 0.78005$ and $\eta_2 = 0.21995$).

ENTROPY, ERROR CORRECTION, AND STATE DISCRIMINATION

From a thermodynamical point of view, QEC may be regarded as a refrigeration process capable of maintaining the quantum system at a constant entropy despite the environmental noisy process whose tendency is to change the entropy of the quantum system itself. Information about the quantum system gathered in quantum measurements is used to keep the system cool. At first sight, it may appear that QEC allows a reduction in the entropy of the quantum system in apparent violation of the Second Law. However, a

careful thermodynamic analysis shows that QEC, like Maxwell's demon, does not violate the Second Law.

Consider a quantum system $\mathcal{Q}$ that is initially in the state ρ with von Neumann entropy $\mathcal{S}(\rho) \stackrel{\text{def}}{=} -\text{Tr}(\rho \log \rho)$. The interaction of $\mathcal{Q}$ with a noisy environment $\mathcal{E}$ takes generally $\mathcal{Q}$ to a new state ρ' with entropy $\mathcal{S}(\rho') > \mathcal{S}(\rho)$. Ideally, when an exact-QEC scheme (optimal information gain) can be employed, the state ρ' with $\mathcal{S}(\rho')$ can return to ρ with $\mathcal{S}(\rho)$. Thus, considering the entropy change of the system $\mathcal{Q}$ just before (when the environmental noise has already acted upon the quantum system of interest $\mathcal{Q}$) and right after QEC, one concludes that

$$\Delta \mathcal{S} \stackrel{\text{def}}{=} \mathcal{S}(\rho) - \mathcal{S}(\rho') < 0. \quad (18)$$

From (18) it seems that QEC violates the Second Law since there is a reduction in entropy of $\mathcal{Q}$ (the total entropy of a closed physical system cannot decrease). However, this is not the case as it turns out from a proper thermodynamical analysis embracing "*all bodies taking part in the process*" ($\mathcal{Q}$ is not a closed system). Omitting technical details, it turns out that the total entropic cost $\Delta \mathcal{S}_{\text{tot}}$ for a single QEC cycle is given by [9],

$$\Delta \mathcal{S}_{\text{tot}} \stackrel{\text{def}}{=} \Delta \mathcal{S} + \mathcal{H}(p_k) = \mathcal{S}(\rho) - \mathcal{S}(\rho') + \mathcal{H}(p_k) \geq 0, \quad (19)$$

where each cycle on average involves the dissipation of $\mathcal{H}(p_k)$ ($\mathcal{H}$ denotes Shannon's entropy function) bits of entropy into the environment when the measurement record k is erased. Eq. (19) implies that exact-QEC does not violate the Second Law because the reduction in the system's entropy ($\Delta \mathcal{S} < 0$) occurs at the expense of an increase in the entropy of the environment ($\mathcal{H}(p_k) \geq 0$) due to the erasure of the demon's measurement record (Landauer's erasure principle, [10]).

How does the thermodynamics of a QEC cycle change when, for instance, the observation (measurement) is not perfect and the information gain is sub-optimal? How does imperfect discrimination of non-orthogonal quantum states affect the entropic analysis of a QEC cycle? An illustrative example may render the idea. Following [3], assume that only two errors A_k with $k = 0, 1$ (operators that characterize the interaction between a quantum system and its environment) need to be corrected and that the imperfect measurement is characterized by two non-orthogonal quantum states of the apparatus given by $|m_1\rangle \stackrel{\text{def}}{=} |0\rangle$ and $|m_2\rangle \stackrel{\text{def}}{=} \xi \sqrt{\delta} |0\rangle + \sqrt{1 - \xi^2 \delta} |1\rangle$ where $\langle m_1 | m_2 \rangle = \xi \sqrt{\delta}$ with $0 \leq \delta \leq 1$ and $0 \leq \xi \leq \delta^{-\frac{1}{2}}$. The negativity of the rate of change of the von Neumann erasure entropy with respect to the quantum overlap $\langle m_1 | m_2 \rangle$,

$$\frac{\partial \mathcal{S} \left(\frac{|m_1\rangle\langle m_1| + |m_2\rangle\langle m_2|}{\text{Tr}(|m_1\rangle\langle m_1| + |m_2\rangle\langle m_2|)} \right)}{\partial (\langle m_1 | m_2 \rangle)} \propto \langle m_1 | m_2 \rangle \log \left(\frac{1 - \langle m_1 | m_2 \rangle}{1 + \langle m_1 | m_2 \rangle} \right) \leq 0, \quad (20)$$

leads to the conclusion that the bigger is the quantum overlap of non-orthogonal quantum apparatus states, the smaller is the erasure entropy and, because of Landauer's principle, the smaller is the quantum information gain.

CONCLUDING COMMENTS

In this Proceeding, our main effort was focused at clarifying the role played by the process of quantum measurement (which cannot perfectly discriminate among non-orthogonal states) in the entropic analysis of an approximate-QEC cycle (sub-optimal information gain). We have provided semi-quantitative reasoning for explaining why the square modulus of the overlap of non-orthogonal quantum states is the essential quantity that limits the effectiveness of discrimination between quantum states when no inconclusive measurement outcome is permitted. Finally, using this point, we have stressed the link among perfect (imperfect) discrimination, optimal (sub-optimal) quantum information gain and exact (approximate)-QEC.

We will deepen our formal understanding and strengthen our quantitative analysis concerning these information-theoretic links in forthcoming efforts. Specifically, we would like to recast both exact and approximate-QEC schemes in a state discrimination formalism for (stabilizer) mixed quantum states represented by density operators (since the entities to be discriminated in QEC are actually subspaces rather than pure quantum states) and, hopefully, quantify analogies and differences between the two schemes in quantum (relative) entropic terms.

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